

---

UNIVERSITY OF TAMPERE  
Faculty of Information Sciences  
Licentiate thesis

---

Jari Kivelä

Correspondences between definability of  
Boolean functions and frame definability  
in modal logic

---

Department of Mathematics, Statistics and Philosophy  
Mathematics  
January 2007

---

# Abstract

We establish a connection between term definability of classes of Boolean functions and definability of finite modal frames. We define a bijective translation between functional terms and uniform degree-1 formulas and show that a class of Boolean functions is defined by functional terms if and only if the corresponding class of finite Scott-Montague frames is defined by the translations of these functional terms, and vice versa. Since clones in particular are term definable, we obtain for each clone a corresponding class of Scott-Montague frames which is defined by uniform degree-1 formulas. As a special case, we get that the clone of all conjunctions and constant functions with the value 1 corresponds to the class of all Kripke frames. We get further correspondences by restricting the binary relation in Kripke frames in a natural way and considering Kripke frames with non-normal worlds. Furthermore, by modifying Kripke semantics, we extend our results to correspondences between linear clones and classes of Kripke frames equipped with modified Kripke semantics. Using these methods, we give, by means of Kripke semantics or modified Kripke semantics, the characterizations of the classes of Scott-Montague frames corresponding to each subclone of the clone of all conjunctions and constants, the clone of all disjunctions and constants, and the clone of linear functions.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>3</b>
2.1	Modal logic . . . . .	3
2.2	Boolean functions . . . . .	5
2.3	Term definable classes of Boolean functions . . . . .	9
<b>3</b>	<b>General correspondence</b>	<b>12</b>
<b>4</b>	<b>Kripke correspondence</b>	<b>20</b>
4.1	Standard Kripke frames . . . . .	20
4.2	Kripke frames with non-normal worlds . . . . .	26
<b>5</b>	<b>Linear clones and modified Kripke semantics</b>	<b>34</b>
5.1	Linear clones . . . . .	35
5.2	The remaining subclones of $L$ . . . . .	44
<b>6</b>	<b>Conclusions</b>	<b>47</b>

# 1 Introduction

The most common semantics for modal logic is the well-known Kripke semantics. However, there are also other interesting semantics for modal logic. Segerberg studied in [11] a semantics which is more general than Kripke semantics and in literature this semantics is often called *neighbourhood semantics*. In [6], Hansson and Gärdenfors gave this semantics an alternative, equivalent definition. Here we adopt their definition and following them, we call it *Scott-Montague semantics*.

A Scott-Montague frame is a structure  $\langle W, F \rangle$ , where  $W$  is a non-empty set and  $F$  is a set function  $\mathcal{P}(W) \rightarrow \mathcal{P}(W)$ , where  $\mathcal{P}(W)$  denotes the power set of  $W$ . If  $\mathcal{M}$  is a model based on a Scott-Montague frame, then the truth condition for the modal operator  $\Box$  in  $\mathcal{M}$  is given by

$$\mathcal{M}, w \models \varphi \text{ if and only if } w \in F(\|\varphi\|^{\mathcal{M}}),$$

where  $\|\varphi\|^{\mathcal{M}}$  is the truth set of  $\varphi$  in  $\mathcal{M}$ . When considering finite Scott-Montague frames  $\langle W, F \rangle$ , the set function  $F$  can be seen as a vector-valued Boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$ , where  $n$  is the cardinality of  $W$ . This gives a connection between finite Scott-Montague frames and Boolean functions. Frames obtained from finite Scott-Montague frames by replacing  $F$  by  $f$  are called *Boolean frames*. The idea of Boolean frames was introduced by Virtanen in [12].

It is well known that certain classes of Boolean functions can be defined by means of functional terms (see e.g. [9]). This approach to definability of Boolean function classes has certain similarities to frame definability in modal logic. In Section 3, we achieve a general correspondence between definability of Boolean functions by functional terms and definability of Scott-Montague frames by modal formulas of a specific form, so-called *uniform degree-1 formulas*. This correspondence is given by a bijective translation between functional terms and uniform degree-1 formulas. We prove that a class of Boolean functions is defined by functional terms if and only if the corresponding class of Scott-Montague frames is defined by the uniform degree-1 formulas which are the translations of the defining functional terms. Conversely, a class of Scott-Montague frames is defined by uniform degree-1 formulas if and only if the corresponding class of Boolean functions is defined by the translations of these formulas.

In this thesis we are especially interested in classes of Boolean functions which are closed under composition and contain all projections. These classes are called *clones* and it is well known that clones are definable by functional terms. As a corollary to our general correspondence result, we obtain for each clone a corresponding class of Scott-Montague frames.

There is a known way to interpret Kripke frames as Scott-Montague frames. Conversely, if a Scott-Montague frame is *augmented* then it can be interpreted as a Kripke frame. In the case of finite frames, being augmented just means that the frame validates the modal axioms  $\Box\top$  and  $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$ . For the definition of augmentation, see [2]. This gives rise to the question: What are the connections between classes of Kripke frames and clones. In Section 4, we prove that the class of all Kripke frames corresponds to the clone  $\Lambda_1$ , i.e. to the clone of all conjunctions. Furthermore, by restricting the binary relation in Kripke frames in natural ways, we get classes of Kripke frames corresponding to some subclones of  $\Lambda_1$ . Also in Section 4, we consider so called *non-normal Kripke frames* and obtain frame classes corresponding to some other subclones of  $\Lambda$ .

Not all the classes of Scott-Montague frames corresponding to clones can be characterized by means of the standard Kripke semantics. However, by modifying Kripke semantics, we are also able to characterize the classes of Scott-Montague frames corresponding to the linear clones  $L$ ,  $L_0$ ,  $L_1$ ,  $L_c$  and  $LS$  in Section 5. The modifications we consider are based on the parity of  $R$ -successors in Kripke frames. For example, one of the modified semantics is given by

$$\mathcal{M}, w \models_o \Box\varphi \text{ if and only if } |R[w] \cap \|\varphi\|_o^{\mathcal{M}}| \text{ is odd.}$$

That is, a formula  $\Box\varphi$  is true at  $w$  if and only if  $\varphi$  is true at an odd number of  $R$ -successors of  $w$ . With respect to this semantics, we show that the class of all Kripke frames corresponds to the clone  $L_0$ , i.e. to the clone of all 0-preserving linear functions. By varying this semantics and giving some natural restrictions to  $R$ , we get classes of Kripke frames with given semantics that correspond to the other linear clones and some subclones of  $L$ .

To summarize, after finding a corresponding class of Scott-Montague frames for each clone in Section 3, we characterize, by means of Kripke or modified Kripke semantics, every class of Scott-Montague frames corresponding to the subclones of  $V$ ,  $\Lambda$  and  $L$  in Sections 4 and 5.

This thesis combines two articles, namely [3] and [7]. The results in Sections 4.2 and 5.2 are from [7], all the other results are from [3].

## 2 Preliminaries

### 2.1 Modal logic

In this thesis we consider *basic modal logic* which is obtained from propositional logic by adding one unary modal operator  $\Box$ . This basic modal logic can be generalized in many ways. In *multi-modal logic* one considers many different unary operations instead of just one. We could also have modal operators of higher arity or make use of fixed points. In the latter case we are dealing with  $\mu$ -*calculus*. For background and further generalizations, see [1]. In the sequel, by modal logic we always mean basic modal logic.

Let  $\Phi$  be a countable set of proposition symbols denoted  $p_1, p_2, \dots$ , or  $p, q, r, \dots$ . The *formulas of modal logic* are defined inductively as follows:

- (1) All proposition symbols  $p \in \Phi$  and constants  $\perp$  and  $\top$  are formulas.
- (2) If  $\varphi$  and  $\psi$  are formulas, then  $\neg\varphi$ ,  $(\varphi \wedge \psi)$  and  $\Box\varphi$  are formulas.

We also make use of other connectives, like  $\vee$ ,  $\rightarrow$  and  $\leftrightarrow$ , and define them in a usual way by using  $\neg$  and  $\wedge$ . The dual operator  $\Diamond$  for  $\Box$  is defined as  $\Diamond = \neg\Box\neg$ . As an analogue to the quantifier rank of a formula in first-order logic we have the concept of *the degree of a formula* in modal logic. The degree of a formula  $\varphi$ , denoted by  $deg(\varphi)$ , is defined inductively as follows:

$$\begin{aligned} deg(\perp) = deg(\top) &= 0, \\ deg(p) &= 0, \text{ for all } p \in \Phi, \\ deg(\neg\varphi) &= deg(\varphi), \\ deg(\varphi \wedge \psi) &= \max\{deg(\varphi), deg(\psi)\}, \\ deg(\Box\varphi) &= deg(\varphi) + 1. \end{aligned}$$

Note that modal formulas of degree 0 are just *propositional formulas*.

In the thesis, we are interested in modal formulas of a certain form. We say that  $\psi$  is a *uniform degree-1 formula* if it is of the form

$$\psi = \varphi(\Box\varphi_1, \dots, \Box\varphi_m)$$

where  $\varphi_1, \dots, \varphi_m$  are propositional formulas,  $\varphi$  is a propositional formula with proposition symbols  $p_1, \dots, p_m$  and  $\psi$  is the formula obtained from  $\varphi$  by replacing every occurrence of  $p_i$  by  $\Box\varphi_i$  for each  $1 \leq i \leq m$ . The formula  $\psi$  here is of degree 1 and it is uniform in the sense that in  $\psi$  connectives are applied only to subformulas of the form  $\Box\theta$ , where  $\theta$  is a propositional formula, or constants. For example,  $\top \wedge \Box\top \wedge \neg\Box p$  is a uniform degree-1 formula, whereas  $\Box p \wedge q$  is not.

The most popular semantics for modal logic is the relational semantics, known as *Kripke semantics*. A *Kripke frame* is a structure  $\mathcal{F} = \langle W, R \rangle$  where  $W$ , called the *universe*, is a non-empty set and  $R$  is a binary relation on  $W$ . From a Kripke frame we get a *Kripke model*  $\mathcal{M} = \langle \mathcal{F}, V \rangle$  (or  $\mathcal{M} = \langle W, R, V \rangle$ ) by adding a *valuation function*  $V : \Phi \rightarrow \mathcal{P}(W)$ , where  $\mathcal{P}(W)$  denotes the set of all subsets of  $W$ . The notion of a formula  $\varphi$  being *true* (or *satisfied*) in the model  $\mathcal{M}$  at a state (or world)  $w \in W$ , denoted  $\mathcal{M}, w \models \varphi$ , is defined inductively as follows:

$$\begin{aligned}
\mathcal{M}, w \models \perp & \quad \text{never,} \\
\mathcal{M}, w \models \top & \quad \text{always,} \\
\mathcal{M}, w \models p & \quad \text{iff } w \in V(p), \text{ for } p \in \Phi, \\
\mathcal{M}, w \models \neg\varphi & \quad \text{iff } \mathcal{M}, w \not\models \varphi, \\
\mathcal{M}, w \models \varphi \wedge \psi & \quad \text{iff } \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi, \\
\mathcal{M}, w \models \Box\varphi & \quad \text{iff for all } v \in W: \text{ if } wRv \text{ then } \mathcal{M}, v \models \varphi.
\end{aligned}$$

It is easy to verify that  $\mathcal{M}, w \models \Diamond\varphi$  if and only if  $\mathcal{M}, v \models \varphi$  for some  $v \in W$  such that  $wRv$ .

In this thesis we are also interested in a more general semantics for modal logic, so-called *Scott-Montague semantics*. A *Scott-Montague frame* is a structure  $\mathcal{F} = \langle W, F \rangle$  where  $W$  is a non-empty set and  $F$  is a *set function*  $\mathcal{P}(W) \rightarrow \mathcal{P}(W)$ . A *Scott-Montague model* is a structure  $\mathcal{M} = \langle \mathcal{F}, V \rangle$  (or  $\mathcal{M} = \langle W, F, V \rangle$ ) where  $\mathcal{F}$  is a Scott-Montague frame and  $V$  is a valuation. The truth conditions for the formulas are the same as in Kripke semantics except for the modal operator  $\Box$ , whose interpretation in such a model  $\mathcal{M}$  is given by

$$\mathcal{M}, w \models \Box\varphi \text{ iff } w \in F(\|\varphi\|^{\mathcal{M}}),$$

where  $\|\varphi\|^{\mathcal{M}}$  denotes the *truth set* of  $\varphi$  in  $\mathcal{M}$ , i.e.  $\|\varphi\|^{\mathcal{M}} = \{v \in W \mid \mathcal{M}, v \models \varphi\}$ . In other words  $F$  assigns to the truth set of  $\varphi$  the truth set of  $\Box\varphi$ , i.e.  $\|\Box\varphi\|^{\mathcal{M}} = F(\|\varphi\|^{\mathcal{M}})$ . For an equivalent, but different, formulation for Scott-Montague semantics, see [2]. Based on that formulation, the semantics is also known as *neighbourhood semantics*.

Let  $\mathcal{M}$  be a Kripke or Scott-Montague model with a universe  $W$  and let  $\mathcal{F}$  be Kripke or Scott-Montague frame with a universe  $W$ . We say that a formula  $\varphi$  is *valid in the model*  $\mathcal{M}$ , denoted  $\mathcal{M} \models \varphi$ , if  $\mathcal{M}, w \models \varphi$  for all  $w \in W$ . A formula  $\varphi$  is *valid in the frame*  $\mathcal{F}$ , denoted  $\mathcal{F} \models \varphi$ , if  $\langle \mathcal{F}, V \rangle \models \varphi$  for all  $V : \Phi \rightarrow \mathcal{P}(W)$ . If  $\Psi$  is a set of formulas such that  $\mathcal{F} \models \varphi$  for all  $\varphi \in \Psi$ , we use the notation  $\mathcal{F} \models \Psi$ . Let  $\mathcal{C}$  be a class of Kripke frames or a class of Scott-Montague frames and let  $\Psi$  be a set of formulas. We say that the class  $\mathcal{C}$  is *defined by*  $\Psi$  (or *axiomatized by*  $\Psi$ ) if

$$\mathcal{F} \in \mathcal{C} \text{ iff } \mathcal{F} \models \Psi.$$

In the sequel, when we talk about definability in modal logic we mean *frame-definability* since we are interested in sets of formulas defining classes of frames, not classes of models.

For example, let  $\mathcal{C}$  be the class of all Kripke frames  $\langle W, R \rangle$  such that  $R$  is transitive. It is well known that for all Kripke frames  $\mathcal{F}$ ,  $\mathcal{F} \in \mathcal{C}$  if and only if  $\mathcal{F} \models \Box p \rightarrow \Box \Box p$ .

As another example, let  $\mathcal{C}$  be the class of all Scott-Montague frames defined by the formula  $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$ . It is straightforward to verify that for all Scott-Montague frames  $\mathcal{F} = \langle W, F \rangle$ ,  $\mathcal{F} \in \mathcal{C}$  if and only if  $F(X) \cap F(Y) = F(X \cap Y)$  for all  $X, Y \subseteq W$ . Note that when considering classes of Scott-Montague -frames  $\langle W, F \rangle$ , it is often easy to see which property of  $F$  corresponds to the fact that the class is defined by some specific formula. This is always the case in this thesis, as we can see in the later sections.

## 2.2 Boolean functions

We denote by  $\mathbb{B}$  the two-element set  $\{0, 1\}$ . The elements of  $\mathbb{B}^n$ , where  $n \geq 1$ , are called *n-vectors* or *n-tuples*. The *all-zero-vector* and the *all-one-vector* are denoted by  $\mathbf{0}$  and  $\mathbf{1}$ , respectively, that is  $\mathbf{0} = (0, \dots, 0)$  and  $\mathbf{1} = (1, \dots, 1)$ . Let  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{B}^n$ . The *complement* of  $\mathbf{a}$ , denoted by  $\bar{\mathbf{a}}$ , is defined component-wise, i.e.  $\bar{\mathbf{a}} = (1 - a_1, \dots, 1 - a_n)$ . We use the notation  $\mathbf{a}[i]$  for the *i*th component of  $\mathbf{a}$ .

A *Boolean function* is a map  $f : \mathbb{B}^n \rightarrow \mathbb{B}$  for some positive integer  $n$ , where  $n$  is called the *arity* of  $f$ . For example, common *Boolean operations*, such as negation (or complement)  $\bar{x}$ , disjunction  $x \vee y$  and conjunction (or multiplication)  $x \wedge y$ , are Boolean functions. The first one is unary, while the other ones are binary Boolean functions. In the sequel, we also make use of addition (mod 2), which is a binary Boolean operation defined by

$$x \oplus y = (x \wedge \bar{y}) \vee (\bar{x} \wedge y).$$

Note that all these binary operations,  $\wedge$ ,  $\vee$  and  $\oplus$ , are both associative and commutative.

For the constant functions (of any arity) having the value 0 and 1 we use the notations  $\mathbf{0}$  and  $\mathbf{1}$ , respectively. Note that we use  $\mathbf{0}$  and  $\mathbf{1}$  to denote two things, but it is always clear from the context whether we mean a vector or a constant function. For each  $n \geq 1$ , the functions  $(x_1, \dots, x_n) \mapsto x_i$ , for some  $1 \leq i \leq n$ , are called *projections*. For an *n*-ary Boolean function  $f$ , the *complement* of  $f$ , denoted by  $\bar{f}$ , is defined by  $\bar{f}(\mathbf{a}) = 1 - f(\mathbf{a})$  for all  $\mathbf{a} \in \mathbb{B}^n$ . Furthermore, the *dual* of  $f$ , denoted by  $f^d$ , is a function defined by  $f^d(\mathbf{a}) = \bar{f}(\bar{\mathbf{a}})$ , for all  $\mathbf{a} \in \mathbb{B}^n$ .



A *vector-valued Boolean function* is a map  $\mathbb{B}^n \rightarrow \mathbb{B}^m$  for some  $n, m \geq 1$ . It is obvious that each vector-valued Boolean function  $f : \mathbb{B}^n \rightarrow \mathbb{B}^m$  can be associated with the sequence  $(f_1, \dots, f_m)$ , where  $f_i$  is a Boolean function  $\mathbb{B}^n \rightarrow \mathbb{B}$  for each  $1 \leq i \leq m$ , such that for all  $\mathbf{a} \in \mathbb{B}^n$ ,

$$f(\mathbf{a})[i] = f_i(\mathbf{a}).$$

These functions  $f_i$  are called the *component functions*, or simply the *components*, of  $f$ . In the sequel, we only consider vector-valued Boolean functions for which  $n = m$ .

Let  $f$  be an  $n$ -ary Boolean function. For each  $1 \leq i \leq n$ , we say that  $x_i$  is an *essential variable* of  $f$  if there are  $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in \mathbb{B}$  such that

$$f(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n).$$

Otherwise, we say that  $x_i$  is a *dummy variable* of  $f$ . Note that the constant functions are the only Boolean functions that have no essential variables.

A *class* of Boolean functions is simply a set  $K \subseteq \bigcup_{n \geq 1} \mathbb{B}^{\mathbb{B}^n}$ , where  $\mathbb{B}^{\mathbb{B}^n}$  denotes the set of all  $n$ -ary Boolean functions. For a vector-valued Boolean function  $f$ , we say that  $f$  *belongs* to the class  $K$ , if every component of  $f$  belongs to  $K$ .

Let  $f$  be an  $n$ -ary Boolean function and let  $g_1, \dots, g_n$  be  $m$ -ary Boolean functions. The *composition* of  $f$  and  $g_1, \dots, g_n$  is an  $m$ -ary Boolean function  $f(g_1, \dots, g_n)$  defined by  $f(g_1, \dots, g_n)(\mathbf{a}) = f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a}))$ , for all  $\mathbf{a} \in \mathbb{B}^m$ . The notion of composition extends naturally to classes of Boolean functions. Let  $K$  and  $K'$  be classes of Boolean functions. The *class composition* of  $K$  and  $K'$ , denoted by  $K \circ K'$ , is defined by

$$K \circ K' = \{f(g_1, \dots, g_n) \mid f \in K \cap \mathbb{B}^{\mathbb{B}^n}, g_1, \dots, g_n \in K' \cap \mathbb{B}^{\mathbb{B}^m}, n, m \geq 1\}.$$

In this thesis our interest is in classes of Boolean functions which are closed under composition of functions and contain all projections. These classes of Boolean functions are called *Boolean clones*. In the sequel, we call them just clones. For example, the class of all Boolean functions and the class of all projections are clearly clones. These are the largest and the smallest clone, respectively, with respect to the inclusion relation of sets. The clones form a lattice in which the meet is defined as the intersection of clones and the join is defined as the smallest clone containing the union of clones. This lattice is called the *Post Lattice* (see Figure 1), named after Emil Post who first classified the set of all clones in [10].

In the thesis, we are especially interested in the following clones:

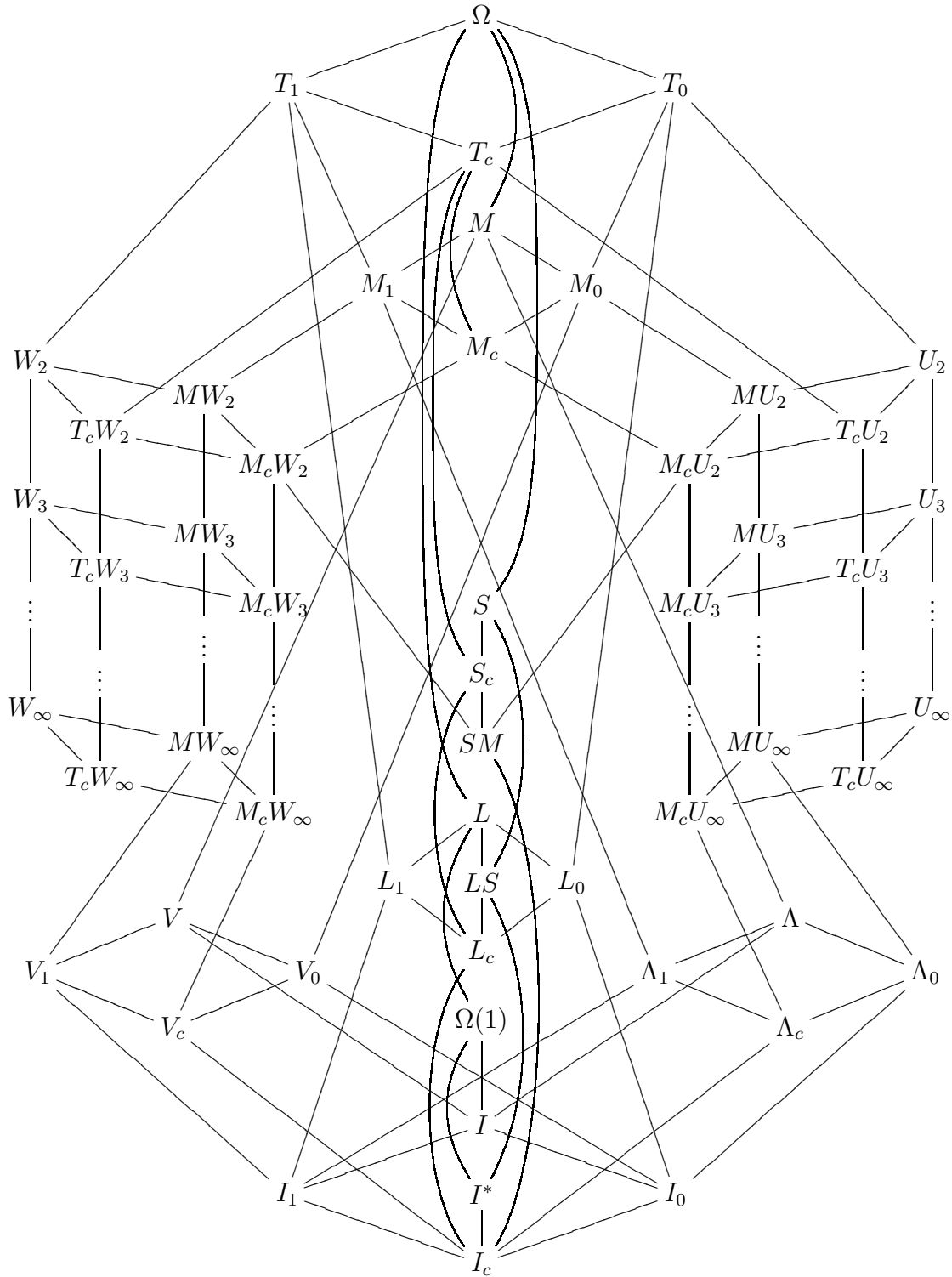


Figure 1: The Post Lattice.

- $\Omega$ , the class of all Boolean functions,
- $T_0$ , the class of all 0-preserving functions, i.e.  
 $T_0 = \{f \in \Omega \mid f(0, \dots, 0) = 0\}$ ,
- $T_1$ , the class of all 1-preserving functions, i.e.  
 $T_1 = \{f \in \Omega \mid f(1, \dots, 1) = 1\}$ ,
- $T_c$ , the class of all constant-preserving functions, i.e.  $T_c = T_0 \cap T_1$ ,
- $S$ , the class of all self-dual functions, i.e.  $S = \{f \in \Omega \mid f^d = f\}$ ,
- $L$ , the class of all linear functions, i.e.  
 $L = \{f \in \Omega \mid f(x_1, \dots, x_n) = c_0 \oplus c_1 x_1 \oplus \dots \oplus c_n x_n, \text{ where } n \geq 1$   
and  $c_i \in \mathbb{B} \text{ for all } 0 \leq i \leq n\}$ ,
- $L_0 = L \cap T_0, L_1 = L \cap T_1, L_c = L \cap T_c, LS = L \cap S$ ,
- $\Lambda$ , the class of all conjunctions and constants, i.e.  
 $\Lambda = \{f \in \Omega \mid f = \mathbf{0}, f = \mathbf{1} \text{ or } f(x_1, \dots, x_n) = x_{i_1} \wedge \dots \wedge x_{i_m}, \text{ where } n \geq 1,$   
 $1 \leq m \leq n \text{ and } x_{i_j} \in \{x_1, \dots, x_n\} \text{ for all } 1 \leq j \leq m\}$ ,
- $\Lambda_0 = \Lambda \cap T_0, \Lambda_1 = \Lambda \cap T_1, \Lambda_c = \Lambda \cap T_c$ ,
- $V$ , the class of all disjunctions and constants, i.e.  
 $V = \{f \in \Omega \mid f = \mathbf{0}, f = \mathbf{1} \text{ or } f(x_1, \dots, x_n) = x_{i_1} \vee \dots \vee x_{i_m}, \text{ where } n \geq 1,$   
 $1 \leq m \leq n \text{ and } x_{i_j} \in \{x_1, \dots, x_n\} \text{ for all } 1 \leq j \leq m\}$ ,
- $V_0 = V \cap T_0, V_1 = V \cap T_1, V_c = V \cap T_c$ ,
- $\Omega(1)$ , the class of all projections, complements of projections and constants,
- $I^*$ , the class of all projections and complements of projections,
- $I$ , the class of all projections and constants,
- $I_0 = I \cap T_0, I_1 = I \cap T_1, I_c = I \cap T_c$ .

Note that  $I_c$  is the smallest clone and it contains only projections.

**Remark 1.** The only functions in  $\Lambda$  having essential variables are conjunctions, and we have that for all  $f \in \Lambda \cap \mathbb{B}^{\mathbb{B}^n}$  and for all  $\mathbf{a} \in \mathbb{B}^n$ ,

$f(\mathbf{a}) = 1$  if and only if  $f \neq \mathbf{0}$  and  $\mathbf{a}[j] = 1$  for all essential variables  $x_j$  of  $f$ .

Since  $\mathbf{0} \notin \Lambda_1$ , we also have that for all  $f \in \Lambda_1 \cap \mathbb{B}^{\mathbb{B}^n}$  and for all  $\mathbf{a} \in \mathbb{B}^n$ ,

$f(\mathbf{a}) = 1$  if and only if  $\mathbf{a}[j] = 1$  for all essential variables  $x_j$  of  $f$ .

## 2.3 Term definable classes of Boolean functions

Let  $X$  be a countable set of *vector-variable symbols*  $\mathbf{x}_1, \mathbf{x}_2, \dots$ . A *Boolean formula* is a formal expression, which is defined inductively as follows:

- (1) All vector-variable symbols  $\mathbf{x}_i \in X$  and constant functions  $\mathbf{0}$  and  $\mathbf{1}$  are Boolean formulas.
- (2) If  $G_1$  and  $G_2$  are Boolean formulas, then  $\neg G_1$  and  $(G_1 \wedge G_2)$  are Boolean formulas.

We also make use of other connectives, like  $\vee$ ,  $\rightarrow$  and  $\leftrightarrow$ , and they are defined by using the connectives  $\neg$  and  $\wedge$  as usually. If the vector-variable symbols occurring in  $G$  are among  $\mathbf{x}_1, \dots, \mathbf{x}_n$ , we write  $G = G(\mathbf{x}_1, \dots, \mathbf{x}_n)$  and say that  $G$  has an arity  $n$ . In the sequel, we may also denote vector-variable symbols by  $\mathbf{x}$  and  $\mathbf{y}$ .

Boolean formulas can be interpreted in the set  $\mathbb{B}^n$  for any  $n \geq 1$ . Each Boolean formula  $G$  defines a Boolean function if  $G$  is interpreted in  $\mathbb{B}$ . For example, let  $G = \mathbf{x}_1 \wedge \mathbf{x}_2$ . If  $G$  is interpreted in  $\mathbb{B}$ , it defines a binary Boolean function  $f$  such that  $f(x_1, x_2) = x_1 \wedge x_2$  for all  $x_1, x_2 \in \mathbb{B}$ . On the other hand, if  $G$  is interpreted in  $\mathbb{B}^n$ , it defines a vector-valued Boolean function  $g : \mathbb{B}^{2n} \rightarrow \mathbb{B}^n$  such that for all  $\mathbf{a}, \mathbf{b} \in \mathbb{B}^n$  and for all  $1 \leq i \leq n$ ,  $g(\mathbf{a}, \mathbf{b})[i] = \mathbf{a}[i] \wedge \mathbf{b}[i]$ . In general, every connective in  $G$  is interpreted component-wise. The negation in Boolean formulas is interpreted as the complement.

A *functional term* is a formal expression

$$T = G(\mathbf{f}(G_1(\mathbf{x}_1, \dots, \mathbf{x}_r)), \dots, \mathbf{f}(G_m(\mathbf{x}_1, \dots, \mathbf{x}_r))),$$

where  $m, r \geq 1$ ,  $G$  is an  $m$ -ary Boolean formula,  $G_1, \dots, G_m$  are  $r$ -ary Boolean formulas,  $\mathbf{x}_1, \dots, \mathbf{x}_r$  are  $r$  distinct vector-variable symbols and  $\mathbf{f}$  is a *function symbol*. For example,  $\mathbf{f}(\mathbf{x}) \rightarrow \mathbf{f}(\mathbf{x} \vee \mathbf{y})$  and  $\neg \mathbf{f}(\mathbf{0}) \wedge (\mathbf{f}(\mathbf{x} \wedge \mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}))$  are functional terms, but  $\mathbf{f}(\mathbf{x} \vee \mathbf{y}) \rightarrow (\mathbf{x} \vee \mathbf{y})$  is not. Note that functional terms and uniform degree-1 formulas are, in a certain sense, of the same form. We will come back to this connection in Section 3. To emphasize this connection, we denote the complement by the negation in the Boolean formulas.

For each  $n \geq 1$ , the functional term  $T$  defines a map  $T : \mathbb{B}^{\mathbb{B}^n} \times \mathbb{B}^{nr} \rightarrow \mathbb{B}$  given by

$$T(f, \mathbf{a}_1, \dots, \mathbf{a}_r) = G(f(G_1(\mathbf{a}_1, \dots, \mathbf{a}_r)), \dots, f(G_m(\mathbf{a}_1, \dots, \mathbf{a}_r))),$$

where we interpret the Boolean formula  $G$  in  $\mathbb{B}$  and the Boolean formulas  $G_i$ ,  $1 \leq i \leq m$ , in  $\mathbb{B}^n$ .

We say that an  $n$ -ary Boolean function  $f$  *satisfies* the functional term  $T$ , denoted by  $T(f) \equiv 1$ , if

$$T(f, \mathbf{a}_1, \dots, \mathbf{a}_r) = 1$$

for all  $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{B}^n$ . For background, see [5, 9]. A class  $K$  of Boolean functions is *defined by* a set  $\mathcal{T}$  of functional terms, denoted by  $K = \langle \mathcal{T} \rangle$ , if  $K$  is the class of all Boolean functions satisfying every member of  $\mathcal{T}$ . If  $\mathcal{T} = \{T\}$ , then we write  $K = \langle T \rangle$ , instead of  $K = \langle \{T\} \rangle$ . A class  $K$  is *term definable* if  $K = \langle \mathcal{T} \rangle$  for some set  $\mathcal{T}$  of functional terms. The following theorem gives a characterization for a class of Boolean functions to be term definable. The result was first obtained by Ekin, Foldes, Hammer and Hellerstein in [4].

**Theorem 1.** *A class  $K$  of Boolean functions is term definable if and only if  $K \circ I_c = K$ .*

The theorem above states that a class  $K$  is term definable if and only if  $K$  is closed under permutation and identification of variables.

Note that, since clones contain all the projections and are closed under composition of functions, clones are in particular term definable. In the following list we give the sets of defining functional terms for the clones that we are considering in this thesis. For a complete list of defining functional terms for each clone, see [5].

- $\Omega = \langle \mathbf{1} \rangle$ ,
- $T_0 = \langle \neg \mathbf{f}(\mathbf{0}) \rangle$ ,  $T_1 = \langle \mathbf{f}(\mathbf{1}) \rangle$ ,
- $T_c = \langle \{ \neg \mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{1}) \} \rangle$ ,
- $S = \langle \neg \mathbf{f}(\mathbf{x}) \leftrightarrow \mathbf{f}(\neg \mathbf{x}) \rangle$ ,
- $L = \langle \mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}) \rangle$ ,
- $L_0 = \langle \mathbf{1} \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}) \rangle$ ,
- $L_1 = \langle \{ \mathbf{f}(\mathbf{1}), \mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}) \} \rangle$ ,
- $L_c = \langle \{ \mathbf{f}(\mathbf{1}), \mathbf{1} \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}) \} \rangle$ ,
- $LS = \langle \{ \mathbf{f}(\mathbf{1}) \oplus \mathbf{f}(\mathbf{0}), \mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}) \} \rangle$ ,
- $\Lambda = \langle \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}) \rangle$ ,
- $\Lambda_0 = \langle \{ \neg \mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}) \} \rangle$ ,
- $\Lambda_1 = \langle \{ \mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}) \} \rangle$ ,

- $\Lambda_c = \langle \{\neg \mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y})\} \rangle$ ,
- $V = \langle \mathbf{f}(\mathbf{x}) \vee \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \vee \mathbf{y}) \rangle$ ,
- $V_0 = \langle \{\neg \mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{x}) \vee \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \vee \mathbf{y})\} \rangle$ ,
- $V_1 = \langle \{\mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \vee \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \vee \mathbf{y})\} \rangle$ ,
- $V_c = \langle \{\neg \mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \vee \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \vee \mathbf{y})\} \rangle$ ,
- $\Omega(1) = \langle \{\mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}),$   
 $(\mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{x} \wedge \mathbf{y})) \rightarrow (\mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}))\} \rangle$ ,
- $I^* = \langle \{\mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{1}), \mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}),$   
 $(\mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{x} \wedge \mathbf{y})) \rightarrow (\mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}))\} \rangle$ ,
- $I = \langle \{\mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}), \mathbf{f}(\mathbf{x}) \vee \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \vee \mathbf{y})\} \rangle$ ,
- $I_0 = \langle \{\neg \mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{x}) \vee \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \vee \mathbf{y}), \mathbf{f}(\neg \mathbf{x}) \rightarrow \neg \mathbf{f}(\mathbf{x})\} \rangle$ ,
- $I_1 = \langle \{\mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}), \neg \mathbf{f}(\mathbf{x}) \rightarrow \mathbf{f}(\neg \mathbf{x})\} \rangle$ ,
- $I_c = \langle \{\mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}), \neg \mathbf{f}(\mathbf{x}) \leftrightarrow \mathbf{f}(\neg \mathbf{x})\} \rangle$ .

Note that if a class  $K$  is defined by a finite set  $\mathcal{T}$  of functional terms, then it is defined by a single functional term which is obtained by taking the conjunction of the functional terms in  $\mathcal{T}$ . For example,

$$\Lambda_1 = \langle \mathbf{f}(\mathbf{1}) \wedge (\mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y})) \rangle.$$

### 3 General correspondence

In this section we will show that definability of Boolean functions by functional terms corresponds exactly to definability of finite Scott-Montague frames by uniform degree-1 formulas. We will make use of the natural bijection between vector-valued Boolean functions  $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$  and set functions  $F : \mathcal{P}(W) \rightarrow \mathcal{P}(W)$ , where  $W$  is a set with  $n$  elements. By using this natural bijection, finite Scott-Montague frames can be interpreted by Boolean functions and this correspondence can be used to define certain classes of frames by using known theories of Boolean function classes. We will only consider finite frames and throughout this section we fix  $W$  to be the  $n$ -element set  $\{w_1, \dots, w_n\}$ .

Let  $X \subseteq W$  and  $\mathbf{x} \in \mathbb{B}^n$ . By  $\mathbf{a}_X$  we denote the characteristic vector of  $X$ , i.e. for all  $1 \leq i \leq n$ ,

$$\mathbf{a}_X[i] = 1 \text{ if and only if } w_i \in X.$$

Conversely, by  $A_{\mathbf{x}}$  we denote the set whose characteristic vector is  $\mathbf{x}$ . It is immediate that  $\mathbf{a}_{A_{\mathbf{x}}} = \mathbf{x}$  and  $A_{\mathbf{a}_X} = X$ .

Let  $f = (f_1, \dots, f_n)$  be a vector-valued Boolean function  $\mathbb{B}^n \rightarrow \mathbb{B}^n$ . We define  $F_f$  to be the function  $\mathcal{P}(W) \rightarrow \mathcal{P}(W)$  such that  $F_f(X) = A_{f(\mathbf{a}_X)}$  for all  $X \subseteq W$ . That is, for all  $X \subseteq W$  and  $w_i \in W$ ,

$$w_i \in F_f(X) \text{ if and only if } f_i(\mathbf{a}_X) = 1.$$

We denote by  $\mathcal{F}_f$  the Scott-Montague frame  $\langle W, F_f \rangle$ .

Let  $\mathcal{F} = \langle W, F \rangle$  be a Scott-Montague frame. We denote by  $f_{\mathcal{F}}$  the vector-valued Boolean function  $\mathbb{B}^n \rightarrow \mathbb{B}^n$  for which  $f_{\mathcal{F}}(\mathbf{x}) = \mathbf{a}_{F(A_{\mathbf{x}})}$  for all  $\mathbf{x} \in \mathbb{B}^n$ . In other words, for all  $\mathbf{x} \in \mathbb{B}^n$  and for all  $1 \leq i \leq n$ ,

$$f_i(\mathbf{x}) = 1 \text{ if and only if } w_i \in F(A_{\mathbf{x}}),$$

where  $f_i$  is the  $i$ th component of  $f_{\mathcal{F}}$ . We also use a notation  $f_{\mathcal{F}} = f_{\mathcal{F}}$ .

It is immediate from these definitions that  $f_{F_f} = f$  and  $F_{f_{\mathcal{F}}} = F$ .

Let  $T$  be a functional term, let  $f = (f_1, \dots, f_n)$  be a vector-valued Boolean function  $\mathbb{B}^n \rightarrow \mathbb{B}^n$  and let  $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{B}^n$ . By  $T(f, \mathbf{a}_1, \dots, \mathbf{a}_r)$  we denote the  $n$ -vector whose  $i$ th component is  $T(f_i, \mathbf{a}_1, \dots, \mathbf{a}_r)$ . We write  $T(f) \equiv \mathbf{1}$  when  $T(f, \mathbf{a}_1, \dots, \mathbf{a}_r) = \mathbf{1}$  for all  $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{B}^n$ . In other words,  $T(f) \equiv \mathbf{1}$  if and only if  $T(f_i, \mathbf{a}_1, \dots, \mathbf{a}_r) = 1$  for all  $1 \leq i \leq n$  and for all  $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{B}^n$ .

In order to establish the desired connection between term definable classes of Boolean functions and classes of Scott-Montague frames definable by uniform degree-1 formulas, we need a translation of functional terms into uniform degree-1 formulas and, conversely, of uniform degree-1 formulas into

functional terms. We will first define a translation of functional terms  $T$  into uniform degree-1 formulas  $\psi_T$  such that for every vector-valued Boolean function  $f$ ,  $T(f) \equiv \mathbf{1}$  if and only if  $\mathcal{F}_f \models \psi_T$ . The idea of the translation is very simple; we just replace vector-variable symbols  $\mathbf{x}_i$  by proposition symbols  $p_i$  and the function symbol  $\mathbf{f}$  by the modal operator  $\Box$ , and all the connectives remain the same. The exact definition of the translation is done in two parts. First we define a translation for Boolean formulas and then, using this, we define a translation for functional terms.

Let  $G = G(\mathbf{x}_1, \dots, \mathbf{x}_r)$  be an  $r$ -ary Boolean formula. The *translation* of  $G$ , denoted by  $\psi_G$ , is defined inductively as follows:

- If  $G = \mathbf{0}$  (or  $\mathbf{1}$ ), then  $\psi_G = \perp$  (or  $\top$  respectively).
- If  $G(\mathbf{x}_1, \dots, \mathbf{x}_r) = \mathbf{x}_i$  for some  $1 \leq i \leq r$ , then  $\psi_G = p_i$ .
- If  $G = \neg H$ , then  $\psi_G = \neg\psi_H$ .
- If  $G = H_1 \wedge H_2$ , then  $\psi_G = \psi_{H_1} \wedge \psi_{H_2}$ .

Let  $T = G(\mathbf{f}(G_1(\mathbf{x}_1, \dots, \mathbf{x}_r)), \dots, \mathbf{f}(G_m(\mathbf{x}_1, \dots, \mathbf{x}_r)))$  be a functional term as defined in Section 2.3, let  $\psi_G(p_1, \dots, p_m)$  be the translation of  $G$  and let  $\psi_{G_i}$  be the translation of  $G_i$  for each  $1 \leq i \leq m$ . The *translation* of  $T$ , denoted by  $\psi_T$ , is defined as

$$\psi_T = \psi_G(\Box\psi_{G_1}, \dots, \Box\psi_{G_m}).$$

For example, let  $T = \mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2) \rightarrow (\mathbf{f}(\mathbf{x}_1) \wedge \mathbf{f}(\mathbf{x}_2))$ . Then

$$\psi_T = \Box(p_1 \wedge p_2) \rightarrow (\Box p_1 \wedge \Box p_2).$$

Clearly, the translation of a functional term yields a uniform degree-1 formula.

Let  $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_r) \in \mathbb{B}^{nr}$ . We denote by  $V_{\mathbf{a}}$  the valuation function  $\Phi \rightarrow \mathcal{P}(W)$  such that  $V_{\mathbf{a}}(p_j) = A_{\mathbf{a}_j}$  for all  $1 \leq j \leq r$ .

**Lemma 1.** *Let  $G$  be an  $r$ -ary Boolean formula and let  $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_r) \in \mathbb{B}^{nr}$ . Let  $\mathcal{M} = \langle W, F, V_{\mathbf{a}} \rangle$  be a Scott-Montague model. Then for all  $1 \leq i \leq n$ ,*

$$G(\mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1 \text{ if and only if } \mathcal{M}, w_i \models \psi_G,$$

*i.e.  $\|\psi_G\|^{\mathcal{M}} = A_{\mathbf{b}}$ , where  $\mathbf{b} = G(\mathbf{a}_1, \dots, \mathbf{a}_r)$ .*



*Proof.* The proof is by induction on the construction of  $G$ . If  $G$  is a constant  $\mathbf{0}$  (or  $\mathbf{1}$ ) then  $\psi_G$  is  $\perp$  (or  $\top$  respectively) and the result is trivial. If  $G(\mathbf{x}_1, \dots, \mathbf{x}_r) = \mathbf{x}_j$  for some  $1 \leq j \leq r$  then  $\psi_G = p_j$  and

$$G(\mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1 \Leftrightarrow \mathbf{a}_j[i] = 1 \Leftrightarrow w_i \in A_{\mathbf{a}_j} \Leftrightarrow \mathcal{M}, w_i \models p_j \Leftrightarrow \mathcal{M}, w_i \models \psi_G.$$

Let  $G$  be of the form  $\neg H$  and suppose that the claim holds for  $H$ . Then

$$\begin{aligned} G(\mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1 &\Leftrightarrow H(\mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 0 \\ &\stackrel{(*)}{\Leftrightarrow} \mathcal{M}, w_i \not\models \psi_H \\ &\Leftrightarrow \mathcal{M}, w_i \models \neg \psi_H \\ &\Leftrightarrow \mathcal{M}, w_i \models \psi_G, \end{aligned}$$

where the equivalence  $(*)$  holds by the induction hypothesis. In the case where  $G$  is of the form  $H_1 \wedge H_2$ , the result also follows straightforwardly from the induction hypothesis.  $\square$

**Theorem 2.** *Let  $T$  be a functional term*

$$G(\mathbf{f}(G_1(\mathbf{x}_1, \dots, \mathbf{x}_r), \dots, \mathbf{f}(G_m(\mathbf{x}_1, \dots, \mathbf{x}_r))))$$

and let  $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_r) \in \mathbb{B}^{nr}$ . Let  $f$  be a vector-valued Boolean function  $\mathbb{B}^n \rightarrow \mathbb{B}^n$ . Then for all  $1 \leq i \leq n$ ,

$$T(f, \mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1 \text{ if and only if } \langle \mathcal{F}_f, V_{\mathbf{a}} \rangle, w_i \models \psi_T.$$

*Proof.* The proof is by induction on the construction of  $G$ . The cases where  $G$  is a constant, or of the form  $\neg H$  or  $H_1 \wedge H_2$ , are handled similarly as in the previous lemma. If  $G(\mathbf{x}_1, \dots, \mathbf{x}_m) = \mathbf{x}_j$  for some  $1 \leq j \leq m$ , then  $T = \mathbf{f}(G_j(\mathbf{x}_1, \dots, \mathbf{x}_r))$  and  $\psi_T = \square \psi_{G_j}$ . Thus

$$\begin{aligned} T(f, \mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1 &\Leftrightarrow f(G_j(\mathbf{a}_1, \dots, \mathbf{a}_r))[i] = 1 \\ &\Leftrightarrow w_i \in A_{f(\mathbf{b})}, \text{ where } \mathbf{b} = G_j(\mathbf{a}_1, \dots, \mathbf{a}_r) \\ &\Leftrightarrow w_i \in F_f(A_{\mathbf{b}}), \text{ where } \mathbf{b} = G_j(\mathbf{a}_1, \dots, \mathbf{a}_r) \\ &\stackrel{(*)}{\Leftrightarrow} w_i \in F_f(\|\psi_{G_j}\|^{\mathcal{M}}), \text{ where } \mathcal{M} = \langle \mathcal{F}_f, V_{\mathbf{a}} \rangle \\ &\Leftrightarrow \langle \mathcal{F}_f, V_{\mathbf{a}} \rangle, w_i \models \square \psi_{G_j} \\ &\Leftrightarrow \langle \mathcal{F}_f, V_{\mathbf{a}} \rangle, w_i \models \psi_T, \end{aligned}$$

where the equivalence  $(*)$  holds by Lemma 1.  $\square$

As an immediate corollary to Theorem 2 we get that the translation  $\psi_T$  works as desired.

**Corollary 1.** *Let  $T$  be a functional term and let  $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_r) \in \mathbb{B}^{nr}$ . Let  $f$  be a vector-valued Boolean function  $\mathbb{B}^n \rightarrow \mathbb{B}^n$ . Then*

$$T(f, \mathbf{a}_1, \dots, \mathbf{a}_r) = \mathbf{1} \text{ if and only if } \langle \mathcal{F}_f, V_{\mathbf{a}} \rangle \models \psi_T,$$

and furthermore

$$T(f) \equiv \mathbf{1} \text{ if and only if } \mathcal{F}_f \models \psi_T. \quad \square$$

Our next aim is to translate uniform degree-1 formulas  $\psi$  into functional terms  $T_\psi$  such that for every Scott-Montague frame  $\mathcal{F}$ ,  $\mathcal{F} \models \psi$  if and only if  $T_\psi(f_{\mathcal{F}}) \equiv \mathbf{1}$ . The idea of the translation is similar to the one introduced above. We just replace the modal operator  $\Box$  in  $\psi$  by the function symbol  $\mathbf{f}$  and proposition symbols  $p_i$  by vector-variable symbols  $\mathbf{x}_i$ . As we will observe later in Theorem 4, the translations  $T \mapsto \psi_T$  and  $\psi \mapsto T_\psi$  are inverses of each other. We will first define a translation of a propositional formula  $\varphi$  into a Boolean formula  $G_\varphi$ , and then using this we define a translation of a uniform degree-1 formula into a functional term.

If  $\varphi$  is a modal formula in which there occur  $r$  different proposition symbols, we always assume for simplicity that those proposition symbols are  $p_1, \dots, p_r$ . Let  $\varphi$  be a propositional formula. The *translation* of  $\varphi$ , denoted by  $G_\varphi$ , is defined inductively as follows:

- If  $\varphi = \perp$  (or  $\top$ ), then  $G_\varphi = \mathbf{0}$  (or  $\mathbf{1}$  respectively).
- If  $\varphi(p_1, \dots, p_r) = p_i$  for some  $1 \leq i \leq r$ , then  $G_\varphi(\mathbf{x}_1, \dots, \mathbf{x}_r) = \mathbf{x}_i$ .
- If  $\varphi = \neg\theta$ , then  $G_\varphi = \neg G_\theta$ .
- If  $\varphi = \theta_1 \wedge \theta_2$ , then  $G_\varphi = G_{\theta_1} \wedge G_{\theta_2}$ .

Let  $\psi = \varphi(\Box\varphi_1, \dots, \Box\varphi_m)$  be a uniform degree-1 formula. The *translation* of  $\psi$ , denoted by  $T_\psi$ , is defined as

$$T_\psi = G_\varphi(\mathbf{f}(G_{\varphi_1}(\mathbf{x}_1, \dots, \mathbf{x}_r)), \dots, \mathbf{f}(G_{\varphi_m}(\mathbf{x}_1, \dots, \mathbf{x}_r))).$$

If, for example,  $\psi = \Box\top \wedge (\Box p_1 \wedge \Box p_2 \leftrightarrow \Box(p_1 \wedge p_2))$ , then

$$T_\psi = \mathbf{f}(\mathbf{1}) \wedge (\mathbf{f}(\mathbf{x}_1) \wedge \mathbf{f}(\mathbf{x}_2) \leftrightarrow \mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2)).$$

Again, it is easy to see that the translation of a uniform degree-1 formula is a functional term.

**Lemma 2.** Let  $\varphi$  be a propositional formula with proposition symbols  $p_1, \dots, p_r$  and let  $\mathcal{M} = \langle W, F, V \rangle$  be a Scott-Montague model. Let  $\mathbf{a}_j = \mathbf{a}_{V(p_j)}$  for all  $1 \leq j \leq r$ . Then for all  $1 \leq i \leq n$ ,

$$\mathcal{M}, w_i \models \varphi \text{ if and only if } G_\varphi(\mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1,$$

i.e.  $G_\varphi(\mathbf{a}_1, \dots, \mathbf{a}_r) = \mathbf{a}_{\|\varphi\|^\mathcal{M}}$ .

*Proof.* The lemma is easily proved by induction on the construction of  $\varphi$ , similarly to the proof of Lemma 1.  $\square$

**Theorem 3.** Let  $\psi$  be a uniform degree-1 formula  $\varphi(\Box\varphi_1, \dots, \Box\varphi_m)$  and let  $p_1, \dots, p_r$  be the proposition symbols which occur in  $\psi$ . Let  $\mathcal{M} = \langle W, F, V \rangle$  be a Scott-Montague model and let  $\mathbf{a}_j = \mathbf{a}_{V(p_j)}$  for all  $1 \leq j \leq r$ . Then for all  $1 \leq i \leq n$ ,

$$\mathcal{M}, w_i \models \psi \text{ if and only if } T_\psi(f_F, \mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1.$$

*Proof.* The proof is by induction on the construction of  $\varphi$ . The cases where  $\varphi$  is a constant  $\perp$  or  $\top$ , or of the form  $\neg\theta$  or  $\theta_1 \wedge \theta_2$ , are straightforward. If  $\varphi(p_1, \dots, p_m) = p_j$  for some  $1 \leq j \leq m$ , then  $\psi = \Box\varphi_j$  and therefore  $T_\psi = \mathbf{f}(G_{\varphi_j}(\mathbf{x}_1, \dots, \mathbf{x}_r))$ . Hence

$$\begin{aligned} \mathcal{M}, w_i \models \psi &\Leftrightarrow \mathcal{M}, w_i \models \Box\varphi_j \\ &\Leftrightarrow w_i \in F(\|\varphi_j\|^\mathcal{M}) \\ &\Leftrightarrow f_F(\mathbf{b})[i] = 1, \text{ where } \mathbf{b} = \mathbf{a}_{\|\varphi_j\|^\mathcal{M}} \\ &\stackrel{(*)}{\Leftrightarrow} f_F(G_{\varphi_j}(\mathbf{a}_1, \dots, \mathbf{a}_r))[i] = 1 \\ &\Leftrightarrow T_\psi(f_F, \mathbf{a}_1, \dots, \mathbf{a}_r)[i] = 1, \end{aligned}$$

where the equivalence  $(*)$  holds by Lemma 2.  $\square$

From Theorem 3 we get immediately the intended corollary:

**Corollary 2.** Let  $\psi$  be a uniform degree-1 formula. Let  $\mathcal{F} = \langle W, F \rangle$  be a Scott-Montague frame and let  $V$  be a function  $\Phi \rightarrow \mathcal{P}(W)$ . Let  $p_1, \dots, p_r$  be the proposition symbols which occur in  $\psi$  and let  $\mathbf{a}_j = \mathbf{a}_{V(p_j)}$  for all  $1 \leq j \leq r$ . Then

$$\langle \mathcal{F}, V \rangle \models \psi \text{ if and only if } T_\psi(f_F, \mathbf{a}_1, \dots, \mathbf{a}_r) = \mathbf{1},$$

and furthermore

$$\mathcal{F} \models \psi \text{ if and only if } T_\psi(f_F) \equiv \mathbf{1}. \quad \square$$

The following lemma and theorem show that the translations  $T \mapsto \psi_T$  and  $\psi \mapsto T_\psi$  are inverses of each other. Thus there is a one-to-one correspondence between functional terms and uniform degree-1 formulas.

**Lemma 3.** *Let  $G$  be a propositional term and let  $\varphi$  be a modal formula of degree 0. Then  $G_{\psi_G} = G$  and  $\psi_{G_\varphi} = \varphi$ .*

*Proof.* Both claims are proved by a straightforward induction.  $\square$

**Theorem 4.** *Let  $T$  be a functional term*

$$G(\mathbf{f}(G_1(\mathbf{x}_1, \dots, \mathbf{x}_r), \dots, \mathbf{f}(G_m(\mathbf{x}_1, \dots, \mathbf{x}_r))))$$

*and let  $\psi$  be a uniform degree-1 formula*

$$\varphi(\Box\varphi_1, \dots, \Box\varphi_m).$$

*Then  $T_{\psi_T} = T$  and  $\psi_{T_\psi} = \psi$ .*

*Proof.* The first claim is proved by induction on the construction of  $G$ . The cases where  $G$  is a constant, or of the form  $\neg H$  or  $H_1 \wedge H_2$ , are straightforward. If  $G(\mathbf{x}_1, \dots, \mathbf{x}_m) = \mathbf{x}_j$  for some  $1 \leq j \leq m$ , then  $T = \mathbf{f}(G_j(\mathbf{x}_1, \dots, \mathbf{x}_r))$  and hence  $\psi_T = \Box\psi_{G_j}$ . Therefore

$$\begin{aligned} T_{\psi_T} &= \mathbf{f}(G_{\psi_{G_j}}(\mathbf{x}_1, \dots, \mathbf{x}_r)) \\ &\stackrel{(*)}{=} \mathbf{f}(G_j(\mathbf{x}_1, \dots, \mathbf{x}_r)) \\ &= T, \end{aligned}$$

where the equality  $(*)$  holds by Lemma 3.

The second claim of the theorem is proved similarly by induction on the construction of  $\varphi$ .  $\square$

Our main result is that definability of Boolean functions by functional terms corresponds exactly to definability of Scott-Montague frames by uniform degree-1 formulas. With this purpose in mind, we define a class of Scott-Montague frames *corresponding* to a class of Boolean functions and a class of Boolean functions *corresponding* to a class of Scott-Montague frames. Let  $K$  be a class of Boolean functions and let  $\mathcal{C}$  be a class of Scott-Montague frames. We denote by  $\mathcal{C}_K$  the class of Scott-Montague frames

$$\{\mathcal{F}_f \mid f \text{ is a vector-valued Boolean function such that } f \in K\},$$

and  $K_{\mathcal{C}}$  denotes the class of Boolean functions

$$\{g \in \Omega \mid g \text{ is a component of } f_{\mathcal{F}}, \text{ where } \mathcal{F} \in \mathcal{C}\}.$$

Note that  $\mathcal{C}_K$  defines a class operation from the classes of Boolean functions to the classes of finite Scott-Montague frames, namely the class operation which assigns to every class  $K$  of Boolean functions the class  $\mathcal{C}_K$  of Scott-Montague frames. Similarly,  $K_{\mathcal{C}}$  defines a class operation from the classes of finite Scott-Montague frames to the classes of Boolean functions.

**Theorem 5.** (a) Let  $K$  be the class of Boolean functions defined by a set  $\mathcal{T}$  of functional terms. Then the class  $\mathcal{C}_K$  is defined by  $\Psi_{\mathcal{T}}$ , where  $\Psi_{\mathcal{T}} = \{\psi_T \mid T \in \mathcal{T}\}$ .

(b) Let  $\mathcal{C}$  be the class of Scott-Montague frames defined by a set  $\Psi$  of uniform degree-1 formulas. Then the class  $K_{\mathcal{C}}$  is defined by  $\mathcal{T}_{\Psi}$ , where  $\mathcal{T}_{\Psi} = \{T_{\psi} \mid \psi \in \Psi\}$ .

*Proof.* For (a), let  $\mathcal{F} = \langle W, F \rangle$  be a Scott-Montague frame. If  $\mathcal{F} \in \mathcal{C}_K$  then  $\mathcal{F} = \mathcal{F}_f$  for some vector-valued Boolean function  $f \in K$ . Now  $T(f) \equiv \mathbf{1}$  for all  $T \in \mathcal{T}$ , and it follows from Corollary 1 that  $\mathcal{F} \models \psi_T$  for all  $T \in \mathcal{T}$ . Therefore  $\mathcal{F} \models \Psi_{\mathcal{T}}$ . Suppose then that  $\mathcal{F} \models \Psi_{\mathcal{T}}$ . Now  $\mathcal{F} \models \psi_T$  for all  $T \in \mathcal{T}$  and we get from Corollary 2 that  $T_{\psi_T}(f_F) \equiv \mathbf{1}$  for all  $T \in \mathcal{T}$ . Since  $T_{\psi_T} = T$  by Theorem 4, we have that  $T(f_F) \equiv \mathbf{1}$  for all  $T \in \mathcal{T}$ . Therefore  $f_F \in K$ , and since  $F = F_{f_F}$  we conclude that  $\mathcal{F} = \mathcal{F}_{f_F} \in \mathcal{C}_K$ .

For (b), suppose first that  $g \in K_{\mathcal{C}}$ . Now  $g$  is a component of a vector-valued Boolean function  $f_{\mathcal{F}}$  where  $\mathcal{F} \in \mathcal{C}$ . Since  $\mathcal{F} \models \psi$  for all  $\psi \in \Psi$ , it follows from Corollary 2 that  $T_{\psi}(f_{\mathcal{F}}) \equiv \mathbf{1}$  for all  $\psi \in \Psi$ . Since  $g$  is a component of  $f_{\mathcal{F}}$ , we get that  $T_{\psi}(g) \equiv \mathbf{1}$  for all  $\psi \in \Psi$ . Suppose then that  $T_{\psi}(g) \equiv \mathbf{1}$  for all  $\psi \in \Psi$ . Let  $\psi \in \Psi$  and let  $g'$  be a vector-valued Boolean function all of which components are  $g$ . Then  $T_{\psi}(g') \equiv \mathbf{1}$  and it follows from Corollary 1 that  $\mathcal{F}_{g'} \models \psi_{T_{\psi}}$ . By Theorem 4, we know that  $\psi_{T_{\psi}} = \psi$ , and therefore  $\mathcal{F}_{g'} \models \psi$ . So  $\mathcal{F}_{g'} \models \Psi$  and hence  $\mathcal{F}_{g'} \in \mathcal{C}$ . Since  $g$  is a component of  $g'$  and  $g' = f_{\mathcal{F}_{g'}}$ , we conclude that  $g \in K_{\mathcal{C}}$ .  $\square$

The following theorem states that the class operations based on  $\mathcal{C}_K$  and  $K_{\mathcal{C}}$  are inverses of each other.

**Theorem 6.** (a) Let  $K$  be a class of Boolean functions defined by a set  $\mathcal{T}$  of functional terms. Then  $K_{\mathcal{C}_K} = K$ .

(b) Let  $\mathcal{C}$  be a class of Scott-Montague frames defined by a set  $\Psi$  of uniform degree-1 formulas. Then  $\mathcal{C}_{K_{\mathcal{C}}} = \mathcal{C}$ .

*Proof.* To prove (a), let  $g$  be a Boolean function. Since the class  $K$  is defined by  $\mathcal{T}$ , it follows from Theorem 5 (a) that the class  $\mathcal{C}_K$  is defined by the set  $\Psi_{\mathcal{T}}$ . Thus

$$\begin{aligned} g \in K_{\mathcal{C}_K} &\stackrel{(1)}{\Leftrightarrow} T_{\psi}(g) \equiv \mathbf{1} \text{ for all } \psi \in \Psi_{\mathcal{T}} \\ &\Leftrightarrow T_{\psi_T}(g) \equiv \mathbf{1} \text{ for all } T \in \mathcal{T} \\ &\stackrel{(2)}{\Leftrightarrow} T(g) \equiv \mathbf{1} \text{ for all } T \in \mathcal{T} \\ &\Leftrightarrow g \in K, \end{aligned}$$

where the equivalence (1) holds by Theorem 5 (b) and (2) by Theorem 4. Hence  $K_{\mathcal{C}_K} = K$ .

The proof of (b) is similar. □

To summarize, Theorems 5 and 6 together with Theorem 4 tell us that a class  $\mathcal{C}$  of Scott-Montague frames is definable by uniform degree-1 formulas if and only if the corresponding class  $K_{\mathcal{C}}$  of Boolean functions is defined by functional terms. Moreover, the defining functional terms are simply the translations of the defining formulas. Similarly a class  $K$  of Boolean functions is definable by functional terms if and only if the corresponding class  $\mathcal{C}_K$  of Scott-Montague frames is definable by uniform degree-1 formulas, and the defining formulas are the translations of the defining functional terms.

Theorem 1 gives a characterization for definability of classes of Boolean functions by functional terms. Using Theorems 1 and 5, one can easily derive a similar characterization for definability of classes of Scott-Montague frames by uniform degree-1 formulas. That is, a class  $\mathcal{C}$  of Scott-Montague frames  $\mathcal{F}$  is definable by a set of uniform degree-1 formulas if and only if  $\mathcal{C}$  satisfies the following closure condition:

If every component  $f_i$  of  $f_{\mathcal{F}}$  can be represented as  $f_i = g(x_{j_1}, \dots, x_{j_n})$  where  $g$  is a component of  $f_{\mathcal{F}'}$  for some  $\mathcal{F}' \in \mathcal{C}$  and  $x_{j_1}, \dots, x_{j_n}$  are projections, then  $\mathcal{F} \in \mathcal{C}$ .

This condition can be formulated in terms of the set function  $F$ , without using the component functions, but in that case the formulation is very technical and therefore more difficult to understand.

## 4 Kripke correspondence

In addition to the general correspondence, there are also connections between classes of Kripke frames and Boolean clones. This is not so surprising since Kripke frames are just special cases of Scott-Montague frames. In Section 4.1 we show that several classes of Kripke frames correspond to classes of Scott-Montague frames which are defined by the translations  $\psi_T$  of functional terms  $T$  defining some Boolean clones. In Section 4.2 we get further correspondences by adding so-called *non-normal worlds* to Kripke frames. Also in this section, we fix  $W$  to be the set  $\{w_1, \dots, w_n\}$ .

### 4.1 Standard Kripke frames

Let  $\mathcal{M}$  and  $\mathcal{M}'$  be Kripke or Scott-Montague models which share a common universe  $W$  and a common valuation function  $V$ . We say that the models  $\mathcal{M}$  and  $\mathcal{M}'$  are *pointwise equivalent* if for all  $w \in W$  and for all formulas  $\varphi$ ,

$$\mathcal{M}, w \models \varphi \text{ if and only if } \mathcal{M}', w \models \varphi.$$

It is well known that there is a one-to-one correspondence between Kripke frames and so-called *augmented* Scott-Montague frames such that for all valuations the models based on these frames are pointwise equivalent. We will give this correspondence on finite frames in the next two lemmas and Proposition 1. In this section we consider only finite frames, in which case augmented Scott-Montague frames are just frames in which the axioms  $\Box\top$  and  $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$  are valid. If we would also consider infinite frames, then we would need an additional condition on Scott-Montague frames. For the correspondence and more information on augmented Scott-Montague frames, see [2].

**Lemma 4.** *Let  $\langle W, R \rangle$  be a Kripke frame and let  $\langle W, F_R \rangle$  be a Scott-Montague frame where  $F_R$  is given by*

$$F_R(X) = \{w \in W \mid \forall v \in W : wRv \Rightarrow v \in X\} \text{ for all } X \subseteq W.$$

*Then for all valuations  $V$ , the models  $\langle W, R, V \rangle$  and  $\langle W, F_R, V \rangle$  are pointwise equivalent. Furthermore the axioms  $\Box\top$  and  $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$  are valid in the frame  $\langle W, F_R \rangle$ .*

*Proof.* Let  $\mathcal{M} = \langle W, R, V \rangle$  and  $\mathcal{M}' = \langle W, F_R, V \rangle$ . To prove the first claim we show that for all formulas  $\varphi$  and for all  $w \in W$ ,

$$\mathcal{M}, w \models \varphi \text{ if and only if } \mathcal{M}', w \models \varphi.$$

The proof is by induction on the construction of  $\varphi$ . It is straightforward to see that the claim holds for constants and proposition symbols. It follows easily from the induction hypothesis that the claim holds also for formulas of the form  $\neg\psi$  and  $\psi_1 \wedge \psi_2$ . Assume that  $\varphi = \Box\psi$ . The induction hypothesis tells us that  $\|\psi\|^{\mathcal{M}} = \|\psi\|^{\mathcal{M}'}$  and hence

$$\begin{aligned} \mathcal{M}, w \models \Box\psi &\Leftrightarrow \forall v \in W : wRv \Rightarrow v \in \|\psi\|^{\mathcal{M}} \\ &\Leftrightarrow \forall v \in W : wRv \Rightarrow v \in \|\psi\|^{\mathcal{M}'} \\ &\Leftrightarrow w \in F_R(\|\psi\|^{\mathcal{M}'}) \\ &\Leftrightarrow \mathcal{M}', w \models \Box\psi. \end{aligned}$$

It follows from the induction principle that the claim holds for every formula  $\varphi$ .

It is well known that the axioms  $\Box\top$  and  $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$  are valid in every Kripke frame. Thus they are valid in the frame  $\langle W, R \rangle$  and it follows from the first claim that they are also valid in the frame  $\langle W, F_R \rangle$ .  $\square$

Let  $\langle W, R \rangle$  be a Kripke frame and let  $w \in W$ . By  $R[w]$  we denote the set of  $R$ -successors of  $w$ , i.e.  $R[w] = \{v \in W \mid wRv\}$ .

**Lemma 5.** *Let  $\langle W, F \rangle$  be a Scott-Montague frame which validates the axioms  $\Box\top$  and  $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$  and let  $f = f_F$ . Let  $\langle W, R_F \rangle$  be a Kripke frame where  $R_F$  is given by*

$$R_F[w_i] = \{w_j \in W \mid x_j \text{ is an essential variable of } f_i\} \text{ for all } w_i \in W,$$

where  $f_i$  is the  $i$ th component function of  $f$ . Then for all valuations  $V$ , the models  $\langle W, F, V \rangle$  and  $\langle W, R_F, V \rangle$  are pointwise equivalent.

*Proof.* The clone  $\Lambda_1$  is defined by the functional terms

$$\mathbf{f}(1), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}),$$

and so, by Theorem 5, the class  $\mathcal{C}_{\Lambda_1}$  of Scott-Montague frames is defined by the axioms  $\Box\top$  and  $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$ . Thus  $\langle W, F \rangle \in \mathcal{C}_{\Lambda_1}$ . Let  $f_i$  be a component function of  $f$ . Then  $f_i \in K_{\mathcal{C}_{\Lambda_1}}$  and it follows from Theorem 6 that  $f_i \in \Lambda_1$ . Hence, as observed in Remark 1, for all  $\mathbf{a} \in \mathbb{B}^n$ ,

$$f_i(\mathbf{a}) = 1 \text{ if and only if } \mathbf{a}[j] = 1 \text{ for all essential variables } x_j \text{ of } f_i.$$

Let  $V$  be a valuation and let  $\mathcal{M} = \langle W, F, V \rangle$  and  $\mathcal{M}' = \langle W, R_F, V \rangle$ . We have to show that for every formula  $\varphi$  and for every  $w_i \in W$ ,

$$\mathcal{M}, w_i \models \varphi \text{ if and only if } \mathcal{M}', w_i \models \varphi.$$



We proceed by induction on the construction of  $\varphi$ . Clearly, the claim holds for constants and proposition symbols. Using the induction hypothesis, it is easy to verify that the claim also holds for  $\varphi = \neg\psi$  and  $\varphi = \psi_1 \wedge \psi_2$ . Suppose that  $\varphi = \Box\psi$ . By the induction hypothesis, we have that  $\|\psi\|^{\mathcal{M}} = \|\psi\|^{\mathcal{M}'}$  and we denote by  $A$  the set  $\|\psi\|^{\mathcal{M}}$ . Now

$$\begin{aligned}
\mathcal{M}, w_i \models \Box\psi &\Leftrightarrow w_i \in F(\|\psi\|^{\mathcal{M}}) \\
&\Leftrightarrow w_i \in F(A) \\
&\Leftrightarrow f_i(\mathbf{a}_A) = 1 \\
&\Leftrightarrow \mathbf{a}_A[j] = 1 \text{ for every essential variable } x_j \text{ of } f_i \\
&\Leftrightarrow w_j \in \|\psi\|^{\mathcal{M}} \text{ for every essential variable } x_j \text{ of } f_i \\
&\Leftrightarrow w_j \in \|\psi\|^{\mathcal{M}'} \text{ for every } w_j \in R_F[w_i] \\
&\Leftrightarrow \mathcal{M}', w_i \models \Box\psi.
\end{aligned}$$

By the induction principle it follows that the claim holds for every formula  $\varphi$ , which completes the proof of the lemma.  $\square$

**Lemma 6.** *Let  $\langle W, R \rangle$  be a Kripke frame and let  $F_R$  be defined as in Lemma 4. Let  $f_i$  be the  $i$ th component of  $f_{F_R}$ . Then*

$$x_j \text{ is an essential variable of } f_i \text{ if and only if } w_j \in R[w_i].$$

*Proof.* By Lemma 4 axioms  $\Box\top$  and  $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$  are valid in  $\langle W, F_R \rangle$ . Thus  $\langle W, F_R \rangle \in \mathcal{C}_{\Lambda_1}$  and hence  $f_i \in \Lambda_1$  as shown in the proof of Lemma 5. From the definition of the function  $F_R$  we get that for every  $1 \leq i \leq n$  and  $\mathbf{a} \in \mathbb{B}^n$ ,

$$f_i(\mathbf{a}) = 1 \Leftrightarrow (\forall k \in \{1, \dots, n\} : w_i R w_k \Rightarrow \mathbf{a}[k] = 1).$$

From this fact, one can see that if  $w_j \notin R[w_i]$  then for all  $\mathbf{a} \in \mathbb{B}^n$ , the value of  $f_i(\mathbf{a})$  does not depend on the  $j$ th component of  $\mathbf{a}$ . Hence, if  $w_j \notin R[w_i]$  then  $x_j$  is not an essential variable of  $f_i$ .

Suppose then that  $w_i R w_j$ . Since  $f_i \in \Lambda_1$ , we have that  $f_i(\mathbf{1}) = 1$ . Let  $\mathbf{a} \in \mathbb{B}^n$  such that  $\mathbf{a}[j] = 0$  and  $\mathbf{a}[k] = 1$  for every  $k \neq j$ . Since  $w_i R w_j$  and  $\mathbf{a}[j] = 0$  we get that  $f_i(\mathbf{a}) = 0$ . Thus  $f_i(\mathbf{1}) \neq f_i(\mathbf{a})$  and so  $x_j$  is an essential variable of  $f_i$ .  $\square$

**Proposition 1.** *Let  $\langle W, R \rangle$  be a Kripke frame, let  $\langle W, F \rangle \in \mathcal{C}_{\Lambda_1}$  and let  $F_R$  and  $R_F$  be defined as in Lemmas 4 and 5. Then the translations  $R \mapsto F_R$  and  $F \mapsto R_F$  are inverses of each other, i.e.  $R_{F_R} = R$  and  $F_{R_F} = F$ .*

*Proof.* We show first that  $R_{F_R} = R$ . Let  $w_i, w_j \in W$  and let  $f_i$  be the  $i$ th component of  $f_{F_R}$ . Then

$$\begin{aligned}
w_j \in R_{F_R}[w_i] &\Leftrightarrow x_j \text{ is an essential variable of } f_i \\
&\Leftrightarrow w_j \in R[w_i],
\end{aligned}$$

where the first equivalence holds by the definition of  $R_{F_R}$  and the second equivalence holds by Lemma 6. Therefore  $R_{F_R} = R$ .

To prove the second claim of the proposition, let  $w_i \in W$  and  $X \subseteq W$ . Let  $g = (g_1, \dots, g_n)$  be the vector-valued Boolean function determined by  $F_{R_F}$ , and let  $f_i$  be the  $i$ th component of  $f_F$ . Since  $\langle W, F \rangle \in \mathcal{C}_{\Lambda_1}$ , we know that  $f_i \in \Lambda_1$ . Thus

$$\begin{aligned}
w_i \in F_{R_F}(X) &\Leftrightarrow g_i(\mathbf{a}_X) = 1 \\
&\stackrel{(1)}{\Leftrightarrow} \forall k \in \{1, \dots, n\} : w_i R_F w_k \Rightarrow \mathbf{a}_X[k] = 1 \\
&\stackrel{(2)}{\Leftrightarrow} \mathbf{a}_X[k] = 1 \text{ for every essential variable } x_k \text{ of } f_i \\
&\stackrel{(3)}{\Leftrightarrow} f_i(\mathbf{a}_X) = 1 \\
&\Leftrightarrow w_i \in F(X).
\end{aligned}$$

The equivalence (1) is explained in the proof of Lemma 6, the equivalence (2) holds by the definition of  $R_F$  and the equivalence (3) follows from Remark 1. Hence, we conclude that  $F_{R_F} = F$ .  $\square$

The previous results give rise to the following definition.

Let  $\mathcal{K}$  be a class of Kripke frames and let  $\mathcal{C}$  be a class of Scott-Montague frames. We say that the class  $\mathcal{K}$  *corresponds* to the class  $\mathcal{C}$  if there are operations  $\Gamma$  from the binary relations to the set functions and  $\Delta$  from the set functions to the binary relations such that the following conditions hold.

- If  $\langle W, R \rangle \in \mathcal{K}$ , then  $\langle W, \Gamma(R) \rangle \in \mathcal{C}$ , and for all valuations  $V$ , the models  $\langle W, R, V \rangle$  and  $\langle W, \Gamma(R), V \rangle$  are pointwise equivalent.
- If  $\langle W, F \rangle \in \mathcal{C}$ , then  $\langle W, \Delta(F) \rangle \in \mathcal{K}$ , and for all valuations  $V$ , the models  $\langle W, F, V \rangle$  and  $\langle W, \Delta(F), V \rangle$  are pointwise equivalent.
- The operations  $\Gamma$  and  $\Delta$  are inverses of each other.

Furthermore, we say that the class  $\mathcal{K}$  *corresponds* to a Boolean clone  $C$  if  $\mathcal{K}$  corresponds to the class  $\mathcal{C}_C$  of Scott-Montague frames. From Lemmas 4 and 5 and Proposition 1 we obtain the following proposition by choosing  $\Gamma(R) = F_R$  and  $\Delta(F) = R_F$ , where  $F_R$  and  $R_F$  are defined as in Lemmas 4 and 5.

**Proposition 2.** *Let  $\mathcal{K}$  be the class of all Kripke frames. Then  $\mathcal{K}$  corresponds to the clone  $\Lambda_1$ .*  $\square$

**Remark 2.** Let  $\langle W, F \rangle$  be a Scott-Montague frame. Since the class  $\mathcal{C}_{\Lambda_1}$  is defined by the axioms  $\square \top$  and  $\square p \wedge \square q \leftrightarrow \square(p \wedge q)$ , it is easy to see that  $\langle W, F \rangle \in \mathcal{C}_{\Lambda_1}$  if and only if the conditions

$$F(W) = W \text{ and}$$

$$F(X) \cap F(Y) = F(X \cap Y)$$

hold for all  $X, Y \subseteq W$ .

We also get interesting correspondence results between some subclasses of  $\mathcal{K}$  and subclones of  $\Lambda_1$ . For example, there is a class of Kripke frames which corresponds to the minimal clone  $I_c$ . This clone plays a key role when characterizing definability of classes of Boolean functions by functional terms, see Theorem 1 and the discussion before it. For some of these correspondence results, we need to consider Kripke frames  $\mathcal{F} = \langle W, R \rangle$  where  $R$  is a *serial* relation, i.e. every  $w \in W$  has an  $R$ -successor. In the following propositions we use the same operations  $\Gamma$  and  $\Delta$  as in Proposition 2. While Lemmas 4 and 5 and Proposition 1 do the most of the work of the proofs, we need to show that the frames obtained from the translations belong to the right classes of Kripke and Scott-Montague frames in question.

**Proposition 3.** *Let  $\mathcal{K}_s$  be the class of all Kripke frames  $\mathcal{F} = \langle W, R \rangle$  such that  $R$  is a serial relation. Then  $\mathcal{K}_s$  corresponds to the clone  $\Lambda_c$ .*

*Proof.* The clone  $\Lambda_c$  is defined by the functional terms

$$\neg \mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}),$$

and so the class  $\mathcal{C}_{\Lambda_c}$  is defined by the axioms  $\neg \Box \perp$ ,  $\Box \top$  and  $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$ . It is easy to verify that a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  validates these axioms if and only if the conditions

- (1)  $F(\emptyset) = \emptyset$ ,
- (2)  $F(W) = W$  and
- (3)  $F(X) \cap F(Y) = F(X \cap Y)$

hold for every  $X, Y \subseteq W$ .

Let  $\langle W, R \rangle \in \mathcal{K}_s$ . Now  $R[w] \neq \emptyset$  for every  $w \in W$  and it is immediate from the definition of  $F_R$  that  $F_R(\emptyset) = \emptyset$ . It follows from Lemma 4 that  $\langle W, F_R \rangle \in \mathcal{C}_{\Lambda_1}$  and thus, by Remark 2,  $F_R$  satisfies conditions (2) and (3). Hence  $\langle W, F_R \rangle \in \mathcal{C}_{\Lambda_c}$ .

Let  $\langle W, F \rangle \in \mathcal{C}_{\Lambda_c}$ . Now  $\langle W, F \rangle \models \neg \Box \perp$  and it follows from Lemma 5 that also  $\langle W, R_F \rangle \models \neg \Box \perp$ . From this fact, one can easily conclude that  $R_F$  has to be a serial relation and hence  $\langle W, R_F \rangle \in \mathcal{K}_s$ .  $\square$

**Proposition 4.** *Let  $\mathcal{K}_f$  be the class of all Kripke frames  $\mathcal{F} = \langle W, R \rangle$  such that  $R$  is a function. Then  $\mathcal{K}_f$  corresponds to the clone  $I_c$ .*

*Proof.* The clone  $I_c$  is defined by the functional terms

$$\mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}), \neg \mathbf{f}(\mathbf{x}) \leftrightarrow \mathbf{f}(\neg \mathbf{x}).$$

Hence the class  $\mathcal{C}_{I_c}$  is defined by the axioms  $\Box \top$ ,  $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$  and  $\neg \Box p \leftrightarrow \Box \neg p$ . It is easy to check that these axioms are valid in a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  if and only if the conditions

- (1)  $F(W) = W$ ,
- (2)  $F(X) \cap F(Y) = F(X \cap Y)$  and
- (3)  $\overline{F(X)} = F(\overline{X})$

hold for every  $X, Y \subseteq W$ .

Let  $\langle W, R \rangle \in \mathcal{K}_f$ . We show that  $\overline{F_R(X)} = F_R(\overline{X})$  for all  $X \subseteq W$ . Let  $w \in W$  and  $X \subseteq W$ . Since  $\langle W, R \rangle \in \mathcal{K}_f$ , the relation  $R$  is a function and we have that

$$\begin{aligned} w \in \overline{F_R(X)} &\Leftrightarrow w \notin F_R(X) \\ &\Leftrightarrow \exists v \in W : wRv \wedge v \in \overline{X} \\ &\Leftrightarrow \forall v \in W : wRv \Rightarrow v \in \overline{X} \\ &\Leftrightarrow w \in F_R(\overline{X}). \end{aligned}$$

Hence  $F_R$  satisfies the condition (3). As before,  $F_R$  satisfies also the conditions (1) and (2), and so  $\langle W, F_R \rangle \in \mathcal{C}_{I_c}$ .

Let  $\langle W, F \rangle \in \mathcal{C}_{I_c}$ . Since  $\langle W, F \rangle \models \neg \Box p \leftrightarrow \Box \neg p$ , it follows from Lemma 5 that  $\langle W, R_F \rangle \models \neg \Box p \leftrightarrow \Box \neg p$ . We have to show that  $R_F$  is a function. Suppose on the contrary that  $R_F$  is not a function. Then either some element of  $W$  does not have an  $R_F$ -successor or some element of  $W$  has at least two of them. Assume first that there is  $w \in W$  which has no  $R_F$ -successor. Let  $\mathcal{M} = \langle W, R_F, V \rangle$  be a Kripke model. Then trivially  $\mathcal{M}, w \models \Box \neg p$  and  $\mathcal{M}, w \not\models \neg \Box p$ . Thus the axiom  $\neg \Box p \leftrightarrow \Box \neg p$  is not valid in the frame  $\langle W, R_F \rangle$ . Assume then that there is  $w \in W$  which has two  $R_F$ -successors  $w_1$  and  $w_2$ . Consider a Kripke model  $\mathcal{M} = \langle W, R_F, V \rangle$  where  $V$  is a valuation such that  $V(p) = \{w_1\}$ . Then  $\mathcal{M}, w \models \neg \Box p$  and  $\mathcal{M}, w \not\models \Box \neg p$ . Hence the axiom  $\neg \Box p \leftrightarrow \Box \neg p$  is not valid in the frame  $\langle W, R_F \rangle$ . So, both cases lead up to a contradiction and therefore  $R_F$  is a function.  $\square$

**Proposition 5.** *Let  $\mathcal{K}_p$  be the class of all Kripke frames  $\mathcal{F} = \langle W, R \rangle$  such that  $R$  is a partial function. Then  $\mathcal{K}_p$  corresponds to the clone  $I_1$ .*

*Proof.* The clone  $I_1$  is defined by the functional terms

$$\mathbf{f}(\mathbf{1}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}), \neg \mathbf{f}(\mathbf{x}) \rightarrow \mathbf{f}(\neg \mathbf{x}).$$

Thus the class  $\mathcal{C}_{I_1}$  of Scott-Montague frames is defined by the axioms  $\Box \top$ ,  $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$  and  $\neg \Box p \rightarrow \Box \neg p$ . These axioms are valid in a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  if and only if the conditions

- (1)  $F(W) = W$ ,
- (2)  $F(X) \cap F(Y) = F(X \cap Y)$  and
- (3)  $\overline{F(X)} \subseteq F(\overline{X})$

hold for every  $X, Y \subseteq W$ .

Let  $\langle W, F \rangle \in \mathcal{K}_p$ . Again  $F_R$  satisfies the conditions (1) and (2). We will show that  $\overline{F_R(X)} \subseteq F_R(\overline{X})$ . Let  $X \subseteq W$  and let  $w \in \overline{F_R(X)}$ . Then  $w \notin F_R(X)$  and thus there exists  $v \in W$  such that  $wRv$  and  $v \notin X$ . Since  $R$  is a partial function, there is only one  $v$  such that  $wRv$ . Hence we have that for all  $v \in W$ ,  $wRv$  implies  $v \in \overline{X}$ , and therefore  $w \in F_R(\overline{X})$ . Thus  $\overline{F_R(X)} \subseteq F_R(\overline{X})$  and we have proved that  $\langle W, F_R \rangle \in \mathcal{C}_{I_1}$ .

Let then  $\langle W, F \rangle \in \mathcal{C}_{I_1}$ . We know that  $\langle W, F \rangle \models \neg \Box p \rightarrow \Box \neg p$  and hence, by Lemma 5,  $\langle W, R_F \rangle \models \neg \Box p \rightarrow \Box \neg p$ . We have to show that  $\langle W, R_F \rangle \in \mathcal{K}_p$ , i.e.  $R_F$  is a partial function. If  $R_F$  is not a partial function then there is  $w \in W$  which has two  $R_F$ -successors. By proceeding as in the proof of Proposition 4, one can show that the axiom  $\neg \Box p \rightarrow \Box \neg p$  is not valid in the frame  $\langle W, R_F \rangle$ . This is a contradiction and thus  $R_F$  has to be a partial function.  $\square$

Note that if we had chosen  $\diamond$  as the basic operator in modal logic and given the truth condition of  $\diamond$  in Scott-Montague semantics by the set function, then the class of all Kripke frames would correspond to the clone  $V_0$ , which is the dual of  $\Lambda_1$ . Similarly, the classes  $\mathcal{K}_s$  and  $\mathcal{K}_p$  would correspond to the duals of  $\Lambda_c$  and  $I_1$ , respectively, i.e. to the clones  $V_c$  and  $I_0$ . Since the clone  $I_c$  is self-dual, the class  $\mathcal{K}_f$  would still correspond to  $I_c$ .

## 4.2 Kripke frames with non-normal worlds

In the previous section, we proved that the class of all Kripke frames corresponds to the clone  $\Lambda_1$ . The clones  $\Lambda$  and  $\Lambda_0$  are quite similar to  $\Lambda_1$ , but they lack the property of 1-preservation. In terms of term definability this property corresponds to the functional term  $\mathbf{f}(\mathbf{1})$ . The translation of this functional term into uniform degree-1 formula is  $\Box \top$ , which is well known to

be valid in every Kripke frame. So, if we want to have any class of “Kripke-kind” frames corresponding to the clone  $\Lambda$  or  $\Lambda_0$ , we have to modify Kripke frames and/or Kripke semantics. In this thesis we modify these concepts by considering so-called *non-normal worlds*, which were introduced by Kripke in [8].

A *non-normal Kripke frame* is a frame  $\mathcal{F} = \langle W, R, N \rangle$  where  $W$  is a non-empty set,  $R$  is a binary relation on  $W$  and  $N$  is a subset of  $W$ . In other words, non-normal Kripke frames are just standard Kripke frames with additional unary relation  $N$ . A *non-normal Kripke model* is a model  $\mathcal{M} = \langle \mathcal{F}, V \rangle$  where  $\mathcal{F}$  is a non-normal Kripke frame and  $V$  is a valuation function  $\Phi \rightarrow \mathcal{P}(W)$ . The semantics for these non-normal Kripke models is given by a truth relation  $\models_N$ , which is defined as the truth relation  $\models$  in the standard Kripke semantics, except for the worlds  $w \in N$ . Let  $w \in N$ . The truth conditions for the propositional formulas are the same as in Kripke semantics but the truth condition for the modal operator  $\Box$  at the world  $w$  is given by

$$\mathcal{M}, w \not\models_N \Box\varphi \text{ for all formulas } \varphi,$$

i.e. every formula of the form  $\Box\varphi$  is false at  $w \in N$ . These  $w \in N$  are called the *non-normal elements of  $\mathcal{M}$*  (and also the non-normal elements of the corresponding frame).

The addition of a unary relation also affects the definition of corresponding classes. Instead of translating a binary relation into a set function and vice versa, we also have to take into account the unary relation defined on the universe. That is, for defining a set function we need a binary relation and also a unary relation. And conversely, from a set function we define both a binary and a unary relation.

Let  $\mathcal{M} = \langle W, R, N, V \rangle$  be a non-normal Kripke model and let  $\mathcal{M}' = \langle W, F, V \rangle$  be a Scott-Montague model. We say that the models  $\mathcal{M}$  and  $\mathcal{M}'$  are *pointwise equivalent with respect to  $\models_N$*  if for all modal formulas  $\varphi$  and for all  $w \in W$ ,

$$\mathcal{M}, w \models_N \varphi \text{ if and only if } \mathcal{M}', w \models \varphi.$$

Let  $\mathcal{K}$  be a class of non-normal Kripke frames and let  $\mathcal{C}$  be a class of Scott-Montague frames. We say that the pair  $(\mathcal{K}, \models_N)$  corresponds to the class  $\mathcal{C}$  if there are operations  $\Gamma$  from the pairs of binary and unary relations to the set functions and  $\Delta$  from the set functions to the pairs of binary and unary relations such that the following conditions hold.

- If  $\langle W, R, N \rangle \in \mathcal{K}$  then  $\langle W, \Gamma(R, N) \rangle \in \mathcal{C}$ , and for all valuations  $V$ , the models  $\langle W, R, N, V \rangle$  and  $\langle W, \Gamma(R, N), V \rangle$  are pointwise equivalent with respect to  $\models_N$ .

- If  $\langle W, F \rangle \in \mathcal{C}$  then  $\langle W, \Delta(F) \rangle \in \mathcal{K}$ , and for all valuations  $V$ , the models  $\langle W, \Delta(F), V \rangle$  and  $\langle W, F, V \rangle$  are pointwise equivalent with respect to  $\models_N$ .
- The operations  $\Gamma$  and  $\Delta$  are inverses of each other.

Furthermore we say that the pair  $(\mathcal{K}, \models_N)$  corresponds to a clone  $\mathcal{C}$  if the pair  $(\mathcal{K}, \models_N)$  corresponds to the class  $\mathcal{C}_\mathcal{C}$ .

Let  $\mathcal{K}^N$  denote the class of all non-normal Kripke frames  $\langle W, R, N \rangle$  such that  $R[w] = \emptyset$  for all  $w \in N$ . In Lemmas 7 and 8 and Proposition 6 we will show that the pair  $(\mathcal{K}^N, \models_N)$  corresponds to the clone  $\Lambda$ . Note that  $\Lambda$  consists of the constants and the non-empty conjunctions, and the only functions in  $\Lambda$  that are not 1-preserving, i.e. are not in  $\Lambda_1$ , are the constants  $\mathbf{0}$ . Let  $W = \{w_1, \dots, w_n\}$  and consider a non-normal Kripke frame  $\langle W, R, N \rangle \in \mathcal{K}^N$  and a Scott-Montague frame  $\langle W, F \rangle \in \mathcal{C}_\Lambda$ . We denote by  $f_i$  the  $i$ th component of the Boolean vector-valued function  $f_F$ . The idea is that the essential variables of  $f_i$  correspond to the  $R$ -successors of  $w_i$ . Thus  $f_i$  is a constant if and only if  $w_i$  does not have  $R$ -successors. Whether  $f_i$  is  $\mathbf{0}$  or  $\mathbf{1}$  depends on whether  $w_i \in N$  or  $w_i \notin N$ .

Since the clone  $\Lambda$  is defined by the functional term

$$\mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}),$$

the class  $\mathcal{C}_\Lambda$  of Scott-Montague frames is defined by the axiom  $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$ . It is easy to see that this axiom is valid in a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  if and only if  $F$  satisfies the condition

$$(1) \quad F(X) \cap F(Y) = F(X \cap Y)$$

for every  $X, Y \subseteq W$ .

**Lemma 7.** *Let  $\langle W, R, N \rangle$  be a non-normal Kripke frame in  $\mathcal{K}^N$  and let  $\langle W, F_{(R,N)} \rangle$  be the Scott-Montague frame where  $F_{(R,N)}$  is given by*

$$F_{(R,N)}(X) = \{w \in W \mid w \notin N \text{ and } \forall v \in W : wRv \Rightarrow v \in X\}$$

for all  $X \subseteq W$ . Then  $\langle W, F_{(R,N)} \rangle \in \mathcal{C}_\Lambda$  and for all valuations  $V$ , the models  $\langle W, R, N, V \rangle$  and  $\langle W, F_{(R,N)}, V \rangle$  are pointwise equivalent with respect to  $\models_N$ .

*Proof.* It is straightforward to verify that  $F_{(R,N)}$  satisfies the condition (1), and therefore  $\langle W, F_{(R,N)} \rangle \in \mathcal{C}_\Lambda$ .

Let  $V$  be a valuation function and let  $\mathcal{M} = \langle W, R, N, V \rangle$  and  $\mathcal{M}' = \langle W, F_{(R,N)}, V \rangle$ . We show that for all formulas  $\varphi$  and for all  $w \in W$ ,

$$\mathcal{M}, w \models_N \varphi \text{ if and only if } \mathcal{M}', w \models \varphi.$$

The proof is by induction on the construction of  $\varphi$ . Clearly, the claim holds for constants and proposition symbols. The cases, where  $\varphi$  is of the form  $\neg\psi$  or  $\psi_1 \wedge \psi_2$ , can be easily verified by making use of the induction hypothesis. So let  $\varphi = \Box\psi$  and suppose that the claim holds for  $\psi$ . Then  $\|\psi\|^{\mathcal{M}} = \|\psi\|^{\mathcal{M}'}$ . Let  $w \in W$  and suppose that  $w \notin N$ . Now

$$\begin{aligned} \mathcal{M}, w \models_N \Box\psi &\Leftrightarrow \forall v \in W : wRv \Rightarrow v \in \|\psi\|^{\mathcal{M}} \\ &\Leftrightarrow w \in F_{(R,N)}(\|\psi\|^{\mathcal{M}'}) \\ &\Leftrightarrow \mathcal{M}', w \models \Box\psi. \end{aligned}$$

If  $w \in N$ , then clearly  $\mathcal{M}, w \not\models_N \Box\psi$  and  $\mathcal{M}', w \not\models \Box\psi$ . Hence the models  $\mathcal{M}$  and  $\mathcal{M}'$  are pointwise equivalent with respect to  $\models_N$ .  $\square$

**Lemma 8.** *Let  $\langle W, F \rangle$  be a Scott-Montague frame in  $\mathcal{C}_\Lambda$  and let  $f_i$  denote the  $i$ th component of  $f_F$ . Let  $\langle W, R_F, N_F \rangle$  be the non-normal Kripke frame where  $R_F$  and  $N_F$  are given by*

$$R_F[w_i] = \{w_j \in W \mid x_j \text{ is an essential variable of } f_i\}$$

for all  $w_i \in W$  and  $N_F = \{w_i \in W \mid f_i = \mathbf{0}\}$ . Then  $\langle W, R_F, N_F \rangle \in \mathcal{K}^N$  and for all valuations  $V$ , the models  $\langle W, R_F, N_F, V \rangle$  and  $\langle W, F, V \rangle$  are pointwise equivalent with respect to  $\models_N$ .

*Proof.* Suppose that  $w_i \in N_F$ . Then  $f_i = \mathbf{0}$  by the definition of  $N_F$ , and since  $\mathbf{0}$  has no essential variables, it follows from the definition of  $R_F$  that  $R_F[w_i] = \emptyset$ . Thus  $\langle W, R_F, N_F \rangle \in \mathcal{K}^N$ .

Since  $\langle W, F \rangle \in \mathcal{C}_\Lambda$ , we have, by the definition of  $K_{\mathcal{C}_\Lambda}$ , that  $f_i \in K_{\mathcal{C}_\Lambda}$  and it follows from Theorem 6 on page 18 that  $f_i \in \Lambda$ . In this case, as noted in Remark 1, for all  $\mathbf{a} \in \mathbb{B}^n$ ,

$f_i(\mathbf{a}) = 1$  if and only if  $f_i \neq \mathbf{0}$  and  $\mathbf{a}[j] = 1$  for all essential variables  $x_j$  of  $f_i$ .

Let  $V$  be a valuation and let  $\mathcal{M} = \langle W, F, V \rangle$  and  $\mathcal{M}' = \langle W, R_F, N_F, V \rangle$ . We have to show that for all formulas  $\varphi$  and for all  $w_i \in W$ ,

$$\mathcal{M}, w_i \models \varphi \text{ if and only if } \mathcal{M}', w_i \models_N \varphi.$$

We proceed by induction on the construction of  $\varphi$ . The cases where  $\varphi$  is a constant, a proposition symbol, or of the form  $\neg\psi$  or  $\psi_1 \wedge \psi_2$ , are straightforward. Let  $\varphi = \Box\psi$  and suppose that  $\psi$  satisfies the claim. Then  $\|\psi\|^{\mathcal{M}} = \|\psi\|^{\mathcal{M}'}$



and we denote by  $A$  the set  $\|\psi\|^{\mathcal{M}}$ . Now

$$\begin{aligned}
\mathcal{M}, w_i \models \Box\psi &\Leftrightarrow w_i \in F(\|\psi\|^{\mathcal{M}}) \\
&\Leftrightarrow w_i \in F(A) \\
&\Leftrightarrow f_i(\mathbf{a}_A) = 1 \\
&\Leftrightarrow f_i \neq \mathbf{0} \text{ and } \mathbf{a}_A[j] = 1 \text{ for all essential variables } x_j \text{ of } f_i \\
&\Leftrightarrow f_i \neq \mathbf{0} \text{ and } w_j \in \|\psi\|^{\mathcal{M}} \text{ for all essential variables } x_j \text{ of } f_i \\
&\Leftrightarrow w_i \notin N_F \text{ and } w_j \in \|\psi\|^{\mathcal{M}'} \text{ for all } w_j \in R_F[w_i] \\
&\Leftrightarrow \mathcal{M}', w_i \models_N \Box\psi.
\end{aligned}$$

By the induction principle, every formula  $\varphi$  satisfies the claim, which completes the proof of the lemma.  $\square$

**Lemma 9.** *Let  $\langle W, R, N \rangle$  be a non-normal Kripke frame in  $\mathcal{K}^N$  and let  $F_{(R,N)}$  be defined as in Lemma 7. Let  $f_i$  be the  $i$ th component of  $f_{F_{(R,N)}}$ . Then for all  $1 \leq j \leq n$ ,*

$$x_j \text{ is an essential variable of } f_i \text{ if and only if } w_j \in R[w_i].$$

Recall that we have fixed  $W$  to be the set  $\{w_1, \dots, w_n\}$ .

*Proof.* From the definition of  $F_{(R,N)}$  we get that for all  $\mathbf{a} \in \mathbb{B}^n$ ,

$$f_i(\mathbf{a}) = 1 \Leftrightarrow w_i \notin N \text{ and } \forall k \in \{1, \dots, n\} : w_i R w_k \Rightarrow \mathbf{a}[k] = 1.$$

From this fact, it is easy to see that if  $w_j \notin R[w_i]$  then for all  $\mathbf{a} \in \mathbb{B}^n$ , the value of  $f_i(\mathbf{a})$  does not depend on the  $j$ th component of  $\mathbf{a}$ . Therefore, if  $w_j \notin R[w_i]$  then  $x_j$  is not an essential variable of  $f_i$ .

Suppose that  $w_j \in R[w_i]$  for some  $1 \leq j \leq n$ . Then  $w_i \notin N$  and it follows from the definition of  $F_{(R,N)}$  that  $w_i \in F_{(R,N)}(W)$ . Thus  $f_i(\mathbf{a}_W) = 1$ , i.e.  $f_i(\mathbf{1}) = 1$ , where  $\mathbf{1} = (1, \dots, 1) \in \mathbb{B}^n$ . Let  $\mathbf{a} \in \mathbb{B}^n$  such that  $\mathbf{a}[j] = 0$  and  $\mathbf{a}[k] = 1$  for all  $k \neq j$ . Since  $w_i R w_j$  and  $\mathbf{a}[j] = 0$ , we have that  $f_i(\mathbf{a}) = 0$ . Thus  $f_i(\mathbf{1}) \neq f_i(\mathbf{a})$ , and hence  $x_j$  is an essential variable of  $f_i$ .  $\square$

**Proposition 6.** *Let  $\langle W, R, N \rangle \in \mathcal{K}^N$  and let  $\langle W, F \rangle \in \mathcal{C}_\Lambda$ . Let  $F_{(R,N)}$ ,  $R_F$  and  $N_F$  be defined as in Lemmas 7 and 8. Then  $R_{F_{(R,N)}} = R$ ,  $N_{F_{(R,N)}} = N$  and  $F_{(R_F, N_F)} = F$ .*

*Proof.* We show first that  $R_{F_{(R,N)}} = R$  and  $N_{F_{(R,N)}} = N$ . Let  $f_i$  be the  $i$ th component of  $f_{F_{(R,N)}}$  and let  $w_i, w_j \in W$ . Then, by Lemma 9,

$$\begin{aligned}
w_j \in R_{F_{(R,N)}}[w_i] &\Leftrightarrow x_j \text{ is an essential variable of } f_i \\
&\Leftrightarrow w_j \in R[w_i].
\end{aligned}$$

Therefore  $R_{F_{(R,N)}} = R$ . If  $w_i \in N$  then it follows from the definition of  $F_{(R,N)}$  that  $f_i = \mathbf{0}$ . If  $w_i \notin N$  then, by the proof of Lemma 9,  $f_i(\mathbf{1}) = 1$  and therefore  $f_i \neq \mathbf{0}$ . Thus  $w_i \in N$  if and only if  $f_i = \mathbf{0}$ , and we have that

$$\begin{aligned} w_i \in N_{F_{(R,N)}} &\Leftrightarrow f_i = \mathbf{0} \\ &\Leftrightarrow w_i \in N. \end{aligned}$$

Hence  $N_{F_{(R,N)}} = N$ .

We show then that  $F_{(R_F, N_F)} = F$ . Let  $f_i$  be the  $i$ th component of  $f_F$ . Since  $\langle W, F \rangle \in \mathcal{C}_\Lambda$ , we know that  $f_i \in \Lambda$ . Let  $X \subseteq W$  and  $w_i \in W$ . Then

$$\begin{aligned} w_i \in F_{(R_F, N_F)}(X) &\Leftrightarrow w_i \notin N_F \text{ and } w_j \in X \text{ for all } w_j \in R_F[w_i] \\ &\Leftrightarrow f_i \neq \mathbf{0} \text{ and } \mathbf{a}_X[j] = 1 \text{ for all essential variables } x_j \text{ of } f_i \\ &\Leftrightarrow f_i(\mathbf{a}_X) = 1 \\ &\Leftrightarrow w_i \in F(X). \end{aligned}$$

Therefore  $F_{(R_F, N_F)} = F$ . □

The following proposition is an immediate consequence of Lemmas 7 and 8 and Proposition 6 by choosing  $\Gamma(R, N) = F_{(R,N)}$  and  $\Delta(F) = (R_F, N_F)$ .

**Proposition 7.** *The pair  $(\mathcal{K}^N, \models_N)$  corresponds to the clone  $\Lambda$ .* □

As in the case of the class of all Kripke frames in Section 4.1, we get further correspondences between some classes of non-normal Kripke frames and subclones of  $\Lambda$  by restricting the binary relation  $R$ . In the next two propositions we use the same operations  $\Gamma$  and  $\Delta$  as in Proposition 7. Since Lemmas 7 and 8 and Proposition 6 still apply, we obtain the proofs with quite a little effort.

**Proposition 8.** *Let  $\mathcal{K}_s^N$  be the class of all non-normal Kripke frames  $\langle W, R, N \rangle$  in  $\mathcal{K}^N$  such that every  $w \in (W \setminus N)$  has an  $R$ -successor. Then the pair  $(\mathcal{K}_s^N, \models_N)$  corresponds to the clone  $\Lambda_0$ .*

*Proof.* The clone  $\Lambda_0$  is defined by the functional terms

$$\neg \mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}).$$

Thus the class  $\mathcal{C}_{\Lambda_0}$  is defined by the axioms  $\neg \Box \perp$  and  $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$ . It is easy to see that these axioms are valid in a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  if and only if the conditions

- (1)  $F(\emptyset) = \emptyset$  and
- (2)  $F(X) \cap F(Y) = F(X \cap Y)$

hold for all  $X, Y \subseteq W$ .

Let  $\langle W, R, N \rangle \in \mathcal{K}_s^N$ . Since every  $w \in (W \setminus N)$  has an  $R$ -successor, it follows straightforwardly from the definition of  $F_{(R,N)}$  that  $F_{(R,N)}(\emptyset) = \emptyset$ . By Lemma 7,  $F_{(R,N)}$  satisfies also the condition (2), and hence  $\langle W, F_{(R,N)} \rangle \in \mathcal{C}_{\Lambda_0}$ .

Let  $\langle W, F \rangle \in \mathcal{C}_{\Lambda_0}$ . Then  $\langle W, F \rangle \models \neg \Box \perp$  and it follows from Lemma 8 that also  $\langle W, R_F, N_F \rangle \models \neg \Box \perp$ . From this, similarly as in Proposition 3 on page 24, we can conclude that every  $w \in (W \setminus N_F)$  has to have an  $R_F$ -successor. Thus  $\langle W, R_F, N_F \rangle \in \mathcal{K}_s^N$ .  $\square$

**Proposition 9.** *Let  $\mathcal{K}_p^N$  be the class of all non-standard Kripke frames  $\langle W, R, N \rangle$  in  $\mathcal{K}^N$  such that  $R$  is a partial function. Then the pair  $(\mathcal{K}_p^N, \models_N)$  corresponds to the clone  $I$ .*

*Proof.* The clone  $I$  is defined by the functional terms

$$\mathbf{f}(\mathbf{x}) \wedge \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \wedge \mathbf{y}), \mathbf{f}(\mathbf{x}) \vee \mathbf{f}(\mathbf{y}) \leftrightarrow \mathbf{f}(\mathbf{x} \vee \mathbf{y}).$$

Therefore the class  $\mathcal{C}_I$  of Scott-Montague frames is defined by the axioms  $\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$  and  $\Box p \vee \Box q \leftrightarrow \Box(p \vee q)$ . These axioms are valid in a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  if and only if the conditions

- (1)  $F(X) \cap F(Y) = F(X \cap Y)$  and
- (2)  $F(X) \cup F(Y) = F(X \cup Y)$

hold for all  $X, Y \subseteq W$ .

Let  $\langle W, R, N \rangle \in \mathcal{K}_p^N$ . Lemma 7 tells us that  $F_{(R,N)}$  satisfies the condition (1). Let  $X, Y \subseteq W$  and  $w \in W$ . Now

$$\begin{aligned} w \in F_{(R,N)}(X) \cup F_{(R,N)}(Y) &\Leftrightarrow w \in F_{(R,N)}(X) \text{ or } w \in F_{(R,N)}(Y) \\ &\Leftrightarrow (w \notin N \text{ and } \forall v \in W : wRv \Rightarrow v \in X) \text{ or} \\ &\quad (w \notin N \text{ and } \forall v \in W : wRv \Rightarrow v \in Y) \\ &\stackrel{(*)}{\Leftrightarrow} w \notin N \text{ and } \forall v \in W : wRv \Rightarrow v \in X \cup Y \\ &\Leftrightarrow w \in F_{(R,N)}(X \cup Y), \end{aligned}$$

where the equivalence  $(*)$  follows from the fact that  $R$  is a partial function. Hence  $F_{(R,N)}$  satisfies the condition (2), and so  $\langle W, F_{(R,N)} \rangle \in \mathcal{C}_I$ .

Let  $\langle W, F \rangle \in \mathcal{C}_I$ . We have to show that  $\langle W, R_F, N_F \rangle \in \mathcal{K}_p^N$ . By Lemma 8, it is sufficient to show that  $R_F$  is a partial function. Since  $\langle W, F \rangle \models \Box p \vee \Box q \leftrightarrow \Box(p \vee q)$ , it follows from Lemma 8 that  $\langle W, R_F, N_F \rangle \models_N \Box p \vee \Box q \leftrightarrow \Box(p \vee q)$ . Suppose that  $R_F$  is not a partial function. Then there are  $w, v_1, v_2 \in W$  such that  $wR_Fv_1$ ,  $wR_Fv_2$  and  $v_1 \neq v_2$ . Clearly  $w \notin N_F$ . Consider a model  $\mathcal{M} = \langle W, R_F, N_F, V \rangle$ , where  $V(p) = W \setminus \{v_2\}$  and

$V(q) = W \setminus \{v_1\}$ . It is easy to see that  $\mathcal{M}, w \models_N \Box(p \vee q)$  but  $\mathcal{M}, w \not\models_N \Box p$  and  $\mathcal{M}, w \not\models_N \Box q$ . This is a contradiction, since  $\mathcal{M} \models_N \Box p \vee \Box q \leftrightarrow \Box(p \vee q)$ . Therefore  $R_F$  is a partial function and  $\langle W, R_F, N_F \rangle \in \mathcal{K}_p^N$ .  $\square$

Similarly, as in the case of Kripke frames in Section 4.1, if we had chosen  $\diamond$  as the basic modal operator and given the truth condition of  $\diamond$  in Scott-Montague semantics by the set function and furthermore defined the truth relation  $\models_N$  in a way that at the non-normal worlds every formula of the form  $\diamond\varphi$  is true, then the pair  $(\mathcal{K}^N, \models_N)$  would correspond to the clone  $V$ , which is the dual clone of  $\Lambda$ . Similarly, the pair  $(\mathcal{K}_s^N, \models_N)$  would correspond to the clone  $V_1$ .

## 5 Linear clones and modified Kripke semantics

In this section we establish further correspondences between classes of Kripke frames and clones of Boolean functions by modifying the standard Kripke semantics. These modifications are based on the parity of the sets of  $R$ -successors. As in the previous section, we consider only finite frames.

Throughout this section we make use of the exclusive-or, denoted by  $\oplus$  and defined by

$$\varphi \oplus \psi = (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi),$$

as a propositional connective in formulas of modal logic. One can easily verify that  $\oplus$  is associative and commutative. Furthermore, for every Kripke or Scott-Montague model  $\mathcal{M}$  with universe  $W$  and for every  $w \in W$ ,  $\mathcal{M}, w \models \varphi_1 \oplus \dots \oplus \varphi_m$  if and only if  $\mathcal{M}, w \models \varphi_i$  for an odd number of indices  $1 \leq i \leq m$ .

Let  $X, Y \subseteq W$ . By  $\bar{X}$  we denote the complement of  $X$  with respect to  $W$ . We use the symbol  $\oplus$  also for the symmetric difference

$$X \oplus Y = (X \setminus Y) \cup (Y \setminus X) = (X \cap \bar{Y}) \cup (Y \cap \bar{X}).$$

Similarly to the exclusive-or, the symmetric difference is associative and commutative.

The frames and models that we shall consider at first in Section 5.1 are essentially the same as those used in Kripke semantics. In fact, the semantics that we will introduce differ from Kripke semantics only in the account for the modal operator  $\Box$ . However, the notion of correspondence between classes of Kripke and Scott-Montague frames must be also modified accordingly. Later in Section 5.1 we will add an unary relation to Kripke frames and in Section 5.2 we will also make use of some natural restrictions of the binary relation  $R$ , as we did in Section 4.

In the sequel we will define three new *truth relations*,  $\models_O$ ,  $\models_S$  and  $\models_M$ , which give the modifications of Kripke semantics that we are going to consider.

Let  $\mathcal{M} = \langle W, R, V \rangle$  be a Kripke model and let  $\mathcal{M}' = \langle W, F, V \rangle$  be a Scott-Montague model. We say that the models  $\mathcal{M}$  and  $\mathcal{M}'$  are *pointwise equivalent with respect to  $\models_\eta$* , where  $\eta \in \{O, S, M\}$ , if for all modal formulas  $\varphi$  and for all  $w \in W$ ,

$$\mathcal{M}, w \models_\eta \varphi \text{ if and only if } \mathcal{M}', w \models \varphi.$$

Let  $(\mathcal{K}, \models_\eta)$  be a pair where  $\mathcal{K}$  is a class of Kripke frames and  $\models_\eta$  is the truth relation referring to the semantics which is being used, and let  $\mathcal{C}$  be

a class of Scott-Montague frames. As in the previous section, we say that the pair  $(\mathcal{K}, \models_\eta)$  *corresponds* to the class  $\mathcal{C}$  if there are operations  $\Gamma$  from the binary relations to the set functions and  $\Delta$  from the set functions to the binary relations such that the following conditions hold.

- If  $\langle W, R \rangle \in \mathcal{K}$ , then  $\langle W, \Gamma(R) \rangle \in \mathcal{C}$ , and for all valuations  $V$ , the models  $\langle W, R, V \rangle$  and  $\langle W, \Gamma(R), V \rangle$  are pointwise equivalent with respect to  $\models_\eta$ .
- If  $\langle W, F \rangle \in \mathcal{C}$ , then  $\langle W, \Delta(F) \rangle \in \mathcal{K}$ , and for all valuations  $V$ , the models  $\langle W, \Delta(F), V \rangle$  and  $\langle W, F, V \rangle$  are pointwise equivalent with respect to  $\models_\eta$ .
- The operations  $\Gamma$  and  $\Delta$  are inverses of each other.

In the Propositions 10, 11 and 12, instead of giving operations  $\Gamma$  and  $\Delta$ , we just define translations  $R \mapsto F_R$  and  $F \mapsto R_F$ .

As in the previous section, we say that the pair  $(\mathcal{K}, \models_\eta)$  *corresponds* to a clone  $C$  if the pair  $(\mathcal{K}, \models_\eta)$  corresponds to the class  $\mathcal{C}_C$ .

## 5.1 Linear clones

In this section we will provide pairs  $(\mathcal{K}, \models_\eta)$  for each of the linear clones  $L$ ,  $L_0$ ,  $L_1$ ,  $L_c$  and  $LS$ . Consider first the clone  $L_0$  which is defined by the functional term

$$\mathbf{1} \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}).$$

It follows from Theorem 5 that the class  $\mathcal{C}_{L_0}$  is defined by the axiom  $\top \oplus \Box p \oplus \Box q \oplus \Box(p \oplus q)$ . It is easy to verify that this axiom is valid in a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  if and only if the condition

$$(1) \quad F(X) \oplus F(Y) = F(X \oplus Y)$$

holds for all  $X, Y \subseteq W$ .

The following lemma gives rise to one modification of Kripke semantics which will, together with the class of all Kripke frames, correspond to the clone  $L_0$ .

**Lemma 10.** *Let  $F : \mathcal{P}(W) \rightarrow \mathcal{P}(W)$  be a set function satisfying the condition (1). Let  $R_F$  be the relation defined by*

$$R_F[w] = \{v \in W \mid w \in F(\{v\})\}$$

*for every  $w \in W$ . Then for all  $X \subseteq W$  and  $w \in W$ ,*

$$w \in F(X) \text{ if and only if } |R_F[w] \cap X| \text{ is odd.}$$

*Proof.* Let  $X \subseteq W$ . First, note that if  $X = X_1 \oplus \cdots \oplus X_k$ , then  $w \in X$  if and only if  $w \in X_i$  for an odd number of indices  $i$ . Since  $F$  satisfies the condition (1) and  $\oplus$  is associative, it is easy to see that  $F(X_1 \oplus \cdots \oplus X_k) = F(X_1) \oplus \cdots \oplus F(X_k)$ . From the condition (1) we also get that  $F(\emptyset) = F(\emptyset \oplus \emptyset) = F(\emptyset) \oplus F(\emptyset) = \emptyset$ .

Let  $w \in W$ . Let  $X'$  denote the set  $R_F[w] \cap X$ , and suppose that  $|X'|$  is even. We show first that  $w \notin F(X')$ . If  $X' = \emptyset$  then  $w \notin F(X')$ , since  $F(\emptyset) = \emptyset$ . Let  $X' = \{w_1, \dots, w_{2m}\}$  for some  $m \geq 1$ . In other words,  $X' = \{w_1\} \oplus \cdots \oplus \{w_{2m}\}$  and it follows from the definitions of  $R_F[w]$  and  $X'$  that  $w \in F(\{w_i\})$  for all  $1 \leq i \leq 2m$ . From this fact and the observations above, we get that

$$w \notin F(\{w_1\}) \oplus \cdots \oplus F(\{w_{2m}\}) = F(\{w_1\} \oplus \cdots \oplus \{w_{2m}\}) = F(X').$$

Now we show that  $w \notin F(X \setminus X')$ . The case  $(X \setminus X') = \emptyset$  is clear. Suppose that  $(X \setminus X') = \{w_1, \dots, w_k\}$  for some  $k \geq 1$ . By the definitions of  $R_F[w]$  and  $X'$  we know that  $w \notin F(\{w_i\})$  for all  $1 \leq i \leq k$ . Thus

$$w \notin F(\{w_1\}) \oplus \cdots \oplus F(\{w_k\}) = F(\{w_1\} \oplus \cdots \oplus \{w_k\}) = F(X \setminus X').$$

So  $w \notin F(X')$  and  $w \notin F(X \setminus X')$ , and hence  $w \notin F(X') \oplus F(X \setminus X')$ . Since  $X = X' \oplus (X \setminus X')$ , it follows from the condition (1) that

$$w \notin F(X') \oplus F(X \setminus X') = F(X' \oplus (X \setminus X')) = F(X).$$

Suppose now that  $|X'|$  is odd. Let  $X' = \{w_1, \dots, w_{2m-1}\}$  for some  $m \geq 1$ . Then  $X' = \{w_1\} \oplus \cdots \oplus \{w_{2m-1}\}$  and since  $w \in F(\{w_i\})$  for all  $1 \leq i \leq 2m-1$ , we have that

$$w \in F(\{w_1\}) \oplus \cdots \oplus F(\{w_{2m-1}\}) = F(\{w_1\} \oplus \cdots \oplus \{w_{2m-1}\}) = F(X').$$

Furthermore, as before,  $w \notin F(X \setminus X')$  and thus

$$w \in F(X') \oplus F(X \setminus X') = F(X' \oplus (X \setminus X')) = F(X).$$

Hence, we have proved the lemma.  $\square$

The previous lemma gives a motivation to the following semantics. Let  $\mathcal{M} = \langle W, R, V \rangle$  be a Kripke model and let  $w \in W$ . We define the truth relation  $\models_O$  inductively for modal formulas. For constants, proposition symbols and Boolean connectives  $\models_O$  is defined as  $\models$  in Kripke semantics.

Let  $\varphi$  be a modal formula. For the modal operator  $\Box$ , the truth relation is defined by

$$\mathcal{M}, w \models_O \Box \varphi \text{ iff } |R[w] \cap \|\varphi\|_O^{\mathcal{M}}| \text{ is odd}$$

where  $\|\varphi\|_O^{\mathcal{M}}$  denotes the truth set of  $\varphi$  in the model  $\mathcal{M}$  with respect to  $\models_O$ .

We denote by  $\mathcal{L}_0$  the pair  $(\mathcal{K}, \models_O)$  where  $\mathcal{K}$  is the class of all Kripke frames. We denote by  $\mathcal{L}_c$  the pair  $(\mathcal{K}_O, \models_O)$  where  $\mathcal{K}_O$  denotes the class of all Kripke frames  $\langle W, R \rangle$  such that every  $w \in W$  has an odd number of  $R$ -successors.

**Proposition 10.** *The pair  $\mathcal{L}_0$  corresponds to the clone  $L_0$ .*

*Proof.* As mentioned before Lemma 10, the class  $\mathcal{C}_{L_0}$  consists of exactly those Scott-Montague frames  $\langle W, F \rangle$  for which

$$(1) F(X) \oplus F(Y) = F(X \oplus Y)$$

holds for all  $X, Y \subseteq W$ .

The translation  $R \mapsto F_R$  is given by

$$F_R(X) = \{w \in W \mid |R[w] \cap X| \text{ is odd}\}$$

for all  $X \subseteq W$  and the translation  $F \mapsto F_F$  is defined as in Lemma 10.

Let  $\langle W, R \rangle \in \mathcal{K}$ . We show that  $\langle W, F_R \rangle \in \mathcal{C}_{L_0}$ , i.e.  $F_R$  satisfies the condition (1). Let  $X, Y \subseteq W$  and let  $w \in W$ . We have to show that  $w \in F_R(X) \oplus F_R(Y)$  if and only if  $w \in F_R(X \oplus Y)$ . It is obvious that the parities of the sets  $R[w] \cap X$ ,  $R[w] \cap Y$  and  $R[w] \cap (X \oplus Y)$  depend only on the parities of the sets  $R[w] \cap (X \setminus Y)$ ,  $R[w] \cap (Y \setminus X)$  and  $R[w] \cap (X \cap Y)$ . So there are eight different cases. We consider here one of them, the other cases follow similarly. So suppose that  $|R[w] \cap (X \setminus Y)|$  and  $|R[w] \cap (Y \setminus X)|$  are odd and  $|R[w] \cap (X \cap Y)|$  is even. Then  $|R[w] \cap X|$  and  $|R[w] \cap Y|$  are odd and  $|R[w] \cap (X \oplus Y)|$  is even. So  $w \in F_R(X)$ ,  $w \in F_R(Y)$  and  $w \notin F_R(X \oplus Y)$ . Thus  $w \notin F_R(X) \oplus F_R(Y)$  and  $w \notin F_R(X \oplus Y)$ , and hence  $w \in F_R(X) \oplus F_R(Y)$  if and only if  $w \in F_R(X \oplus Y)$ . The rest of the cases can be proved similarly, and therefore  $\langle W, F_R \rangle \in \mathcal{C}_{L_0}$ .

Let  $V$  be a valuation and let  $\mathcal{M} = \langle W, R, V \rangle$  and  $\mathcal{M}' = \langle W, F_R, V \rangle$ . We show that the claim

$$\mathcal{M}, w \models_O \varphi \text{ if and only if } \mathcal{M}', w \models \varphi$$

holds for every modal formula  $\varphi$  and  $w \in W$ . This is done by induction on the construction of  $\varphi$ . The base cases,  $\varphi = \perp$ ,  $\varphi = \top$  or  $\varphi = p$ , follow immediately from the definition of  $\models_O$ , and the cases, where  $\varphi = \neg\psi$  or  $\varphi = \psi_1 \wedge \psi_2$ , can be easily verified by making use of the induction hypothesis. So let  $\varphi = \Box\psi$  and suppose that  $\psi$  satisfies the claim. Then  $\|\psi\|_O^{\mathcal{M}} = \|\psi\|^{\mathcal{M}'}$  and

$$\begin{aligned} \mathcal{M}, w \models_O \Box\psi &\Leftrightarrow |R[w] \cap \|\psi\|_O^{\mathcal{M}}| \text{ is odd} \\ &\Leftrightarrow |R[w] \cap \|\psi\|^{\mathcal{M}'}| \text{ is odd} \\ &\Leftrightarrow w \in F_R(\|\psi\|^{\mathcal{M}'}) \\ &\Leftrightarrow \mathcal{M}', w \models \Box\psi. \end{aligned}$$



Thus the models  $\mathcal{M}$  and  $\mathcal{M}'$  are pointwise equivalent with respect to  $\models_O$ .

Let then  $\langle W, F \rangle \in \mathcal{C}_{L_0}$ . Since  $R_F$  is a binary relation on  $W$ , we have that  $\langle W, R_F \rangle \in \mathcal{K}$ .

Let  $V$  be a valuation and let  $\mathcal{M} = \langle W, F, V \rangle$  and  $\mathcal{M}' = \langle W, R_F, V \rangle$ . We show that for all formulas  $\varphi$  and for all  $w \in W$ ,

$$\mathcal{M}, w \models \varphi \text{ if and only if } \mathcal{M}', w \models_O \varphi.$$

This is again done by induction. We consider only the case  $\varphi = \Box\psi$ , the other cases are straightforward. So let  $\varphi = \Box\psi$  where  $\psi$  satisfies the claim above. Then  $\|\psi\|^{\mathcal{M}} = \|\psi\|_O^{\mathcal{M}'}$  and

$$\begin{aligned} \mathcal{M}, w \models \Box\psi &\Leftrightarrow w \in F(\|\psi\|^{\mathcal{M}}) \\ &\stackrel{(*)}{\Leftrightarrow} |R_F[w] \cap \|\psi\|^{\mathcal{M}}| \text{ is odd} \\ &\Leftrightarrow |R_F[w] \cap \|\psi\|_O^{\mathcal{M}'}| \text{ is odd} \\ &\Leftrightarrow \mathcal{M}', w \models_O \Box\psi, \end{aligned}$$

where the equivalence  $(*)$  holds by Lemma 10. Hence the models  $\mathcal{M}'$  and  $\mathcal{M}$  are pointwise equivalent with respect to  $\models_O$ .

Let  $\langle W, R \rangle \in \mathcal{K}$  and let  $\langle W, F \rangle \in \mathcal{C}_{L_0}$ . We still have to prove that  $R_{F_R} = R$  and  $F_{R_F} = F$ . Let  $w \in W$  and  $X \subseteq W$ . Since for all  $v \in W$ ,

$$\begin{aligned} v \in R_{F_R}[w] &\Leftrightarrow w \in F_R(\{v\}) \\ &\Leftrightarrow |R[w] \cap \{v\}| \text{ is odd} \\ &\Leftrightarrow v \in R[w], \end{aligned}$$

and

$$\begin{aligned} w \in F_{R_F}(X) &\Leftrightarrow |R_F[w] \cap X| \text{ is odd} \\ &\stackrel{(*)}{\Leftrightarrow} w \in F(X), \end{aligned}$$

where the equivalence  $(*)$  holds again by Lemma 10, we can conclude that  $R_{F_R} = R$  and  $F_{R_F} = F$ .  $\square$

**Proposition 11.** *The pair  $\mathcal{L}_c$  corresponds to the clone  $L_c$ .*

*Proof.* The proof of Proposition 11 follows exactly the same steps as in the proof of Proposition 10. The clone  $L_c$  is defined by the functional terms

$$\mathbf{f}(\mathbf{1}), \mathbf{1} \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}),$$

and so the class  $\mathcal{C}_{L_c}$  is defined by the axioms  $\Box\top$  and  $\top \oplus \Box p \oplus \Box q \oplus \Box(p \oplus q)$ . These axioms are valid in a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  if and only if the conditions

- (i)  $F(W) = W$  and
- (ii)  $F(X) \oplus F(Y) = F(X \oplus Y)$

hold for all  $X, Y \subseteq W$ .

The translations  $R \mapsto F_R$  and  $F \mapsto R_F$  are the same as in Proposition 10.

Let  $\langle W, R \rangle \in \mathcal{K}_O$ . We show that  $\langle W, F_R \rangle \in \mathcal{C}_{L_c}$ . Since  $|R[w]|$  is odd for every  $w \in W$  it is clear that  $F_R(W) = W$ . By Lemma 10, the function  $F_R$  also satisfies the condition (ii). Thus  $\langle W, F_R \rangle \in \mathcal{C}_{L_c}$ .

Let  $\langle W, F \rangle \in \mathcal{C}_{L_c}$ . We have to show that  $|R_F[w]|$  is odd for every  $w \in W$ . Let  $w \in W$ . Since  $F(W) = W$ , it follows from Lemma 10 that  $R_F[w] \cap W$  has an odd number of elements. Since  $R_F[w] \cap W = R_F[w]$ , we have that  $|R_F[w]|$  is odd, and therefore  $\langle W, R_F \rangle \in \mathcal{K}_O$ .

The remainder of the proof is similar to the proof of Proposition 10.  $\square$

Let  $F$  be a set function  $\mathcal{P}(W) \rightarrow \mathcal{P}(W)$ . For the rest of the section we define the translation  $F \mapsto R_F$  by

$$R_F[w] = \{v \in W \mid w \in F(\{v\}) \Leftrightarrow w \notin F(\emptyset)\}$$

for all  $w \in W$ . Note that this definition coincides with that in Lemma 10 because in that case  $F(\emptyset) = \emptyset$ .

The following lemma will be useful in the sequel.

**Lemma 11.** *Let  $F$  be a set function  $\mathcal{P}(W) \rightarrow \mathcal{P}(W)$  which satisfies the condition*

$$(2) \quad F(\emptyset) \oplus F(X) \oplus F(Y) = F(X \oplus Y)$$

for all  $X, Y \subseteq W$ . Let  $X \subseteq W$  and let  $w \in W$ .

(a) *If  $w \notin F(\emptyset)$ , then  $w \in F(X)$  if and only if  $|R_F[w] \cap X|$  is odd.*

(b) *If  $w \in F(\emptyset)$ , then  $w \in F(X)$  if and only if  $|R_F[w] \cap X|$  is even.*

*Proof.* As mentioned before, if  $X = X_1 \oplus \dots \oplus X_k$  then  $w \in X$  if and only if  $w \in X_i$  for odd number of indices  $i$ . If  $w \notin F(\emptyset)$  then  $R_F[w] = \{v \in W \mid w \in F(\{v\})\}$ . We denote by  $X'$  the set  $X \cap \{v \in W \mid w \in F(\{v\})\}$ . And if  $w \in F(\emptyset)$  then  $R_F[w] = \{v \in W \mid w \notin F(\{v\})\}$ . For the set  $X \cap \{v \in W \mid w \notin F(\{v\})\}$  we use a notation  $X''$ .

To prove (a), suppose that  $w \notin F(\emptyset)$ . It follows from the condition (2) that

$$w \in F(X) \oplus F(Y) \text{ if and only if } w \in F(X \oplus Y)$$

and from this, it is easy to see that  $w \in F(X_1 \oplus \cdots \oplus X_k)$  if and only if  $w \in F(X_1) \oplus \cdots \oplus F(X_k)$ . Now the rest of the proof of (a) is essentially the same as the proof of Lemma 10.

For the part (b), suppose that  $w \in F(\emptyset)$ . By using the condition (2), it is easy to see that  $w \in F(X')$ . In addition, an easy induction on the cardinality of  $X''$  shows that  $w \in F(X'')$  if and only if  $|X''|$  is even. Since  $w \in F(\emptyset)$  and  $w \in F(X')$ , we have that  $w \in F(\emptyset) \oplus F(X') \oplus F(X'')$  if and only if  $w \in F(X'')$ . Since  $X' \oplus X'' = X$ , we get by the condition (2) that

$$\begin{aligned} w \in F(X) &\Leftrightarrow w \in F(X' \oplus X'') \\ &\Leftrightarrow w \in F(\emptyset) \oplus F(X') \oplus F(X'') \\ &\Leftrightarrow w \in F(X'') \\ &\Leftrightarrow |X''| \text{ is even.} \end{aligned}$$

This proves (b). □

Let  $\mathcal{M} = \langle W, R, V \rangle$  be a Kripke model and let  $w \in W$ . The truth relation  $\models_S$  is defined as  $\models_O$  except for the modal operator  $\Box$ , for which we define

$$\mathcal{M}, w \models_S \Box\varphi \text{ iff the sets } R[w] \text{ and } R[w] \cap \|\varphi\|_S^{\mathcal{M}} \text{ have the same parity,}$$

where  $\|\varphi\|_S^{\mathcal{M}}$  denotes the truth set of  $\varphi$  in the model  $\mathcal{M}$  with respect to  $\models_S$ .

We denote by  $\mathcal{L}_1$  the pair  $(\mathcal{K}, \models_S)$  where  $\mathcal{K}$  is the class of all Kripke frames.

**Proposition 12.** *The pair  $\mathcal{L}_1$  corresponds to the clone  $L_1$ .*

*Proof.* The clone  $L_1$  is defined by the functional terms

$$\mathbf{f}(\mathbf{1}), \mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}).$$

Thus the class  $\mathcal{C}_{L_1}$  is defined by the axioms  $\Box\top$  and  $\top \oplus \Box\perp \oplus \Box p \oplus \Box q \oplus \Box(p \oplus q)$ . It is easy to verify that these axioms are valid in a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  if and only if the conditions

- (i)  $F(W) = W$  and
- (ii)  $F(\emptyset) \oplus F(X) \oplus F(Y) = F(X \oplus Y)$

hold for all  $X, Y \subseteq W$ .

The translation  $R \mapsto F_R$  is given by

$$F_R(X) = \{w \in W \mid |R[w] \cap X| \text{ is odd} \Leftrightarrow |R[w]| \text{ is odd}\}$$

for all  $X \subseteq W$ .

Let  $\langle W, R \rangle \in \mathcal{K}$ . We have to show that  $\langle W, F_R \rangle \in \mathcal{C}_{L_1}$ , i.e.  $F_R$  satisfies the conditions (i) and (ii). It is immediate from the definition of  $F_R$  that  $F_R(W) = W$ . It also follows straightforwardly from the definition of  $F_R$  that for all  $w \in W$ ,  $w \in F_R(\emptyset)$  if and only if  $|R[w]|$  is even. Using this fact one can prove, similarly as in the proof of Proposition 10, that for all  $X, Y \subseteq W$  and for all  $w \in W$ ,

$$w \in F_R(\emptyset) \oplus F_R(X) \oplus F_R(Y) \text{ if and only if } w \in F_R(X \oplus Y).$$

Thus  $F_R$  satisfies also the condition (ii) and therefore  $\langle W, F_R \rangle \in \mathcal{C}_{L_1}$ .

Let  $V$  be a valuation and let  $\mathcal{M} = \langle W, R, V \rangle$  and  $\mathcal{M}' = \langle W, F_R, V \rangle$ . Similarly, as in the proof of Proposition 10, it can be proved that the models  $\mathcal{M}$  and  $\mathcal{M}'$  are pointwise equivalent with respect to  $\models_S$ .

Let  $\langle W, F \rangle \in \mathcal{C}_{L_1}$ . Then  $R_F$  is a relation on  $W$  and  $\langle W, R_F \rangle \in \mathcal{K}$ .

Let  $V$  be a valuation and let  $\mathcal{M} = \langle W, F, V \rangle$  and  $\mathcal{M}' = \langle W, R_F, V \rangle$ . By using Lemma 11, one can easily show that the models  $\mathcal{M}'$  and  $\mathcal{M}$  are pointwise equivalent with respect to  $\models_S$ .

Let  $\langle W, R \rangle \in \mathcal{K}$  and  $\langle W, F \rangle \in \mathcal{C}_{L_1}$ . Our final task is to prove that  $R_{F_R} = R$  and  $F_{R_F} = F$ . Let  $w, v \in W$ . Suppose that  $v \in R_{F_R}[w]$  and  $w \in F_R(\emptyset)$ . Now it follows from the definition of  $R_{F_R}$  that  $w \notin F_R(\{v\})$ . From the definition of  $F_R$  we get that  $|R[w] \cap \{v\}|$  is odd if and only if  $|R[w]|$  is even. Since  $w \in F_R(\emptyset)$  and  $|R[w] \cap \emptyset|$  is even, it has to be the case that  $|R[w]|$  is even. Thus  $|R[w] \cap \{v\}|$  is odd and hence  $v \in R[w]$ . A similar proof shows that  $v \in R[w]$  if  $w \notin F_R(\emptyset)$ .

Now suppose that  $v \in R[w]$ , i.e.  $|R[w] \cap \{v\}|$  is odd. If  $|R[w]|$  is also odd then  $w \in F_R(\{v\})$  and  $w \notin F_R(\emptyset)$  and thus  $v \in R_{F_R}[w]$  by the definition of  $R_{F_R}$ . Also, if  $|R[w]|$  is even, it is easy to see that  $v \in R_{F_R}[w]$ . Hence we have proved that  $R_{F_R} = R$ .

Similarly we can show that  $F_{R_F} = F$ . □

For the remaining two linear clones,  $L$  and  $LS$ , we consider Kripke frames with an additional unary relation  $U \subseteq W$ . Let  $\mathcal{K}^U$  be the class of all frames  $\mathcal{F} = \langle W, R, U \rangle$  where  $U \subseteq W$ . For the models based on these frames, we define a truth relation  $\models_M$  inductively as follows. Let  $\mathcal{M} = \langle \mathcal{F}, V \rangle$  be a model where  $\mathcal{F} \in \mathcal{K}^U$  and  $V$  is a valuation function, and let  $w \in W$ . For constants, proposition symbols and Boolean connectives  $\models_M$  is defined as  $\models$  in Kripke semantics. Let  $\varphi$  be a modal formula. If  $w \in U$  then

$$\mathcal{M}, w \models_M \Box \varphi \text{ if and only if } |R[w] \cap \|\varphi\|_M^{\mathcal{M}}| \text{ is odd,}$$

and if  $w \notin U$  then

$$\mathcal{M}, w \models_M \Box \varphi \text{ if and only if } |R[w] \cap \|\varphi\|_M^{\mathcal{M}}| \text{ is even.}$$

Here  $\|\varphi\|_M^{\mathcal{M}}$  denotes the truth set of  $\varphi$  in the model  $\mathcal{M}$  with respect to  $\models_M$ . Note that the frames  $\langle W, R, U \rangle$  are actually the same as the non-normal Kripke frames in Section 4.2. However, here we use the symbol  $U$  instead of  $N$  to make it clear that now we are not dealing with non-normal worlds.

We denote by  $\mathcal{L}$  the pair  $(\mathcal{K}^U, \models_M)$ . Let  $\mathcal{K}_O^U$  be the class of all frames  $\langle W, R, U \rangle$  in  $\mathcal{K}^U$  such that every  $w \in W$  has an odd number of  $R$ -successors. We denote by  $\mathcal{L}_S$  the pair  $(\mathcal{K}_O^U, \models_M)$ .

The notion of a pair corresponding to a clone is defined similarly as in Section 4.2. In the sequel, instead of giving operations  $\Gamma$  and  $\Delta$ , we only define translations  $(R, U) \mapsto F_{(R,U)}$  and  $F \mapsto (R_F, U_F)$ .

**Proposition 13.** *The pair  $\mathcal{L}$  corresponds to the clone  $L$ .*

*Proof.* The clone  $L$  is defined by the functional term

$$\mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}).$$

Thus the class  $\mathcal{C}_L$  of Scott-Montague frames is defined by the axiom  $\top \oplus \Box \perp \oplus \Box p \oplus \Box q \oplus \Box(p \oplus q)$ . This axiom is valid in a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  if and only if the condition

$$(i) \quad F(\emptyset) \oplus F(X) \oplus F(Y) = F(X \oplus Y)$$

holds for all  $X, Y \subseteq W$ .

The translations  $(R, U) \mapsto F_{(R,U)}$  and  $F \mapsto (R_F, U_F)$  are defined as follows. For all  $X \subseteq W$ ,

$$F_{(R,U)}(X) = \{w \in W \mid w \in U \text{ and } |R[w] \cap X| \text{ is odd}\} \cup \{w \in W \mid w \notin U \text{ and } |R[w] \cap X| \text{ is even}\},$$

$R_F$  is defined as before Lemma 11 and  $U_F = \{w \in W \mid w \notin F(\emptyset)\}$ .

Let  $\langle W, R, U \rangle \in \mathcal{K}^U$ . As in the proof of Proposition 10, we can show that for all  $X, Y \subseteq W$  and for all  $w \in W$ ,

$$w \in F_{(R,U)}(\emptyset) \oplus F_{(R,U)}(X) \oplus F_{(R,U)}(Y) \text{ if and only if } w \in F_{(R,U)}(X \oplus Y).$$

Thus  $F_{(R,U)}$  satisfies the condition (i) and  $\langle W, F_{(R,U)} \rangle \in \mathcal{C}_L$ .

Let  $V$  be a valuation function and let  $\mathcal{M} = \langle W, R, U, V \rangle$  and  $\mathcal{M}' = \langle W, F_{(R,U)}, V \rangle$ . We show that for all modal formulas  $\varphi$  and for all  $w \in W$ ,

$$\mathcal{M}, w \models_M \varphi \text{ if and only if } \mathcal{M}', w \models \varphi.$$

The proof is by induction on the construction of  $\varphi$ . We consider only the case where  $\varphi$  is of the form  $\Box\psi$ , the other ones are straightforward. So let

$\varphi = \Box\psi$  and suppose that the claim holds for  $\psi$ , whence  $\|\psi\|_M^{\mathcal{M}} = \|\psi\|_M^{\mathcal{M}'}$ . Let  $w \in U$ . Then

$$\begin{aligned} \mathcal{M}, w \models_M \Box\psi &\Leftrightarrow |R[w] \cap \|\psi\|_M^{\mathcal{M}}| \text{ is odd} \\ &\Leftrightarrow |R[w] \cap \|\psi\|_M^{\mathcal{M}'}| \text{ is odd} \\ &\Leftrightarrow w \in F_{(R,U)}(\|\psi\|_M^{\mathcal{M}'}) \\ &\Leftrightarrow \mathcal{M}', w \models \Box\psi. \end{aligned}$$

Similarly,  $\mathcal{M}, w \models_M \Box\psi$  if and only if  $\mathcal{M}', w \models \Box\psi$ , whenever  $w \notin U$ . Thus the models  $\mathcal{M}$  and  $\mathcal{M}'$  are pointwise equivalent with respect to  $\models_M$ .

Let  $\langle W, F \rangle \in \mathcal{C}_L$ . Since  $R_F$  is a binary relation on  $W$  and  $U_F$  is a subset of  $W$ , we have that  $\langle W, R_F, U_F \rangle \in \mathcal{K}^U$ .

Let  $V$  be a valuation and let  $\mathcal{M} = \langle W, F \rangle$  and  $\mathcal{M}' = \langle W, R_F, U_F \rangle$ . As above, it can be shown that the models  $\mathcal{M}'$  and  $\mathcal{M}$  are pointwise equivalent with respect to  $\models_M$ .

Let  $\langle W, R, U \rangle \in \mathcal{K}^U$  and  $\langle W, F \rangle \in \mathcal{C}_L$ . We still have to show that  $R_{F_{(R,U)}} = R$ ,  $U_{F_{(R,U)}} = U$  and  $F_{(R_F, U_F)} = F$ . Let  $w, v \in W$ . It is easy to see that

$$w \in U \Leftrightarrow w \notin F_{(R,U)}(\emptyset) \Leftrightarrow w \in U_{F_{(R,U)}}.$$

Also, it is not difficult to see that

$$\begin{aligned} v \in R[w] &\Leftrightarrow w \in \overline{U} \oplus F_{(R,U)}(\{v\}) \\ &\Leftrightarrow w \in F_{(R,U)}(\emptyset) \oplus F_{(R,U)}(\{v\}) \\ &\Leftrightarrow v \in R_{F_{(R,U)}}[w]. \end{aligned}$$

Thus  $R_{F_{(R,U)}} = R$  and  $U_{F_{(R,U)}} = U$ .

Let  $X \subseteq W$  and  $w \in W$ . Suppose that  $w \in F_{(R_F, U_F)}(X)$ . Let  $w \in F_{(R_F, U_F)}(\emptyset)$ . Then  $w \notin U_F$  and furthermore  $w \in F(\emptyset)$ . Since  $w \notin U_F$  and  $w \in F_{(R_F, U_F)}(X)$ , we can conclude that  $|R_F[w] \cap X|$  is even. Since  $w \in F(\emptyset)$ , it follows from Lemma 11 that  $w \in F(X)$ . Similarly we can prove that  $w \in F(X)$  if  $w \notin F_{(R_F, U_F)}(\emptyset)$ .

Let  $w \in F(X)$ . Assume first that  $w \in F(\emptyset)$ . Then  $w \notin U_F$ . Since  $w \in F(\emptyset)$  and  $w \in F(X)$ , it follows from Lemma 11 that  $|R_F[w] \cap X|$  is even, and hence we get that  $w \in F_{(R_F, U_F)}(X)$ . A similar proof shows that  $w \in F_{(R_F, U_F)}(X)$  when  $w \notin F(\emptyset)$ . Thus  $F_{(R_F, U_F)} = F$ .  $\square$

With similar methods, using the same translations  $(R, U) \mapsto F_{(R,U)}$  and  $F \mapsto (R_F, U_F)$  as in the proof of Proposition 13, we can also prove the following proposition which completes the cases of linear clones.

**Proposition 14.** *The pair  $\mathcal{L}_S$  corresponds to the clone  $LS$ .*  $\square$

## 5.2 The remaining subclones of $L$

In this section we will give the characterizations of the classes of Scott-Montague frames corresponding to the subclones of  $\Omega(1)$  and  $I^*$  of  $L$  by means of modified Kripke semantics. This will complete our aim to characterize all the classes of Scott-Montague frames corresponding to the subclones of  $V$ ,  $\Lambda$  and  $L$ . We will consider Kripke frames with an additional unary relation as in the previous section and we use  $\models_M$  as semantics. Furthermore, we will use some natural restrictions of the binary relation  $R$ .

Let  $\mathcal{K}^U$  be the class of all frames  $\langle W, R, U \rangle$ , where  $U \subseteq W$ , as in the previous section. We denote by  $\mathcal{K}_p^U$  the class of all frames  $\langle W, R, U \rangle$  in  $\mathcal{K}^U$  such that  $R$  is a partial function and we denote by  $\mathcal{K}_f^U$  the class of all frames  $\langle W, R, U \rangle$  in  $\mathcal{K}^U$  such that  $R$  is a function.

**Proposition 15.** *The pair  $(\mathcal{K}_p^U, \models_M)$  corresponds to the clone  $\Omega(1)$ .*

*Proof.* The clone  $\Omega(1)$  is defined by the functional terms

$$\mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y}), (\mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{x} \wedge \mathbf{y})) \rightarrow (\mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y})).$$

Thus, by Theorem 5, the class  $\mathcal{C}_{\Omega(1)}$  of Scott-Montague frames is defined by the axioms  $\top \oplus \Box \perp \oplus \Box p \oplus \Box q \oplus \Box(p \oplus q)$  and  $(\Box p \oplus \Box(p \wedge q)) \rightarrow (\Box p \oplus \Box q)$ . One can easily verify that these axioms are valid in a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  if and only if the conditions

- (1)  $F(\emptyset) \oplus F(X) \oplus F(Y) = F(X \oplus Y)$  and
- (2)  $F(X) \oplus F(X \cap Y) \subseteq F(X) \oplus F(Y)$

hold for every  $X, Y \subseteq W$  and for every  $w \in W$ .

We define the translations  $(R, U) \mapsto F_{(R,U)}$  and  $F \mapsto (R_F, U_F)$  in the following way. For all  $X \subseteq W$ ,

$$F_{(R,U)}(X) = \{w \in W \mid w \in U \text{ and } |R[w] \cap X| \text{ is odd}\} \cup \{w \in W \mid w \notin U \text{ and } |R[w] \cap X| \text{ is even}\},$$

for all  $w \in W$ ,

$$R_F[w] = \{v \in W \mid w \in F(\{v\}) \Leftrightarrow w \notin F(\emptyset)\}$$

and  $U_F = \{w \in W \mid w \notin F(\emptyset)\}$ .

Let  $\langle W, R, U \rangle \in \mathcal{K}_p^U$ . Now  $\langle W, R, U \rangle \in \mathcal{K}^U$  and it was proved in Proposition 13 that in this case  $F_{(R,U)}$  satisfies the condition (1). We will show that

$F_{(R,U)}$  satisfies also the condition (2). Let  $X, Y \subseteq W$  and  $w \in W$ . Suppose that  $w \in F_{(R,U)}(X) \oplus F_{(R,U)}(X \cap Y)$ , that is

$$w \in F_{(R,U)}(X) \text{ if and only if } w \notin F_{(R,U)}(X \cap Y).$$

First, consider that  $w \in U$ . Then it has to be the case that  $w \in F_{(R,U)}(X)$ . If  $w \notin F_{(R,U)}(X)$  then, by the definition of  $F_{(R,U)}$ , the cardinality of the set  $R[w] \cap X$  would be even. Since  $R$  is a partial function,  $R[w] = \emptyset$  or  $R[w]$  is a singleton, and hence  $R[w] \cap X = \emptyset$ . Now also  $R[w] \cap (X \cap Y) = \emptyset$  and therefore  $w \notin F_{(R,U)}(X \cap Y)$ . Thus  $w \notin F_{(R,U)}(X) \oplus F_{(R,U)}(X \cap Y)$ , which would be a contradiction. So,  $w \in F_{(R,U)}(X)$  and hence  $|R[w] \cap X|$  is odd. Since  $R$  is a partial function,  $R[w] = R[w] \cap X = \{v\}$  for some  $v \in W$ . Since  $w \in F_{(R,U)}(X)$ , it follows from the assumption that  $w \notin F_{(R,U)}(X \cap Y)$ . Therefore  $|R[w] \cap (X \cap Y)|$  is even, which means that  $R[w] \cap (X \cap Y) = \emptyset$ . Clearly,  $R[w] \cap Y = \emptyset$  and hence  $w \notin F_{(R,U)}(Y)$ . Thus  $w \in F_{(R,U)}(X) \oplus F_{(R,U)}(Y)$ .

Suppose then that  $w \notin U$ . Now we can conclude, similarly as above, that  $w \notin F_{(R,U)}(X)$ . Thus, by the assumption,  $w \in F_{(R,U)}(X \cap Y)$ . It follows from the definition of  $F_{(R,U)}$  that  $|R[w] \cap (X \cap Y)|$  is even. Since  $R$  is a partial function, we have that  $R[w] \cap (X \cap Y) = \emptyset$ . Since  $w \notin F_{(R,U)}(X)$ ,  $|R[w] \cap X|$  is odd and, similarly as above,  $R[w] \cap Y = \emptyset$ . Thus  $w \in F_{(R,U)}(Y)$  and therefore  $w \in F_{(R,U)}(X) \oplus F_{(R,U)}(Y)$ . So we have proved that  $F_{(R,U)}$  satisfies also the condition (2) and hence  $\langle W, F_{(R,U)} \rangle \in \mathcal{C}_{\Omega(1)}$ .

Let  $V$  be a valuation function and let  $\mathcal{M} = \langle W, R, U, V \rangle$  and  $\mathcal{M}' = \langle W, F_{(R,U)}, V \rangle$ . Since  $\langle W, R, U \rangle \in \mathcal{K}_p^U \subseteq \mathcal{K}^U$ , we know by Proposition 13 that the models  $\mathcal{M}$  and  $\mathcal{M}'$  are pointwise equivalent with respect to  $\models_M$ .

Let  $\langle W, F \rangle \in \mathcal{C}_{\Omega(1)}$ . Now  $\langle W, F \rangle \in \mathcal{C}_L$  and it was proved also in Proposition 13 that in this case, for all valuations  $V$ , the models  $\langle W, R_F, U_F, V \rangle$  and  $\langle W, F, V \rangle$  are pointwise equivalent with respect to  $\models_M$ . We will show that  $\langle W, R_F, U_F \rangle \in \mathcal{K}_p^U$ . Clearly  $U_F$  is a subset of  $W$  and  $R_F$  is a binary relation on  $W$ , and hence it is sufficient to show that  $R_F$  is a partial function. Suppose that  $R_F$  is not a partial function, i.e. there exist  $w, v_1, v_2 \in W$  such that  $v_1 \neq v_2$  and  $wR_Fv_1$  and  $wR_Fv_2$ . Consider a model  $\mathcal{M} = \langle W, R_F, U_F, V \rangle$  where  $V(p) = \{v_1\}$  and  $V(q) = \{v_2\}$ . Suppose first that  $w \notin F(\emptyset)$ . Now  $w \in U_F$ ,  $R_F[w] \cap \|p\|^{\mathcal{M}} = \{v_1\}$ ,  $R_F[w] \cap \|q\|^{\mathcal{M}} = \{v_2\}$  and  $R_F[w] \cap \|p \wedge q\|^{\mathcal{M}} = \emptyset$ . Thus, by the definition of  $\models_M$ ,  $\mathcal{M}, w \models_M \Box p$ ,  $\mathcal{M}, w \models_M \Box q$  and  $\mathcal{M}, w \not\models_M \Box(p \wedge q)$ . Therefore  $\mathcal{M}, w \models_M \Box p \oplus \Box(p \wedge q)$  and  $\mathcal{M}, w \not\models_M \Box p \oplus \Box q$ , and thus

$$\mathcal{M}, w \not\models_M (\Box p \oplus \Box(p \wedge q)) \rightarrow (\Box p \oplus \Box q).$$

A similar proof shows that also if  $w \in F(\emptyset)$ , then  $\mathcal{M}, w \not\models_M (\Box p \oplus \Box(p \wedge q)) \rightarrow (\Box p \oplus \Box q)$ . This is a contradiction, since the frame  $\langle W, F \rangle$  validates the



formula above and, by pointwise equivalence, so does the frame  $\langle W, R_F, U_F \rangle$ . Thus  $R_F$  has to be a partial function, and therefore  $\langle W, R_F, U_F \rangle \in \mathcal{K}_p^U$ .

In addition, we proved in Proposition 13 that  $R_{F_{(R,U)}} = R$ ,  $U_{F_{(R,U)}} = U$  and  $F_{(R_F, U_F)} = F$ .  $\square$

**Proposition 16.** *The pair  $(\mathcal{K}_f^U, \models_M)$  corresponds to the clone  $I^*$ .*

*Proof.* The clone  $I^*$  is defined by the functional terms

- (i)  $\mathbf{1} \oplus \mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}) \oplus \mathbf{f}(\mathbf{x} \oplus \mathbf{y})$ ,
- (ii)  $(\mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{x} \wedge \mathbf{y})) \rightarrow (\mathbf{f}(\mathbf{x}) \oplus \mathbf{f}(\mathbf{y}))$  and
- (iii)  $\mathbf{f}(\mathbf{0}) \oplus \mathbf{f}(\mathbf{1})$ .

Therefore the class  $\mathcal{C}_{I^*}$  is defined by the corresponding axioms  $\top \oplus \square \perp \oplus \square p \oplus \square q \oplus \square(p \oplus q)$ ,  $(\square p \oplus \square(p \wedge q)) \rightarrow (\square p \oplus \square q)$  and  $\square \perp \oplus \square \top$ . These axioms are valid in a Scott-Montague frame  $\mathcal{F} = \langle W, F \rangle$  if and only if the conditions

- (1)  $F(\emptyset) \oplus F(X) \oplus F(Y) = F(X \oplus Y)$ ,
- (2)  $F(X) \oplus F(X \cap Y) \subseteq F(X) \oplus F(Y)$  and
- (3)  $F(\emptyset) \oplus F(W) = W$

hold for every  $X, Y \subseteq W$ .

Let  $(R, U) \mapsto F_{(R,U)}$  and  $F \mapsto (R_F, U_F)$  be the translations defined in the proof of Proposition 15. In addition to the proof of Proposition 15, all we need to show is that if  $\langle W, R, U \rangle \in \mathcal{K}_f^U$  then  $F_{(R,U)}$  satisfies the condition (3), and if  $\langle W, F \rangle \in \mathcal{C}_{I^*}$  then  $R_F$  is a function.

Let  $\langle W, R, U \rangle \in \mathcal{K}_f^U$  and let  $w \in W$ . Suppose that  $w \in U$ . Then clearly  $w \notin F_{(R,U)}(\emptyset)$ . Since  $R$  is a function,  $R[w]$  is a singleton and thus  $w \in F_{(R,U)}(W)$ . Thus  $w \in F_{(R,U)}(\emptyset) \oplus F_{(R,U)}(W)$ . Also, if  $w \notin U$ , it is easy to see that  $w \in F_{(R,U)}(\emptyset) \oplus F_{(R,U)}(W)$ . Hence  $F_{(R,U)}$  satisfies the condition (3).

Let then  $\langle W, F \rangle \in \mathcal{C}_{I^*}$ . We know by the proof of Proposition 15 that  $R_F$  is a partial function. So we have show that  $R_F[w] \neq \emptyset$  for all  $w \in W$ . Suppose that there exists  $w \in W$  such that  $R_F[w] = \emptyset$ . Consider some model  $\mathcal{M} = \langle W, R_F, U_F, V \rangle$ . If  $w \in U_F$  then  $\mathcal{M}, w \not\models_M \square \perp$  and  $\mathcal{M}, w \not\models_M \square \top$ , and hence  $\mathcal{M}, w \not\models_M \square \perp \oplus \square \top$ . Similarly, if  $w \notin U_F$  then  $\mathcal{M}, w \not\models_M \square \perp \oplus \square \top$ . This is a contradiction, since we can show, by using the pointwise equivalence of the models  $\mathcal{M}$  and  $\langle W, F, V \rangle$ , that the model  $\mathcal{M}$  validates the formula  $\square \perp \oplus \square \top$ . Therefore  $R_F$  is a function.  $\square$

## 6 Conclusions

We have established a complete correspondence between definability of Boolean functions by functional terms and definability of finite Scott-Montague frames by uniform degree-1 formulas. We introduced a bijective translation between functional terms and uniform degree-1 formulas and, based on this, we showed that a class of Boolean functions is defined by functional terms if and only if the corresponding class of Scott-Montague frames is defined by the translations of these functional terms, and vice versa. Furthermore, we characterized classes of Scott-Montague frames corresponding to the clone  $\Lambda_1$  and to some particular subclones of  $\Lambda_1$  by classes of Kripke frames. By modifying Kripke semantics, we were also able to characterize the classes of Scott-Montague frames corresponding to the other subclones of  $V$ ,  $\Lambda$ , and  $L$ .

At this point the obvious question is: For which of the remaining clones the corresponding class of Scott-Montague frames can be characterized by further modifications of Kripke semantics? Some interesting clones, such as  $S$ ,  $M$  and  $SM$ , seem to be quite difficult to handle. In the case of the subclones of  $V$ ,  $\Lambda$  and  $L$ , the associativity of the generating functions of the clones seems to be crucial. There is a qualitative difference when associativity does not hold. For example, the clone  $SM$  is generated by the majority of three variables, which is not an associative function. Nevertheless, it would be interesting to see whether the classes of Scott-Montague frames corresponding to the clones in the top of the Post Lattice can also be characterized by modifying Kripke semantics in a different way.

## References

- [1] P. Blackburn, M. de Rijke, Y. Venema. *Modal Logic*, Cambridge University Press, Cambridge, 2001.
- [2] B. F. Chellas. *Modal Logic, an Introduction*, Cambridge University Press, 1980.
- [3] M. Couceiro, L. Hella, J. Kivelä. “Term definable classes of Boolean functions and frame definability in modal logic”, Report A 361 Dec 2005, University of Tampere, <http://www.math.tut.fi/algebra/>. To appear in *Logic Journal of the IGPL*.
- [4] O. Ekin, S. Foldes, P. L. Hammer, L. Hellerstein. “Equational Characterizations of Boolean Functions Classes”, *Discrete Mathematics*, **211** (2000) 27–51.
- [5] S. Foldes, G. Pogoyan. “Post Classes Characterized by Functional Terms”, *Discrete Applied Mathematics* **142** (2004) 35–51.
- [6] B. Hansson, P. Gärdenfors “A Guide to Intensional Semantics”, *Modality, Morality and Other Problems of Sense and Nonsense: Essays Dedicated to Sören Halldén*, ss. 151-67, Gleerup, Lund, 1973.
- [7] J. Kivelä. “Frame Classes Equipped with Modified Kripke Semantics Corresponding to the Subclones of  $V$ ,  $\Lambda$  and  $L$ ”, Report A 369 Aug 2006, University of Tampere.
- [8] S. Kripke. “Semantical Analysis of Modal Logic II. Non-normal Modal Propositional Calculi”, *The Theory of Models*, North-Holland, Amsterdam, 1965, pp. 206–220.
- [9] G. Pogoyan. “Classes of Boolean Functions Defined by Functional Terms”, *Multiple Valued Logic*, **7** (2002) 417–448.
- [10] E. L. Post. “The Two-Valued Iterative Systems of Mathematical Logic”, *Annals of Mathematical Studies* **5** (1941) 1–122.
- [11] K. Segerberg. “An Essay in Classical Modal Logic”, *Filosofiska Studier utgivna av Filosofiska Föreningen och Filosofiska Institutionen vid Uppsala Universitet, nr 13*, Uppsala, 1971.
- [12] A. Virtanen. “Essee Deskriptiivisestä Modaalilogiikasta”, manuscript, Tampere, 1994