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Tiivistelmä

Tämä työ käsittelee multimodaalilogiikkoja, joiden kieleen liittyy periaatteessa mielivaltainen algebra siten että jokaista modaalioperaattoria vastaa kieleen liittyvän algebran kaava. Syntaktista teoriaa ei käsitellä, lukuunottamatta muutamaa esimerkinomaista deduktiota liitteessä. Sen sijaan tässä työssä keskitytään malliteoriaan, erityisesti algebrallisten (syntaktisten) operaattoreiden tulkintaan. Aluksi esitellään työn alaan kuuluvien modaalikielten sekä mallien yleinen määrittely ja käsitellään joukko-opillisten operaatioiden modaalista määriteltävyyttä. Tässä yhteydessä esitellään *perustan* käsite, joka on kehystä yleisempi struktuuri, missä mallien perusjoukkoon W on liitetty kielen syntaktisten operaattorien tulkinta funktiona $(W \times W)^n \to (W \times W)$. Tämän jälkeen käsitellään malliteoreettisia työkaluja (kehysten erilliset yhdisteet ja kehysten väliset pseudo-epimorfismit), joiden avulla osoitetaan tiettyjä yleisiä ehtoja operaatioiden modaaliselle määriteltävyydelle kehystasolla.

Käsiteltävien logiikkojen nimeksi on tässä annettu *Propositional State Transition Logics* (propositionaalinen tilasiirtymien logiikka). Kyseessä on työnimi, joka jäi lopullisen vedoksen otsikoksi. *Prosessien logiikka* olisi ehkä ollut tässä yhteydessä osuvampi, mutta tämä nimitys on jo käytössä ja viittaa lähinnä dynaamista logiikkaa ja aikalogiikoita yhdistelevään multimodaalilogiikkaan. Käyttöön jäänyt nimitys lienee kuitenkin perusteltu, sillä tässä tekstissä käsiteltävät loogiset kielet soveltunevat parhaiten mallintamaan erilaisia systeemejä, joissa mielenkiinnon kohteena ovat systeemin tilasta toiseen tapahtuvat epätriviaalit (siis, ei-atomaariset) siirtymät. Esimerkiksi ohjelmat, joissa algoritmeja ketjutetaan, iteroidaan ja muulla tavoin yhdistellään, ovat tämänkaltaisia systeemejä.

Tässä työssä ei kuitenkaan keskitytä mallintamiseen eikä käsiteltävien logiikkojen intuitiiviseen tulkintaan. Tekstin keskeisin sisältö koostuu määriteltävyystuloksista, joissa todistetaan vastaavuuksia kaavajoukkojen ja näiden sisältämien kaavojen syntaktisten operaattorien tulkintojen välillä. Määriteltävyystulokset on jaettu kahteen lukuun, joista ensimäinen käsittelee kehysmääriteltävyyttä ja toinen perustamääriteltävyyttä. Osoittautuu, että kaikki relaatiokalkyylin operaattorit ovat perustamääriteltäviä ja tyypilliset sulkeumat (refleksiivinen, symmetrinen ja transitiivinen) sekä perusta- että kehysmääriteltäviä.

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Abstract

This paper is an introduction to a class of multimodal logics with an algebraic structure associated with modal operators. The discussion focuses on model theory, especially on the interpretation of several syntactic operators on modalities. First part of the paper discusses definitions of languages and models and considers conceptual issues on modal definability of operations on relations. Also in this context the concept of *foundations*, which is an abstraction level between domains and frames, is introduced. Then some general results on the expressive power of the class of logics in question is covered.

The core of the article consists of several definability results establishing correspondences between formulae and interpretations of syntactic operators within various languages. The correspondence results are divided into two sections discussing operator definability on the level of frames and foundations respectively. It will be demonstrated that all the operations of the calculus of relations are foundation definable, and moreover that all the common closures (reflexive, symmetric and transitive) are definable on the level of frames and foundations.

Contents

1 Introduction

In this paper we consider a class of multimodal logics called *Propositional State Transition Logics* (henceforth $PSTL$). The class contains multimodal logics that allow operations on modalities such that syntactic operators with atomic modalities form an algebraic structure. The expressions of the resulting algebra are then associated with modal operators. Well-known example of this kind of logic is *Propositional Dynamic Logic* (PDL).[1] In fact, $PSTL$ can been seen as a generalization of the PDL. Moreover, the fragment of PDL that does not include the test operator (usually denoted $\langle \varphi ? \rangle$, wherein φ is a formula of PDL) is a propositional state transition logic. A rather recent study in related spirit can be found in (Broersen, 2003) with a discussion of informal semantics of logics that resemble $PSTL$.

The name *propositional state transition logics* conveys the idea that this class of logics is designed to reason about systems that involve processes that transform the state of the systems, for example programs, automata, dynamic epistemic systems, epistemic actions of agents, etc. The research behind this paper originates in a seminar held at the University of Tampere at summer 2005 on an article entitled "Learning and Epistemic Logic" by Veikko Rantala (Rantala, 2006). The article discusses a brand of epistemic logic that incorporates learning processes and the possibility to apply operations on the processes. Although the focal idea in (Rantala, 2006) is not directly involved in the operations on modalities,[2] Ari Virtanen (the seminar instructor) became interested in generalizing the concept of operations on the learning processes. Thus, at the initial stage of the research the possible epistemic interpretations of operations were considered, although the focus rather quickly shifted towards pure mathematics without much contemplation about informal semantics of $PSTL$. Modeling of, or reasoning about, some (re)active, dynamic, or generally any causal system within the formal framework requires that the (possible) compositional structure of the processes involved can be properly expressed in the chosen language. That is, the language assigned for the task should be expressive enough to capture the essential properties of the structure of the processes in question. Moreover, formal expressions should be specific in the sense that they do not allow inappropriate interpretations. In what follows, modeling issues are not further elaborated but hopefully these superficial remarks reveal some of the motivations underlying the research.

Sections 6 and 7 consist of several definability results which establish correspondences between formulae of the $PSTL$ languages and interpretations of syntactic operators on modalities. The referred sections form the core of this report, wherein two different logics are covered. Section 6 discusses the more common variety of modal definability, i.e. modal definability issues on the level of frames. In Section 7 modal definability on the level of *foundations* is considered. *Foundation logics* is rather novel approach that constitutes a general and elegant framework for studying modal definability of operations on binary relations. Indeed, within the multimodal logics without such operations the concept of

[1] For exposition, see e.g. (Harel et al., 2000) or virtually any standard textbook on multimodal logics.

[2] To be sure, (Rantala, 2006) considers only one operation that is not even well-defined in the sense of definition 3.1.

foundations does not make much sense, but within the $PSTL$ and related logics the foundation framework offers a natural and expressive level of abstraction over frames. It will be demonstrated that all the operations of the calculus of relations are foundation definable.

Section 3 discusses the concept of definability in the $PSTL$ frame logics and foundation logics respectively. During the research, it has become clear that the concept of definability in the context of $PSTL$ is not necessarily too trivial. The discussion is intentionally kept on a rather intuitive level. Sections 4 and 5 provide general model theoretic tools for resolving issues in frame logic definability.

Next section begins to explore the $PSTL$ by defining the $PSTL$ languages and models. The fundamental definitions are originally proposed by Ari Virtanen in an unpublished manuscript at the above mentioned seminar in 2005. The research has been carried out in more or less active periods during 2005–2007 in close collaboration with Virtanen and Antti Kuusisto. Specifying the actual contribution of Virtanen and Kuusisto to the accomplishment of this work would flood the paper with footnotes. Some focal contributions are explicitly mentioned, but above all I wish to thank them for their valuable effort through several meetings and discussions, and also for their tolerance of my rather loose working pace. I also would like to express my gratitude to Mirja Hartimo for several helpful comments on the penultimate draft of the text and especially for correcting my English.

The paper is written with an aspiration to make it as accessible as possible to everyone with an entry-level competence in modal logics. Thus, extensive background information on the subject is not required, but at least cursory acquaintance with some standard textbook is recommended.[3]

[3] For example (Blackburn et al., 2002), or (Rantala & Virtanen, 2004) for Finnish readers.

2 Syntax and semantics

2.1 Syntax

Definition 2.1. The basic elements of any given language in the context of $PSTL$ are the following:

A Set of *propositional symbols* $\Pi \subseteq \{p_0, p_1, p_2, \ldots\}$

A Set of *atomic process symbols* $\mathcal{A} \subseteq \{a_0, a_1, a_2, \ldots\}$

A Set of *syntactic operators* $\mathcal{F} \subseteq \{f_0, f_1, f_2, \ldots\}$

Logical Connectives $\neg, \wedge$ and parentheses (,), [,]

Sets Π, $\mathcal{A}$ and $\mathcal{F}$ are assumed as nonempty and countable.

Definition 2.2. Provided that sets $\mathcal{A}$ and $\mathcal{F}$ are given, the set of all modal terms Λ is defined in the following way:

Every process symbol $a \in \mathcal{A}$ is modal a term, thus for all $a \in \mathcal{A} : a \in \Lambda$

If $f \in \mathcal{F}$ is k-ary operator and $\alpha_1, \ldots, \alpha_k \in \Lambda$, then $f(\alpha_1, \ldots, \alpha_k) \in \Lambda$

For the modal term constructed from k-ary syntactic operator $f \in \mathcal{F}$ with parameter array $\alpha_1, \ldots, \alpha_k \in \Lambda$, we use the prefix notation $f(\alpha_1, \ldots, \alpha_k)$. For the sake of legibility, we typically use the infix notation for binary operators (e.g. $\alpha \otimes \beta$, if α, β are modal terms and $\otimes$ is binary syntactic operator), and denotation of the form $\alpha^\dagger$ for a unary operator $\dagger \in \mathcal{F}$. If denotation of the form $\tilde{\cdot}$ is used for unary operator $\tilde{\cdot} \in \mathcal{F}$, modal term $\tilde{\cdot}(\alpha)$ is denoted $\tilde{\alpha}$. This is because of standard denotational conventions for some common (e.g. set-theoretical) operators. In the following, the term *process* refers to any modal term $\alpha \in \Lambda$, whether atomic or molecular.

At this point it should also be noted that if the syntactic expression of a formula or other such entity contains a chain of (associative) binary operators, connectives, etc. with an ambivalent structure, e.g. $R \cup S \cup T$, the order of execution is from left to right; that is, the syntactic structure is assumed to be $((R \cup S) \cup T)$ in the case of the example.

Definition 2.3. If sets $\Pi, \mathcal{A}$ and $\mathcal{F}$ are given, particular $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$ is the set of formulae generated by the following recursive formula generation rules:

If $p_i \in \Pi$, then (p_i) is a formula

If (φ) is a formula, then $(\neg\varphi)$ is a formula

If (φ) and (ψ) are formulae, then $(\varphi \land \psi)$ is a formula

If (φ) is a formula and $\alpha \in \Lambda$, then $([\alpha]\varphi)$ is a formula

In what follows, we conform to the standard practice and omit the outermost and otherwise unnecessary brackets from the formulae.

Definition 2.4. We use logical connectives $\lor, \to, \leftrightarrow$, modal operator $\langle\alpha\rangle$ and constants $\bot$ and $\top$ as abbreviations defined in the standard way: provided that φ and ψ are formulae and $\alpha \in \Lambda$,

$$\varphi \lor \psi \equiv_{def} \neg(\neg\varphi \land \neg\psi)$$

$$\varphi \to \psi \equiv_{def} \neg\varphi \lor \psi$$

$$\varphi \leftrightarrow \psi \equiv_{def} (\varphi \to \psi) \land (\psi \to \varphi)$$

$$\langle\alpha\rangle\varphi \equiv_{def} \neg[\alpha]\neg\varphi$$

$$\bot \equiv_{def} (\varphi \land \neg\varphi)$$

$$\top \equiv_{def} (\varphi \lor \neg\varphi)$$

2.2 Semantics

2.2.1 *PSTL* structures

Definition 2.5. The quadruple $\langle W, I, \tilde{R}, P\rangle$ is a model of language $L(\Pi, \mathcal{A}, \mathcal{F})$, if it satisfies the following definitions.

The set $W = \{w_0, w_1, w_2, \ldots\}$ is the *domain* consisting all states w_i of the model. Only restriction on the domain W is that in every model, W is a non-empty set.

Structure $\langle W, I\rangle$ is called *a foundation.* An element I is an interpretation mapping, $\mathrm{dom}(I) = \mathcal{F}$, defined as follows: Let $f \in \mathcal{F}$ be a k-ary syntactic operator, then[4]

$$I(f) = F : (\mathcal{P}(W \times W))^k \to \mathcal{P}(W \times W).$$

Thus, the interpretation I of k-ary operator f maps every k-tuple of binary relations of the domain to a single binary relation, hence the interpretation mapping is a function

$$I : \mathcal{F} \to \bigcup_{n=0}^{\infty} \{F \mid F : (\mathcal{P}(W \times W))^n \to \mathcal{P}(W \times W)\}.$$

We will not make much use of the above technical definition of I, but instead will use it to generalize a map $\tilde{R}$ to a more intuitive mapping R, defined as follows:

[4]Provided that A is a set, we use the notation A^n for the n-fold Cartesian product of A and the notation $\mathcal{P}(A)$ to denote the power set of A.

Structure $\langle W, I, \tilde{R}\rangle$ is a *frame* of the model, if $\tilde{R}$ is a mapping $\tilde{R} : \mathcal{A} \rightarrow \mathcal{P}(W \times W)$, whence $\tilde{R}(a) \subseteq W \times W$, $a \in \mathcal{A}$. For a given frame $\langle W, I, \tilde{R}\rangle$, function $R : \Lambda \rightarrow \mathcal{P}(W \times W)$ is a generalization of $\tilde{R}$ defined in the following way. Let $f \in \mathcal{F}$ be k-ary operator, then

For all $a \in \mathcal{A} : R(a) = \tilde{R}(a)$

For all $\alpha_1, \ldots, \alpha_k \in \Lambda : R(f(\alpha_1, \ldots, \alpha_k)) = I(f)(R(\alpha_1), \ldots, R(\alpha_k))$.

Finally, *the model* $\langle W, I, \tilde{R}, P\rangle$ is obtained by adding *a valuation function* $P : \Pi \rightarrow \mathcal{P}(W)$ into the frame $\langle W, I, \tilde{R}\rangle$.

We say that *the model* $M = \langle W, I, \tilde{R}, P\rangle$ *is a* $PSTL$*-model for language* $L(\Pi, \mathcal{A}, \mathcal{F})$ (or $L(\Pi, \mathcal{A}, \mathcal{F})$*-model*), if there is an interpretation in M for the syntactic operator f, process a and proposition p exactly when $f \in \mathcal{F}$, $a \in \mathcal{A}$ and $p \in \Pi$. Likewise, we say that *frame* $F = \langle W, I, \tilde{R}\rangle$ *is* $L(\Pi, \mathcal{A}, \mathcal{F})$*-frame* if the above defining condition is met, save the reference to the set of propositional symbols Π, and that *the foundation* $\langle W, I\rangle$ *is* $L(\Pi, \mathcal{A}, \mathcal{F})$*-foundation*, if the interpretation $I(f)$ is defined for every operator symbol $f \in \mathcal{F}$.

2.2.2 Truth definitions

Definition 2.6. Let $M = \langle W, I, \tilde{R}, P\rangle$ be a $PSTL$-model for the language $L(\Pi, \mathcal{A}, \mathcal{F})$, $w \in W$, $p \in \Pi$, $\alpha \in \Lambda$ and $\varphi, \psi \in L(\Pi, \mathcal{A}, \mathcal{F})$. The truth definitions for formulae in the state $w \in W$ are the following:

$M, w \vDash p$ if and only if $w \in P(p)$

$M, w \vDash \neg\varphi$ if and only if $M, w \nvDash \varphi$

$M, w \vDash \varphi \wedge \psi$ if and only if $M, w \vDash \varphi$ *and* $M, w \vDash \psi$

$M, w \vDash [\alpha]\varphi$ if and only if $\forall w' \in W : wR(\alpha)w' \Rightarrow M, w' \vDash \varphi$

We say that φ *is true in the state* w *of the model* M*, if* $M, w \vDash \varphi$. For the formulae abbreviations presented in 2.4, the truth definitions are implied in the above. Since in this paper the semantics of modal operators are of special importance, an explicit presentation of the truth definition for the operator formulae of the form $\langle\alpha\rangle\varphi$ is given here:

$M, w \vDash \langle\alpha\rangle\varphi$ if and only if $\exists w' \in W : wR(\alpha)w'$ *and* $M, w' \vDash \varphi$

We write $M \vDash \varphi$, if *the formula* φ *is valid in the model* M, that is $\forall w \in W : M, w \vDash \varphi$. Similarly we write $F \vDash \varphi$, if *the formula* φ *is valid in every model of the frame* $F = \langle W, I, \tilde{R}\rangle$, and $\langle W, I\rangle \vDash \varphi$, if φ *is valid in every frame of the foundation* $\langle W, I\rangle$.

Let $M = \langle W, I, \tilde{R}, P\rangle$ be a model for the $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$ and $\Gamma \subseteq L(\Pi, \mathcal{A}, \mathcal{F})$. We say that M is a Γ-model (or model of Γ), if $M \vDash \Gamma$; that is, $\forall \varphi \in \Gamma : M \vDash \varphi$. Correspondingly, we say that the frame $F = \langle W, I, \tilde{R}\rangle$ is a Γ-frame, if $F \vDash \Gamma$, and that the foundation $\langle W, I\rangle$ is a Γ-foundation, if $\langle W, I\rangle \vDash \Gamma$.

3 The concept of definability in $PSTL$

We use the notation $\mathcal{C}(R)$ to denote that the relation R satisfies characterization $\mathcal{C}$. By *characterization $\mathcal{C}$ of k-ary syntactic operator f*, denoted $\mathcal{C}(I(f)(R_1, R_2, \ldots, R_k))$, we mean that relation $I(f)(R_1, R_2, \ldots, R_k)$ has an effective dependence on the relations $R_1, R_2, \ldots, R_k$ under description $\mathcal{C}$. Thus, basically we consider $\mathcal{C}$ to be a higher-order predicate. It is convenient to use similar notation for the relation and operator characterizations, since these concepts coincide: $R(f(\alpha_1, \ldots, \alpha_k)) = I(f)(R(\alpha_1), \ldots, R(\alpha_k))$, hence $\mathcal{C}(R(f(\alpha_1, \ldots, \alpha_k))) \Leftrightarrow \mathcal{C}(I(f)(R(\alpha_1), \ldots, R(\alpha_k)))$.

We write $I(f) \simeq G$, if G is well-defined operator such that $I(f) = G \restriction \mathrm{dom}(I(f))$; i.e. $\forall x_1, \ldots, x_n \in \mathrm{dom}(I(f)) : I(f)(x_1, \ldots, x_n) = G(x_1, \ldots, x_n)$.[5]

Definition 3.1. We say that the operator characterization $\mathcal{C}$ is *frame definable*, if there is a $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$ with k-ary operator $f \in \mathcal{F}$ and a set of formulae $\Gamma \subseteq L(\Pi, \mathcal{A}, \mathcal{F})$ such that the following correspondence holds:

$$\langle W, I, \tilde{R}\rangle \vDash \Gamma \textit{ if and only if } \forall \alpha_1, \ldots, \alpha_k \in \Lambda : \mathcal{C}(R(f(\alpha_1, \ldots, \alpha_k))).$$

We say that *the characterization $\mathcal{C}$ of the operator f is well defined*, if for all frames $\langle W, I, \tilde{R}\rangle$, $\langle W, J, \tilde{R}\rangle$ it is the case that $\forall \alpha_1, \ldots, \alpha_k \in \Lambda$:[6]

$$\textit{if } \mathcal{C}(I(f)(R(\alpha_1), \ldots, R(\alpha_k))) \textit{ and } \mathcal{C}(J(f)(R(\alpha_1), \ldots, R(\alpha_k))), \textit{ then}$$
$$I(f)(R(\alpha_1), \ldots, R(\alpha_k)) = J(f)(R(\alpha_1), \ldots, R(\alpha_k))$$

For example, if $\mathcal{C}(x(y, z)) ='' x$ is the union of (the relations) y, z'', then $\forall \alpha, \beta \in \Lambda : \mathcal{C}(I(+)(R(\alpha), R(\beta))) \Leftrightarrow R(\alpha) \cup R(\beta)$, thus the characterization $\mathcal{C}$ is well defined since the extension (output) of the union operation is unambiguous.

An example of *an ill-defined characterization* is $I(\sqcap)(R(\alpha), R(\beta)) \subseteq R(\alpha) \cap R(\beta)$. Clearly, if $R(\alpha) \cap R(\beta) \neq \emptyset$, there are at least two relations S that satisfy the condition $S \subseteq R(\alpha) \cap R(\beta)$, namely $S = \emptyset$ and $S = R(\alpha) \cap R(\beta)$. Therefore, in the general case there exists interpretations $I(\sqcap)$ and $J(\sqcap)$ such that $\exists \alpha, \beta \in \Lambda : I(\sqcap)(R(\alpha), R(\beta)) \neq J(\sqcap)(R(\alpha), R(\beta))$, but $I(\sqcap)(R(\alpha), R(\beta)) \subseteq R(\alpha) \cap R(\beta)$ and $J(\sqcap)(R(\alpha), R(\beta)) \subseteq R(\alpha) \cap R(\beta)$.

[5] Thus, $(I(f) = G) \Rightarrow (I(f) \simeq G)$; and $(I(f) = G) \Leftrightarrow (I(f) \simeq G$ *and* $\mathrm{dom}(I(f)) = \mathrm{dom}(G))$.

[6] In fact, this is an *iff*-clause, the other direction being trivial.

Definition 3.2. We say that the operator characterization $\mathcal{C}$ is *foundation definable*, if there exists a $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$ with a k-ary operator $f \in \mathcal{F}$ and a set of formulae $\Gamma \subseteq L(\Pi, \mathcal{A}, \mathcal{F})$ such that the following correspondence holds:

$$\langle W, I \rangle \vDash \Gamma \text{ if and only if } \forall R_1, \ldots, R_k \subseteq W \times W : \mathcal{C}(I(f)(R_1, \ldots, R_k)).$$

The well-definedness condition for the characterization $\mathcal{C}$ of k-ary operator $f \in \mathcal{F}$ in the context of the foundation definability is the following. If $\langle W, I \rangle$ and $\langle W, J \rangle$ are $L(\Pi, \mathcal{A}, \mathcal{F})$-foundations, then $\forall R_1, \ldots, R_k \subseteq W \times W$:

$$\textit{if } \mathcal{C}(I(f)(R_1, \ldots, R_k)) \textit{ and } \mathcal{C}(J(f)(R_1, \ldots, R_k)), \textit{ then}$$
$$I(f)(R_1, \ldots, R_k) = J(f)(R_1, \ldots, R_k).$$

Thus, $I(f) = J(f)$. Moreover $I = J$, if the condition holds for every $f \in \mathcal{F}$.

Therefore, the interpretation $I(f)$ satisfying the constraints of a well-defined characterization $\mathcal{C}$ of f is unique (satisfied by exactly one interpretation mapping).

Naturally, this means that we interpret the operator f under the interpretation I as a well-defined operation on relations $R_1, \ldots, R_k$. For example, if $\forall R, S \subseteq W \times W : I(+)(R, S) = R \cup S$, then we say that $I(+)$ satisfies the characterization (or the predicate)

$$\mathcal{C}(x) ='' x \text{ is the union operation}'' \text{ (or within our notation, } \mathcal{C}(x) ='' x \simeq \cup'').$$

Note that if an operation is definable in $PSTL$-language $L(\Pi, \mathcal{A}, \{f_1, \ldots, f_n\})$, the operation in question is always definable in any $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$ with $\{f_1, \ldots, f_n\} \subseteq \mathcal{F}$. On the other hand, if an operator *is not definable* in language $L(\Pi, \mathcal{A}, \{f_1, \ldots, f_n\})$ it might be definable in some other language $L(\Pi, \mathcal{A}, \mathcal{F}^*)$, for example, with $\{f_1, \ldots, f_n\} \subset \mathcal{F}^*$; that is, in language with more syntactic operators. At this point it is unfortunately not completely clear to what extent expressive power increases (besides syntactical definability) with the introduction of several operators in the frame and foundation logics respectively.[7]

Example 3.3. If there is a $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$ with binary operator $f \in \mathcal{F}$ and a set of formulae $\Gamma \subseteq L(\Pi, \mathcal{A}, \mathcal{F})$ such that

$$\langle W, I, \tilde{R} \rangle \vDash \Gamma \Leftrightarrow \forall \alpha, \beta \in \Lambda : R(f(\alpha, \beta)) = R(\alpha) \cup R(\beta),$$

as it actually turns out to be (cf. theorem 6.8), we say that *the property of being the union operation on the relations* $R(\alpha), R(\beta); \forall \alpha, \beta \in \Lambda$ *is definable.* Or, preferably, that

[7] Although, it *seems* to be the case that in the frame logics every definable operation is definable in one-operator language and in the foundation logics some operations require language with multiple operators. For example relative complementation seems to be definable only with the union and the intersection, and thus apparently requires language with three binary syntactic operators.

the operator returning the union of relations $R(\alpha), R(\beta); \forall \alpha, \beta \in \Lambda$ *is frame definable*, provided that the correspondence holds in every $L(\Pi, \mathcal{A}, \mathcal{F})$-frame.

If on the other hand we have a correspondence result:

$$\langle W, I \rangle \models \Gamma \Leftrightarrow \forall R, S \subseteq W \times W : I(f)(R, S) = R \cup S,$$

this clearly means that $I(f) \simeq \cup$. Then we say that *the union is (foundation) definable in* $PSTL$, provided that the correspondence holds in every $L(\Pi, \mathcal{A}, \mathcal{F})$-foundation.

Other weaker notions of definability may be useful since there are correspondences that do not define proper operators, as in the following example:

Example 3.4. There are correspondence theorems that do not define any operators (as in lemma 3.7 and lemma 3.8). Also, we can define *pseudo-operators*. For example, consider language $L(\Pi, \mathcal{A}, \{\tilde{\cdot}\})$ and let $\tilde{\cdot}$ be a unary operator. We have the following correspondence (cf. lemma 3.7):

$$\langle W, I, \tilde{R} \rangle \models [\alpha]\varphi \to [\tilde{\alpha}]\varphi \textit{ if and only if } \forall \alpha \in \Lambda : R(\tilde{\alpha}) \subseteq R(\alpha),$$

The operation $I(\tilde{\cdot})$ returns a subrelation of $R(\alpha)$. Since there are always relations that conform to $R(\tilde{\alpha})$ ($\emptyset$ and $R(\alpha)$, for example), the operation $I(\tilde{\cdot})$ is defined by the axiom of choice, but does not correspond to a proper operator since it is not generally well-defined in the sense of definition 3.1.

Example 3.5. Assume that the binary union is frame definable with a set of formulae Γ and consider the frame $F = \langle W, I, \tilde{R} \rangle$ of the language at issue such that

$$\begin{cases} W = \{w_1, w_2, w_3\}, \\ I(f)(R, S) = \begin{cases} R \cup S, & \textit{if } R, S \subseteq W \times W : R \neq \{\langle w_1, w_3\} \textit{ or } S \neq \{\langle w_3, w_1 \rangle\}, \\ \{\langle w_2, w_2 \rangle\}, & \textit{if } R = \{\langle w_1, w_3 \rangle\} \textit{ and } S = \{\langle w_3, w_1 \rangle\}, \end{cases} \\ \tilde{R}(a_1) = \{\langle w_1, w_2 \rangle\}, \tilde{R}(a_2) = \{\langle w_2, w_3 \rangle\}. \end{cases}$$

Remember that we defined $I(f)$ such that $I(f) = F : (\mathcal{P}(W \times W))^2 \to \mathcal{P}(W \times W)$, in the case, where f is a binary operator. Now, since there are no processes $\alpha, \beta \in \Lambda$ with an access to relations $\{\langle w_1, w_3 \rangle\}$ and $\{\langle w_3, w_1 \rangle\}$,[8] we have that $I(f)(\{\langle w_1, w_3 \rangle\}, \{\langle w_3, w_1 \rangle\}) = \{\langle w_2, w_2 \rangle\}$, and still $I(f)(R(\alpha), R(\beta)) = R(\alpha) \cup R(\beta); \forall \alpha, \beta \in \Lambda$. By the latter notion, $F \models \Gamma$ regardless of the fact that $\exists S, T \subseteq W \times W : I(f)(S, T) \neq S \cup T$.

The above means just that although $F = \langle W, I, \tilde{R} \rangle$ is a Γ-frame, $\langle W, I \rangle$ is not a Γ-foundation. If we also require that the foundation of $F = \langle W, I, \tilde{R} \rangle$ conforms to $\langle W, I \rangle \models \Gamma$, the definition of the interpretation $I(f)$ given above is not acceptable. This is because there exists a frame $F^* = \langle W, I, \tilde{R}^* \rangle$ of the foundation $\langle W, I \rangle$ such that $\tilde{R}^*(a_1) = \{\langle w_1, w_3 \rangle\}$ and $\tilde{R}^*(a_2) = \{\langle w_3, w_1 \rangle\}$. Since F^* satisfies $I(f)(R^*(a_1), R^*(a_2)) = \{\langle w_2, w_2 \rangle\} \neq R^*(a_1) \cup R^*(a_2)$, by assumption that Γ defines the binary union we infer $F^* \not\models \Gamma$, hence $\langle W, I \rangle \not\models \Gamma$.

[8]That is, there are no processes $\alpha, \beta \in \Lambda$ such that $R(\alpha) = \{\langle w_1, w_3 \rangle\}$, *and* $R(\beta) = \{\langle w_3, w_1 \rangle\}$.

The morale of Example 3.5 is that the relation of frames and foundations is similar to the relation of models and frames, i.e. every frame of a Γ-foundation is automatically a Γ-frame, but not the other way around. The reason for this is that in frame definability theorems the right-hand side of the equivalence pertain only to relations that are accessible by processes in the set Λ in the frames under consideration, not every relation of the foundation in general.

That said, frame definability is a sufficient condition for the foundation definability, since if $\langle W, I, \tilde{R}\rangle \vDash \Gamma \Leftrightarrow \forall \alpha_1, \ldots, \alpha_k \in \Lambda : \mathcal{C}(I(f)(R(\alpha_1), \ldots, R(\alpha_k)))$, then

$$\langle W, I\rangle \vDash \Gamma \Leftrightarrow \forall R_1, \ldots, R_k \subseteq W \times W : \mathcal{C}(I(f)(R_1, \ldots, R_k)).$$

This is because there is always a frame $F = \langle W, I, \tilde{R}\rangle$ such that $R_1 = R(a_{i_1}), \ldots, R_k = R(a_{i_k})$ for some $a_{i_1}, \ldots, a_{i_k} \in \mathcal{A}$. Therefore, every array of relations in the $\mathrm{dom}(I(f))$ is accessible by some set of processes in some frame of any given foundation.

This last remark comes with one technical proviso. If we consider a $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$ with a k-ary operator $f \in \mathcal{F}$ and $|\mathcal{A}| < k$, there is a foundation $\langle W, I\rangle$ such that $|\mathcal{A}| < k \leq |\mathcal{P}(W \times W)|$. In this case the claim, that every array of relations $R_1, \ldots, R_k \in \mathrm{dom}(I(f)) = (W \times W)^k$ is accessible in some frame of foundation $\langle W, I\rangle$, is clearly false. For example, consider a language $L(\Pi, \{a\}, \{+\})$ wherein $+$ is binary operator. There is a set of formulae $\Gamma \subseteq L(\Pi, \{a\}, \{+\})$ that defines the union on the level of frames (cf. theorem 6.8). Consider then a foundation $\langle W, I\rangle$ such that $W = \{w_1, w_2\}$ and

$$I(+)(S,T) = \begin{cases} S, & \textit{if } S = T \\ \emptyset, & \textit{if } S \neq T. \end{cases}$$

Now, clearly $R(\alpha + \beta)) = R(\alpha) \cup R(\beta); \forall \alpha, \beta \in \Lambda$ holds in every frame of the foundation, since $\forall \alpha, \beta \in \Lambda : R(\alpha) = R(\beta) = R(a)$. Hence, the foundation satisfies $\langle W, I\rangle \vDash \Gamma$ (cf. theorem 3.6). Regardless of this, there are relations $R, S \subseteq W \times W$ such that $I(+)(R, S) \neq R \cup S$ (for example $R = \{\langle w_1, w_1\rangle\}$ and $S = \{\langle w_2, w_2\rangle\}$, thus $I(+)(R, S) = \emptyset \neq R \cup S$).

Any language that contains a k-ary operator f such that $|\mathcal{A}| < k$ is a pathological special case. This issue fortunately does not cause problems in what follows. It should be kept in mind, however, that k-ary operations are *generally not foundation definable in any language* $L(\Pi, \mathcal{A}, \mathcal{F})$, *if* $|\mathcal{A}| < k$. Hence, the cardinality of a set of atomic processes $\mathcal{A}$ could have an effect on the expressive power of the language. This is because on the level of frames every process $\alpha \in \Lambda$ corresponds to fixed relation, whereas on the level of foundations atomic processes are essentially free variables, whose binding is tantamount to choosing a frame, and therefore the cardinality of a set $\mathcal{A}$ amounts to the number of variables within the foundation logics. This is one notable difference between the frame and the foundation logics.

The next theorem formalizes some of the preceding considerations.

Theorem 3.6. *Let $L(\Pi, \mathcal{A}, \mathcal{F})$ be a $PSTL$ language with k-ary operator $k \in \mathcal{F}$. Let $\mathcal{C}$ be any well-defined operator characterization and assume that there is a set of formulae*

$\Gamma \subseteq L(\Pi, \mathcal{A}, \mathcal{F})$ *such that* $\langle W, I, \tilde{R} \rangle \vDash \Gamma \Leftrightarrow \forall \alpha_1, \ldots, \alpha_k \in \Lambda : \mathcal{C}(I(f)(R(\alpha_1), \ldots, R(\alpha_k)))$,

$$\textit{then } \langle W, I \rangle \vDash \Gamma \Leftrightarrow \forall R_1, \ldots, R_k \subseteq W \times W : \mathcal{C}(I(f)(R_1, \ldots, R_k)), \textit{ provided } |\mathcal{A}| \geq k.$$

That is, every frame definable operator characterization is foundation definable with the same set of formulae, and hence within the same language.

Proof. Let $\mathcal{C}$ be an operator characterization. Let $L(\Pi, \mathcal{A}, \mathcal{F})$ be a $PSTL$-language with k-ary operator $f \in \mathcal{F}$, $|\mathcal{A}| \geq k$ and a set of formulae $\Gamma \subseteq L(\Pi, \mathcal{A}, \mathcal{F})$ such that

$$\langle W, I, \tilde{R} \rangle \vDash \Gamma \Leftrightarrow \forall \alpha_1, \ldots, \alpha_k \in \Lambda : \mathcal{C}(I(f)(R(\alpha_1), \ldots, R(\alpha_k)))$$

holds in every $L(\Pi, \mathcal{A}, \mathcal{F})$-frame $\langle W, I, \tilde{R} \rangle$.

Consider a foundation $\langle W, I \rangle$ and let $R_1, \ldots, R_k \subseteq W \times W$ be such relations that $\not\mathcal{C}(I(f)(R_1, \ldots, R_k))$. Then there is a frame $\langle W, I, \tilde{R} \rangle$ such that $R(a_1) = R_1, \ldots, R(a_k) = R_k$. Now, $(R_1, \ldots, R_k) = (R(a_1), \ldots, R(a_k))$, therefore $\not\mathcal{C}(I(f)(R(a_1), \ldots, R(a_k)))$. Hence, $\exists \alpha_1, \ldots, \alpha_k \in \Lambda : \not\mathcal{C}(I(f)(R(a_1), \ldots, R(a_k)))$, thus by the initial frame definability assumption we infer $\langle W, I, \tilde{R} \rangle \nvDash \Gamma$, hence $\langle W, I \rangle \nvDash \Gamma$. Therefore, by the contraposition we have that the frame definability assumption implies $\langle W, I \rangle \vDash \Gamma \Rightarrow \forall R_1, \ldots, R_k \subseteq W \times W : \mathcal{C}(I(f)(R_1, \ldots, R_k))$.

Assume then that $\langle W, I \rangle$ is $L(\Pi, \mathcal{A}, \mathcal{F})$-foundation such that $\forall R_1, \ldots, R_k \subseteq W \times W : \mathcal{C}(I(f)(R_1, \ldots, R_k))$. Then pick any frame $\langle W, I, \tilde{R} \rangle$ of the given foundation. Since $\forall R_1, \ldots, R_k \subseteq W \times W : \mathcal{C}(I(f)(R_1, \ldots, R_k))$, especially then $\mathcal{C}(I(f)(R(\alpha_1), \ldots, R(\alpha_k)))$, for all $\alpha_1, \ldots, \alpha_k \in \Lambda$. Hence, we infer by the frame definability assumption that $\langle W, I, \tilde{R} \rangle \vDash \Gamma$. Since this holds in every frame of the foundation, we have that if the frame definability holds, $\forall R_1, \ldots, R_k \subseteq W \times W : \mathcal{C}(I(f)(R_1, \ldots, R_k)) \Rightarrow \langle W, I \rangle \vDash \Gamma$. □

Therefore, every frame definable operator is foundation definable. However, there are foundation definable operators that are not frame definable, for example operator that returns the universal relation of the domain. (Cf. theorems 6.3 and 7.1.[9]) Thus, we are considering two different $PSTL$ logics with definability as the primary semantic concept, *the $PSTL$ frame logic* and *the $PSTL$ foundation logic*.

In what follows, we consider both logics, i.e. definability with respect to frames and foundations. By theorem 3.6, we only need to prove that $\mathcal{C}$ is frame definable or that $\mathcal{C}$ is not foundation definable and we get the other definability issue resolved by default. We treat both definability issues explicitly only if it happens that $\mathcal{C}$ is foundation definable, but not frame definable property.

We close this section by proving two simple lemmas with one substantial corollary.

Lemma 3.7. *If* $\alpha, \beta \in \Lambda$, *then* $\langle W, I, \tilde{R} \rangle \vDash [\alpha]\varphi \rightarrow [\beta]\varphi$ *if and only if* $R(\beta) \subseteq R(\alpha)$.

[9]This issue, with the proof of theorem 7.1, was first pointed out to the author by Antti Kuusisto at the University of Tampere, 2006.

Proof. Let $\langle W, I, \tilde{R}\rangle$ be a $PSTL$-frame; $\alpha, \beta \in \Lambda$ processes that satisfy $R(\beta) \subseteq R(\alpha)$ and P an arbitrary valuation function. Now, the structure $M = \langle W, I, \tilde{R}, P\rangle$ is $PSTL$-model with $R(\beta) \subseteq R(\alpha)$. Consider state $w \in W$ and assume $M, w \vDash [\alpha]\varphi$, i.e. $\forall w' \in W : wR(\alpha)w' \Rightarrow M, w' \vDash \varphi$. Hence, by the assumption $R(\beta) \subseteq R(\alpha)$, clearly $\forall w' \in W : wR(\beta)w' \Rightarrow M, w' \vDash \varphi$, thus $M, w \vDash [\beta]\varphi$. Therefore, if $R(\beta) \subseteq R(\alpha)$, then $M, w \vDash [\alpha]\varphi \rightarrow [\beta]\varphi$.

Then assume $R(\beta) \nsubseteq R(\alpha)$, thus $\exists w, w' \in W : wR(\beta)w'$ *and* $w\not{R}(\alpha)w'$. Choose valuation function P such that $P(p) = W \setminus \{w'\}$, for some $p \in \Pi$. Since $w\not{R}(\alpha)w'$, the choice of $P(p)$ directly implies $M, w \vDash [\alpha]p$. On the other hand, $w' \notin P(p)$, thus $M, w' \nvDash p$, which, with $wR(\beta)w'$, implies $M, w \nvDash [\beta]p$. Therefore, $M, w \nvDash [\alpha]p \rightarrow [\beta]p$. Hence, if $R(\beta) \nsubseteq R(\alpha)$ holds in frame $\langle W, I, \tilde{R}\rangle$, there is a valuation P and a formula φ such that $\langle W, I, \tilde{R}\rangle \nvDash [\alpha]\varphi \rightarrow [\beta]\varphi$. Therefore, if $\langle W, I, \tilde{R}\rangle \vDash [\alpha]\varphi \rightarrow [\beta]\varphi$, then $R(\beta) \subseteq R(\alpha)$. ☐

Lemma 3.8. *If $\alpha, \beta \in \Lambda$, then $\langle W, I, \tilde{R}\rangle \vDash [\alpha]\varphi \leftrightarrow [\beta]\varphi \Leftrightarrow R(\alpha) = R(\beta)$.*

Proof. Consider an arbitrary $PSTL$-frame $\langle W, I, \tilde{R}\rangle$. By lemma 3.7, the proof is a trivial chain of equivalences:

$$\begin{aligned}\langle W, I, \tilde{R}\rangle \vDash [\alpha]\varphi \leftrightarrow [\beta]\varphi \;&\Leftrightarrow\; \langle W, I, \tilde{R}\rangle \vDash [\alpha]\varphi \rightarrow [\beta]\varphi \text{ and } \langle W, I, \tilde{R}\rangle \vDash [\beta]\varphi \rightarrow [\alpha]\varphi\\ &\Leftrightarrow\; R(\beta) \subseteq R(\alpha) \text{ and } R(\alpha) \subseteq R(\beta)\\ &\Leftrightarrow\; R(\alpha) = R(\beta)\end{aligned}$$

☐

Corollary 3.9. *Let $L(\Pi, \mathcal{A}, \mathcal{F})$ be a $PSTL$-language such that $f, g \in \mathcal{F}$ wherein $\mathrm{dom}(f) = \Lambda^m$, $\mathrm{dom}(g) = \Lambda^n$ and let $|\mathcal{A}| \geq m + n$. Then,*

$$\langle W, I\rangle \vDash [f(\alpha_1, \ldots, \alpha_m)]\varphi \leftrightarrow [g(\beta_1, \ldots, \beta_n)]\varphi$$

if and only if

$$\forall S_1, \ldots S_m, T_1, \ldots, T_n \subseteq W \times W : I(f)(S_1, \ldots S_m) = I(g)(T_1, \ldots, T_n).$$

Proof. Assume that the foundation $\langle W, I\rangle$ satisfies $\forall S_1, \ldots S_m, T_1, \ldots, T_n \subseteq W \times W : I(f)(S_1, \ldots S_m) = I(g)(T_1, \ldots, T_n)$. Then, especially $\forall \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \Lambda : I(f)(R(\alpha_1), \ldots, R(\alpha_m)) = I(g)(R(\beta_1), \ldots, R(\beta_n))$ holds in every frame $\langle W, I, \tilde{R}\rangle$. Thus, by Lemma 3.8, $\langle W, I, \tilde{R}\rangle \vDash [f(\alpha_1, \ldots, \alpha_m)]\varphi \leftrightarrow [g(\beta_1, \ldots, \beta)]\varphi$ holds in every frame of $\langle W, I\rangle$ for every $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \Lambda$, i.e. $\langle W, I\rangle \vDash [f(\alpha_1, \ldots, \alpha_m)]\varphi \leftrightarrow [g(\beta_1, \ldots, \beta_n)]\varphi$.

Assume then that $\exists S_1, \ldots, S_m, T_1, \ldots, T_n \subseteq W \times W : I(f)(S_1, \ldots, S_m) \neq I(g)(T_1, \ldots, T_n)$. By the assumption $|\mathcal{A}| \geq m + n$, there is a frame $\langle W, I, \tilde{R}\rangle$ such that $\tilde{R}(a_1) = S_1, \ldots, \tilde{R}(a_m) = S_m$ and $\tilde{R}(b_1) = T_1, \ldots, \tilde{R}(b_n) = T_n$; $a_1, \ldots a_m, b_1, \ldots b_n \in \mathcal{A}$. Thus, $R(f(a_1, \ldots, a_m)) \neq R(g(b_1, \ldots, b_n))$, hence, by lemma 3.8, $\langle W, I, \tilde{R}\rangle \nvDash [f(a_1, \ldots, a_m)]\varphi \leftrightarrow [g(b_1, \ldots b_n)]\varphi$. Therefore, $\langle W, I\rangle \nvDash [f(\alpha_1, \ldots, \alpha_m)]\varphi \leftrightarrow [g(\beta_1, \ldots, \beta_n)]\varphi$. ☐

4 The union of the disjoint $PSTL$ models and frames

In this section, we define an operation on frames called *the union of the disjoint frames.* The operation on Kripke-frames should be found in any textbook considering modal definability.[10] Since the standard Kripke-frames do not contain the interpretation mapping I, we also need to take care of that the interpretation mapping in the unified $PSTL$ frame is both well defined and provides for the most essential property of the operation; that is, I should be defined such that all the frame validities are conserved in the union of disjoint frames.

Definition 4.1. Let $M_i = \langle W_i, I_i, \tilde{R}_i, P_i \rangle$ and $M_j = \langle W_j, I_j, \tilde{R}_j, P_j \rangle$ be models of the same language $L(\Pi, \mathcal{A}, \mathcal{F})$, i.e. there is an interpretation for every $p \in \Pi, f \in \mathcal{F}$ and $a \in \mathcal{A}$ in both models M_i and M_j. If $W_i \cap W_j = \emptyset$, the models M_i and M_j are said to be *(pairwise) disjoint.*

If necessary, we can always assume that any two models $M_i = \langle W_i, I_i, \tilde{R}_i, P_i \rangle$ and $M_j = \langle W_j, I_j, \tilde{R}_j, P_j \rangle$ are pairwise disjoint, since there is a canonical way to force the models to satisfy the disjointness condition. This is done by renaming both, the domain elements $w_n \in W_i$ and the domain W_i itself, by tagging the initial indices with new a index k. Thus, if for example $W_i = \{w_0, w_1, w_2\}$, tagging yields $W_{i_k} = \{w_{0_k}, w_{1_k}, w_{2_k}\}$. Renaming is then applied to any element in M_i, thus e.g. from $w_0 \tilde{R}_i(\alpha) w_1$, we get $w_{0_k} \tilde{R}_{i_k}(\alpha) w_{1_k}$, etc. Call the resulting model M_{i_k}. Then the procedure is applied to M_j with a new index $k' \neq k$. The net result is that necessarily $W_{i_k} \cap W_{j_{k'}} = \emptyset$, since the elements of sets W_{i_k} and $W_{j_{k'}}$ are syntactically divergent. Of course, the process can be applied to any number of models under consideration.

Next we define an operation on disjoint $PSTL$-models that returns a single unified model and prove several properties of the operation. Then we generalize the operation to apply for frames and prove the most important result of the section: every frame definable property is conserved within the union of disjoint frames. We use the result mainly to prove that some operators are not frame definable in any $PSTL$-language.

Definition 4.2. Let M_i and M_j be disjoint models. The disjoint union $M_i \uplus M_j$ of the models M_j, M_i is the structure $\langle W_i \uplus W_j, I_i \uplus I_j, \tilde{R}_i \uplus \tilde{R}_j, P_i \uplus P_j \rangle$ with the following definitions:

(i) $W_i \uplus W_j = W_i \cup W_j$

(ii) $(\tilde{R}_i \uplus \tilde{R}_j) : \mathcal{A} \to \mathcal{P}((W_i \uplus W_j) \times (W_i \uplus W_j))$ s.t. $(\tilde{R}_i \uplus \tilde{R}_j)(a) = \tilde{R}_i(a) \cup \tilde{R}_j(a), \forall a \in \mathcal{A}$

(iii) $(P_i \uplus P_j) : \Pi \to \mathcal{P}(W_i \uplus W_j)$ s.t. $(P_i \uplus P_j)(p) = P_i(p) \cup P_j(p), \forall p \in \Pi$

The interpretation function

$$(I_i \uplus I_j) : \mathcal{F} \to \bigcup_{n=0}^{\infty} \{F | F : (\mathcal{P}((W_i \uplus W_j) \times (W_i \uplus W_j)))^n \to \mathcal{P}((W_i \uplus W_j) \times (W_i \uplus W_j))\}$$

[10]See for example (Blackburn et al., 2002, pp.138–139) or (Rantala & Virtanen, 2004, pg.219).

is likewise as follows: Let $f \in \mathcal{F}$ be k-ary operator, then

(iv) $(I_i \uplus I_j)(f) = F : V \subseteq ((W_i \uplus W_j) \times (W_i \uplus W_j))^k \mapsto U \subseteq (W_i \uplus W_j) \times (W_i \uplus W_j)$

wherein if $V = ((S_1 \cup T_1), \ldots (S_k \cup T_k))$ such that $S_r \subseteq W_i \times W_i, T_r \subseteq W_j \times W_j; 1 \leq r \leq k$, then $U = I_i(f)(S_1, \ldots, S_k) \cup I_j(f)(T_1, \ldots T_k)$; otherwise $U = W_i \cup W_j$.

Note that the choice to map relations, that are not properly included in the initial frames, to $W_i \cup W_j$ is arbitrary; this is just to ensure that the resulting unified frame is well-defined.

By the above definition, mapping $R_i \uplus R_j : \Lambda \rightarrow \mathcal{P}(W_i \uplus W_j)$ satisfying

$$
\begin{array}{lrcl}
\text{(v)} & (R_i \uplus R_j)(a) = (\tilde{R}_i \uplus \tilde{R}_j)(a) & & \\
\text{(vi)} & (R_i \uplus R_j)(f(\alpha_1, \ldots, \alpha_k)) & = & (I_i \uplus I_j)(f)((R_i \uplus R_j)(\alpha_1), \ldots, (R_i \uplus R_j)(\alpha_k)) \\
 & & = & I_i(f)(R_i(\alpha_1), \ldots, R_i(\alpha_k)) \cup I_j(f)(R_j(\alpha_1), \ldots, R_j(\alpha_k))
\end{array}
$$

is well defined and consequently $\forall \alpha \in \Lambda : (R_i \uplus R_j)(\alpha) = R_i(\alpha) \cup R_j(\alpha)$.

We adopt a more compact notation, $M_{(i;j)} = \langle W_{(i;j)}, I_{(i;j)}, \tilde{R}_{(i;j)}, P_{(i;j)} \rangle$, for structure $M_i \uplus M_j = \langle W_i \uplus W_j, I_i \uplus I_j, \tilde{R}_i \uplus \tilde{R}_j, P_i \uplus P_j \rangle$.

Lemma 4.3. *Let $M_i = \langle W_i, I_i, \tilde{R}_i, P_i \rangle$ and $M_j = \langle W_j, I_j, \tilde{R}_j, P_j \rangle$ be $L(\Pi, \mathcal{A}, \mathcal{F})$-models. Every formula $\varphi \in L(\Pi, \mathcal{A}, \mathcal{F})$ and every state $w \in W_i \cup W_j$ satisfies*

(i) $M_{(i;j)}, w \vDash \varphi \Leftrightarrow M_i, w \vDash \varphi$, *if* $w \in W_i$ *and*

(ii) $M_{(i;j)}, w \vDash \varphi \Leftrightarrow M_j, w \vDash \varphi$, *if* $w \in W_j$.

Proof. The proof is a typical induction process over the structure of an arbitrary formula φ. Assume first that $\varphi = p \in \Pi$ and consider the state $w \in W_{(i;j)}$. Since $W_{(i;j)} = W_i \cup W_j$, $w \in W_i$ or $w \in W_j$. If $w \in W_i$, by disjointness of the domains W_i and W_j, we readily have that $w \notin W_j$. Therefore, the part (ii) of the claim, $M_{(i;j)}, w \vDash p \Leftrightarrow M_j, w \vDash p$; *if* $w \in W_j$, holds trivially since the condition $w \in W_j$ is not satisfied. Recall that valuation function $P_{(i;j)}$ is defined such that $\forall p_k \in \Pi : P_{(i;j)}(p_k) = P_i(p_k) \cup P_j(p_k)$. The definition has an immediate reformulation: $\forall w \in W_i \cup W_j : w \in P_{(i;j)}(p_k) \Leftrightarrow (w \in P_i(p_k)$ *or* $w \in P_j(p_k))$. Now $w \notin P_j(p)$, since by the assumption $w \in W_i$; hence we have that $w \in P_{(i;j)}(p) \Leftrightarrow w \in P_i(p)$. By Definition 2.6, the latter is exactly the part (i) of the claim, namely that $M_{(i;j)}, w \vDash p \Leftrightarrow M_i, w \vDash p$.

If, on the other hand, $w \notin W_i$ then $w \in W_j$ and the proof for the basis of the induction is identical with the previous demonstration. Therefore the claim holds, if $\varphi = p \in \Pi$.

Then assume that the claim holds for the formulae ξ and ζ. The connectives are treated in the standard way, thus we bypass the demonstrations as trivial and consider next the operator case; that is, let $\varphi = [\alpha]\xi$, for $\alpha \in \Lambda$.

Pick an arbitrary state $w \in W_i \cup W_j$. By symmetry, we may assume that $w \in W_i$. Since $W_i \cap W_j = \emptyset$, we again have $w \notin W_j$ and the part (ii) of the claim, $M_{(i;j)}, w \models [\alpha]\xi \Leftrightarrow M_j, w \models [\alpha]\xi;\ \textit{if}\ w \in W_j$, holds trivially.

Next assume $M_i, w \not\models [\alpha]\xi$. Then $\exists w' \in W_i : wR_i(\alpha)w'\ \textit{and}\ M_i, w' \not\models \xi$. Since $w' \in W_i$, $w' \in W_i \cup W_j = W_i \uplus W_j$, and because $M_i, w' \not\models \xi$ by the induction hypothesis $M_{(i;j)}, w' \not\models \xi$. Now by Definition of $R_{(i;j)}$ we have that $wR_{(i;j)}(\alpha)w'$, hence $M_{(i;j)}, w \not\models [\alpha]\xi$. So if $M_{(i;j)}, w \models [\alpha]\xi$, then $M_i, w \models [\alpha]\xi$, provided $w \in W_i$.

Assume then that $M_{(i;j)}, w \not\models [\alpha]\xi$, i.e. $\exists w' \in W_{(i;j)} : wR_{(i;j)}(\alpha)w'\ \textit{and}\ M_{(i;j)}, w' \not\models \xi$. Since $R_{(i;j)}(\alpha) = R_i(\alpha) \cup R_j(\alpha)$ and $W_i \cap W_j = \emptyset$ we infer, by the initial assumption $w \in W_i$, that $w' \in W_i$ and $wR_i(\alpha)w'$. Now by induction hypothesis $M_i, w' \not\models \xi$, hence $M_i, w \not\models [\alpha]\xi$. Therefore if $M_i, w \models [\alpha]\xi$, then $M_{(i;j)}, w \models [\alpha]\xi$. □

Lemma 4.3 has an array of useful corollaries.

Corollary 4.4. *$M_{(i;j)} \models \varphi \Leftrightarrow M_i \models \varphi$ and $M_j \models \varphi$.*

Proof.

$$\begin{array}{rcl} M_{(i;j)} \models \varphi & \Leftrightarrow & \forall w \in W_{(i;j)} : M_{(i;j)}, w \models \varphi \\ \text{(By definition 4.2)} & \Leftrightarrow & \forall w \in W_i \cup W_j : M_{(i;j)}, w \models \varphi \\ \text{(By lemma 4.3)} & \Leftrightarrow & \forall w' \in W_i : M_i, w' \models \varphi\ \textit{and}\ \forall w'' \in W_j : M_j, w'' \models \varphi \\ & \Leftrightarrow & M_i \models \varphi \text{ and } M_j \models \varphi \end{array}$$

□

Corollary 4.5. *Corollary 4.4 holds for the disjoint union of indefinite number of models. That is, let*

$$M_{(0;1;\ldots;n)} = \biguplus_{i=0}^{n} M_i,\ \textit{then}\ M_{(0;1;\ldots;n)} \models \varphi \Leftrightarrow M_0 \models \varphi, M_1 \models \varphi, \ldots, M_n \models \varphi.$$

Proof. This is an induction over the index set $\{0, 1, \ldots, n\}$ with the basis provided in Corollary 4.4, hence we begin by assuming that the claim holds for $M_{(0;1;\ldots;n-1)}$. Now,

$$\begin{array}{rcl} M_0 \models \varphi, M_1 \models \varphi, \ldots, M_{n-1} \models \varphi, M_n \models \varphi & \Leftrightarrow & M_{(0;1;\ldots;n-1)} \models \varphi\ \textit{and}\ M_n \models \varphi \\ & \Leftrightarrow & M_{(0;1;\ldots;n-1)} \models \varphi \uplus M_n \models \varphi \\ & \Leftrightarrow & M_{(0;1;\ldots;n)} \models \varphi, \end{array}$$

where the first equivalence is due to the induction hypothesis and the second is implied by Corollary 4.4. □

Definition 4.6. We get the definition for *the union of disjoint frames* easily from Definition 4.2 by simply omitting the references to valuations: Let $F_i = \langle W_i, I_i, \tilde{R}_i \rangle$ and $F_j = \langle W_j, I_j, \tilde{R}_j \rangle$ be disjoint frames of the same $PSTL$-language, i.e. the domains of F_i, F_j satisfy $W_i \cap W_j = \emptyset$, then structure $F_i \uplus F_j = \langle W_i \uplus W_j, I_i \uplus I_j, \tilde{R}_i \uplus \tilde{R}_j \rangle$ is the structure that satisfies the conditions (i),(ii) and (iv) (and therefore, effectively, the conditions (v) and (vi)) of Definition 4.2. Again, we denote $F_{(0;1;\ldots;n)} = F_1 \uplus F_1 \uplus \ldots \uplus F_n$.

Corollary 4.7. *Frame validity is conserved within the union of disjoint frames; that is, let* $L(\Pi, \mathcal{A}, \mathcal{F})$ *be a* $PSTL$*-language and* F_i, F_j $L(\Pi, \mathcal{A}, \mathcal{F})$*-frames, then*

$$\forall \varphi \in L(\Pi, \mathcal{A}, \mathcal{F}) : F_{(i;j)} \vDash \varphi \Leftrightarrow F_i \vDash \varphi \text{ and } F_j \vDash \varphi.$$

Proof. Assume $F_i \vDash \varphi$ and $F_j \vDash \varphi$. Now if $F_{(i;j)} \nvDash \varphi$, there is a valuation $P_{(i;j)}$ such that $\langle F_{(i;j)}, P_{(i;j)} \rangle \nvDash \varphi$; i.e. $\exists w \in W_{(i;j)} : \langle F_{(i;j)}, P_{(i;j)} \rangle, w \nvDash \varphi$. If $w \in W_i$, by lemma 4.3 $\langle F_i, P_i \rangle, w \nvDash \varphi$. Likewise if $w \in W_j$, then $\langle F_j, P_j \rangle, w \nvDash \varphi$. In any case $F_i \nvDash \varphi$ or $F_j \nvDash \varphi$, which contradicts the assumption. Thus, $F_i \vDash \varphi$ *and* $F_j \vDash \varphi \Rightarrow F_{(i;j)} \vDash \varphi$.

Next assume that $F_i \nvDash \varphi$ or $F_j \nvDash \varphi$. Then, exploiting Definition 2.6, directly by Lemma 4.3, we have that $F_{(i;j)} \nvDash \varphi$. □

Lemma 4.8. *Let* $L(\Pi, \mathcal{A}, \mathcal{F})$ *be a* $PSTL$*-language. Any definable well-defined characterization of an interpretation* $I(f)$*, for all* $f \in \mathcal{F}$*, is conserved in the union of disjoint* $L(\Pi, \mathcal{A}, \mathcal{F})$*-frames.*

To be exact, the lemma states that given any $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$, if there is a set of formulae $\Gamma \subseteq L(\Pi, \mathcal{A}, \mathcal{F})$ such that Γ defines some well-defined characterization $\mathcal{C}$ of k-ary operator $f \in \mathcal{F}$ and the following holds:

(i) Frame $F_i = \langle W_i, I_i, \tilde{R}_i \rangle$ satisfies $\mathcal{C}(R_i(f(\alpha_1, \ldots, \alpha_k))), \forall \alpha_1, \ldots, \alpha_k \in \Lambda$

(ii) Frame $F_j = \langle W_j, I_j, \tilde{R}_j \rangle$ satisfies $\mathcal{C}(R_j(f(\alpha_1, \ldots, \alpha_k))), \forall \alpha_1, \ldots, \alpha_k \in \Lambda$,

then $F_{(i;j)} = \langle W_{(i;j)}, I_{(i;j)}, \tilde{R}_{(i;j)} \rangle$ satisfies $\mathcal{C}(R_{(i;j)}(f(\alpha_1, \ldots, \alpha_k))), \forall \alpha_1, \ldots, \alpha_k \in \Lambda$.

Proof. Let $L(\Pi, \mathcal{A}, \mathcal{F})$ be a $PSTL$-language with a k-ary operator $f \in \mathcal{F}$. Assume first that $\exists \Gamma \subseteq L(\Pi, \mathcal{A}, \mathcal{F})$ such that the following correspondence holds for every $L(\Pi, \mathcal{A}, \mathcal{F})$-frame $\langle W, I, \tilde{R} \rangle$:

$$1^\circ : \quad \langle W, I, \tilde{R} \rangle \vDash \Gamma \text{ if and only if } \mathcal{C}(R(f(\alpha_1, \ldots, \alpha_k))), \forall \alpha_1, \ldots, \alpha_k \in \Lambda.$$

Let $F_i = \langle W_i, I_i, \tilde{R}_i \rangle$ and $F_j = \langle W_j, I_j, \tilde{R}_j \rangle$ be $L(\Pi, \mathcal{A}, \mathcal{F})$-frames such that $\forall \alpha_1, \ldots, \alpha_k \in \Lambda : \mathcal{C}(R_i(f(\alpha_1, \ldots, \alpha_k)))$ *and* $\mathcal{C}(R_j(f(\alpha_1, \ldots, \alpha_k)))$. By the assumption 1°, $F_i \vDash \Gamma$ and $F_j \vDash \Gamma$. Now, by Corollary 4.7, $F_{(i;j)} \vDash \Gamma$, therefore, again by the assumption 1°, $\mathcal{C}(R_{(i;j)}(f(\alpha_1, \ldots, \alpha_k))), \forall \alpha_1, \ldots, \alpha_k \in \Lambda$. □

Since the concept of characterization is left somewhat intuitive in the opening of the chapter 3, it will be worthwhile to elaborate on actual content of Lemma 4.8. The reader should notice that the lemma actually has a non-trivial and applicable content, since the concept of the frame definability and the constraints assigned in Definition 2.5 to the well-defined characterizations should be clear and straightforward.

Consider what it would mean if Lemma 4.8 did not hold. For contradiction, consider again the unified frame $F_{(i;j)} = \langle W_{(i;j)}, I_{(i;j)}, \tilde{R}_{(i;j)} \rangle$ and assume that there is a $L(\Pi, \mathcal{A}, \mathcal{F})$-frame $F = \langle W_{(i;j)}, I, \tilde{R}_{(i;j)} \rangle$ such that

$$2^\circ : \quad \mathcal{C}(I(f)(R_{(i;j)}(\alpha_1), \ldots, R_{(i;j)}(\alpha_k))), \forall \alpha_1, \ldots, \alpha_k \in \Lambda \textit{ and } I_{(i;j)}(f) \neq I(f).$$

The latter part of the condition 2° means that

$$\exists \beta_1, \ldots, \beta_k \in \Lambda : I_{(i;j)}(f)(R_{(i;j)}(\beta_1), \ldots, R_{(i;j)}(\beta_k)) \neq I(f)(R_{(i;j)}(\beta_1), \ldots, R_{(i;)}(\beta_k)).$$

Therefore, according to Definition 3.1, the first part of 2° implies that either, contrary to assumption, $\mathcal{C}$ is ill-defined property since it is satisfied with two divergent mappings with the same argument list, or $I_{(i;j)}(f)(R_{(i;j)}(\beta_1), \ldots, R_{(i;j)}(\beta_k))$ fails to satisfy $\mathcal{C}$. The latter in turn implies with 1° that $F_{(i;j)} \not\vDash \Gamma$, which contradicts the assumptions $F_i \vDash \Gamma$, $F_j \vDash \Gamma$ and Lemma 4.7. This last remark demonstrates that actually Lemma 4.8 is a reformulation of Lemma 4.7 that allows us to bypass several mechanical steps in the inference required to apply Lemma 4.7 properly.

5 Pseudoepimorphic $PSTL$-frames

For some non-frame definability results we need a different approach than with the union of disjoint frames. Next, we introduce standard model theoretical tool called *pseudoepimorphism*, henceforth *p-morphism* for short. (Cf. e.g. Blackburn et al. (2002, pp.60–62)). P-morphisms form a subclass of natural homomorphisms over Kripke-frames called *bounded morphisms*. We start by defining Λ-bounded morphisms; that is, bounded morphisms suited for $PSTL$-frames.

Definition 5.1. Let $F = \langle W, I, \tilde{R} \rangle$ and $F^* = \langle W^*, I^*, \tilde{R}^* \rangle$ be frames of the same $PSTL$-language. Function $g : W \to W^*$ is a *Λ-bounded morphism from F to F^**, if

(i) $\forall \alpha \in \Lambda : wR(\alpha)v \Rightarrow g(w)R^*(\alpha)g(v)$ and

(ii) $\forall \alpha \in \Lambda : g(w)R^*(\alpha)v' \Rightarrow \exists v \in W : wR(\alpha)v \textit{ and } g(v) = v'$.

If $g : W \to W^*$ is a surjective Λ-bounded morphism, we say that g is a *Λ-p-morphism* and that *F^* is a Λ-p-morphic image of F*.

We extrapolate the above definition to define Λ-bounded morphisms over $PSTL$-models: Let $M = \langle W, I, \tilde{R}, P \rangle$ and $M^* = \langle W^*, I^*, \tilde{R}^*, P^* \rangle$ be $L(\Pi, \mathcal{A}, \mathcal{F})$-models. Function $g : W \to W^*$ is *a Λ-bounded morphism from M to M^**, if g satisfies conditions (i), (ii) and

(iii) $\forall p \in \Pi : \forall w \in W : M, w \vDash p \Leftrightarrow M^*, g(w) \vDash p$.

Lemma 5.2. *Let $M = \langle W, I, \tilde{R}, P\rangle$ and $M^* = \langle W^*, I^*, \tilde{R}^*, P^*\rangle$ be models of the same PSTL-language $L(\Pi, \mathcal{A}, \mathcal{F})$ such that there exists Λ-bounded morphism $g : W \to W^*$. Then, $\forall w \in W : M, w \vDash \varphi$ if and only if $M^*, g(w) \vDash \varphi$.*

Proof. Choose any state $w \in W$. We prove the claim by induction over the structure of formula φ. Let $\varphi = p \in \Pi$. Then the claim holds directly by the above condition (iii) for the Λ-bounded morphisms on models.

Assume that the claim holds for formulae ξ and ζ. As usual, the connective steps are straightforward:

$$M, w \vDash \neg\xi \Leftrightarrow M, w \nvDash \xi \Leftrightarrow_{(i.h.)} M^*, g(w) \nvDash \xi \Leftrightarrow M^*, g(w) \vDash \neg\xi$$

$$\begin{array}{rcl} M, w \vDash \xi \wedge \zeta & \Leftrightarrow & M, w \vDash \xi \text{ and } M, w \vDash \zeta \\ \text{(By the induction hypothesis)} & \Leftrightarrow & M^*, g(w) \vDash \xi \text{ and } M^*, g(w) \vDash \zeta \\ & \Leftrightarrow & M^*, g(w) \vDash \xi \wedge \zeta \end{array}$$

Then let $\varphi = [\alpha]\xi$, for arbitrary $\alpha \in \Lambda$. First assume $M, w \nvDash [\alpha]\xi$, i.e. $\exists v \in W : wR(\alpha)v$ *and* $M, v \nvDash \xi$. Since g is a Λ-bounded morphism from M to M^*, by the condition (i), $g(w)R^*(\alpha)g(v)$ and by the induction hypothesis $M^*, g(v) \nvDash \xi$. Therefore $M^*, g(w) \nvDash [\alpha]\xi$.

Next assume $M^*, g(w) \nvDash [\alpha]\xi$. Then $\exists v' \in W^* : g(w)R^*(\alpha)v'$ *and* $M^*, v' \nvDash \xi$. Therefore, by the condition (ii), $\exists v \in W : wR(\alpha)v$ and $g(v) = v'$. Since $M^*, v' \nvDash \xi$, by the induction hypothesis we have $M, v \nvDash \xi$, thus $M, w \nvDash [\alpha]\xi$. Since the choice of process $\alpha \in \Lambda$ was arbitrary, we infer from the preceding steps that $\forall \alpha \in \Lambda : M, w \vDash [\alpha]\xi \Leftrightarrow M^*, g(w) \vDash [\alpha]\xi$. This completes the induction and the proof of Lemma 5.2. □

We close this short section with a theorem concerning the most important property of Λ-p-morphisms in respect of frame definability issues.

Theorem 5.3. *Let $L(\Pi, \mathcal{A}, \mathcal{F})$ be a PSTL-language and $F = \langle W, I, \tilde{R}\rangle$, $F^* = \langle W^*, I^*, \tilde{R}^*\rangle$ frames of the named language such that F^* is a Λ-p-morphic image of F, i.e. there is a Λ-p-morphism $g : W \to W^*$. Then $\forall \varphi \in L(\Pi, \mathcal{A}, \mathcal{F}) : F \vDash \varphi \Rightarrow F^* \vDash \varphi$.*

Proof. For a contradiction, assume that $F \vDash \varphi$ *and* $F^* \nvDash \varphi$. Then there is a model $M^* = \langle W^*, I^*, \tilde{R}^*, P^*\rangle$ with state $w' \in W$ such that $M^*, w' \nvDash \varphi$. Consider a valuation P such that $\forall p \in \Pi : P(p) = \{w \in W | g(w) \in P^*(p)\}$. Clearly P is well-defined and also, in effect, now $g : W \to W^*$ is a Λ-bounded morphism from M onto M^*. Also, since $g : W \to W^*$ is surjective, we have that $\exists w \in W : w' = g(w)$. Now, lemma 5.2 states that $M, w \vDash \varphi \Leftrightarrow M^*, w' \vDash \varphi$, if $w' = g(w)$; hence we infer $M, w \nvDash \varphi$ from $M^*, w' \nvDash \varphi$. But then we have that $F \nvDash \varphi$, a contradiction. Therefore the counter-assumption is necessarily false. □

6 Operator definability in $PSTL$ frame logics

6.1 Constant operators

This section discusses some typical 0-ary operators, i.e. operators with an empty argument list. The class of 0-ary operators is a natural member in our family of operators and implied in Definition 2.5: If $f \in \mathcal{F}$ is 0-ary, then $I(f) = F : \{\emptyset\} \to \mathcal{P}(W \times W)$, since $(\mathcal{P}(W \times W))^0 = \{\emptyset\}$.[11] For 0-ary operator f, we associate the interpretation $I(f)$ directly with the target relation S of mapping F. Thus, if $I(f) = F : \emptyset \mapsto S$, we write $I(f) = S = R(f)$.[12]

Actually, by *a constant operator* we mean an operator with any arity that returns fixed relation $R \subseteq W \times W$ with any parameter array. By definition, any 0-ary operator f is automatically a constant operator. (Constant in relation to the given foundation, that is.) On the other hand, any k-ary constant operator could be assigned any arity. Whence, it is natural to generally associate constant operators with 0-ary operators, for the output of a constant operator is not dependent on its parametric relations. For example, if we have a k-ary operator f such that $\exists R \subseteq W \times W : \forall R_1, R_2, \ldots, R_k \subseteq W \times W : I(f)(R_1, R_2, \ldots, R_k) = R$, we can always introduce a 0-ary operator f_0, and on the other hand a $k+1$-ary operator f_{k+1}, such that $I(f_0) = R$ and $\forall R_1, R_2, \ldots, R_{k+1} \subseteq W \times W : I(f_{k+1})(R_1, R_2, \ldots, R_{k+1}) = R$. Therefore, generally speaking, it is pointless to define k-ary constant operators with $k > 0$. Regardless of this fact, we formulate non-definability theorems for constant operators with an arbitrary arity to emphasize the point that the results are valid for any $PSTL$-language.

Finally, it should be noted that if $\mathcal{C}(R)$ is definable in Kripke semantics, then $\mathcal{C}(R(f))$ is frame definable in $PSTL$ language $L(\Pi, \mathcal{A}, \{f\})$ wherein f is 0-ary operator. It should be clear that if there is a set of formulae Γ of standard modal logic (i.e. modal logic that contains single modal operator $\Box$) such that $\langle W, R \rangle \models \Gamma \Leftrightarrow \mathcal{C}(R)$, wherein $\langle W, R \rangle$ is a Kripke-frame, then $\langle W, I, \tilde{R} \rangle \models \Gamma^* \Leftrightarrow \mathcal{C}(R(f))$, where Γ^* is obtained by replacing the occurrences of the modal operator $\Box$ with the modal operator $[f]$. Thus, the following definability results are familiar from Kripke semantics. (For example, the proof of Theorem 6.2 is left as an exercise in Rantala & Virtanen (2004, in pg.212).)

Theorem 6.1. *Let $L(\Pi, \mathcal{A}, \{\underline{0}\})$ be a $PSTL$-language such that $\underline{0}$ is 0-ary operator. The operator returning the empty relation $\emptyset$ is frame definable within $L(\Pi, \mathcal{A}, \{\underline{0}\})$ since the following correspondence holds:*

$$\langle W, I, \tilde{R} \rangle \models [\underline{0}]\bot \text{ if and only if } R(\underline{0}) = \emptyset.$$

[11]Technically, A^0 is the set that has the empty tuple as its only member. We use the notation $\emptyset$ to denote the empty tuple in the context of the constant operators.

[12]This is a minor technicality: Actually $I(f)$ is a function from the empty tuple to $\mathcal{P}(W \times W)$, but as in the Definition 2.5 on page 5 we can write $I(f)(\emptyset) = R(f(\emptyset))$. Thus, if we omit the redundant reference to the empty tuple $\emptyset$, the expression $I(f)(\emptyset) = R(f(\emptyset))$ reduces to $I(f) = R(f)$.

Proof. The proof is trivial. If $R(\underline{0}) = \emptyset$ holds in the frame $F = \langle W, I, \tilde{R} \rangle$, directly by the truth definition of the operator formula $[\underline{0}]\bot$, we have that $F \vDash [\underline{0}]\bot$.

If, on the other hand, $R(\underline{0}) \neq \emptyset$, there exists states $w, w' \in W$ such that $wR(\underline{0})w'$. Now, $M, w' \nvDash \bot$ by the definition of the constant $\bot$, hence $M, w \nvDash [\underline{0}]\bot$. Thus, $F \nvDash [\underline{0}]\bot$. □

Theorem 6.2. *Let $L(\Pi, \mathcal{A}, \{D\})$ be a $PSTL$-language with 0-ary operator D. An operator returning the diagonal relation $\{\langle w, w \rangle \mid w \in W\}$ (i.e. the identity map of W) is frame definable in language $L(\Pi, \mathcal{A}, \{D\})$ since the following holds:*

$$\langle W, I, \tilde{R} \rangle \vDash [D]\varphi \leftrightarrow \varphi \text{ if and only if } R(D) = \{\langle w, w \rangle \mid w \in W\}.$$

Proof. The proof for the implication from right to left is quite straightforward. Let $M = \langle W, I, \tilde{R}, P \rangle$ be a $L(\Pi, \mathcal{A}, \{\mathrm{D}\})$-model satisfying $R(\mathrm{D}) = \{\langle w, w \rangle \mid w \in W\}$. Now $\forall w \in W$: *if* $M, w \vDash [\mathrm{D}]\varphi$, *then* $M, w \vDash \varphi$, because $wR(\mathrm{D})w$. Similarly, if $M, w \nvDash [\mathrm{D}]\varphi$, then $\exists w, w' \in W : wR(\mathrm{D})w'$ *and* $M, w' \nvDash \varphi$. By the definition of $R(\mathrm{D})$, necessarily $w = w'$, thus $M, w \nvDash \varphi$. Hence, we conclude $R(\mathrm{D}) = \{\langle w, w \rangle | w \in W\} \Rightarrow \langle W, I, \tilde{R} \rangle \vDash [\mathrm{D}]\varphi \leftrightarrow \varphi$.

Then, let $F = \langle W, I, \tilde{R} \rangle$ be a $L(\Pi, \mathcal{A}, \mathcal{F})$-frame such that $R(\mathrm{D}) \neq \{\langle w, w \rangle | w \in W\}$. Now F satisfies either

(i) $\exists w \in W : w \not R(\mathrm{D})w$, or
(ii) $\exists w, w' \in W : wR(\mathrm{D})w'$ *and* $w \neq w'$.

First assume that the frame F satisfies the condition (i). Let $M = \langle W, I, \tilde{R}, P \rangle$ be a model with a valuation mapping P for some propositional symbol $p \in \Pi$ defined such that $P(p) = W \setminus \{w\}$. Then, by the definition of $P(p)$, we readily have $M, w \nvDash p$. On the same grounds, with the condition (i), $\forall w' \in W : wR(\mathrm{D})w' \Rightarrow M, w' \vDash p$; that is, $M, w \vDash [\mathrm{D}]p$. Therefore $M, w \nvDash [\mathrm{D}]p \leftrightarrow p$.

Next assume that F satisfies the condition (ii). Let $M = \langle W, I, \tilde{R}, P \rangle$ be structure with valuation $P(p) = \{w\}$, for some $p \in \Pi$; i.e. $M, w \vDash p$. By condition (ii), $wR(\mathrm{D})w'$ and $w \neq w'$. By the choice of $P(p)$ we have $M, w' \nvDash p$, thus $M, w \nvDash [\mathrm{D}]p$. Therefore $M, w \nvDash [\mathrm{D}]p \leftrightarrow p$.

Hence, we have that $\langle W, I, \tilde{R} \rangle \vDash [\mathrm{D}]\varphi \leftrightarrow \varphi \Rightarrow R(\mathrm{D}) = \{\langle w, w \rangle | w \in W\}$. □

Results of the section 4 are very useful for proving that several operators are not definable in any $PSTL$-language, especially in the case of domain dependent operators. Since several standard constant operators are inherently domain dependent, we find powerful applications for Lemma 4.4 and its corollaries within the non-frame definable constant operators.

Theorem 6.3. *The operator that returns the universal relation $W \times W$ of an arbitrary frame $\langle W, I, \tilde{R} \rangle$, is not frame definable in any $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$.*

The theorem is directly implied by Lemma 4.8 but for the sake of illustration we consider the proof in this case in more detail.

Proof. Assume the contrary; that is, there is a $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$ and a set of formulae $\Gamma \subseteq L(\Pi, \mathcal{A}, \mathcal{F})$ such that Γ defines the universal relation on frames. Thus, there is a k-ary operator $f \in \mathcal{F}$ such that

$$\langle W, I, \tilde{R} \rangle \models \Gamma \Leftrightarrow \forall \alpha_1, \ldots, \alpha_k \in \Lambda : R(f(\alpha_1, \ldots, \alpha_k)) = W \times W.$$

Let $F_1 = \langle W_1, I_1, \tilde{R}_1 \rangle$ and $F_2 = \langle W_2, I_2, \tilde{R}_2 \rangle$ be Γ-frames (i.e. $F_1 \models \Gamma$ and $F_2 \models \Gamma$) defined such that $W_1 = \{w_1\}$, $W_2 = \{w_2\}$ and $\forall a \in \mathcal{A} : \tilde{R}_1(a) = \emptyset = \tilde{R}_2(a)$. We assume that $I_1(g)$ and $I_2(g)$ are well defined for all $g \in \mathcal{F}$, possibly as required by Γ. Especially, since F_1 and F_2 are Γ-frames, $I_1(f)(R_1(\alpha_1), \ldots, R_1(\alpha_k)) = W_1 \times W_1$ and $I_2(f)(R_2(\alpha_1), \ldots, R_2(\alpha_k)) = W_2 \times W_2$.

Now, by Definition 4.6, the structure $F_{(1;2)} = \langle W_{(1;2)}, I_{(1;2)}, \tilde{R}_{(1;2)} \rangle$ is the following:

$$\begin{cases} W_{(1;2)} = \{w_1, w_2\} \\ \forall \alpha_1, \ldots, \alpha_k \in \Lambda : I_{(1;2)}(f)(R_{(1;2)}(\alpha_1), \ldots, R_{(1;2)}(\alpha_k)) = \{\langle w_1, w_1 \rangle, \langle w_2, w_2 \rangle\} \\ (\forall g \in \mathcal{F} : I_{(1;2)}(g) \text{ defined by } I_1, I_2 \text{ and the operation } \uplus) \\ \forall a_i \in \mathcal{A} : \tilde{R}_{(1;2)}(a_i) = \emptyset \end{cases}$$

Now, $\exists \alpha_1, \ldots, \alpha_k \in \Lambda : R_{(1;2)}(f(\alpha_1, \ldots, \alpha_k)) \neq W_{(1;2)} \times W_{(1;2)}$ (indeed, we can choose any list of processes $\alpha_1, \ldots, \alpha_k \in \Lambda$ we like). Thus, by the counter-assumption, $F_{(1;2)} \not\models \Gamma$. Consequently, by Corollary 4.7, either $F_1 \not\models \Gamma$ or $F_2 \not\models \Gamma$. Hence, the counter-assumption implies contradiction and therefore is false. □

Theorem 6.4. *The operation that maps to the diversity relation of the domain W, i.e. relation $\{\langle w, w' \rangle | w, w' \in W : w \neq w'\}$, is not frame definable in any $PSTL$-language.*

Proof. The proof should be obvious. Consider frames $F_1 = \langle \{w_1\}, I_1, \tilde{R}_1 \rangle$ and $F_2 = \langle \{w_1\}, I_2, \tilde{R}_2 \rangle$. The frames have the same diversity relation, namely the empty relation. Now, if we have an operator $f \in \mathcal{F}$ such that $\forall \alpha_1, \ldots, \alpha_k \in \Lambda : R_1(f(\alpha_1, \ldots, \alpha_k)) = \emptyset$ and $R_2(f(\alpha_1, \ldots, \alpha_k)) = \emptyset$, then $\forall \alpha_1, \ldots, \alpha_k \in \Lambda : R_{(1;2)}(f(\alpha_1, \ldots, \alpha_k)) = \emptyset$. But the diversity relation of the frame $F_{(1:2)}$ is $\{\langle w_1, w_2 \rangle, \langle w_2, w_1 \rangle\}$, therefore the theorem holds by Lemma 4.8. □

6.2 Unary operators

In this section we shall discuss only two operators, where one is frame definable and the other not. We shall begin with the latter exploiting Lemma 4.8.

Theorem 6.5. *The operator that returns the complement of the relation $R(\alpha), \forall \alpha \in \Lambda$, with respect to the universal relation of a given domain, i.e. the relation $\overline{R(\alpha)} = (W \times W) \setminus R(\alpha)$, is not frame definable in any $PSTL$-language.*

Since the complementation is a domain dependent operation and the scope of the application of Lemma 4.8 clearly coincides with the domain dependent properties, this should be enough to inform the reader that the complementation is not frame definable in any $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$. Anyhow, we look at the proof briefly with some more detail.

Proof. Consider frames $F_1 = \langle W_1, I_1, \tilde{R}_1 \rangle$ and $F_2 = \langle W_2, I_2, \tilde{R}_2 \rangle$ of any $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$ with a unary operator $f \in \mathcal{F}$ such that

$$\begin{cases} W_1 = \{w_1\} \\ I_1(f)(R_1(\alpha)) = \overline{R_1(\alpha)}, \forall \alpha \in \Lambda \\ \tilde{R}_1(a) = \emptyset, \forall a \in \mathcal{A} \end{cases} \quad \textit{and} \quad \begin{cases} W_2 = \{w_2\} \\ I_2(f)(R(\alpha)) = \overline{R(\alpha)}, \forall \alpha \in \Lambda \\ \tilde{R}_2(a) = \emptyset, \forall a \in \mathcal{A} \end{cases}$$

Thus, for any $a \in \mathcal{A}$, $I_1(f)(R_1(a)) = \{\langle w_1, w_1 \rangle\}$ and $I_2(f)(R_2(a)) = \{\langle w_2, w_2 \rangle\}$. Now, consider the frame $F_{(1;2)} = \langle W_{(1;2)}, I_{(1;2)}, \tilde{R}_{(1;2)} \rangle$ and pick any $a \in \mathcal{A}$. Since $R_{(1;2)}(a) = \emptyset$ and $I_{(1;2)}(f)(R_{(1;2)}(a)) = \{\langle w_1, w_1 \rangle, \langle w_2, w_2 \rangle\} \neq W_{(1;2)} \times W_{(1;2)} = \overline{R_{(1;2)}(a)}$, the complementation property is not conserved within the interpretation $I_{(1;2)}(f)$. Therefore, by Lemma 4.8, the operator carrying out the complementation is not frame definable in any $PSTL$-language. □

Note two details in the previous proof. First, the set of syntactic operators $\mathcal{F}$ could contain a set of operators $\{f_1, f_2, f_3, \ldots\} \subseteq \mathcal{F} : f \notin \{f_1, f_2, f_3, \ldots\}$ and should this be the case, interpretations $I_k(f_i), k \in \{1, 2, (1;2)\}, i \in \{1, 2, 3, \ldots\}$ were not explicitly defined in frames F_1, F_2 and $F_{(1;2)}$. In any case, we can assume that the possible extraneous operators are assigned any interpretation since in the crucial step of the proof operator f operates directly on the atomic process. Therefore, whatever interpretations operators $f_1, f_2, f_3 \ldots$ should have, this does not affect the conclusion. Secondly, we assumed that f is a unary operator. This, again, does not affect the conclusion for the following reason. Clearly the arity of the operator f should be $n \geq 1$, and if the arity is $n > 1$, we need a projection function to pick out the relation $R(\alpha)$ from the argument list in $I(f)(R(\alpha_1), \ldots, R(\alpha), \ldots R(\alpha_n))$, if we want f to return the complement of $R(\alpha)$ with respect to $W \times W$. Otherwise the output of f would not be effectively dependent on the argument list, thus rendering f ill-defined. And this, in turn, effectively reduces the n-ary operator f to a unary operator. Hence, the proof is valid regardless the actual arity assigned to the operator f, or the arity of any other operator in $\mathcal{F}$ for that matter.

Theorem 6.6. *Let $L(\Pi, \mathcal{A}, \{\breve{\cdot}\})$ be a $PSTL$-language and $\breve{\cdot}$ a unary operator. The operation returning the inverse of relation $R(\alpha)$, i.e. $R(\alpha)^{-1} = \{\langle w, w' \rangle | \langle w', w \rangle \in R(\alpha)\}$, is frame definable in $L(\Pi, \mathcal{A}, \{\breve{\cdot}\})$, since the following correspondence holds:*

$$\langle W, I, \tilde{R} \rangle \vDash \varphi \to [\alpha]\langle \breve{\alpha} \rangle \varphi \wedge [\breve{\alpha}]\langle \alpha \rangle \varphi \textit{ if and only if } \forall \alpha \in \Lambda : R(\breve{\alpha}) = R(\alpha)^{-1}.$$

Proof. [13] Let $\langle W, I, \tilde{R} \rangle$ be a $L(\Pi, \mathcal{A}, \{\breve{\cdot}\})$-frame such that $\forall \alpha \in \Lambda : R(\breve{\alpha}) = R(\alpha)^{-1}$. Consider an arbitrary model $M = \langle W, I, \tilde{R}, P \rangle$ of a frame F and let $w \in W$ be such a state

[13] For partial result, see (Harel et al., 2000, pp.177–178).

that $M, w \vDash \varphi$. If $wR(\alpha)w'$, then, by the definition of an inverse relation, $w'R(\alpha)^{-1}w$. By assumption $R(\breve{\alpha}) = R(\alpha)^{-1}$, we infer $M, w' \vDash \langle\breve{\alpha}\rangle\varphi$. All we assumed about state w' was that $wR(\alpha)w'$, whence the conclusion holds for all $w' \in W : wR(\alpha)w'$; that is, $\forall w' \in W : wR(\alpha)w' \Rightarrow M, w' \vDash \langle\breve{\alpha}\rangle\varphi$. Therefore, $M, w \vDash \varphi \Rightarrow M, w \vDash [\alpha]\langle\breve{\alpha}\rangle\varphi$. If on the other hand $wR(\breve{\alpha})w'$, similarly as above then $w'R(\alpha)w$, since $R(\breve{\alpha}) = R(\alpha)^{-1}$, hence $M, w' \vDash \langle\alpha\rangle\varphi$. Thus, $\forall w' \in W : wR(\breve{\alpha})w' \Rightarrow M, w' \vDash \langle\alpha\rangle\varphi$, i.e. $M, w \vDash [\breve{\alpha}]\langle\alpha\rangle\varphi$. From the above results we infer $M, w \vDash \varphi \to [\alpha]\langle\breve{\alpha}\rangle\varphi \wedge [\breve{\alpha}]\langle\alpha\rangle\varphi$.

Therefore, *if* $\forall\alpha \in \Lambda : R(\breve{\alpha}) = R(\alpha)^{-1}$, *then* $\langle W, I, \tilde{R}\rangle \vDash \varphi \to [\alpha]\langle\breve{\alpha}\rangle\varphi \wedge [\breve{\alpha}]\langle\alpha\rangle\varphi$.

Next, consider the frame $F = \langle W, I, \tilde{R}\rangle$ and assume that $\exists\alpha \in \Lambda : R(\breve{\alpha}) \neq R(\alpha)^{-1}$. Then either (i) $R(\breve{\alpha}) \nsubseteq R(\alpha)^{-1}$ or (ii) $R(\alpha)^{-1} \nsubseteq R(\breve{\alpha})$; that is,

$$\begin{array}{ll} \text{(i)} & \exists w, w' \in W : wR(\breve{\alpha})w' \textit{ and } w \not{R}(\alpha)^{-1}w', \text{ or} \\ \text{(ii)} & \exists w, w' \in W : wR(\alpha)^{-1}w' \textit{ and } w\not{R}(\breve{\alpha})w' \end{array}$$

First, assume that F satisfies the condition (i). Let $M = \langle W, I, \tilde{R}, P\rangle$ be model with valuation $P(p) = \{w\}$, for some $p \in \Pi$. Thus, $M, w \vDash p$. Since $w\not{R}(\alpha)^{-1}w'$ by condition (i), by the definition of inverse relation we then have $w'\not{R}(\alpha)w$. Therefore, $M, w' \nvDash \langle\alpha\rangle p$ by the choice of $P(p)$. Again, $wR(\breve{\alpha})w'$ by condition (i), whence $\exists w' \in W : wR(\breve{\alpha})w'$ *and* $M, w' \nvDash \langle\alpha\rangle p$, i.e. $M, w \nvDash [\breve{\alpha}]\langle\alpha\rangle p$. Therefore, we have $M, w \nvDash [\alpha]\langle\breve{\alpha}\rangle p \wedge [\breve{\alpha}]\langle\alpha\rangle p$. Hence, we conclude that $M, w \nvDash p \to [\alpha]\langle\breve{\alpha}\rangle p \wedge [\breve{\alpha}]\langle\alpha\rangle p$.

Second, assume that F satisfies (ii). Let $M = \langle W, I, \tilde{R}, P\rangle$ be a model such that $P(p) = \{w'\}$, i.e. $M, w' \vDash p$. Now, $w'R(\alpha)w$, because by (ii) $wR(\alpha)^{-1}w'$. On the other hand, $w\not{R}(\breve{\alpha})w'$ and $P(p) = \{w'\}$, whence $M, w \nvDash \langle\breve{\alpha}\rangle p$. Thus, $w'R(\alpha)w$ and $M, w \nvDash \langle\breve{\alpha}\rangle p$, hence $M, w' \nvDash [\alpha]\langle\breve{\alpha}\rangle p$ and we conclude that $M, w' \nvDash p \to [\alpha]\langle\breve{\alpha}\rangle p \wedge [\breve{\alpha}]\langle\alpha\rangle p$

Thus, if a frame F satisfies $R(\breve{\alpha}) \neq R(\alpha)^{-1}$, then $F \nvDash \varphi \to [\alpha]\langle\breve{\alpha}\rangle\varphi \wedge [\breve{\alpha}]\langle\alpha\rangle\varphi$. Therefore, *if* $\langle W, I, \tilde{R}\rangle \vDash \varphi \to [\alpha]\langle\breve{\alpha}\rangle\varphi \wedge [\breve{\alpha}]\langle\alpha\rangle\varphi$, *then* $\forall\alpha \in \Lambda : R(\breve{\alpha}) = R(\alpha)^{-1}$. □

6.3 Binary operators

The composition and the union of binary relations are basic operations in the interpretation of the regular *PDL*. Thus, the results 6.7 and 6.8 are well-known.[14]

Theorem 6.7. *Let* $L(\Pi, \mathcal{A}, \{;\})$ *be a PSTL-language such that* ; *is a binary syntactic operator. The operator that returns the composition of relations* $R(\alpha), R(\beta); \alpha, \beta \in \Lambda$, *i.e.* $R(\alpha) \circ R(\beta) = \{\langle w, w'\rangle | \exists w'' \in W : wR(\alpha)w'' \textit{ and } w''R(\beta)w'\}$, *is frame definable in* $L(\Pi, \mathcal{A}, \{;\})$, *since*

$$\langle W, I, \tilde{R}\rangle \vDash \langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi \textit{ if and only if } \forall\alpha, \beta \in \Lambda : R(\alpha;\beta) = R(\alpha) \circ R(\beta).$$

[14]See for example (Blackburn et al., 2002, pp.132–133) or (Harel et al., 2000, pp.175–176).

Proof. Let $\langle W, I, \tilde{R}\rangle$ be a $L(\Pi, \mathcal{A}, \{;\})$-frame. Assume $\forall \alpha, \beta \in \Lambda : R(\alpha;\beta) = R(\alpha) \circ R(\beta)$. Choose a model $M = \langle W, I, \tilde{R}, P\rangle$ and any state $w \in W$. Now,

$$\begin{array}{rcl}
M, w \vDash \langle \alpha;\beta\rangle\varphi & \Leftrightarrow & \exists w' \in W : wR(\alpha;\beta)w' \textit{ and } M, w' \vDash \varphi \\
(\text{Since } I(;) \simeq \circ) & \Leftrightarrow & \exists w' \in W : w(R(\alpha) \circ R(\beta))w' \textit{ and } M, w' \vDash \varphi \\
(\text{By the definition of } \circ) & \Leftrightarrow & \exists w' \in W : \exists w'' \in W : wR(\alpha)w'', w''R(\beta)w' \textit{ and } M, w' \vDash \varphi \\
 & \Leftrightarrow & \exists w'' \in W : wR(\alpha)w'' \textit{ and } M, w'' \vDash \langle\beta\rangle\varphi \\
 & \Leftrightarrow & M, w \vDash \langle\alpha\rangle\langle\beta\rangle\varphi
\end{array}$$

Therefore, $\forall \alpha, \beta \in \Lambda : R(\alpha;\beta) = R(\alpha) \circ R(\beta) \Rightarrow \langle W, I, \tilde{R}\rangle \vDash \langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$.

Then let $\langle W, I, \tilde{R}\rangle$ be a $L(\Pi, \mathcal{A}, \{;\})$-frame such that $\exists \alpha, \beta \in \Lambda : R(\alpha;\beta) \neq R(\alpha) \circ R(\beta)$. Now the frame satisfies either:[15]

$$\begin{array}{rl}
\text{(i)} & \exists w, w' \in W : \langle w, w'\rangle \in R(\alpha;\beta) \textit{ and } \langle w, w'\rangle \notin R(\alpha) \circ R(\beta), \text{ or} \\
\text{(ii)} & \exists w, w' \in W : \langle w, w'\rangle \notin R(\alpha;\beta) \textit{ and } \langle w, w'\rangle \in R(\alpha) \circ R(\beta)
\end{array}$$

Assume that $\langle W, I, \tilde{R}\rangle$ satisfies (i) and let $M = \langle W, I, \tilde{R}, P\rangle$ be such model that $P(p) = \{w'\}$, for some $p \in \Pi$. Since $\langle w, w'\rangle \notin R(\alpha) \circ R(\beta)$, the choice of $P(p)$ implies that $\nexists w'' \in W : \langle w, w''\rangle \in R(\alpha) \circ R(\beta) \textit{ and } M, w'' \vDash p$. By the above equivalence chain, $\exists w'' \in W : \langle w, w''\rangle \in R(\alpha) \circ R(\beta) \textit{ and } M, w'' \vDash p \Leftrightarrow M, w \vDash \langle\alpha\rangle\langle\beta\rangle p$, thus we infer $M, w \nvDash \langle\alpha\rangle\langle\beta\rangle p$. On the other hand, $\langle w, w'\rangle \in R(\alpha;\beta)$, hence by the choice of $P(p)$ we have that $M, w \vDash \langle\alpha;\beta\rangle p$. Therefore, $M, w \nvDash \langle\alpha;\beta\rangle p \leftrightarrow \langle\alpha\rangle\langle\beta\rangle p$.

Next consider the case that the frame $\langle W, I, \tilde{R}\rangle$ satisfies the condition (ii). Again, we choose a valuation $P(p) = \{w'\}$ for some $p \in \Pi$ and thus obtain a model $M = \langle W, I, \tilde{R}, P\rangle$ such that $M, w \nvDash \langle\alpha;\beta\rangle p$, since $\langle w, w'\rangle \notin R(\alpha;\beta)$. By the condition (ii), $\langle w, w'\rangle \in R(\alpha) \circ R(\beta)$, and by the choice of $P(p)$ we have that $M, w' \vDash p$. Therefore, we infer by the above equivalence chain that $M, w \vDash \langle\alpha\rangle\langle\beta\rangle p$. Hence, $M, w \nvDash \langle\alpha;\beta\rangle p \leftrightarrow \langle\alpha\rangle\langle\beta\rangle p$.

Therefore, either way, if the frame $\langle W, I, \tilde{R}\rangle$ satisfies $\exists \alpha, \beta \in \Lambda : R(\alpha;\beta) \neq R(\alpha) \circ R(\beta)$, then $\langle W, I, \tilde{R}\rangle \nvDash \langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$. □

Theorem 6.8. *Consider PSTL-language $L(\Pi, \mathcal{A}, \{+\})$ such that $+$ is a binary operator. Then,*

$$\langle W, I, \tilde{R}\rangle \vDash [\alpha+\beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi \textit{ if and only if } \forall \alpha, \beta \in \Lambda : R(\alpha+\beta) = R(\alpha) \cup R(\beta).$$

That is, operator $I(+) \simeq \cup$ is frame definable in a PSTL-language with a single binary syntactic operator.

[15]A remark on notation: As usual, $\langle w, w'\rangle \in R(\alpha)$ denotes $wR(\alpha)w'$ and $\langle w, w'\rangle \notin R(\alpha)$ denotes $w\not{R}(\alpha)w'$. For the notational simplicity, denotation $wR(\alpha)w'$ is used throughout the text when appropriate. For proofs considering complex relations, however, the set-inclusion notation is adopted for legibility, especially in the case of the proofs about non-inclusion relations.

Proof. Let $\langle W, I, \tilde{R}\rangle$ be $L(\Pi, \mathcal{A}, \{+\})$-frame such that $\forall \alpha, \beta \in \Lambda : R(\alpha+\beta) = R(\alpha) \cup R(\beta)$. Consider a model $M = \langle W, I, \tilde{R}, P\rangle$ and $w \in W$. Then $M, w \vDash [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$, since

$$\begin{aligned} M, w \vDash [\alpha + \beta]\varphi &\Leftrightarrow \forall w' \in W : wR(\alpha + \beta)w' \Rightarrow M, w' \vDash \varphi \\ &\Leftrightarrow \forall w' \in W : w(R(\alpha) \cup R(\beta))w' \Rightarrow M, w' \vDash \varphi \\ &\Leftrightarrow \forall v \in W : wR(\alpha)v \Rightarrow M, v \vDash \varphi \text{ and } \forall v' \in W : wR(\beta)v' \Rightarrow M, v' \vDash \varphi \\ &\Leftrightarrow M, w \vDash [\alpha]\varphi \text{ and } M, w \vDash [\beta]\varphi \\ &\Leftrightarrow M, w \vDash [\alpha]\varphi \wedge [\beta]\varphi. \end{aligned}$$

Thus, we infer $\forall \alpha, \beta \in \Lambda : R(\alpha+\beta) = R(\alpha) \cup R(\beta) \Rightarrow \langle W, I, \tilde{R}\rangle \vDash [\alpha+\beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$.

We then consider the frame $\langle W, I, \tilde{R}\rangle$ such that $\exists \alpha, \beta \in \Lambda : R(\alpha + \beta) \neq R(\alpha) \cup R(\beta)$. Hence, the frame satisfies either:

(i) $\exists w, w' \in W : \langle w, w'\rangle \in R(\alpha) \cup R(\beta)$ *and* $\langle w, w'\rangle \notin R(\alpha + \beta)$, or
(ii) $\exists w, w' \in W : \langle w, w'\rangle \notin R(\alpha) \cup R(\beta)$ *and* $\langle w, w'\rangle \in R(\alpha + \beta)$

Assume first that the frame satisfies the condition (i). Let P be a valuation function with a definition $P(p) = W \setminus \{w'\}$ for some $p \in \Pi$. Since $\langle w, w'\rangle \in R(\alpha) \cup R(\beta)$, we have $\langle w, w'\rangle \in R(\alpha)$ or $\langle w, w'\rangle \in R(\beta)$. If $\langle w, w'\rangle \in R(\alpha)$, clearly $M, w \nvDash [\alpha]p$. If $\langle w, w'\rangle \in R(\beta)$, likewise $M, w \nvDash [\beta]\varphi$. At any rate, condition (i) with a valuation $P(p) = W \setminus \{w'\}$ implies $M, w \nvDash [\alpha]\varphi \wedge [\beta]\varphi$. On the other hand, $\langle w, w'\rangle \notin R(\alpha+\beta)$. Thus, by the definition of valuation $P(p)$, we have that $\forall w'' \in W : \langle w, w''\rangle \in R(\alpha + \beta) \Rightarrow M, w'' \vDash \varphi$, hence $M, w \vDash [\alpha + \beta]p$. Therefore, $M, w \nvDash [\alpha + \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p$.

Then assume that $\langle W, I, \tilde{R}\rangle$ satisfies (ii). This part of the proof proceeds as the previous one. Thus, let $M = \langle W, I, \tilde{R}, P\rangle$ be a model such that $P(p) = W \setminus \{w'\}$, for some $p \in \Pi$. Now $\langle w, w'\rangle \notin R(\alpha)$ and $\langle w, w'\rangle \notin R(\beta)$, since $\langle w, w'\rangle \notin R(\alpha) \cup R(\beta)$. Evidently, then, $M, w \vDash [\alpha]p$ and $M, w \vDash [\beta]p$, whence $M, w \vDash [\alpha]p \wedge [\beta]p$. Since $\langle w, w'\rangle \in R(\alpha + \beta)$ and $M, w' \nvDash p$, we have $M, w \nvDash [\alpha + \beta]p$. Thus, $M, w \nvDash [\alpha + \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p$.

Therefore, $\langle W, I, \tilde{R}\rangle \vDash [\alpha+\beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi \Rightarrow \forall \alpha, \beta \in \Lambda : R(\alpha+\beta) = R(\alpha) \cup R(\beta)$. □

The relative union of the relations R, S is rather untypical domain dependent operation on binary relations presented in the Tarski's paper (Tarski, 1941, pg.76). The operation is defined as follows.

Definition 6.9. Let $R \dotplus S$ denote the relative union of binary relations R, S on domain W, i.e. $R, S \subseteq W \times W$. Then,

$$R \dotplus S = \{\langle w, w'\rangle | \forall w'' \in W : wRw'' \textit{ or } w''Sw'\}.$$

For clarification it should be noted that $R \dotplus S = \overline{\overline{R} \circ \overline{S}}$, since

$$\begin{aligned} \langle w, w'\rangle \in \overline{\overline{R} \circ \overline{S}} &\Leftrightarrow \langle w, w'\rangle \notin \overline{R} \circ \overline{S} \\ &\Leftrightarrow \nexists w'' \in W : \langle w, w''\rangle \in \overline{R} \textit{ and } \langle w'', w'\rangle \in \overline{S} \\ &\Leftrightarrow \forall w'' \in W : \langle w, w''\rangle \notin \overline{R} \textit{ or } \langle w'', w'\rangle \notin \overline{S} \\ &\Leftrightarrow \forall w'' \in W : \langle w, w''\rangle \in R \textit{ or } \langle w'', w'\rangle \in S \\ &\Leftrightarrow \langle w, w'\rangle \in R \dotplus S. \end{aligned}$$

Theorem 6.10. *The operator that returns the relative union of $R(\alpha), R(\beta); \forall\alpha, \beta \in \Lambda$ is not frame definable in any $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$.*

Proof. Pick any $PSTL$-language $L(\Pi, \mathcal{A}, \mathcal{F})$ with a binary syntactic operator $f \in \mathcal{F}$. Let $F_1 = \langle W_1, I_1, \tilde{R}_1\rangle$ and $F_2 = \langle W_2, I_2, \tilde{R}_2\rangle$ be such $L(\Pi, \mathcal{A}, \mathcal{F})$-frames that

$$\begin{cases} W_1 = \{w_1\} \\ \tilde{R}_1(a) = \{\langle w_1, w_1\rangle\}, \forall a \in \mathcal{A} \\ I_1(f)(R_1(\alpha), R_1(\beta)) = R_1(\alpha) \dotplus R_1(\beta), \forall\alpha, \beta \in \Lambda \end{cases}$$

$$\begin{cases} W_2 = \{w_2\} \\ R_2(a) = \{\langle w_2, w_2\rangle\}, \forall a \in \mathcal{A} \\ I_2(f)(R_2(\alpha), R_2(\beta)) = R_2(\alpha) \dotplus R_2(\beta), \forall\alpha, \beta \in \Lambda \end{cases}$$

As usual, we also assume that $I_1(g)$ and $I_2(g)$ are defined for all $g \in \mathcal{F}$.

Choose any $a \in \mathcal{A}$. By the definitions of I_1 and I_2, we have that $R_1(f(a, a)) = R_1(a) \dotplus R_1(a) = \{\langle w_1, w_1\rangle\}$ and $R_2(f(a, a)) = R_2(a) \dotplus R_2(a) = \{\langle w_2, w_2\rangle\}$. Consider then the structure $F_{(1;2)} = \langle W_{(1;2)}, I_{(1;2)}, \tilde{R}_{(1;2)}\rangle$. Now, $R_{(1;2)}(f(a, a)) = \{\langle w_1, w_1\rangle, \langle w_2, w_2\rangle\}$, by Definition 4.2. On the other hand, $R_{(1;2)}(a) \dotplus R_{(1;2)}(a) = \{\langle w_1, w_2\rangle, \langle w_2, w_1\rangle\}$. Thus, $R_{(1;2)}(f(a, a)) \neq R_{(1;2)}(a) \dotplus R_{(1;2)}(a)$ (whence, $I_{(1;2)}(f) \not\simeq \dotplus$), though $I_1(f) \simeq \dotplus$ and $I_2(f) \simeq \dotplus$. Therefore, by Lemma 4.8, the operator that returns the relative union is not frame definable in $L(\Pi, \mathcal{A}, \mathcal{F})$, since the relative unification is not conserved in the union of disjoint frames. □

So far, we have carried out the non-definability proofs with respect to the whole class of $PSTL$-languages. Next we proceed to demonstrate two weaker non-definability results: First, that the intersection is not frame definable in any $PSTL$ language $L(\Pi, \mathcal{A}, \{+, \cdot\})$, wherein $+$ and $\cdot$ are binary, with respect to the frame class $\mathbf{C}_{(+,\cup)} = \{\langle W, I, \tilde{R}\rangle | I(+) \simeq \cup\}$; that is, the intersection is not frame definable with the union. We go over the definability issue of the intersection with the union operation since $I(\cdot) \simeq \cap$ turns out to be foundation definable within the language in question, given interpretation $I(+) \simeq \cup$. Second, that the relative complementation (i.e. the set-theoretical difference) is not frame definable in any $PSTL$ language $L(\Pi, \mathcal{A}, \{/\})$ wherein $/$ is a binary operator.

We assume that the candidate syntactic operators for carrying out the target operations are binary. This should not effect the generality of the following theorems;[16] the restriction is to simplify the following discussion. But the possibility, that there is a more expressive language $L(\Pi, \mathcal{A}, \mathcal{F})$, (that is, a language that contains more syntactic operators) such that, for instance, relative complementation is definable in $L(\Pi, \mathcal{A}, \mathcal{F})$, is left open. Frame definability of the operators in question seems unlikely, but the resolution of the issue is presently open for the lack of proper general frame definability results.

Theorem 6.11. *The intersection operator of the relations $R(\alpha), R(\beta)$ for every $\alpha, \beta \in \Lambda$ is not frame definable with respect to the frame class $\mathbf{C}_{(+,\cup)} = \{\langle W, I, \tilde{R}\rangle | I(+) \simeq \cup\}$*

[16] Similarly as in the proofs of theorems 6.5 and 6.10. Although we lack a formal proof in this context, the reasoning is on the lines with the comments after the proof of theorem 6.5 in page 6.2.

within language $L(\Pi, \mathcal{A}, \{+, \cdot\})$, wherein $+$ and $\cdot$ are binary operators. Thus, in brief, the intersection is not frame definable (even) with the union operator.

Proof. Let $L(\Pi, \mathcal{A}, \{+, \cdot\})$ be a $PSTL$-language such that $|\mathcal{A}| \geq 2$ and $+, \cdot$ are binary syntactic operators.

Consider frames $F_1 = \langle W_1, I_2, \tilde{R}_1 \rangle$ and $F_2 = \langle W_2, I_2, \tilde{R}_2 \rangle$ such that

$$\begin{cases} W_1 = \{w_0, w_1, w_2, w_3\} \\ I_1(+)(S,T) = S \cup T; \forall S, T \subseteq W_1 \times W_1 \\ I_1(\cdot)(S,T) = \begin{cases} \emptyset; & S = \{\langle w_0, w_1 \rangle, \langle w_0, w_2 \rangle\}, T = \{\langle w_0, w_2 \rangle, \langle w_0, w_3 \rangle\} \\ \emptyset; & S = \{\langle w_0, w_2 \rangle, \langle w_0, w_3 \rangle\}, T = \{\langle w_0, w_1 \rangle, \langle w_0, w_2 \rangle\} \\ S \cap T; & otherwise \end{cases} \\ \tilde{R}_1(a_i) = \begin{cases} \{\langle w_0, w_1 \rangle, \langle w_0, w_2 \rangle\}; & i \text{ is odd} \\ \{\langle w_0, w_2 \rangle, \langle w_0, w_3 \rangle\}; & i \text{ is even} \end{cases} \end{cases}$$

$$\begin{cases} W_2 = \{w_0, w_1, w_2, w_3, w_4\} \\ I_2(+)(S,T) = S \cup T; \forall S, T \subseteq W_2 \times W_2 \\ I_2(\cdot)(S,T) = S \cap T; \forall S, T \subseteq W_2 \times W_2 \\ \tilde{R}_2(a_i) = \begin{cases} \{\langle w_0, w_1 \rangle, \langle w_0, w_2 \rangle\}; & i \text{ is odd} \\ \{\langle w_0, w_3 \rangle, \langle w_0, w_4 \rangle\}; & i \text{ is even} \end{cases} \end{cases}$$

Clearly, $F_1, F_2 \in \mathbf{C}_{(+,\cup)}$. We proceed to demonstrate that there is a surjective Λ-bounded morphisms from W_2 to W_1.

Let $A_i \subseteq W_1 \times W_1$ and $B_i \subseteq W_2 \times W_2, i \in \{1, 2, 3, 4\}$ be the following relations:

$$\begin{array}{ll} A_1 = \{\langle w_0, w_1 \rangle, \langle w_0, w_2 \rangle\} & B_1 = \{\langle w_0, w_1 \rangle, \langle w_0, w_2 \rangle\} \\ A_2 = \{\langle w_0, w_2 \rangle, \langle w_0, w_3 \rangle\} & B_2 = \{\langle w_0, w_3 \rangle, \langle w_0, w_4 \rangle\} \\ A_3 = A_1 \cup A_2 & B_3 = B_1 \cup B_2 \\ A_4 = \emptyset & B_4 = \emptyset \end{array}$$

Let $\mathcal{I} = \{1, 2, 3, 4\}$. With the following matrices, where $entry_{jk} = I(f)(C_j, C_k)$, we can verify that $\forall i, j \in \mathcal{I} : \exists k \in \mathcal{I} : \Big(I_1(f)(A_i, A_j) = A_k \ and \ I_2(f)(B_i, B_j) = B_k\Big); f \in \{+, \cdot\}$.

$I_1(+)$	A_1	A_2	A_3	A_4
A_1	A_1	A_3	A_3	A_1
A_2	A_3	A_2	A_3	A_2
A_3	A_3	A_3	A_3	A_3
A_4	A_1	A_2	A_3	A_4

$I_2(+)$	B_1	B_2	B_3	B_4
B_1	B_1	B_3	B_3	B_1
B_2	B_3	B_2	B_3	B_2
B_3	B_3	B_3	B_3	B_3
B_4	B_1	B_2	B_3	B_4

$I_1(\cdot)$	A_1	A_2	A_3	A_4
A_1	A_1	A_4	A_1	A_4
A_2	A_4	A_2	A_2	A_4
A_3	A_1	A_2	A_3	A_4
A_4	A_4	A_4	A_4	A_4

$I_2(\cdot)$	B_1	B_2	B_3	B_4
B_1	B_1	B_4	B_1	B_4
B_2	B_4	B_2	B_2	B_4
B_3	B_1	B_2	B_3	B_4
B_4	B_4	B_4	B_4	B_4

Next, we show that $\forall \alpha \in \Lambda : \exists i \in \mathcal{I} : R_1(\alpha) = A_i \textit{ and } R_2(\alpha) = B_i$. Call this claim (*).

Let $\alpha = a_i$ such that $a_i \in \mathcal{A}$. If i is odd, by the definitions of $\tilde{R}_1$ and $\tilde{R}_2$ then $R_1(a_i) = A_1$ and $R_2(a_i) = B_1$. If on the other hand i is even, on the same grounds $R_1(a_i) = A_2$ and $R_1(a_i) = B_2$. Thus, the claim holds for atomic processes.

Assume then that the claim holds for processes β and γ; that is,

$$R_1(\beta) = A_i \textit{ and } R_2(\beta) = B_i, \text{ and } R_1(\gamma) = A_j \textit{ and } R_2(\gamma) = B_j; \text{ for some } i, j \in \mathcal{J},$$

and let $\alpha = \beta + \gamma$. Now, by the induction hypothesis, $R_1(\beta + \gamma) = I(+)(A_j, A_k)$ and $R_2(\beta + \gamma) = I(+)(B_j, B_k)$, thus, as stated above, we can compute with the matrices that $\exists i \in \mathcal{I} : (I(+)(A_j, A_k) = A_i \textit{ and } I(+)(B_j, B_k) = B_i)$. Therefore, $\exists i \in \mathcal{I} : R_1(\alpha) = A_i \textit{ and } R_2(\alpha) = A_i$. In case $\alpha = \beta \cdot \gamma$ similar argument clearly proves the induction claim. Thus, by induction, claim (*) holds.

Next, let $g : W_2 \to W_1$ be the following mapping:

$$g : \begin{cases} w_0 \mapsto w_0 \\ w_1 \mapsto w_1 \\ w_2 \mapsto w_2 \\ w_3 \mapsto w_2 \\ w_4 \mapsto w_3 \end{cases}$$

Recall the conditions of Λ-bounded morphisms from W_2 to W_1 (cf. definition 5.1):

(i) $\forall \alpha \in \Lambda : wR_2(\alpha)v \Rightarrow g(w)R_1(\alpha)g(v)$,
(ii) $\forall \alpha \in \Lambda : g(w)R_1(\alpha)v' \Rightarrow \exists v \in W : wR_2(\alpha)v \textit{ and } g(v) = v'$.

Since, by claim (*), $\forall \alpha \in \Lambda : \exists i \in \mathcal{I} : R_1(\alpha) = A_i \Leftrightarrow R_2(\alpha) = B_i$, it is rather straightforward to verify that g is a Λ-bounded morphism. Since mapping g is also surjective, F_1 is a Λ-p-morphic image of F_2.

Now that we have in place the Λ-bounded morphism from W_2 onto W_1, we introduce counter-assumption to the theorem; that is, assume there is a set of formulae $\Gamma \subseteq L(\Pi, \mathcal{A}, \{+, \cdot\})$ such that $F \models \Gamma \Leftrightarrow I(\cdot) \simeq \cap$, provided $F \in \mathbf{C}_{(+,\cup)}$. Now, by the definition of $I_1(\cdot)$, we have that $R_1(a_1 \cdot a_2) = \emptyset \neq R_1(a_1) \cap R_1(a_2) = \{w_0, w_2\}$. Therefore, by the counter-assumption, $F_1 \not\models \Gamma$. Also we have $F_2 \models \Gamma$, since $I_2(\cdot) \simeq \cap$. But then, since $F_2 \models \Gamma$ and F_1 is Λ-p-morphic image of F_2, we have that $F_1 \models \Gamma$ by Theorem 5.3. Therefore, the counter-assumption cannot hold. □

Corollary 6.12. *The operator returning the intersection of relations $R(\alpha), R(\beta); \alpha, \beta \in \Lambda$ is not frame definable in any $PSTL$-language that contains only one syntactic operator.*

Proof. Clear by Theorem 6.11. □

The relative complementation is the only operation covered in this paper that does not appear in Tarski's list of basic operators in (Tarski, 1941). We consider it here since the relative complementation turns out to be rather useful foundation definable operation. (Cf. Theorem 7.3 and its corollaries in section 7.)

Theorem 6.13. *Let $L(\Pi, \mathcal{A}, \{/\})$ be a PSTL-language wherein $|\mathcal{A}| \geq 2$ and $/$ is a binary syntactic operator. The operator that returns the relative complementation of relations $R(\alpha), R(\beta); \alpha, \beta \in \Lambda$, i.e. the set-theoretical difference $R(\alpha) \setminus R(\beta) = \{\langle w, w'\rangle | \langle w, w'\rangle \in R(\alpha)$ and $\langle w, w'\rangle \notin R(\beta)\}$, cannot be defined in $L(\Pi, \mathcal{A}, \{/\})$.*

Proof. Assume to the contrary, i.e. $\exists \Gamma \subseteq L(\Pi, \mathcal{A}, \{/\})$ such that for every $L(\Pi, \mathcal{A}, \{/\})$-frame $\langle W, I, \tilde{R}\rangle$ it is the case that $\langle W, I, \tilde{R}\rangle \vDash \Gamma \Leftrightarrow \forall \alpha, \beta \in \Lambda : R(\alpha/\beta) = R(\alpha) \setminus R(\beta)$.

We demonstrate that the counter-assumption implies contradiction by applying the same method and the same kind of frames as in the proof of Theorem 6.11. Thus, let $F_1 = \langle W_1, I_1, \tilde{R}_1\rangle$ and $F_2 = \langle W_2, I_2, \tilde{R}_2\rangle$ be $L(\Pi, \mathcal{A}, \{/\})$-frames such that

$$\begin{cases} W_1 = \{w_0, w_1, w_2, w_3\} \\ I_1(/)(S,T) = \begin{cases} S; & S \neq T \\ \emptyset; & S = T \end{cases} \\ \tilde{R}_1(a_i) = \begin{cases} \{\langle w_0, w_1\rangle, \langle w_0, w_2\rangle\}; & i \text{ is even} \\ \{\langle w_0, w_2\rangle, \langle w_0, w_3\rangle\}; & i \text{ is odd} \end{cases} \end{cases}$$

$$\begin{cases} W_1 = \{w_0, w_1, w_2, w_3, w_4\} \\ I_1(/)(S,T) = S \setminus T; \forall S, T \subseteq W_2 \times W_2 \\ \tilde{R}_2(a_i) = \begin{cases} \{\langle w_0, w_1\rangle, \langle w_0, w_2\rangle\}; & i \text{ is even} \\ \{\langle w_0, w_3\rangle, \langle w_0, w_4\rangle\}; & i \text{ is odd} \end{cases} \end{cases}$$

Let $A_1, A_2, A_3 \subseteq W_1 \times W_1$ and $B_1, B_2, B_3 \subseteq W_2 \times W_2$ be the following relations:

$$\begin{array}{ll} A_1 = \{\langle w_0, w_1\rangle, \langle w_0, w_2\rangle\} & B_1 = \{\langle w_0, w_1\rangle, \langle w_0, w_2\rangle\} \\ A_2 = \{\langle w_0, w_2\rangle, \langle w_0, w_3\rangle\} & B_2 = \{\langle w_0, w_3\rangle, \langle w_0, w_4\rangle\} \\ A_3 = \emptyset & B_3 = \emptyset \end{array}$$

and $\mathcal{I} = \{1, 2, 3\}$ the corresponding index-set. Then, $\forall \alpha \in \Lambda : \exists i \in \mathcal{I} : (R_1(\alpha) = A_i$ *and* $R_2(\alpha) = B_i)$. We demonstrate this by an induction over the structure of the process $\alpha \in \Lambda$.

Let $\alpha = a_i \in \mathcal{A}$. Then the claim clearly holds, since if i is even, $R_1(\alpha) = A_1$ *and* $R_2(\alpha) = B_1$ and similarly if i is odd, $R_1(\alpha) = A_2$ *and* $R_2(\alpha) = B_2$. Then assume that the claim holds for the processes $\beta, \gamma \in \Lambda$.

Consider the following matrices with $entry_{jk} = I(/)(C_j, C_k)$:

$I_1(/)$	A_1	A_2	A_3
A_1	A_3	A_1	A_1
A_2	A_2	A_3	A_2
A_3	A_3	A_3	A_3

$I_2(/)$	B_1	B_2	B_3
B_1	B_3	B_1	B_1
B_2	B_2	B_3	B_2
B_3	B_3	B_3	B_3

Since, by the induction hypothesis, $\exists i \in \mathcal{I} : R_1(\beta) = A_i$ *and* $R_2(\beta) = B_i$ and $\exists j \in \mathcal{I} : R_1(\gamma) = A_j$ *and* $R_2(\gamma) = B_j$, the matrices clearly demonstrate that process β/γ is mapped under I_1 and I_2 such that $\exists i \in \mathcal{I} : R_1(\beta/\gamma) = A_i$ *and* $R_2(\beta/\gamma) = B_i$. Therefore the claim holds for $\beta/\gamma \in \Lambda$, and by induction we then infer that the claim holds for every $\alpha \in \Lambda$.

Next, we set function $g : W_2 \to W_1$ such that

$$g : \begin{cases} w_0 \mapsto w_0 \\ w_1 \mapsto w_1 \\ w_2 \mapsto w_2 \\ w_3 \mapsto w_2 \\ w_4 \mapsto w_3 \end{cases}$$

By resorting to the previous induction-claim, we need to check only relations $A_i, B_i; i \in \mathcal{I}$ to confirm that g is surjective Λ-bounded morphism, and clearly this is the case. Thus, F_1 is Λ-p-morphic image of F_2. Since $I_1(/)(R(a_1), R(a_2)) = A_1 \neq \{\langle w_0, w_1\rangle\} = R_1(a_1) \setminus R_1(a_2)$, by the counter-assumption we infer $F_1 \nvDash \Gamma$. But then, by the very same assumption, $F_2 \vDash \Gamma$ since $I_2(/) \simeq \setminus$. Whence, by Theorem 5.3. $F_1 \vDash \Gamma$ since F_1 is a Λ-p-morphic image of F_2. Therefore, the counter-assumption is necessarily false. □

6.4 Closures

Although the closure operations are unary, we shall discuss common closures in a separate section for their special status. We shall prove that operators that return the transitive and the reflexive closure of a given relation $R(\alpha), \alpha \in \Lambda$, are frame definable respectively. This implies frame definability of the reflexive transitive closure within a language with two unary syntactic operators. Since the reflexive transitive closure corresponds to the interpretation of *Kleene-closure* within, for example, the contexts of PDL and regular languages, frame definability of reflexive transitive closure with single unary syntactic operator is covered.

Definition 6.14. Herein dealt closures of relations are defined as follows: Let R be a relation over domain W, then[17]

$$\begin{array}{rcll} r(R) & = & R \cup R^{[0]} & \text{(reflexive closure)} \\ s(R) & = & R \cup R^{-1} & \text{(symmetric closure)} \\ t(R) & = & \bigcup_{n=1}^{\infty} R^{[n]} & \text{(transitive closure} \\ tr(R) & = & \bigcup_{n=0}^{\infty} R^{[n]} & \text{(reflexive transitive closure)} \end{array}$$

Theorem 6.15. *The operator that returns the reflexive closure of $R(\alpha), \forall\alpha \in \Lambda$ is frame definable in language $L(\Pi, \mathcal{A}, \{r\})$ in which r is a unary syntactic operator, since*

$$\langle W, I, \tilde{R}\rangle \vDash [\alpha^r]\varphi \leftrightarrow [\alpha]\varphi \wedge \varphi \text{ if and only if } \forall\alpha \in \Lambda : R(\alpha^r) = r(R(\alpha)).$$

[17]We denote the n-fold composition of relation R, i.e. $R_1 \circ R_2 \circ \ldots \circ R_n$, as $R^{[n]}$. Whence, recursive definition of $R^{[n]}$ is that $R^{[0]} = \{\langle w, w\rangle | w \in W\}$ and $R^{[n+1]} = R^{[n]} \circ R$.

Proof. Let $M = \langle W, I, \tilde{R}, P\rangle$ be a model of $L(\Pi, \mathcal{A}, \{r\})$-frame $\langle W, I, \tilde{R}\rangle$. Assume $\forall \alpha \in \Lambda : R(\alpha^r) = r(R(\alpha))$ and consider an arbitrary state $w \in W$. It is trivial to conclude that $M, w \models [\alpha^r]\varphi \leftrightarrow [\alpha]\varphi \wedge \varphi$, since

$$\begin{aligned} M, w \models [\alpha^r]\varphi \quad &\Leftrightarrow \quad \forall w' \in W : wR(\alpha^r)w' \Rightarrow M, w' \models \varphi \\ (\text{Since } R(\alpha^r) = r(R(\alpha))) \quad &\Leftrightarrow \quad \forall w'' \in W : wR(\alpha)w'' \Rightarrow M, w'' \models \varphi \textit{ and } M, w \models \varphi \\ &\Leftrightarrow \quad M, w \models [\alpha]\varphi \textit{ and } M, w \models \varphi \\ &\Leftrightarrow \quad M, w \models [\alpha]\varphi \wedge \varphi. \end{aligned}$$

So, if $\forall \alpha \in \Lambda : R(\alpha^r) = r(R(\alpha))$, then $\langle W, I, \tilde{R}\rangle \models [\alpha^r]\varphi \leftrightarrow [\alpha]\varphi \wedge \varphi$.

Then consider $L(\Pi, \mathcal{A}, \mathcal{F})$-frame $\langle W, I, \tilde{R}\rangle$ that satisfies $R(\alpha^r) \neq r(R(\alpha))$ for some $\alpha \in \Lambda$, i.e. either $R(\alpha^r) \nsubseteq r(R(\alpha))$ or $r(R(\alpha)) \nsubseteq R(\alpha^r)$.

Assume first that $R(\alpha^r) \nsubseteq r(R(\alpha))$. Then, $\exists w, w' \in W : \langle w, w'\rangle \in R(\alpha^r)$ *and* $\langle w, w'\rangle \notin r(R(\alpha))$. Choose a valuation P such that $P(p) = W \setminus \{w'\}$, for some $p \in \Pi$. Now the resulting model $M = \langle W, I, \tilde{R}, P\rangle$ satisfies $M, w \not\models [\alpha^r]p$. That said, $\langle w, w'\rangle \notin R(\alpha)$, since $\langle w, w'\rangle \notin r(R(\alpha)) = R(\alpha) \cup \{\langle v, v\rangle | v \in W\}$, thus, by the choice of valuation $P(p)$, $M, w \models [\alpha]p$. Also $w \neq w'$ since $r(R(\alpha))$ is reflexive and $\langle w, w'\rangle \notin r(R(\alpha))$, whence $M, w \models p$. Therefore, $M, w \models [\alpha]p \wedge p$, furthermore $M, w \not\models [\alpha^r]p \leftrightarrow [\alpha]p \wedge p$, whence $\langle W, I, \tilde{R}\rangle \not\models [\alpha^r]\varphi \leftrightarrow [\alpha]\varphi \wedge \varphi$.

Assume next $r(R(\alpha)) \nsubseteq R(\alpha^r)$. Then, $\exists w, w' \in W : \langle w, w'\rangle \in r(R(\alpha))$ *and* $\langle w, w'\rangle \notin R(\alpha^r)$. Consider then a model $M = \langle W, I, \tilde{R}, P\rangle$ where $P(p) = W \setminus \{w'\}$, for some $p \in \Pi$. Now $M, w' \models [\alpha^r]p$. Since $r(R(\alpha)) = R(\alpha) \cup \{\langle v, v\rangle | v \in W\}$, either $\langle w, w'\rangle \in R(\alpha)$ or $w = w'$. Thus, either we have $\langle w, w'\rangle \in R(\alpha)$ which implies $M, w \not\models [\alpha]p$ with $P(p)$, or $\langle w, w'\rangle \notin R(\alpha)$, whence $w = w'$ and by the choice of $P(p)$ then $M, w \not\models p$. In either case, $M, w \not\models [\alpha]p \wedge p$. Therefore, $M, w \not\models [\alpha^r]p \leftrightarrow [\alpha]p \wedge p$; that is, $\langle W, I, \tilde{R}\rangle \not\models [\alpha^r]\varphi \leftrightarrow [\alpha]\varphi \wedge \varphi$.

Therefore, if $\langle W, I, \tilde{R}\rangle \models [\alpha^r]\varphi \leftrightarrow [\alpha]\varphi \wedge \varphi$, then $\forall \alpha \in \Lambda : R(\alpha^r) = r(R(\alpha))$ □

Theorem 6.16. *Let $L(\Pi, \mathcal{A}, \{s\})$ be a PSTL-language such that s is a unary operator and $|\Pi| \geq 2$. Then*

$$\langle W, I, \tilde{R}\rangle \models \Sigma \textit{ if and only if } \forall \alpha \in \Lambda : R(\alpha^s) = s(R(\alpha)),$$

wherein $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ such that

$$\begin{aligned} \sigma_1 &= [\alpha^s]\varphi \rightarrow [\alpha]\varphi \\ \sigma_2 &= \varphi \rightarrow [\alpha]\langle\alpha^s\rangle\varphi \\ \sigma_3 &= \langle\alpha^s\rangle([\alpha]\varphi \wedge \psi) \rightarrow \varphi \vee \langle\alpha\rangle\psi \end{aligned}$$

Proof. Let $\langle W, I, \tilde{R}\rangle$ be such $L(\Pi, \mathcal{A}, \{s\})$-frame that $\forall \alpha \in \Lambda : R(\alpha^s) = s(R(\alpha))$ and choose a model $M = \langle W, I, \tilde{R}, P\rangle$ with an arbitrary valuation P. Pick any $w \in W$ and assume $M, w \models [\alpha^s]\varphi$. Now, $\forall w' \in W : wR(\alpha^s)w' \Rightarrow M, w' \models \varphi$. Since $R(\alpha^s) = s(R(\alpha)) = R(\alpha) \cup R(\alpha)^{-1}$, clearly also $\forall w'' \in W : wR(\alpha)w'' \Rightarrow M, w'' \models \varphi$, i.e. $M, w \models [\alpha]\varphi$. Whence, $M, w \models [\alpha^s]\varphi \rightarrow [\alpha]\varphi$.

Assume then that $M, w \models \varphi$. Now, if $wR(\alpha)w'$, then $w'R(\alpha)^{-1}w$. Since $R(\alpha^s) = R(\alpha) \cup R(\alpha)^{-1}$, clearly then $w'R(\alpha^s)w$ also. Hence, $M, w' \models \langle\alpha^s\rangle\varphi$ since $M, w \models \varphi$. As usual, this holds for every state $w' \in W$ whenever $wR(\alpha)w'$, thus we have $M, w \models [\alpha]\langle\alpha^s\rangle\varphi$. Hence, $M, w \models \varphi \to [\alpha]\langle\alpha^s\rangle\varphi$.

Finally, assume $M, w \models \langle\alpha^s\rangle([\alpha]\varphi \wedge \psi)$. Then, $\exists w' \in W : wR(\alpha^s)w'$ *and* $M, w' \models [\alpha]\varphi \wedge \psi$, i.e. $M, w' \models [\alpha]\varphi$ and $M, w' \models \psi$. Since $wR(\alpha^s)w'$, by the assumption $R(\alpha^s) = R(\alpha) \cup R(\alpha)^{-1}$, either $wR(\alpha)w'$ or $wR(\alpha)^{-1}w'$. If $wR(\alpha)^{-1}w'$, then $w'R(\alpha)w$ and furthermore $M, w \models \varphi$, since $M, w' \models [\alpha]\varphi$. If on the other hand $wR(\alpha)w'$, then evidently $M, w \models \langle\alpha\rangle\psi$. In either case $M, w \models \varphi \vee \langle\alpha\rangle\psi$, hence $M, w \models \langle\alpha^s\rangle([\alpha]\varphi \wedge \varphi) \to \varphi \vee \langle\alpha\rangle\psi$.

Therefore, we conclude that if $R(\alpha^s) = s(R(\alpha))$, then $\langle W, I, \tilde{R}\rangle \models \Sigma$.

Next, consider $L(\Pi, \mathcal{A}, \{s\})$-frame $\langle W, I, \tilde{R}\rangle$ wherein $\exists\alpha \in \Lambda$ such that either $R(\alpha^s) \nsubseteq s(R(\alpha))$ or $s(R(\alpha)) \nsubseteq R(\alpha^s)$.

If $\exists\alpha \in \Lambda : R(\alpha^s) \nsubseteq s(R(\alpha))$, then there are states $w, w' \in W$ such that $\langle w, w'\rangle \in R(\alpha^s)$, $\langle w, w'\rangle \notin R(\alpha)$ and $\langle w, w'\rangle \notin R(\alpha)^{-1}$. Consider then model $M = \langle W, I, \tilde{R}, P\rangle$ such that $P(p) = \{w'' | \langle w', w''\rangle \in R(\alpha)\}$ and $P(q) = \{w'\}$ for some $p, q \in \Pi : p \neq q$. Then, $M, w' \models [\alpha]p$ and $M, w' \models q$, and consequently $M, w \models \langle\alpha^s\rangle([\alpha]p \wedge q)$. Since $\langle w, w'\rangle \notin R(\alpha)^{-1}$, we have that $\langle w', w\rangle \notin R(\alpha)$. Thus, $w \notin P(p)$, i.e. $M, w \not\models p$. Since also $\langle w, w'\rangle \notin R(\alpha)$ and $P(q) = \{w'\}$, we have $M, w \not\models \langle\alpha\rangle q$, hence $M, w \not\models p \vee \langle\alpha\rangle q$. Therefore, $M, w \not\models \langle\alpha^s\rangle([\alpha]p \wedge q) \to p \vee \langle\alpha\rangle q$.

Consider then the other possibility: $\exists\alpha \in \Lambda : s(R(\alpha)) \nsubseteq R(\alpha^s)$. Then, $\exists w, w' \in W : \langle w, w'\rangle \notin R(\alpha^s)$, and $\langle w, w'\rangle \in R(\alpha)$ *or* $\langle w, w'\rangle \in R(\alpha)^{-1}$.

If $\langle w, w'\rangle \in R(\alpha)$, a model $M = \langle W, I, \tilde{R}, P\rangle$, with $P(p) = W \setminus \{w'\}$, clearly satisfies $M, w \models [\alpha^s]p$ and $M, w \not\models [\alpha]p$, whence $M, w \not\models [\alpha^s]p \to [\alpha]p$.

Assume next $\langle w, w'\rangle \in R(\alpha)^{-1}$ and consider a model $M = \langle W, I, \tilde{R}, P\rangle$ with $P(p) = \{w'\}$, i.e. $M, w' \models p$. Now $M, w \not\models \langle\alpha^s\rangle p$, since $\langle w, w'\rangle \notin R(\alpha^s)$, and $\langle w', w\rangle \in R(\alpha)$, since $\langle w, w'\rangle \in R(\alpha)^{-1}$. Therefore, $M, w' \not\models [\alpha]\langle\alpha^s\rangle p$, whence $M, w' \not\models p \to [\alpha]\langle\alpha^s\rangle p$.

Thus, in any case, if $\langle W, I, \tilde{R}\rangle \models \Sigma$, then $R(\alpha^s) = s(R(\alpha))$. □

Currently, it is unknown (at least to the author, that is) whether a language $L(\{p\}, \mathcal{A}, \{s\})$, i.e. a $PSTL$-language with a single unary operator and $|\Pi| = 1$, is expressive enough to define the operator that picks out the symmetric closure on the level of frames. That said, the operator in question is definable within a language that contains a single propositional symbol and one unary and one binary operator. Next we show that (1°) reflexive closure is syntactically frame definable in $L(\Pi, \mathcal{A}, \{\mathrm{D}, +\})$-frame class

$$\{F \mid F \models [\mathrm{D}]\varphi \leftrightarrow \varphi \textit{ and } F \models [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi\}.$$

and similarly, (2°) symmetric closure is syntactically frame definable in $L(\Pi, \mathcal{A}, \{\breve{\cdot}, +\})$-frame class

$$\{F \mid F \models \varphi \to [\alpha]\langle\breve{\alpha}\rangle\varphi \wedge [\breve{\alpha}]\langle\alpha\rangle\varphi \textit{ and } F \models [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi\}.$$

The claim (1°) is due to the fact that if $F = \langle W, I, \tilde{R}\rangle$ is a $L(\Pi, \mathcal{A}, \{\mathrm{D}, +\})$-frame such that $F \vDash [\mathrm{D}]\alpha \leftrightarrow \alpha$ and $F \vDash [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$, by Theorems 6.2 and 6.8 $R(\mathrm{D}) = \{\langle w, w\rangle | w \in W\}$ and $\forall \alpha, \beta \in \Lambda : R(\alpha + \beta) = R(\alpha) \cup R(\beta)$. Therefore,

$$\begin{array}{rcl} R(\alpha + \mathrm{D}) & = & R(\alpha) \cup R(\mathrm{D}) \\ & = & R(\alpha) \cup \{\langle w, w\rangle | w \in W\} \\ & = & r(R(\alpha)). \end{array}$$

Correspondingly, if the frame $F = \langle W, I, \tilde{R}\rangle$ satisfies $F \vDash \varphi \rightarrow [\alpha]\langle\breve{\alpha}\rangle\varphi \wedge [\breve{\alpha}]\langle\alpha\rangle\varphi$ and $F \vDash [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$, then, by Theorems 6.2 and 6.8, $\forall \alpha \in \Lambda : R(\breve{\alpha}) = R(\alpha)^{-1}$ and $\forall \alpha, \beta \in \Lambda : R(\alpha + \beta) = R(\alpha) \cup R(\beta)$. Thus, the claim (2°) holds, since

$$\begin{array}{rcl} R(\alpha + \breve{\alpha}) & = & R(\alpha) \cup R(\breve{\alpha}) \\ & = & R(\alpha) \cup R(\alpha)^{-1} \\ & = & s(R(\alpha)). \end{array}$$

Note that the latter case holds even if $|\Pi| = 1$, since one propositional symbol is sufficient to define the operator that returns the inverse relation and the operator that returns the union of $R(\alpha), R(\beta); \alpha, \beta \in \Lambda$, although in this case $|\mathcal{F}| \geq 2$.

Theorem 6.17. *Consider PSTL-language $L(\Pi, \mathcal{A}, \{t\})$ where t is unary operator. Then,*

$$\langle W, I, \tilde{R}\rangle \vDash \Theta \textit{ if and only if } \forall \alpha \in \Lambda : R(\alpha^t) = t(R(\alpha)),$$

wherein $\Theta = \{\theta_1, \theta_2\}$ such that

$$\begin{array}{l} \theta_1 = [\alpha^t]\varphi \rightarrow [\alpha]\varphi \wedge [\alpha^t][\alpha]\varphi \\ \theta_2 = [\alpha^t](\varphi \rightarrow [\alpha]\varphi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha^t]\varphi) \end{array}$$

Proof. Consider any $L(\Pi, \mathcal{A}, \{t\})$-frame $\langle W, I, \tilde{R}\rangle$ that satisfies $\forall \alpha \in \Lambda : R(\alpha^t) = t(R(\alpha))$ and let $M = \langle W, I, \tilde{R}, P\rangle$ be a model with an arbitrary valuation mapping P. Pick $w \in W$ and assume that $M, w \vDash [\alpha^t]\varphi$. Then, contrary to the claim of the theorem, assume that $M, w \nvDash [\alpha]\varphi \wedge [\alpha^t][\alpha]\varphi$. Thus, (1°) $M, w \nvDash [\alpha]\varphi$ or (2°) $M, w \nvDash [\alpha^t][\alpha]\varphi$.

(1°): If $M, w \nvDash [\alpha]\varphi$, then $\exists w' \in W : wR(\alpha)w'$ *and* $M, w' \nvDash \varphi$. Since $R(\alpha) \subseteq t(R(\alpha)) = R(\alpha^t)$, now $wR(\alpha^t)w'$ which implies $M, w \nvDash [\alpha^t]\varphi$.

(2°): If $M, w \nvDash [\alpha^t][\alpha]\varphi$, then $\exists w' \in W : wR(\alpha^t)w'$ *and* $M, w' \nvDash [\alpha]\varphi$, i.e. $\exists w'' \in W : w'R(\alpha)w''$ *and* $M, w'' \nvDash \varphi$. Therefore, $\exists w'' \in W : w(R(\alpha^t) \circ R(\alpha))w''$ *and* $M, w'' \nvDash \varphi$. Recall that by Definition 6.14,

$$t(R(\alpha)) = \bigcup_{n=1}^{\infty} R(\alpha)^{[n]}.$$

Thus, since $R(\alpha^t) = t(R(\alpha))$, we now have that $w(R(\alpha)^{[z]} \circ R(\alpha))w''$, for some $z \in \mathbb{Z}_+$. Furthermore, then $wR(\alpha)^{[z+1]}w'$, since $R(\alpha)^{[z]} \circ R(\alpha) = R(\alpha)^{[z+1]}$. Again, directly by

Definition 6.14, $\forall z \in \mathbb{Z}_+ : R(\alpha)^{[z+1]} \subseteq t(R(\alpha))$, therefore $w(t(R(\alpha)))w''$. Clearly, then, $M, w \nvDash [\alpha^t]\varphi$, since $M, w'' \nvDash \varphi$.

Thus, in any case, if $M, w \vDash [\alpha^t]$, then $M, w \vDash [\alpha]\varphi \wedge [\alpha^t][\alpha]\varphi$; that is, $M, w \vDash [\alpha^t]\varphi \rightarrow [\alpha]\varphi \wedge [\alpha^t][\alpha]\varphi$.

Assume next $M, w \nvDash [\alpha]\varphi \rightarrow [\alpha^t]\varphi$; that is $M, w \vDash [\alpha]\varphi$ and $M, w \nvDash [\alpha^t]\varphi$. As in the preceding argument, since $R(\alpha^t) = t(R(\alpha))$, $M, w \nvDash [\alpha^t]\varphi$ implies (i) $\exists z' \in \mathbb{Z}_+ : \exists w' \in W : wR(\alpha)^{[z']}w'$ *and* $M, w' \nvDash \varphi$. Since $M, w \vDash [\alpha]\varphi$, $\forall v \in W : wR(\alpha)v \Rightarrow M, v \vDash \varphi$, and furthermore $R(\alpha) = R(\alpha)^{[1]}$, hence necessarily $z' > 1$. Thus, we write $z' = z + 1$ and reformulate the corollary (i) as:

$$\text{(ii) } \exists z \in \mathbb{Z}_+ : \exists w' \in W : wR(\alpha)^{[z+1]}w' \textit{ and } M, w' \nvDash \varphi.$$

In addition, we are permitted to assume that z in question is the smallest positive integer that satisfies the claim (ii); that is, by the minimality of z, $\forall y \in \mathbb{Z}_+ : y \leq z \Rightarrow \forall v \in W : wR(\alpha)^{[y]}v \Rightarrow M, v \vDash \varphi$. Again, since $R(\alpha)^{[z+1]} = R(\alpha)^{[z]} \circ R(\alpha)$, from (ii) we infer that $w(R(\alpha)^{[z]} \circ R(\alpha))w'$, whence $\exists w'' \in W : wR(\alpha)^{[z]}w''$ *and* $w''R(\alpha)w'$. By the minimality of z, now $M, w'' \vDash \varphi$. Furthermore $M, w'' \nvDash [\alpha]\varphi$, since $w''R(\alpha)w'$ and $M, w' \nvDash \varphi$, whence $M, w'' \nvDash \varphi \rightarrow [\alpha]\varphi$. Thus, $M, w \nvDash [\alpha^t](\varphi \rightarrow [\alpha]\varphi)$, since we have $wR(\alpha)^{[z]}w''$, $M, w'' \nvDash \varphi \rightarrow [\alpha]\varphi$ and $R(\alpha)^{[z]} \subseteq R(\alpha^t)$. In conclusion, if $M, w \nvDash [\alpha]\varphi \rightarrow [\alpha^t]\varphi$, then $M, w \nvDash [\alpha^t](\varphi \rightarrow [\alpha]\varphi)$, therefore $M, w \vDash [\alpha^t](\varphi \rightarrow [\alpha]\varphi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha^t]\varphi)$.

On the grounds of the above results, if $\forall \alpha \in \Lambda : R(\alpha^t) = t(R(\alpha))$, then $\langle W, I, \tilde{R} \rangle \vDash \Theta$.

Next, let $\langle W, I, \tilde{R} \rangle$ be a $L(\Pi, \mathcal{A}, \{t\})$-frame such that $\exists \alpha \in \Lambda : t(R(\alpha)) \nsubseteq R(\alpha^t)$ or $\exists \alpha \in \Lambda : R(\alpha^t) \nsubseteq t(R(\alpha))$.

To begin with, assume $\exists \alpha \in \Lambda : t(R(\alpha)) \nsubseteq R(\alpha^t)$, i.e. $(R(\alpha)^{[1]} \cup R(\alpha)^{[2]} \cup R(\alpha)^{[3]} \cup \ldots) \nsubseteq R(\alpha^t)$. Then $\exists z \in \mathbb{Z}_+ : R(\alpha)^{[z]} \nsubseteq R(\alpha^t)$ *and* $\forall y \in \mathbb{Z}_+ : y < z \Rightarrow R(\alpha)^{[y]} \subseteq R(\alpha^t)$; that is, z is the smallest positive integer such that there is a pair $\langle w, w' \rangle \in W \times W$ such that $\langle w, w' \rangle \in R(\alpha)^{[z]}$ *and* $\langle w, w' \rangle \notin R(\alpha^t)$. Then, let $M = \langle W, I, \tilde{R}, P \rangle$ be a model with $P(p) = W \setminus \{w'\}$. If $z = 1$, then $\langle w, w' \rangle \in R(\alpha)$, thus $M, w \nvDash [\alpha]p$. If $z > 1$, we write $z' + 1 = z$; that is, now $\langle w, w' \rangle \in R(\alpha)^{[z'+1]}$. It should be clear, on the basis of the considerations carried out previously, that then $\exists w'' \in W : \langle w, w'' \rangle \in R(\alpha)^{[z']}$ *and* $\langle w'', w' \rangle \in R(\alpha)$. Hence, $M, w'' \nvDash [\alpha]p$, since $M, w' \nvDash p$. By the minimality of z, $R(\alpha)^{[z']} \subseteq R(\alpha^t)$, thus $\langle w, w'' \rangle \in R(\alpha^t)$. Therefore, $M, w \nvDash [\alpha^t][\alpha]p$. Consequently, in any case $M, w \nvDash [\alpha]p$ or $M, w \nvDash [\alpha^t][\alpha]p$, thus $M, w \nvDash [\alpha]p \wedge [\alpha^t][\alpha]p$. Nevertheless, $M, w \vDash [\alpha^t]p$, since $P(p) = W \setminus \{w'\}$ and $\langle w, w' \rangle \notin R(\alpha^t)$, thus $M, w \nvDash [\alpha^t]p \rightarrow [\alpha]p \wedge [\alpha^t][\alpha]p$. Whence, $\langle W, I, \tilde{R} \rangle \nvDash [\alpha^t]\varphi \rightarrow [\alpha]\varphi \wedge [\alpha^t][\alpha]\varphi$. Accordingly, if $\langle W, I, \tilde{R} \rangle \vDash \theta_1$, then $\forall \alpha \in \Lambda : t(R(\alpha)) \subseteq R(\alpha^t)$.

Consider then $L(\Pi, \mathcal{A}, \{t\})$-frame $\langle W, I, \tilde{R} \rangle$ such that $\exists \alpha \in \Lambda : R(\alpha^t) \nsubseteq t(R(\alpha))$, i.e. $\exists w, w' \in W : \langle w, w' \rangle \in R(\alpha^t)$ *and* $\langle w, w' \rangle \notin t(R(\alpha))$. Set a valuation $P(p) = \{ v \mid \langle w, v \rangle \in t(R(\alpha))\}$ for some $p \in \Pi$. Consider then the resulting model $M = \langle W, I, \tilde{R}, P \rangle$ and pick an arbitrary state $w'' \in W : \langle w, w'' \rangle \in R(\alpha^t)$. If $\langle w, w'' \rangle \notin t(R(\alpha))$, then $M, w'' \nvDash p$ by the definition of $P(p)$, thus $M, w'' \vDash p \rightarrow [\alpha]p$. Assume then that $\langle w, w'' \rangle \in t(R(\alpha))$. Now if $\exists v \in W : \langle w'', v \rangle \in R(\alpha)$, also $\langle w, v \rangle \in t(R(\alpha))$ by the definition 6.14, and furthermore then $M, v \models p$ by the definition of $P(p)$. Thus, $\forall v \in W : \langle w'', v \rangle \in R(\alpha) \Rightarrow M, v \models p$, i.e. $M, w'' \models [\alpha]p$, whence $M, w'' \models p \rightarrow [\alpha]p$. Therefore, if $\langle w, w'' \rangle \in R(\alpha^t)$, in any case $M, w'' \models p \rightarrow [\alpha]p$, thus $M, w \models [\alpha^t](p \rightarrow [\alpha]p)$. Now, clearly $M, w \models [\alpha]p$ since $R(\alpha) \subseteq t(R(\alpha))$ and thus by the definition of $P(p)$, for all $v \in W : \langle w, v \rangle \in R(\alpha) \Rightarrow M, v \models p$. However, $\langle w, w' \rangle \notin t(R(\alpha))$, hence $M, w' \not\models p$. Furthermore then $M, w \not\models [\alpha^t]p$, since $\langle w, w' \rangle \in R(\alpha^t)$. Therefore, $M, w \not\models [\alpha]p \rightarrow [\alpha^t]p$, hence we finally have that $M, w \not\models [\alpha^t](p \rightarrow [\alpha]p) \rightarrow ([\alpha]p \rightarrow [\alpha^t]p)$. Thus, if $\langle W, I, \tilde{R} \rangle \models \theta_2$, then $\forall \alpha \in \Lambda : R(\alpha)^t \subseteq t(R(\alpha))$. □

Now we have covered the model theoretic results that enable us to define syntactically the operator that maps to the reflexive transitive closure of $R(\alpha)$, within particular frame-classes of languages $L(\Pi, \mathcal{A}, \{r, t\})$ and $L(\Pi, \mathcal{A}, \{\mathrm{D}, t, +\})$. Clearly, if we consider the $L(\Pi, \mathcal{A}, \{\mathrm{D}, t, +\})$ frame-class

$$\mathbf{C} = \{\ F \mid F \models [\mathrm{D}]\varphi \leftrightarrow \varphi \ \textit{and}\ F \models [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi \ \textit{and}\ F \models \Theta\},$$

then in every frame $F \in \mathbf{C} : \forall \alpha \in \Lambda : R(\mathrm{D} + \alpha^t) = tr(R(\alpha))$, since

$$\begin{aligned} R(\mathrm{D} + \alpha^t) &= R(\mathrm{D}) \cup R(\alpha^t) \\ &= R(\alpha)^0 \cup \left(\textstyle\bigcup_{n=1}^{\infty} R(\alpha)^{[n]} \right) \\ &= \textstyle\bigcup_{n=0}^{\infty} R(\alpha)^{[n]} \\ &= tr(R(\alpha)), \end{aligned}$$

by Theorems 6.2, 6.8 and 6.17. Similarly, within the $L(\Pi, \mathcal{A}, \{r, t\})$ frame-class

$$\mathbf{C}^* = \{\ F \mid F \models [\alpha^r]\varphi \leftrightarrow [\alpha]\varphi \wedge \varphi \ \textit{and}\ F \models \Theta\},$$

clearly $R((\alpha^r)^t) = t(R(\alpha^r)) = tr(R(\alpha))$, by Theorems 6.15 and 6.17.

Anyhow, reflexive transitive closure can also be defined independently within $PSTL$-language with one unary operator, as we shall next demonstrate. The following theorem is also familiar from the regular PDL (cf. Blackburn et al., 2002, pg.132).

Theorem 6.18. *Let $L(\Pi, \mathcal{A}, \{^*\})$ be a $PSTL$-language with unary operator *. Then,*

$$\langle W, I, \tilde{R} \rangle \models \Upsilon \ \textit{if and only if}\ \forall \alpha \in \Lambda : R(\alpha^*) = tr(R(\alpha)),$$

wherein $\Upsilon = \{\upsilon_1, \upsilon_2\}$ such that

$$\begin{aligned} \upsilon_1 &= \langle \alpha^* \rangle \varphi \leftrightarrow \varphi \vee \langle \alpha \rangle \langle \alpha^* \rangle \varphi \\ \upsilon_2 &= [\alpha^*](\varphi \rightarrow [\alpha]\varphi) \rightarrow (\varphi \rightarrow [\alpha^*]\varphi) \end{aligned}$$

Proof. First recall that

$$\begin{aligned} 1^\circ &: \ tr(R(\alpha)) = \textstyle\bigcup_{n=0}^{\infty} R(\alpha)^{[n]} = R(\alpha)^{[0]} \cup R(\alpha)^{[1]} \cup R(\alpha)^{[2]} \cup \ldots, \\ 2^\circ &: \ \forall n, m \in \mathbb{N} : R(\alpha)^{[n]} \circ R(\alpha)^{[m]} = R(\alpha)^{[n+m]} \ \textit{and} \\ 3^\circ &: \ \forall n \in \mathbb{N} : R(\alpha)^{[n+1]} \subseteq \textstyle\bigcup_{i=1}^{\infty} R(\alpha)^{[i]} \subseteq \textstyle\bigcup_{j=0}^{\infty} R(\alpha)^{[j]}, \end{aligned}$$

$$\begin{array}{llll} \text{hence} \quad w(R(\alpha)\circ tr(R(\alpha)))w' & \Leftrightarrow & \exists n\in\mathbb{N} : w(R(\alpha)\circ R(\alpha)^{[n]})w' & (1^\circ) \\ & \Leftrightarrow & \exists n\in\mathbb{N} : wR(\alpha)^{[1+n]}w' & (2^\circ) \\ & \Rightarrow & w(tr(R(\alpha)))w' & (3^\circ). \end{array}$$

Therefore, 4°: $w(R(\alpha)\circ tr(R(\alpha)))w' \Leftrightarrow \exists n\in\mathbb{N} : wR(\alpha)^{[1+n]}w'$, and
5°: $R(\alpha)\circ tr(R(\alpha)) \subseteq tr(R(\alpha))$.

Let $M = \langle W, I, \tilde{R}, P\rangle$ be $L(\Pi, \mathcal{A}, \{^*\})$-model that satisfies $\forall\alpha\in\Lambda : R(\alpha^*) = tr(R(\alpha))$ and pick an arbitrary state $w\in W$. If $M, w \vDash \langle\alpha^*\rangle\varphi$, then $\exists w'\in W : wR(\alpha^*)w'$ *and* $M, w' \vDash \varphi$. Since $R(\alpha^*) = tr(R(\alpha))$, by ($1^\circ$) now (i) $\exists n\in\mathbb{N} : \exists w'\in W : wR(\alpha)^{[n]}w'$. If the former holds when $n = 0$, then $M, w \vDash \varphi$. If $n > 0$, we set $n = 1 + m$ and reformulate (i) as (ii) $\exists m\in\mathbb{N} : wR(\alpha)^{[1+m]}w'$. Then by ($4^\circ$), $w(R(\alpha)\circ tr(R(\alpha)))w'$, thus $w(R(\alpha)\circ R(\alpha^*))w'$ by the assumption $R(\alpha^*) = tr(R(\alpha))$. Therefore, $M, w \vDash \langle\alpha\rangle\langle\alpha^*\rangle\varphi$, since $w(R(\alpha)\circ R(\alpha^*)w'$ *and* $M, w' \vDash \varphi$ is equivalent to $M, w \vDash \langle\alpha\rangle\langle\alpha^*\rangle\varphi$ (cf. the proof of theorem 6.7). Thus in any case, if $M, w \vDash \langle\alpha^*\rangle\varphi$, then $M, w \vDash \varphi$ *or* $M, w \vDash \langle\alpha\rangle\langle\alpha^*\rangle\varphi$, i.e. $M, w \vDash \langle\alpha^*\rangle\varphi \to \varphi \vee \langle\alpha\rangle\langle\alpha^*\rangle\varphi$.

Assume then that $M, w \nvDash \langle\alpha^*\rangle\varphi$; that is, $\forall w'\in W : wR(\alpha^*)w' \Rightarrow M, w' \nvDash \varphi$. Since $R(\alpha^*) = tr(R(\alpha))$, by the reflexivity of relation $R(\alpha^*)$, we have $M, w \nvDash \varphi$. Furthermore, by (5°), also $\forall w''\in W : w(R(\alpha)\circ R(\alpha^*))w' \Rightarrow M, w' \nvDash \varphi$. By the inference presented in the proof of Theorem 6.7, $\forall w''\in W : w(R(\alpha)\circ R(\alpha^*))w' \Rightarrow M, w' \nvDash \varphi$ is equivalent to $M, w \nvDash \langle\alpha\rangle\langle\alpha^*\rangle\varphi$. Therefore, if $M, w \nvDash \langle\alpha^*\rangle\varphi$, then $M, w \nvDash \varphi$ *and* $N, w \nvDash \langle\alpha\rangle\langle\alpha^*\rangle\varphi$, i.e. $M, w \vDash \varphi \vee \langle\alpha\rangle\langle\alpha^*\rangle\varphi \to \langle\alpha^*\rangle\varphi$.

From the above results we infer that if $R(\alpha^*) = tr(R(\alpha))$, then $\langle W, I, \tilde{R}\rangle \vDash \upsilon_1$.

Assume next $M, w \nvDash \varphi \to [\alpha^*]\varphi$; that is, $M, w \vDash \varphi$ and $M, w \nvDash [\alpha^*]\varphi$, thus $\exists w'\in W : w(tr(R(\alpha)))w'$ *and* $M, w' \nvDash \varphi$. As usual, then $\exists n\in\mathbb{N} : wR(\alpha)^{[n]}w'$ and $\forall m\in\mathbb{N} : m < n \Rightarrow (\forall v\in W : wR(\alpha)^{[m]} \Rightarrow M, v \vDash \varphi)$. Since $M, w \vDash \varphi$, necessarily $n > 0$, i.e. $w \neq w'$. Hence, we write $n = m + 1$. Then, $\exists w''\in W : wR(\alpha)^{[m]}w''$ *and* $w''R(\alpha)w'$, and by the minimality of $n = m + 1$, we have that $M, w'' \vDash \varphi$. (Note that now it is possible that $m = 0$, i.e. $w'' = w$.) Since $w''R(\alpha)w'$ and $M, w' \nvDash \varphi$, we have $M, w'' \nvDash [\alpha]\varphi$. Thus, $M, w'' \nvDash \varphi \to [\alpha]\varphi$. Since $wR(\alpha)^{[m]}w''$ and $R(\alpha)^{[m]} \subseteq tr(R(\alpha)) = R(\alpha^*)$, finally $M, w \nvDash [\alpha^*](\varphi \to [\alpha]\varphi)$. Therefore, if $M, w \nvDash \varphi \to [\alpha^*]\varphi$, then $M, w \nvDash [\alpha^*](\varphi \to [\alpha]\varphi)$, i.e. $M, w \vDash [\alpha^*](\varphi \to [\alpha]\varphi) \to (\varphi \to [\alpha^*]\varphi)$. Thus, $\langle W, I, \tilde{R}\rangle \vDash \upsilon_2$, provided $R(\alpha^*) = tr(R(\alpha))$.

To recap the previous, we now have that $\forall\alpha\in\Lambda : R(\alpha^*) = tr(R(\alpha)) \Rightarrow \langle W, I, \tilde{R}\rangle \vDash \Upsilon$.

Consider then such $L(\Pi, \mathcal{A}, \{^*\})$-frame $\langle W, I, \tilde{R}\rangle$ that $\exists\alpha\in\Lambda : tr(R(\alpha)) \nsubseteq R(\alpha^*)$. Then, $\exists n\in\mathbb{N} : \exists w, w'\in W : \langle w, w'\rangle \in R(\alpha)^{[n]}$ *and* $\langle w, w'\rangle \notin R(\alpha^*)$ and moreover $\forall m\in\mathbb{N} : m < n \Rightarrow R(\alpha)^{[m]} \subseteq R(\alpha^*)$, i.e. n is minimal.

If $n = 0$, then $w = w'$, thus $\langle w, w\rangle \notin R(\alpha^*)$. Then the model $M = \langle W, I, \tilde{R}, P\rangle$, with $P(p) = \{w\}$, clearly satisfies $M, w \vDash p \vee \langle\alpha\rangle\langle\alpha^*\rangle p$ and $M, w \nvDash \langle\alpha^*\rangle p$, thus $M, w \nvDash \langle\alpha^*\rangle p \leftrightarrow p \vee \langle\alpha\rangle\langle\alpha^*\rangle p$.

If $n > 1$, we write $n = 1 + m$; thus, $\langle w, w' \rangle \in R(\alpha)^{[m+1]}$. Consider then a model $M = \langle W, I, \tilde{R}, P \rangle$ such that $P(p) = \{w'\}$, for some $p \in \Pi$; that is, $M, w' \vDash p$. Since $R(\alpha)^{[m]} \subseteq R(\alpha^*)$, by the minimality of $n = 1 + m$, we infer with the fact

$$\langle w, w' \rangle \in R(\alpha)^{[1+m]} \textit{ and } M, w' \vDash p$$
$$\Leftrightarrow$$
$$\exists w'' \in W : \langle w, w'' \rangle \in R(\alpha) \textit{ and } \langle w'', w' \rangle \in R(\alpha)^{[m]} \textit{ and } M, w' \vDash p,$$

that first of all $\langle w'', w' \rangle \in R(\alpha^*)$, thus $M, w'' \vDash \langle \alpha^* \rangle p$. Secondly, $M, w \vDash \langle \alpha \rangle \langle \alpha^* \rangle p$, since $\langle w, w'' \rangle \in R(\alpha)$. However, $\langle w, w' \rangle \notin R(\alpha^*)$, therefore $M, w \nvDash \langle \alpha^* \rangle p$ by the choice of $P(p)$. Accordingly then $M, w \nvDash \langle \alpha^* \rangle p \leftrightarrow p \vee \langle \alpha \rangle \langle \alpha^* \rangle p$, and consequently $\langle W, I, \tilde{R} \rangle \nvDash \upsilon_1$.

Therefore, if $\langle W, I, \tilde{R} \rangle \vDash \upsilon_1$, then $\forall \alpha \in \Lambda : tr(R(\alpha) \subseteq R(\alpha^*))$.

Next, let $\langle W, I, \tilde{R} \rangle$ be $L(\Pi, \mathcal{A}\{^*\})$-frame and assume that $\exists \alpha \in \Lambda : R(\alpha^*) \nsubseteq tr(R(\alpha))$, i.e. $\exists w, w' \in W : \langle w, w' \rangle \in R(\alpha^*) \textit{ and } \langle w, w' \rangle \notin tr(R(\alpha))$. Consider a model $M = \langle W, I, \tilde{R}, P \rangle$ in which $P(p) = \{ v \mid \langle w, v \rangle \in tr(R(\alpha))\}$, for some $p \in \Pi$. Then, $M, w \vDash p$ by the reflexivity of $tr(R(\alpha))$. Also we have that $\forall v \in W : \langle w, v \rangle \in tr(R(\alpha)) \Rightarrow M, v \vDash [\alpha]p$. This is because if $\exists v \in W : \langle w, v \rangle \in tr(R(\alpha)) \textit{ and } M, v \nvDash [\alpha]p$, then $\exists v' \in W : \langle v, v' \rangle \in R(\alpha) \textit{ and } M, v' \nvDash p$. Moreover, if $\langle v, v' \rangle \in R(\alpha)$, then $\langle v, v' \rangle \in tr(R(\alpha))$. Thus, we have that $\langle w, v \rangle \in tr(R(\alpha))$ and $\langle v, v' \rangle \in tr(R(\alpha))$, whence $\langle w, v' \rangle \in tr(R(\alpha))$ by the transitivity of $tr(R(\alpha))$. Then, $v' \in \{v \mid \langle w, v \rangle \in tr(R(\alpha))\} = P(p)$, i.e. $M, v' \vDash p$, a contradiction. Therefore, (i) if $\langle w, v \rangle \in R(\alpha^*) \textit{ and } \langle w, v \rangle \in tr(R(\alpha))$, then $M, v \vDash p \rightarrow [\alpha]p$, since $M, v \vDash [\alpha]p$. If on the other hand (ii) $\langle w, v \rangle \in R(\alpha^*) \textit{ and } \langle w, v \rangle \notin tr(R(\alpha))$, then $M, v \nvDash p$ by the choice of $P(p)$, therefore $M, v \vDash p \rightarrow [\alpha]p$.

By (i) and (ii), $\forall v \in W : \langle w, v \rangle \in R(\alpha^*) \Rightarrow M, v \vDash p \rightarrow [\alpha]p$, hence $M, w \vDash [\alpha^*](p \rightarrow [\alpha]p)$. Now, $M, w' \nvDash p$ since by the assumption $\langle w, w' \rangle \notin tr(R(\alpha))$, hence $w' \notin P(p)$. Therefore by the assumption $\langle w, w' \rangle \in R(\alpha^*)$, we have $M, w \nvDash [\alpha^*]p$. Previously we concluded that $M, w \vDash p$, therefore $M, w \nvDash (p \rightarrow [\alpha^*]p)$. Thus, $M, w \nvDash [\alpha^*](p \rightarrow [\alpha]p) \rightarrow (p \rightarrow [\alpha^*]p)$. Therefore, if $\langle W, I, \tilde{R} \rangle \vDash \upsilon_2$, then $\forall \alpha \in \Lambda : R(\alpha^*) \subseteq tr(R(\alpha))$.

In conclusion, if $\langle W, I, \tilde{R} \rangle \vDash \Upsilon$, then $\forall \alpha \in \Lambda : R(\alpha^*) = tr(R(\alpha))$. □

Notice that on the basis of the above results, it is possible on the level of frames to syntactically define the equivalence relation determined by the relation $R(\alpha)$; that is, the smallest equivalence relation $S \subseteq W \times W$ that contains $R(\alpha)$. Relation $trs(R(\alpha))$ is the equivalence determined by $R(\alpha)$, and by the results in this section relation $trs(R(\alpha))$ is frame definable, for example, within the language $L(\Pi, \mathcal{A}, \{t, r, s\})$ wherein t, r and s are unary syntactic operators.

7 Definability in $PSTL$ foundation logics

In this chapter we move on to address definability issues in the foundation logics. Recall that according to Theorem 3.6, every frame definable operation is immediately foundation definable, but not the other way around. Therefore, we need to go through only few operations covered in Chapter 6 that proved not to be frame definable.

It would be interesting to sort out the general properties that discriminate between frame definable and exclusively foundation definable operators. If the target operation makes possible on the level of frames to define operation that maps to the universal relation, this clearly renders the operation as non-frame definable.[18] The same does not hold for foundation logics, as is manifest by Theorem 7.1. Of course, this casts only one necessary condition for the frame definability. In addition, by Theorems 6.11 and 7.2, there seems to be deeper issues that govern this division, since the intersection is completely domain independent operation and moreover generally *downsizes* the extension of the parametric relations. Regrettably, these issues are not discussed here, for they remain so far unresolved. Indeed, it is unclear what are the general restrictions of the expressive power of $PSTL$ frame and foundation logics respectively.

In this chapter we do not seek the minimal definability results; that is, consider whether some given operation is definable for instance within a language that contains only one syntactic operator. Rather, we show how to define various operators by resorting to the definability of other operations. It is more or less an open question to what extent the operations are independently definable.

It should be noted, that in what follows we use schematic symbols $\alpha, \beta, \gamma \ldots$ for process variables in formulae that define operators, but in fact we could use proper atomic process symbols $a, b, c \ldots$ instead and the results would remain the same.[19] Also, we could use propositional symbols $p, q \ldots$ instead of schemata $\varphi, \psi \ldots$ In Chapter 6, in connection with the frame logic definability, we could use propositional symbols instead of schematic formulae as well, but *not* proper atomic process symbols instead of schematic processes. For consistency, the schematic symbols are used in the following, but it could be noteworthy that in the case of foundation definability we actually could use *proper formulae* and that is something we cannot do in the case of frame definability.

Theorem 7.1. *Let $L(\Pi, \mathcal{A}, \{\times\})$ be a $PSTL$-language such that $\times$ is a 0-ary operator. The operator that returns the universal relation $W \times W$ of foundation $\langle W, I \rangle$ is foundation definable within $L(\Pi, \mathcal{A}, \{\times\})$. We have the following correspondence:*

$$\langle W, I \rangle \vDash [\times]\varphi \to [\alpha]\varphi \text{ if and only if } I(\times) = W \times W.$$

Proof. Let $\langle W, I \rangle$ be a $L(\Pi, \mathcal{A}, \{\times\})$-foundation. Assume $I(\times) = W \times W$. Then, clearly by Lemma 3.7, $\forall \tilde{R} : \langle W, I, \tilde{R} \rangle \vDash [\times]\varphi \to [\alpha]\varphi$ since $\forall \alpha \in \Lambda : R(\alpha) \subseteq R(\times)$. Therefore, $I(\times) = W \times W \Rightarrow \langle W, I \rangle \vDash [\times]\varphi \to [\alpha]\varphi$.

Assume then $I(\times) \neq W \times W$, i.e. $I(\times) \subset W \times W$. Then, there is a frame $\langle W, I, \tilde{R} \rangle$ such that $R(a) = W \times W$ for some $a \in \mathcal{A}$. Since $R(\times) \neq W \times W$, there are states

[18] For example, complementation (with the empty relation), the diversity relation (with the union and the diagonal relation), etc.

[19] As long as we take care that in case of any k-ary operator, we use distinct atomic processes $a_1, \ldots, a_k$. See example 3.5 and the following remarks in page 9 for explanation.

$w, w' \in W$ such that $w \not R(\times) w'$. Let $M = \langle W, I, \tilde{R}, P \rangle$ be a model with $P(p) = W \setminus \{w'\}$. Now, $M, w \vDash [\times]p$ and $M, w \nvDash [a]p$, thus $M, w \nvDash [\times]p \to [a]p$, and furthermore then $\langle W, I \rangle \nvDash [\times]\varphi \to [\alpha]\varphi$. Therefore, $\langle W, I \rangle \vDash [\times]\varphi \to [\alpha]\varphi \Rightarrow I(\times) = W \times W$. □

Next we proceed to demonstrate that the intersection operation is foundation definable with the union. The defining set of formulae Ψ with partial sketch of the following proof was first provided by Ari Virtanen at the University of Tampere in 2005. This particular logic has been under some discussion during the research into $PSTL$ conducted by the author, Virtanen and Kuusisto. Kuusisto (2007) has proven that there is an elegant complete axiomatization for logic $\{\varphi \mid \langle W, I, \rangle \vDash \varphi,$ wherein $I(+) \simeq \cup$ *and* $I(\cdot) \simeq \cap\}$. In fact, axiomatization $\Psi \cup \{[(\alpha \cdot (\beta \cdot \gamma)]\varphi \leftrightarrow [(\alpha \cdot \beta) \cdot \gamma]\varphi\}$ is deductively equivalent within a natural $PSTL$ deduction system with the axiomatization used by Kuusisto. (Cf. appendix A for a brief presentation.)

Theorem 7.2. *Consider $PSTL$ language $L(\Pi, \mathcal{A}, \{+, \cdot\})$ wherein $+$ and $\cdot$ are binary syntactic operators and $|\mathcal{A}| \geq 2$. Let $\Psi = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$ be the following set of schemata:*

$$\begin{aligned}
\psi_1 &= [\alpha]\varphi \vee [\beta]\varphi \to [\alpha \cdot \beta]\varphi \\
\psi_2 &= [\alpha]\varphi \leftrightarrow [\alpha \cdot \alpha]\varphi \\
\psi_3 &= [\alpha \cdot \beta]\varphi \leftrightarrow [\beta \cdot \alpha]\varphi \\
\psi_4 &= [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi \\
\psi_5 &= [(\alpha + \beta) \cdot \gamma]\varphi \leftrightarrow [\alpha \cdot \gamma]\varphi \wedge [\beta \cdot \gamma]\varphi
\end{aligned}$$

Then, $\langle W, I \rangle \vDash \Psi$ if and only if $I(+) \simeq \cup$ and $I(\cdot) \simeq \cap$.

Proof. First, consider a foundation $\langle W, I \rangle$ such that $I(+) \simeq \cup$ and $I(\cdot) \simeq \cap$. By Theorem 6.8 and Lemmata 3.7 and 3.8, it is relatively easy to see that $\langle W, I \rangle \vDash \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$. Choose any frame $\langle W, I, \tilde{R} \rangle$, then

$\langle W, I, \tilde{R} \rangle \vDash \psi_1$: Since $R(\alpha) \cap R(\beta) \subseteq R(\alpha)$, $R(\alpha) \cap R(\beta) \subseteq R(\beta)$ and $R(\alpha \cdot \beta) = R(\alpha) \cap R(\beta)$, clearly $R(\alpha \cdot \beta) \subseteq R(\alpha)$ and $R(\alpha \cdot \beta) \subseteq R(\beta)$. Hence, by Lemma 3.7 we have that $\langle W, I, \tilde{R} \rangle \vDash [\alpha]\varphi \to [\alpha \cdot \beta]\varphi$ and $\langle W, I, \tilde{R} \rangle \vDash [\beta]\varphi \to [\alpha \cdot \beta]\varphi$. Therefore, $\langle W, I, \tilde{R} \rangle \vDash [\alpha]\varphi \vee [\alpha]\beta \to [\alpha \cdot \beta]\varphi$.

$\langle W, I, \tilde{R} \rangle \vDash \psi_2$: Since $R(\alpha) = R(\alpha) \cap R(\alpha) = R(\alpha \cdot \alpha)$, we have $\langle W, I, \tilde{R} \rangle \vDash [\alpha]\varphi \leftrightarrow [\alpha \cdot \alpha]\varphi$ by Lemma 3.8.

$\langle W, I, \tilde{R} \rangle \vDash \psi_3$: Since $R(\alpha \cdot \beta) = R(\alpha) \cap R(\beta) = R(\beta) \cap R(\alpha) = R(\beta \cdot \alpha)$, Lemma 3.8 again directly implies that $\langle W, I, \tilde{R} \rangle \vDash [\alpha \cdot \beta]\varphi \leftrightarrow [\beta \cdot \alpha]\varphi$.

$\langle W, I, \tilde{R} \rangle \vDash \psi_4$: By Theorem 6.8.

$$\begin{aligned}
\langle W, I, \tilde{R} \rangle \vDash \psi_5 : \quad R((\alpha + \beta) \cdot \gamma) &= (R(\alpha) \cup R(\beta)) \cap R(\gamma) \\
&= (R(\alpha) \cap R(\gamma)) \cup (R(\beta) \cap R(\gamma)) \\
&= R((\alpha \cdot \gamma) + (\beta \cdot \gamma)).
\end{aligned}$$

Hence, by Lemma 3.8, $\langle W, I, \tilde{R} \rangle \vDash [(\alpha + \beta) \cdot \gamma]\varphi \leftrightarrow [(\alpha \cdot \gamma) + (\beta \cdot \gamma)]\varphi$. Therefore, $\langle W, I, \tilde{R} \rangle \vDash [(\alpha + \beta) \cdot \gamma]\varphi \leftrightarrow [\alpha \cdot \gamma]\varphi \wedge [\beta \cdot \gamma]\varphi$, since $\langle W, I, \tilde{R} \rangle \vDash \psi_4$.

Since the choice of mapping $\tilde{R}$ was arbitrary, we conclude that that the results are valid in all frames of foundation $\langle W, I\rangle$. Therefore, if $I(+) \simeq \cup$ and $I(\cdot) \simeq \cap$, then $\langle W, I\rangle \vDash \Psi$.

Consider then $L(\Pi, \mathcal{A}, \{+, \cdot\})$-foundation $\langle W, I\rangle$ and assume that $\langle W, I\rangle \vDash \Psi$. By theorems 6.8 and 3.6 we readily get the other demonstrable, i.e. $I(+) \simeq \cup$, since $\langle W, I\rangle \vDash \psi_4$. To show that also $I(\cdot) \simeq \cap$ we first need some intermediate results.

7.2.a: $\langle W, I\rangle \vDash \psi_1 \Rightarrow \forall S, T \subseteq W \times W : I(\cdot)(S, T) \subseteq S \cap T$. To prove this, assume that $\exists S, T \subseteq W \times W : I(\cdot)(S, T) \nsubseteq S \cap T$. Then there is frame $\langle W, I, \tilde{R}\rangle$ such that $R(a) = S$ and $R(b) = T$, for some $a, b \in \mathcal{A}$, whence $R(a \cdot b) \nsubseteq R(a) \cap R(b)$. Thus, $\exists w, w' \in W : \langle w, w'\rangle \in R(a \cdot b)$ and $\langle w, w'\rangle \notin R(a)$ *or* $\langle w, w'\rangle \notin R(b)$. Consider then a model $M = \langle W, I, \tilde{R}, P\rangle$ wherein $P(p) = W \backslash \{w'\}$. Since $\langle w, w'\rangle \notin R(a)$ or $\langle w, w'\rangle \notin R(b)$, either $M, w \vDash [a]p$ or $M, w \vDash [b]p$. In either case, $M, w \vDash [a]p \vee [b]p$. On the other hand, $\langle w, w'\rangle \in R(a \cdot b)$, whence $M, w \nvDash [a \cdot b]p$. Therefore, $M, w \nvDash [a]p \vee [b]p \rightarrow [a \cdot b]p$, whence $\langle W, I\rangle \nvDash [\alpha]\varphi \vee [\beta]\varphi \rightarrow [\alpha \cdot \beta]\varphi$. Hence, $\langle W, I\rangle \vDash \psi_1 \Rightarrow \forall S, T \subseteq W \times W : I(\cdot)(S, T) \subseteq S \cap T$.

The following correspondences hold by Lemma 3.8:

$$\langle W, I, \tilde{R}\rangle \vDash [\alpha]\varphi \leftrightarrow [\alpha \cdot \alpha]\varphi \Leftrightarrow R(\alpha) = R(\alpha \cdot \alpha),$$
$$\langle W, I, \tilde{R}\rangle \vDash [\alpha \cdot \beta]\varphi \leftrightarrow [\beta \cdot \alpha]\varphi \Leftrightarrow R(\alpha \cdot \beta) = R(\beta \cdot \alpha).$$

Thus, we conclude by Theorem 3.6 that

7.2.b: $\langle W, I\rangle \vDash \psi_2 \Leftrightarrow \forall S \subseteq W \times W : S = I(\cdot)(S, S)$,
7.2.c: $\langle W, I\rangle \vDash \psi_3 \Leftrightarrow \forall S, T \subseteq W \times W : I(\cdot)(S, T) = I(\cdot)(T, S)$.

7.2.d: $\langle W, I\rangle \vDash \psi_5 \Rightarrow \forall R, S, T \subseteq W \times W : I(\cdot)((R \cup S), T) = (I(\cdot)(R, T)) \cup (I(\cdot)(S, T))$, provided $\langle W, I\rangle \vDash \psi_4$. Assume

$$\langle W, I\rangle \vDash [(\alpha + \beta) \cdot \gamma]\varphi \leftrightarrow [\alpha \cdot \gamma]\varphi \wedge [\beta \cdot \gamma]\varphi \textit{ and } \langle W, I\rangle \vDash [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta \cdot \gamma]\varphi.$$

Since $PSTL$ foundation logic respects standard propositional logic, we can readily infer from the assumptions that $\langle W, I\rangle \vDash [(\alpha + \beta) \cdot \gamma]\varphi \leftrightarrow [(\alpha \cdot \gamma) + (\beta \cdot \gamma)]\varphi$. Then, by corollary 3.9, we infer that

$$\forall R, S, T \subseteq W \times W : I(\cdot)((I(+)(R, S)), T) = I(+)((I(\cdot)(R, T)), (I(\cdot)(S, T))).$$

Since assumption $\langle W, I\rangle \vDash \psi_4$ implies $I(+) \simeq \cup$, we conclude that

$$\forall R, S, T \subseteq W \times W : I(\cdot)((R \cup S), T) = (I(\cdot)(R, T)) \cup (I(\cdot)(S, T)).$$

We denote $S \sqcap T = I(\cdot)(S, T)$ for readability of the proof to follow. Now, to recap the results 7.2.(a-d) we have the following:

1° $\forall S, T \subseteq W \times W : S \sqcap T \subseteq S \cap T$, by 7.2.a.

2° The operation $\sqcap$ is idempotent: $\forall S \subseteq W \times W : S = S \sqcap S$, by 7.2.b.

3° The operation $\sqcap$ is commutative: $\forall S, T \subseteq W \times W : S \sqcap T = T \sqcap S$, by 7.2.c.

4° By 7.2.d, the operation $\sqcap$ respects the distinct distributive law with union, namely $\forall R, S, T \subseteq W \times W : (R \cup S) \sqcap T = (R \sqcap T) \cup (S \sqcap T)$.

5° $\forall R, S, T \subseteq: W \times W : T \sqcap (R \cup S) = (T \sqcap R) \cup (T \sqcap S)$. This is a commutative reformulation of 4°, permitted by 3°.

Then pick any two relations $A, B \subseteq W \times W$. We denote

$$A' = A \setminus B,\ B' = B \setminus A \ \text{ and } C = A \cap B.$$

Hence $6^\circ : A = (A' \cup C)$ and $7^\circ : B = (B' \cup C)$.

Also, by 1°, we have that $A' \sqcap B' \subseteq A' \cap B' = \emptyset$, $A' \sqcap C \subseteq A' \cap C = \emptyset$ and $C \sqcap B' \subseteq C \cap B' = \emptyset$; that is,

$$8^\circ : A' \sqcap B' = \emptyset,\ 9^\circ : A' \sqcap C = \emptyset \ \text{ and } \ 10^\circ : C \sqcap B' = \emptyset.$$

Finally, we are ready to commit the the concluding inference:

$$\begin{array}{rcll} I(\cdot)(A, B) & = & A \sqcap B & \\ & = & (A' \cup C) \sqcap (B' \cup C) & 6^\circ, 7^\circ \\ & = & (A' \sqcap (B' \cup C)) \cup (C \sqcap (B' \cup C)) & 4^\circ \\ & = & ((A' \sqcap B') \cup (A' \sqcap C)) \cup ((C \sqcap B') \cup (C \sqcap C)) & 5^\circ \\ & = & (\emptyset \cup \emptyset) \cup (\emptyset \cup (C \sqcap C)) & 8^\circ, 9^\circ, 10^\circ \\ & = & C \sqcap C & \\ & = & C & 2^\circ \\ & = & A \cap B & \end{array}$$

Therefore, $\forall S, T \subseteq W \times W : I(\cdot)(S, T) = S \cap T$. This concludes the proof of the other implication in the theorem; that is, if $\langle W, I \rangle \vDash \Psi$, then $I(+) \simeq \cup$ and $I(\cdot) \simeq \cap$. □

Theorem 7.3. *Let $L(\Pi, \mathcal{A}, \{+, \cdot, /\})$ be a PSTL-language such that $+, \cdot$ and $/$ are binary syntactic operators and $|\mathcal{A}| \geq 2$. Then,*

$$\langle W, I \rangle \vDash \Psi \cup K \text{ if and only if } I(+) \simeq \cup, I(\cdot) \simeq \cap \text{ and } I(/) \simeq \setminus,$$

wherein set Ψ is as defined in page 38 and

$$\kappa_1 = [\alpha/\beta]\varphi \wedge [\alpha \cdot \beta]\varphi \leftrightarrow [\alpha]\varphi$$
$$\kappa_2 = [\beta \cdot (\alpha/\beta)]\bot$$

Proof. Consider $L(\Pi, \mathcal{A}, \{+, \cdot, /\})$-foundation $\langle W, I \rangle$ such that $I(+) \simeq \cup$, $I(\cdot) \simeq \cap$ and $I(/) \simeq \setminus$. Immediately by Theorem 7.2, $\langle W, I \rangle \vDash \Psi$.

Then choose any frame $\langle W, I, \tilde{R}\rangle$ of the foundation under consideration. Evidently $\forall \alpha, \beta \in \Lambda : (R(\alpha) \setminus R(\beta)) \cup (R(\alpha) \cap R(\beta)) = R(\alpha)$, therefore $\langle W, I, \tilde{R}\rangle \vDash [(\alpha/\beta) + (\alpha \cdot \beta)]\varphi \leftrightarrow [\alpha]\varphi$ by Lemma 3.8. By Theorem 6.8, $\langle W, I, \tilde{R}\rangle \vDash [(\alpha/\beta) + (\alpha \cdot \beta)]\varphi \leftrightarrow [\alpha/\beta]\varphi \wedge [\alpha \cdot \beta]\varphi$, thus we now have that $\langle W, I, \tilde{R}\rangle \vDash [\alpha/\beta]\varphi \wedge [\alpha \cdot \beta]\varphi \leftrightarrow [\alpha]\varphi$.

Next, $R(\beta \cdot (\alpha/\beta)) = \emptyset$, since $\forall \alpha, \beta \in \Lambda : R(\beta) \cap (R(\alpha) \setminus R(\beta)) = \emptyset$. Thus, $\langle W, I, \tilde{R}\rangle \vDash [\beta \cdot (\alpha/\beta)]\bot$.

Therefore, $\langle W, I, \tilde{R}\rangle \vDash \{\kappa_1, \kappa_2\}$. Since the choice of mapping $\tilde{R}$ was arbitrary, we conclude that $\langle W, I\rangle \vDash K$, hence if $I(+) \simeq \cup$, $I(\cdot) \simeq \cap$ and $I(/) \simeq \setminus$, then $\langle W, I\rangle \vDash \Psi \cup K$.

Assume then $\langle W, I\rangle \vDash \Psi \cup K$. Since $\langle W, I\rangle \vDash \Psi$, by Theorem 7.2 $I(+) \simeq \cup$ and $I(\cdot) \simeq \cap$. We still need to prove that $I(/) \simeq \setminus$, i.e. $\forall S, T \subseteq W \times W : I(/)(S, T) = S \setminus T$.

Assume to the contrary; that is, either $I(/)(S, T) \nsubseteq S \setminus T$ or $S \setminus T \nsubseteq I(/)(S, T)$ holds for some relations $S, T \subseteq W \times W$.

Assume $I(/)(S, T) \nsubseteq S \setminus T$ and consider a frame $\langle W, I, \tilde{R}\rangle$ such that $\tilde{R}(a) = S$ and $\tilde{R}(b) = T$ for some $a, b \in \mathcal{A}$, whence $\exists w, w' \in W : \langle w, w'\rangle \in R(a/b)$ *and* $\langle w, w'\rangle \notin R(a) \setminus R(b)$. Since $\langle w, w'\rangle \notin R(a) \setminus R(b)$, either $\langle w, w'\rangle \notin R(a)$ or $\langle w, w'\rangle \in R(a) \cap R(b)$.

Assume first that $\langle w, w'\rangle \notin R(a)$. Consider then a model $M = \langle W, I, \tilde{R}, P\rangle$ with $P(p) = W \setminus \{w'\}$. Since $\langle w, w'\rangle \notin R(a)$, clearly $M, w \vDash [a]p$. However, $\langle w, w'\rangle \in R(a/b)$, thus $M, w \nvDash [a/b]p$, whence $M, w \nvDash [a/b]p \wedge [a \cdot b]p$. Therefore, $\langle W, I\rangle \nvDash [a/b]p \wedge [a \cdot b]p \leftrightarrow [a]p$, contrary to the assumption $\langle W, I\rangle \vDash \kappa_1$.

Consider then the other possibility, namely $\langle w, w'\rangle \in R(a) \cap R(b)$. Then $\langle w, w'\rangle \in R(b)$, and $\langle w, w'\rangle \in R(a/b)$ by the initial assumption, thus $\langle w, w'\rangle \in R(b) \cap R(a/b)$. Therefore, $\langle w, w'\rangle \in R(b \cdot (a/b))$, since $I(\cdot) \simeq \cap$ by the assumption $\langle W, I, \tilde{R}\rangle \vDash \Psi$. Whence, by the definition of constant $\bot$, clearly $M, w \nvDash [b \cdot (a/b)]\bot$. Thus, $\langle W, I\rangle \nvDash \kappa_2$, contrary to the assumption $\langle W, I\rangle \vDash \Psi \cup K$.

Therefore, $\langle W, I\rangle \vDash \Psi \cup K \Rightarrow \forall S, T \subseteq W \times W : I(/)(S, T) \subseteq S \setminus T$.

Next, assume $S \setminus T \nsubseteq I(/)(S, T)$. Then there is a frame $\langle W, I, \tilde{R}\rangle$ wherein $\tilde{R}(a) = S$ and $\tilde{R}(b) = T$, for some $a, b \in \mathcal{A}$, such that $\exists w, w' \in W : \langle w, w'\rangle \in R(a) \setminus R(b)$ and $\langle w, w'\rangle \notin R(a/b)$. Since $\langle w, w'\rangle \in R(a) \setminus R(b)$, we have $\langle w, w'\rangle \in R(a)$ and $\langle w, w'\rangle \notin R(b)$; that is, $\langle w, w'\rangle \notin R(a) \cap R(b)$. Consider a model $M = \langle W, I, \tilde{R}, P\rangle$ wherein $P(p) = W \setminus \{w'\}$. Since $\langle w, w'\rangle \in R(a)$, we have $M, w \nvDash [a]p$. On the other hand, we have that $\langle w, w'\rangle \notin R(a/b)$ and $\langle w, w'\rangle \notin R(a) \cap R(b)$, i.e. $\langle w, w'\rangle \notin R(a \cdot b)$. Thus, by the choice of valuation $P(p)$, now $M, w \vDash [a/b]p$ and $M, w \vDash [a \cdot b]p$; that is, $M, w \vDash [a/b]p \wedge [a \cdot b]p$. Hence $M, w \nvDash [a/b]p \wedge [a \cdot b]p \leftrightarrow [a]p$, which contradicts the assumption $\langle W, I\rangle \vDash \kappa_1$. Therefore $\langle W, I\rangle \vDash \Psi \cup \{\kappa_1\} \Rightarrow \forall S, T \subseteq W \times W : S \setminus T \subseteq I(/)(S, T)$.

Thus, the proof for $\langle W, I\rangle \vDash \Psi \cup K \Rightarrow \forall S, T \subseteq W \times W : I(/)(S, T) = S \setminus T$ is complete. □

Theorem 7.4. *Consider a PSTL-language $L(\Pi, \mathcal{A}, \{\times, +, \cdot, /\})$ wherein $|\mathcal{A}| \geq 2$ and* $\mathrm{dom}(\times) = \Lambda^0$ *and* $\mathrm{dom}(+) = \mathrm{dom}(\cdot) = \mathrm{dom}(/) = \Lambda^2$. *The complementation is syntactically foundation definable in $L(\Pi, \mathcal{A}, \{\times, +, \cdot, /\})$, since* *if* $\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\}$*, then* $\forall S \subseteq W \times W : I(/)(I(\times), S) = \overline{S}$*.*

Proof. Cf. the proof of corollary 7.5 below. □

Corollary 7.5. *Let* $L(\Pi, \mathcal{A}, \{\times, \bar{\cdot}, +, \cdot, /\})$ *be a PSTL-language with* $|\mathcal{A}| \geq 2$ *and* $\mathrm{dom}(\times) = \Lambda^0$, $\mathrm{dom}(\bar{\cdot}) = \Lambda^1$ *and* $\mathrm{dom}(+) = \mathrm{dom}(\cdot) = \mathrm{dom}(/) = \Lambda^2$*. The operation that returns the complement of* $S \subseteq W \times W$ *is foundation definable in the language at issue, since the following correspondence holds:*

$$\langle W, I\rangle \vDash (\Psi \cup K \cup \{[\times]\varphi \to [\alpha]\}) \cup \{[\overline{\alpha}]\varphi \leftrightarrow [\times/\alpha]\varphi\}$$

if and only if

$$I(\times) = W \times W, I(+) \simeq \cup, I(\cdot) \simeq \cap, I(/) \simeq \setminus \text{ and } \forall S \subseteq W \times W : I(\bar{\cdot})(S) = \overline{S}.$$

The essential content of Theorem 7.5 is actually the following: Recall that by Theorems 7.1,7.2 and 7.3, $\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\}$ is equivalent to that $I(\times) = W \times W$, $I(+) \simeq \cup$, $I(\cdot) \simeq \cap$ and $I(/) \simeq \setminus$. Now, if $\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\}$ the following holds:

$$\langle W, I\rangle \vDash [\overline{\alpha}]\varphi \leftrightarrow [\times/\alpha]\varphi \text{ if and only if } \forall S \subseteq W \times W : I(\bar{\cdot})(S) = \overline{S}.$$

Proof. Assume that $\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\}$. Thus $I(\times) = W \times W$, $I(+) \simeq \cup$, $I(\cdot) \simeq \cap$ and $I(/) \simeq \setminus$, by Theorems 7.1, 7.2 and 7.3. Therefore $\forall S \subseteq W \times W : I(/)(I(\times), S) = \overline{S}$.

By Corollary 3.9, $\langle W, I\rangle \vDash [\overline{\alpha}]\varphi \leftrightarrow [\times/\alpha]\varphi \Leftrightarrow \forall S \subseteq W \times W : I(\bar{\cdot})(S) = I(/)(I(\times), S)$.[20] Thus, $\langle W, I\rangle \vDash [\overline{\alpha}]\varphi \leftrightarrow [\times/\alpha]\varphi \Leftrightarrow \forall S \subseteq W \times W : I(\bar{\cdot})(S) = \overline{S}$, for $I(/)(I(\times), S) = \overline{S}$. □

Theorem 7.6. *Let* $L(\Pi, \mathcal{A}, \{\times, ;, +, \cdot, /\})$ *be a PSTL-language such that* $|\mathcal{A}| \geq 2$ *and* $\mathrm{dom}(\times) = \Lambda^0, \mathrm{dom}(\bar{\cdot}) = \Lambda^1$ *and* $\mathrm{dom}(;) = \mathrm{dom}(+) = \mathrm{dom}(\cdot) = \mathrm{dom}(/) = \Lambda^2$*. Then, the relative union is syntactically foundation definable in the language since*

if $\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\} \cup \{\langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi\}$*, then*

$\forall S, T \subseteq W \times W : I(/)(I(\times), (I(;)(I(/)(I(\times), S), I(/)(I(\times), T)))) = S \dotplus T$.[21]

[20]Note that this is a somewhat nontrivial application of 3.9. Of course the trivial implication of the corollary is that if $\langle W, I\rangle$ is $L(\Pi, \mathcal{A}, \mathcal{F})$-foundation with $|\mathcal{A}| \geq n$ and $f, g \in \mathcal{F}$, then

$$\langle W, I\rangle \vDash [f(\alpha_1, \ldots, \alpha_n)]\varphi \leftrightarrow [g(\alpha_1, \ldots, \alpha_n)]\varphi \Leftrightarrow \forall S_1, \ldots, S_n \subseteq W \times W : I(f)(S_1, \ldots, S_n) = I(g)(S_1, \ldots, S_n)$$

We can treat $(\times/x)$ as a unary compound operator where x denotes the place of an argument, that is $(\times/x)$ stands for instances of $I(/) : ((W \times W), S) \mapsto (W \times W) \setminus S$ for any $S \subseteq W \times W$. Therefore, the commented step in the proof of theorem 7.5 holds. Indeed, the proof shows that there is no logical difference between operators $I(\bar{\cdot})$ and $I(/)(\times/x)$ in the considered foundation class.

[21]Notice that if we relax our denotational conventions and just write $I(f) = f$ and use infix notation, the overly complex statement $I(/)(I(\times), (I(;)(I(/)(I(\times), S), I(/)(I(\times), T)))) = S \dotplus T$ reduces to $(\times/((\times/S); (\times/T))) = S \dotplus T$.

Proof. Recall that $S \dot{+} T = \overline{\overline{S} \circ \overline{T}}$ (cf. Definition 6.9). If $\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\}$, by Theorem 7.4: (1°) $I(/)(I(\times), S) = \overline{S}$. Furthermore, Theorem 6.7 with Theorem 3.6 implies (2°) $I(;)(S,T) = S \circ T$, if $\langle W, I\rangle \vDash \langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$.

Hence, if $\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\} \cup \{\langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi\}$, the following chain of equivalences holds:

$$\begin{aligned}
I(/)\Big(I(\times), (I(;)(I(/)(I(\times), S), I(/)(I(\times), T)))\Big) &= I(/)\Big(I(\times), (I(;)(\overline{S}, \overline{T}))\Big) && 1° \\
&= I(/)\Big(I(\times), ((\overline{S} \circ \overline{T}))\Big) && 2° \\
&= \overline{\overline{S} \circ \overline{T}} && 1° \\
&= S \dot{+} T && \text{def.6.9}
\end{aligned}$$

□

If we want to reserve a distinct syntactic operator to carry out the relative union, we can always assign one for the task. This requires that we introduce a language with an additional binary syntactic operator in regard to language $L(\Pi, \mathcal{A}, \{\times, ;, +, \cdot, /\})$. For illustration, one in the following corollary of Theorem 7.6 is considered.

Corollary 7.7. *Let* $L(\Pi, \mathcal{A}, \{\times, \bar{\cdot}, ;, +, \cdot, /, \oplus\})$ *be a PSTL-language such that* $|\mathcal{A}| \geq 2$ *and* $\mathrm{dom}(\times) = \Lambda^0$, $\mathrm{dom}(\bar{\cdot}) = \Lambda^1$ *and* $\mathrm{dom}(+) = \mathrm{dom}(\cdot) = \mathrm{dom}(/) = \mathrm{dom}(;) = \mathrm{dom}(\oplus) = \Lambda^2$. *Now,*

if $\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\} \cup \{[\overline{\alpha}]\varphi \leftrightarrow [\times/\alpha]\varphi\} \cup \{\langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi\}$, *then*

$\langle W, I\rangle \vDash [\alpha \oplus \beta]\varphi \leftrightarrow [\overline{\overline{\alpha};\overline{\beta}}]\varphi$ *if and only if* $\forall S, T \subseteq W \times W : I(\oplus)(S,T) = S \dot{+} T$.

Notice that operator $\bar{\cdot}$ is actually redundant here, but its introduction makes the demonstration far more easier to follow.

Proof. Let $\langle W, I\rangle$ be $L(\Pi, \mathcal{A}, \{\times, \bar{\cdot}, +, \cdot, /, ;, \oplus\})$-foundation such that

$$\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\} \cup \{[\overline{\alpha}]\varphi \leftrightarrow [\times/\alpha]\varphi\} \cup \{\langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi\}.$$

Since $\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\} \cup \{[\overline{\alpha}]\varphi \leftrightarrow [\times/\alpha]\varphi\}$, we have $I(\bar{\cdot})(S) = \overline{S}$ by Theorem 7.5. Also, since $\langle W, I\rangle \vDash \langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$, we have $I(;)(S,T) = S \circ T$ by Theorems 6.7 and 3.6. Therefore,

$$\forall S, T \subseteq W \times W : I(\bar{\cdot})(I(;)(I(\bar{\cdot})(S), I(\bar{\cdot})(T))) = \overline{\overline{S} \circ \overline{T}} = S \dot{+} T.$$

Thus, if $\langle W, I\rangle \vDash [\alpha \oplus \beta]\varphi \leftrightarrow [\overline{\overline{\alpha};\overline{\beta}}]\varphi$, then

$$\forall S, T \subseteq W \times W : I(\oplus)(S,T) = I(\bar{\cdot})(I(;)(I(\bar{\cdot})(S), I(\bar{\cdot})(T))),$$

by Theorem 3.9. Whence, we conclude that $\forall S, T \subseteq W \times W : I(\oplus)(S,T) = S \dot{+} T$. □

Finally, we have one more operation to be considered: the constant operator that returns the diversity relation over the domain of given foundation $\langle W, I\rangle$. Rather less surprisingly, the operation is definable.

Theorem 7.8. *Consider PSTL-language* $L(\Pi, \mathcal{A}, \{D, \times, +, \cdot, /\})$ *wherein* $|\mathcal{A}| \geq 2$ *and* $\mathrm{dom}(D) = \mathrm{dom}(\times) = \Lambda^0$ *and* $\mathrm{dom}(+) = \mathrm{dom}(\cdot) = \mathrm{dom}(/) = \Lambda^2$*. The operator that returns the diversity relation of a given domain is syntactically foundation definable in the language. Let* $\langle W, I\rangle$ *be a* $L(\Pi, \mathcal{A}, \{D, \otimes, +, \cdot, /\})$*-foundation. Now,*

$$\text{if } \langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\} \cup \{[D]\varphi \leftrightarrow \varphi\}, \text{ then}$$
$$I(\times/D) = \{\langle w, w'\rangle | w, w' \in W : w \neq w'\}.$$

Proof. The proof is trivial. Assume $\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\} \cup \{[\mathrm{D}]\varphi \leftrightarrow \varphi\}$. Since $\langle W, I\rangle \vDash [\mathrm{D}]\varphi \leftrightarrow \varphi$, we have that $I(\mathrm{D}) = \{\langle w, w\rangle | w \in W\}$ by Theorems 6.2 and 3.6. In addition, $\langle W, I\rangle \vDash \Psi \cup K \cup \{[\times]\varphi \to [\alpha]\varphi\}$ implies $I(\times) = W \times W$ and $I(S/T) = S \setminus T$ by Theorems 7.1 and 7.4 respectively. Therefore, $I(\times/\mathrm{D}) = I(\times)\setminus I(\mathrm{D}) = (W \times W) \setminus I(\mathrm{D}) = \overline{I(\mathrm{D})} = \{\langle w, w'\rangle | w, w' \in W : w \neq w'\}$. □

Moreover, we can introduce a distinct (0-ary) syntactic operator that maps to the diversity relation if needed; that is, we evidently have the corresponding corollary to Theorem 7.8 with Theorems 7.4 and 7.6.

8 Summary

In this paper we have considered several particular operation definability issues in various *PSTL* languages in the level of frames and foundations respectively. The results are recapitulated in the table below.[22]

Table representing frame and foundation definability results.

UNARY	frame	found.
complement	no	yes
inverse	yes	yes

BINARY	frame	found.
composition	yes	yes
union	yes	yes
relative union	no	yes
set substraction	no*	yes
intersection	no*	yes

CLOSURES	frame	found.
reflexive	yes	yes
symmetric	yes	yes
transitive	yes	yes
trans. refl.	yes	yes

CONSTANTS	frame	found.
empty	yes	yes
diagonal	yes	yes
universal	no	yes
diversity	no	yes

[22]Results tagged with asterisk (*) are unresolved in general case. See section 6, page 6.3 for explanation.

In Sections 4 and 5, we discussed some general issues in frame logic definability. The covered model theoretic tools were the union of disjoint frames and bounded morphisms, which both are familiar tools of any normal modal logic semantics, generalized for the context of $PSTL$.

It is unfortunate that thus far we lack similar powerful model theoretical tools for foundation logics. Of course, one could define for example *the union of disjoint foundations* by just omitting the references to mapping $\tilde{R}_i \uplus \tilde{R}_j$ in the definition of the union of disjoint frames (cf. Definition 4.2). The problem, however, resides in the fact that while the fundamental idea of the union of disjoint frames is that the operation does not always preserve operator characterizations, but always preserves frame validities, the described operation on disjoint foundations most definitely does not generally preserve *either* property of foundations.

At this point we also lack general characterization of expressiveness of $PSTL$. It should be clear that $PSTL$ frame logic is not as expressive as the *First-Order Logic* (FOL), since we cannot, for example, define the complementation on the level of frames within any $PSTL$ language. On the other hand, transitive closure is frame definable in $PSTL$ language with one unary operator while the closure is not definable in FOL. Therefore, the class of definable operations in FOL and $PSTL$ frame logics intersect, but neither language is strictly more expressive than the other in this respect.

It would be even more interesting to pin down the class of foundation definable operations, although for the reason mentioned above we have even less to say about what is not definable in foundation logics than in frame logics. For some overview on the positive results, we know that the calculus of relations can be expressed completely within the $PSTL$ foundation logics. The calculus of relations, for one, is equivalent in expressive and deductive power with FOL fragment with three variables. It is rather surprising that this fragment is adequate to formulate virtually any first-order set theory and first-order number theory (Givant, 2006). Moreover, since the transitive closure is also definable in foundation logics, the proofs presented in this paper disclose that the $PSTL$ foundation logics provide a highly expressive framework. Unfortunately it is not yet established whether every first-order definable operation is foundation definable or not; that is, whether $PSTL$ foundation logics framework is strictly more expressive than FOL.

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A Brief review on the syntactic theory

Definition A.1. We call a deduction system S_Λ as *the minimal deduction system of $PSTL$ language* $L(\Pi, \mathcal{A}, \mathcal{F})$ defined in the following.

The minimal deduction system S_Λ contains standard axiomatization of *Propositional Logic* (PL), i.e. axioms

$$\begin{array}{ll} Ax_1: & \varphi \to (\psi \to \varphi) \\ Ax_2: & (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \\ Ax_3: & (\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi) \end{array}$$

and is closed under inference rule *Modus Ponens*:

$$(MP): \quad \textit{if } \vdash \varphi \to \psi \textit{ and } \vdash \varphi, \textit{ then } \vdash \psi.$$

In what follows, we do not apply axioms Ax_i or rule (MP) explicitly but instead resort to an inference rule (PL) that allows us to introduce any formula $\varphi \in L(\Pi, \mathcal{A}, \mathcal{F})$ that can be deduced from the subset of preceding formulae by using classical propositional logic. Rule (PL) is justified simply by the fact that system consisting Ax_1, Ax_2, Ax_3 and (MP) contains every tautology as its theorem, i.e. constitutes a complete axiomatization of the propositional logic. Note that we consider also *modal tautologies* (e.g. $[\alpha]\varphi \to [\alpha]\varphi \vee [\beta]\psi$) to be classical tautologies, although they are not contained in language of PL proper.

In addition, S_Λ contains generalization of the standard axiom K of normal modal logic:

$$(K\Lambda) \quad : \quad [\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$$

and generalization of inference rule (RN):

$$(RN\Lambda) \quad : \quad \forall \alpha \in \Lambda: \textit{ if } \vdash \varphi, \textit{ then } \vdash [\alpha]\varphi$$

We proceed to demonstrate the deductive equivalence of $\Psi^* = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6\}$ and $\Omega = \{\psi_3, \psi_4, \psi_6, \omega_1, \omega_2, \omega_3\}$, wherein

$$\begin{array}{lcl} \psi_1 & = & [\alpha]\varphi \vee [\beta]\varphi \to [\alpha \cdot \beta]\varphi \\ \psi_2 & = & [\alpha]\varphi \leftrightarrow [\alpha \cdot \alpha]\varphi \\ \psi_3 & = & [\alpha \cdot \beta]\varphi \leftrightarrow [\beta \cdot \alpha]\varphi \\ \psi_4 & = & [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi \\ \psi_5 & = & [(\alpha + \beta) \cdot \gamma]\varphi \leftrightarrow [\alpha \cdot \gamma]\varphi \wedge [\beta \cdot \gamma]\varphi \\ \psi_6 & = & [\alpha \cdot (\beta \cdot \gamma)]\varphi \leftrightarrow [(\alpha \cdot \beta) \cdot \gamma]\varphi \\ \omega_1 & = & [\alpha + (\alpha \cdot \beta)]\varphi \leftrightarrow [\alpha]\varphi \\ \omega_2 & = & [\alpha \cdot (\alpha + \beta)]\varphi \leftrightarrow [\alpha]\varphi \\ \omega_3 & = & [\alpha \cdot (\beta + \gamma)]\varphi \leftrightarrow [(\alpha \cdot \beta) + (\alpha \cdot \gamma)]\varphi \end{array}$$

The motivation behind this is that, as mentioned in the opening of section 7, Antti Kuusisto has proven that a slightly expanded deduction system S^*_Λ with axioms Ω constitute a complete axiomatization of the $PSTL$ foundation logic of language $L(\Pi, \mathcal{A}, \{+, \cdot\})$ with interpretations $I(+) \simeq \cup$ and $I(\cdot) \simeq \cap$. The minimal deduction system S_Λ is not adequate to establish completeness, but it is sufficient to carry out deductions $\Psi^* \vdash \Omega$ and $\Omega \vdash \Psi^*$. Interested reader is referred to consult (Kuusisto, 2007).

We begin with $\Psi^* \vdash \Omega$. All we need to prove is $\Psi^* \vdash \{\omega_1, \omega_2, \omega_3\}$, since $\psi_3, \psi_4, \psi_6 \in \Psi^*$.

$\Psi^* \vdash \omega_1$:	1.	$[\alpha]\varphi \rightarrow [\alpha]\varphi \vee [\beta]\varphi$	(PL)
	2.	$[\alpha]\varphi \vee [\beta]\varphi \rightarrow [\alpha \cdot \beta]\varphi$	ψ_1
	3.	$[\alpha]\varphi \rightarrow [\alpha \cdot \beta]\varphi$	$1, 2\ (PL)$
	4.	$[\alpha]\varphi \rightarrow [\alpha]\varphi \wedge [\alpha \cdot \beta]\varphi$	$3\ (PL)$
	5.	$[\alpha]\varphi \wedge [\alpha \cdot \beta]\varphi \leftrightarrow [\alpha + (\alpha \cdot \beta)]\varphi$	ψ_4
	6.	$[\alpha]\varphi \rightarrow [\alpha + (\alpha \cdot \beta)]\varphi$	$4, 5\ (PL)$
	7.	$[\alpha + (\alpha \cdot \beta)]\varphi \leftrightarrow [\alpha]\varphi \wedge [\alpha \cdot \beta]\varphi$	ψ_4
	8.	$[\alpha]\varphi \wedge [\alpha \cdot \beta]\varphi \rightarrow [\alpha]\varphi$	(PL)
	9.	$[\alpha + (\alpha \cdot \beta)]\varphi \rightarrow [\alpha]\varphi$	$7, 8\ (PL)$
	10.	$[\alpha + (\alpha \cdot \beta)]\varphi \leftrightarrow [\alpha]\varphi$	$6, 9\ (PL)$

$\Psi^* \vdash \omega_2$:	1.	$[\alpha]\varphi \rightarrow [\alpha]\varphi \vee [\alpha + \beta]\varphi$	(PL)
	2.	$[\alpha]\varphi \vee [\alpha + \beta]\varphi \rightarrow [\alpha \cdot (\alpha + \beta)]\varphi$	ψ_1
	3.	$[\alpha]\varphi \rightarrow [\alpha \cdot (\alpha + \beta)]\varphi$	$1, 2\ (PL)$
	4.	$[\alpha \cdot (\alpha + \beta)]\varphi \leftrightarrow [(\alpha + \beta) \cdot \alpha]\varphi$	ψ_3
	5.	$[(\alpha + \beta) \cdot \alpha]\varphi \leftrightarrow [\alpha \cdot \alpha]\varphi \wedge [\beta \cdot \alpha]\varphi$	ψ_5
	6.	$[\alpha \cdot \alpha]\varphi \wedge [\beta \cdot \alpha]\varphi \rightarrow [\alpha \cdot \alpha]\varphi$	(PL)
	7.	$[\alpha \cdot \alpha]\varphi \leftrightarrow [\alpha]\varphi$	ψ_2
	8.	$[\alpha \cdot (\alpha + \beta)]\varphi \rightarrow [\alpha]\varphi$	$4 - 7\ (PL)$
	9.	$[\alpha \cdot (\alpha + \beta)]\varphi \leftrightarrow [\alpha]\varphi$	$3, 8\ (PL)$

$\Psi^* \vdash \omega_3$:	1.	$[\alpha \cdot (\beta + \gamma)]\varphi \leftrightarrow [(\beta + \gamma) \cdot \alpha]\varphi$	ψ_3
	2.	$[(\beta + \gamma) \cdot \alpha]\varphi \leftrightarrow [\beta \cdot \alpha]\varphi \wedge [\gamma \cdot \alpha]\varphi$	ψ_5
	3.	$[\alpha \cdot (\beta + \gamma)]\varphi \leftrightarrow [\beta \cdot \alpha]\varphi \wedge [\gamma \cdot \alpha]\varphi$	$1, 2\ (PL)$
	4.	$[\beta \cdot \alpha]\varphi \leftrightarrow [\alpha \cdot \beta]\varphi$	ψ_3
	5.	$[\gamma \cdot \alpha]\varphi \leftrightarrow [\alpha \cdot \gamma]\varphi$	ψ_3
	6.	$[\beta \cdot \alpha]\varphi \wedge [\gamma \cdot \alpha]\varphi \leftrightarrow [\alpha \cdot \beta]\varphi \wedge [\alpha \cdot \gamma]\varphi$	$4, 5\ (PL)$
	7.	$[\alpha \cdot (\beta + \gamma)]\varphi \leftrightarrow [\alpha \cdot \beta]\varphi \wedge [\alpha \cdot \gamma]\varphi$	$3, 6\ (PL)$
	8.	$[(\alpha \cdot \beta) + (\alpha \cdot \gamma)]\varphi \leftrightarrow [\alpha \cdot \beta]\varphi \wedge [\alpha \cdot \gamma]\varphi$	ψ_4
	9.	$[\alpha \cdot (\beta + \gamma)]\varphi \leftrightarrow [(\alpha \cdot \beta) + (\alpha \cdot \gamma)]\varphi$	$7, 8\ (PL)$

To establish $\Omega \vdash \Psi^*$, we need to carry out deductions for $\Omega \vdash \{\psi_1, \psi_2, \psi_5\}$, since $\psi_3, \psi_4, \psi_6 \in \Psi^*$.

$$
\begin{array}{lrll}
\Omega \vdash \psi_1: & 1. & [\alpha]\varphi \leftrightarrow [\alpha + (\alpha \cdot \beta)]\varphi & \omega_1 \\
& 2. & [\alpha + (\alpha \cdot \beta)]\varphi \leftrightarrow [\alpha]\varphi \wedge [\alpha \cdot \beta]\varphi & \psi_4 \\
& 3. & [\alpha]\varphi \wedge [\alpha \cdot \beta]\varphi \rightarrow [\alpha \cdot \beta]\varphi & (PL) \\
& 4. & [\alpha]\varphi \rightarrow [\alpha \cdot \beta]\varphi & 1-3\ (PL) \\
& 5. & [\beta]\varphi \leftrightarrow [\beta + (\beta \cdot \alpha)]\varphi & \omega_1 \\
& 6. & [\beta + (\beta \cdot \alpha)]\varphi \leftrightarrow [\beta]\varphi \wedge [\beta \cdot \alpha]\varphi & \psi_4 \\
& 7. & [\beta]\varphi \wedge [\beta \cdot \alpha]\varphi \rightarrow [\beta \cdot \alpha]\varphi & (PL) \\
& 8. & [\beta \cdot \alpha]\varphi \leftrightarrow [\alpha \cdot \beta]\varphi & \psi_3 \\
& 9. & [\beta]\varphi \rightarrow [\alpha \cdot \beta]\varphi & 5-8\ (PL) \\
& 10. & [\alpha]\varphi \vee [\beta]\varphi \rightarrow [\alpha \cdot \beta]\varphi & 4, 9\ (PL)
\end{array}
$$

$$
\begin{array}{lrll}
\Omega \vdash \psi_5: & 1. & [\gamma \cdot (\alpha + \beta)]\varphi \leftrightarrow [\gamma \cdot \alpha]\varphi \wedge [\gamma \cdot \beta]\varphi & \omega_3 \\
& 2. & [\gamma \cdot \alpha]\varphi \leftrightarrow [\alpha \cdot \gamma]\varphi & \psi_3 \\
& 3. & [\gamma \cdot \beta]\varphi \leftrightarrow [\beta \cdot \gamma]\varphi & \psi_3 \\
& 4. & [\gamma \cdot \alpha]\varphi \wedge [\gamma \cdot \beta]\varphi \leftrightarrow [\alpha \cdot \gamma]\varphi \wedge [\beta \cdot \gamma]\varphi & 2, 3\ (PL) \\
& 5. & [\gamma \cdot (\alpha + \beta)]\varphi \leftrightarrow [\alpha \cdot \gamma]\varphi \wedge [\beta \cdot \gamma]\varphi & 1, 4\ (PL) \\
& 6. & [\gamma \cdot (\alpha + \beta)]\varphi \leftrightarrow [(\alpha + \beta) \cdot \gamma]\varphi & \psi_3 \\
& 7. & [(\alpha + \beta) \cdot \gamma]\varphi \leftrightarrow [\alpha \cdot \gamma]\varphi \wedge [\beta \cdot \gamma]\varphi & 5, 6\ (PL)
\end{array}
$$

$$
\begin{array}{lrll}
\Omega \vdash \psi_2: & 1. & [\alpha]\varphi \leftrightarrow [\alpha \cdot (\alpha + \alpha)]\varphi & \omega_2 \\
& 2. & [\alpha \cdot (\alpha + \alpha)]\varphi \leftrightarrow [(\alpha + \alpha) \cdot \alpha]\varphi & \psi_3 \\
& 3. & [(\alpha + \alpha) \cdot \alpha]\varphi \leftrightarrow [\alpha \cdot \alpha]\varphi \wedge [\alpha \cdot \alpha]\varphi & \psi_5 \\
& 4. & [\alpha \cdot \alpha]\varphi \wedge [\alpha \cdot \alpha]\varphi \leftrightarrow [\alpha \cdot \alpha]\varphi & (PL) \\
& 5. & [\alpha]\varphi \leftrightarrow [\alpha \cdot \alpha]\varphi & 1-4\ (PL)
\end{array}
$$

Therefore, $\vdash_{S_\Lambda} \Psi^* \leftrightarrow \Omega$, whence the set of formulae Ψ (see page 38) with a formula ψ_6 provides the complete axiomatization of logic $\{\varphi | \langle W, I \rangle \vDash \varphi, I(+) \simeq \cup \ and \ I(\cdot) \simeq \cap\}$ within the deduction system S_Λ^* provided by Kuusisto. Whether ψ_6 is derivable from Ψ in S_Λ^* is currently unknown. This seems unlikely, since then ψ_6 would be redundant in Ω also and we have reasons to believe that this is not the case (again, for elaboration see (Kuusisto, 2007)).