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MIXED LATTICE GROUPS

LICENTIATE OF SCIENCE THESIS

Abstract

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A mixed lattice group is a generalization of a lattice ordered group. The theory of mixed lattice semigroups dates back to the 1970s, but the corresponding theory for groups has been relatively unexplored. In this thesis we investigate the basic structure of mixed lattice groups, and study how some of the fundamental concepts in Riesz spaces and lattice ordered groups, such as the absolute value and other related ideas, can be extended to mixed lattice groups. We give a fundamental classification of mixed lattice groups based on their order properties. We define the generalized absolute values and derive various related identities and inequalities. We then introduce the concept of an ideal in a mixed lattice group and prove some basic results related to them. Using these ideas, we begin a study of homomorphisms in mixed lattice groups and their elementary properties. In this connection, we also investigate the quotient group construction, and show that under certain conditions, the quotient group will also be a mixed lattice group. Finally, we briefly consider topologies in mixed lattice groups and give a few sufficient conditions for a group topology to be compatible with the mixed lattice structure.

Contents

Chapter 1

Introduction

1.1 Historical background

The theory of mixed lattice semigroups was developed by Maynard Arsove and Heinz Leutwiler, mainly during the 1970s, in connection with an axiomatic treatment of potential theory. Their goal was to find a suitable mathematical framework in which the essential concepts and ideas of classical and axiomatic potential theory could be formulated in a purely algebraic way. They published their theory in a series of research articles ([4], [5], [6]) and a monograph [7], in which the authors presented the algebraic theory of mixed lattice semigroups. Their work resulted in a unified theory which generalized some of the earlier axiomatizations of potential theory, such as the theory of cones of potentials, developed by Mokobodzki and Sibony (see [7] and references therein), and the theory of H-cones developed by the Romanian group of mathematicians, Boboc, Bucur and Cornea [9]. These theories can be regarded as special cases of the Arsove–Leutwiler theory.

The main difference between these earlier developments and the Arsove–Leutwiler theory is that the former are formulated in Riesz spaces, or vector lattices, while the latter theory is built around a different type of algebraic structure, called a mixed lattice semigroup, which is a positive additive semigroup with two distinct partial orderings. The usual symmetrical upper and lower envelopes (that is, the supremum and the infimum) of Riesz space theory are replaced by unsymmetrical "mixed" envelopes which are formed with respect to the two partial orderings. If the two partial orderings are identical, then the mixed envelopes become the ordinary supremum and infimum, and the structure is thus reduced to a lattice ordered semigroup. In this sense, the mixed lattice semigroup can be viewed as a generalization of a lattice ordered semigroup.

A mixed lattice semigroup is, by definition, a positive partially ordered semigroup. Obviously, the positivity condition does not hold in a group setting; this is in contrast with the general theory of groups, where every group is also a semigroup. Therefore, as the basic theory of mixed lattice semigroups was fully developed in [7], it would seem natural to consider the mixed lattice structure also in groups or even in vector spaces. Interestingly, Arsove and Leutwiler did not extend their study in this direction. In fact, they only briefly mentioned this possibility in [7], while adding that "there is no assurance that the mixed lattice structure can be carried over to this group [of formal differences]". It wasn't until 1991, when Eriksson–Bique studied group extensions of mixed lattice semigroups in [11], and showed that most of the basic properties of a mixed lattice semigroup are indeed preserved in such extensions. Later, in 1999, the same author gave a more general definition of a mixed lattice group in [12], and discussed their fundamental structure and algebraic properties. However, after this the research on the subject has been virtually nonexistent, and consequently, the theory of mixed lattice groups remains relatively unexplored. The purpose of this thesis is to fill this gap, and initiate a systematic study of mixed lattice groups.

1.2 Aims and contributions of this research

The main theme of this work is to study the similarities between mixed lattice groups and lattice ordered groups, and to examine which aspects of the theory of lattice ordered groups can be generalized to the mixed lattice group setting. Many of the results in this thesis are generalizations of the corresponding results in Riesz spaces, and in many cases, the methods of proof are also similar. There are, however, some significant differences too. The main difference – and also the main difficulty in generalizing certain aspects of Riesz space theory in mixed lattice groups – is the fact that the mixed envelopes are not commutative, associative or distributive, unlike supremum and infimum in Riesz spaces and lattice ordered groups.

The most important contributions of this research are the generalized absolute values, and the classification of mixed lattice groups into regular, almost regular and pre-regular cases. With this classification we can isolate those properties that are essential assumptions for various results presented in this work. The most fundamental definition based on the generalized absolute values is the notion of an ideal. Almost all the new theory we have developed here rests on these concepts.

Even though the concept of a mixed lattice group has its roots in potential theory, it is also an interesting mathematical structure in its own right, and one can study its different properties without actually referring to its potential theoretic origins. This is exactly the approach we have chosen in this work. In other words, this work is not about potential theory, neither do we attempt to give any potential theoretic interpretations to our results. This is a study about the structure and properties of mixed lattice groups, and although we give many concrete examples to illustrate our results, the examples are not related to potential theory.

What follows is a brief survey of the contents of the thesis.

Chapter 2 is a short introduction to the basic definitions and results in theory of Riesz spaces and lattice ordered groups. In later chapters we will often draw comparisons between the definitions and results of this introductory chapter and our new theory.

The concept of a mixed lattice semigroup is introduced in Chapter 3. Here we will not attempt to give a detailed account of the theory of mixed lattice semigroups. We will just introduce the fundamental ideas that will be needed in the sequel. Arsove and Leutwiler have studied the theory of mixed lattice semigroups in much greater depth in their monograph [7]. We have, however, given most of the proofs here – and also in Chapter 4 – for two reasons. Firstly, we wanted to illustrate the basic ideas and techniques of the theory in detail. In many cases, we have added details that are missing from the original proofs found in the references. Secondly, some of the material presented here is not very easily available and we wanted to make the presentation of the essential theory as self-contained as possible.

Chapter 4 presents the definition and the basic theory of mixed lattice groups, developed mainly by Eriksson–Bique in [11] and [12]. Section 4.1 introduces the general properties of mixed lattice groups, while section 4.2 discusses the group extensions of mixed lattice semigroups. These are special cases of mixed lattice groups which enjoy some particularly nice properties.

Most of the new results of this research are presented in Chapters 5 to 7. All the definitions, results and examples in these chapters are new and original to this work, unless otherwise stated. In Chapter 5 we introduce the notions of regular, almost regular and pre-regular mixed lattice groups, which turn out to be important for the further development of the theory. We also present several examples and counterexamples related to these concepts. The absolute value of an element is a fundamental concept in Riesz space theory. In section 5.2 we will introduce the generalized absolute values, which is the corresponding idea in mixed lattice groups, and study their basic properties. Section 5.4 investigates the notion of an ideal in the mixed lattice group setting. In Riesz spaces, ideals are subspaces that have certain order-related properties, and they play a fundamental role in the theory concerning the structure of Riesz spaces. In mixed lattice groups we define analogous concepts accordingly, as subgroups with certain order properties. The definitions are based on the concept of the generalized absolute value. The ideas of this section will be important for the theory presented in the following sections.

The basic theory of mixed lattice group homomorphisms is developed in Chapter 6. We will show that mixed lattice homomorphisms have many properties that are similar to the lattice homomorphisms in Riesz spaces. Homomorphisms, or structure preserving maps, are important in many different areas of mathematics, especially in connection with different kinds of representation theorems. In theory of Riesz spaces the representation theorems (such as those by Kakutani, Krein and Nakano, see [2], Theorems 4.27 and 4.28) allow us to represent certain types of Riesz spaces as concrete function spaces. Such representations have become a very powerful tool in the further development of the theory, and they have some rather far reaching consequences. Similar representation results for mixed lattice groups haven't been discovered yet, but it is reasonable to expect that homomorphisms will play a crucial role if such theorems are to be found. In section 6.2 we apply the results of the preceding sections to study the quotient group construction in mixed lattice groups. There is a close connection between mixed lattice homomorphisms, their kernels and quotient mixed lattice groups. In Riesz space theory, quotient spaces – just as lattice homomorphisms – are important for certain theorems concerning the structural properties of Riesz spaces. As further examples we could mention the Dedekind completion of certain Archimedean Riesz spaces ([1], Theorem 2.18), and also the proof of the fundamental Dodds–Fremlin domination theorem for compact operators in Banach lattices incorporates the idea of the quotient space construction ([10]), to mention just a few of the many applications.

Finally, in Chapter 7 we explore the idea of a topological mixed lattice group. Section 7.1 presents the basic facts from the theory of topological groups. Then in Section 7.2, using topological Riesz spaces as our model, we give conditions for the topology under which the mixed lattice operations will be continuous. There are different possibilities for defining a compatible topology and we study some of them. The aforementioned representation theorems in Riesz spaces usually require a suitable norm, which introduces additional topological structure. Of course, introduction of a topology will open doors to many other possibilities as well, such as duality theory, which has been one of the main motivations for studying topologies in groups and vector spaces.

We are really only scratching the surface here, but it is hoped that this work would serve as a good starting point for further research on the subject.

Chapter 2

Riesz spaces and lattice ordered groups

The purpose of this work is to study the similarities between Riesz spaces and mixed lattice groups, and to investigate to what extent the basic theory of Riesz spaces can be extended to the more general structure of mixed lattice groups. To compare the corresponding theories and results, we will begin by giving the basic definitions and theorems of Riesz spaces and lattice ordered groups. For proofs of the theorems in this chapter we refer to [14] and [15].

Definition 2.0.1. Let G be a commutative additive group and $\leq$ a partial order on G. Then G is called a *partially ordered group* (or a *po-group*), if the following condition holds for all $x, y \in G$:

(i) If $x \leq y$, then $x + z \leq y + z$ for every $z \in G$.

A *lattice ordered group* (or an *l-group*) is a partially ordered group G, which is a lattice, that is, the elements $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist for all $x, y \in G$.
A real vector space E together with a partial order $\leq$ is called an *ordered vector space*, if the following conditions hold for all $x, y \in E$:

(i) If $x \leq y$, then $x + z \leq y + z$ for every $z \in E$,

(ii) If $x \leq y$, then $ax \leq ay$ for every real number $a \geq 0$.

A *Riesz space* (or a *vector lattice*) is an ordered vector space E, which is a lattice, that is, the elements $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist for all $x, y \in E$.

We will now introduce some additional basic concepts together with various results that hold in all Riesz spaces and lattice ordered groups. The results will be formulated for Riesz spaces, but observe that none of them makes use of the scalar multiplication, and so all these results hold in lattice ordered groups as well. In most cases, the proofs are also the same.

Definition 2.0.2. Let E be a Riesz space. Define $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$. The elements x^+, x^- and $|x|$ are called the x *positive part*, the *negative part* and the *absolute value* of x, respectively. The set $E_+ = \{x \in E \ : \ x \geq 0\}$ is called the *positive cone*.

The next theorem lists some basic facts about positive and negative parts and the absolute value.

Theorem 2.0.3. *The following hold in every Riesz space:*

(a) $x^+, x^- \in E_+$, $x^+ = (-x)^-$, $x^- = (-x)^+$ *and* $|x| = |-x|$

(b) $|\,|x|\,| = |x|$, $|x| = x$ *if and only if* $x \geq 0$, *and* $|x| = 0$ *if and only if* $x = 0$

(c) $|x| = x^+ + x^- = x^+ \vee x^-$

(d) $x = x^+ - x^-$

(e) $x^+ \wedge x^- = 0$

(f) $x \vee y + x \wedge y = x + y$

(g) $x \vee y - x \wedge y = |x - y|$

(h) $2(x \vee y) = x + y + |x - y|$

(i) $(x + y)^+ \leq x^+ + y^+$, $(x + y)^- \leq x^- + y^-$ *and* $|x + y| \leq |x| + |y|$.

(j) *If $x = u - v$ and $u \wedge v = 0$, then $u = x^+$ and $v = x^-$.*

Definition 2.0.4. Let E be a Riesz space. A subspace $F \subseteq E$ is called a *Riesz subspace* if F is a Riesz space in its own right, that is, if $x \vee y \in F$ and $x \wedge y \in F$ whenever $x, y \in F$. A subset $A \subseteq E$ is called *solid*, if $x \in A$ and $|y| \leq |x|$ imply $y \in A$. A solid subspace $I \subseteq E$ is called an *ideal*.

Theorem 2.0.5. *The following hold in every Riesz space:*

(a) *A partially ordered space E is a Riesz space if and only if x^+ (or equivalently, x^-) exists for all $x \in E$.*

(b) *Every ideal in a Riesz space E is a Riesz subspace of E.*

(c) *An intersection of Riesz subspaces (respectively, ideals) is a Riesz subspace (respectively, an ideal).*

Definition 2.0.6. Let E and F be Riesz spaces. A linear mapping $T : E \to F$ is a *Riesz homomorphism* if $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$. A bijective Riesz homomorphism is called a *Riesz isomorphism*.

Riesz homomorphisms can be characterized as follows.

Theorem 2.0.7. *Let $T : E \to F$ be a linear operator between two Riesz spaces E and F. Then the following statements are equivalent.*

(a) *T is a Riesz homomorphism.*

(b) *$T(x \vee y) = Tx \vee Ty$ for all $x, y \in E$.*

(c) *$T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$.*

(d) *$T(x^-) = (Tx)^-$ for all $x \in E$.*

(e) *$T(x^+) = (Tx)^+$ for all $x \in E$.*

(f) *$T(|x|) = |Tx|$ for all $x \in E$.*

(g) *If* $x \wedge y = 0$ *in* E*, then* $Tx \wedge Ty = 0$ *holds in* F*.*

The proof of the preceding theorem in lattice ordered groups is slightly different than in Riesz spaces (for details, see [13], Theorem 4.5).

The next theorem gives some additional basic properties of Riesz homomorphisms.

Theorem 2.0.8. *Let* $T : E \to F$ *be a Riesz homomorphism. Then the following hold.*

(a) T *is increasing, that is,* $x \leq y$ *in* E *implies* $Tx \leq Ty$ *in* F*.*

(b) *The set* $T(E)$ *is a Riesz subspace of* F*.*

(c) *The kernel of* T *is an ideal in* E*.*

For Riesz isomorphisms, we have the following characterization.

Theorem 2.0.9. *Let* E *and* F *be Riesz spaces and* $T : E \to F$ *a bijective linear operator. Then* T *is a Riesz isomorphism if and only if both* T *and* T^{-1} *are increasing.*

Homomorphisms and quotient spaces are closely related, and in case of a Riesz space E, the quotient space E/I will also be a Riesz space if I is an ideal in E.

Theorem 2.0.10. *If* I *is an ideal in a Riesz space* E*, then* E/I *is a Riesz space, and the canonical projection* $x \mapsto [x]$ *is a Riesz homomorphism whose kernel is* I*.*

Chapter 3

Mixed lattice semigroups

3.1 Definitions and basic properties

We begin by stating the definitions of the basic structures. Let $(\mathcal{S}, +, \leq)$ be a positive partially ordered commutative semigroup. That is, we require that $0 \in \mathcal{S}$ and $u \geq 0$ for all $u \in \mathcal{S}$. In addition, we assume that the partial order satisfies the cancellation law

$$u \leq v \quad \iff \quad u + w \leq v + w \tag{3.1.1}$$

for all $u, v, w \in \mathcal{S}$. This partial order $\leq$ is called the *initial order*. Next we define another partial order on $\mathcal{S}$, called the *specific order*, as follows

$$u \preccurlyeq v \quad \iff \quad v = u + w \quad \text{for some } w \in \mathcal{S}. \tag{3.1.2}$$

To see that this is indeed a partial ordering, we need to check that $\preccurlyeq$ is reflexive, transitive and antisymmetric. Reflexivity is clear, since for every $u \in \mathcal{S}$ we have $u = u + 0$, and since $0 \in \mathcal{S}$, this is equivalent to $u \preccurlyeq u$ by definition (3.1.2). For transitivity, assume that $w \preccurlyeq v$ and $v \preccurlyeq u$. Then $u = v + v'$ and $v = w + w'$ for some $v', w' \in \mathcal{S}$. These imply that $u = v + v' = (w + w') + v' = w + (w' + v')$. Since $w' + v' \in \mathcal{S}$, it follows by (3.1.2) that $w \preccurlyeq u$. Finally, to prove antisymmetry, assume $u \preccurlyeq v$ and $v \preccurlyeq u$. Then $u = v + v'$ and $v = u + u'$ for some $u', v' \in \mathcal{S}$. Adding v' to the last equality gives $u = v + v' = u + u' + v'$. As $u' + v' \in \mathcal{S}$, it follows that $u' + v' = 0$. But this implies that $u' = v' = 0$. Indeed, since

$u' + v' = 0 \leq v'$, it follows by the cancellation law that $u' \leq 0$. But $u' \in \mathcal{S}$ and so $u' \geq 0$. Since $\leq$ is a partial order, it follows that $u' = 0$. Similarly we get $v' = 0$. Hence, $u = v$. This proves that $\preccurlyeq$ is a partial order.

If we equip the semigroup $\mathcal{S}$ with the partial order $\preccurlyeq$, then the set $(\mathcal{S}, \leq, \preccurlyeq)$ will be a positive partially ordered semigroup with respect to both partial orderings $\leq$ and $\preccurlyeq$ and the specific order $\preccurlyeq$ obeys the cancellation law. To see this, we first note that if $u \in \mathcal{S}$ then $u = 0 + u$, and by (3.1.2) this implies $u \succcurlyeq 0$. Thus, $\mathcal{S}$ is positive with respect to $\preccurlyeq$. Moreover, if $u \preccurlyeq v$ then $v = u + u'$ for some $u' \in \mathcal{S}$. It follows that $v + w = u + w + u'$ for any element $w \in \mathcal{S}$, and so $u + w \preccurlyeq v + w$. Conversely, if $u + w \preccurlyeq v + w$, then $v + w = u + w + u'$ for some $u' \in \mathcal{S}$. Since equality is just a special case of the relation $\leq$, it follows by the cancellation law (3.1.1) that $v = u + u'$, or equivalently, $u \preccurlyeq v$. Hence, the cancellation law holds for $\preccurlyeq$ as well.

With these two partial orders $\leq$ and $\preccurlyeq$ we define the *mixed lower envelope*

$$u \curlywedge v \;=\; \max\{\, w \in \mathcal{S} :\ w \preccurlyeq u \ \text{ and }\ w \leq v \,\}$$

and the *mixed upper envelope*

$$u \curlyvee v \;=\; \min\{\, w \in \mathcal{S} :\ w \succcurlyeq u \ \text{ and }\ w \geq v \,\},$$

where the minimum and maximum are taken with respect to the initial order $\leq$. The following definition was given by Arsove and Leutwiler in [7].

Definition 3.1.1. Let $(\mathcal{S}, +, \leq, \preccurlyeq)$ be a positive partially ordered commutative semigroup with two partial orders $\leq$ and $\preccurlyeq$ as defined above. If the mixed upper and lower envelopes $u \curlyvee v$ and $u \curlywedge v$ exist for all $u, v \in \mathcal{S}$, and they satisfy the identity

$$u \curlyvee v \,+\, v \curlywedge u \;=\; u + v$$

then $(\mathcal{S}, +, \leq, \preccurlyeq)$ is called a *mixed lattice semigroup*.

We immediately observe the similarities between the above definition and the definition of a lattice ordered group given in Chapter 2. The last identity in the definition has its counterpart in Riesz spaces and lattice ordered groups, where it is not included in the definitions, but is a consequence of them (see Theorem 2.0.3 (e)).

Next we will present the most important basic properties of mixed lattice semigroups and show how to derive them from the definitions.

Theorem 3.1.2. *Let $\mathcal{S}$ be a mixed lattice semigroup. Then the following hold.*

(a) *The inequality $u \preccurlyeq v$ implies $u \leq v$.*

(b) *$u \preccurlyeq v$ if and only if there exists a unique difference element $v - u \in \mathcal{S}$ satisfying $u + (v - u) = v$.*

(c) $u \curlywedge v \preccurlyeq u \preccurlyeq u \curlyvee v$ *and* $u \curlywedge v \leq v \leq u \curlyvee v$ *for all* $u, v \in \mathcal{S}$.

(d) $x \preccurlyeq u$ *and* $y \leq v \implies x \curlyvee y \leq u \curlyvee v$ *and* $x \curlywedge y \leq u \curlywedge v$.

(e) $u \preccurlyeq w$ *and* $v \leq w \implies u \curlyvee v \leq w$ *and* $u \curlywedge v \leq w$.

(f) $u \leq v \iff u \curlywedge v = u \iff v \curlyvee u = v$.

(g) $v \preccurlyeq u \iff u \curlywedge v = v \iff v \curlyvee u = u$.

Proof. (a) If $u \preccurlyeq v$ then $v = u + u'$ for some $u' \in \mathcal{S}$. Since $w \geq 0$ for all $w \in \mathcal{S}$, we have $u' \geq 0$, and it follows that $v = u + u' \geq u + 0 = u$.

(b) Assume that $u \preccurlyeq v$. Then $v = u + w$ for some $w \in \mathcal{S}$. Assume there is some other element $w' \in \mathcal{S}$ such that $v = u + w'$. Then, since $v \leq v$, we have $u + w \leq u + w'$ and $u + w' \leq u + w$, or equivalently, $w \leq w'$ and $w' \leq w$. Hence $w = w'$ is the unique difference element. The converse implication follows immediately from the definition of specific order.

(c) These follow immediately from the definitions of mixed envelopes.

(d) Assume that $x \preccurlyeq u$ and $y \leq v$ and denote $A = \{w \in \mathcal{S} : w \succcurlyeq x \text{ and } w \geq y\}$ and $B = \{w \in \mathcal{S} : w \succcurlyeq u \succcurlyeq x \text{ and } w \geq v \geq y\}$. Then certainly $B \subset A$, and hence $\min A \leq \min B$, that is, $x \curlyvee y \leq u \curlyvee v$. Similar arguments show that $x \curlywedge y \leq u \curlywedge v$.

(e) Observe that $w \curlywedge w = w \curlyvee w = w$ for all $w \in \mathcal{S}$. Hence, this is just a special case of (d).

(f) Assume first that $u \leq v$. Since $v \preccurlyeq v$, by (d) we have $v \curlyvee u \leq v \curlyvee v = v$, and so by Definition 3.1.1 we get $u+v = u \curlywedge v + v \curlyvee u \leq u \curlywedge v + v$. This implies $u \leq u \curlywedge v$. On the other hand, by (c) we have $u \succcurlyeq u \curlywedge v$, and so by (a) we have $u \geq u \curlywedge v$. Hence $u = u \curlywedge v$. Conversely, if $u = u \curlywedge v$ then $u \leq v$ by (c). For the other equivalence, note that if $u \curlywedge v = u$ then the identity $u + v = u \curlywedge v + v \curlyvee u$ implies $v \curlyvee u = v$. The reverse implication is proved similarly.

(g) This is similar to (f). The only minor difference is in the proof of the first implication, where we don't have to use the property given in (a). This small difference will be important later, so we will give the details. Assume that $v \preccurlyeq u$. Since $u \leq u$, by (d) we have $v \curlyvee u \leq u \curlyvee u = u$, and so by definition 3.1.1 we get $u+v = u \curlywedge v+v \curlyvee u \leq u \curlywedge v + u$. This implies $v \leq u \curlywedge v$. On the other hand, by (c) we have $u \curlywedge v \leq v$. Hence $v = u \curlywedge v$.

■

There are several different conditions which are equivalent to the definition of a mixed lattice semigroup. We will not need them all in this work, but the next theorem presents those which will be useful later. The other equivalent conditions not given here are discussed in [7] and [12]. The proof of the next theorem is a combination of ideas from [7] and [12].

Theorem 3.1.3. *Let $(\mathcal{S}, +, \leq, \preccurlyeq)$ be a positive partially ordered commutative semigroup with respect to both partial orderings $\leq$ and $\preccurlyeq$. The following statements are equivalent.*

(a) $(\mathcal{S}, +, \leq, \preccurlyeq)$ is a mixed lattice semigroup.

(b) The element $m = \min\{w \in \mathcal{S} : v \leq w+u\}$ exists for all $u, v \in \mathcal{S}$ and satisfies $m \preccurlyeq v$.

(c) The mixed upper envelope $u \curlyvee v$ exists for all $u, v \in \mathcal{S}$ and it satisfies $u \curlyvee v \preccurlyeq u + v$.

Proof. (a) $\Longrightarrow$ (b) Since $\mathcal{S}$ is a mixed lattice semigroup, the element $u \curlyvee v$ exists and $u \curlyvee v \geq v$. It follows that the element $m = u \curlyvee v - u$ satisfies the condition $v \leq m + u$. Assume that $w \in \mathcal{S}$ is another element such that $v \leq w + u$. Then, since $u \preccurlyeq u + w$, we have $w + u \geq u \curlyvee v$. This implies that $w \geq u \curlyvee v - u = m$, and so we have proved that $m = \min\{w \in \mathcal{S} : v \leq w + u\}$. By hypothesis, the element $v \curlywedge u$ also exists in $\mathcal{S}$ and the identity $u \curlyvee v + v \curlywedge u = u + v$ holds. It now follows that $m = u \curlyvee v - u = v - v \curlywedge u$, which implies that $v = m + v \curlywedge u$. Since $v \curlywedge u \in \mathcal{S}$, this means that $m \preccurlyeq v$ by the definition of specific order.
(b) $\Longrightarrow$ (c) Let $u, v \in \mathcal{S}$. Assume that $m = \min\{w \in \mathcal{S} : v \leq w + u\}$ exists in $\mathcal{S}$ and $m \preccurlyeq v$. Then certainly $v \leq m + u$, and the definition of specific order implies $u \preccurlyeq u + m$. Let $w \in \mathcal{S}$ be any other element such that $w \succcurlyeq u$ and $w \geq v$. Then $w = u + u'$ for some $u' \in \mathcal{S}$, and so $v \leq w = u + u'$. Therefore, we have $m \leq u'$ and so $w = u + u' \geq u + m$. This proves that $m + u = u \curlyvee v$. By assumption we have $m \preccurlyeq v$, that is, $m = u \curlyvee v - u \preccurlyeq v$. It follows that $u \curlyvee v \preccurlyeq u + v$.
(c) $\Longrightarrow$ (a) First, let $u, v \in \mathcal{S}$ be fixed elements, and note that if w and w' are elements of $\mathcal{S}$ such that $w + w' = u + v$, then the condition $w \preccurlyeq u$ implies $u + w' \succcurlyeq w + w' = u + v$, which implies $w' \succcurlyeq v$. Conversely, $w' \succcurlyeq v$ implies $u + w' \succcurlyeq u + v = w + w'$, or $w \preccurlyeq u$. Hence $w \preccurlyeq u$ if and only if $w' \succcurlyeq v$. Similarly, $w \leq v$ if and only if $w' \geq u$. (Note that these kind of elements always exist, for we can choose $w = 0$ and $w' = u + v$.) Thus, if we make w larger, then we must make w' smaller for the equality $w + w' = u + v$ to hold. Assume now that $v \curlyvee u$ exists and satisfies $v \curlyvee u \preccurlyeq u + v$. This means that $u + v = v \curlyvee u + m$ for some $m \in \mathcal{S}$, and since $v \curlyvee u \succcurlyeq v$ and $v \curlyvee u \geq u$, the above discussion shows that $m \preccurlyeq u$ and $m \leq v$. But by definition $v \curlyvee u$ is the smallest of all those elements $w' \in \mathcal{S}$ that satisfy $w' \succcurlyeq v$ and $w' \geq u$. Hence, the element m must be the largest of all those elements $w \in \mathcal{S}$ that satisfy $w \preccurlyeq u$ and $w \leq v$, that is, $m = u \curlywedge v$. This shows that $\mathcal{S}$ is a mixed lattice semigroup. ∎

With Theorem 3.1.3 we can prove the next result, which gives an important translation invariance property of the mixed envelopes. The proof given here is from [12].

Theorem 3.1.4. *Let $\mathcal{S}$ be a mixed lattice semigroup and $u, v, a \in \mathcal{S}$. Then*

$$(u+a) \curlyvee (v+a) = u \curlyvee v + a$$

and

$$(u+a) \curlywedge (v+a) = u \curlywedge v + a.$$

Proof. By Theorem 3.1.3 the existence of the mixed upper envelope $u \curlyvee v$ is equivalent to the existence of the element $m = \min\{w \in \mathcal{S} : v \leq w + u\}$, and in the proof of Theorem 3.1.3 it was shown that $m = u \curlyvee v - u$. Since $v \leq w + u$ if and only if $v + a \leq w + u + a$, it follows that $m = \min\{w \in \mathcal{S} : v \leq w + u\} = \min\{w \in \mathcal{S} : v + a \leq w + u + a\}$ and $m = u \curlyvee v - u = (u+a) \curlyvee (v+a) - (u+a)$. This implies that $(u+a) \curlyvee (v+a) = u \curlyvee v + a$. To prove the second identity, we note that

$$(u+a) \curlywedge (v+a) + (v+a) \curlyvee (u+a) = u + v + a + a.$$

From the first identity we substitute $(v+a) \curlyvee (u+a) = v \curlyvee u + a$ on the left hand side, and on the right hand side we substitute the identity $u + v = v \curlyvee u + u \curlywedge v$ to get

$$(u+a) \curlywedge (v+a) + v \curlyvee u + a \ = \ v \curlyvee u + u \curlywedge v + a + a.$$

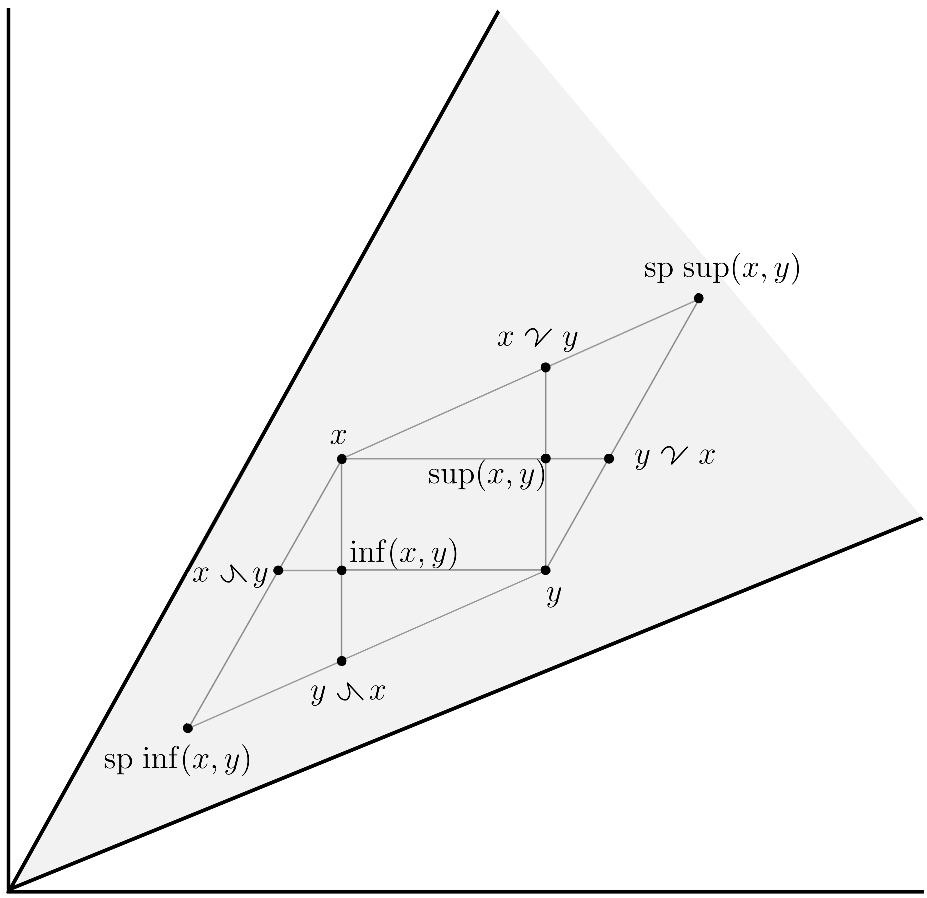
The second identity now follows from this by cancelling the term $v \curlyvee u + a$. ∎

3.2 Examples of mixed lattice semigroups

We will present some examples of mixed lattice semigroups. The first one is due to Arsove and Leutwiler ([7], pp. 8–9).

Example 3.2.1. Our first example is a simple geometric example which helps to visualize the mixed envelopes. The setting is the plane $\mathbb{R}^2$, where the initial order is defined as the partial order with the positive cone $\{(u, v) : u \geq 0 \text{ and } v \geq 0\}$. Geometrically, this is just the region bounded by the positive coordinate axes. The specific order is determined by another cone, which is the shaded region in the figure below. The mixed lattice semigroup is thus the shaded region, that is, the set $\{x : x \succcurlyeq 0\}$.

Here, and also in the sequel, we denote the supremum and infimum formed with respect to initial order by $\sup$ and $\inf$, respectively (whenever they exist), and the supremum and infimum formed with respect to specific order by $\mathrm{sp}\sup$ and $\mathrm{sp}\inf$, respectively (whenever they exist). In the figure, we have two elements x and y and the related elements:



It is useful to keep this geometric picture in mind when we discuss examples of mixed lattice groups in Section 5.1. ▲

In [7], the authors gave several examples of mixed lattice semigroups, such as the semigroup of all nonnegative superharmonic functions on some region, and the semigroup of all nonnegative concave real functions on some interval. Another interesting example of a mixed lattice semigroup is the set of all positive nondecreasing real functions on some interval. We close this section with a detailed discussion of this example, which is also due to Arsove and Leutwiler ([7], Theorem 21.2).

Example 3.2.2. Let $\mathcal{S}$ be the set of all positive increasing real functions on interval $[a, b]$ with the usual pointwise addition. We define initial order in $\mathcal{S}$ by

$$f \leq g \qquad \Longleftrightarrow \qquad f(x) \leq g(x) \quad \text{for all } x \in [a, b]$$

and specific order by

$$f \preccurlyeq g \quad \Longleftrightarrow \quad f(x) \leq g(x) \ \text{ for all } x \in [a, b] \ \text{ and } \ g - f \ \text{ is increasing on } [a, b].$$

We now claim that $(\mathcal{S}, \leq, \preccurlyeq)$ is a mixed lattice semigroup, and for any f and g in $\mathcal{S}$, the mixed lower and upper envelopes are given by

$$(f \curlywedge g)(u) = \inf \{f(u) - (f(x) - g(x))^{+} \, : \, x \in [a, u]\}$$

and

$$(f \curlyvee g)(u) = \sup \{f(u) + (g(x) - f(x))^{+} \, : \, x \in [a, u]\},$$

where $c^{+} = \max\{0, c\}$ is the positive part of the real number c.

To prove this claim, we first note that $f \geq 0$ for all $f \in \mathcal{S}$ and obviously the condition $f \leq g \iff f + h \leq g + h$ holds for all $f, g, h \in \mathcal{S}$. Also, it is evident that $f \preccurlyeq g \iff g = f + (g - f)$, where $g - f \in \mathcal{S}$, and so $\preccurlyeq$ is indeed a specific order in $\mathcal{S}$.

Next we need to show that the expressions for $f \curlywedge g$ and $f \curlyvee g$ actually are the lower and upper envelopes of f and g in $\mathcal{S}$. It is easy to see that they are both nonnegative. In fact, since $(g(x) - f(x))^{+} \geq 0$ for all $x \in [a, u]$, it follows that $(f \curlyvee g)(u) \geq f(u) \geq 0$ for all $u \in [a, b]$. Moreover, since f and g are positive, we have $(f(x) - g(x))^{+} \leq f(x)$ for all $x \in [a, u]$, and since f is increasing and $x \leq u$ we have

$$f(u) - (f(x) - g(x))^{+} \geq f(u) - f(x) \geq 0.$$

Hence $(f \curlywedge g)(u) \geq 0$ for all $u \in [a,b]$. In addition, $f \curlyvee g$ and $f \curlywedge g$ are increasing. To see that $f \curlyvee g$ is increasing, let $a \leq u \leq v \leq b$ and note that

$$\begin{aligned}(f \curlyvee g)(u) &= \sup\{f(u) + (g(x) - f(x))^+ \,:\, x \in [a,u]\} \\ &\leq \sup\{f(u) + (g(x) - f(x))^+ \,:\, x \in [a,v]\} \\ &= (f \curlyvee g)(v),\end{aligned}$$

since the supremum is taken over a larger set in the latter expression. To show that $f \curlywedge g$ is also increasing, let $a \leq u \leq v \leq b$ and choose any $x \in [a,b]$ such that $x \leq v$. Then we have either $x \leq u$ or $u \leq x \leq v$. If $x \leq u$ then, since f is increasing, we have $f(u) \leq f(v)$ and so

$$(f \curlywedge g)(u) \leq f(u) - (f(x) - g(x))^+ \leq f(v) - (f(x) - g(x))^+.$$

Assume then that $u \leq x \leq v$ and consider the function $h(x) = f(x) - (f(x) - g(x))^+$. If $f(x) \geq g(x)$ then $(f(x) - g(x))^+ = f(x) - g(x)$ and $h(x) = g(x)$. On the other hand, if $f(x) < g(x)$ then $(f(x) - g(x))^+ = 0$ and we have $h(x) = f(x)$. Thus, at every point h equals either f or g, which are both increasing functions. Therefore, h is increasing and we have

$$\begin{aligned}(f \curlywedge g)(u) &= \inf\{f(u) - (f(y) - g(y))^+ \,:\, y \in [a,u]\} \\ &\leq f(u) - (f(u) - g(u))^+ \\ &\leq f(x) - (f(x) - g(x))^+ \\ &\leq f(v) - (f(x) - g(x))^+.\end{aligned}$$

Hence, in either case we have $(f \curlywedge g)(u) \leq (f \curlywedge g)(v)$ and so $f \curlywedge g$ is increasing.

We still need to show that the expressions for $f \curlywedge g$ and $f \curlyvee g$ actually give the desired maximum and minimum elements, as required in the definition of mixed envelopes. First we observe as above, that if $u \in [a,b]$ and $f(u) \geq g(u)$ then $(f(u) - g(u))^+ = f(u) - g(u)$ and we have

$$(f \curlywedge g)(u) \leq f(u) - (f(u) - g(u))^+ = f(u) - (f(u) - g(u)) = g(u).$$

If $f(u) < g(u)$ then $(f(u) - g(u))^+ = 0$ and we have

$$(f \curlywedge g)(u) \leq f(u) - (f(u) - g(u))^+ = f(u) - 0 = f(u) \leq g(u),$$

and so $f \curlywedge g \leq g$. Similar arguments show that $f \curlyvee g \geq g$. Next we want to show that $f \curlywedge g \preccurlyeq f$. For this, let $a \leq u \leq v \leq b$. Then, since $[a,u] \subset [a,v]$, we have

$$\inf \{-(f(x)-g(x))^+ \,:\, x \in [a,v]\} \;\leq\; \inf \{-(f(x)-g(x))^+ \,:\, x \in [a,u]\}$$

and it follows that

$$\begin{aligned}
& (f \curlywedge g)(v) - (f \curlywedge g)(u) \\
= \;& \inf_{x\in[a,v]} \{f(v) - (f(x)-g(x))^+\} - \inf_{x\in[a,u]} \{f(u) - (f(x)-g(x))^+\} \\
= \;& f(v) - f(u) + \inf_{x\in[a,v]} \{-(f(x)-g(x))^+\} - \inf_{x\in[a,u]} \{-(f(x)-g(x))^+\} \\
\leq \;& f(v) - f(u).
\end{aligned}$$

But this means that on any subinterval $[u,v] \subset [a,b]$ the value of f increases more than the value of $f \curlywedge g$, and hence $f \curlywedge g \preccurlyeq f$. By similar arguments we can establish that $f \curlyvee g \succcurlyeq f$. Assume then that h is any function in $\mathcal{S}$ such that $h \preccurlyeq f$ and $h \leq g$. If $a \leq x \leq u$ then $f(x) - g(x) \leq f(x) - h(x)$. Now $h \preccurlyeq f$, or $f - h$ is increasing, and so it follows that $f(x) - h(x) \leq f(u) - h(u)$. Therefore we have

$$f(x) - g(x) \leq f(x) - h(x) \leq f(u) - h(u)$$

and this implies that $h(u) \leq f(u) - (f(x) - g(x))$. Now if $f(x) \geq g(x)$ then we have $h(u) \leq f(u) - (f(x)-g(x)) = f(u) - (f(x)-g(x))^+$. If $f(x) < g(x)$ then $(f(x) - g(x))^+ = 0$. But $h \preccurlyeq f$ and by Theorem 3.1.2 (a) this implies that $h \leq f$ and so we have $h(u) \leq f(u) = f(u) + 0 = f(u) - (f(x)-g(x))^+$. In either case, the inequality $h(u) \leq f(u) - (f(x)-g(x))^+$ holds and so $h \leq f \curlywedge g$. This proves that $f \curlywedge g$ indeed gives the required maximum element, that is

$$f \curlywedge g = \max \{h \in \mathcal{S} : h \preccurlyeq f \;\text{ and }\; h \leq g\}$$

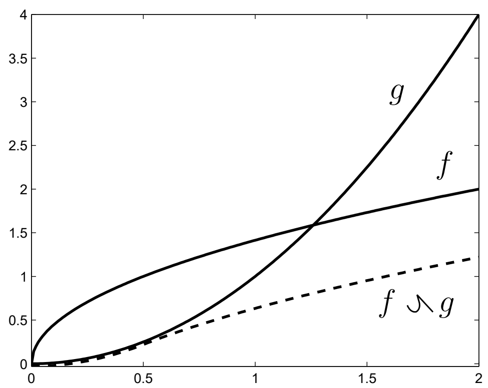
and so it is the mixed lower envelope of f and g. A similar argument shows that $f \curlyvee g$ is the mixed upper envelope of f and g. It only remains to verify that $f \curlywedge g + g \curlyvee f = f + g$.

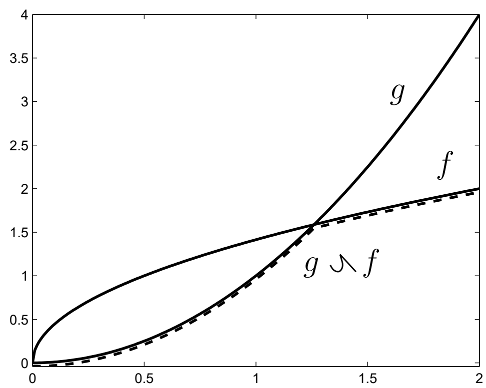
This holds, since

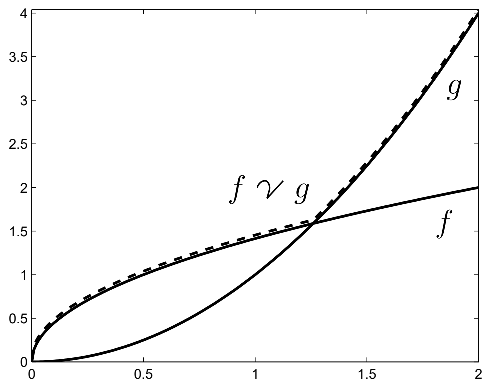
$$
\begin{aligned}
& (f \curlywedge g)(u) + (g \curlyvee f)(u) \\
= \; & \inf_{x\in[a,u]} \{f(u) - (f(x) - g(x))^+\} + \sup_{x\in[a,u]} \{g(u) + (f(x) - g(x))^+\} \\
= \; & f(u) + g(u) + \inf_{x\in[a,u]} \{-(f(x) - g(x))^+\} + \sup_{x\in[a,u]} \{(f(x) - g(x))^+\} \\
= \; & f(u) + g(u)
\end{aligned}
$$

for all $u \in [a, b]$ (here we recall the fact that $\sup A = -\inf(-A)$ for every set A for which the supremum exists). We have now shown that every condition of Definition 3.1.1 is satisfied, and hence $\mathcal{S}$ is a mixed lattice semigroup.

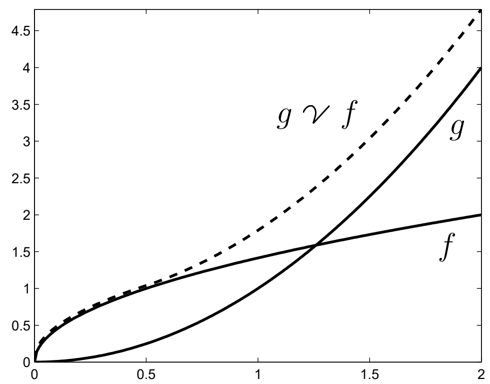
The following figures illustrate the mixed envelopes of two functions. The functions $f(x) = \sqrt{2x}$ and $g(x) = x^2$ are plotted with solid lines and their mixed envelopes with dashed lines.







$f \curlyvee g$
g
f



$g \curlyvee f$
g
f

▲

Chapter 4

Mixed lattice groups

In this chapter we extend the study of mixed lattice structure to groups. The main difference is that now we have also negative elements. As a result, some of the proofs become easier while the structure itself becomes richer and we can consider certain features, such as the absolute value, which are not present in mixed lattice semigroups.

4.1 General properties

The following definition of a mixed lattice group was given in [12].

Definition 4.1.1. Let $(\mathcal{G}, +, \leq, \preccurlyeq)$ be a partially ordered commutative group with respect to two partial orders $\leq$ and $\preccurlyeq$ that satisfy the cancellation law (3.1.1). If the mixed lower envelope $x \curlywedge y$ (or, equivalently, the mixed upper envelope $x \curlyvee y$) exists for all $x, y \in \mathcal{G}$, then $(\mathcal{G}, +, \leq, \preccurlyeq)$ is called a *mixed lattice group*.

We note that the above definition is not as restrictive as the definition of a mixed lattice semigroup. Most importantly, the two partial orders are not required to be related in any way. In particular, $x \preccurlyeq y$ does not necessarily imply $x \leq y$. It should also be emphasized that a mixed lattice group $\mathcal{G}$ is not a mixed lattice semigroup, except in the trivial case $\mathcal{G} = \{0\}$. Definition 4.1.1 also clearly shows that a lattice ordered group is just a special case of a mixed lattice group. Indeed, if the two partial orderings $\leq$ and $\preccurlyeq$ coincide, then the mixed lattice group reduces to an ordinary lattice ordered group.

In this section it will be shown that almost all of the properties of mixed lattice semigroups are carried over to the group setting. Note that in the definition of mixed lattice group we only require the existence of the mixed lower (or upper) envelope. The existence of the mixed upper (resp. lower) envelope follows from this due to the presence of negative elements.

Theorem 4.1.2. *Let $\mathcal{G}$ be a mixed lattice group. Then the mixed upper envelope $x \curlyvee y$ exists and the following identity holds*

$$x \curlyvee y = -(-x \curlywedge -y)$$

for all $x, y \in \mathcal{G}$.

Proof. The given identity follows immediately from the definitions of mixed envelopes and the fact that $\max A = -\min(-A)$ for every set A for which the maximum exists. Hence, by the identity, the existence of the mixed lower envelope implies the existence of the mixed upper envelope. ■

In a mixed lattice group the mixed envelopes are defined the same way as in mixed lattice semigroups. Consequently, those calculation rules that follow immediately from the definitions will be valid also in every mixed lattice group, and the proofs are exactly the same as in the semigroup case (see Theorem 3.1.2 (c), (d), (e) and (g)). Thus we have:

Theorem 4.1.3. *Let $\mathcal{G}$ be a mixed lattice group. Then the following hold.*

(a) $x \curlywedge y \preccurlyeq x \preccurlyeq x \curlyvee y$ *and* $x \curlywedge y \leq y \leq x \curlyvee y$ *for all* $x, y \in \mathcal{G}$.

(b) $x \preccurlyeq u$ *and* $y \leq v \implies x \curlyvee y \leq u \curlyvee v$ *and* $x \curlywedge y \leq u \curlywedge v$.

(c) $x \preccurlyeq z$ *and* $y \leq z \implies x \curlyvee y \leq z$ *and* $x \curlywedge y \leq z$.

(d) $y \preccurlyeq x \iff x \curlywedge y = y \iff y \curlyvee x = x$.

The following calculation rules for the mixed envelopes hold as well. These were proved in [12], Lemma 3.1.

Theorem 4.1.4. *Let $\mathcal{G}$ be a mixed lattice group. Then the following identities hold for all $x, y, z \in \mathcal{G}$.*

(a) $\quad x \curlyvee y \, + \, y \curlywedge x \; = \; x + y$

(b) $\quad z \, + \, x \curlywedge y \; = \; (x + z) \curlywedge (y + z)$

(c) $\quad z \, + \, x \curlyvee y \; = \; (x + z) \curlyvee (y + z)$

Proof. We will first prove the identity (b). From $x \curlywedge y \preccurlyeq x$ and $x \curlywedge y \leq y$ it follows that $z + x \curlywedge y \preccurlyeq z + x$ and $z + x \curlywedge y \leq z + y$. Hence by 4.1.3 (b) we have $z + x \curlywedge y \leq (x + z) \curlywedge (y + z)$. If w is some element of $\mathcal{G}$ satisfying $w \preccurlyeq x + z$ and $w \leq y + z$ then the cancellation property implies $w - z \preccurlyeq x$ and $w - z \leq y$. Again by 4.1.3 (b) we have $w - z \leq x \curlywedge y$ and so $w \leq z + x \curlywedge y$. Thus we have shown that

$$z + x \curlywedge y = \max\{w \in \mathcal{G} : w \preccurlyeq x + z \text{ and } w \leq y + z\} = (x + z) \curlywedge (y + z),$$

and the proof of (b) is complete. The identity (c) now follows from (b) and Theorem 4.1.2 since

$$(x + z) \curlyvee (y + z) = -((-x - z) \curlywedge (-y - z)) = -(-z + (-x) \curlywedge (-y)) = z + x \curlyvee y.$$

To prove (a) we observe that $x + y \preccurlyeq x + y \curlyvee x$ and $x + y \leq y + y \curlyvee x$. Applying (b) and 4.1.3 (b) gives

$$x + y \leq (x + y \curlyvee x) \curlywedge (y + y \curlyvee x) = y \curlyvee x + x \curlywedge y.$$

Similar reasoning together with (c) gives the reverse inequality

$$x + y \geq y \curlyvee x + x \curlywedge y$$

which completes the proof of (a). ∎

Note that the identity in Theorem 4.1.4 (a) is included in the definition of mixed lattice semigroup, but in a mixed lattice group it follows from the existence of the mixed envelopes.

The next result helps to determine whether some structure is actually a mixed lattice group. This theorem is an extension of Theorem 3.3 of [12], in which the equivalence *(a)* $\Longleftrightarrow$ *(e)* was proved.

Theorem 4.1.5. *Let $\mathcal{G}$ be a partially ordered group. Then the following conditions are equivalent:*

(a) *$\mathcal{G}$ is a mixed lattice group.*

(b) *The mixed lower envelope $0 \curlywedge x$ exists for all $x \in \mathcal{G}$.*

(c) *The mixed upper envelope $0 \curlyvee x$ exists for all $x \in \mathcal{G}$.*

(d) *The mixed lower envelope $x \curlywedge 0$ exists for all $x \in \mathcal{G}$.*

(e) *The mixed upper envelope $x \curlyvee 0$ exists for all $x \in \mathcal{G}$.*

Proof. *(a)* $\Longrightarrow$ *(b)* Trivial.
(b) $\Longrightarrow$ *(c)* If $0 \curlywedge x$ exists for all $x \in \mathcal{G}$ then by Theorem 4.1.2 $0 \curlyvee x = -(0 \curlywedge (-x))$.
(c) $\Longrightarrow$ *(d)* If $0 \curlyvee x$ exists then by Theorem 4.1.4 (a) we have $x \curlywedge 0 = x - 0 \curlyvee x$.
(d) $\Longrightarrow$ *(e)* Follows again from Theorem 4.1.2.
(e) $\Longrightarrow$ *(a)* Assume that $x \curlyvee 0$ exists for all $x \in \mathcal{G}$ and let $x, y \in \mathcal{G}$. Since $(x-y) \curlyvee 0 \succcurlyeq x-y$ and $(x-y) \curlyvee 0 \geq 0$ it follows by 4.1.3 (b) that $x-(x-y) \curlyvee 0 \preccurlyeq x-(x-y) = y$ and $x-(x-y) \curlyvee 0 \leq x$. Hence by 4.1.3 (b) we have $x-(x-y) \curlyvee 0 \leq y \curlywedge x$. If $w \in \mathcal{G}$ is any other element such that $w \preccurlyeq y$ and $w \leq x$ then $x - w \succcurlyeq x - y$ and $x - w \geq 0$. Therefore, by 4.1.3 (b) we have $x - w \geq (x - y) \curlyvee 0$ and so $w \leq x - (x - y) \curlyvee 0$. Thus we have shown that

$$x - (x - y) \curlyvee 0 = \max\{w \in \mathcal{G} : w \preccurlyeq y \text{ and } w \leq x\} = y \curlywedge x,$$

and so the mixed lower envelope $y \curlywedge x$ exists for all $x, y \in \mathcal{G}$ and $\mathcal{G}$ is a mixed lattice group. ■

Some additional properties hold in mixed lattice groups under certain conditions. The following characterization is from [12], Theorem 3.5.

Theorem 4.1.6. *Let $\mathcal{G}$ be a mixed lattice group and denote $\mathcal{U} = \{w \in \mathcal{G} : w \succcurlyeq 0\}$. Then the following conditions are equivalent.*

(a) $\mathcal{U}$ *is a mixed lattice semigroup.*

(b) $x \curlywedge y \succcurlyeq 0$ *for all* $x, y \in \mathcal{U}$*, where* $\curlywedge$ *is the mixed lower envelope in* $\mathcal{G}$.

(c) *For any* $x, y, z \in \mathcal{G}$*, the conditions* $z \preccurlyeq x$ *and* $z \preccurlyeq y$ *imply* $z \preccurlyeq x \curlywedge y$.

(d) *For any* $x, y, z \in \mathcal{G}$*, the conditions* $z \succcurlyeq x$ *and* $z \succcurlyeq y$ *imply* $z \succcurlyeq x \curlyvee y$.

Proof. (a) $\Longrightarrow$ (b) Assume that $\mathcal{U}$ is a mixed lattice semigroup. Let $x, y \in \mathcal{U}$. We need to show that the mixed lower envelope $x \curlywedge y$ in $\mathcal{G}$ is the same as in $\mathcal{U}$. If we form the mixed upper envelope $x \curlyvee y$ in $\mathcal{G}$, then $x \curlyvee y \succcurlyeq x \succcurlyeq 0$. Therefore, the mixed upper envelope of x and y in $\mathcal{G}$ belongs to $\mathcal{U}$ and thus the mixed upper envelopes in $\mathcal{G}$ and in $\mathcal{U}$ are equal. By the definition of mixed lattice semigroup, the identity $x \curlyvee y + y \curlywedge x = x + y$ holds for all x and y in $\mathcal{U}$, but it also holds for all x and y in $\mathcal{G}$ by Theorem 4.1.4. Since $x \curlyvee y$ is the same in $\mathcal{G}$ and in $\mathcal{U}$, it follows that the mixed lower envelope $y \curlywedge x$ is also the same in $\mathcal{G}$ and in $\mathcal{U}$. Since $y \curlywedge x \succcurlyeq 0$ for all $x, y \in \mathcal{U}$, we have proved (b).
(b) $\Longrightarrow$ (c) Suppose that (b) holds and let $x, y, z \in \mathcal{G}$ be such that $z \preccurlyeq x$ and $z \preccurlyeq y$. Then $x - z \succcurlyeq 0$ and $y - z \succcurlyeq 0$, and by assumption we have $(x - z) \curlywedge (y - z) \succcurlyeq 0$. Applying Theorem 4.1.4 we get $x \curlywedge y - z \succcurlyeq 0$, or $x \curlywedge y \succcurlyeq z$.
(c) $\Longrightarrow$ (d) Assume that (c) holds and let $x, y, z \in \mathcal{G}$ be such that $z \succcurlyeq x$ and $z \succcurlyeq y$. Then $-z \preccurlyeq -x$ and $-z \preccurlyeq -y$. By assumption we have $-z \preccurlyeq (-x) \curlywedge (-y)$, and applying Theorem 4.1.2 gives $-z \preccurlyeq -(x \curlyvee y)$. This implies that $z \succcurlyeq x \curlyvee y$, which establishes (d).
(d) $\Longrightarrow$ (a) Assuming that (d) holds, let $x, y \in \mathcal{U}$. Then $x + y \succcurlyeq x$ and $x + y \succcurlyeq y$. We then apply (d) to get $x + y \succcurlyeq x \curlyvee y$. Since $x \curlyvee y \succcurlyeq x \succcurlyeq 0$, the element $x \curlyvee y$ belongs to $\mathcal{U}$. Hence, by Theorem 3.1.3 $\mathcal{U}$ is a mixed lattice semigroup and the proof is complete. ■

To finish this section, we will show that the following version of the Riesz decomposition property holds in every mixed lattice group.

Theorem 4.1.7. *Let $\mathcal{G}$ be a mixed lattice group and let $u \succcurlyeq 0,\ v_1 \geq 0$ and $v_2 \succcurlyeq 0$ be elements of $\mathcal{G}$ satisfying $u \leq v_1 + v_2$. Then there exist elements $u_1 \geq 0$ and $u_2 \succcurlyeq 0$ such that $u_1 \leq v_1,\ u_2 \leq v_2$ and $u = u_1 + u_2$.*

Proof. The element $u_1 = u \curlywedge v_1$ satisfies $u_1 \leq v_1$ and $u_1 \geq 0$, since $u \succcurlyeq 0$ and $v_1 \geq 0$. Let $u_2 = u - u_1$. Then $u = u_1 + u_2$ and $u_2 \succcurlyeq 0$, since $u_1 \preccurlyeq u$. It remains to show that $u_2 \leq v_2$. For this, we note that $0 \preccurlyeq 0$ and $u - v_1 \leq v_2$, and so we have

$$u_2 = u - u \curlywedge v_1 = u + (-u) \curlyvee (-v_1) = 0 \curlyvee (u - v_1) \leq 0 \curlyvee v_2 = v_2,$$

where the last equality follows by Theorem 4.1.3 (d), since $v_2 \succcurlyeq 0$. ∎

4.2 Group extensions of mixed lattice semigroups

Next we will discuss the group extensions of mixed lattice semigroups. It will be shown that every mixed lattice semigroup $\mathcal{S}$ can be extended to a group of formal differences of elements of $\mathcal{S}$ and the mixed lattice structure is preserved in this extension. Many important examples of mixed lattice groups arise in this way. All the results in this section are from [11].

We will first give a precise meaning to the idea of ”group of formal differences”.

Let $\mathcal{S}$ be a mixed lattice semigroup and define an equivalence relation on $\mathcal{S} \times \mathcal{S}$ by putting

$$(u, v) \sim (x, y) \qquad \Longleftrightarrow \qquad u + y = v + x.$$

The equivalence class generated by (u, v) is denoted by $[(u, v)]$ and the set of all equivalence classes by $[(\mathcal{S}, \mathcal{S})]$. We define addition in $[(\mathcal{S}, \mathcal{S})]$ by

$$[(u, v)] + [(x, y)] = [(u + x, v + y)]$$

and the partial order by

$$[(u, v)] \leq [(x, y)] \qquad \Longleftrightarrow \qquad u + y \leq x + v.$$

Now the zero element of $[(\mathcal{S},\mathcal{S})]$ is $0 = [(0,0)]$ and the above definitions imply that the property

$$[(u,v)] \leq [(x,y)] \qquad \Longleftrightarrow \qquad [(u,v)] + [(a,b)] \leq [(x,y)] + [(a,b)]$$

holds in $[(\mathcal{S},\mathcal{S})]$. We observe that the sets $\mathcal{S}$ and $\{[(u,0)] : u \in \mathcal{S}\}$ are isomorphic. We also have $[(u,0)] + [(0,u)] = [(0,0)]$, and so $[(u,0)] = -[(0,u)]$ for all $u \in \mathcal{S}$. Thus, $[(\mathcal{S},\mathcal{S})]$ is a group and we can identify $[(u,0)]$ with u. We now denote $[(\mathcal{S},\mathcal{S})] = \mathcal{S} - \mathcal{S}$ and we call $\mathcal{S} - \mathcal{S}$ the *group of formal differences* of elements of $\mathcal{S}$.

Finally, we define the specific order in $\mathcal{S} - \mathcal{S}$ by

$$u - v \preccurlyeq x - y \quad \text{in } \mathcal{S} - \mathcal{S} \qquad \Longleftrightarrow \qquad u + y \preccurlyeq x + v \quad \text{in } \mathcal{S}.$$

We will show that this specific order is indeed a partial order and it has the same properties as the original specific order in $\mathcal{S}$.

Theorem 4.2.1. *The specific order in $\mathcal{S} - \mathcal{S}$ defined above is a partial order satisfying the cancellation law. Moreover, it has the property*

$$u - v \preccurlyeq x - y \qquad \Longleftrightarrow \qquad x - y = u - v + w \quad \textit{for some } w \in \mathcal{S},$$

for all $u, v, x, y \in \mathcal{S}$.

Proof. The given relation is obviously reflexive and antisymmetric.

To prove the transitivity, let $u - v$, $x - y$ and $z - w$ be elements of $\mathcal{S} - \mathcal{S}$. Then $u - v \preccurlyeq x - y \preccurlyeq z - w$ if and only if $u + y \preccurlyeq x + v$ and $x + w \preccurlyeq z + y$. It follows that $u + y + w \preccurlyeq x + v + w$ and $x + w + v \preccurlyeq z + y + v$ and since the specific order in $\mathcal{S}$ is transitive, we have

$$u + y + w \preccurlyeq x + v + w \preccurlyeq z + y + v.$$

The specific order in $\mathcal{S}$ obeys the cancellation law, so the above inequality implies that $u + w \preccurlyeq z + v$, and so $u - v \preccurlyeq z - w$. This shows that the specific order in $\mathcal{S} - \mathcal{S}$ is a partial order.

Next we want to show that the specific order in $\mathcal{S}-\mathcal{S}$ also obeys the cancellation law. Indeed, since the specific order in $\mathcal{S}$ has the cancellation property, we have

$$\begin{aligned} u-v \preccurlyeq x-y \ \text{ in } \mathcal{S}-\mathcal{S} \quad &\Longleftrightarrow \quad u+y \preccurlyeq x+v \ \text{ in } \mathcal{S} \\ &\Longleftrightarrow \quad u+y+z+w \preccurlyeq x+v+z+w \ \text{ in } \mathcal{S} \\ &\Longleftrightarrow \quad (u-v)+(z-w) \preccurlyeq (x-y)+(z-w) \ \text{ in } \mathcal{S}-\mathcal{S}. \end{aligned}$$

Finally, by the definition of specific order we have

$$\begin{aligned} u-v \preccurlyeq x-y \ \text{ in } \mathcal{S}-\mathcal{S} \quad &\Longleftrightarrow \quad u+y \preccurlyeq x+v \ \text{ in } \mathcal{S} \\ &\Longleftrightarrow \quad x+v = u+v+w \ \text{ for some } w \text{ in } \mathcal{S} \\ &\Longleftrightarrow \quad x-y = u-v+w \ \text{ for some } w \text{ in } \mathcal{S}. \end{aligned}$$

■

We define the mixed envelopes in $\mathcal{S}-\mathcal{S}$ the same way as they were defined in $\mathcal{S}$. The next theorem shows that the mixed envelopes always exist in $\mathcal{S}-\mathcal{S}$, and thus $\mathcal{S}-\mathcal{S}$ is actually a mixed lattice group.

Theorem 4.2.2. *Let $\mathcal{S}$ be a mixed lattice semigroup and $u_1, u_2, v_1, v_2 \in \mathcal{S}$. Then the mixed envelopes of elements u_1-u_2 and v_1-v_2 in $\mathcal{S}-\mathcal{S}$ exist and they are given by*

$$(u_1-u_2) \curlywedge (v_1-v_2) = (u_1+v_2) \curlywedge (v_1+u_2) - (u_2+v_2)$$

and

$$(u_1-u_2) \curlyvee (v_1-v_2) = (u_1+v_2) \curlyvee (v_1+u_2) - (u_2+v_2).$$

In particular, $\mathcal{S}-\mathcal{S}$ is a mixed lattice group.

Proof. If $\mathcal{S}$ is a mixed lattice semigroup and $u_1, u_2, v_1, v_2 \in \mathcal{S}$ then the mixed lower envelope $(u_1+v_2) \curlywedge (v_1+u_2)$ exists in $\mathcal{S}$ and we have

$$(u_1+v_2) \curlywedge (v_1+u_2) - (u_2+v_2) \preccurlyeq (u_1+v_2) - (u_2+v_2) = u_1 - u_2$$

and also

$$(u_1+v_2) \curlywedge (v_1+u_2) - (u_2+v_2) \leq (v_1+u_2) - (u_2+v_2) = v_1 - v_2.$$

Assume then that $w_1 - w_2 \in \mathcal{S} - \mathcal{S}$ is any element that satisfies $w_1 - w_2 \preccurlyeq u_1 - u_2$ and $w_1 - w_2 \leq v_1 - v_2$. It follows that $w_1 + u_2 \preccurlyeq u_1 + w_2$ and $w_1 + v_2 \leq v_1 + w_2$. Adding v_2 and u_2 to these inequalities, respectively, we get $w_1 + u_2 + v_2 \preccurlyeq u_1 + w_2 + v_2$ and $w_1 + u_2 + v_2 \leq v_1 + w_2 + u_2$. These inequalities together with Theorems 3.1.2 (d) and 3.1.4 imply that

$$w_1 + u_2 + v_2 \leq (u_1 + w_2 + v_2) \curlywedge (v_1 + w_2 + u_2) = (u_1 + v_2) \curlywedge (v_1 + u_2) + w_2.$$

From this we conclude that

$$w_1 - w_2 \leq (u_1 + v_2) \curlywedge (v_1 + u_2) - (u_2 + v_2),$$

and so the first equality is proved. The second equality follows by similar arguments. It follows now from the definition of mixed lattice group, that the group of formal differences of a mixed lattice semigroup $\mathcal{S}$ is a mixed lattice group. ■

We finish this chapter with an interesting example of a group extension of a mixed lattice semigroup to which we will return again later. This example is based on Example 3.2.2 of Chapter 2, and it was given in [12] (Example 1).

Example 4.2.3. Let $\mathcal{G} = BV([a, b])$ be the set of all functions of bounded variation on an interval $[a, b]$. Functions of bounded variation have the following well known characterization (see for example [3], Theorem 6.13):

$$f \in BV([a,b]) \quad \Longleftrightarrow \quad f = f_1 - f_2, \text{ where } f_1 \text{ and } f_2 \text{ are positive increasing functions.}$$

We showed in Example 3.2.2 that the set $\mathcal{S}$ of all positive increasing functions is a mixed lattice semigroup. Hence, we can write $\mathcal{G} = \mathcal{S} - \mathcal{S}$ and so $\mathcal{G}$ is a mixed lattice group where the initial and specific orders are defined by

$$f \leq g \quad \Longleftrightarrow \quad f(x) \leq g(x) \quad \text{for all } x \in [a, b]$$

and

$$f \preccurlyeq g \quad \Longleftrightarrow \quad g - f \text{ is increasing and } g - f \geq 0,$$

respectively. ▲

Chapter 5

Algebraic structure of mixed lattice groups

We will investigate which features of the theory of Riesz spaces can be carried over to mixed lattice groups. Many of the results given here have their counterparts in the theory of Riesz spaces and lattice ordered groups. In fact, to prove these results, we can often apply the same type of techniques as in Riesz spaces. Here the main difference is due to the fact that the mixed envelopes are neither commutative nor distributive, unlike the operations of forming the suprema and infima. In some cases, we can work around these difficulties. However, there are also results that hold in lattice ordered groups but cannot be generalized to mixed lattice groups. All the definitions, results and examples given in this and the following chapters are new and original to this work, unless otherwise stated.

5.1 Regular, almost regular and pre-regular mixed lattice groups

We will first discuss the order structure of mixed lattice groups and introduce some new important related concepts. In the preceding chapter we have proved the basic properties and calculation rules that hold in all mixed lattice groups. We saw that most of the properties of mixed lattice semigroups also hold in mixed lattice groups. We will frequently refer to these later, and for easy reference, we list all the fundamental properties of mixed lattice groups below. The following properties hold for all elements x, y and z in an arbitrary mixed lattice group, apart from (P4), (P5a) and (P7), which require certain additional assumptions and will be discussed below.

$$
\begin{array}{ll}
(P1) & x \curlyvee y + y \curlywedge x = x + y \\
(P2a) & z + x \curlyvee y = (x+z) \curlyvee (y+z) \\
(P2b) & z + x \curlywedge y = (x+z) \curlywedge (y+z) \\
(P3) & x \curlyvee y = -(-x \curlywedge -y) \\
(P4) & x \preccurlyeq y \implies x \leq y \\
(P5a) & x \leq y \iff y \curlyvee x = y \iff x \curlywedge y = x \\
(P5b) & x \preccurlyeq y \iff x \curlyvee y = y \iff y \curlywedge x = x \\
(P6a) & x \preccurlyeq u \quad \text{and} \quad y \leq v \implies x \curlyvee y \leq u \curlyvee v \quad \text{and} \quad x \curlywedge y \leq u \curlywedge v \\
(P6b) & x \preccurlyeq z \quad \text{and} \quad y \leq z \implies x \curlyvee y \leq z \quad \text{and} \quad x \curlywedge y \leq z \\
(P7a) & x \preccurlyeq z \quad \text{and} \quad y \preccurlyeq z \implies x \curlyvee y \preccurlyeq z \\
(P7b) & z \preccurlyeq x \quad \text{and} \quad z \preccurlyeq y \implies z \preccurlyeq x \curlywedge y
\end{array}
$$

The property (P7) was discussed in Theorem 4.1.6. It remains to consider the semigroup properties (P4) and (P5a). As we have seen, in mixed lattice semigroups the property (P4) follows from the definition of specific order. However, in a mixed lattice group there is no such close connection between the two partial orders, since the specific order is defined as any partial order which obeys the cancellation law. As a consequence, (P4) does not hold in every mixed lattice group.

As we develop our theory further in the following sections, it turns out that this property is quite essential in many situations. For example, the generalized absolute values, which we introduce in the next sections, do not always behave nicely if the mixed lattice group does not have the property (P4). Consequently, many results will be valid only if (P4) holds. Moreover, property (P7) is also essential for certain results and, in many cases, a mixed lattice group can be viewed as a group extension of some mixed lattice semigroup. In such cases, the mixed lattice group has algebraic properties which are closest to those of a mixed lattice semigroup, and as we shall see, this situation is also closely related to (P4). Due to the importance of these properties, we introduce the following classification of mixed lattice groups.

Definition 5.1.1. A mixed lattice group $\mathcal{G}$ is called *regular* if there exists a mixed lattice semigroup $\mathcal{S}$ such that $\mathcal{G} = \mathcal{S} - \mathcal{S}$. A mixed lattice group $\mathcal{G}$ is called *almost regular* if the set $\{w \in \mathcal{G} : w \succcurlyeq 0\}$ is a mixed lattice semigroup. A mixed lattice group $\mathcal{G}$ is called *pre-regular* if $x \preccurlyeq y$ implies $x \leq y$ in $\mathcal{G}$. A mixed lattice group that is not pre-regular is called *irregular*.

In Theorem 4.1.6 it was shown that $\mathcal{G}$ is almost regular if and only if the properties (P7a) and (P7b) hold in $\mathcal{G}$. It is now a simple matter to verify that every regular mixed lattice group is almost regular, and every almost regular mixed lattice group is pre-regular. Moreover, pre-regularity implies property (P5a). The converse implications do not hold in general (see examples below).

Theorem 5.1.2. *The following hold.*

(a) Every regular mixed lattice group is almost regular.

(b) Every almost regular mixed lattice group is pre-regular.

(c) Every pre-regular mixed lattice group has the property (P5a).

Proof. (a) Let $\mathcal{S}$ be a mixed lattice semigroup such that $\mathcal{G} = \mathcal{S} - \mathcal{S}$ and let $\mathcal{U} = \{w \in \mathcal{G} : w \succcurlyeq 0\}$. It is sufficient to show that $\mathcal{S} = \mathcal{U}$. If $x \in \mathcal{S}$ then $x = 0 + x$ and so by the definition of specific order we have $x \succcurlyeq 0$. Thus, the inclusion $\mathcal{S} \subset \mathcal{U}$ holds. To prove the converse inclusion, assume that $x \in \mathcal{U}$, or $x \succcurlyeq 0$. By Theorem 4.2.1 we have $x = 0 + w = w$ for some $w \in \mathcal{S}$, and so $x \in \mathcal{S}$. Hence $\mathcal{U} \subset \mathcal{S}$. Consequently, $\mathcal{S} = \mathcal{U}$ and so $\mathcal{G}$ is almost regular.

(b) Assume that $\mathcal{G}$ is almost regular and $x \succcurlyeq y$. Then $x - y \succcurlyeq 0$, and the set $\{w \in \mathcal{G} : w \succcurlyeq 0\}$ is a mixed lattice semigroup, so by Theorem 3.1.2 (a) we have $x - y \geq 0$, or $x \geq y$. Hence, $\mathcal{G}$ is pre-regular.

(c) The proof for this is the same as in the semigroup case. Note that only the first implication in (P5a) requires (P4), as can be seen from the proof of Theorem 3.1.2 (f).

■

In preceding theorem we have proved the following implications

$$\text{regular} \quad \Longrightarrow \quad \text{almost regular} \quad \Longrightarrow \quad \text{pre-regular.}$$

None of these implications can be reversed, as the following examples show.

Example 5.1.3. Let $\mathcal{G} = \mathbb{R}^2$ and define $\leq$ as the partial order induced by the positive cone $C_1 = \{(x, y) : x \geq 0 \,\text{and}\, y \geq 0\}$. Define specific order $\preccurlyeq$ as the partial order with the positive cone $C_2 = \{(x, x) : x \geq 0\}$. Both of these are partial orders in $\mathbb{R}^2$ such that the cancellation law holds. Now if $x = (x_1, y_1)$ then we can easily check that if $x_1 \geq y_1$, then $0 \curlyvee x = (x_1, x_1)$ if $x_1 \geq 0$, and $0 \curlyvee x = (0, 0)$ if $x_1 < 0$. On the other hand, if $y_1 > x_1$, then $0 \curlyvee x = (y_1, y_1)$ if $y_1 \geq 0$, and $0 \curlyvee x = (0, 0)$ if $y_1 < 0$. Thus, the element $0 \curlyvee x$ exists for all $x \in \mathcal{G}$. Geometrically, the existence of $0 \curlyvee x$ is obvious, since the sets $x + C_1$ and C_2 always intersect and there is a minimal point of intersection. Thus, by Theorem 4.1.5, $(\mathcal{G}, \leq, \preccurlyeq)$ is a mixed lattice group which is clearly pre-regular, since $C_2 \subset C_1$.

Now assume that there exists a mixed lattice semigroup $\mathcal{S}$ such that $\mathcal{G} = \mathcal{S} - \mathcal{S}$. By definition, $\mathcal{S}$ must be a positive semigroup and so we must have $\mathcal{S} \subset C_1$. On the other hand, by (P4) we must have $C_2 \subset \mathcal{S}$ and since the cone C_2 does not generate the whole space $\mathbb{R}^2$, we must also have $\mathcal{S} \neq C_2$. Thus, there exists an element $x \in \mathcal{S}$ such that $x \in C_1$ and $x \notin C_2$. But then the element $x \curlywedge 0$ does not exist in C_1, and therefore, it does not exist in $\mathcal{S}$. This is a contradiction which shows that $\mathcal{G}$ is pre-regular, but not regular.

Note however, that C_2 is a mixed lattice semigroup and thus by Theorem 4.1.6 $\mathcal{G}$ is almost regular. Indeed, the two partial orders $\leq$ and $\preccurlyeq$ coincide on C_2, that is, whenever x and y are in C_2, we have $x \preccurlyeq y$ if and only if $x \leq y$. In fact, the set $C_2 = \{x \in \mathcal{G} : x \succcurlyeq 0\}$ is totally ordered, and hence it can be identified with the set of positive real numbers with the usual ordering. This set is obviously a very special case of a mixed lattice semigroup. ▲

Our next example shows that there are pre-regular mixed lattice groups that are not almost regular.

Example 5.1.4. Let $\mathcal{G} = (\mathbb{Z} \times \mathbb{Z}, \leq, \preccurlyeq)$ and define partial orders $\leq$ and $\preccurlyeq$ as follows. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ then $x \leq y$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$. In addition, $x \preccurlyeq y$ iff $x = y$ or $y_1 \geq x_1 + 1$ and $y_2 \geq x_2 + 1$. Using Theorem 4.1.5 it is easy to check that $\mathcal{G}$ is a mixed lattice group. Indeed, if $x_1 \geq 0$ and $x_2 \geq 0$ then $x \curlyvee 0 = x$. If $x_1 < 0$ and $x_2 < 0$ then $x \curlyvee 0 = 0$. If $x_1 < 0$ and $x_2 \geq 0$ then $x \curlyvee 0 = (1, x_2 + 1)$. Finally, if $x_2 < 0$ and $x_1 \geq 0$ then $x \curlyvee 0 = (x_1 + 1, 1)$. Hence, $\mathcal{G}$ is a mixed lattice group which is clearly pre-regular. Now let $x = (1, 1)$ and $y = (1, 2)$, and denote $W = \{w \in \mathcal{G} : w \succcurlyeq 0\}$. Then $x \in W$ and $y \in W$ but $y \curlywedge x = (0, 1) \notin W$. By Theorem 4.1.6 the set W is not a mixed lattice semigroup and so $\mathcal{G}$ is not almost regular. ▲

Next we give an example of a mixed lattice group which is irregular, and hence does not have properties (P4), (P5a) and (P7).

Example 5.1.5. Let $\mathcal{G} = (\mathbb{Z} \times \mathbb{Z}, \leq, \preccurlyeq)$, where $\leq$ and $\preccurlyeq$ are lexicographic orders defined as

$$x \leq y \quad \Longleftrightarrow \quad x_1 \leq y_1 \quad \text{or} \quad (\ x_1 = y_1 \quad \text{and} \quad x_2 \leq y_2\ )$$

and

$$x \preccurlyeq y \quad \Longleftrightarrow \quad x_2 \leq y_2 \quad \text{or} \quad (\ x_2 = y_2 \quad \text{and} \quad x_1 \leq y_1\ ).$$

Then $\mathcal{G}$ is a partially ordered group with the usual coordinatewise addition, and since both partial orders are total orders they obviously have the properties

$$x \leq y \iff x + z \leq y + z \quad \text{and} \quad x \preccurlyeq y \iff x + z \preccurlyeq y + z.$$

It is also easy to see that $x \curlyvee 0$ exists for every $x \in \mathcal{G}$. Indeed, if $x = (x_1, x_2)$ with $x_2 < 0$ then $x \curlyvee 0 = 0$. If $x_2 \geq 0$ and $x_1 \leq 0$ then $x \curlyvee 0 = (0, x_2)$. Finally, if $x_2 \geq 0$ and $x_1 > 0$ then $x \curlyvee 0 = (0, x_2 + 1)$. Hence, by Theorem 4.1.5 $\mathcal{G}$ is a mixed lattice group. However, it is clear that neither of the implications $x \preccurlyeq y \implies x \leq y$ or $x \leq y \implies x \preccurlyeq y$ hold in $\mathcal{G}$. In view of Theorem 5.1.2, this shows that there is no such mixed lattice semigroup $\mathcal{S}$ for which $\mathcal{G} = \mathcal{S} - \mathcal{S}$. In addition, it is evident that every $x \in \mathcal{G}$ can be written as $x = u - v$ where $u, v \succcurlyeq 0$. Hence, in particular, the set $\{w \in \mathcal{G} : w \succcurlyeq 0\}$ is not a mixed lattice semigroup, and so the property (P7) does not hold in $\mathcal{G}$ either. Indeed, if $x = (1,1)$, $y = (-1,1)$ and $z = (-2,2)$, then $x \preccurlyeq z$ and $y \preccurlyeq z$ but $x \curlyvee y = (-1,2) \not\succcurlyeq z$. Finally, we note that (P5a) does not hold in $\mathcal{G}$. To see this, let $x = (0,1)$ and $y = (1,0)$. Then $x \leq y$ but $x \curlywedge y = y \neq x$. ▲

Finally, an example of a regular mixed lattice group was given in Example 4.2.3.

As for the actual lattice properties of mixed lattice groups, there are examples which show that a mixed lattice group is not necessarily a lattice with respect to either the specific order or the initial order. We give some such examples below.

Example 5.1.6. Consider again the mixed lattice group $(\mathcal{G}, \leq, \preccurlyeq)$ of Example 5.1.3. Observe that the element $\operatorname{sp}\sup\{x, y\}$ does not exist, unless the points x and y both lie on the same line, which is parallel to the line $y = x$. Hence $\mathcal{G}$ is not a lattice with respect to ordering $\preccurlyeq$. If we interchange the partial orders $\leq$ and $\preccurlyeq$ such that we choose $\preccurlyeq$ to be the initial order, and $\leq$ the specific order, then we get another mixed lattice group $(\mathcal{G}, \preccurlyeq, \leq)$, which is not pre-regular and not a lattice with respect to initial order $\preccurlyeq$. These facts can be checked similarly as above. ▲

Example 5.1.7. This example is a three-dimensional version of Example 5.1.3. Let $\mathcal{G} = \mathbb{R}^3$ and define $\leq$ as the partial order induced by the positive cone $C_1 = \{(x,y,z) : z \geq \sqrt{x^2 + y^2}\}$. Define specific order $\preccurlyeq$ as the partial order with the positive cone $C_2 = \{(x,y,z) : x = y = 0,\ z \geq 0\}$. Now both of these are partial orders in $\mathbb{R}^3$ such that the cancellation law holds. Geometrically it is obvious that the element $x \curlyvee 0$ exists for all $x \in \mathbb{R}^3$, since the cone C_1 and any line parallel to z-axis always intersect. Thus $(\mathcal{G}, \leq, \preccurlyeq)$ is a mixed lattice group which is clearly pre-regular, since $C_2 \subset C_1$. This is an example of a mixed lattice group which is not a lattice with respect to either partial ordering. Again, this is easy to see geometrically for partial order $\leq$, since the intersection of two circular cones of type C_1 is not necessarily a circular cone and thus the element $\sup\{x, y\}$ does not exists if the elements x and y are not comparable. $\mathcal{G}$ is also not a lattice with respect to $\preccurlyeq$, since the element $\mathrm{sp}\,\sup\{x, y\}$ does not exist, unless the points x and y both lie on the same line, which is parallel to the z-axis. ▲

The following example was given in [12] (Example 2). We shall return to this example later to illustrate our other results, so we will state it here briefly.

Example 5.1.8. Consider the partially ordered group $(\mathbb{Z}, \leq, \preccurlyeq)$, where $\leq$ is the usual order. Let p be a strictly positive integer and define specific order $\preccurlyeq$ by setting

$$n \preccurlyeq m \quad \text{if } m - n \geq 0 \text{ and } m - n \text{ is divisible by } p.$$

Then $(\mathbb{Z}, \leq, \preccurlyeq)$ is a mixed lattice group.

This is another example of a mixed lattice group which is almost regular but not regular. It is also easy to see that this mixed lattice group is not a lattice with respect to specific order. For example, if we choose $p = 3$ then the element $s = \mathrm{sp}\,\sup\{1, -1\}$ does not exists in $\mathcal{G}$. For if s existed, then the numbers $s - 1$ and $s - (-1) = s + 1$ would be both divisible by 3, which is impossible. ▲

To illustrate some of the fundamental differences between mixed lattice groups and lattice ordered groups, let us recall the following result from the theory of lattice ordered groups (see [8], Corollary 2, pp. 294).

Theorem 5.1.9. *If $(\mathcal{G}, \leq)$ is a commutative lattice ordered group with $a, b \in \mathcal{G}$ and $n \in \mathbb{N}$, then $na \leq nb$ implies $a \leq b$.*

Our last example shows that this result does not hold in mixed lattice groups. This example is a modification of Example 5.1.4.

Example 5.1.10. Let $\mathcal{G} = (\mathbb{Z} \times \mathbb{Z}, \leq, \preccurlyeq)$ and define partial orders $\leq$ and $\preccurlyeq$ as follows. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ then $x \leq y$ iff $x = y$, or $y = (x_1 + 1, x_2 + 1)$, or $y_1 \geq x_1 + 2$ and $y_2 \geq x_2 + 2$. In addition, $x \preccurlyeq y$ iff $y_1 - x_1 = y_2 - x_2 \geq 0$. Similar arguments as in

Example 5.1.4 show that $\mathcal{G}$ is a mixed lattice group, and $\mathcal{G}$ is not a lattice with respect to either partial order (choose, for example, $x = (1,0)$. It is then easy to see that $\sup\{0,x\}$ and $\mathrm{sp}\,\sup\{0,x\}$ do not exist in $\mathcal{G}$). Now if $x = (1,5)$ and $y = (0,3)$, then $2x \geq 2y$ but $x \ngeq y$. ▲

5.2 Generalized absolute values

One of the fundamental concepts in the theory of Riesz spaces and lattice ordered groups is the absolute value of an element. However, in the present context this idea is completely unexplored. It is therefore natural to incorporate the notion of absolute value into the theory of mixed lattice groups. We shall now begin to develop our theory in this direction.

We start by introducing the upper and lower parts of an element which play the roles of the positive and negative parts of an element in a Riesz space. They could be referred to as positive, negative, specifically positive and specifically negative parts of an element. However, we will not use this terminology in order to avoid confusion with ordinary positive and negative parts that may sometimes appear if the mixed lattice group is a lattice with respect to one or both of the partial orderings. With the upper and lower parts we can also generalize the absolute value of an element.

Definition 5.2.1. Let $\mathcal{G}$ be a mixed lattice group and let $x \in \mathcal{G}$. The elements ${}^u x = x \curlyvee 0$ and ${}^l x = (-x) \curlyvee 0$ are called the *upper part* and *lower part* of x, respectively. Similarly, the elements $x^u = 0 \curlyvee x$ and $x^l = 0 \curlyvee (-x)$ are called the *specific upper part* and *specific lower part* of x, respectively.

From the above definitions we observe that for the upper and lower parts we have ${}^u x \geq 0$ and ${}^l x \geq 0$, and for the specific upper and lower parts $x^u \succcurlyeq 0$ and $x^l \succcurlyeq 0$. By Theorem 4.1.5, a partially ordered group $\mathcal{G}$ is a mixed lattice group if and only if one of the upper or lower parts exists for all $x \in \mathcal{G}$ (compare this with Theorem 2.0.5 (a)).

The upper and lower parts have the following basic properties.

Theorem 5.2.2. *Let $\mathcal{G}$ be a mixed lattice group and $x \in \mathcal{G}$. Then we have*

(a) $\quad {}^u x \,=\, {}^l(-x) \quad$ *and* $\quad x^u \,=\, (-x)^l.$

(b) $\quad {}^u x \curlyvee x^l \,=\, {}^u x \,+\, x^l \quad$ *and* $\quad {}^l x \curlyvee x^u \,=\, {}^l x \,+\, x^u.$

(c) $\quad x \,=\, x^u \,-\, {}^l x \,=\, {}^u x \,-\, x^l.$

(d) $\quad x^u \curlywedge {}^l x \,=\, 0 \,=\, x^l \curlywedge {}^u x.$

(e) $\quad x^u \curlyvee {}^l x \,=\, {}^u x \,+\, {}^l x \,=\, x^l \,+\, x^u \,=\, x^l \curlyvee {}^u x.$

(f) $\quad {}^u x - {}^l x = x + {}^u x \curlywedge x^l \quad$ *and* $\quad x^u - x^l = x + {}^l x \curlywedge x^u.$

Proof. (a) These follow immediately from the definitions.

(b) Using (P2a) repeatedly we get

$$\begin{aligned} {}^u x + x^l &= x \curlyvee 0 + 0 \curlyvee (-x) \\ &= (x + 0 \curlyvee (-x)) \curlyvee (0 + 0 \curlyvee (-x)) \\ &= (x \curlyvee 0) \curlyvee (0 \curlyvee (-x)) \\ &= {}^u x \curlyvee x^l. \end{aligned}$$

The proof for the second equality is similar.

(c) Using (P1) and (P3) we get

$${}^u x - x^l = x \curlyvee 0 - 0 \curlyvee (-x) = x \curlyvee 0 + 0 \curlywedge x = x + 0 = x.$$

Similarly, $x^u - {}^l x = x.$

(d) Combining (P1) and (b) gives

$${}^u x \curlyvee x^l + x^l \curlywedge {}^u x = {}^u x + x^l = {}^u x \curlyvee x^l.$$

From this it follows that $x^l \curlywedge {}^u x = 0$. The other equality is similar.

(e) For the first equality, we use (P2a) twice to get

$$\begin{aligned}
{}^u x + {}^l x &= x \curlyvee 0 + (-x) \curlyvee 0 \\
&= (x + (-x) \curlyvee 0) \curlyvee (0 + (-x) \curlyvee 0) \\
&= (0 \curlyvee x) \curlyvee ((-x) \curlyvee 0) \\
&= x^u \curlyvee {}^l x.
\end{aligned}$$

The last equality is proved similarly. The middle equality follows from (c).

(f) First we note that by (P2a) and (P3)

$$(x^u \curlyvee {}^l x) - x^u - {}^l x = (-{}^l x) \curlyvee (-x^u) = -({}^l x \curlywedge x^u)$$

and

$$(x^l \curlyvee {}^u x) - {}^u x - x^l = (-{}^u x) \curlyvee (-x^l) = -({}^u x \curlywedge x^l).$$

So by adding $-x^u - {}^l x$ and $-{}^u x - x^l$, respectively, to the two equations in (e) we get

$${}^u x \curlywedge x^l = {}^u x - x^u \qquad \text{and} \qquad -({}^l x \curlywedge x^u) = x^l - {}^l x.$$

Now we use (c) and substitute $x^u = x + {}^l x$ into the first equality and $x^l = {}^u x - x$ into the second equality to get

$${}^u x - {}^l x = x + {}^u x \curlywedge x^l \qquad \text{and} \qquad x^u - x^l = x + {}^l x \curlywedge x^u.$$

■

We make some important observations concerning Theorem 5.2.2. First, we should compare Theorem 5.2.2 (a)–(d) and the corresponding identities in Theorem 2.0.3. Secondly, we should also note the similarity between Theorem 5.2.2 (e)–(f) and the properties of the absolute value in Riesz spaces and lattice ordered groups in Theorem 2.0.3. Both the ”unsymmetrical” expressions ${}^u x \curlyvee x^l$ and ${}^l x \curlyvee x^u$ as well as the ”symmetrical” expressions $x^u \curlyvee {}^l x$ and $x^l \curlyvee {}^u x$ (which are equal) have properties similar to the absolute value of an element in lattice ordered groups, and they can be expressed as the sum of upper and lower parts, just as the absolute value can be expressed as the sum of positive and negative parts. These observations motivate the following definition.

Definition 5.2.3. Let $\mathcal{G}$ be a mixed lattice group and $x \in \mathcal{G}$. The elements ${}^u x^l = {}^u x + x^l$ and ${}^l x^u = {}^l x + x^u$ are called the *unsymmetrical generalized absolute values* of x. We shall call the element ${}^u x + {}^l x = x^u + x^l$ the *symmetrical generalized absolute value* of x and denote it by $s(x)$.

Due to the unsymmetrical nature of the mixed envelopes, each element has three distinct generalized absolute values instead of just one absolute value. We recall that in Riesz spaces and lattice ordered groups the absolute value is usually defined as $|x| = x \vee (-x)$. The generalized absolute values can also be given similar expressions, but this cannot be done without certain additional assumptions. We will discuss this later, but let us first examine the basic properties of the generalized absolute values. As we might expect, they turn out to have other similarities with the ordinary absolute value, as the next few theorems show.

Theorem 5.2.4. *Let $\mathcal{G}$ be a mixed lattice group and $x \in \mathcal{G}$. Then we have*

(a) ${}^u x^l = {}^u x \curlyvee x^l = {}^u x + x^l$ *and* ${}^l x^u = {}^l x \curlyvee x^u = {}^l x + x^u$.

(b) $s(x) = x^u \curlyvee {}^l x = {}^u x + {}^l x = x^l + x^u = x^l \curlyvee {}^u x$.

(c) ${}^u x^l = {}^l(-x)^u$ *and* $s(-x) = s(x)$.

(d) ${}^u x^l + {}^l x^u = 2s(x)$.

(e) $s(x) \succcurlyeq 0$ *and* $s(x) \geq 0$. *Moreover,* $s(x) = 0$ *if and only if* $x = 0$.

(f) $x \succcurlyeq 0$ *if and only if* $x = {}^l x^u = x^u$ *and* ${}^l x = 0$. *In this case,* $s(x) = {}^u x$.

(g) $(x^u)^u = {}^l(x^u)^u = x^u$ *and* $(x^l)^u = {}^l(x^l)^u = x^l$.

(h) $s(s(x)) = s(x)$.

(i) *If* $0 \preccurlyeq x \preccurlyeq y$ *then* $s(x) \leq s(y)$.

Proof. (a) and (b) are just restatements of Theorem 5.2.2.

(c) By Theorem 5.2.2 (a) we have ${}^{l}(-x)^{u} = {}^{l}(-x) \curlyvee (-x)^{u} = {}^{u}x \curlyvee x^{l} = {}^{u}x^{l}$. The second identity follows easily from (b) and Theorem 5.2.2 (a). Indeed, we have

$$s(-x) = {}^{u}(-x) + {}^{l}(-x) = {}^{l}x + {}^{u}x = s(x).$$

(d) This follows immediately from (a) and (b).

(e) Since $x^{u} \succcurlyeq 0$ and $x^{l} \succcurlyeq 0$ we have $s(x) = x^{u} + x^{l} \succcurlyeq 0$. Also, ${}^{u}x \geq 0$ and ${}^{l}x \geq 0$ imply $s(x) = {}^{u}x + {}^{l}x \geq 0$. It is clear that $x = 0$ implies $s(x) = 0$. Assume then that $s(x) = 0$. Now $s(x) = {}^{u}x + {}^{l}x = 0,$ or ${}^{u}x = -{}^{l}x$. Hence $0 \leq {}^{u}x = -{}^{l}x \leq 0$, which implies ${}^{u}x = 0$. On the other hand, $s(x) = x^{u} + x^{l} = 0,$ or $x^{l} = -x^{u}$. So we have $0 \preccurlyeq x^{l} = -x^{u} \preccurlyeq 0$, which in turn implies $x^{l} = 0$. Consequently, $x = {}^{u}x - x^{l} = 0$.

(f) If $x = x^{u}$ then $x \succcurlyeq 0$. Conversely, let $x \succcurlyeq 0$. Then ${}^{l}x = (-x) \curlyvee 0 \geq 0$ and since $-x \preccurlyeq 0$ and $0 \leq 0$, it follows by (P6) that ${}^{l}x = (-x) \curlyvee 0 \leq 0 \curlyvee 0 = 0$. Hence, ${}^{l}x = 0$. This implies that $x = x^{u} - {}^{l}x = x^{u}$ and ${}^{l}x^{u} = {}^{l}x + x^{u} = x^{u} = x$.

(g) These two follow immediately from (f), since $x^{u} \succcurlyeq 0$ and $x^{l} \succcurlyeq 0$.

(h) (e) and (f) imply that $s(x)^{u} = s(x)$ and $s(x)^{l} = 0$. Hence $s(s(x)) = s(x)^{u} + s(x)^{l} = s(x)$.

(i) If $0 \preccurlyeq x \preccurlyeq y$ then by (f) we have $s(x) = {}^{u}x$ and $s(y) = {}^{u}y$. Now by (P6) the inequalities $x \preccurlyeq y$ and $0 \leq 0$ imply $s(x) = {}^{u}x = x \curlyvee 0 \leq y \curlyvee 0 = {}^{u}y = s(y)$.

■

Geometrically – whenever this kind of geometric interpretation is meaningful – the formula in Theorem 5.2.4 (d) means that the symmetrical absolute value is the midpoint of the line connecting the unsymmetrical absolute values. We also observe that there is an interesting symmetry between the upper and lower envelopes of the upper and lower parts of an element. From Theorems 5.2.2 and 5.2.4 it follows that

$$\begin{aligned} {}^{u}x \curlyvee x^{l} + {}^{l}x \curlyvee x^{u} &= 2\,(x^{u} \curlyvee {}^{l}x) \quad \text{and} \\ {}^{u}x \curlywedge x^{l} + {}^{l}x \curlywedge x^{u} &= 2\,(x^{u} \curlywedge {}^{l}x). \end{aligned}$$

By looking at Definitions 5.2.1 and 5.2.3, and Theorem 5.2.2, it is obvious that if the partial orders $\leq$ and $\preccurlyeq$ coincide then the mixed lattice group $\mathcal{G}$ becomes an ordinary lattice ordered group, and in this case, the upper parts become equal to the ordinary positive part. Similarly, the lower parts become equal to the ordinary negative part and the generalized absolute values become equal to the ordinary absolute value. This gives further evidence that our definition of the generalized absolute values are compatible with the definition of the ordinary absolute value.

The next theorem gives some useful additional properties of the generalized absolute values. The first three are the triangle inequalities.

Theorem 5.2.5. *Let $\mathcal{G}$ be a mixed lattice group. For all $x, y \in \mathcal{G}$ the following hold.*

(a) $\quad {}^u x + {}^u y \geq {}^u(x+y), \quad x^l + y^l \geq (x+y)^l$ *and* ${}^u x^l + {}^u y^l \geq {}^u(x+y)^l$.

(b) $\quad x^u + y^u \geq (x+y)^u, \quad {}^l x + {}^l y \geq {}^l(x+y)$ *and* ${}^l x^u + {}^l y^u \geq {}^l(x+y)^u$.

(c) $\quad s(x+y) \leq s(x) + s(y)$.

(d) $\quad s(x^u - y^u) \leq s(x-y)$.

(e) $\quad x \curlyvee y - y \curlywedge x = {}^u(x-y)^l$ *and* $y \curlyvee x - x \curlywedge y = {}^l(x-y)^u$.

(f) $\quad x \curlyvee y + x \curlyvee y = x + y + {}^u(x-y)^l$.

Proof. (a) Since ${}^u x = x \curlyvee 0 \succcurlyeq x$ and ${}^u x = x \curlyvee 0 \geq 0$ and similarly ${}^u y = y \curlyvee 0 \succcurlyeq y$ and ${}^u y = y \curlyvee 0 \geq 0$, it follows by (P6) that ${}^u x + {}^u y \geq (x+y) \curlyvee 0 = {}^u(x+y)$. The proof for the second inequality is similar. It then follows that

$$ {}^u(x+y)^l = {}^u(x+y) + (x+y)^l \leq {}^u x + {}^u y + x^l + y^l = {}^u x^l + {}^u y^l. $$

(b) Just repeat the arguments in (a).

(c) Using (a) and (b) together with Theorem 5.2.2 (e) we get

$$ s(x+y) = {}^u(x+y) + {}^l(x+y) \leq {}^u x + {}^u y + {}^l x + {}^l y = ({}^u x + {}^l x) + ({}^u y + {}^l y) = s(x) + s(y). $$

(d) From the identity $x = y + (x - y)$ it follows by (b) that $x^u \le y^u + (x - y)^u$ and so we have $x^u - y^u \le (x - y)^u$. Since $0 \preccurlyeq 0$, it follows by (P6) and Theorem 5.2.2 that

$$(x^u - y^u)^u = 0 \curlyvee (x^u - y^u) \le 0 \curlyvee (x - y)^u = ((x - y)^u)^u = (x - y)^u.$$

Exchanging x and y gives similarly $(y^u - x^u)^u \le (y - x)^u$. But by Theorem 5.2.2(a) $(y^u - x^u)^u = (x^u - y^u)^l$ and $(y - x)^u = (x - y)^l$, hence $(x^u - y^u)^l \le (x - y)^l$. Adding the two inequalities gives

$$(x^u - y^u)^u + (x^u - y^u)^l \le (x - y)^u + (x - y)^l,$$

or equivalently, $s(x^u - y^u) \le s(x - y)$.

(e) Applying Theorem 5.2.2 we get

$$\begin{aligned} y \curlyvee x - x \curlywedge y &= y \curlyvee x + (-x \curlyvee -y) \\ &= [y + (-x \curlyvee -y)] \curlyvee [x + (-x \curlyvee -y)] \\ &= [(y - x) \curlyvee 0] \curlyvee [0 \curlyvee (x - y)] \\ &= {}^u(y - x) \curlyvee (y - x)^l \\ &= {}^u(y - x)^l. \end{aligned}$$

The second equality follows from the first one by exchanging x and y and applying Theorem 5.2.2 (a).

(f) From (P1) we get $y \curlywedge x = x + y - (x \curlyvee y)$. The desired result follows by substituting this into (e).

■

As we already mentioned, parts (a), (b) and (c) of the last Theorem are just mixed lattice versions of the triangle inequality (Theorem 2.0.3 (i)). We should also compare the identities in (e) and (f) with the corresponding result in Riesz spaces (Theorem 2.0.3 (g) and (h)).

All the results presented so far hold in an arbitrary mixed lattice group. However, the generalized absolute values can behave somewhat unexpectedly in general, as can be seen by the examples that follow. In pre-regular mixed lattice groups we can say more about the upper and lower parts as well as the generalized absolute values. In fact, under the pre-regularity condition the generalized absolute values turn out to have certain natural properties of the absolute value, as the following result shows.

Theorem 5.2.6. *Let $\mathcal{G}$ be a pre-regular mixed lattice group and $x \in \mathcal{G}$. Then we have*

(a) $x \succcurlyeq 0$ *if and only if* $x = s(x) = {}^{l}x^{u} = {}^{u}x^{l} = {}^{u}x = x^{u}$ *and* ${}^{l}x = x^{l} = 0$.

(b) $x \geq 0$ *if and only if* $x = {}^{u}x^{l} = {}^{u}x$. *In this case,* $x^{l} = 0$.

(c) ${}^{u}x^{l} \geq 0$ *and* ${}^{l}x^{u} \geq 0$. *Moreover,* ${}^{u}x^{l} = {}^{l}x^{u} = 0$ *if and only if* $x = 0$.

(d) ${}^{u}({}^{u}x) = {}^{u}({}^{u}x)^{l} = {}^{u}x$ *and* ${}^{u}({}^{l}x) = {}^{u}({}^{l}x)^{l} = {}^{l}x$.

(e) ${}^{u}({}^{u}x^{l})^{l} = {}^{u}x^{l}$ *and* ${}^{u}({}^{l}x^{u})^{l} = {}^{l}x^{u}$.

(f) *If* $0 \preccurlyeq x \preccurlyeq y$ *then* $s(x) \preccurlyeq s(y)$.

Proof. (a) Let $x \succcurlyeq 0$. By (P5b) it follows that $x = 0 \curlyvee x = x^{u}$. Then we have $x = x^{u} = x^{u} - {}^{l}x$ and it follows that ${}^{l}x = 0$, and ${}^{l}x^{u} = {}^{l}x + x^{u} = x^{u} = x$. Furthermore, we have $x^{l} = 0 \curlyvee (-x) \succcurlyeq 0$, which implies that $x^{l} \geq 0$ by (P4). But if $x \succcurlyeq 0$ then $-x \preccurlyeq 0$, which implies that $-x \leq 0$. Hence, since $0 \preccurlyeq 0$, by (P6) we have $x^{l} = 0 \curlyvee (-x) \leq 0 \curlyvee 0 = 0$. Thus $x^{l} = 0$ and so $x = {}^{u}x - x^{l} = {}^{u}x$ and ${}^{u}x^{l} = {}^{u}x + x^{l} = {}^{u}x = x$. Conversely, if $x = {}^{l}x^{u} = {}^{l}x \curlyvee x^{u}$ then $x \geq x^{u} = 0 \curlyvee x \geq x$ and so $x = x^{u} = 0 \curlyvee x$ and it follows from (P5b) that $x \succcurlyeq 0$.

(b) Similar.

(c) Obviously ${}^{u}x^{l} \geq 0$ and ${}^{l}x^{u} \geq 0$. If $x = 0$ then evidently ${}^{u}x^{l} = {}^{l}x^{u} = 0$. Conversely, assume that ${}^{l}x^{u} = {}^{l}x + x^{u} = 0$. Then $x^{u} = -{}^{l}x$, which implies $2x^{u} = x^{u} - {}^{l}x = x$. Now $x = 2x^{u} \succcurlyeq 0$, and so by (a) we have $x = {}^{l}x^{u} = 0$. If ${}^{u}x^{l} = 0$ then similar arguments show that $x = 0$.

(d) These follow immediately from (b), since ${}^{u}x \geq 0$ and ${}^{l}x \geq 0$.

(e) This follows immediately from (a) and (b).

(f) If $0 \preccurlyeq x \preccurlyeq y$ then by (a) we have $x = s(x)$ and $y = s(y)$. Hence, $s(x) \preccurlyeq s(y)$.

■

The next example shows that the last theorem is not valid without the pre-regularity condition, and that the generalized absolute value may even be negative.

Example 5.2.7. Consider the lexicographically ordered mixed lattice group of Example 5.1.5. If $x = (1, -1)$ then $x \geq 0$ but ${}^u x = (0, 0) \leq x$. Moreover, $x^l = -x \leq 0$ and so ${}^u x^l = {}^u x + x^l = -x \leq 0$. This shows that, in general, one of the generalized absolute values as well as the upper or lower part can in fact be negative with respect to one of the partial orders. However, Theorem 5.2.4 (d) and (e) imply that both unsymmetrical absolute values cannot be negative at the same time. ▲

If the mixed lattice group $\mathcal{G}$ is almost regular, then the following versions of the triangle inequalities hold also for the specific order.

Theorem 5.2.8. *Let $\mathcal{G}$ be an almost regular mixed lattice group. Then for all $x, y \in \mathcal{G}$ the following hold.*

$$ {}^u x + {}^u y \succcurlyeq {}^u(x + y), \qquad {}^l x + {}^l y \succcurlyeq {}^l(x + y) \quad \textit{and} \quad s(x + y) \preccurlyeq s(x) + s(y). $$

In particular, these inequalities hold if $\mathcal{G}$ is regular.

Proof. We will only indicate the proof of the first inequality. The others are proved similarly, as in Theorem 5.2.5. We have ${}^u x + {}^u y \succcurlyeq x + y$ and $0 \succcurlyeq 0$, and so by (P7) we get ${}^u x + {}^u y \succcurlyeq (x + y) \curlyvee 0 = {}^u(x + y)$. The rest then follows as in Theorem 5.2.5. ■

The next lemma will be useful later, in section 7.2.

Lemma 5.2.9. *Let $\mathcal{G}$ be an almost regular mixed lattice group. Then for all $x, y \in \mathcal{G}$ the following hold.*

(a) $\quad {}^u({}^u x - {}^u y) \preccurlyeq {}^u(x - y)^l \quad$ *and* $\quad ({}^u x - {}^u y)^l \leq {}^l({}^l(x - y)^u)^u.$

(b) $\quad s({}^u x - {}^u y) \preccurlyeq s(x-y)$

Proof. (a) As in the proof of Theorem 5.2.5 (d), it follows from the identity $x = y+(x-y)$ that

$$ {}^u x - {}^u y \preccurlyeq {}^u(x-y) \preccurlyeq {}^u(x-y) + (x-y)^l = {}^u(x-y)^l $$

(here we have applied Theorem 5.2.8). Exchanging x and y above gives

$$ {}^u y - {}^u x \preccurlyeq {}^u(y-x)^l = {}^l(x-y)^u. $$

Now an application of (P7) and Theorem 5.2.6 (b) and (c) gives

$$ {}^u({}^u x - {}^u y) = ({}^u x - {}^u y) \curlyvee 0 \preccurlyeq {}^u(x-y)^l \curlyvee 0 = {}^u({}^u(x-y)^l) = {}^u(x-y)^l. $$

From the second inequality we obtain similarly

$$ \begin{aligned} ({}^u x - {}^u y)^l = 0 \curlyvee ({}^u y - {}^u x) \preccurlyeq ({}^l(x-y)^u)^u &\leq ({}^l(x-y)^u)^u + {}^l({}^l(x-y)^u) \\ &= {}^l({}^l(x-y)^u)^u. \end{aligned} $$

(b) The proof is similar to proof of Theorem 5.2.5 (d), so we will only outline the main ideas. Starting with $x = y + (x-y)$ we get (using Theorem 5.2.8) ${}^u x - {}^u y \preccurlyeq {}^u(x-y)$. Applying property (P7) to this gives ${}^u({}^u x - {}^u y) \preccurlyeq {}^u(x-y)$. Exchanging x and y gives ${}^l({}^u x - {}^u y) \preccurlyeq {}^l(x-y)$. The result follows by adding these inequalities. ■

5.3 Additional results on generalized absolute values

In this section we show that under certain conditions the generalized absolute values have some additional properties that do not hold in general. The results in this section – although interesting in their own right – are not needed elsewhere in this work.

For the discussion that follows, we will need the next definition.

Definition 5.3.1. A mixed lattice group $\mathcal{G}$ is called *Archimedean* if the condition $nx \leq y$ for all $n \in \mathbb{N}$ implies $x \leq 0$. Similarly, $\mathcal{G}$ is called *specifically Archimedean* if the condition $nx \preccurlyeq y$ for all $n \in \mathbb{N}$ implies $x \preccurlyeq 0$.

In [11, Theorem 4.1] Eriksson–Bique has proved the following result concerning Archimedean mixed lattice semigroups.

Theorem 5.3.2. *If $\mathcal{S}$ is an Archimedean mixed lattice semigroup, then*

$$nx \curlywedge ny = n(x \curlywedge y) \text{ for all } n \in \mathbb{N} \text{ and } x, y \in \mathcal{S}.$$

Combining the result of Theorem 5.3.2 with the identities in Theorems 4.2.2 and 4.1.2 yields the following corollary.

Corollary 5.3.3. *If $\mathcal{G}$ is a regular Archimedean mixed lattice group, then the following hold for all $x, y \in \mathcal{G}$.*

(a) $nx \curlywedge ny = n(x \curlywedge y)$ *and* $nx \curlyvee ny = n(x \curlyvee y)$ *for* $n = 0, 1, 2, \ldots$

(b) $nx \curlywedge ny = n(x \curlyvee y)$ *and* $nx \curlyvee ny = n(x \curlywedge y)$ *for* $n = -1, -2, \ldots$.

It should be noted that in the preceding theorem the Archimedean property is not a necessary condition, as the result holds in every non-Archimedean Riesz space.

Now, if $\mathcal{G}$ is a regular Archimedean mixed lattice group, we have the following expressions for the generalized absolute values. These correspond to the definition of absolute value in Riesz spaces, that is $|x| = x \vee (-x)$.

Theorem 5.3.4. *Let $\mathcal{G}$ be a regular Archimedean mixed lattice group. Then* ${}^u x^l = x \curlyvee (-x)$ *and* ${}^l x^u = (-x) \curlyvee x$.

Proof. By Theorems 5.2.2 and 5.3.3 we have

$$\begin{aligned}
x \curlyvee (-x) &= ({}^u x - x^l) \curlyvee (x^l - {}^u x) \\
&= (-2x^l) \curlyvee (-2\,{}^u x) + {}^u x + x^l \\
&= -2 \underbrace{(x^l \curlywedge {}^u x)}_{=0} + {}^u x^l \\
&= {}^u x^l.
\end{aligned}$$

The proof for the other identity is similar. ■

In general, the identities in Corollary 5.3.3 and Theorem 5.3.4 do not hold as the following example shows.

Example 5.3.5. Consider the (non-regular) mixed lattice group $(\mathbb{Z}, \leq, \preccurlyeq)$ of Example 5.1.8. This mixed lattice group will also provide us with a counterexample to show that Corollary 5.3.3 and Theorem 5.3.4 do not hold in every mixed lattice group. To see this, choose $p = 3$. Then we find that $-1 \curlyvee 1 = 2$ but $-3 \curlyvee 3 = 3 \neq 3(-1 \curlyvee 1) = 6$ and so 5.3.3 fails to hold. Next, let $x = -1$ and compute ${}^u x = -1 \curlyvee 0 = 2$ and $x^l = 0 \curlyvee 1 = 3$. Then by Theorem 5.2.2 we have ${}^u x^l = 2 + 3 = 5$. However, if we try to use the formula of Theorem 5.3.4 we get $x \curlyvee (-x) = -1 \curlyvee 1 = 2 \neq {}^u x^l$. ▲

In Theorem 5.2.2 we showed that any $x \in \mathcal{G}$ can be written as a difference of upper and lower parts. These decompositions $x = x^u - {}^l x$ and $x = {}^u x - x^l$ with $x^u \curlywedge {}^l x = 0 = x^l \curlywedge {}^u x$ are minimal in the sense of the following theorem.

Theorem 5.3.6. *Let $\mathcal{G}$ be a mixed lattice group. If $x \in \mathcal{G}$ and $x = u - v$ then the following hold.*

(a) *If $u \succcurlyeq 0$ and $v \geq 0$ then $u \geq x^u$ and $v \geq {}^l x$.*

(b) *If $u \geq 0$ and $v \succcurlyeq 0$ then $u \geq {}^u x$ and $v \geq x^l$.*

and conversely,

(c) *If $x = u - v$ with $u \curlywedge v = 0$ then $u = x^u$ and $v = {}^l x$.*

(d) *If $x = u - v$ with $v \curlywedge u = 0$ then $u = {}^u x$ and $v = x^l$.*

Proof. (a) Let $x = u - v$ with $u \succcurlyeq 0$ and $v \geq 0$. Then $u = x + v \geq x$ and it follows by (P6a) and (P5b) that $x^u = 0 \curlyvee x \leq 0 \curlyvee u = u$. Then, using (P6) again we get

$$ {}^l x = (-x) \curlyvee 0 = (v - u) \curlyvee 0 = v + (\underbrace{-u}_{\preccurlyeq 0} \curlyvee \underbrace{-v}_{\leq 0}) \leq v + 0 = v. $$

(b) Similar.

(c) Assume that $x = u - v$ with $u \curlywedge v = 0$. Then we have

$$0 = u \curlywedge v = u + 0 \curlywedge (v - u) = u - 0 \curlyvee (u - v) = u - (u - v)^u = u - x^u,$$

and so $u = x^u$. It now follows that $v = u - x = x^u - x = {}^l x$.

(d) Similar.

■

The preceding theorem also has its counterpart in Riesz spaces (Theorem 2.0.3 (j)). Hence, the above result corresponds to the representation of elements in a Riesz space as a difference of two disjoint elements. The theory of disjointness in mixed lattice semigroups has been studied by Arsove and Leutwiler [7].

If a mixed lattice group $\mathcal{G}$ is a lattice with respect to specific order $\preccurlyeq$ then for all $x \in \mathcal{G}$ the absolute value with respect to $\preccurlyeq$ exist and is defined in the usual way. We denote it by $\mathrm{sp}\,|x| = \mathrm{sp}\,\sup\{x, -x\}$. If $\mathcal{G}$ is also almost regular, we have the following relationship between the specific absolute value and the generalized absolute values.

Theorem 5.3.7. *Let $\mathcal{G}$ be a almost regular mixed lattice group that is a lattice with respect to its specific order. Then* $\mathrm{sp}\,|x| = \mathrm{sp}\,\sup\{x, -x\} = \mathrm{sp}\,\sup\{{}^u x^l, {}^l x^u\}$ *for all* $x \in \mathcal{G}$.

Proof. We have ${}^u x^l = {}^u x \curlyvee x^l \succcurlyeq {}^u x \succcurlyeq x$ and ${}^l x^u = {}^l x \curlyvee x^u \succcurlyeq {}^l x \succcurlyeq -x$. From this it follows that $\mathrm{sp}\,\sup\{{}^u x^l, {}^l x^u\} \succcurlyeq \mathrm{sp}\,\sup\{x, -x\} = \mathrm{sp}\,|x|$. On the other hand, we have $\mathrm{sp}\,|x| \succcurlyeq x$, $\mathrm{sp}\,|x| \succcurlyeq -x$ and $\mathrm{sp}\,|x| \succcurlyeq 0$. Then by (P7) $\mathrm{sp}\,|x| \succcurlyeq 0 \curlyvee x = x^u$ and $\mathrm{sp}\,|x| \succcurlyeq (-x) \curlyvee 0 = {}^l x$. So $\mathrm{sp}\,|x| \succcurlyeq {}^l x \curlyvee x^u = {}^l x^u$. Similarly we show that $\mathrm{sp}\,|x| \succcurlyeq {}^u x^l$. Hence, $\mathrm{sp}\,\sup\{{}^u x^l, {}^l x^u\} \preccurlyeq \mathrm{sp}\,|x|$ and the proof is complete. ■

5.4 Solid sets and ideals

In the theory of Riesz spaces the notion of a solid set plays a fundamental role. Therefore, in order to develop the corresponding theory in mixed lattice groups, we need to find a suitable definition of solidness in the present context. Since we have different types of generalizations of absolute value available, as well as two distinct partial orders, we can define different types of solidness properties accordingly.

Definition 5.4.1. Let $\mathcal{G}$ be a mixed lattice group. A subset $U \subset \mathcal{G}$ is called *UL-solid* if ${}^u x^l \leq {}^u y^l$ and $y \in U$ together imply $x \in U$. A subset $U \subset \mathcal{G}$ is called *LU-solid* if ${}^l x^u \leq {}^l y^u$ and $y \in U$ together imply $x \in U$. A subset $U \subset \mathcal{G}$ is called *symmetrically solid* if $s(x) \leq s(y)$ and $y \in U$ together imply $x \in U$. *Specifically UL-solid*, *specifically LU-solid* and *specifically symmetrically solid* sets are defined analogously with respect to specific order. If U is both UL-solid and LU-solid, then U is called an *unsymmetrically solid* set, that is, if $y \in U$ then ${}^u x^l \leq {}^u y^l$ implies $x \in U$ and ${}^l z^u \leq {}^l y^u$ implies $z \in U$. We define *specifically unsymmetrically solid* set similarly with respect to specific order.

If $\mathcal{G}$ is a pre-regular mixed lattice group and $U \in \mathcal{G}$ is a nonempty subset with one of the above solidness properties, then it is clear that $0 \in U$. In pre-regular mixed lattice groups we also have the following simple result.

Theorem 5.4.2. *Let $\mathcal{G}$ be a pre-regular mixed lattice group and $S \subset \mathcal{G}$. If S is UL-solid, LU-solid, unsymmetrically solid or symmetrically solid, then it is specifically UL-solid, specifically LU-solid, specifically unsymmetrically solid or specifically symmetrically solid, respectively.*

Proof. Let $S \subset \mathcal{G}$ be a UL-solid set. If $x \in S$ and ${}^u y^l \preccurlyeq {}^u x^l$, then by pre-regularity we have ${}^u y^l \leq {}^u x^l$. Since S is UL-solid, it follows that $y \in S$ and hence S is specifically UL-solid. The other cases are proved similarly. ■

In general, none of the solidness properties imply the others. Consider the following example.

Example 5.4.3. Consider again the mixed lattice group of Example 5.1.8,that is, $(\mathbb{Z}, \leq, \preccurlyeq)$, where $\leq$ is the usual order. Let p be a strictly positive integer and define specific order $\preccurlyeq$ by setting

$$n \preccurlyeq m \quad \text{if } m - n \geq 0 \text{ and } m - n \text{ is divisible by } p.$$

Choose again $p = 3$ and consider the set $U = \{0, 1\}$. Then U is UL-solid but not LU-solid. For example, $1 \in U$ and ${}^l(-1)^u = 1 \leq {}^l1^u = 5$ but $-1 \notin U$. Also, U in not symmetrically solid, since $s(2) = 3 \leq s(1) = 3$ and $1 \in U$ but $2 \notin U$.

Next, let $V = \{-1, 0\}$. Then V is LU-solid but not UL-solid, since $-1 \in V$ and ${}^u1^l = 1 \leq {}^u(-1)^l = 5$ but $1 \notin V$. Furthermore, V is not symmetrically solid since $s(1) = 3 \leq s(-1) = 3$ and $-1 \in V$ but $1 \notin V$.

Finally, let $W = \{-3, -2, -1, 0, 1, 2, 3\}$. Now W is symmetrically solid but it is not UL-solid, as $-1 \in W$ and ${}^u4^l = 4 \leq {}^u(-1)^l = 5$ but $4 \notin W$. W is not LU-solid either, since $1 \in W$ and ${}^l(-4)^u = 4 \leq {}^l1^u = 5$ but $-4 \notin W$. ▲

However, the situation is different with solid subgroups.

Theorem 5.4.4. *A subgroup $\mathcal{S}$ of a mixed lattice group $\mathcal{G}$ is (specifically) UL-solid if and only if it is (specifically) LU-solid.*

Proof. Let $\mathcal{S}$ be a UL-solid subgroup and assume that ${}^ly^u \leq {}^lx^u$ with $x \in \mathcal{S}$. Since $\mathcal{S}$ is a subgroup, we have $-x \in \mathcal{S}$ and using the identity of Theorem 5.2.4 (a) we get

$$ {}^u(-y)^l = {}^ly^u \leq {}^lx^u = {}^u(-x)^l. $$

Since $\mathcal{S}$ is UL-solid, it follows that $-y \in \mathcal{S}$. Again, since $\mathcal{S}$ is a subgroup, this implies that $y \in \mathcal{S}$. Hence $\mathcal{S}$ is LU-solid. The reverse implication is proved similarly, and the proof for specifically solid case is essentially the same, just replace $\leq$ by $\preccurlyeq$. ■

By virtue of the last theorem, we shall call a UL-solid (or equivalently, LU-solid) subgroup an *unsymmetrically solid subgroup*.

Solid subgroups are obviously related to Riesz subspaces and ideals, which are important concepts in Riesz space theory. Next we give the corresponding definitions in mixed lattice groups.

Definition 5.4.5. A subgroup $\mathcal{S}$ of a mixed lattice group $\mathcal{G}$ is called a *mixed lattice subgroup* of $\mathcal{G}$ if $\mathcal{S}$ is a mixed lattice group in its own right, that is, if $x \curlyvee y$ belongs to $\mathcal{S}$ whenever $x, y \in \mathcal{S}$. If $\mathcal{A}$ is an unsymmetrically solid or a symmetrically solid mixed lattice subgroup of $\mathcal{G}$ then $\mathcal{A}$ is called an *unsymmetrical ideal* or a *symmetrical ideal* of $\mathcal{G}$, respectively. We define a *specific unsymmetrical ideal* and a *specific symmetrical ideal* similarly with respect to specific order.

Note that in our definition of an ideal we require that $\mathcal{A}$ must be a mixed lattice subgroup. The situation is therefore different than in Riesz spaces, where an ideal is only required to be a solid subspace. In fact, a solid subspace is necessarily a Riesz subspace, whereas in our theory things are a bit more complicated. In general, if $\mathcal{A}$ is a specifically solid subgroup, then there is no guarantee that $\mathcal{A}$ is also a mixed lattice subgroup. So, to avoid any unnecessary complications, we have included this condition in our definition.

However, we can show that solid subgroups are necessarily mixed lattice subgroups. Moreover, if $\mathcal{G}$ is pre-regular, then also specifically solid subgroups are mixed lattice subgroups of $\mathcal{G}$. This result makes it easier to determine whether a given subgroup is actually an ideal, since we only need to check the solidness.

Theorem 5.4.6. *Let $\mathcal{G}$ be a mixed lattice group. The following statements hold.*

(a) Every unsymmetrically solid subgroup of $\mathcal{G}$ is a mixed lattice subgroup, and hence an unsymmetrical ideal in $\mathcal{G}$.

(b) Every symmetrically solid subgroup of $\mathcal{G}$ is a mixed lattice subgroup, and hence a symmetrical ideal in $\mathcal{G}$.

(c) If $\mathcal{G}$ is pre-regular, then every specifically unsymmetrically solid subgroup of $\mathcal{G}$ is a mixed lattice subgroup, and hence a specific unsymmetrical ideal in $\mathcal{G}$.

(d) If $\mathcal{G}$ is pre-regular, then every specifically symmetrically solid subgroup of $\mathcal{G}$ is a mixed lattice subgroup, and hence a specific symmetrical ideal in $\mathcal{G}$.

Proof. (a) Let $\mathcal{A}$ be an unsymmetrically solid subgroup and $x \in \mathcal{A}$. Since $x^u \succcurlyeq 0$, it follows from Theorem 5.2.4 (g) that ${}^l(x^u)^u = x^u \leq {}^l x + x^u = {}^l x^u$. Then by hypothesis $x^u \in \mathcal{A}$, and by Theorem 4.1.5 $\mathcal{A}$ is a mixed lattice subgroup of $\mathcal{G}$.

(b) Let $\mathcal{A}$ be a symmetrically solid subgroup and $x \in \mathcal{A}$. Since $0 \preccurlyeq x^u \preccurlyeq x^u + x^l = s(x)$, it follows from Theorem 5.2.4 (h) and (i) that $s(x^u) \leq s(s(x)) = s(x)$. Then by assumption $x^u \in \mathcal{A}$, and by Theorem 4.1.5 $\mathcal{A}$ is a mixed lattice subgroup of $\mathcal{G}$.

(c) Let $\mathcal{A}$ be a specifically unsymmetrically solid subgroup and $x \in \mathcal{A}$. If $\mathcal{G}$ is pre-regular, then by 5.2.6 (d) we have ${}^u({}^u x)^l = {}^u x \preccurlyeq {}^u x + x^l = {}^u x^l$, since $x^l \succcurlyeq 0$. By hypothesis ${}^u x \in \mathcal{A}$ and so $\mathcal{A}$ is a mixed lattice subgroup of $\mathcal{G}$.

(d) The proof for this is the same as in (b), just apply Theorem 5.2.6 (f) instead of 5.2.4 (i). ∎

The next Lemma gives a useful characterization for ideals in a pre-regular mixed lattice group.

Lemma 5.4.7. *If $\mathcal{G}$ is a pre-regular mixed lattice group, then the following hold.*

(a) $\mathcal{A}$ is a specific symmetrical ideal in $\mathcal{G}$ if and only if the following conditions hold

(i) $x \in \mathcal{A}$ if and only if $s(x) \in \mathcal{A}$,

(ii) $0 \preccurlyeq x \preccurlyeq y$ with $y \in \mathcal{A}$ implies $x \in \mathcal{A}$.

(b) $\mathcal{A}$ is an unsymmetrical ideal in $\mathcal{G}$ if and only if the following conditions hold

(i) $x \in \mathcal{A}$ if and only if ${}^u x^l \in \mathcal{A}$,

(ii) $0 \leq x \leq y$ with $y \in \mathcal{A}$ implies $x \in \mathcal{A}$.

Proof. (a) Assume first that $\mathcal{A}$ is a specific symmetrical ideal. Condition (i) follows from the identity $s(x) = s(s(x))$. For (ii), assume that $0 \preccurlyeq x \preccurlyeq y$ with $y \in \mathcal{A}$. Then by Theorem 5.2.6 (f) we have $0 \preccurlyeq s(x) \preccurlyeq s(y)$ with $y \in \mathcal{A}$. By hypothesis, this implies that $x \in \mathcal{A}$. Conversely, assume that conditions (i) and (ii) hold. If $0 \preccurlyeq s(x) \preccurlyeq s(y)$ and $y \in \mathcal{A}$, then also $s(y) \in \mathcal{A}$ by (i), and by Theorem 5.2.6 (a) we have $0 \preccurlyeq x \preccurlyeq s(s(y))$ with $s(y) \in \mathcal{A}$. By (ii) this implies $x \in \mathcal{A}$, and hence $\mathcal{A}$ is a specific symmetrical ideal.

(b) Similar.

■

We state some further basic facts concerning ideals.

Theorem 5.4.8. *In a mixed lattice group $\mathcal{G}$ the following hold.*

(a) If $\mathcal{G}$ is pre-regular and $\mathcal{A}$ is an unsymmetrical ideal or a symmetrical ideal in $\mathcal{G}$, then it is a specific unsymmetrical ideal or a specific symmetrical ideal, respectively.

(b) Intersections of (specific) symmetrical ideals are (specific) symmetrical ideals, and intersections of (specific) unsymmetrical ideals are (specific) unsymmetrical ideals.

Proof. (a) This follows immediately from Theorem 5.4.2.

(b) We will only prove the assertion for symmetrical ideals; the other cases are similar. Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of symmetrical ideals in $\mathcal{G}$. If $x \in \bigcap_{i\in I} \mathcal{A}_i$ and $s(y) \leq s(x)$, then $x \in \mathcal{A}_i$ for all $i \in I$, and it follows by the hypothesis that $y \in \mathcal{A}_i$ for all $i \in I$. Hence $y \in \bigcap_{i\in I} \mathcal{A}_i$ and so $\bigcap_{i\in I} \mathcal{A}_i$ is a symmetrical ideal in $\mathcal{G}$.

■

We close this section with some examples.

Example 5.4.9. Consider again the mixed lattice group $\mathcal{G}$ of Example 5.1.8. In this mixed lattice group, the only unsymmetrical ideals are $\{0\}$ and $\mathcal{G}$. Indeed, if $\mathcal{S}$ is an unsymmetrical ideal in $\mathcal{G}$ and $x \in \mathcal{S}$ with $x \geq 1 \geq 0$, then since $\mathcal{G}$ is pre-regular, we have ${}^u x^l = x \geq 1$, and so ${}^u 1^l = 1 \leq {}^u x^l$. Hence $1 \in \mathcal{S}$, and consequently, $n \cdot 1 = n \in \mathcal{S}$ for every $n \in \mathbb{Z}$. Thus $\mathcal{S} = \mathcal{G}$. Similar arguments show that if $\mathcal{S}$ is a symmetrical ideal, then either $\mathcal{S} = \{0\}$ or $\mathcal{S} = \mathcal{G}$. It is also easy to see that there are no nontrivial specific symmetrical ideals in $\mathcal{G}$. The subgroup $\mathcal{A} = \{pn : n \in \mathbb{Z}\}$ is a specific unsymmetrical ideal in $\mathcal{G}$. To see this, let $x \in \mathcal{A}$. Then either $x \succcurlyeq 0$ or $x \preccurlyeq 0$. If $x \succcurlyeq 0$ then ${}^u x^l = x \succcurlyeq 0$. If $x \preccurlyeq 0$ then $-x \succcurlyeq 0$ and ${}^u x^l = {}^l(-x)^u = -x$. In both cases, ${}^u x^l \succcurlyeq 0$ and so ${}^u x^l$ is divisible by p. Now if ${}^u y^l \preccurlyeq {}^u x^l$ then ${}^u y^l$ is also divisible by p. Since $y^l \succcurlyeq 0$, which means that y^l is divisible by p, then also ${}^u y = {}^u y^l - y^l$ is divisible by p. Therefore, $y = {}^u y - y^l$ is divisible by p, and so $y \in \mathcal{A}$. ▲

Example 5.4.10. Let us return to Example 5.1.3. In $(\mathcal{G}, \leq, \preccurlyeq)$, the set $\mathcal{A} = \{(x, y) : x = y\}$ is a specific unsymmetrical ideal, but not an unsymmetrical ideal or a (specific) symmetrical ideal. We saw in Example 5.1.3 that the set $\mathcal{S} = \{x \in \mathcal{G} : x \succcurlyeq 0\}$ is a mixed lattice semigroup. Now $\mathcal{A} = \mathcal{S} - \mathcal{S}$, and so $\mathcal{A}$ is a mixed lattice subgroup of $\mathcal{G}$. Let $u = (x, y)$. In Example 5.1.3 we also computed the element u^u, and we can find the element ${}^l u$ as well. If $x > y$ we have

$$u^u = \begin{cases} (x, x) & \text{if } x \geq 0, \\ (0, 0) & \text{if } x < 0, \end{cases} \qquad \text{and} \qquad {}^l u = \begin{cases} (0, x - y) & \text{if } x \geq 0, \\ (-x, -y) & \text{if } x < 0. \end{cases}$$

For $y < x$ we have

$$u^u = \begin{cases} (y, y) & \text{if } y \geq 0, \\ (0, 0) & \text{if } y < 0, \end{cases} \qquad \text{and} \qquad {}^l u = \begin{cases} (y - x, 0) & \text{if } y \geq 0, \\ (-x, -y) & \text{if } y < 0. \end{cases}$$

Finally, if $x = y$, then $u^u = u$ and ${}^l u = 0$ if $x, y \geq 0$, and $u^u = 0$ and ${}^l u = -u$ if $x, y < 0$.

Now it is a straightforward task to check all the different cases.

Case 1: If $x > y$ and $x \geq 0$, then ${}^l u^u = u^u + {}^l u = (x, x + x - y) \notin \mathcal{A}$, since $x - y > 0$.
Case 2: If $x > y$ and $x < 0$, then ${}^l u^u = u^u + {}^l u = (-x, -y) \notin \mathcal{A}$, since $-x < -y$.
Case 3: If $x < y$ and $y \geq 0$, then ${}^l u^u = u^u + {}^l u = (y + y - x, y) \notin \mathcal{A}$, since $y - x > 0$.
Case 4: If $x < y$ and $x < 0$, then ${}^l u^u = u^u + {}^l u = (-x, -y) \notin \mathcal{A}$, since $-x > -y$.
Case 5: If $x = y$, then ${}^l u^u = u \in \mathcal{A}$ if $u \geq 0$, and ${}^l u^u = -u \in \mathcal{A}$ if $u \leq 0$.

This shows that $u \in \mathcal{A}$ if and only if ${}^l u^u \in \mathcal{A}$. Therefore, if $u \in \mathcal{A}$ and ${}^l v^u \preccurlyeq {}^l u^u$, then ${}^l v^u \in \mathcal{A}$, which implies that $v \in \mathcal{A}$. Hence $\mathcal{A}$ is a specific unsymmetrical ideal in $\mathcal{G}$.

To see that $\mathcal{A}$ is not a specific symmetrical ideal, let $x \notin \mathcal{A}$. Then by 5.2.4 we have $s(x) \in \mathcal{A}$, and so $s(x) \preccurlyeq s(s(x))$, but $x \notin \mathcal{A}$. We can show similarly that $\mathcal{A}$ is not a symmetrical ideal. Finally, $\mathcal{A}$ is not an unsymmetrical ideal. For example, if $x = (1, 1)$ and $y = (1, 0)$, then $x \in \mathcal{A}$ and $y = {}^u y^l \leq {}^u x^l = x$, but $y \notin \mathcal{A}$.

Interchanging the partial orders $\leq$ and $\preccurlyeq$, just as was done in Example 5.1.6, we get another mixed lattice group $(\mathcal{G}, \preccurlyeq, \leq)$. The situation is similar to the one above, except that now the set $\mathcal{A}$ is an unsymmetrical ideal, but not a specific unsymmetrical ideal or a (specific) symmetrical ideal. This can be seen by using essentially the same arguments as above. ▲

Example 5.4.11. As another example, consider $\mathcal{G} = BV([0,1])$, the functions of bounded variation on $[0,1]$ (see Example 4.2.3). Let $\mathcal{S}$ be the set of all constant functions. Then $\mathcal{S}$ is obviously a subgroup of $\mathcal{G}$, and it is also a mixed lattice subgroup, where the upper and lower envelopes are given by $f \curlyvee g = \max\{f,g\}$ and $f \curlywedge g = \min\{f,g\}$. However, $\mathcal{S}$ is not a (specific) symmetrical or a (specific) unsymmetrical ideal in $\mathcal{G}$. This can be seen by choosing $f(x) = 1$ and $g(x) = 1 - x$. Then $g \geq 0$ and $f \succcurlyeq 0$, and since $\mathcal{G}$ is regular, we have $g = {}^u g^l = {}^u g$ and $f = {}^u f^l = s(f)$. Now ${}^u g^l \leq {}^u f^l$ and ${}^u g^l \preccurlyeq {}^u f^l$, and $f \in \mathcal{S}$ but $g \notin \mathcal{S}$. Hence $\mathcal{S}$ is not an unsymmetrical ideal or a specific unsymmetrical ideal in $\mathcal{G}$. Observe then that ${}^l g(x) = x$, and so $s(g) = 1 - x + x = 1 = f$. Therefore we have $s(g) \leq s(f)$ and $s(g) \preccurlyeq s(f)$ with $f \in \mathcal{S}$, but $g \notin \mathcal{S}$. Hence $\mathcal{S}$ is not a symmetrical ideal or a specific symmetrical ideal in $\mathcal{G}$.

Next, consider the subset $\mathcal{A} = \{f \in \mathcal{G} : f(0) = 0\}$. This is both (specific) symmetrical and (specific) unsymmetrical ideal in $\mathcal{G}$. To see this, let $f \in \mathcal{A}$ and ${}^u g^l \leq {}^u f^l$. By the definitions of initial and specific orders (see Example 4.2.3), it is clear that ${}^u f(0) = f^l(0) = 0$, and so ${}^u f^l \in \mathcal{A}$. Moreover, since $\mathcal{G}$ is regular, we have $0 \leq {}^u g^l$ and hence $0 \leq {}^u g^l(0) \leq {}^u f^l(0) = 0$. This implies that ${}^u g^l(0) = 0$ and so $g \in \mathcal{A}$. It follows by Theorem 5.4.6 that $\mathcal{A}$ is an unsymmetrical ideal in $\mathcal{G}$. Similar arguments show that $\mathcal{A}$ is a symmetrical ideal in $\mathcal{G}$, and it then follows from Theorem 5.4.2 that $\mathcal{A}$ is also specific symmetrical and unsymmetrical ideal in $\mathcal{G}$. ▲

Chapter 6

Mixed lattice homomorphisms and quotient groups

6.1 Mixed lattice homomorphisms

Next we use the results of preceding sections to study mixed lattice group homomorphisms.

Definition 6.1.1. Let $\mathcal{G}$ and $\mathcal{H}$ be mixed lattice (semi)groups. An additive mapping $T : \mathcal{G} \to \mathcal{H}$ is a *mixed lattice (semi)group homomorphism* if $T(x \curlyvee y) = Tx \curlyvee Ty$ and $T(x \curlywedge y) = Tx \curlywedge Ty$ for all $x, y \in \mathcal{G}$.

If T is a mixed lattice group homomorphism, then by additivity $T(0) = 0$ and $T(-x) = -Tx$ for all $x \in \mathcal{G}$.

We give the following characterization of mixed lattice group homomorphisms (compare this with Theorem 2.0.7).

Theorem 6.1.2. *Let $T : \mathcal{G} \to \mathcal{H}$ be an additive operator between two mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$. Then the following statements are equivalent.*

(a) T *is a mixed lattice group homomorphism.*

(b) $T(x \curlyvee y) = Tx \curlyvee Ty$ *for all* $x, y \in \mathcal{G}$.

(c) $T(x^l) = (Tx)^l$ *for all* $x \in \mathcal{G}$.

(d) $T(x^u) = (Tx)^u$ *for all* $x \in \mathcal{G}$.

(e) $T({}^l x) = {}^l(Tx)$ *for all* $x \in \mathcal{G}$.

(f) $T({}^u x) = {}^u(Tx)$ *for all* $x \in \mathcal{G}$.

(g) $T(x \curlywedge y) = Tx \curlywedge Ty$ *for all* $x, y \in \mathcal{G}$.

(h) *If* $x \curlywedge y = 0$ *in* $\mathcal{G}$*, then* $Tx \curlywedge Ty = 0$ *holds in* $\mathcal{H}$.

Proof. *(a)* $\Longrightarrow$ *(b)* Obvious.

(b) $\Longrightarrow$ *(c)* Assuming that (b) holds, we have

$$T(x^l) = T(0 \curlyvee (-x)) = T(0) \curlyvee T(-x) = 0 \curlyvee (-Tx) = (Tx)^l.$$

(c) $\Longrightarrow$ *(d)* If (c) holds, then by Theorem 5.2.2 (a) we have

$$T(x^u) = T((-x)^l) = (T(-x))^l = (-Tx)^l = Tx^u.$$

(d) $\Longrightarrow$ *(e)* Assume (d) holds. From the identity ${}^l x = x^u - x$ it follows that

$$T({}^l x) = T(x^u - x) = T(x^u) - Tx = (Tx)^u - Tx = {}^l(Tx).$$

(e) $\Longrightarrow$ *(f)* If (e) holds, then

$$T({}^u x) = T({}^l(-x)) = {}^l(T(-x)) = {}^l(-Tx) = {}^u Tx.$$

(f) $\Longrightarrow$ *(g)* First we note that for any elements u and v in a mixed lattice group we have

$$u - {}^u(u - v) = u - (u - v) \curlyvee 0 = u + (v - u) \curlywedge 0 = v \curlywedge u,$$

so by additivity of T and part (f) we have

$$T(y \curlywedge x) = T(x - {}^u(x - y)) = Tx - {}^u(Tx - Ty) = Ty \curlywedge Tx.$$

(g) $\Longrightarrow$ *(h)* If $x \curlywedge y = 0$ then (g) implies $Tx \curlywedge Ty = T(x \curlywedge y) = T(0) = 0.$

(h) $\Longrightarrow$ *(g)* First we note that $0 = (x \curlywedge y) - (x \curlywedge y) = (x - x \curlywedge y) \curlywedge (y - x \curlywedge y),$ so by (h) we have

$$\begin{aligned} 0 &= T(x - x \curlywedge y) \curlywedge T(y - x \curlywedge y) \\ &= (Tx - T(x \curlywedge y)) \curlywedge (Ty - T(x \curlywedge y)) \\ &= Tx \curlywedge Ty \, - \, T(x \curlywedge y). \end{aligned}$$

Hence, $T(x \curlywedge y) = Tx \curlywedge Ty.$

(g) $\Longrightarrow$ *(a)* If *(g)* holds, then

$$\begin{aligned} T(x \curlyvee y) &= T(-(-x \curlywedge -y)) \\ &= -T(-x \curlywedge -y) \\ &= -(T(-x) \curlywedge T(-y)) \\ &= -(-Tx \curlywedge -Ty) \\ &= Tx \curlyvee Ty. \end{aligned}$$

■

In addition, mixed lattice homomorphisms have the following properties concerning the generalized absolute values.

Theorem 6.1.3. *Let $T : \mathcal{G} \to \mathcal{H}$ be a mixed lattice group homomorphism between two mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$. Then*

$$T({}^u x^l) = {}^u(Tx)^l\,, \quad T({}^l x^u) = {}^l(Tx)^u \quad \textit{and} \quad T(s(x)) = s(Tx) \;\; \textit{for all}\, x \in \mathcal{G}.$$

Proof. Applying Theorem 6.1.2 we get

$$T({}^l x^u) = T({}^l x + x^u) = T({}^l x) + T(x^u) = {}^l(Tx) + (Tx)^u = {}^l(Tx)^u.$$

The other identities are proved similarly. ■

If $\mathcal{H}$ happens to be a lattice with respect to one of the partial orderings, then the first two conditions in the last theorem are actually equivalent to T being a homomorphism. To prove this, we need the following result (see [8], Corollary 1, pp. 294).

Lemma 6.1.4. *If G is a lattice ordered group with $x \in G$ and $n \in \mathbb{N}$, then $nx = 0$ implies $x = 0$.*

Theorem 6.1.5. *Let $T : \mathcal{G} \to \mathcal{H}$ be an additive operator between two mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$. If $\mathcal{H}$ is a lattice with respect to one of the partial orderings, then the following conditions are equivalent.*

(a) T *is a mixed lattice group homomorphism.*

(b) $T({}^l x^u) = {}^l(Tx)^u$ *for all* $x \in G$.

(c) $T({}^u x^l) = {}^u(Tx)^l$ *for all* $x \in G$.

Proof. (a) $\Longrightarrow$ *(b)* This was proved in Theorem 6.1.3.
(b) $\Longrightarrow$ *(c)* Follows by the identity ${}^u x^l = {}^l(-x)^u$ in Theorem 5.2.4.
(c) $\Longrightarrow$ *(a)* By Theorem 5.2.5 (f) we have $2(x \curlyvee y) = x + y + {}^u(x - y)^l$, and so applying this together with the hypothesis we obtain

$$\begin{aligned} 2T(x \curlyvee y) = T(2(x \curlyvee y)) &= Tx + Ty + T({}^u(x-y)^l) \\ &= Tx + Ty + {}^u(Tx - Ty)^l \\ &= 2(Tx \curlyvee Ty). \end{aligned}$$

Since $\mathcal{H}$ is a lattice, it follows by Lemma 6.1.4 that $T(x \curlyvee y) = Tx \curlyvee Ty$, that is, T is a mixed lattice group homomorphism. ∎

Note that the assumption in Lemma 6.1.4 that $\mathcal{G}$ is a lattice is not a necessary condition. In Examples 5.1.7 and 5.1.10 we showed that there exists mixed lattice groups that are not lattices with respect to either partial ordering.

We shall say that a mapping $T : \mathcal{G} \to \mathcal{H}$ between two mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$ is *increasing* if $x \leq y$ in $\mathcal{G}$ implies $Tx \leq Ty$ in $\mathcal{H}$. Similarly, T is *specifically increasing*, if $x \preccurlyeq y$ in $\mathcal{G}$ implies $Tx \preccurlyeq Ty$ in $\mathcal{H}$. We now prove the following result which gives some additional basic facts about mixed lattice homomorphisms.

Theorem 6.1.6. *Let $T : \mathcal{G} \to \mathcal{H}$ be a mixed lattice group homomorphism between two mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$. Then the following hold.*

(a) T is specifically increasing. In addition, if $\mathcal{G}$ is pre-regular then T is increasing.

(b) If $\mathcal{S} \subset \mathcal{G}$ is a mixed lattice subgroup (or subsemigroup) of $\mathcal{G}$, then $T(\mathcal{S})$ is a mixed lattice subgroup (or subsemigroup) of $\mathcal{H}$.

(c) If $\mathcal{G}$ is pre-regular then $T(\mathcal{G})$ is pre-regular.

(d) If $\mathcal{G}$ is (almost) regular then $T(\mathcal{G})$ is (almost) regular.

(e) If $\mathcal{G}$ is pre-regular and $\mathcal{A}$ is a specific symmetrical ideal in $\mathcal{G}$, then $T(\mathcal{A})$ is a specific symmetrical ideal in $T(\mathcal{G})$. Moreover, if $\mathcal{G}$ is pre-regular and $\mathcal{A}$ is an unsymmetrical ideal in $\mathcal{G}$, then $T(\mathcal{A})$ is an unsymmetrical ideal in $T(\mathcal{G})$.

Proof. (a) If $x \succcurlyeq y$ then $x - y \succcurlyeq 0$, and by Theorem 5.2.4 we have $(x-y)^u = x - y$. Since T is a homomorphism, it follows that

$$Tx - Ty = T(x-y) = T((x-y)^u) = [T(x-y)]^u \succcurlyeq 0,$$

and so $Tx \succcurlyeq Ty$.

If $\mathcal{G}$ is pre-regular, then $x \geq 0$ if and only if $x = {}^u x$, by 5.2.6. Similar arguments as above now establish that $x \geq y$ implies $Tx \geq Ty$.

(b) These statements are evident by the definition of a mixed lattice homomorphism.

(c) It is sufficient to show that if $y \in T(\mathcal{G})$, then $y \succcurlyeq 0$ implies $y \geq 0$. If $y \succcurlyeq 0$ then by Theorem 5.2.4 (f) $y = y^u$. There exists $x \in \mathcal{G}$ such that $y = Tx$, and we have $y = y^u = (Tx)^u = T(x^u)$. Now $x^u \succcurlyeq 0$ and $\mathcal{G}$ is pre-regular, so it follows that $x^u \geq 0$. By (a) T is increasing, and so $y = T(x^u) \geq 0$. Hence $T(\mathcal{G})$ is pre-regular.

(d) Assume that $\mathcal{G}$ is almost regular. Let $\mathcal{U} = \{w \in T(\mathcal{G}) : w \succcurlyeq 0\}$. We need to show that $\mathcal{U}$ is a mixed lattice semigroup. If $u, v \in \mathcal{U}$, then $u = u^u$ and $v = v^u$ by Theorem 5.2.4, and there exist elements $x, y \in \mathcal{G}$ such that $u = Tx$ and $v = Ty$. Since $x^u \succcurlyeq 0$, $y^u \succcurlyeq 0$ and $\mathcal{G}$ is almost regular, it follows by Theorem 4.1.6 that $x^u \curlywedge y^u \succcurlyeq 0$. By (a) T is specifically increasing, so we have

$$u \curlywedge v = u^u \curlywedge v^u = (Tx)^u \curlywedge (Ty)^u = T(x^u \curlywedge y^u) \succcurlyeq 0.$$

Hence, by Theorem 4.1.6 the set $\mathcal{U}$ is a mixed lattice semigroup and so $T(\mathcal{G})$ is almost regular.

If $\mathcal{G}$ is regular, then the conclusion follows from (b), since if $\mathcal{S}$ is a mixed lattice semigroup such that $\mathcal{G} = \mathcal{S} - \mathcal{S}$, then $T(\mathcal{G}) = T(\mathcal{S}) - T(\mathcal{S})$, where $T(\mathcal{S})$ is a mixed lattice semigroup.

(e) Both statements are proved similarly, so we will only give the proof for the case where $\mathcal{G}$ is pre-regular and $\mathcal{A}$ is an unsymmetrical ideal. By Lemma 5.4.7 it is sufficient to show that if $0 \leq y \in \mathcal{A}$ and $0 \leq Tx \leq Ty$ with $Ty \in T(\mathcal{A})$ and $x \in \mathcal{G}$, then $Tx \in T(\mathcal{A})$. Since $Tx \geq 0$ it follows that $Tx = {}^u(Tx)$. Applying the same arguments as in (d) we can show that $x^l = 0$, and so $x \geq 0$. Moreover, $0 \leq Tx \leq Ty$ implies $T(y - x) \geq 0$, and again similar reasoning shows that $y - x \geq 0$. Hence we have $0 \leq x \leq y$ with $y \in \mathcal{A}$, and since $\mathcal{A}$ is an unsymmetrical ideal, it follows that $x \in \mathcal{A}$ and so $Tx \in T(\mathcal{A})$. Hence, $T(\mathcal{A})$ is an unsymmetrical ideal.

■

Let $T : \mathcal{G} \to \mathcal{H}$ be a mapping, and recall that the *kernel* of T is the set $N(T) = \{x \in \mathcal{G} : Tx = 0\}$. We have the following result concerning the kernel of a mixed lattice homomorphism.

Theorem 6.1.7. *Let $T : \mathcal{G} \to \mathcal{H}$ be a mixed lattice group homomorphism between two mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$. Then the kernel of T is a specific symmetrical ideal. In addition, if $\mathcal{G}$ is pre-regular then the kernel $N(T)$ is also a symmetrical ideal and an unsymmetrical ideal, and hence a specific unsymmetrical ideal.*

Proof. The kernel $N(T)$ is clearly a subgroup of $\mathcal{G}$, and it is also a mixed lattice subgroup of $\mathcal{G}$, for if $x, y \in N(T)$ then $T(x \curlyvee y) = Tx \curlyvee Ty = 0 \curlyvee 0 = 0$, hence $x \curlyvee y \in N(T)$. Assume then that $0 \preccurlyeq s(y) \preccurlyeq s(x)$ and $x \in N(T)$. Since T is specifically increasing, it follows that

$$T(0) = 0 \preccurlyeq T(s(y)) \preccurlyeq T(s(x)) = s(Tx) = s(0) = 0.$$

This implies that $T(s(y)) = s(Ty) = 0$, and hence (by 5.2.4 (a)) $Ty = 0$, i.e. $y \in N(T)$. Thus, $N(T)$ is a specific symmetrical ideal.

Next, assume that $\mathcal{G}$ is pre-regular. Then by 5.2.6 we have ${}^u x^l \geq 0$ for all $x \in \mathcal{G}$. Suppose now that $0 \leq {}^u y^l \leq {}^u x^l$ with $x \in N(T)$. By Theorem 6.1.6 T is increasing, and so we have

$$T(0) = 0 \leq T({}^u y^l) \leq T({}^u x^l) = {}^u (Tx)^l = {}^u 0^l = 0.$$

Hence, $T({}^u y^l) = {}^u (Ty)^l = 0$, and it follows by Theorem 5.2.6 (c) that $Ty = 0$, or $y \in N(T)$. This shows that $N(T)$ is an unsymmetrical ideal in $\mathcal{G}$, and hence a specific unsymmetrical ideal by Theorem 5.4.2. Similar arguments show that $N(T)$ is also a symmetrical ideal in $\mathcal{G}$. ■

The converse of the preceding theorem holds too. That is, if $\mathcal{A}$ is an unsymmetrical ideal in a pre-regular mixed lattice group $\mathcal{G}$, then there exists a mixed lattice homomorphism whose kernel is $\mathcal{A}$. We will prove this later in Theorem 6.2.3. The last two results correspond to Theorem 2.0.8 in Riesz spaces.

If $T : \mathcal{G} \to \mathcal{H}$ is a bijective mixed lattice homomorphism, then T is called a *mixed lattice isomorphism*, and the mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$ are said to be *isomorphic*. In pre-regular mixed lattice groups, isomorphisms have the following characterization which is analogous to the characterization of Riesz isomorphisms between two Riesz spaces (Theorem 2.0.9).

Theorem 6.1.8. *Let $\mathcal{G}$ and $\mathcal{H}$ be pre-regular mixed lattice groups and $T : \mathcal{G} \to \mathcal{H}$ a bijective additive mapping. Then T is a mixed lattice isomorphism if and only if both T and T^{-1} are increasing and specifically increasing.*

Proof. If T is an isomorphism, then T^{-1} is also an isomorphism and they are both increasing and specifically increasing by Theorem 6.1.6. Conversely, assume that T and T^{-1} are increasing and specifically increasing. If $x, y \in \mathcal{G}$, then $x \preccurlyeq x \curlyvee y$ and $y \leq x \curlyvee y$, which implies that $Tx \preccurlyeq T(x \curlyvee y)$ and $Ty \leq T(x \curlyvee y)$. Hence, $Tx \curlyvee Ty \leq T(x \curlyvee y)$. Similar reasoning shows that $T^{-1}(u) \curlyvee T^{-1}(v) \leq T^{-1}(u \curlyvee v)$ for all $u, v \in \mathcal{H}$. Since T is bijective, there exist unique u and v in $\mathcal{H}$ such that $u = Tx$ and $v = Ty$. So we have $x \curlyvee y \leq T^{-1}(Tx \curlyvee Ty)$. Now since T is increasing, we can apply T and obtain

$T(x \curlyvee y) \leq Tx \curlyvee Ty$. Hence, $T(x \curlyvee y) = Tx \curlyvee Ty$, and so T is a mixed lattice homomorphism. Since T was assumed to be bijective, T is an isomorphism. ■

Mixed lattice homomorphisms also have a domination property described in the next theorem. For a similar result in Riesz spaces, see [14], 2.6.9.

Theorem 6.1.9. *Let $\mathcal{G}$ and $\mathcal{H}$ be mixed lattice groups and $T : \mathcal{G} \to \mathcal{H}$ a mixed lattice homomorphism. If $S : \mathcal{G} \to \mathcal{H}$ is an additive operator such that S and $T - S$ are both increasing and specifically increasing, then S is a mixed lattice homomorphism.*

Proof. Let x and y be elements of $\mathcal{G}$ satisfying $x \curlywedge y = 0$. This implies that $x \succcurlyeq 0$ and $y \geq 0$. To see this, we write $x \curlywedge y = x + 0 \curlywedge (y - x) = x - 0 \curlyvee (x - y) = x - (x-y)^u = 0$, or $x = (x-y)^u \succcurlyeq 0$. Similarly we get $y = {}^u(y-x) \geq 0$. By our hypothesis it follows that $0 \preccurlyeq Sx \preccurlyeq Tx$ and $0 \leq Sy \leq Ty$. Since T is a homomorphism, we have $0 \leq Sx \curlywedge Sy \leq Tx \curlywedge Ty = T(x \curlywedge y) = T(0) = 0$. Hence $Sx \curlywedge Sy = 0$ and so by Theorem 6.1.2 S is a mixed lattice homomorphism. ■

Next we will examine the extensions of mixed lattice semigroup homomorphisms to mixed lattice group homomorphisms. The following lemma is a slight modification of Kantorovich's classical extension lemma in Riesz spaces (see [2], Theorem 1.10).

Lemma 6.1.10. *Let $\mathcal{S}$ and $\mathcal{T}$ be mixed lattice semigroups and $T : \mathcal{S} \to \mathcal{T}$ an additive operator. If $\mathcal{G}$ and $\mathcal{H}$ are the regular mixed lattice groups generated by $\mathcal{S}$ and $\mathcal{T}$, respectively (i.e. $\mathcal{G} = \mathcal{S} - \mathcal{S}$ and $\mathcal{H} = \mathcal{T} - \mathcal{T}$), then there exists a unique additive operator $S : \mathcal{G} \to \mathcal{H}$ such that S extends T, that is, $Sx = Tx$ for all $x \in \mathcal{S}$.*

Proof. Let $T : \mathcal{S} \to \mathcal{T}$ be an additive mapping. Let $w \in \mathcal{G}$. Then $w = u - v$ where $u, v \in \mathcal{S}$. This representation may not be unique but if we define $S : \mathcal{G} \to \mathcal{H}$ by $Sw = Tu - Tv$ then Sw does not depend on the particular representation of w. Indeed, if $u_1 - u_2 = v_1 - v_2 = w \quad (u_1, u_2, v_1, v_2 \in \mathcal{S})$ are two representations of w, then $u_1 + v_2 = u_2 + v_1$ and

$$T(u_1) + T(v_2) = T(u_1 + v_2) = T(u_2 + v_1) = T(u_2) + T(v_1)$$

and so $T(u_1) - T(u_2) = T(v_1) - T(v_2)$, that is, $S(u_1 - u_2) = S(v_1 - v_2)$. Now, if $u \in \mathcal{S}$ then $u = u - 0$ and $Su = Tu - T0 = Tu$, so S is indeed an extension of T to $\mathcal{G}$. To prove the additivity of S, let $u, v \in \mathcal{G}$ and write $u = u_1 - u_2$ and $v = v_1 - v_2$ where $u_1, u_2, v_1, v_2 \in \mathcal{S}$. Then we have

$$\begin{aligned} S(u+v) &= S(u_1 + v_1 - (u_2 + v_2)) \\ &= T(u_1 + v_1) - T(u_2 + v_2) \\ &= (Tu_1 - Tu_2) + (Tv_1 - Tv_2) \\ &= Su + Sv. \end{aligned}$$

Thus S is additive on $\mathcal{G}$. For the uniqueness, assume that $R : \mathcal{G} \to \mathcal{H}$ is another extension of T. If $u = u_1 - u_2$ with $u_1, u_2 \in \mathcal{S}$, then since R and T must be identical on $\mathcal{S}$ we have $Ru = Ru_1 - Ru_2 = Tu_1 - Tu_2 = Su$. Thus $R = S$ and the proof is complete. ∎

The next result shows that if a mixed lattice group is extended to a group of formal differences, then a mixed lattice semigroup homomorphism can be extended to a mixed lattice group homomorphism.

Theorem 6.1.11. *Let $T : \mathcal{S} \to \mathcal{T}$ be a mixed lattice semigroup homomorphism between two mixed lattice semigroups $\mathcal{S}$ and $\mathcal{T}$. If $\mathcal{G} = \mathcal{S} - \mathcal{S}$ and $\mathcal{H} = \mathcal{T} - \mathcal{T}$ are the mixed lattice group extensions of $\mathcal{S}$ and $\mathcal{T}$, then T can be extended to a mixed lattice group homomorphism $S : \mathcal{G} \to \mathcal{H}$.*

Proof. Let $T : \mathcal{S} \to \mathcal{T}$ be a mixed lattice semigroup homomorphism. Since T is an additive map, by Lemma 6.1.10 it has an additive extension $S : \mathcal{G} \to \mathcal{H}$ where $\mathcal{G} = \mathcal{S} - \mathcal{S}$ and $\mathcal{H} = \mathcal{T} - \mathcal{T}$ are the mixed lattice group extensions of $\mathcal{S}$ and $\mathcal{T}$, respectively. We need to show that the extension S preserves the mixed lattice operations. Let $w, z \in \mathcal{G}$ and write $w = u - v$ and $z = x - y$ where $u, v, x, y \in \mathcal{S}$. Then by Theorem 4.2.2

$$w \curlyvee z = (u - v) \curlyvee (x - y) = \underbrace{(u + y)}_{\in \mathcal{S}} \curlyvee \underbrace{(x + v)}_{\in \mathcal{S}} - \underbrace{(v + y)}_{\in \mathcal{S}}.$$

Since $S = T$ on $\mathcal{S}$, we get

$$\begin{aligned} S(w \curlyvee z) &= S(\underbrace{(u+y) \curlyvee (x+v)}_{\in \mathcal{S}}) - S(\underbrace{(v+y)}_{\in \mathcal{S}}) \\ &= T((u+y) \curlyvee (x+v)) - T(v+y) \\ &= T(u+y) \curlyvee T(x+v) - T(v+y) \\ &= (Tu+Ty) \curlyvee (Tx+Tv) - Tv - Ty \\ &= (Tu-Tv) \curlyvee (Tx-Ty) \\ &= Sw \curlyvee Sz \end{aligned}$$

for all $w, z \in \mathcal{G}$. By Theorem 6.1.2 S is a mixed lattice group homomorphism. ■

6.2 Quotient mixed lattice groups

In this section we examine how quotient group constructions work in mixed lattice groups, and we obtain results that have analogues in Riesz space theory. We have seen that the generalized absolute values may exhibit somewhat counter-intuitive behaviour in general mixed lattice groups (see Example 5.2.7). To avoid such complications, we will restrict our study to pre-regular mixed lattice groups.

First we recall some basic facts from group theory. Let $\mathcal{G}$ be an additive group and $\mathcal{S}$ a subgroup of $\mathcal{G}$. We can introduce an equivalence relation $\sim$ in $\mathcal{G}$ by defining $x \sim y$ iff $x - y \in \mathcal{S}$. The set of all elements in $\mathcal{G}$ equivalent to some $x \in \mathcal{G}$ is called the equivalence class (or coset) of x and denoted by $[x]$. The subgroup $\mathcal{S}$ itself is one of the equivalence classes, namely $\mathcal{S} = [0]$. The set of all equivalence classes is called the quotient group of $\mathcal{G}$ modulo $\mathcal{S}$ and it is denoted by $\mathcal{G}/\mathcal{S}$. We can define addition in the set $\mathcal{G}/\mathcal{S}$ by

$$[x] + [y] = [x+y] \quad \text{for all } [x] \text{ and } [y],$$

and the set $\mathcal{G}/\mathcal{S}$ then becomes a group together with this addition, where the inverse element is $-[x] = [-x]$.

Assume now that $\mathcal{G}$ is a pre-regular mixed lattice group. We will prove that if $\mathcal{A}$ is an unsymmetrical ideal in $\mathcal{G}$ then we can define initial and specific orders in $\mathcal{G}/\mathcal{A}$ in such manner that $\mathcal{G}/\mathcal{A}$ becomes a mixed lattice group. First we define the partial orderings.

Definition 6.2.1. Let $\mathcal{A}$ be an unsymmetrical ideal in a pre-regular mixed lattice group $\mathcal{G}$. If $[x], [y] \in \mathcal{G}/\mathcal{A}$ then we define $[x] \leq [y]$ whenever there exists elements $x \in [x]$ and $y \in [y]$ such that $x \leq y$ in $\mathcal{G}$. Similarly, $[x] \preccurlyeq [y]$ whenever there exists elements $x \in [x]$ and $y \in [y]$ such that $x \preccurlyeq y$ in $\mathcal{G}$.

Before we proceed we need to verify that the above definition actually defines partial orderings in $\mathcal{G}/\mathcal{A}$. We will begin with the initial order.

It is obvious that $[x] \leq [x]$ for every $[x]$. For transitivity, assume that $[x] \leq [y]$ and $[y] \leq [z]$. Let $x \in [x]$, $y_1, y_2 \in [y]$ and $z \in [z]$ be such that $x \leq y_1$ and $y_2 \leq z$. Then

$$x \leq y_1 = y_2 + (y_1 - y_2) \leq z + (y_1 - y_2),$$

but by definition $y_1 - y_2 \in \mathcal{A}$, and so $z + (y_1 - y_2) \in [z]$. This shows that $[x] \leq [z]$.

Finally, to prove antisymmetry, let $[x] \leq [y]$ and $[y] \leq [x]$. Then there exist $x_1, x_2 \in [x]$ and $y_1, y_2 \in [y]$ such that $x_1 \leq y_1$ and $y_2 \leq x_2$. Since $\mathcal{A}$ is a subgroup, it follows that

$$0 \leq y_1 - x_1 \leq (y_1 - x_1) + (x_2 - y_2) = (x_2 - x_1) + (y_1 - y_2) \in \mathcal{A}.$$

By the pre-regularity assumption we now have

$${}^{u}(y_1 - x_1)^{l} = y_1 - x_1 \leq (x_2 - x_1) + (y_1 - y_2) = {}^{u}(x_2 - x_1 + y_1 - y_2)^{l}.$$

Since $\mathcal{A}$ is an unsymmetrical ideal, it follows that $y_1 - x_1 \in \mathcal{A}$, that is, $[x] = [y]$. Noting that $\mathcal{A}$ is also a specific unsymmetrical ideal (by Theorem 5.4.2), we can show similarly that the specific order given in Definition 6.2.1 is likewise a partial ordering in $\mathcal{G}/\mathcal{A}$.

We will now show that with these orderings the set $\mathcal{G}/\mathcal{A}$ becomes a pre-regular mixed lattice group.

Theorem 6.2.2. *If $\mathcal{A}$ is an unsymmetrical ideal in a pre-regular mixed lattice group $\mathcal{G}$, then $\mathcal{G}/\mathcal{A}$ is a pre-regular mixed lattice group with respect to initial and specific orders given in Definition 6.2.1.*

Proof. First we must show that these partial orderings are compatible with the group structure, that is, $\mathcal{G}/\mathcal{A}$ is a partially ordered group with respect to orderings $\leq$ and $\preccurlyeq$. To this end, assume that $[x] \leq [y]$ and choose $x \in [x]$ and $y \in [y]$ such that $x \leq y$, and let $z \in [z]$. Then $x+z \leq y+z$, and so $[x+z] \leq [y+z]$. By definition of addition in $\mathcal{G}/\mathcal{A}$, this is equivalent to $[x]+[z] \leq [y]+[z]$. Conversely, assume that $[x]+[z] \leq [y]+[z]$, or equivalently, $[x+z] \leq [y+z]$. Then by what was just proved we have $[x+z]+[-z] \leq [y+z]+[-z]$, or $[x] \leq [y]$. Hence, the cancellation law holds for the partial order $\leq$. The specific order $\preccurlyeq$ is treated similarly.

It remains to show that $\mathcal{G}/\mathcal{A}$ is a mixed lattice group. We will do this by showing that $[x]^u$ exists for all $x \in \mathcal{G}$ and it is equal to $[x^u]$. Let $x \in \mathcal{G}$ and notice that $x^u \geq x$ and $x^u \succcurlyeq 0$, so we have $[x] \leq [x^u]$ and $[0] \preccurlyeq [x^u]$. Assume then that $[x] \leq [y]$ and $[0] \preccurlyeq [y]$. Then there exist $x_1 \in [x]$ and $y_1, y_2 \in [y]$ such that $x_1 \leq y_1$ and $0 \preccurlyeq y_2$. It follows that $x_1 \leq y_1 \leq y_2 \curlyvee y_1$ and $x - x_1 \leq 0 \curlyvee (x - x_1) = (x - x_1)^u$. This implies that

$$x = x_1 + (x - x_1) \leq y_2 \curlyvee y_1 + (x - x_1)^u = y_2 + (y_1 - y_2)^u + (x - x_1)^u.$$

Now we also have

$$y_2 + (y_1 - y_2)^u + (x - x_1)^u \succcurlyeq 0,$$

and therefore

$$x^u = 0 \curlyvee x \leq y_2 + (y_1 - y_2)^u + (x - x_1)^u.$$

Now, since $y_1, y_2 \in [y]$ and $x, x_1 \in [x]$, we have $y_1 - y_2 \in \mathcal{A}$ and $x - x_1 \in \mathcal{A}$, and since $\mathcal{A}$ is a mixed lattice subgroup, it follows that $(y_1 - y_2)^u \in \mathcal{A}$ and $(x - x_1)^u \in \mathcal{A}$. Hence,

$$x^u \leq y_2 + (y_1 - y_2)^u + (x - x_1)^u \in y_2 + \mathcal{A} = [y_2] = [y].$$

Thus, $[x^u] \leq [y]$ and so we have proved that

$$[x^u] = \min \{[w] \in \mathcal{G}/\mathcal{A} : [0] \preccurlyeq [w] \text{ and } [x] \leq [w]\} = [0] \curlyvee [x] = [x]^u,$$

and so $\mathcal{G}/\mathcal{A}$ is a mixed lattice group, by Theorem 4.1.5.

Finally, let $[x] \preccurlyeq [y]$. Then there exist x and y in $\mathcal{G}$ satisfying $x \preccurlyeq y$. Since $\mathcal{G}$ is pre-regular, this implies that $x \leq y$. Consequently, $[x] \leq [y]$ and so $\mathcal{G}/\mathcal{A}$ is pre-regular. ■

The mixed lattice group $\mathcal{G}/\mathcal{A}$ is called the *quotient mixed lattice group of $\mathcal{G}$ modulo $\mathcal{A}$*.

Now we observe that there is a close connection between mixed lattice homomorphisms and quotient mixed lattice groups.

Theorem 6.2.3. *If $\mathcal{A}$ is an unsymmetrical ideal in a pre-regular mixed lattice group $\mathcal{G}$, then $\mathcal{G}/\mathcal{A}$ is a mixed lattice homomorphic image of $\mathcal{G}$ under the canonical mapping $x \mapsto [x]$. Conversely, if $\mathcal{G}$ is pre-regular and $T : \mathcal{G} \to \mathcal{H}$ is a surjective mixed lattice homomorphism with kernel $N(T)$, then $\mathcal{G}/N(T)$ and $\mathcal{H}$ are isomorphic.*

Proof. Let $\mathcal{G}$ be a pre-regular mixed lattice group and $\mathcal{A}$ an unsymmetrical ideal in $\mathcal{G}$. Inspecting the proof of Theorem 6.2.2 reveals that $[x^u] = [x]^u$ in $\mathcal{G}/\mathcal{A}$, and so by Theorem 6.1.2 the mapping $x \mapsto [x]$ is a mixed lattice homomorphism of $\mathcal{G}$ onto $\mathcal{G}/\mathcal{A}$, and $\mathcal{A}$ is the kernel of this homomorphism.

Conversely, let $T : \mathcal{G} \to \mathcal{H}$ be a surjective mixed lattice homomorphism with kernel $N(T)$. Then by Theorem 6.1.7, $N(T)$ is an unsymmetrical ideal in $\mathcal{G}$, and $\mathcal{G}/N(T)$ is a mixed lattice group by Theorem 6.2.2. Now the mapping $f : \mathcal{G}/N(T) \to \mathcal{H}$, where $f([x]) = Tx$, is a mixed lattice isomorphism of $\mathcal{G}/N(T)$ onto $\mathcal{H}$. To prove this, we need to show that f is a bijective mixed lattice homomorphism. First, if $Tx = Ty$ then $T(x - y) = 0$, and so $x - y \in N(T)$. But this means that x and y belong to the same equivalence class in $\mathcal{G}/N(T)$, that is, $[x] = [y]$. Thus f is injective. Next, let $z \in \mathcal{H}$. Since T is surjective, there exists an element $x \in \mathcal{G}$ such that $z = Tx$. Now we have $f([x]) = Tx = z$, and so f is surjective. Finally, since $[x^u] = [x]^u$ in $\mathcal{G}/N(T)$, and T is a homomorphism, it follows that

$$f([x]^u) = f([x^u]) = T(x^u) = (Tx)^u = (f([x]))^u.$$

Hence, f is a bijective homomorphism, and therefore an isomorphism. ∎

We give two examples.

Example 6.2.4. Let $\mathcal{G} = BV([0,1])$ be the set of all functions of bounded variation on the interval $[0,1]$ (see Example 4.2.3). In Example 5.4.11 we saw that the set $\mathcal{S}$ of all constant functions is a mixed lattice subgroup of $\mathcal{G}$, but not an ideal. In Example 5.4.11 we also considered the set $\mathcal{A} = \{f \in \mathcal{G} : f(0) = 0\}$, which was seen to be a (specific) symmetrical ideal as well as a (specific) unsymmetrical ideal in $\mathcal{G}$. As a group, the quotient group $\mathcal{G}/\mathcal{S}$ can be identified with the subgroup $\mathcal{A}$ of $\mathcal{G}$. If we define a mapping $T : \mathcal{G}/\mathcal{S} \to \mathcal{A}$ by

$T([f]) = f - f(0)$, then T is clearly an additive map (that is, a group homomorphism), which is bijective. Indeed, if $T([f]) = T([g])$ and choose any $f \in [f]$ and $g \in [g]$, then $f - f(0) = g - g(0)$, or $f - g = f(0) - g(0)$. Now $f(0) - g(0)$ is a constant and so $f - g \in \mathcal{S}$, or $[f] = [g]$. Thus T is injective. Let $f \in \mathcal{A}$. Then $f \in \mathcal{G}$ and so $f \in [f] \in \mathcal{G}/\mathcal{S}$, and we have $T([f]) = f - f(0) = f$, since $f(0) = 0$ for all $f \in \mathcal{A}$. Hence T is surjective, and therefore bijective. This shows that the groups $\mathcal{G}/\mathcal{S}$ and $\mathcal{A}$ are isomorphic.

Now, since $\mathcal{S}$ is not an ideal, Theorem 6.2.1 does not apply here. In particular, the canonical mapping $x \mapsto [x]$ is not a mixed lattice homomorphism, because the kernel of this mapping is $\mathcal{S}$, and the kernel of a mixed lattice homomorphism is necessarily an ideal by Theorem 6.1.7. ▲

Example 6.2.5. Consider the $\mathcal{G}$, $\mathcal{A}$ and $\mathcal{S}$ of the preceding example. Since $\mathcal{A}$ is an unsymmetrical ideal in $\mathcal{G}$ (see Example 5.4.11), the quotient group $\mathcal{G}/\mathcal{A}$ is a mixed lattice group and the canonical map $f \mapsto [f]$ is a mixed lattice homomorphism. We observe that the elements of $\mathcal{G}/\mathcal{A}$ are the equivalence classes determined by constant functions, that is, $\mathcal{G}/\mathcal{A} = \{[f] : f \text{ is constant}\}$. Indeed, if f and g are in $[f]$, then $f - g \in \mathcal{A}$, or $(f - g)(0) = 0$. This implies $f(0) = g(0)$ and hence $[f] = [g]$ if and only if $f(0) = g(0)$. Since $f(0)$ is a constant, we have $[f] = [f(0)]$.

Now the mapping $T : \mathcal{G} \to \mathcal{S}$ defined by $Tf = f(0)$ is a mixed lattice homomorphism, since

$$T(f \curlyvee g) = (f \curlyvee g)(0) = \max\{f(0), g(0)\} = Tf \curlyvee Tg.$$

Clearly, T is surjective, since $\mathcal{S} \subset \mathcal{G}$ and $T(\mathcal{S}) = \mathcal{S}$. Moreover, $\mathcal{A}$ is the kernel of T, since $T(f) = f(0) = 0$ if and only if $f \in \mathcal{A}$. Hence, by Theorem 6.2.3, $\mathcal{G}/\mathcal{A}$ and $\mathcal{S}$ are isomorphic mixed lattice groups. ▲

Chapter 7

Topological mixed lattice groups

So far the theory of mixed lattice groups has been developed only from purely algebraic point of view. We will now explore the idea of a topological mixed lattice group, and derive some basic facts about mixed lattice group topologies using the theory of topological Riesz spaces ([1]) and lattice ordered groups ([13]) as our model.

7.1 Topological groups

In this section we present the terminology and some basic facts concerning topological groups. For proofs of the statements in this section we refer to [16].

Let G be an additive commutative group. If τ is a topology on G such that

(i) the map $x \mapsto -x$ of G into G is continuous,

(ii) the map $(x, y) \mapsto x + y$ of $G \times G$ into G is continuous,

then (G, τ) is called a *topological group*. In condition (ii) it is understood that $G \times G$ is equipped with the product topology. The above definition can be also expressed by saying that a group topology is compatible with the algebraic structure of the group.

Equivalently, the conditions (i) and (ii) can be given as follows:

(i) If $x \in G$ and U is a neighborhood of $-x$, then there is a neighborhood V of x such that $-V \subset U$.

(ii) If $x, y \in G$ and U is a neighborhood of $x + y$, then there are neighborhoods V of x and W of y such that $W + V \subset U$.

A collection $\mathcal{B}$ of neighborhoods of zero is called a *neighborhood base at zero* if for every neighborhood U of zero there exists $V \in \mathcal{B}$ such that $V \subset U$.

Every group topology is translation invariant, that is, for any $y \in G$ the map $x \mapsto x + y$ is a homeomorphism. It follows that if $x \in G$ and $\mathcal{B}$ is a neighborhood base at zero, then $x + \mathcal{B} = \{x + B : b \in \mathcal{B}\}$ is a neighborhood base at x. This means that the topology is completely determined if the neighborhoods of zero are given, and for this reason, we usually only need to work with neighborhoods of zero.

A subset V of a group G is called *symmetric*, if $V = -V$. Every topological group has a base at zero consisting of symmetric sets. Furthermore, if U is any neighborhood of zero, then there is a symmetric neighborhood V of zero such that $V + V \subset U$.

Let G and H be topological groups. A map $f : G \to H$ is called *uniformly continuous*, if for every neighborhood V of zero in H there exists a neighborhood U of zero in G such that $x - y \in U$ implies $f(x) - f(y) \in V$. For example, the inversion map $x \mapsto -x$, addition and translation are all uniformly continuous. Uniform continuity implies continuity, and the composition of uniformly continuous maps is uniformly continuous.

7.2 Topologies in mixed lattice groups

We will now turn our attention to the problem of finding a suitable topology in a mixed lattice group. Again, if we wish to obtain any useful results then the topology should be compatible with the underlying algebraic structure. The usual approach is to require that the fundamental algebraic operations must be continuous. Therefore, the mixed lattice operations should be continuous, and this leads us to the following definition.

Definition 7.2.1. Let $\mathcal{G}$ be a mixed lattice group. Then $(\mathcal{G}, \tau)$ is called a *topological mixed lattice group* if τ is a group topology such that the mixed upper and lower envelopes are continuous mappings from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{G}$.

In Riesz spaces the compatibility is achieved by requiring that the topology has a local base consisting of solid sets. In mixed lattice groups things are more complicated, for as we have seen, there are different notions of solidness, and therefore, different possible ways to approach the problem of defining a compatible topology. We proceed to explore some of these possibilities but let us first consider the uniform continuity of the mixed lattice operations.

Theorem 7.2.2. *Let $(\mathcal{G}, \tau)$ be a topological mixed lattice group. Then the following conditions are mutually equivalent.*

(a) The map $(x, y) \mapsto x \curlyvee y$ from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{G}$ is uniformly continuous.

(b) The map $(x, y) \mapsto x \curlywedge y$ from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{G}$ is uniformly continuous.

(c) The map $x \mapsto {}^{l}x$ from $\mathcal{G}$ to $\mathcal{G}$ is uniformly continuous.

(d) The map $x \mapsto {}^{u}x$ from $\mathcal{G}$ to $\mathcal{G}$ is uniformly continuous.

(e) The map $x \mapsto x^{l}$ from $\mathcal{G}$ to $\mathcal{G}$ is uniformly continuous.

(f) The map $x \mapsto x^{u}$ from $\mathcal{G}$ to $\mathcal{G}$ is uniformly continuous.

Proof. *(a)* $\Longrightarrow$ *(b)* This follows from the identity $x \curlywedge y = -(-x \curlyvee -y)$ and uniform continuity of the map $x \mapsto -x$.
(b) $\Longrightarrow$ *(c)* Follows from the identity ${}^{l}x = -(x \curlywedge 0)$.
(c) $\Longrightarrow$ *(d)* Follows from the identity ${}^{u}x = {}^{l}(-x)$.
(d) $\Longrightarrow$ *(e)* This follows since $x^{l} = {}^{u}x - x$ and addition is uniformly continuous.
(e) $\Longrightarrow$ *(f)* Follows from the identity $x^{u} = (-x)^{l}$.
(f) $\Longrightarrow$ *(a)* If $x \mapsto x^{u}$ is uniformly continuous then, since addition is also uniformly continuous, it follows that

$$x \curlyvee y = x + 0 \curlyvee (y - x) = x + (y - x)^{u}$$

is also uniformly continuous as a composition of uniformly continuous mappings. ∎

It would be desirable to add the continuity of the generalized absolute values to the list of conditions in preceding theorem, but the main problem with this is the fact that in a group the multiplication by rational numbers is not available. Otherwise the result would follow easily from the identities $2x^l = {}^u x^l - x$ and $2^l x = {}^l x^u - x$ and Theorem 5.2.4 (d).

However, in any topological mixed lattice group all the generalized absolute values are also continuous mappings from $\mathcal{G}$ to $\mathcal{G}$ (but their continuity does not necessarily imply the conditions in Theorem 7.2.2).

Theorem 7.2.3. *Let $(\mathcal{G}, \tau)$ be a topological mixed lattice group. Then the maps $x \mapsto {}^l x^u$, $x \mapsto {}^u x^l$ and $x \mapsto s(x)$ from $\mathcal{G}$ to $\mathcal{G}$ are continuous.*

Proof. It follows from the definition of topological mixed lattice group that the upper and lower parts are continuous, so ${}^u x^l = {}^u x + x^l$, ${}^l x^u = {}^l x + x^u$ and $s(x) = {}^u x + {}^l x$ are also continuous. ■

Next we give some sufficient conditions for a group topology to be compatible with the mixed lattice structure. We will first focus on the situation where the local base consists of symmetrically solid sets. In this case we can obtain the most general results.

Definition 7.2.4. A group topology τ on a mixed lattice group is called *locally symmetrically solid* if τ has a base at zero consisting of symmetrically solid sets.

It is important to note that the above definition is in agreement with general properties of a group topology. Recall that a set V is called *symmetric* if $V = -V$, and every group topology has a base at zero consisting of symmetric sets. Every symmetrically solid set V is also necessarily symmetric, since $s(x) = s(-x)$, and so $x \in V$ implies $-x \in V$. Hence, it is possible for a group topology to be locally symmetrically solid.

Our next Theorem shows that locally symmetrically solid topologies have the desired properties.

Theorem 7.2.5. *Let $(\mathcal{G}, \tau)$ be a mixed lattice group with a locally symmetrically solid group topology τ. Then the equivalent conditions given in Theorem 7.2.2 hold in $(\mathcal{G}, \tau)$.*

Proof. We will show that the assumption implies uniform continuity of the map $x \mapsto x^u$. Let U be a neighborhood of zero. Then there exists a symmetrically solid neighborhood V of zero such that $V \subset U$. If $x - y \in V$, then $s(x-y) \in V$ and by Theorem 5.2.5 (d) we have $s(x^u - y^u) \leq s(x-y)$ and since V is symmetrically solid we have $x^u - y^u \in V \subset U$. This shows that the map $x \mapsto x^u$ is uniformly continuous. ■

Next we will look at the case where the local base consists of unsymmetrically solid sets. In this case, we need to make some additional assumptions for the underlying mixed lattice group. We recall from Section 5.4 that a set S is called unsymmetrically solid, if it is both UL-solid and LU-solid.

Definition 7.2.6. A group topology τ on a mixed lattice group is called *locally unsymmetrically solid* if τ has a base at zero consisting of unsymmetrically solid sets.

Our next theorem shows that locally unsymmetrically solid topologies have the required continuity properties if $\mathcal{G}$ is almost regular.

Theorem 7.2.7. *Let $(\mathcal{G}, \tau)$ be a almost regular mixed lattice group equipped with a locally unsymmetrically solid group topology τ. Then the equivalent conditions given in Theorem 7.2.2 hold in $(\mathcal{G}, \tau)$.*

Proof. We will show that the assumption implies uniform continuity of the map $x \mapsto {}^u x$. Let U be a neighborhood of zero. Then there exists an unsymmetrically solid neighborhood V of zero such that $V \subset U$. Furthermore, there exists an unsymmetrically solid neighborhood W of zero such that $W + W \subset V$. Next, choose a symmetric neighborhood W_1 of zero such that $W_1 \subset W$. Let $x - y \in W_1$. Then, since ${}^u({}^u x - {}^u y) \geq 0$ it follows by Theorem 5.2.6 (b) and Lemma 5.2.9 (a) that

$$^u({}^u({}^u x - {}^u y))^l = {}^u({}^u x - {}^u y) \preccurlyeq {}^u(x-y)^l.$$

This implies that ${}^u({}^u x - {}^u y) \in W$, since W is specifically UL-solid (by Theorem 5.4.2). Now W_1 is symmetric, so $y - x \in W_1 \subset W$, and we have

$$^u({}^l(x-y)^u)^l = {}^u({}^u(y-x)^l)^l = {}^u(y-x)^l \preccurlyeq {}^u(y-x)^l.$$

Hence ${}^{l}(x-y)^{u} \in W$, since W is specifically UL-solid. Now we have $({}^{u}x - {}^{u}y)^{l} \succcurlyeq 0$, and so it follows from 5.2.6 (a) and Lemma 5.2.9 (a) that

$${}^{l}(({}^{u}x - {}^{u}y)^{l})^{u} = ({}^{u}x - {}^{u}y)^{l} \leq {}^{l}({}^{l}(x-y)^{u})^{u}.$$

This implies that $({}^{u}x - {}^{u}y)^{l} \in W$, by LU-solidness of W. Hence we have

$${}^{u}({}^{u}x - {}^{u}y)^{l} = {}^{u}({}^{u}x - {}^{u}y) + ({}^{u}x - {}^{u}y)^{l} \in W + W \subset V,$$

and so ${}^{u}x - {}^{u}y \in V \subset U$, since V is UL-solid. This shows that the map $x \mapsto {}^{u}x$ is uniformly continuous. ∎

If $\mathcal{G}$ is almost regular then the equivalent conditions in the above theorem certainly hold if τ is locally symmetrically solid (by Theorem 7.2.5). But now we can actually prove the same result under slightly weaker assumptions. In fact, it is sufficient that τ has a base of neighborhoods at zero consisting of specifically symmetrically solid sets.

Theorem 7.2.8. *Let $(\mathcal{G}, \tau)$ be a almost regular mixed lattice group equipped with a locally specifically symmetrically solid group topology τ. Then the equivalent conditions given in Theorem 7.2.2 hold in $(\mathcal{G}, \tau)$.*

Proof. The proof is exactly the same as in Theorem 7.2.5, except that here we prove the uniform continuity of the map $x \mapsto {}^{u}x$. In the proof, replace $\leq$ by $\preccurlyeq$ and apply Lemma 5.2.9 (b) in place of Theorem 5.2.5 (d). ∎

As our final example, we give a concrete example of a locally symmetrically solid topology in $BV([0,1])$.

Example 7.2.9. Let us consider again the functions of bounded variation (Example 4.2.3). We will show that $d(f,g) = \sup\{s(f-g)(x) : x \in [0,1]\}$ is a metric on $BV([0,1])$ and that d generates a locally symmetrically solid topology. First we note that $d(f,g) < \infty$, since functions of bounded variations are bounded ([3], Theorem 6.7). The properties of a metric now follow from the properties of symmetrical absolute value. Indeed, that $d(f,g) \geq 0$ and $d(f,g) = 0$ if and only if $f = g$ follow from Theorem 5.2.4 (e). The symmetry property $d(f, g) = d(g, f)$ is evident by Theorem 5.2.4 (c). Finally, by Theorem 5.2.5 (c) we have $s(f - g) = s(f - h + (h - g)) \leq s(f - h) + s(h - g)$, and this implies that $d(f, g) \leq d(f, h) + d(h, g)$. Thus, d is a metric.

Moreover, d is a translation-invariant metric, that is, $d(f + h, g + h) = d(f, g)$ for all $f, g, h \in BV([0, 1])$. This follows immediately from the definition of d. Hence, the topology induced by d is a group topology (see [16], Section 2.7). It only remains to prove that this topology is locally symmetrically solid. For this, it is sufficient to show that every ball $B(0, r)$ is symmetrically solid. Assume that $f \in B(0, r)$, that is, $d(f, 0) < r$. Now, if $s(g) \leq s(f)$ then $s(g)(x) \leq s(f)(x)$ for all $x \in [0, 1]$ and this implies that $d(g, 0) \leq d(f, 0)$, or $g \in B(0, r)$. Hence, the metric topology induced by d is locally symmetrically solid. ▲

To conclude, we will make a few remarks to indicate how we could take the above example, and indeed the whole theory of mixed lattice structures, one step further. It has been shown that an Archimedean mixed lattice semigroup can be extended to an ordered convex cone which is also a mixed lattice semigroup ([11], Theorems 4.5 and 4.10). In such extensions, the multiplication by real numbers is defined. Furthermore, an ordered convex cone can be extended to an ordered vector space. Hence, we can talk about linear mixed lattices, and the set $BV([0, 1])$ could be seen as an Archimedean regular linear mixed lattice. The result of Corollary 5.3.3 could also be extended to cover multiplication by real numbers, and so it follows easily that the metric d in the above example has the homogeneity property $d(cf, cg) = |c|d(f, g)$ for all $c \in \mathbb{R}$. In this case, the metric d defines a norm on $BV([0, 1])$, by $||f|| = d(f, 0)$, and the norm topology is locally symmetrically solid.

More generally, we could define the concept of a linear mixed lattice in a similar way as the mixed lattice group was defined in Definition 4.1.1. That is, a linear mixed lattice is simply an ordered vector space with two partial orderings such that the mixed envelopes exist for each pair of elements. Naturally, the basic theory developed in this work then applies to linear mixed lattices as well, but the scalar multiplication would introduce some additional properties that are not present in the group setting. In any case, this should open up some interesting possibilities for further research.

Chapter 8

Conclusions and open questions

In this work we have generalized some basic theory of Riesz spaces and lattice ordered groups to mixed lattice groups, including the theory of absolute values and ideals, homomorphisms, quotients and some topological considerations. As we have pointed out, most of these concepts have their counterparts in Riesz space theory as well as in the theory of mixed lattice semigroups. Since this work represents the first attempt to investigate these ideas, there are naturally many open questions left, and even those aspects of the theory we have presented here are nowhere near complete. Indeed, there are many possible directions for future research.

Theorem 5.3.6 hints at the concept of disjoint elements in mixed lattice groups. In Riesz spaces, the theory concerning disjointness is well developed, and it has also been studied in mixed lattice semigroups by Arsove and Leutwiler. We have not pursued these ideas any further in this work, and a great deal of research could be done in this direction. Disjointness is closely related to the theory of bands, which are ideals with certain order completeness properties. We did not study the theory of bands here either, although they are very important in theory of Riesz spaces and mixed lattice semigroups. For further research, the theory of bands and related topics (such as Riesz decompositions in mixed lattice groups) should be among the first things to consider.

Another worthwhile topic of research would be operator theory in mixed lattice groups. Under what conditions does the set of additive operators between mixed lattice groups have a mixed lattice structure, and what kind of order completeness properties does the set of operators possess? These two questions have been discussed by Arsove and Leutwiler for additive operators in mixed lattice semigroups. Similar results are also well known for linear operators in Riesz spaces.

In Chapter 7 we gave conditions under which a group topology is compatible with the mixed lattice structure. However, our results provided only sufficient conditions. It would be interesting to find out whether it is possible to obtain characterizations similar to locally solid topologies in Riesz spaces. In the last section, after Example 7.2.9 we also briefly discussed the mixed lattice structure on vector spaces. This approach could open up a whole new set of possibilities. One could then study how the mixed lattice structure behaves under the scalar multiplication, and develop a theory of linear mixed lattices that would be analogous to the theory of Riesz spaces. Furthermore, Example 7.2.9 suggests that it should be possible to define a compatible norm in such vector spaces. This could enable us to apply the tools of functional analysis to study linear mixed lattices. Perhaps this kind of approach could make the theory more suited for applications too. Nevertheless, the results of Chapter 7 should provide a good starting point for these investigations.

One particularly interesting question is related to the regularity properties of linear mixed lattices. In Example 5.1.4 we showed that there are pre-regular mixed lattice groups that are not almost regular. Our counterexample was based on the fact that the set of positive elements in a mixed lattice group is not necessarily convex. However, in an ordered vector space the positive cone is always convex, so this raises the following question: Does there exist a pre-regular linear mixed lattice that is not almost regular? A negative answer to this question would result in a somewhat simplified theory.

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