# Resource Conscious Quantification and Ontologies with Degrees of Significance

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We discuss formal ontologies where objects can be ranked according to the modes of existence they are associated with. Mathematical objects, for instance, may be ranked according to how admissible they are; it is natural to consider the positive integers more admissible than the very large cardinals, for example. We also give a *resource conscious* interpretation of the existential quantifier and investigate systems with formal ontologies in the resulting resource conscious setting. In this setting a formula $\exists x \varphi(x)$ is true if we have enough resources to construct a witness $c$ for the variable $x$ such that $\varphi(c)$ holds. The nature of the required resources can be interpreted in various ways, depending on the system studied.

## 1 Introduction

In this article we consider a perspective that assigns mathematical objects different *degrees of significance*. For example, it might be desirable to assign finite binary strings a higher degree of significance than infinite binary strings simply because finite ones are physically realizable (at least in principle). The aim of the article is to discuss issues related to such perspectives and to briefly investigate properties of related formal systems. We define formal ontologies that assign each object of the domain of an ontology a value in a partially ordered set. An object $a$ with a value higher than that of another object $b$ is considered in some sense more significant than the object $b$. Significance can mean various different things here. In most cases we define an object $a$ to be more significant than $b$ iff $a$ is more admissible, in one sense or another, than $b$. For example, $a$ could be a decidable set and $b$ a recursively enumerable but not decidable set.

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The point of the article is not to investigate the notion of significance. The word "significance" is simply an umbrella term that works better or worse for the systems we define. Perhaps "admissibility" would be a better term, at least for most of the examples we consider in this article. Admissibility and significance are not wholly unrelated notions. For example one might put together a computer system where a computer $A$ produces binary strings that another computer $B$ then reads. For rather obvious reasons, one *might* wish to regard outputs of $A$ that are generated after a short period of computation as more *significant* than those that would make $A$ run for, say, 1000 years. When optimizing $B$, it would be a good choice to take into account that $B$ is not likely to receive from $A$ strings of the latter variety.

Even though some of the considerations below are philosophical in nature, I do not wish to argue in favor of or against any position in the philosophy of mathematics – institutionalized or not. However, I believe that one can deepen one's understanding of mathematics by gaining more insight into issues arising from considerations related to different perspectives on the foundations of mathematics. Also, importantly, considerations that may appear philosophical can often quite justifiably be regarded also as considerations with the objective to design (formal) tools for artificial intelligence and computer science. Such tools that have their origins in philosophical considerations are quite numerous indeed, and various industrial applications exist. Modal logic is a prime example of a field with its origins in philosophy that has recently found a very wide range of real-life applications, most of these applications being related to computer science. See [2] and [3] for example.

In Section 2 below we define a framework for constructing formal ontologies with degrees of significance. We then discuss a couple of toy-systems where such an ontology is applied. Within the framework it is relatively natural to define for example foundation systems for mathematics that take into account considerations related to the limitations on available physical resources. Among other things, such considerations seem to motivate various finitist and ultrafinitist attitudes. Examples 2.2, 2.3 and 3.2 below briefly elaborate such ideas.

In Section 3 we define a semantics for first-order logic that attempts to take into account some aspects of considerations related to limitations of available resources. The idea is to give the existential quantifier a *resource conscious* interpretation. Roughly speaking, $\exists x\, \varphi(x)$ is true if and only if we have enough resources to construct a witness $c$ for $x$ such that $\varphi(c)$ holds. We use this new logic to investigate properties of formal ontologies with degrees of significance. The nature of the resources involved can be interpreted in different ways, depending on the issues modeled. The formal system we develop is not supposed to model a single fully specified framework, such as a foundation system for mathematics. The idea is to investigate issues related to limitations of available resources from a general point of view, and most of the tools defined are only of a tenatative nature.

Logics that take into account issues related to limitations on available resources have of course been consired before. See [7] for example. And of course there are various logics that attempt to account for issues related to different aspects of the notion of constructibility. Intuitionistic logic and the system considered in [5] are examples of such logics.

When developing formal systems, the background assumptions we base our approach on are classical.

## 2 Mathematics and Ontologies with Degrees of Significance

When putting together a foundation system for mathematics, one natural approach is to attempt to define the following.

1. A collection of mathematical objects.

2. A language to make assertions about the objects, together with a suitable semantics for the language.

This is of course only one approach, and radically different approaches may lead to systems that elucidate some issues much better than systems based on the above approach. In this article we do not wish to define a foundation system for mathematics, but we do something that is not wholly unrelated to such considerations. In this section we define a framework for specifying formal ontologies whose objects are ranked by assigning them a value in a partial order. This kind of a device can be used for various different purposes. For one reason or another, one may wish to consider some object $c$ as somehow more *significant* than another object $d$. In that case the object $c$ would be assigned a rank greater than that of $d$. Some other pair of objects could receive incomparable ranks, perhaps because they are considered to have modes of existence that are rather different. For example *numbers* and *proofs* could, *perhaps*, be assigned incomparable ranks. The criteria by which degrees of significance are assigned can vary; the word "significance" can mean different things in different applications. Indeed, in one system the object $c$ could receive a higher degree of signifance the object $d$, and in another system a lower one.

After defining the framework for degree ontologies, we discuss a range of toy-systems that employ such an ontology. In the next section we investigate these systems using a formal language.

**Definition 2.1** (Degree ontologies)**.** Let $\mathcal{A} = (A, \leq)$ be a partial order, and let $C$ be a set. Let $f : C \to A$ be function. The structure $O = (C, \mathcal{A}, f)$ is a *poset valued ontology*, or alternatively, a *degree ontology*. We call $f$ the *rank function* of the ontology $O$.

We then consider a toy-system that can be motivated by considerations related to various finitist and ultrafinitist attitudes. Let me repeat that I do not wish to defend or attack any perspective on the philosophy of mathematics. The aim here is simply to describe a possible perspective that is not incoherent. Understanding different approaches to the foundations of mathematics can increase one's mathematical capacity, and furthermore, considerations motivating different kinds of foundations can be useful from the point of view of real-life applications.

**Example 2.2.** Let $\mathbb{N}_{\geq 1}$ denote the set of positive integers and let

$$\mathcal{R} = (\ [0,1],\ \leq^{\mathcal{R}}\ ),$$

where $\leq^{\mathcal{R}}$ is the usual ordering of the real numbers restricted to the interval $[0,1]$. Define the function $f : \mathbb{N}_{\geq 1} \rightarrow [0,1]$ such that $f(x) = \frac{1}{x}$. Define the ontology $(\mathbb{N}_{\geq 1}, \mathcal{R}, f)$.

The ontology $(\mathbb{N}_{\geq 1}, \mathcal{R}, f)$ models an attitude from which positive integers become increasingly insignificant as they become greater and greater. Here the positive integers are in a sense identified with the cardinality they represent. The larger integers exist out there, but the greater they are, the less we care about their properties. This kind of an attitude could be related to a real-life application where one can construct representations of positive integers encoded in unary. Perhaps the application is a computer system that allows for the dynamic increase of resources, so there is no *clear* limit on the size of integers that could be encountered. However, assume the time required for the construction of a representation of an integer is roughly proportional to the cardinality of the integer in question. Then it can be quite natural to adopt the attitude that the significance of integers decreases as they grow larger and larger. After all, in the real-life use of the system, one is not likely to encounter the very large integers at all. Of course it could be the case that, say, the integer 5 occurs much more frequently than 4, and therefore 5 should perhaps be assigned a much higher degree of significance than 4. In other words, the desired properties of the rank function depend very much on the particularities of the system we wish to model, and also on the set of properties of the system that we want to take into account. There are various sensible ways of defining a degree ontology for a set of objects. The ranking of the objects can be based on various diffent kinds of principles.

**Example 2.3.** Let $C = \mathbb{N} \cup \{\mathbb{N}\}$, and let $l \in \mathbb{N}_{\geq 1}$ be a positive integer. Let $\mathcal{R}$ be as in the previous example. Define the function $h : C \rightarrow [0,1]$ as follows.

$$h(x) = \begin{cases} \frac{l}{x+l} & \text{if } x \in \mathbb{N}, \\ 0 & \text{if } x = \mathbb{N}. \end{cases}$$

Define the ontology $(C, \mathcal{R}, h)$.

This example is rather similar to Example 2.2. One novel feature here is the parameter $l$. Especially ultrafinitist considerations are often motivated on the grounds of physical limitations on available resources. For example, Sazonov's article [11] takes the number $2^{1000}$ to be large enough to be excluded from the list of numbers to be considered. If we define $l = 2^{1000}$, then the integer $2^{1000}$ receives the degree $\frac{1}{2}$ of significance. This *could* be taken to mean that the *degree of existence*, whatever that is, of numbers below $l$ is higher than their degree of non-existence; numbers above $l$ would be considered more non-existent than existent. The number $l = 2^{1000}$ would strike a middle ground between these two regions. Finally, the ontology $(C, \mathcal{R}, h)$ assigns the set $\mathbb{N}$ the degree 0. This could be taken to mean that the set $\mathbb{N}$ exists but with the degree zero of existence. Perhaps "subsistence" would be a better term for such a mode of existence.

A funny feature, perhaps, of the above system is that only the number 0 gets assigned the degree 1 of existence. It is, of course, easy to change this in case one so desires. On various psychological and even historical grounds one might wish to assign 0 a smaller degree of existence than the degree of, say, the number 1. It all depends, of course, on what one wishes to model exactly, and how.

The toy ontologies of Examples 2.3 and 2.4 avoid commitment to there existing some greatest natural number, for example $2^{1000} - 1$, while still retaining their ultrafinitist character.[1] This fact could be one of the strong points of the attitude modeled by the ontologies defined in the examples. The systems show the (rather obvious) fact that one does not have to commit to the existence of the largest natural number in order to entertain the perspective that in some sense very large numbers are quite inaccessible and therefore even appear *in some sense* non-existent. All too often this kind of a perspective is naïvely assumed to necessarily commit to the existence of the largest natural number.[2]

**Example 2.4.** Consider sets $S \subseteq \mathbb{N}$ with the elements $s \in S$ represented in binary. Let $\mathbb{N}(bin)$ denote the set of binary representations of the numbers in $\mathbb{N}$. There are continuum many sets $S \subseteq \mathbb{N}(bin)$. Some of these sets $S$ are rather "tame" in the sense that there is a Turing machine that decides for each $n \in \mathbb{N}(bin)$ whether $n$ is in the set in question. However, there are only $\aleph_0$ many Turing algorithms, so most of the sets $S \subseteq \mathbb{N}(bin)$ are not tame in the above sense. What kind of a status should we assign to a subset $S \subseteq \mathbb{N}(bin)$ that is not Turing-decidable? What kind of an object is $S$, if we believe we cannot systematically specify (i.e., decide), even in principle, which objects $n \in \mathbb{N}(bin)$ belong to $S$? Indeed, there seems to be nothing directly incoherent about adopting the attitude (towards the notion

[1] Of course, a serious and fully committed radical finitist might not like the fact that we referred to the set of *real numbers* in the formulation of the toy-systems.

[2] For very interesting related considerations, see [8].

of existence) that this kind of a set does not exist at all. Perhaps a somewhat more appealing attitude, however, would be that the mode of existence of an undecicable set is in some sense weaker than that of a decidable set. There are various natural ways to proceed, of course. For example we could define an ontology satisfying the following conditions.

1. Decidable sets receive a higher degree of existence than those that are not decidable.

2. Recursively enumerable but undecidable sets receive a degree of existence that is somewhere between the degree(s) of existence of decidable sets and the degree(s) of existence of sets that are not recursively enumerable.

Also, it could perhaps be desirable to make the degree of existence of a set $S \subseteq \mathbb{N}(bin)$ depend on other complexity measures, such as complexity classes of computational complexity theory [9].

Let us then consider the nature of the set $Pow(\mathbb{N}(bin))$ and the degree of existence of the cardinal $\aleph_1$. There are, of course, only countably many recursively enumerable sets $S \subseteq \mathbb{N}(bin)$. Perhaps it would be natural to assign the set $\aleph_1$ a degree equal to that of the sets $S \subseteq Pow(\mathbb{N}(bin))$ that are not recursively enumerable, or perhaps even a lower degree.

Finally, it would be natural to assign finite sets $S \subseteq \mathbb{N}(bin)$ higher degrees of existence than infinite sets. Finite sets could be ranked such that the smaller finite sets receive higher degrees than larger ones.

The currently paradigmatic practise of doing mathematics within the framework provided by ZFC also shows an attitude where mathematical objects have different modes of existence associated with them: mathematicians casually talk about sets and proper classes. Sets are treated as first-class citizens and proper classes as second-class citizens existing on the meta-level. Even though one might not wish to characterize sets as somehow *more* important than proper classes, sets and proper classes do have *different* rights as citizens in the mathematical realm. This practise of distinguishing between sets and proper classes is somewhat analogous to the attitude described above in Example 2.4 where undecidable sets were assigned the status of second-class citizens. A carfully defined foundation system where only decidable sets are considered to exist as object-level objects and undecidable sets receive the status analoguous to that of proper classes, can be quite natural and well-motivated indeed.

**Example 2.5.** Let $\Sigma$ be an alphabet consisting of, say, 30 symbols. The set $\Sigma$ could contain for example the English alphabet plus some additional symbols. Let us define an encryption scheme in order to encrypt finite strings in $\Sigma^*$. Let $n$ be some fixed positive integer large enough so that we are happy always sending a message (i.e., a string) with exactly $n$ symbols. So, we always write messages with $n$ symbols. However, after encryption, what we actually ultimately send, is a string with exactly $30n$ symbols. Let

$$s : \{1, ..., n\} \longrightarrow \{1, ..., 30n\}$$

be a sequence without repetition. In other words, $s$ is an injection. Let $u \in \Sigma^*$ be a message with $n$ symbols we wish to send. We first fill up a tape of $30n$ slots in the following way. We write the first symbol of $u$ to the slot $s(1)$, the second one to the slot $s(2)$, and so on. The $n$-th symbol of $u$ of course goes to the slot $s(n)$. Then we use the remaining $29n$ slots such that once we are finished, each of the $30n$ slots has exactly one symbol in it, and each symbol in $\Sigma$ is written in exactly $n$ slots of the tape of $30n$ slots. We then send the string on the tape to the receiver. The receiver of course knows the key $s$.

Assume there is a code breaker who knows more or less everything there is to know about us, the receiver and the encryption scheme we use. He might even know the method we use in order to fill up the tape of $30n$ slots after a message with $n$ symbols has been written on it. However, he does not know $s$. Assume we never change the key $s$. Assume also that the breaker knows that we never change $s$. The breaker investigates more and more sent strings with $30n$ symbols and compares the strings to the courses of action that we and the receiver take. As he gathers more data, the breaker begins considering some sequences as more likely key candidates than others. Some sequences become more *significant* and important to the breaker than others. Perhaps he devotes more time and energy to the set of more significant sequences.

The breaker entertains a degreed ranking system that resembles a degree ontology. (Of course one might not wish to talk about *ontologies* in this case.) The domain of the related formal degree ontology would be the set of injective sequences mapping $\{1, ..., n\}$ to $\{1, ..., 30n\}$. This could be a rather large set, or course. The ranking of the breaker's system evolves as the breaker analyzes more and more sent strings. It may not always be an easy task to choose a suitable ranking. For example, it is an interesting question whether the breaker should consider the sequence $t$ such that $t(x) = x$ for all $x \in \{1, ..., n\}$ as a rather unlikely sequence even before any messages have been sent.

It is indeed natural to rank objects according to one's state of knowledge and understanding of a system under investigation. Such a ranking then evolves as one gathers more data and/or obtains a deeper understanding of the system.

We end the current section by briefly visiting the realm of mythomatics. Let us first turn our attention to questions concerning issues related to mathematical realism.

Is there a set whose cardinality is strictly between those of the sets $\mathbb{N}$ and $\mathbb{R}$? Does this question make sense? Does the question have a definite answer that we simply do not know yet? Questions of this kind might arise in ordinary mathematical work more easily than is often thought. It is easy to write a sentence of second-order logic that states something along the following lines.

"There is an infinite $N \subseteq D$, where $D$ is the domain of the current model, and for all infinite $S \subseteq D$, there is a surjective map from $S$ to $N$. Furthermore, for each infinite $T \subseteq D$, there is a bijective map from $D$ to $T$, or alternatively, there is a bijective map from $N$ to $T$."

Now, what is the truth value of this sentence in a model with the domain $\mathbb{R}$? Do we wish to assign it a third truth value $i$ standing for *indeterminate* (rather than unknown)? How exactly do we wish to treat second-order quantification? Exactly what sort of a collection of objects do we want second-order quantifiers to range over?

Is the continuum hypothesis true? Do we wish to be realist about the question and simply say that we do not know the answer to that question, but there necessarily is an answer in $\{\,yes, no\}$ that forces itself upon us? Perhaps we are not satisfied with such an attitude and therefore decide to take the problem of vagueness seriously in the case of sets. The "notion" of a *table* is useful and partially determined, but there are borderline cases where we seem to enter the twilight zone, and possibly sometimes there simply is no one single correct definite answer to the question whether an object is a table. No matter how natural some notion might seem at first, perhaps there are cases that cannot be considered to be in any sense fully determined. We might perhaps even end up in a situation where we feel that some notion should satisfy some property $P$, but other individuals have created a different kind of understanding of the notion in question suggesting that $P$ does not hold. Perhaps their understanding of the notion in question is very similar to ours, but not the same. In ZFC we of course have $x \notin x$ for all sets $x$, but is this necessarily true of our informal conception of a set? Is there any problem in entertaining two different notions of a set, one for well-founded sets and another one for non-well-founded ones?

Perhaps one day there appears an argument establishing that in some reasonable sense the options "CH is true" and "CH is not true" are *equally natural.* It is not uncommon in the practise of mathematics that one fiddles about with a collection of reasonably well understood but not fully specified objects, and when faced with a choice whether the objects should satisfy some property, there is no obvious single correct way to go. However, it also seems possible that one day there appears a reasonably convincing argument establishing that the continuum hypothesis *does* have a truth value in $\{yes, no\}$. Perhaps that argument even establishes which one of the two truth values the hypothesis has. Why should the continuum hypothesis not have a definite truth value? The fact that the continuum hypothesis is independent of ZFC does not seem to definitely imply that the hypothesis is somehow vague. After all, why should any first-order axiomatization of a binary relation provide a good enough framework for specifying what sets are?

Maybe all we wish to say about the continuum hypothesis is that perhaps it has a truth value in $\{\, yes, no\}$ that forces itself upon us, and perhaps not. After all, some brilliant guys seem to say that the continuum hypothesis has a determinate truth value,[3] and some equally brilliant guys seem to say that continuum hypothesis is a vague statement.[4] The (possibly vague) question about vagueness of the continuum hypothesis is interesting. And of course vagueness of the notion of vagueness is interesting too...

Assume we want to put such considerations aside and simply consider the description "$X$ is a set whose cardinality is strictly between those of $\mathbb{N}$ and $\mathbb{R}$". Let us assume we are happy about not forming an opinion about whether such a set exists and whether the question about its existence even makes sense. We might wish to define for ourselves an ontology, possibly a provisional one, where such sets exist even though we know that perhaps in the future we might have to seriously revise our views. It would, then, be quite natural to assign to those exotic entities a lower degree of existence than to objects to whose existence we have a high degree of commitment to. A degree ontology seems to work relatively fine here. Of course if we wished to define a formal degree ontology within the framework provided by ZFC, the way to go could be to let the objects in the domain of the ontology corresponding to the exotic entities be encodings of some sort, perhaps some rather familiar sets.

Let us dig deeper into the mythomatical realm. Philosophers of mathematics often discuss the so called Russell Set, i.e., the "set of all sets that are not members of themselves" familiar from naïve set theory. It is of course natural to adopt the attitude that this entity is not an object at all despite the fact that we have even given it a proper name. After all, this set seems to satisfy a contradiction. However, one is also free to entertain the perspective that, indeed, such objects as the Russell Set and perhaps a round square exist,[5] at least as objects of thought or reflection. The Russell Set has a *description* associated with it, and we can reflect on the description. However, it would be quite natural to admit that an object of thought that satisfies a contradiction should be associated with a weaker degree of existence than an object that does not satisfy a contradiction. (And, of course, one might not wish to use the term "existence" here at all.) Finally, one might wish

[3]See the assertion attributed to Cohen at the beginning of Chapter 11 of [4].

[4]See Feferman's article in [6].

[5]Here some word other than "exist" could perhaps be a much better choice. For example, one encounters the term "absistence" in discussions related to the thought of Alexius Meinong.

to call the Russell Set a *concept* or an *entity* rather than an object. After all, it is a somewhat weird object. Perhaps one would wish to characterize such objects as having an intension but no extension associated with them.

This kind of an ontology might not be that strange and useless after all. For example, in a proof by contradiction we may provisionally talk about, say, the greatest prime number $p$. (Assume we do this without any radical ultrafinitist overtones.) It then makes sense to talk about all primes smaller than $p$. However, does it make sense to talk about all circles smaller than $p$? In *some* sense contradictory "objects" can be thought to have some properties and lack others. Ontologies with such entities could be useful for some purpose. Again, if we wished to define a formal degree ontology with contradictory objects, an easy way out could be to let the objects in the domain of the formal ontology that correspond to contradictory objects be encodings of some sort. For example they could be sets that encode formulae that describe the contradictory objects.

The idea of a contradictory object might seem weird at first, but is there really anything very deep about such a conception? Is there any *real* problem in employing such a conception?

## 3 A Logic for Models with a Degree Ontology

In this section we define a semantics for first-order logic that attempts to take into account aspects behind attitudes that assign different degrees of significance to different objects. There are, of course, various other ways of defining systems of formal logic that model such attitudes, and I do not wish to claim that the system we define in this section is particularly canonical. The objective of the section is simply to introduce a tentative system of logic that with some degree of naturality models at least *some* aspects of the phenomena related to degrees of significance. The main idea behind the system is to give the existential quantifier a *resource conscious reading*. Roughly, the sentence $\exists x(Px)$ is true if we have enough resources to construct (a representation of) an object that has the property $P$. The reader is free to replace the symbol $\exists$ with, say, $C$, if (s)he does not like the connotation that existence is somehow strongly related to the possibility of constructing an object. I do not wish to defend (or attack) such a view.

Let $V$ be a vocabulary containing relation symbols only. Let $\mathrm{FO}_{\mathrm{V}}$ denote the set of first-order $V$-formulae. Let $\mathfrak{A}$ be a first-order $V$-model with the domain $A$, and let $f : A \longrightarrow \mathbb{R}_{\geq 0}$ be a function. Here $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. Let $r \in \mathbb{R}_{\geq 0}$ and $\varphi \in \mathrm{FO}_V$. Let VAR be a countably infinite set of first-order variable symbols. Let $U \subseteq \mathrm{VAR}$ and let $v : U \longrightarrow A$ be a *variable assignment function.* Note that the set $U$ is allowed be finite, even empty. If $a \in A$ and $x \in \mathrm{VAR}$, we define

$$v[x \mapsto a] \;=\; \begin{cases} \big(v \setminus \{\,(x, v(x))\}\big) \;\cup\; \{(x,a)\} & \text{if } x \in \mathit{Dom}(v), \\ v \cup \{(x,a)\} & \text{if } x \notin \mathit{Dom}(v). \end{cases}$$

We define truth of $\mathrm{FO_V}$ formulae with respect to objects of the type $(\mathfrak{A}, f, r, v)$. Intuitively, the function $f$ assigns each object $a \in A$ a *cost* $f(a) \in \mathbb{R}_{\geq 0}$, which is the amount of resources needed in order to construct the object $a$, or perhaps a representation of $a$. The number $r \in \mathbb{R}_{\geq 0}$ denotes the amount of resources available. The range $\mathit{Ran}(v)$ of the variable assignment $v$ is the collection of objects that we already have constructed.[6] The variable symbols in $\mathit{Dom}(v)$ are the the names (or pointers, if you like) that we have assigned to the objects in $\mathit{Ran}(v)$. We call the objects of the type $(\mathfrak{A}, f, r, v)$ *resource interpretations.*

If $\psi$ is an atomic formula, we define

$$(\mathfrak{A}, f, r, v) \models \psi \text{ iff } \mathfrak{A}, v \models \psi.$$

We choose to give the Boolean connectives the classical interpretation. However, we define

$$\big(\mathfrak{A}, f, r, v\big) \;\models\; \exists x(\varphi(x))$$

if and only if at least one of the following conditions is satisfied.

1. $\exists a \in \big(A \setminus \mathit{Ran}(v)\big)\Big(r - f(a) \geq 0 \text{ and } \big(\mathfrak{A}, f, r - f(a), v[x \mapsto a]\big) \models \varphi(x)\Big).$
2. $\exists a \in \mathit{Ran}(v)\Big(\; \big(\mathfrak{A}, f, r, v[x \mapsto a]\big) \models \varphi(x) \;\Big).$

The quantifier $\exists x$ could be understood to assert something along the following lines.

"There exists an object, call it $x$, such that we can first construct a representation of the object and then observe that..."

Notice that by the latter satisfaction clause for the existential quantifier, if we already have a representation of an object, we do not have to consume any more resources in order to construct a new representation for it. The old one will do.[7]

We consider a few examples that illustrate the properties of the logic.

[6] There are cases where this intuition is not exactly correct, however, as we shall see.

[7] Quantification of variables already in the domain of the variable assignment function problematizes this intuition, however. Evaluating the formula

$$\exists x\big(Px \wedge \exists x(\neg Px \wedge \exists x(Px))\big)$$

in a suitable resource interpretation should directly illustrate the issue. Notice the nested quantification of the same variable in the formula. If we construct an element $a$, tag it with $x$, and then construct some other element $b$ and use $x$ to tag $b$, we loose the free access to $a$. The element $a$ does not have our tag on it any more.

**Example 3.1.** Let $P$ and $Q$ be unary relation symbols and consider the model $\mathfrak{A} = (A, P^{\mathfrak{A}}, Q^{\mathfrak{A}})$, where $P^{\mathfrak{A}}$ and $Q^{\mathfrak{A}}$ are non-empty subsets of the domain $A$ such that $P^{\mathfrak{A}} \cap Q^{\mathfrak{A}} = \emptyset$. Let $f(a) = f(b) = 100$. We have

$$(\mathfrak{A}, f, 150, \emptyset) \models \exists x(Px) \wedge \exists y(Qy).$$

Notice, however, that

$$(\mathfrak{A}, f, 150, \emptyset) \not\models \exists x \exists y(Px \wedge Qy).$$

In the standard first-order setting, the two sentences are of course equivalent.

The domain $A$ of the model $\mathfrak{A}$ in the above example could be, for example, the set of items in a shop, and $r$ could be the amount of money we have. $P^{\mathfrak{A}}$ could be the set of shirts and $Q^{\mathfrak{A}}$ the set of ties. The function $f$ would simply give the prize of the items in $A$. The sentence $\exists x(Px) \wedge \exists y(Qy)$ would then correctly assert that we have enough money to buy a shirt and we also have enough money to buy a tie. However, this time we would not be able to buy *both* a shirt and a tie. The fact that the sentence $\exists x \exists y(Px \wedge Qy)$ is not satisfied reflects this sad state of affairs.

In Example 2.2 we defined an ontology where the degree of significance of a natural number depended on its cardinality. We now consider an example related to this kind of an attitude.

**Example 3.2.** Define the cost function $g : \mathbb{N}_{\geq 1} \to \mathbb{R}_{\geq 0}$ such that $g(x) = x$. Here the amount of resources it takes to construct a representation of a positive integer is the cardinality the integer represents. The cost function $g$ could quite naturally give rise to the ontology $(\mathbb{N}_{\geq 1}, \mathcal{R}, f)$ defined in Example 2.2. The degree of significance of a positive integer assigned by $f$ is inversely proportional to the cost given by $g$. Our resource conscious interpretation of first-order logic captures some quite interesting features related to these kinds of the ultrafinitist attitudes. Let $\mathfrak{N} = (\mathbb{N}_{\geq 1}, P^{\mathfrak{N}}, Q^{\mathfrak{N}}, S^{\mathfrak{N}})$, where

1. $P^{\mathfrak{N}} = \{2\}$,
2. $Q^{\mathfrak{N}} = \{2^{600} - 1\}$,
3. $S^{\mathfrak{N}} = \{2^{600}\}$.

Let $r = 2^{600}$ and call $\mathfrak{M} = (\mathfrak{N}, g, r, \emptyset)$. We have

$$\mathfrak{M} \models \exists x(Sx).$$

However,

$$\mathfrak{M} \not\models \exists x \exists y\big(Px \wedge Qx\big).$$

The state of affairs here resembles the situation in the previous example. We can just construct a representation for the number $2^{600}$ within the system $\mathfrak{M}$.

We can also construct a representation for each smaller number. However, we cannot construct representations for $2^{600} - 1$ and 2 and have them exist simultaneously. Similar phenomena are quite common indeed; being able to feed a village of 100 people does not imply that you can feed two villages of 90 people each.

**Example 3.3.** Let us define the model $\mathfrak{S} = (S, R^{\mathfrak{S}})$, where $S = Dom(\mathfrak{S})$ is a set of finite binary strings and $R^{\mathfrak{S}} \subseteq S \times S$ a binary relation. Assume we are modeling a computer system where each file requires some kind of a support file in order for the system to work correctly. Let the set $S$ correspond to a set of files, and define the relation $R^{\mathfrak{S}}$ such that $R^{\mathfrak{S}}(x, y)$ if and only if $y$ is a support file of $x$. Let $f : S \longrightarrow \mathbb{R}_{\geq 0}$ be a function such that for each $u \in S$, $f(u)$ denotes the amount of memory that $u$ requires. Let $r \in \mathbb{R}_{\geq 0}$ be the amount of memory available. The first-order sentence $\forall x \exists y (Rxy)$ can be seen as a kind of a safety specification asserting that if we can store a file $x$, then there remains enough memory for a support file $y$ of $x$. Here $\forall x$ stands for $\neg \exists x \neg$. Assume that we have

$$(\mathfrak{S}, f, r, \emptyset) \models \forall x \exists y (Rxy).$$

Now, there could exist a file $s \in S$ that does not have support file associated with it it all, but the system cannot store $s$ in the first place (i.e., $f(s) > r$), and therefore $s$ poses no problem to us. The safety specification asserts that whatever we *can* store, we can also support.

Recall the sentence $\exists x \exists y (Px \wedge Qy)$ that was not satisfied by the resource interpretation $(\mathfrak{A}, f, 150, \emptyset)$ defined in Example 3.1. We can make the sentence true by increasing the amount of available resources: for example $(\mathfrak{A}, f, 200, \emptyset)$ satisfies the sentence. Consider then the sentence $\varphi = \forall x \exists y (Rxy)$ encountered in Example 3.3. It is easy to define a model $\mathfrak{A}$ and a cost function $f : Dom(\mathfrak{A}) \longrightarrow \mathbb{R}_{\geq 0}$ such that the resource interpretation $(\mathfrak{A}, f, r, \emptyset)$ does not satisfy $\varphi$, no matter how large the amount $r \in \mathbb{R}_{\geq 0}$ is made. These considerations motivate the following observations.

**Proposition 3.4.** *An existential first-order sentence of a relational vocabulary is satisfiable by some resource interpretation iff it is satisfiable in the standard first-order sense.*

**Proposition 3.5.** *The following holds if and only if we are investigating a finite model $\mathfrak{A}$. For each first-order sentence $\varphi$ of a relational vocabulary the following are equivalent.*

1. *For each cost function $f : Dom(\mathfrak{A}) \longrightarrow \mathbb{R}_{\geq 0}$, there exists an amount $r \in \mathbb{R}_{\geq 0}$ of resources such that $(\mathfrak{A}, f, r, \emptyset) \models \varphi$.*

2. *$\mathfrak{A} \models \varphi$ in the standard first-order sense.*

In most cases when a formal system is defined, it is too easy to point out features that *perhaps* appear undesirable or strange. We identify sentences that are satisfiable in the resource conscious setting even though when giving them the standard first-order reading, they are certainly not satisfiable. Perhaps some of the strangeness here is due to the decision to keep the symbols $\exists$ and $\forall$ in our language. For example, the sentence $\forall x(x \neq x)$ is satisfied by any resource interpretation where each object has a non-zero cost, the amount of available resources is zero, and the variable assignment is empty. Of course the sentence can also be satisfied in the ordinary first-order sense if one allows for models with the empty domain to exist. However, also the sentence

$$\exists x \exists y(x \neq y) \wedge \exists x \forall y(x = y)$$

is easily seen to be satisfiable. It is also possible to define a resource interpretation that satisfies the sentence $\forall x \exists y(Rxy)$, but not the sentence $\exists z \forall x \exists y(Rxy)$.

We defined our system such that determining the truth of an atomic formula consumes no resources whatsoever. This was done mainly for the sake of simplicity. One could, of course, wish to consider a system where checking the truth of an atom also consumes available resources. Also, we restricted our attention to languages with a relational vocabulary. With a function symbol $s$, one can write terms of the type $s(x)$. It may happen that the interpretation of $s(x)$ is an element whose construction consumes a large amount of resources even though the construction of $x$ consumes very little. This could, perhaps, be an undesirable feature. One can of course always attempt to alter the semantics of a formal system in order to deal with undesirable features. Here, however, I have wished to keep the picture simple. After all, these investigations should be regarded as tentative in nature.

## 4 Concluding Remarks

We have defined a framework for specifying formal ontologies where objects are ranked according to how significant we consider them. Such ontologies can be used for various different kinds of purposes. For instance, we have toyed with an ultrafinitist ontology where the degree of existence of the positive integers decreases as they get greater and greater. We have also defined an alternative interpretation of the existential quantifier that reflects attitudes related to such degree ontologies, and investigated the properties of the resulting logic.

As it always seems to be the case with articles similar in character to this paper, it is probably rather easy to identify incoherencies and dubious implicit assumptions from the ideas presented, and various more natural approaches to the problems considered undoubtedly exist. The aim of this article, however, has not been to search for a single canonical approach to any of the issues discussed. Instead, the approach in the article has been tentative in nature. Indeed, perhaps a more natural approach for example to modeling intuitions that lead to ultrafinitist considerations would build on some kind of a fuzzy identity such that from some large number $n$ onwards, numbers are regarded in some sense *increasingly identical*. Indeed, this idea seems to have some psychological motivation to it. For example think of two heaps of sand, one having some large number $n$ of grains and the other one $n + 1$ grains. For investigations along such lines, some kind of a fuzzy logic (see [1],[10]) could turn out to be a natural choice if the use of a formal language was desired.

Questions related to the role of resources have, of course, received a lot of well-deserved attention in various fields of computer science. Unfortunately, however, such issues have attracted relatively little attention from people working on the foundations and philosophy of mathematics. This is likely to change in the future. There is no harm in developing formal systems that model different kinds of perspectives on the foundational issues in mathematics. In the best case, the process of developing such systems can lead to discoveries that significantly elucidate the nature of mathematics and have direct industrial value.

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