The College Mathematics Journal

ISSN: 0746-8342 (Print) 1931-1346 (Online) Journal homepage: www.tandfonline.com/journals/ucmj20

Characterizing the Logarithm Through Continuity, Monotonicity, and Functional Equations

Pentti Haukkanen & Timo Tossavainen

To cite this article: Pentti Haukkanen & Timo Tossavainen (16 Jan 2026): Characterizing the Logarithm Through Continuity, Monotonicity, and Functional Equations, The College Mathematics Journal, DOI: 10.1080/07468342.2025.2600267

To link to this article: https://doi.org/10.1080/07468342.2025.2600267

Published online: 16 Jan 2026.

Characterizing the Logarithm Through Continuity, Monotonicity, and Functional Equations

Pentti Haukkanen and Timo Tossavainen

Pentti Haukkanen (pentti.haukkanen@tuni.fi) received his Ph.D. in mathematics from Tampere University, Finland, under the supervision of Seppo Hyyrö. He served, among other positions, as a university lecturer in mathematics at Tampere University and retired after a full career in 2024. He enjoys various sports and culture.

Timo Tossavainen (timo.tossavainen@ltu.se) is professor of mathematics and science education at Luleå University of Technology in Sweden. He earned his Ph.D. in mathematics from the University of Jyväskylä, Finland, under the supervision of Pekka Koskela. Outside of academia, he enjoys cooking for his family, cross-country skiing, jogging, reading nonfiction, and listening to good music across all genres.

The logarithm function is a familiar concept to all mathematics students, but not all may be aware that it can be defined in several different ways. An intriguing historical and pedagogical introduction to this topic was recently given by Ricardo [1], who surveyed four distinct yet equivalent characterizations of the natural logarithm function: as the inverse of the natural exponential function, as the integral $\int_1^x \frac{dt}{t}$, as the limit $\lim_{n\to\infty} n(x^{1/n} - 1)$, and as the continuous solution of the functional equation

$$f(xy) = f(x) + f(y),\ f(e) = 1. \tag{1}$$

From the perspective of a college student, the last approach—established by Cauchy [2] in the early 19th century—may seem the most accessible, as it does not require knowledge of integration, limits, or other advanced calculus tools. However, the logarithm can also be defined using a functional equation different from (1). For instance, it can be characterized using the functional equation

$$f(x + y) - f(x) - f(y) = f(1/x + 1/y)$$

or, equivalently,

$$f(x + y) - f(xy) = f(1/x + 1/y)$$

together with the continuity of f [3].

doi.org/10.1080/07468342.2025.2600267

In this paper, we extend the discussion of the fourth approach presented in [1], by considering additional alternative definitions of the logarithm based on functional equations supplemented with an extra condition. Such a supplementary condition is necessary; for instance, a function $f : \mathbb{R}_+ \to \mathbb{R}$ that satisfies only the product rule

$$f(xy) = f(x) + f(y) \tag{2}$$

for all $x, y \in \mathbb{R}_+$ need not be the logarithm function.

Another common way to define the logarithm is to assume a power rule, i.e., a functional equation of the form

$$f(x^p) = pf(x). \tag{3}$$

The first significant result involving a power rule for the logarithm of real numbers originates from the work of Fubini [4] at the end of the 19th century. He showed that the only function $f : (1, \infty) \to \mathbb{R}$ satisfying the functional equation

$$f(x^2) = 2f(x)$$

together with the condition

$$\lim_{x \to 1^+} \frac{f(x)}{x - 1} = 1 \tag{4}$$

is the logarithm.

Half a century later, Milkman [5] presented further notable results on this topic. He proved, among other things, that if $f(x)$ is an increasing function defined for all positive real numbers x and satisfies (3) for all positive integers p, then $f(x) = c \log x$ for some constant c. Milkman also observed that it is sufficient for (3) to hold on a dense subset of the positive reals and for all prime numbers p.

One of the objectives of this paper is to improve upon the results of Fubini and Milkman. Following [6] and [7], we replace the analytical condition (4) with an alternative functional equation of the form (3), together with a weaker analytical condition—continuity or monotonicity. Additionally, our approach allows us to extend the domain of f from $(1, \infty)$ to $(0, \infty)$. We shall also show that it suffices for (3) to hold for two suitable numbers p, provided that at least one them is negative. Furthermore, if we restrict the domain to either $(1, \infty)$ or $(0, 1)$, it suffices that (3) holds, for example, for two distinct primes. Ultimately, we will establish a power rule characterization of the logarithm function on the entire domain $(0, \infty)$.

Characterizations of the logarithm are also relevant from the perspective of *arithmetic functions*, i.e., mappings from the set of positive integers to the set of real numbers. For arithmetic functions, (2) becomes

$$f(mn) = f(m) + f(n), \tag{5}$$

where m and n are positive integers. If (5) holds for all positive integers m and n such that $\gcd(m, n) = 1$, then f is said to be an *additive* function. If (5) holds for all positive integers m, n, then f is said to be a *completely additive* function [8–10].

The arithmetic logarithm function is precious to number theorists due to its connection with the Prime Number Theorem, which states that the nth prime is asymptotically $n \log n$. Furthermore, the derivative of the Riemann zeta function is

$$\zeta'(z) = -\sum_{n=2}^{\infty} \frac{\log n}{n^z}.$$

It was P. Erdős who first identified conditions under which an additive or a completely additive arithmetic function must be the logarithm function (up to a constant multiple). He proved that if an additive function f satisfies

$$f(n+1) - f(n) \geq 0 \quad (n = 1, 2, \ldots)$$

or

$$\lim_{n\to\infty} \big(f(n+1) - f(n)\big) = 0,$$

then $f(n) = c \log n$ [11]. A few years later Milkman [5] provided an elementary proof of Erdős' result.

We conclude this paper with a result characterizing the arithmetic logarithm using (3). While our result is relatively modest, it is the first of its kind in this direction and may be considered an analogue of Milkman's results.

Preliminaries

Before presenting our results, we recall a few definitions and record a theorem.

A function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$, is said to be *monotone* if it is increasing or decreasing; i.e., it preserves or reverses the order of its arguments. Specifically,

- f is *increasing* if

$$x_1 \leq x_2 \quad \Rightarrow \quad f(x_1) \leq f(x_2) \quad \text{for all } x_1, x_2 \in D;$$

- f is *decreasing* if

$$x_1 \leq x_2 \quad \Rightarrow \quad f(x_1) \geq f(x_2) \quad \text{for all } x_1, x_2 \in D.$$

A subset $A \subset \mathbb{R}$ is said to be *dense* in the set of real numbers if between any two distinct real numbers there exists at least one element of A. Equivalently, A is dense in $\mathbb{R}$ if for every $x \in \mathbb{R}$ and every $\varepsilon > 0$ there exists $a \in A$ such that

$$|x - a| < \varepsilon.$$

A key tool to be used in our proofs is Kronecker's Approximation Theorem (see, e.g., [12, Chapter 7] or [13, Chapter XXIII]), which describes how combinations of integers and an irrational number can approximate any real number. In its simplest form, the theorem states that if α is an irrational number, then the set

$$\{m + n\alpha : m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. In other words, for every $x \in \mathbb{R}$ and every $\varepsilon > 0$ there is an integer combination of 1 and α such that its distance to x is less than ε.

Characterization with two or more functional equations

In the following two theorems, we assume that $p \neq \pm 1$ and $q \neq \pm 1$ are two distinct nonzero real numbers such that at least one of them is negative, and that $p^m \neq q^n$ for all nonzero integers m and n. In Theorem 1, it would suffice to assume that p and q are distinct prime numbers if the domain of h is taken to be $(0, \infty)$.

Theorem 1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous or monotone function such that*

$$h(px) = ph(x) \text{ and } h(qx) = qh(x) \quad \text{for all } x \in \mathbb{R}. \tag{6}$$

Then

$$h(x) = h(1)x \quad \text{for all } x \in \mathbb{R}. \tag{7}$$

Proof. From (6) we obtain

$$h(p^m q^n) = h(1) p^m q^n, \; m, n \in \mathbb{Z}.$$

In the beginning of the section, we assumed that $p < 0$ or $q < 0$; without loss of generality we may suppose that $q < 0$. Let $S_+ = \{p^{2m} q^{2n} \mid m, n \in \mathbb{Z}\}$ and $S_- = \{p^{2m} q^{2n+1} \mid m, n \in \mathbb{Z}\}$. Clearly, (7) holds for all $x \in S_+ \cup S_-$.

First, assume that h is continuous. Since continuous functions on $\mathbb{R}$ that agree on a dense subset are equal [**14**, Problem 8.6], it suffices to show that $S_+ \cup S_-$ is dense in $\mathbb{R}$.

In order to show that S_+ is dense in the set of positive real numbers, we first take logarithms of the elements of S_+ and obtain an equivalent claim stating that $T = \{m \log p^2 + n \log q^2 \mid m, n \in \mathbb{Z}\}$ is dense in the set of real numbers (compare [**6**], Theorem 5). Dividing the elements of T by $\log p^2$, we obtain another equivalent claim stating that the set $U = \{m + n \log q^2 / \log p^2 \mid m, n \in \mathbb{Z}\}$ is dense in the set of real numbers. Since $\log q^2 / \log p^2$ is irrational (if it were rational r/s, then $r \log p^2 = s \log q^2$ and further $p^{2r} = q^{2s}$), the claim follows now from Kronecker's Approximation Theorem: S_+ is dense in the set of positive real numbers.

In a similar way we can show that S_- is dense in the set of negative real numbers. Consequently, $S_+ \cup S_-$ is dense in the set of real numbers, and therefore (7) holds for all $x \in \mathbb{R}$.

Second, assume that h is monotone. Denote $c = h(1)$. If $c = 0$, then $h(x) = 0$ in the dense set $S_+ \cup S_-$ of $\mathbb{R}$ described in the first part of this proof. Thus, by monotonicity, $h(x) = 0$ for every $x \in \mathbb{R}$, and (7) holds.

Assume now that $c > 0$. Then h is increasing. Suppose that there exists $x_0 \in \mathbb{R}$ such that $h(x_0) \neq cx_0$. Then $h(x_0) = cx_0 + d$, where $d \neq 0$. Consider the case $d > 0$. By the density of $S_+ \cup S_-$, there exist integers m and n such that $x_1 = p^m q^n > x_0$ and $x_1 - x_0 < d/c$. But then

$$h(x_1) = cx_1 = cx_0 + c(x_1 - x_0) < cx_0 + d = h(x_0),$$

contradicting the fact that h is increasing. Thus (7) follows. In the case $d < 0$, we choose $x_1 < x_0$ and proceed analogously. If $c < 0$, then h is decreasing, and the similar argument as above holds. ∎

Next we give our power rule characterization of the logarithm function on $(0, \infty)$.

Theorem 2. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous or monotone function such that*

$$f(x^p) = pf(x) \text{ and } f(x^q) = qf(x) \quad \text{for all } x > 0.$$

Then

$$f(x) = f(e) \log x \quad \text{for all } x > 0.$$

Proof. Now, let us consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = f(e^t)$. Then h is a continuous or monotone function such that

$$h(pt) = f(e^{pt}) = pf(e^t) = ph(t),$$

and

$$h(qt) = f(e^{qt}) = qf(e^t) = qh(t)$$

for all real t. By Theorem 1, we obtain

$$f(e^t) = h(t) = h(1)t = f(e)t,\ t > 0.$$

The proof is completed upon setting $x = e^t$. ■

Characterizations with three functional equations also exist. For instance, there is a modification of Theorem 1, where the extension to $\mathbb{R}_-$ is carried out by the functional equation $h(-x) = -h(x)$.

Arithmetic logarithm function and arithmetic derivative

To the best of our knowledge, the following theorem is the first attempt to characterize the logarithmic arithmetic function using the power rule. While the result is quite straightforward, it is not entirely trivial. We encourage the reader to explore this topic further.

To prove our theorem, we need a lemma that follows by induction from the Fundamental Theorem of Arithmetic, see [**9**, Proposition 1].

Lemma 1. *A completely additive function is determined by its values at the primes.*

Theorem 3. *Assume that f is an additive arithmetic function possessing the power property*

$$f(p^k) = kf(p) \quad \text{for all primes } p \text{ and positive integers } k. \tag{8}$$

Further, assume that there exists a real constant c such that $f(p)/\log p = c$ for all primes p. Then $f(n) = c \log n$.

Proof. Since f is additive possessing the property (8), it is completely additive. The function $c \log n$ is also completely additive. Now the functions $f(n)$ and $c \log n$ agree at primes. By Lemma 1, they are equal. ■

We conclude this paper by briefly discussing a connection between our topic, the Leibniz rule, and the study of differentiating numbers; see, e.g., [**15**, **16**]. Namely, a function $f(x)$ satisfies the *Leibniz rule*

$$f(xy) = yf(x) + xf(y) \tag{9}$$

for all $x, y \in \mathbb{R}_+$ if and only if $g(x) = f(x)/x$ satisfies the product rule (2). If $g(x)$ is continuous, this holds if and only if $g(x) = c \log x$ for some constant c [**5**]. This implies that if $f : \mathbb{R}_+ \to \mathbb{R}$ is continuous, then f satisfies the condition (9) if and only if $f(x) = cx \log x$ for some constant c.

For arithmetic functions, the Leibniz rule naturally takes the form

$$f(mn) = nf(m) + mf(n) \tag{10}$$

for all $m, n \in \mathbb{Z}_+$. The Leibniz rule (10) and the product rule (5) for arithmetic functions can be viewed as recurrence relations. The *arithmetic derivative* $D(n)$ is an arithmetic function satisfying the Leibniz rule (10) with initial conditions $D(p) = 1$ for all prime numbers p. A related function, the *logarithmic derivative* $\mathrm{ld}(n) = D(n)/n$, satisfies the product rule (5) for all positive integers m, n with initial conditions $\mathrm{ld}(p) = 1/p$ for all prime numbers p. Therefore, $\mathrm{ld}(n)$ is a completely additive arithmetic function.

Acknowledgment. The authors wish to thank the editor and the reviewers for their valuable comments.

Disclosure statement. No potential conflict of interest was reported by the author(s).

Summary. The logarithm function on the positive real numbers can be characterized in several ways. In this paper, we focus on classical characterizations based on the product rule or the power rule, and we improve upon some results by Fubini and Milkman. Our main result can be stated as follows: If a continuous or monotone function on the positive real numbers satisfies a power rule for both a negative and a positive exponent, then it is the logarithm function or the zero function. We also briefly consider the arithmetic logarithm function and its connection to the concept of arithmetic derivative.

References

[1] Ricardo H. The equivalence of definitions of the natural logarithm function. Coll Math J. 2022;53(3): 190–196.
[2] Cauchy AL. Cours d'analyse de l'Ecole royale polytechnique. 1. re partie, Analyse algébrique. de l'Imprimerie royale; 1821.
[3] Heuvers KJ, Kannappan P. A third logarithmic functional equation and Pexider generalizations. Aequationes Math. 2005;70(1–2):117–1121.
[4] Fubini G. Di una nuova successione di numeri. Period Mat. 1899;14:147–149.
[5] Milkman J. Note on the functional equations $f(xy) = f(x) + f(y)$, $f(x^n) = nf(x)$. Proc Amer Math Soc. 1950;1(4):505–508.
[6] Halmetoja M, Haukkanen P, Laurinolli T, Merikoski JK, Tossavainen T, Virtanen A. On direct and inverse proportionality. Amer Math Monthly. 2007;114(10):871–881.
[7] Tossavainen T, Haukkanen P. The Schröder equation and some elementary functions. Int J Math Anal. 2007;1(3):101–106.
[8] Laohakosol V, Tangsupphathawat P. Characterizations of additive functions. Lith Math J. 2016;56(4): 518–528.
[9] Merikoski JK, Haukkanen P, Tossavainen T. Complete additivity, complete multiplicativity, and Leibniz-additivity on rationals. Integers. 2021;21:Paper No. A33.
[10] Schwab ED. Dirichlet product and completely additive arithmetical functions. Nieuw Arch Wisk. 1995;13(2):187–193.
[11] Erdős P. On the distribution function of additive functions. Ann Math. 1946;47:1–20.
[12] Apostol TM. Modular functions and Dirichlet series in number theory. New York: Springer-Verlag; 1976.
[13] Hardy GH, Wright EM. An introduction to the theory of numbers. 6th ed. Oxford: Oxford University Press; 2008.
[14] Spivak M. Calculus. 2nd ed. Berkeley: Publish or Perish; 1980.
[15] Pasten H. A derivation of the infinitude of primes. Amer Math Monthly. 2024;131(1):66–73.
[16] Tossavainen T, Haukkanen P, Merikoski JK, Mattila M. We can differentiate numbers, too. Coll Math J. 2024;55(2):100–108.

