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REMARKS ON THE ASSIGNMENT HEURISTIC FOR DRAWING BIPARTITE GRAPHS

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Abstract. We show that the error done by the assignment heuristic for drawing bipartite graphs has no constant bound. Contrary to earlier results published in the literature, there are bipartite graphs for which the assignment heuristic outputs drawings with edge crossings although there exist drawings without edge crossings.

Key Words and Phrases: graph drawing, barycenter heuristic, median heuristic, assignment heuristic.

1. Introduction

In a bipartite graph $G = (V_1 \cup V_2, E)$, $V_1 \cap V_2 = \emptyset$, each edge in E connects a vertex of V_1 to a vertex of V_2. Suppose a linear order is defined in V_1 and the vertices are placed according to this (fixed) order to the line $y = 1$. We want to define a linear order in V_2 such that placing the vertices of V_2 according to this order to the line $y = 2$ minimizes the number of edge crossings. This problem is known to be NP-complete [4].

Several heuristics for ordering the vertices are defined in the literature [3, 6, 8, 9]. The following two heuristics are easy to implement. For each vertex x in V_2, let bc(x) be the average of the positions of the vertices incident with x and let me(x) be the median of the same positions. In the *barycenter heuristic* (*b-heuristic*, for short) the vertices are ordered according to increasing bc-values, while in the *median heuristic* (*m-heuristic*, for short) they are ordered according to increasing me-values. Tests have shown that the b-heuristic usually performs slightly better than the m-heuristic [8], although only in the m-heuristic the relative error done has a constant upper bound [5-7].

2. The assignment heuristic

Catarci [1,2] has introduced the *assignment heuristic* (*a-heuristic*, for short) which works as follows. Suppose V_2 has m vertices arbitrarily numbered from 1 to m. Each vertex in V_2 can be placed to any of the m positions on the line x = 2. Let c_{ij}, i, j = 1,.., m, be the number of edge crossings obtained by placing the vertex x_i to the position j and a vertex incident with all vertices of V_1 to all other m - 1 positions. Our task is to choose exactly one element from each row and each column of the m×m-matrix C = c_{ij} such that the sum of the elements chosen is minimized. If the chosen element in the column j is c_{ij}, then the vertex x_i is placed to the position j. We could also state the heuristic more formally as an assignment problem

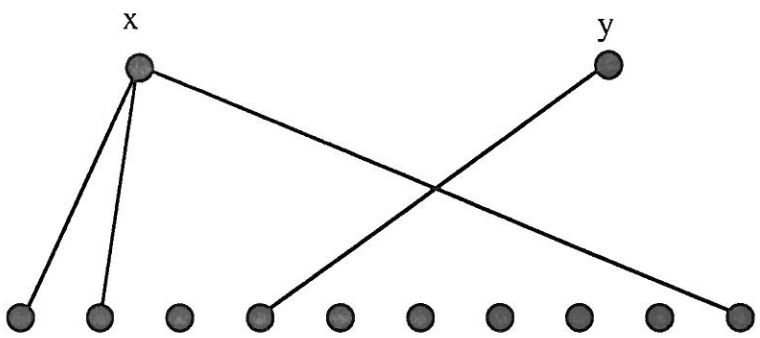
$$\min \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij}$$

subject to the constrains $\sum_{i=1}^{m} x_{ij} = 1$ and $\sum_{j=1}^{m} x_{ij} = 1$ [1,2].

Catarci has tested her heuristic against the m-heuristic [1]. The tests indicated that the a-heuristic outperforms the m-heuristic.

3. The a-heuristic fails to find some drawings with no edge crossings

This chapter demonstrates some differences between the heuristics. Our first example shows that there are cases where the b-heuristic fails to find the optimal order, while the a-heuristic outputs the optimal one. Consider the graph G_1 and the corresponding matrix C in figure 1.



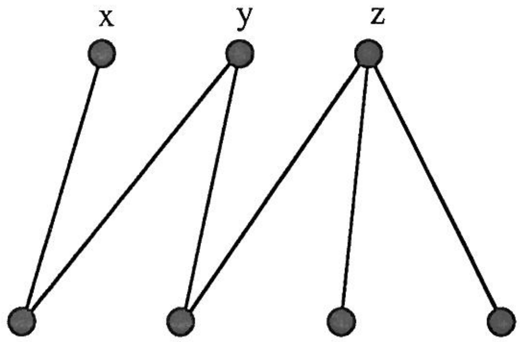
x	10	17
y	3	6

Figure 1. The graph G_1 and the corresponding matrix C.

Catarci [1-2] considers connected graphs only. The graph G_1 could be augmented connected e.g. by adding to V_2 a vertex incident with all vertices in V_1.

It follows that the a-heuristic correctly places the vertex x to the left and the vertex y to the right. Moreover, we have bc(x) = 4.33 and bc(y) = 4, and thus, the b-heuristic erroneously places x to the right and y to the left. The construction of the graph G_1 can be used to show that the relative error done by the b-heuristic has no constant bound [8].

According to Catarci [1-2], the a-heuristic always finds an order with no edge crossings if such an order exists. Our next example shows that this does not hold true. Consider the graph G_2 and the corresponding matrix C in figure 2.



$$
\begin{array}{c|ccc}
x & 0 & 3 & 6 \\
y & 2 & 6 & 10 \\
z & 12 & 9 & 6
\end{array}
$$

Figure 2. The graph G_2 and the corresponding matrix C.

Hence, the a-heuristic produces the order $y < x < z$ with one edge crossing, although there is an order with no edge crossings as shown in figure 2. The b-heuristic finds the correct order.

We can even generalize the above observation. Consider connected bipartite graphs $H_k = (V_1 \cup V_2, E)$, k = 1,2,..., where V_1 consists of 2k + 2 vertices, $V_2 = \{x, y, z\}$, and x is incident with the first vertex in V_1, y is incident with the vertices from 1 to k + 1, and z is incident with the vertices form k + 1 to 2k + 2. The matrix C of H_k is shown in figure 3a and in a more readable form in figure 3b.

x	0	$2k+1$	$4k+2$
y	$2\sum_{i=1}^{k} i$	$(k+1)^2 + 2\sum_{i=1}^{k} i$	$2(k+1)^2 + 2\sum_{i=1}^{k} i$
z	$2k(k+2) + 2\sum_{i=1}^{k+1} i$	$k(k+2) + 2\sum_{i=1}^{k+1} i$	$2\sum_{i=1}^{k+1} i$

Figure 3a. The matrix C of the graph H_k.

x	0	$2k + 1$	$4k + 2$
y	$k^2 + k$	$2k^2 + 3k + 1$	$3k^2 + 5k + 2$
z	$3k^2 + 7k + 2$	$2k^2 + 5k + 2$	$k^2 + 3k + 2$

Figure 3b. The matrix of figure 3a in a reduced form.

The assignment heuristic gives the erroneous order $y < x < z$ with k edge crossings, while the correct order $x < y < z$ has no edge crossings. By increasing the number of vertices in V_1, we can increase without the error done by the a-heuristic.

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