



UNIVERSITY
OF TAMPERE

This document has been downloaded from
Tampub – The Institutional Repository of University of Tampere

Publisher's version

Authors: Haukkanen Pentti
Name of article: Some characterizations of specially multiplicative functions
Year of publication: 2003
Name of journal: International Journal of Mathematics and Mathematical Sciences
Volume: 2003
Number of issue: 37
Pages: 2335-2344
ISSN: 1687-0425
Discipline: Natural sciences / Mathematics
Language: en
School/Other Unit: School of Information Sciences

URL: <http://www.hindawi.com/journals/ijmms/2003/515979/abs/>

URN: <http://urn.fi/urn:nbn:uta-3-736>

DOI: <http://dx.doi.org/10.1155/S0161171203301139>

All material supplied via TamPub is protected by copyright and other intellectual property rights, and duplication or sale of all part of any of the repository collections is not permitted, except that material may be duplicated by you for your research use or educational purposes in electronic or print form. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone who is not an authorized user.

SOME CHARACTERIZATIONS OF SPECIALLY MULTIPLICATIVE FUNCTIONS

PENTTI HAUKKANEN

Received 10 January 2003

A multiplicative function f is said to be specially multiplicative if there is a completely multiplicative function f_A such that $f(m)f(n) = \sum_{d|(m,n)} f(mn/d^2)f_A(d)$ for all m and n . For example, the divisor functions and Ramanujan's τ -function are specially multiplicative functions. Some characterizations of specially multiplicative functions are given in the literature. In this paper, we provide some further characterizations of specially multiplicative functions.

2000 Mathematics Subject Classification: 11A25.

1. Introduction. An arithmetical function f is said to be multiplicative if $f(1) = 1$ and

$$f(m)f(n) = f(mn) \tag{1.1}$$

whenever $(m, n) = 1$. If (1.1) holds for all m and n , then f is said to be completely multiplicative. A multiplicative function is known if the values $f(p^n)$ are known for all prime numbers p and positive integers n . A completely multiplicative function is known if the values $f(p)$ are known for all prime numbers p .

A multiplicative function f is said to be specially multiplicative if there is a completely multiplicative function f_A such that

$$f(m)f(n) = \sum_{d|(m,n)} f\left(\frac{mn}{d^2}\right)f_A(d) \tag{1.2}$$

for all m and n , or equivalently

$$f(mn) = \sum_{d|(m,n)} f\left(\frac{m}{d}\right)f\left(\frac{n}{d}\right)\mu(d)f_A(d) \tag{1.3}$$

for all m and n , where μ is the Möbius function. If $f_A = \delta$, where $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$, then (1.2) reduces to (1.1). Therefore, the class of completely multiplicative functions is a subclass of the class of specially multiplicative functions.

The study of specially multiplicative functions was initiated in [7], and arose in an effort to understand the identity

$$\sigma_\alpha(mn) = \sum_{d|(m,n)} \sigma_\alpha\left(\frac{m}{d}\right)\sigma_\alpha\left(\frac{n}{d}\right)\mu(d)d^\alpha, \tag{1.4}$$

where $\sigma_\alpha(n)$ denotes the sum of the α th powers of the positive divisors of n . Vaidyanathaswamy used the term “quadratic function,” while the term “specially multiplicative function” was coined by Lehmer [3]. For more background information, reference is made to the books by McCarthy [4] and Sivaramakrishnan [6].

The Dirichlet convolution of two arithmetical functions f and g is defined as

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right). \tag{1.5}$$

The function δ serves as the identity under the Dirichlet convolution. An arithmetical function f possesses a Dirichlet inverse f^{-1} if and only if $f(1) \neq 0$.

We next review some basic characterizations of specially multiplicative functions, see [4, 6].

PROPOSITION 1.1. *The following statements are equivalent.*

- (1) *The function f is a specially multiplicative function.*
- (2) *The function f is the Dirichlet convolution of two completely multiplicative functions a and b . (In this case $f_A = ab$, the usual product of a and b .)*
- (3) *The function f is a multiplicative function, and for each prime number p ,*

$$f^{-1}(p^n) = 0, \quad n \geq 3. \tag{1.6}$$

(In this case $f_A(p) = f^{-1}(p^2)$ for all prime numbers p .)

- (4) *The function f is a multiplicative function, and for each prime number p , there exists a complex number $g(p)$ such that*

$$f(p^{n+1}) = f(p)f(p^n) - g(p)f(p^{n-1}), \quad n \geq 1. \tag{1.7}$$

(In this case $f_A(p) = g(p)$ for all prime numbers p .)

- (5) *The function f is a multiplicative function, and for each prime number p , there exists a complex number $g(p)$ such that*

$$f(p^n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} [f(p)]^{n-2k} [g(p)]^k, \quad n \geq 0. \tag{1.8}$$

(In this case $f_A(p) = g(p)$ for all prime numbers p .)

REMARK 1.2. Completely multiplicative functions a and b in part 2 need not be unique. The usual product ab , however, is unique. For example, let a, b, c , and d be completely multiplicative functions such that $a(p) = 1$ and $b(p) = 2$ for all prime numbers p , and $c(2) = 2, c(p) = 1, d(2) = 1$, and $d(p) = 2$ for all prime numbers $p \neq 2$. Then $a * b = c * d$, but $a, b \neq c$ and $a, b \neq d$. However, $ab = cd$.

The purpose of this paper is to provide some further characterizations of specially multiplicative functions. As applications, we obtain formulas for the usual products $\sigma_\alpha \phi_\beta, \sigma_\alpha \sigma_\beta$, and $\sigma_\alpha \tau$, where ϕ_β is a generalized Euler totient function and τ is Ramanujan's τ -function. The function ϕ_β is given by $\phi_\beta = N^\beta * \mu$, where $N^\beta(n) = n^\beta$ for all n . In particular, we denote $N^1 = N, N^0 = \zeta$, and $\phi_1 = \phi$, where ϕ is the Euler totient function. Ramanujan's τ -function is a specially multiplicative function with $\tau_A = N^{11}$.

In the characterizations, we need the concepts of the unitary convolution and the k th convolute. The unitary convolution of two arithmetical functions f and g is defined as

$$(f \oplus g)(n) = \sum_{d \parallel n} f(d)g\left(\frac{n}{d}\right), \tag{1.9}$$

where $d \parallel n$ means that $d|n, (d, n/d) = 1$. The k th convolute of an arithmetical function f is defined as $\Omega_k(f)(n) = f(n^{1/k})$ if n is a k th power, and $\Omega_k(f)(n) = 0$ otherwise.

2. Characterizations

THEOREM 2.1. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$h * f(g * \mu) = fg, \tag{2.1}$$

where h is the specially multiplicative function such that

$$h(p) = f(p), \quad h_A(p) = g(p)f_A(p) \tag{2.2}$$

for all prime numbers p . Conversely, if $f(1) = 1$ and there exist completely multiplicative functions a, b, g , and k such that

$$a * b * f(g * \mu) = fg, \tag{2.3}$$

where

$$a(p) + b(p) = f(p), \quad a(p)b(p) = g(p)k(p), \quad (g * \mu)(n) \neq g(n) \tag{2.4}$$

for all prime numbers p and integers $n (\geq 2)$, then f is a specially multiplicative function with $f_A = k$.

PROOF. By multiplicativity, it suffices to show that (2.1) holds at prime powers, that is,

$$[f(g * \mu)](p^e) = (fg * h^{-1})(p^e) \quad (2.5)$$

for all prime powers p^e . If $e = 1$, then both sides of (2.5) are equal to $f(p)g(p) - f(p)$. Assume that $e \geq 2$. Then

$$\begin{aligned} (fg * h^{-1})(p^e) &= f(p^e)g(p^e) + f(p^{e-1})g(p^{e-1})h^{-1}(p) \\ &\quad + f(p^{e-2})g(p^{e-2})h^{-1}(p^2) \\ &= f(p^e)g(p^e) - f(p^{e-1})g(p^{e-1})f(p) \\ &\quad + f(p^{e-2})g(p^{e-2})g(p)f_A(p). \end{aligned} \quad (2.6)$$

By (1.7), we obtain

$$(fg * h^{-1})(p^e) = f(p^e)g(p^e) - f(p^e)g(p^{e-1}) = f(p^e)(g * \mu)(p^e). \quad (2.7)$$

Thus we have proved (2.5).

To prove the converse, we write (2.3) in the form

$$(f(g * \mu))(n) = (fg * a^{-1} * b^{-1})(n). \quad (2.8)$$

We write $n = p^{e+1}$ ($e \geq 1$) and, after some simplifications, obtain

$$f(p^{e+1}) = f(p^e)f(p) - f(p^{e-1})k(p). \quad (2.9)$$

Therefore, by (1.7), it remains to prove that f is multiplicative. Denote $n = p_1^{e_1} \cdots p_r^{e_r} p_{r+1} \cdots p_{r+s}$, where $e_i > 1$ ($i = 1, 2, \dots, r$). We proceed by induction on $e_1 + \cdots + e_r + s$ to prove that

$$f(n) = f(p_1^{e_1}) \cdots f(p_r^{e_r})f(p_{r+1}) \cdots f(p_{r+s}). \quad (2.10)$$

If $e_1 + \cdots + e_r + s = 1$, then (2.10) holds. Suppose that (2.10) holds when $e_1 + \cdots + e_r + s < m$. Then for $e_1 + \cdots + e_r + s = m$, we have after some manipulation

$$\begin{aligned} f(n)(g * \mu)(n) &= (fg * a^{-1} * b^{-1})(n) \\ &= f(n)g(n) + \sum_{\substack{d|n \\ d>1}} f\left(\frac{n}{d}\right)g\left(\frac{n}{d}\right)(a^{-1} * b^{-1})(d) \\ &= f(n)g(n) - \prod_{p^e \| n} f(p^e)g(p^e) + \prod_{p^e \| n} (fg * a^{-1} * b^{-1})(p^e) \end{aligned}$$

$$\begin{aligned}
 &= f(n)g(n) - \prod_{p^e \parallel n} f(p^e)g(p^e) \\
 &+ \prod_{i=1}^r \left[f(p_i^{e_i})g(p_i^{e_i}) - f(p_i^{e_i-1})f(p_i)g(p_i^{e_i-1}) + f(p_i^{e_i-2})k(p_i)g(p_i^{e_i-1}) \right] \\
 &\times \prod_{i=1}^s (f(p_{r+i})g(p_{r+i}) - f(p_{r+i})).
 \end{aligned}
 \tag{2.11}$$

Using (2.9), we obtain

$$f(n)(g * \mu)(n) = f(n)g(n) - g(n) \prod_{p^e \parallel n} f(p^e) + (g * \mu)(n) \prod_{p^e \parallel n} f(p^e).
 \tag{2.12}$$

This gives (2.10). □

REMARK 2.2. The converse part of [Theorem 2.1](#) can also be written as follows. If $f(1) = 1$ and there exist completely multiplicative functions g and k , and a specially multiplicative function h such that

$$h * f(g * \mu) = fg,
 \tag{2.13}$$

where

$$h(p) = f(p), \quad h_A(p) = g(p)k(p), \quad (g * \mu)(n) \neq g(n)
 \tag{2.14}$$

for all prime numbers p and integers $n (\geq 2)$, then f is a specially multiplicative function with $f_A = k$.

COROLLARY 2.3. *If f is a specially multiplicative function, then*

$$h * f\phi = fN,
 \tag{2.15}$$

where h is the specially multiplicative function such that

$$h(p) = f(p), \quad h_A(p) = pf_A(p)
 \tag{2.16}$$

for all prime numbers p . Conversely, if $f(1) = 1$ and if there exist completely multiplicative functions a , b , and k such that

$$a * b * f\phi = fN,
 \tag{2.17}$$

where

$$a(p) + b(p) = f(p), \quad a(p)b(p) = pk(p)
 \tag{2.18}$$

for all prime numbers p , then f is a specially multiplicative function with $f_A = k$.

COROLLARY 2.4. *If f and g are completely multiplicative functions, then*

$$f * f(g * \mu) = fg. \tag{2.19}$$

Conversely, if $f(1) = 1$ and if there exists a completely multiplicative function g such that

$$f * f(g * \mu) = fg, \tag{2.20}$$

where

$$(g * \mu)(n) \neq g(n) \tag{2.21}$$

for all integers $n (\geq 2)$, then f is a completely multiplicative function.

COROLLARY 2.5 (Sivaramakrishnan [5]). *If $f(1) = 1$, then f is a completely multiplicative function if and only if*

$$f * f\phi = fN. \tag{2.22}$$

EXAMPLE 2.6. We have

$$\sigma_\alpha \phi_\beta = \sigma_\alpha N^\beta * h^{-1}, \tag{2.23}$$

where h is the specially multiplicative function such that

$$h(p) = \sigma_\alpha(p) = p^\alpha + 1, \quad h_A(p) = p^\beta p^\alpha = p^{\alpha+\beta} \tag{2.24}$$

for all prime numbers p .

THEOREM 2.7. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$f(g * \mu) = fg * (\mu f \oplus \Omega_2(\mu^2 f_A g)). \tag{2.25}$$

Conversely, if $f(1) \neq 0$ and if there exist completely multiplicative functions g and k such that

$$f(g * \mu) = fg * (\mu f \oplus \Omega_2(\mu^2 kg)), \tag{2.26}$$

where

$$(g * \mu)(n) \neq g(n) \tag{2.27}$$

for all n , then f is a specially multiplicative function with $f_A = k$.

PROOF. We observe that

$$\begin{aligned} (\mu f \oplus \Omega_2(\mu^2 f_A g))(p) &= -f(p), \\ (\mu f \oplus \Omega_2(\mu^2 f_A g))(p^2) &= f_A(p)g(p), \\ (\mu f \oplus \Omega_2(\mu^2 f_A g))(p^n) &= 0 \end{aligned} \tag{2.28}$$

for all prime numbers p and integers $n (\geq 3)$. Therefore $\mu f \oplus \Omega_2(\mu^2 f_A g) = h^{-1}$, where h is the specially multiplicative function in [Theorem 2.1](#). Thus [\(2.25\)](#) follows from [\(2.1\)](#).

The converse follows from [Theorem 2.1](#) since $\mu f \oplus \Omega_2(\mu^2 g k) = a^{-1} * b^{-1}$, where a and b are completely multiplicative functions as given in [Theorem 2.1](#). □

THEOREM 2.8. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$f(g * \mu) = fg * (f^{-1} \oplus \Omega_2(\mu^2 f_A (g \oplus \mu))). \tag{2.29}$$

Conversely, if $f(1) = 1$ and there exist completely multiplicative functions c, d , and g such that

$$f(g * \mu) = fg * ((c * d)^{-1} \oplus \Omega_2(\mu^2 cd(g \oplus \mu))), \tag{2.30}$$

where

$$c(p) + d(p) = f(p), \quad (g * \mu)(n) \neq g(n) \tag{2.31}$$

for all prime numbers p and integers $n (\geq 2)$, then f is the specially multiplicative function given as $f = c * d$.

PROOF. Proof of [Theorem 2.8](#) is similar to that of [Theorem 2.7](#). □

EXAMPLE 2.9. We have

$$\begin{aligned} \sigma_\alpha \phi_\beta &= \sigma_\alpha N^\beta * (\mu \sigma_\alpha \oplus \Omega_2(\mu^2 N^{\alpha+\beta})), \\ \sigma_\alpha \phi_\beta &= \sigma_\alpha N^\beta * (\sigma_\alpha^{-1} \oplus \Omega_2(\mu^2 N^\alpha (N^\beta \oplus \mu))). \end{aligned} \tag{2.32}$$

LEMMA 2.10. *Suppose that f is an arithmetical function such that $f(1) = 1$ and $f^{-1}(p^i) = 0$ for $3 \leq i < k$ ($k \geq 4$). Then*

$$f(p^k) = f(p)f(p^{k-1}) - f^{-1}(p^2)f(p^{k-2}) - f^{-1}(p^k). \tag{2.33}$$

PROOF. [Lemma 2.10](#) follows from the equation

$$\sum_{i=0}^k f^{-1}(p^i)f(p^{k-i}) = 0. \tag{2.34}$$

□

THEOREM 2.11. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$f(g * \zeta) = fg * f * \Omega_2(f_A g)^{-1}. \tag{2.35}$$

Conversely, if f is a multiplicative function such that

$$f(g * \zeta) = fg * f * \Omega_2(hg)^{-1}, \quad (2.36)$$

where g is a completely multiplicative function with $g(p)(g * \zeta)(p^e) \neq 0$ for all prime powers p^e and where h is a completely multiplicative function, then f is a specially multiplicative function with $f_A = h$.

PROOF. Let $f = a * b$, where a and b are completely multiplicative functions. It is known [7] that

$$f(g * \zeta) = (a * b)(g * \zeta) = ag * a\zeta * bg * b\zeta * \Omega_2(abg\zeta)^{-1}. \quad (2.37)$$

Using elementary properties of arithmetical functions, we obtain

$$f(g * \zeta) = (a * b)g * (a * b) * \Omega_2(f_A g)^{-1} = fg * f * \Omega_2(f_A g)^{-1}. \quad (2.38)$$

This proves (2.35).

Assume that (2.36) holds. Then (2.36) at p^2 gives

$$h(p) = f(p)^2 - f(p^2). \quad (2.39)$$

Since $f^{-1}(p^2) = f(p)^2 - f(p^2)$ for all multiplicative functions, we obtain

$$h(p) = f^{-1}(p^2). \quad (2.40)$$

We next prove that

$$f^{-1}(p^i) = 0 \quad \forall i \geq 3. \quad (2.41)$$

We proceed by induction on i . Calculating (2.36) at p^3 and using (2.40) gives

$$f(p^3) = f(p)f(p^2) - f(p)f^{-1}(p^2). \quad (2.42)$$

Since

$$f(p^3) - f(p^2)f(p) + f(p)f^{-1}(p^2) + f^{-1}(p^3) = 0, \quad (2.43)$$

we see that $f^{-1}(p^3) = 0$.

Suppose that $f^{-1}(p^i) = 0$ for all $3 \leq i < k$ ($k > 3$). We write (2.36) as

$$f(g * \zeta) * f^{-1} = fg * \Omega_2(hg)^{-1}. \quad (2.44)$$

Suppose that k is even, say $k = 2e$ ($e > 1$). At p^{2e} , the left-hand side of (2.44) becomes

$$\begin{aligned} & \sum_{i=0}^{2e} f(p^i)(g * \zeta)(p^i)f^{-1}(p^{2e-i}) \\ &= f^{-1}(p^{2e}) + f(p^{2e-2})(g * \zeta)(p^{2e-2})f^{-1}(p^2) \\ & \quad + f(p^{2e-1})(g * \zeta)(p^{2e-1})f^{-1}(p) + f(p^{2e})(g * \zeta)(p^{2e}) \\ &= f^{-1}(p^{2e}) - f^{-1}(p^{2e})(g * \zeta)(p^{2e-2}) \\ & \quad - f(p)f(p^{2e-1})g(p^{2e-1}) + f(p^{2e})g(p^{2e-1}) + f(p^{2e})g(p^{2e}) \\ &= f^{-1}(p^{2e}) - f^{-1}(p^{2e})(g * \zeta)(p^{2e-1}) \\ & \quad - f^{-1}(p^2)f(p^{2e-2})g(p^{2e-1}) + f(p^{2e})g(p^{2e}), \end{aligned} \tag{2.45}$$

where the last two equations are derived by Lemma 2.10. Further, at p^{2e} , the right-hand side of (2.44) becomes

$$\begin{aligned} & \sum_{i=0}^{2e} f(p^{2e-i})g(p^{2e-i})\Omega_2(hg)^{-1}(p^i) \\ &= \sum_{i=0}^e f(p^{2(e-i)})g(p^{2(e-i)})\mu(p^i)h(p^i)g(p^i) \\ &= f(p^{2e})g(p^{2e}) - f(p^{2(e-1)})g(p^{2(e-1)})h(p)g(p). \end{aligned} \tag{2.46}$$

Now, we see that $f^{-1}(p^{2e}) = 0$, that is, $f^{-1}(p^k) = 0$.

If k is odd, a similar argument applies. Thus (2.41) holds and therefore, by (1.6), f is a specially multiplicative function with $f_A = h$. □

COROLLARY 2.12. *If f is a specially multiplicative function, then*

$$f\sigma_0 = f * f * \Omega_2(f_A)^{-1}. \tag{2.47}$$

Conversely, if f is a multiplicative function such that

$$f\sigma_0 = f * f * \Omega_2(h)^{-1}, \tag{2.48}$$

where h is a completely multiplicative function, then f is a specially multiplicative function with $f_A = h$.

COROLLARY 2.13 (Apostol [1]). *If f and g are completely multiplicative functions, then*

$$f(g * \zeta) = fg * f. \tag{2.49}$$

Conversely, if f is a multiplicative function such that

$$f(g * \zeta) = fg * f, \tag{2.50}$$

where g is a completely multiplicative function with $g(p)(g * \zeta)(p^e) \neq 0$ for all prime powers p^e , then f is a completely multiplicative function.

COROLLARY 2.14 (Carlitz [2]). *Suppose that f is a multiplicative function. Then f is a completely multiplicative function if and only if*

$$f\sigma_0 = f * f. \quad (2.51)$$

COROLLARY 2.15. *There exist*

$$\begin{aligned} \tau\sigma_\alpha &= \tau N^\alpha * \tau * \Omega_2(N^{\alpha+11})^{-1}, \\ \sigma_\alpha\sigma_\beta &= \sigma_\alpha N^\beta * \sigma_\alpha * \Omega_2(N^{\alpha+\beta})^{-1}. \end{aligned} \quad (2.52)$$

REFERENCES

- [1] T. M. Apostol, *Some properties of completely multiplicative arithmetical functions*, Amer. Math. Monthly **78** (1971), 266–271.
- [2] L. Carlitz, *Completely multiplicative function*, Amer. Math. Monthly **78** (1971), 1140.
- [3] D. H. Lehmer, *Some functions of Ramanujan*, Math. Student **27** (1959), 105–116.
- [4] P. J. McCarthy, *Introduction to Arithmetical Functions*, Universitext, Springer-Verlag, New York, 1986.
- [5] R. Sivaramakrishnan, *Multiplicative function and its Dirichlet inverse*, Amer. Math. Monthly **77** (1970), 772–773.
- [6] ———, *Classical Theory of Arithmetic Functions*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 126, Marcel Dekker, New York, 1989.
- [7] R. Vaidyanathaswamy, *The theory of multiplicative arithmetic functions*, Trans. Amer. Math. Soc. **33** (1931), no. 2, 579–662.

Pentti Haukkanen: Department of Mathematics, Statistics and Philosophy, University of Tampere, FIN-33014, Finland

E-mail address: Pentti.Haukkanen@uta.fi