

Modal Logics and Definability

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Abstract. In recent years, research into the mathematical foundations of modal logic has become increasingly popular. One of the main reasons for this is the fact that modal logic seems to adapt well to the requirements of a wide range of different fields of application. This paper is a summary of some of the author's contributions to modal *definability theory*. Much of the paper should be accessible to non-specialists.

1 Introduction

The formal study of modal notions, such as possibility and necessity, has a long history, reaching back all the way to Aristotle. During the latter half of the 20th century, modal logic developed fast, and currently the field has a wide range of applications in different disciplines ranging from computer science to economics. In fact, modal logic has turned out to be very successful from the point of view of applications. The following extract is from [2].

”Modal logic today is a vast family of studies of modal notions, with the original philosophical and mathematical motivations still alive, but with an increasing symbiosis with other fields, and in particular, with computer science. Indeed, its interface with computer science (and more generally, informatics) is extremely broad, ranging from hardware and software verification, to ontologies in medical and bio-informatics, and the analysis of query languages for XML documents.”

The success of modal logic in relation to applications is due to the following factors.

1. Modal logics are *natural* (from a variety of points of view) for a wide range of applications. The language of modal logic often greatly resembles the natural language used in order to speak about a class of phenomena.
2. Reasoning with modal logics can very often be fully *automated*. Furthermore, the algorithms involved tend to be relatively *efficient*.

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A sufficiently developed framework tends to require a strong mathematical background theory. Research into the mathematical foundations of modal logic has developed fast in the 20th century, especially after the development of Kripke semantics (see [3]). This paper is a summary of the author’s main contributions to the understanding of model theory of modal logic. The focus of the author’s research has been on the definability theory of very expressive modal systems.

The objective in definability theory is to classify formal logics according to their expressive power. For example, using the standard formal logic FO (first-order logic), one can write the sentence $\exists x(P(x))$ which states that “there exists an element x in the set named P ”. In other words, the sentence states that the set named P is not empty. However, one cannot write a sentence of FO asserting that the set named P is infinite. In the more expressive system SO (second-order logic), one *can* write a sentence asserting that the set named P is infinite. There are various reasons why the use of a *less* expressive logic rather than a more expressive one is often desired. One of the most important of these reasons is the trade-off between the expressive power of a logic and the computational resources needed for automated reasoning with the logic. Reasoning with a more expressive logic tends to require more computation time. The results discussed in this paper are related to the classification of different logics according to their expressive power. Such results are often useful when choosing a logic suitable for the formalization of a system that requires automated reasoning tools.

2 Preliminary Considerations

In this section, we informally describe the logics that the results in the following section are related to. However, we will *not give a complete formal definition* of the systems here. The interested reader is referred to [9] and [10] for the related definitions and to [3] and [5] for background theory.

1. Propositional logic is a system where we have for example statements of the type $P \wedge Q$ (read “ P and Q ”) and $\neg P$ (read “not P ”). The statements P and Q are atomic statements. Depending on the application, the atomic statements could be statements such as “ $\log y \geq 100$ ” or “Program `check_status` will not terminate.”. In propositional logic, atomic statements are combined with the connectives \neg , \wedge , \vee , \rightarrow and \leftrightarrow (read “not”, “and”, “or”, “implies” and “if and only if”, respectively).
2. The system ML (multimodal logic) extends propositional logic by statements of the type $\langle R \rangle \varphi$. The statement can be read for example as “ φ can be obtained by doing R ” or “ φ is possible via R ”. Here the statement φ always corresponds to some set of elements and R corresponds to a binary relation between elements.
3. The system SOPML (second-order propositional modal logic) is a very expressive extension of ML with statements such as $\exists P(\langle R \rangle P)$, which states that *there exists some set P* such that $\langle R \rangle P$ holds.
4. The systems $\Sigma_1^1(\text{ML})$ and $\Sigma_1^1(\text{BML}^\equiv)$ are very expressive extensions of the system ML. In the system $\Sigma_1^1(\text{ML})$ we can make assertions such as $\exists R(\langle R \rangle \varphi)$, stating that

there exists some binary relation R such that $\langle R \rangle \varphi$ can be made true. The system $\Sigma_1^1(\text{BML}^\equiv)$ is even stronger than $\Sigma_1^1(\text{ML})$. In $\Sigma_1^1(\text{BML}^\equiv)$ we can for example make statements such as $\exists R(\langle R' \rangle \varphi)$, which can be read as " φ is possible via the complement of R " or " φ is possible by doing not R ".

5. The system $\exists\text{MSO}$ is a very expressive system where we can make statements such as $\forall x \exists y (R(x, y) \wedge P(y))$, which states that "for all elements x there is some element y such that R connects x to y and y is in the set named P ". Furthermore, we can even make statements of the type $\exists P(\varphi(P))$, which states that *there exists some set, let us call this set P* , such that the statement $\varphi(P)$ is true. Here, the statement $\varphi(P)$ is some statement about P allowed by the system. For example, $\varphi(P)$ could be the statement $\forall x \exists y (R(x, y) \wedge P(y))$ from above.

3 A Sample of Results

The results in this section are from [9] and [10]. The interested reader finds a detailed presentation of the results from there. For background theory, see [3] and [5]. The article [9] is joint work with Lauri Hella.

The following theorem states that the expressive power of SOPML (second-order propositional modal logic) grows without bound as the number of alternations between blocks of universal and existential quantifiers is increased in a quantifier prefix of a modal formula. The theorem answers an open problem from [1] (also addressed in [4]).

Theorem 3.1 ([10]). *The alternation hierarchy of second-order propositional modal logic (SOPML) is infinite.*

The following theorem shows that prenex quantified *binary* relations can be eliminated from modal formulae when the universal modality is added to the system.

Theorem 3.2 ([9]). *There is an effective translation from the system $\Sigma_1^1(\text{ML})$ into the system $\exists\text{MSO}(\text{MLE})$.*

The theorem gives a method for establishing decidability results in multimodal logic. The following corollary is an immediate consequence of the theorem.

Corollary 3.3. Let \mathcal{D} be a class of Kripke frames (W, R_0) . Consider the class $\mathcal{C} = \{ (W, \{R_i\}_{i \in \mathbb{N}}) \mid (W, R_0) \in \mathcal{D} \}$ of multimodal Kripke frames. Now, if the satisfiability problem for MLE w.r.t. \mathcal{D} is decidable, then the satisfiability problem for ML w.r.t. \mathcal{C} is decidable.

The following result is similar to Theorem 3.2.

Theorem 3.4 ([9]). *There is an effective translation from the system $\Sigma_1^1(\text{BML}^\equiv)$ into the system $\exists\text{MSO}$.*

Again we obtain a method for establishing decidability results.

Corollary 3.5. *Let V and $U \subseteq V$ be sets of indices. Let \mathcal{D} be a class of Kripke frames $(W, \{R_j\}_{j \in U})$. Consider the class*

$$\mathcal{C} = \{ (W, \{R_i\}_{i \in V}) \mid (W, \{R_j\}_{j \in U}) \in \mathcal{D} \}$$

of Kripke frames. Now, if the \forall MMSO-theory of \mathcal{D} is decidable, then the satisfiability problem for BML^\forall w.r.t. \mathcal{C} is decidable.

We note that BML^\forall subsumes a large number of typical extensions of multimodal logic such as modal logic with universal modality [7], modal logic with difference modality [12] and Boolean Modal Logic [6]. Therefore, Theorem 3.4 and Corollary 3.5 can be applied to a whole range of typical modal logics. The following example illustrates the use of Corollary 3.5.

Example 3.6. Consider the class \mathcal{C} of Kripke frames $\mathfrak{F} = (W, R_0, R_1, \dots)$, where W is countably infinite and $R_0^\mathfrak{F}$ is a dense linear ordering of W without endpoints. Assume we wish to know whether the satisfiability problem of, say, multimodal logic with the difference modality w.r.t. \mathcal{C} is decidable. The MSO-theory of $(\mathbb{Q}, <)$ is known to be decidable [11]. Thus, by Corollary 3.5, we can immediately answer "yes" to our question about decidability of the satisfiability problem.

4 Concluding Remarks

Nowadays, non-classical logics such as modal logic have a wide range of applications. Non-classical logics often offer a more suitable framework for an application than classical first-order logic. In addition to their applicability, non-classical logics are also fascinating for a variety of different reasons. For example, Gödel's (in)famous first incompleteness theorem leads to the realization that the first-order theory of arithmetic is not effectively axiomatizable. However, it is immediate that by considering a sufficiently weaker logic, the related theory of arithmetic can be made even finitely axiomatizable.

Currently, the main field of application of modal logic is computer science. Definability theory of modal logic plays a role in the building of the mathematical foundations of modal logic. In this paper we have had a look at the author's main contributions to the understanding of modal definability theory. The results are related to very expressive modal systems. In the future, the author intends to generalize some of the results. In particular, it remains to be investigated whether the alternation hierarchy of SOPML is *strict*. Also, the theorem concerning $\Sigma_1^1(\text{BML}^\forall)$ can perhaps serve as a stepping stone towards settling the open question of Grädel and Rosen (see [8]) whether a certain system $\Sigma_1^1(\text{FO}^2)$ is weaker in expressive power (over the class of directed graphs) than the system $\exists\text{MSO}$. [*End of the original TISE seminar article*]

SOPML can more or less directly be used for example in a distributed computing context. Second-order quantifiers roughly correspond to random input bits to nodes, and $\langle R \rangle P$ means that the evaluation node receives the message P from some neighbour. This extends the modal framework for constant-time distributed computation initiated by Hella and co-authors in a PODC 2012 paper and extended to non-constant-time contexts with finite and infinite state spaces by the author (Kuusisto 2013, submitted). Distributed systems can rather flexibly be used to model a wide range of phenomena. Crystal lattices,

the brain, traffic systems, and indeed, more or less everything, can more or less naturally be regarded as a distributed communication system.

The approach where everything is modelled by a distributed system or a multiagent system—or more generally, a computation model—provides a reasonably unified and highly sensible algorithmic approach to science. In an ideal case, discrete (toy) models can—in some reasonable sense—explain phenomena that can otherwise be only described. Compared to traditional ad hoc approaches, discrete systems are more likely to lead to more transparent perspectives on infinities, undecidability, and other burdens on the underlying mathematical level, thereby shedding light on the underlying information theoretic issues.

The approach is full of philosophically interesting issues, some of which are rather delightful. The computing universe of Zuse is sometimes understood in the funny way that the Universe *is* a computer. While this sounds like a nice sci-fi idea, it is certainly true that the Universe (and more or less everything else) can be sensibly *modelled by* discrete computation models. And ideally this can be fruitful. But of course nothing is ever perfect. Weinberg has criticized Wolfram's CA approach for being inherently local. While a CA is a local device, it is rather trivial to define similar non-local devices. One can consider for example different kinds of automata walking on (infinite) grids and directed graphs. These are similar to Turing machine heads. There is no problem in defining an automaton consisting of multiple heads and a single control unit. The heads can move around and different heads can be on different nodes. Still, the heads can be controlled by a single unit and therefore be in some a kind of non-local contact with each other. Furthermore, indeterminism—which is in some sense inherently related to non-closedness of (discrete) systems—can be modelled by external inputs to nodes. For example (infinite) bit strings can do well in this context. The possibilities are endless indeed. And logic can play a cool role in the related background theory.

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