12^{th} International Conference on Fracture Mechanics of Concrete and Concrete Structures
FraMCoS-12
B.L.A. Pichler, Ch. Hellmich, P. Preinstorfer (Eds)

https://doi.org/10.21012/FC12.1377
TT-C2:2

TENSORIAL CONTINUUM DAMAGE MODEL FOR CONCRETE

SULATA DHAKAL*, JUHA HARTIKAINEN*, REIJO KOUHIA*, TIMO SAKSALA*, JANI VILPPO*, KIM CALONIUS†, ALEXIS FEDOROFF† AND KARI KOLARI†

* Tampere University, Faculty of Built Environment, Structural Mechanics
P.O. Box 600, FI-33101 Tampere, Finland
e-mail: sulata.dhakal@tuni.fi, juha.hartikainen@tuni.fi, reijo.kouhia@tuni.fi, timo.saksala@tuni.fi, jani.vilppo@tuni.fi

†VTT Technical Research Centre of Finland
P.O. Box 1000, FI-02044 VTT Espoo, Finland e-mail: kim.calonius@vtt.fi, alexis.fedoroff@vtt.fi, kari.kolari@vtt.fi

Key words: anisotropic damage; axial splitting; elastic-brittle material; concrete; Ottosen's 4-parameter criterion; constitutive equations; the specific Gibbs free energy; dissipation potential; failure modes, finite element implementation

Abstract. A tensorial continuum damage model for concrete which can correctly predict the failure stress states and failure modes in general multiaxial stress states is presented. The model is thermodynamically consistent and is based on proper expressions for the specific Gibbs free energy and the complementary form of the dissipation potential. Damaging of the material is described by a symmetric positive definite second order damage tensor. Invariant theory is used in construction of the potential functions which guarantees that the proper symmetry behaviour is satisfied and no artificial symmetrization operations need not to be done. The failure surface is formulated in a way that it mimics the behaviour of the well-known Ottosen's four parameter failure surface. The predictions of the proposed model are compared to the Concrete Damaged Plasticity (CDP) model available in the commercial finite element software Abaqus in uniaxial and equibiaxial cases. The CDP model is calibrated against the uniaxial test results. However, for the CDP model the strain in the loading direction in the biaxial case starts to deviate from the experimental results already before the peak stress, while the present model yields accurate prediction. The constitutive model is implemented as an UMAT subroutine to be used in Abaqus.

1 INTRODUCTION

Concrete is one of the most commonly used construction materials, and has been subjected to a significant amount of research including failure behaviour in consideration of different loading conditions and micro-structural defects such as voids, inhomogeneities and micro-cracks. Typical characteristics of the failure behaviour of concrete include gradual loss of the elasticity, volume dilatancy and strain softening, which are consequences of the propagation and coalescence of micro-cracks leading ultimately to the material failure [10, 41]. The mechanical failure of concrete is induced mainly by cracking in tension due to propagation of the most critical micro-crack, and by crushing in compression due to the interaction of distributed micro-cracks [2]. This fact shows up in uniaxial compression as axial splitting along the direction of compression. Schreyer [38] observed that the classical stress criteria do not have the flexibility to reflect the failure modes

for various stress states and thus cannot predict axial-splitting. Only few continuum models can predict the compressive axial-splitting, e.g. [2, 17, 24, 33, 38, 42, 46], although the physics of axial-splitting have been studied quite extensively, see e.g. [18] and the references therein.

A huge number of different constitutive models based on plasticity theory have been presented for concrete, see e.g. [11, 15, 44]. Damage descriptions by means of scalar, vectorial or tensorial damage variables have been utilized. Scalar damage variables are easy to implement, and hence widely applied, e.g. [14, 20, 27, 34]. Mazars and Pijaudier-Cabot [31] presented relationships between isotropic damage and fracture mechanics theories. The damage of rock-like materials, however, is definitely anisotropic due to the orientation of micro-cracks depending on the stress state. This feature can be described only by vectorial or tensorial damage variables. Vectorial damage variables have been used in [4, 5, 9, 24, 32] and second- or higher-order damage tensors in [1, 10, 12, 23, 28, 33, 36, 39, 45, 47, 49].

Several authors have combined plasticity and damage to model the failure of concrete [6, 14, 19, 21, 22, 43, 47]. On the other hand, since the inelastic behaviour of concrete is rather due to damage and micro-cracking than plastic deformations, as presented by [30, 31], models based merely on damage and micro-cracking have been formulated regularly.

The well-known Barcelona model, using a scalar damage variable for the degradation of both the volumetric and distortional elasticity, separately, was presented in [29]. Lee and Fenves [26, 27] revised the model by using two independent scalar damage variables to represent properly the cyclic behaviour of concrete. Their model has been implemented in the commercial finite element software Abaqus by [8]. A similar approach is carried out by [16], and a selection of 3D concrete models is reviewed in [35, 40]. As an alternative approach the microplane approach has got much attention, see e.g. [3, 40].

Ottosen proposed in 1977 [37] a model which captures the relevant features of concrete failure under various multiaxial stress states. In a recent study [48], the Ottosen model is formulated as an elasto-plastic model with a non-associated flow rule. Combination of the Ottosen model with damage mechanics with two damage parameters is given by Contrafatto and Cuomo [7].

In the present paper, a model for concrete, which was proposed in [42], is briefly summarized. In developing this model the following items were emphasized. The model should be able to capture the basic brittle failure modes: a) axial splitting along the direction of unconfined uniaxial compression, b) damaging perpendicular to the direction of uniaxial unconfined tension. It should also predict similar failure stresses to the Ottosen model and be thermodynamically consistent.

2 Constitutive model

This conference paper gives a brief overview of the recent continuum damage based constitutive model proposed in [42]. The model can capture the basic brittle failure modes and predict multiaxial failure stresses identical to those predicted by the Ottosen failure surface. Formulation of the model is based on continuum thermodynamics and pertinent choices for the two potential functions: the specific Gibbs free energy (Gibbs function) and the dual form of the dissipation potential, which are developed by using the invariant theory of scalar tensor-valued functions.

The reversible behaviour is modelled by the free energy function which depends on two symmetric second order tensors, the stress $\boldsymbol{\sigma}$ and the damage tensor $\boldsymbol{D}$. In this particular case the integrity basis contains ten invariants Adopting only the terms which are linear in $\boldsymbol{D}$ and quadratic in $\boldsymbol{\sigma}$, the mechanical part of the

free energy function is given as

$$\rho_0\psi^{\mathrm{c}}(\boldsymbol{\sigma},\boldsymbol{D},\kappa) = \frac{1}{4G}\left[\mathrm{tr}(\boldsymbol{s}^2)+\mathrm{tr}(\boldsymbol{s}^2\boldsymbol{D})\right] + \frac{1}{18K_{\mathrm{b}}}(1+\chi\,\mathrm{tr}\,\boldsymbol{D})(\mathrm{tr}\,\boldsymbol{\sigma})^2 + \int_0^{\kappa}\int_0^{\kappa'} g(\kappa'')\mathrm{d}\kappa''\mathrm{d}\kappa', \quad (1)$$

where G and K_{b} stand for the shear and bulk modulus, χ is a dimensionless positive material parameter, κ is a scalar internal variable describing the hardening/softening behaviour and ρ_0 is the density. The deviatoric stress tensor is denoted as $\boldsymbol{s} = \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}$. The integrand $g(\kappa)$ is a four-parameter rational function [42]

$$g(\kappa) = \frac{H_0}{\kappa_0}\frac{h_2\left(\kappa/\kappa_0\right)^2 - 2h_1\left(\kappa/\kappa_0\right) - 1}{\left[h_2\left(\kappa/\kappa_0\right)^2+1\right]^2}. \quad (2)$$

For description of the irreversible damaging behaviour, the dissipation potential is defined by the thermodynamic forces $\boldsymbol{Y}$ and K describing the damage and hardening, respectively. The dissipation potential is now defined as

$$\varphi(\boldsymbol{Y},K;\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \mathrm{I}(\boldsymbol{Y},K;\boldsymbol{\sigma},\boldsymbol{\varepsilon}), \quad (3)$$

where I denotes the indicator function that restricts the stresses inside the elastic domain [13], and it is defined as

$$\mathrm{I}(\boldsymbol{Y},\boldsymbol{K};\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \begin{cases} 0, & \text{if } (\boldsymbol{Y},K)\in\Sigma \\ +\infty, & \text{if } (\boldsymbol{Y},K)\notin\Sigma, \end{cases} \quad (4)$$

where Σ is a convex set defining the admissible elastic domain for $(\boldsymbol{Y},K)$ and bounded by the damage criterion $f(\boldsymbol{Y},K;\boldsymbol{\sigma},\boldsymbol{\varepsilon})\leq 0$.

The failure surface is in line with the Ottosen model [37] and since the failure modes of deformation are essential, the damage surface is now formulated as

$$f(\boldsymbol{Y},K;\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \frac{AJ_2}{\sigma_{\mathrm{c}}} + \Lambda(\theta)\sqrt{J_2} + BI_1 - (\sigma_{\mathrm{c0}}+K) + \mathrm{tr}\left[\boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{Y}-\boldsymbol{Y}(\boldsymbol{\sigma})\right)\right], \quad (5)$$

where $J_2 = \frac{1}{2}\mathrm{tr}(\boldsymbol{s}^2)$, $I_1 = \mathrm{tr}\,\boldsymbol{\sigma}$ and σ_{c}, σ_{c0} are the uniaxial compressive strength and damage initiation stress. The shape factor $\Lambda(\theta)$ depends on the Lode angle $\theta = (1/3)\arccos(3\sqrt{3}\,J_3/(2J_2^{3/2}))$ and determines the size and shape of the failure surface on the deviatoric plane [37, 42] and $J_3 = \det(\boldsymbol{s})$. Shape of the failure locus changes from almost triangular shape at low hydrostatic compressive stress states to more rounded shape at high hydrostatic compressive stress states, see Fig. 1. The symmetric positive definite second-order tensor $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{\sigma},\boldsymbol{\varepsilon})$ is defined as

$$\boldsymbol{A} = \frac{1}{1+\beta_1\langle\mathrm{tr}\,\boldsymbol{\sigma}\rangle/\sigma_{\mathrm{t}}}\left(\frac{\boldsymbol{\varepsilon}_+}{\|\boldsymbol{\varepsilon}_+\|}+\beta_2\boldsymbol{I}\right) \quad (6)$$

and $\boldsymbol{\varepsilon}_+$ is the positive part of the elastic strain tensor, i.e. $\boldsymbol{\varepsilon}_+ = \sum_{i=1}^{3}\langle\varepsilon_i\rangle\boldsymbol{\phi}_i\otimes\boldsymbol{\phi}_i$, where ε_i, $i = 1,\ldots,3$ are the eigenvalues of the elastic strain tensor, $\boldsymbol{\phi}_i$ stands for the corresponding eigenvectors, β_1,β_2 are positive non-dimensional parameters and σ_{t} is the uniaxial tensile strength. The norm of a second order tensor tensor is defined in a standard way, i.e. $\|\boldsymbol{\varepsilon}_+\| = \sqrt{\mathrm{tr}(\boldsymbol{\varepsilon}_+^2)}$, and $\langle\bullet\rangle$ denotes the Macaulay brackets.

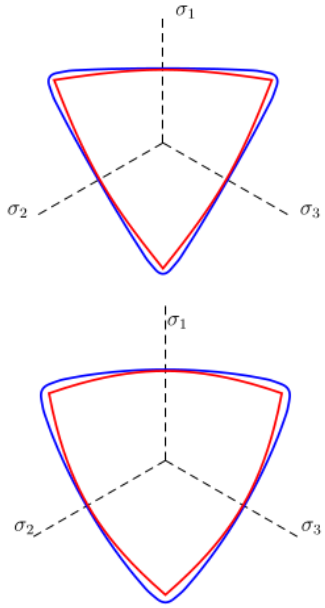

Figure 1: Shape of the failure locus on the deviatoric plane. Above on the π-plane ($\sigma_{\mathrm{m}} = 0$) and at $\sigma_{\mathrm{m}} = -\sigma_{\mathrm{c}}$ below. Blue curve corresponds to the Ottosen model and red one to the Barcelona model.

The thermodynamic force $\boldsymbol{Y}$ dual to the damage rate is obtained from the definition

$$\boldsymbol{Y}(\boldsymbol{\sigma}) = \rho_0 \frac{\partial \psi^c}{\partial \boldsymbol{D}} = \frac{1}{4G}\boldsymbol{s}^2 + \frac{\chi}{18K_\mathrm{b}}(\mathrm{tr}\ \boldsymbol{\sigma})^2 \boldsymbol{I}. \tag{7}$$

and the hardening variable K is

$$K = -\rho_0 \frac{\partial \psi^c}{\partial \kappa} = -\int_0^\kappa g(\kappa')\mathrm{d}\kappa' = H_0 \frac{h_1 \left(\kappa/\kappa_0\right)^2 + \left(\kappa/\kappa_0\right)}{h_2 \left(\kappa/\kappa_0\right)^2 + 1}, \tag{8}$$

Positive definiteness of tensor $\boldsymbol{A}$ in (5) together with the requirement that $\chi \geq 0$ in (7) guarantees that the dissipation power is non-negative.

Using the approach presented in [42] results in the constitutive equations

$$\boldsymbol{\varepsilon} = \rho_0 \frac{\partial \psi^c}{\partial \boldsymbol{\sigma}}, \quad \dot{\boldsymbol{D}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{Y}} = \dot{\lambda}\boldsymbol{A}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \qquad \dot{\kappa} = -\dot{\lambda}\frac{\partial f}{\partial K} = \dot{\lambda}. \tag{9}$$

The multiplier $\dot{\lambda}$ is determined from the consistency condition $\dot{f}(\boldsymbol{Y}, K; \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = 0$. It can be easily seen that the failure mode is correct in unconfined uniaxial tension and compression, see [42, Sec. 3.2].

3 IDENTIFICATION OF MATERIAL PARAMETERS

The Ottosen parameter set A, B, k_1 and k_2 in the failure surface are adjusted to represent the damage surface corresponding with the four failure states [37, 42]. Since damage starts to develop at fairly low stress levels, the damage initiation stress in compression σ_{c0} needs to be selected. To obtain the parameters χ, κ_0, β_1 and β_2, the uniaxial compression and tension as well as the biaxial test results are needed. It should be emphasized that the twelve model parameters $(A, B, k_1, k_2, K_\infty, H_0, h_1, h_2, \kappa_0, \chi, \beta_1, \beta_2)$ can be obtained by using closed form expressions from the data shown in [42, Table 1] (strength values in uniaxial compression and tension, biaxial compression, one extra point from compressive meridian (I_1, J_2)-values, damage initiation stress in uniaxial compression, strain values at the peak stress in uniaxial compression, strain at uniaxial tension failure and one stress and corresponding strain value on the post-peak range. Complete procedure to obtain the parameter values is described in [42, Sect. 3.3].

4 RESULTS

Predictions of the model are compared to the well documented experimental data for concrete by Kupfer et al. [25] and the values used in calibration of the model are shown in [42, Tables 1 and 2] as well as the current model parameters. The biaxial tests in [25] were performed with specimens having dimensions 20 cm $\times$ 20 cm $\times$ 5 cm and the model results are compared to the tests having unconfined uniaxial compressive strength of 30.9 MPa, maximum aggregate size 15 mm and the water-cement ratio 0.9.

Results for the equibiaxial compression test ($\sigma_{11} = \sigma_{22}$) are shown in Fig. 2.

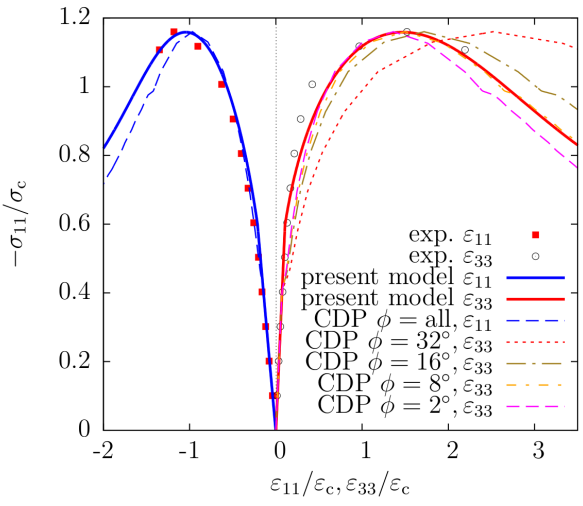

Figure 2: Stress-strain behaviour in equibiaxial compression ($\sigma_{11} = \sigma_{22}$) [42, Fig. 8] with experimental results from [25]. The Abaqus CDP model responses are shown for four values of the dilatation angle. It should be noticed that different dilatation angle gives the best fit to experimental data in comparison to unconfined uniaxial compression (see [42, Fig. 4]) for the CDP model.

The best similarity to the experimental results for the Abaqus CDP model is obtained when the dilatation angle is 8°, in which case the $\sigma_{11}, \epsilon_{33}$-curve is almost identical to the response of the present model. However, for the CPD model the strain in the loading direction starts to deviate from the experimental results already before the peak stress.

As a result of the present model, the dominant damage develops perpendicularly to the plane of loading as it can be seen from Fig. 3.

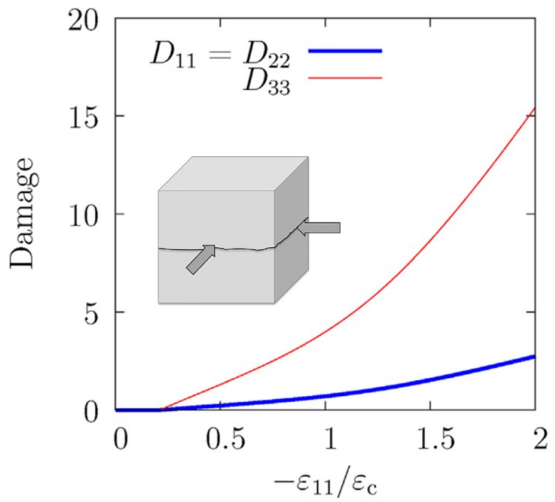

Figure 3: Damage-strain behaviour, damage is the largest in the 33-direction, i.e. the fracture mode corresponds splitting along the compressive plane illustrated [42, Fig.9].

5 CONCLUSIONS

A thermodynamically consistent formulation to model anisotropic damage for concrete and other quasi-brittle materials is presented. The model is based on proper expressions for the specific Gibbs free energy and the complementary form of the dissipation potential. Damaging of the material is described by a symmetric positive definite second order damage tensor. In this approach the values of the components in the damage tensor do not have an upper bound, thus facilitating a continuous numerical implementation. Especially, the failure surface is formulated in such a way that it will mimic the behaviour of the well-known Ottosen's four parameter failure surface. Although the formulation is basically non-associated, it follows closely the one for the standard dissipative solid.

One significant feature of the proposed model is its ability to model the failure modes correctly. In uniaxial compression, the mode of failure is axial splitting, i.e. the damaged zones are aligned parallel to the direction of the compressive stress. In tension, the failure takes place in a plane perpendicular to the applied stress. The obtained results are in accordance with the well-known experimental results found in literature.

To obtain a well behaving numerical model avoiding pathological mesh-dependency, some regularization strategy should be included in the model. Fassin et al. [12] regularized a CDM-model where damage is described by a second order tensor by using only a scalar variable.

Future research will also be focused on describing the plastic and cyclic behaviour of the material.

5.1 Acknowledgements

This work is funded by the Research Council of Finland, project ConSus - Towards sustainable carbon free concrete construction. Decision number 356502.

REFERENCES

[1] BADEL, P., GODARD, V., AND LEBLOND, J.-B. Application of some anisotropic damage model to the prediction of the failure of some complex industrial concrete structure. *International Journal of Solids and Structures 44*, 18 (2007), 5848 – 5874.

[2] BASISTA, M. Micromechanics of damage in brittle solids. In *Anisotropic Behaviour of Damaged Materials*, J. Skrzypek and A. Ganczarski, Eds., vol. 9 of *Lecture Notes in Applied and Computational Mechanics*. Springer-Verlag, 2003, pp. 221–258.

[3] BAŽANT, Z., CANER, F., CAROL, I., ADLEY, M., AND AKERS, S. Microplane model M4 for concrete. I: Formulation with work-

conjugate deviatoric stress. *Journal of Engineering Mechanics 126*, 9 (2000), 944–953.

[4] CHABOCHE, J.-L. Damage induced anisotropy: On the difficulties associated with the active/passive unilateral condition. *International Journal of Damage Mechanics 1*, 2 (1992), 148–171.

[5] CHABOCHE, J.-L. Development of continuum damage mechanics for elastic solids sustaining anisotropic and unilateral damage. *International Journal of Damage Mechanics 2*, 4 (1993), 311–329.

[6] CICEKLI, U., VOYIADJIS, G., AND AL-RUB, R. A. A plasticity and anisotropic damage model for plain concrete. *International Journal of Plasticity 23* (2007), 1874–1900.

[7] CONTRAFATTO, L., AND CUOMO, M. Comparison of two forms of strain decomposition in an elastic-plastic damaging model for concrete. *Modelling and Simulation in Materials Science and Engineering 15*, 4 (may 2007), S405–S423.

[8] DASSAULT SYSTÉMES. Abaqus unified FEA, 2020. Simulia.

[9] DAVISON, L., AND STEVENS, A. Thermodynamical constitution of spalling elastic bodies. *Journal of Applied Physics 44* (1973), 668–674.

[10] DRAGON, A., HALM, D., AND DÉSOYER, T. Anisotropic damage in quasi-brittle solids: modelling, computational issues and applications. *Computer Methods in Applied Mechanics and Engineering 183*, 3-4 (2000), 331–352.

[11] DRAGON, A., AND MRÓZ, Z. A continuum model for plastic-brittle behavior of rock and concrete. *International Journal of Engineering Science 17* (1979), 121–137.

[12] FASSIN, M., EGGERSMANN, R., WULFINGHOFF, S., AND REESE, S. Gradient-extended anisotropic brittle damage modeling using a second order damage tensor – theory, implementation and numerical examples. *International Journal of Solids and Structures 167* (2019), 93–126.

[13] FRÉMOND, M. *Non-Smooth Thermomechanics*. Springer, Berlin, 2002.

[14] GRASSL, P., AND JIRÁSEK, M. Damage-plastic model for concrete failure. *International Journal of Solids and Structures 43* (2006), 7166–7196.

[15] GRASSL, P., LUNDGREN, K., AND GYLLTOFT, K. Concrete in compression, a plasticity theory with a novel hardening law. *International Journal of Solids and Structures 39* (2002), 5205–5223.

[16] GRASSL, P., XENOS, D., NYSTRÖM, U., REMPLING, R., AND GYLLTOFT, K. CDPM2: A damage-plasticity approach to modelling te failure of concrete. *International Journal of Solids and Structures 50*, 24 (2013), 3805–3816.

[17] HALM, D., DRAGON, A., AND CHARLES, Y. Damage model for quasi-brittle solids: Coupled effects of induced and initial anisotropy. *J. Phys. IV France 105* (2003), 313–320.

[18] HORII, H., AND NEMAT-NASSER, S. Compression induced nonplanar crack extension with application to splitting, exfoliation, and rockburst. *Journal of Geophysical Research 87*, B8 (1982), 6,805–6,821.

[19] IBRAHIMBEGOVIĆ, A., MARCOVIC, D., AND GATUINGT, F. Constitutive model of coupled damage-plasticity and its finite element implementation. *Revue Européenne des Eléments Finis 12*, 4 (2003), 381–405.

[20] JASON, L., HUERTA, A., PIJAUDIER-CABOT, G., AND GHAVAMIAN, S. An elastic plastic damage formulation for concrete: Application to elementary tests and comparison with an anisotropic damage model. *Computer Methods in Applied Mechanics and Engineering 195* (2006), 7077–7092.

[21] JEFFERSON, A. Craft - a plastic-damage-contact model for concrete. i. model theory and thermodynamic considerations. *International Journal of Solids and Structures 40*, 22 (2003), 5973–5999.

[22] JEFFERSON, A., MIHAI, I., TENCHEV, R., ALNAAS, W., COLE, G., AND LYONS, P. A

plastic-damage-contact constitutive model for concrete with smoothed evolution functions. *Computers & Structures 169* (2016), 40–56.

[23] JU, J. W. Isotropic and anisotropic damage variables in continuum damage mechanics. *Journal of Engineering Mechanics 116*, 12 (1990), 2764–2770.

[24] KOLARI, K. Damage mechanics model for brittle failure of transversely isotropic solids - finite element implementation. Tech. Rep. 628, VTT Publications, Espoo, 2007.

[25] KUPFER, H., HILSDORF, H., AND RÜSCH, H. Behaviour of concrete under biaxial stresses. *Journal of the American Concrete Institute 66*, 8 (August 1969), 656–666.

[26] LEE, J. *Theory and implementation of plastic-damage model for concrete structures under cyclic and dynamic loading*. PhD thesis, University of California, Berkeley, 1996.

[27] LEE, J., AND FENVES, G. Plastic-damage model for cyclic loading of concrete structures. *Journal of the Engineering Mechanics, ASCE 124*, 8 (August 1998), 892–900.

[28] LUBARDA, V., KRAJCINOVIC, D., AND S.MASTILOVIC. Damage model for brittle elastic solids with unequal tensile and compressive strengths. *Engineering Fracture Mechanics 49*, 5 (1994), 681–697.

[29] LUBLINER, J., OLIVER, J., OLLER, S., AND OÑATE, E. A plastic-damage model for concrete. *International Journal of Solids and Structures 25*, 3 (1989), 299–326.

[30] MAZARS, J., AND PIJAUDIER-CABOT, G. Continuum damage theory – application to concrete. *Journal of the Engineering Mechanics, ASCE 115*, 2 (1989), 345–365.

[31] MAZARS, J., AND PIJAUDIER-CABOT, G. From damage to fracture mechanics and conversely: a combined approach. *International Journal of Solids and Structures 33*, 20-22 (1996), 3327–3342.

[32] MIKKOLA, M., AND PIILA, P. Nonlinear response of concrete by use of the damage theory. In *Proceedings of the International Conference on Computer-Aided Analysis and design of Concrete Structures* (Swansea, 1984), Pineridge Press, pp. 179–189.

[33] MURAKAMI, S., AND KAMIYA, K. Constitutive and damage evolution equations of elastic-brittle materials based on irreversible thermodynamics. *International Journal of Mechanical Sciences 39*, 4 (1997), 473–486.

[34] NGUYEN, G., AND KORSUNSKY, A. Development of an approach to constitutive modelling of concrete: Isotropic damage coupled with plasticity. *International Journal of Solids and Structures 45* (2008), 5483–5501.

[35] OLIVEIRA, D., PENNA, S., AND PITANGUEIRA, R. Elastoplastic constitutive modeling for concrete: a theoretical and computatioal approach. *IBRACON Structures and Materials Journal 13*, 1 (2020), 171–182.

[36] ORTIZ, M. A constitutive theory for the ielastic behavior of concrete. *Mechanics of Materials 4*, 1 (1985), 67–93.

[37] OTTOSEN, N. A failure criterion for concrete. *Journal of the Engineering Mechanics, ASCE 103*, EM4 (August 1977), 527–535.

[38] SCHREYER, H. L. Modelling surface orientation and stress at failure of concrete and geological materials. *International Journal for Numerical and Analytical Methods in Geomechanics 31*, 2 (2007), 147–171.

[39] SIMO, J., AND JU, J. Strain- and stress based continuum damage models-i. formulation. *International Journal of Solids and Structures 23*, 7 (1987), 821–840.

[40] VALENTINI, B., AND HOFSTETTER, G. Review and enhancement of 3d concrete models for large-scale numerical simulations of concrete structures. *International Journal for Numerical and Analytical Methods in Geomechanics 37*, 3 (2013), 221–246.

[41] VAN MIER, J. Fracture of concrete under complex stress. *Heron 31*, 3 (1986), 1–90.

[42] VILPPO, J., KOUHIA, R., HARTIKAINEN, J., KOLARI, K., FEDOROFF, A., AND CALONIUS, K. Anisotropic damage model for con-

crete and other quasi-brittle materials. *International Journal of Solids and Structures 225* (2021), 111048.

[43] VOYIADJIS, G., TAQIEDDIN, Z., AND KATTAN, P. Anisotropic damage-plasticity model for concrete. *International Journal of Plasticity 24* (2008), 1946–1965.

[44] WILLAM, K., AND WARNKE, E. Constitutive model for the triaxial behaviour of concrete. In *IABSE Proceedings* (1975), vol. 19, pp. 1–30. Seminar on Concrete Structures Subjected to Triaxial Stresses, Bergamo, Italy May 17-19, 1974.

[45] WULFINGHOFF, S., FASSIN, M., AND REESE, S. A damage growth criterion for anisotropic damage models motivated from micromechanics. *International Journal of Solids and Structures 121* (2017), 21–32.

[46] YAZDANI, S., AND SCHREYER, H. An anisotropic damage model with dilatation for concrete. *Mechanics of Materials 7*, 3 (1988), 231–244.

[47] YAZDANI, S., AND SCHREYER, H. Combined plasticity and damage mechanics model for plain concrete. *Journal of the Engineering Mechanics, ASCE 116* (1990), 1435–1450.

[48] ZHANG, X., WU, H., LI, J., PI, A., AND HUANG, F. A constitutive model of concrete based on Ottosen yield criterion. *International Journal of Solids and Structures 193-194* (2020), 79 – 89.

[49] ZHU, Q., KONDO, D., AND SHAO, J. Micromechanical analysis of coupling between anisotropic damage and friction in quasi brittle materials: Role of the homogenization scheme. *International Journal of Solids and Structures 45*, 5 (2008), 1385–1405.