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Aspects on the optimal decoupling point setting problem

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ABSTRACT

Demand-Driven Material Requirement Planning (DDMRP) represents a combination of traditional Material Requirements Planning (MRP) and the reorder point method. A key consideration in DDMRP revolves around determining the optimal position of decoupling points, also referred to as strategic inventory positions. This article addresses the question of where these decoupling points should be strategically positioned, utilizing a directed universal graph derived from the Bill of Materials (BOM) to formalize the optimal decoupling point setting problem. To address this challenge, analytical formulas are developed. The analytical formulas utilize parameters such as delivery time, demand variance, replenishment time, lot sizes, holding costs, and service levels. These formulas provide insights into key characteristics of optimal decoupling points. The obtained results can be categorized into arguments advocating for decoupling points to be positioned either more upstream or more downstream. Furthermore, we derive specific characteristics that an optimal decoupling point position should possess. This research contributes valuable knowledge for practitioners seeking to enhance the efficiency and effectiveness of their DDMRP implementation.

1. Introduction

The strategic positioning of the decoupling point has been discussed by several authors. Sharman [50] or Olhager [44] defined the order penetration point as the stage in the value chain where a particular product is linked to a specific customer order. Olhager [44] identified two major factors that affect the strategic positioning of the order penetration point: the ratio production to delivery lead time and the coefficient of variation of the demand. Make-to-order strategies (order penetration point at the first production stage) are only possible if the production lead time is shorter than the customer's required delivery lead time, see [7]. If the demand variability is very high, especially with a huge number of product variants, it is not reasonable to use a make-to-stock policy, since this would lead to excessive inventories, see [44,15]. Vollman [55] and Jodlbauer [25] addressed the complexity of the bill of material and the number of variants and their influence on the decoupling point, suggesting that it is advantageous to have a customer order reference before most of the variants arise along the production path. Naim and Gosling among others [42,45,16,35,44,37,43] differentiate between pre-order penetration point operations (upstream, forecast-driven) and post-order penetration point operations (downstream,

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customer-order-driven), suggesting consumption-based approaches (e.g., KANBAN), lean methods, and focusing on cost efficiency as well as productivity for pre-order penetration point operations and agile approaches, time-phased approaches, e.g., material required planning (MRP), and focusing on delivery speed, market responsiveness as well as flexibility for post-order penetration point operations. According to [15,25], most product variants should be built in post-order penetration point operations. Rudberg [49] uses the degree of demand certainty for the definition of the customer order decoupling point: a customer order decoupling point separates decisions made under certainty (called above customer-order-driven) from decisions made under uncertainty (called above forecast-driven) concerning customer demand. The designation *customer order decoupling point* has prevailed over *order penetration point* in the literature, see, e.g., [45]. Sharman [50] and Olhager [45] emphasize that the customer order decoupling point is the last point (nearest to the customer) at which the inventory is held and from where the customer order process starts. Practical experience supports the existence of multiple decoupling points, especially in supply chains, see [13,5]. Sun [51] discuss multiple decoupling points for manufacturing based on the BOM. Chung [8] presents multiple decoupling points in supply chains with three types of postponement strategies: form-based, time-based, and place-based. Harfeldt-Berg and Olhager [18] give a good overview of customer order decoupling point frameworks. Wang et al. [56] discusses customer order decoupling point integration in deteriorating supply chains. Supply chain literature stresses the importance of supply chain integration in the sense of shared information. Donk [54] discusses that different locations of the customer order decoupling point are related to different emphases in supply chain integration: for make-to-order downstream, for assembly-to-order internal, and for make-to-stock upstream. Several authors address mathematical optimization models to set customer order decoupling points. Gupta [15] minimizes the sum of inventory cost and redesign cost, subject to a service level constraint, by using a queuing approach. Their focus is on delayed product differentiation assuming that the product differentiation is always performed in the make to order part of the production system. Sun [51] position multiple decoupling points in a complicated supply network instead of a supply chain. Their model is based on the BOM and determine a so-called partition line dividing the supply network into a make-to-stock and a make-to-order part. Jeong [22] considers the positioning of the decoupling point, the production planning, as well as inventory strategy simultaneously. This model is an optimal control theory model, with the objective function being the total cost of deviation from the target production rate, and the target inventory level. The optimal positioning of the decoupling point, the optimal path of the production rate and the inventory level are the decision variables. Rim and Jiang [48,23,24] discuss optimal inventory positioning (answering where should be how much inventory) for make-to-order production systems and under demand-driven MRP conditions [47]. They extend their basic model, see [48], to a generalized BOM structures, see [23], and stochastic processing times, see [24]. The inventory is modeled according to the heuristics proposed in [46], which aim to minimize total inventory cost while meeting the lead time required by customers, using a genetic algorithm. Liu [36] develop a combined optimization two-echelon problem for setting a customer order decoupling point and rescheduling caused by order insertion. In [28] and [33], an order insertion strategy combined with due date quoting and rescheduling is presented. Most of the above approaches use MRP, with customer orders for upstream of the customer order decoupling point (customer-order-driven) and the reorder point method downstream of the customer order decoupling point (make-to-stock environment). MRP is a planning method based on customer orders or forecasts by exploring the BOM explosion and backward time-phasing (start time equals due date minus planned lead time), see [30,32]. The reorder point method is a consumption-based approach, with the inventory monitored continuously. When the inventory position goes below the predefined reorder point, a replenishment or reproduction order of a predefined batch size is placed, see [21]. The expected delivery date (due date) is determined by forward time-phasing (delivery date equals start time plus planned lead time). Several authors address the determination of the optimal reorder point quantity, Heydari [20] present a two-stage supply chain model focusing on stochastic and long lead times using, ordering cost, holding cost, and excess inventory cost (needed for ensuring the predefined service level) as optimization criteria. A multi-item approach is discussed in [31]. The need to change the perspective in supply chains from supply-driven to demand-driven is postulated in [26]. The concept of demand-driven MRP (DDMRP) developed by Ptak and Smith, see [46,47]. In [34], the transition from MRP to DDMRP is motivated and exemplified. DDMRP is a multi-echelon approach based on MRP and strategic inventory positions. Some items from the BOM have been defined as strategic inventory positions, which must fulfill three main tasks: ensure short enough response times (meet the customer-requested or expected delivery lead time), protect from uncertainty, and control the dispersion of demand variability, see [4]. The items held in the strategic inventory positions are reorder-point planned, see [27], while all other items MRP-planned. Thus, DDMRP can be regarded as a combination of MRP and the customer reorder point concept. DDMRP uses only customer orders for independent demand, no forecasts are applied at the operational level. Strategic inventory positions with the lowest level code along a BOM path function as the customer order decoupling point. Downstream operations to those decoupling points are MRP-planned based on customer orders (MRP for make-to-order). All upstream operations performed to those decoupling points contribute to filling up the inventory, independent of whether MRP (for make to stock) or the reorder point method is used. Important differences from traditional planning concepts in the DDMRP concept include that child items of reorder-point-planned items (strategic inventory position) can be MRP-planned items (MRP for make-to-stock) and that more than one decoupling point (strategic inventory position) along a BOM path is possible. Ptak [46] propose heuristics for strategic inventory positioning (determining which items should be reorder-point-planned) and for dimensioning target inventory levels (determining the reorder point quantity) based on variability factors, planned adjustment factors, lead time factors, average daily usage, and several lead times. Miclo [38,39] compare a particular situation of material requirements planning (MRP) with DDMRP using discrete event simulation. Their model is based on the heuristics and suggestions of Ptak and Smith [46], and they conclude that MRP needs more average inventory than DDMRP to attain the same on-time delivery level. Furthermore, DDMRP is better able to counteract variability, reduce the risk on spike demand, and maintains buffers in less risky zones than MRP. Some of these advantages are gained through buffer management based on the theory of constraints [14] and by anticipating future spike demand with the reorder point method [47]. In [1], the applicability of DDMRP in complex manufacturing systems is analyzed. The terms “decoupling point” and “strategic inventory position” are used synonymously throughout the article. Furthermore, strategic inventory positions for which only MRP-planned items occur on all possible paths to end items are called “customer decoupling points”.

The main contribution of this paper is the development of a model to determine nearly optimal solutions for the combined tasks of decoupling positioning and reorder point dimensioning in a DDMRP environment. The positioning sub problem is computationally demanding, see [22,23]. To tackle the problem of dimensioning, we utilize sub-problem methods for predefined service levels. As a result, we classify BOM structures into known graph classes and find that BOM structures represent hierarchical and directed graphs that have already been studied theoretically by Dehmer and Emmert-Streib [9]. Also, we provide analytical results to better understand the optimal decoupling point setting problem.

2. Methods and results

In this section, we start with providing some theoretical background on the optimal decoupling point setting problem. Afterwards, we present important definitions, including the graph-theoretical classification of BOM structures as directed, hierarchical graphs. In Section 2.3, we prove analytical results and draw conclusions.

2.1. Background and problem description

Commercial Enterprise Resource Planning (ERP) Systems and their production planning models are based on the BOM structure, see [21]. In the master data, the planning method must be defined for each item in the BOM. The BOM represents a specialized hierarchical and directed graph consisting of items (vertices) that share the same low-level code. For a detailed graph-theoretical description of BOM structures, see Section 2.2. The low level code indicates the lowest level in a BOM; we refer to these vertices as end items whose low-level code equals zero, see section 2.2.

The two governing planning methods are the MRP approach and the consumption-based reorder point method, see [25]. MRP performs well when future demand is well-known. If the resulting order lead time is shorter than the customer-required delivery lead time, MRP can achieve a good service level with low inventory. However, for unknown future demand or when the resulting order lead time exceeds the customer-required delivery lead time, MRP results in either poor service levels or high inventory levels, see [29]. In [53], DDMRP is compared to MRP, KANBAN, and DBR. The reorder point method does not require knowledge of near-future demand to ensure a high service level. The required inventory level for the reorder point method increases with higher demand fluctuations and longer supply lead times. In the DDMRP approach, see [47], both planning methods are combined: the reorder point method is applied to strategic inventory positions, while the MRP method is used for all other items. A MRP-planned item requires the following master data: safety stock, lot size, and planned lead time. A reorder point planned item requires the reorder point, lot size, and planned lead time. In the DDMRP concept, strategic inventory positions protect the system against variations and uncertainties. For simplicity, we neglect the spike anticipation concept of DDMRP. Unlike in [47], the inventory position in our reorder model is defined as the stock on hand plus outstanding orders minus backorders (without accounting for future known demand). The safety stock for MRP-planned items is set to zero. Furthermore, for MRP-planned items, the lot-for-lot strategy is applied, meaning the lot size equals the net demand per time bucket. We assume there is no inventory for MRP-planned items because they are only produced on demand (either based on customer orders or the supply requirements of strategic inventory positions) and in lot-for-lot quantities. A comparable assumption is made in [51]. The replenishment of MRP-planned items requires a certain lead time. Furthermore, we assume that the service level of MRP-planned items depends only on the availability of the required child items (quality, capacity, breakdown, and so forth are not considered). Reorder point planned items are stock, cause inventory costs and are immediately available. Depending on the reorder point and the consumption during the replenishment lead time, a particular service level less than one hundred percent is yielded. Like in commercial ERP master data files, the lot sizes for reorder point-planned items and planned lead times for both MRP as well as reorder point-planned items are predefined. For simplicity, variable production cost and capacity constraints are neglected.

2.2. Graph-theoretical definitions and classification of BOM structures

In this section, we present graph-theoretical preliminaries to formally introduce BOM-structures [23]. For the first time, we classify BOM structures into known graph classes [6] to deepen our understanding of these complex graphs. We emphasize that the analysis of complex structures represented as graphs (networks) has been a long-standing problem across various disciplines, see [11,41,52]. Graph structure analysis generally follows two main directions: descriptive graph theory [17] and quantitative graph theory [10]. In this paper, we break new ground by providing a proper graph-theoretical definition of BOM structures.

The BOM structure-, demand pattern (average demand, demand deviation), customer order pattern (customer-required delivery lead time), and target service level for end items are drivers in determining which items should be planned using which method (inventory positioning task) and how these items should be parameterized (reorder point dimensioning task). In this paper, the objective is to determine the strategic positions of decoupling points (reorder-point-planned items) using graphs to ensure customer-required service levels and minimum inventory costs. To start, we provide a graph-theoretical classification of BOM structures and proceed with a definition according to Dehmer and Emmert-Streib [9].

Definition 2.1. Let

$$V := \{v_{0,1}, v_{0,2}, \ldots, v_{0,|V_0|}, v_{1,1}, v_{1,2}, \ldots, v_{1,|V_1|}, v_{2,1}, v_{2,2}, \ldots, v_{2,|V_2|}, \ldots, v_{|\mathcal{L}|,1}, v_{|\mathcal{L}|,2}, \ldots, v_{|\mathcal{L}|,|V_{|\mathcal{L}|}|}\},$$

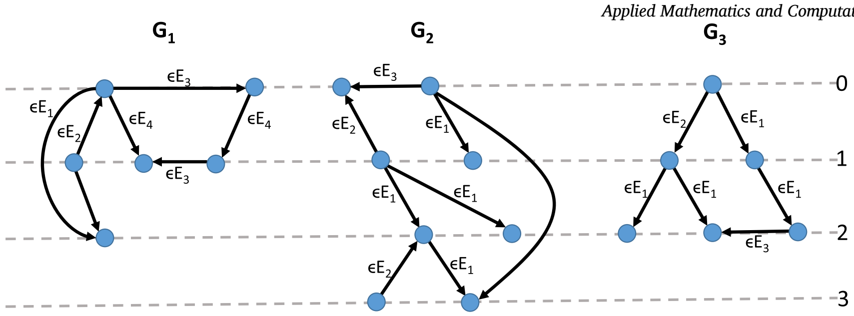

Fig. 1. Three directed universal graphs.

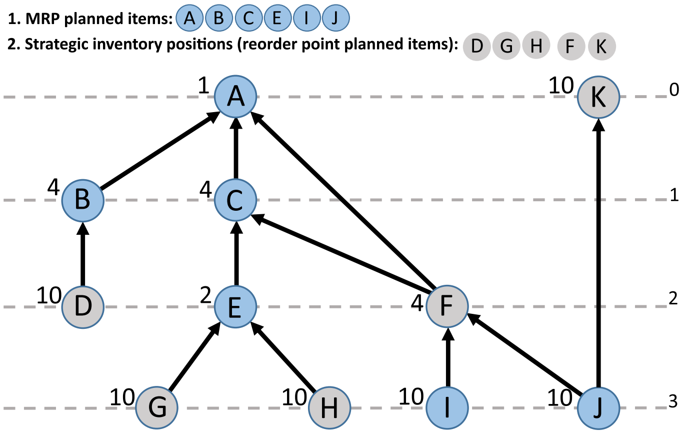

Fig. 2. Example of a BOM-structure representing a special directed universal graph.

$|V| < \infty$ be the vertex set. $L : V \longrightarrow \mathcal{L}$ is a multi-level function [9]. L assigns a level to all vertices $\in V$ which forms the level set $\mathcal{L}$. $v_{i,j}$ denotes the j-th vertex on the i-th level, $0 \leq i \leq |\mathcal{L}|, 1 \leq j \leq |V_i|$. $|V_i|$ denotes the number of vertices on level i. The edge set $E := E_1 \cup E_2 \cup E_3$ is defined as follows [9]:

- (E_1) Down-edges are edges which change at least one level.
- (E_2) Up-edges are edges which change at least one level.
- (E_3) Across-edges do not change levels.

We call $G = (V, E)$ a finite, directed universal graph.

We see that a directed universal graph, according to Definition 2.1, is unlabeled. This definition can be easily extended by defining vertex or edge alphabets and labeling functions [17]. In this paper, we consider BOM structures that are vertex-labeled with these labels represented as natural numbers.

Fig. 1 shows three unlabeled *directed universal graphs*. The edge types of these three example graphs are classified according to Definition 2.1. Note that G_3, shown in 1, represents the degenerate case of a directed universal graphs, as it has only a single root vertex; such graphs are referred to as *generalized trees* [12]. However, this case won't occur when dealing with BOM structures in practice [23].

By considering Fig. 2, we observe that a vertex-labeled BOM structure represents a special directed universal graph. More precisely, the BOM does possess any Down and Across-edges. We see that for all BOM-edges $e \in E$, it holds that $e \in E_2 = E$. Hence, E_1 and E_3 are empty sets. We emphasize that we use the term *item* instead of *vertex* to further analyze BOM structures. Moreover, we adopt the notation and terminology from Jiang and Rim [23] and consider general BOM structures, where items can have multiple immediate parents [23].

It is evident that we can classify the three important classes of BOM-items (vertices)– namely, end items, intermediate items, and purchased items– using simple graph-theoretical measures [17] as shown in Fig. 2. Let $\delta_{\text{out}}(v_i)$ denote the out-degree and $\delta_{\text{in}}(v_i)$ denotes the in-degree of $v_i \in V$, respectively. We call v_i an end item if $\delta_{\text{out}}(v_i) = 0$. Purchased items are vertices $v_i \in V$ whose in-degrees vanish, i.e., $\delta_{\text{in}}(v_i) = 0$. In order to characterize intermediate items in the same way, we yield $\delta_{\text{out}}(v_i) \geq 1$ and $\delta_{\text{in}}(v_i) \geq 1$.

Now, let us we explain Fig. 2 in more detail. The depicted BOM-structure consists of two end items (zero low-level code), A and K. The intermediate items are B, C, E, and F, whereas the purchased items are given by the vertices D, G, H, I, and J. The shown vertex label is the planned lead time of an item. For instance, it takes 4 days to replenish item B. The customer-requested lead time for both finished goods A and K is 8 days. The grey items are strategic inventory positions regarding reorder point planned items. Item K must be reorder point planned as its planned lead time is longer than the customer required delivery lead time. Item A can be MRP-planned because its planned lead time is shorter than the customer requested lead time. The purchased items D, G, and H must be reorder point planned because their resulting production lead time is longer than the customer-requested lead time. Additionally, their parent vertices A, B, C, and E have shorter resulting lead times than the customer-requested delivery lead time. Item F must be reorder point planned because the path (F, A) has a resulting production lead time that exceeds the customer-requested delivery lead time.

We will now continue with further definitions to analyze BOM-structures.

Definition 2.2 *(BOM-path length)*. The BOM-path length $pl(A, B)$ between two items A and B is defined as the maximum possible sum of planned lead times along a directed path from item A downstream to item B. The planned lead time of B is included in the sum, whereas the planned lead time of A is not. B has to be a parent, grand-parent, or higher order parent of A, and both must be connected by at least one path. By definition $pl(A, A)$ equals the planned lead time of A, if A and B are unconnected $pl(A, B) = \infty$.

Generally, there exists more than one path between two items, e.g., the items F and A in Fig. 2. According to Fig. 2, we yield $pl(F, A) = \max(1 + 4, 1) = 5$. The BOM-path length always refers to the maximum possible length. For a BOM-sequence of MRP-planned items, the resulting production lead time equals the sum of the planned lead times of the items, see Ptak and Smith [46]. For a path from B to A, the production lead time can be determined by:

$$\begin{aligned}
&\ell_{B,A} = \textstyle\sum_{i=1}^{n} \ell_{p(i)} \\
&\text{The } \ell_{B,A} \text{ production lead time for replinishing item A and all intermediate items } p(2), \dots, p(n) = B, \\
&\ell_{p(i)} \text{is the planned lead time of item } p(i), \\
&p(n) = B, \\
&p(1) = A, \\
&p(i+1) \text{ is a sub} - \text{item of } p(i) \text{ for } i = 1 \text{ to } n-1, \\
&p(i) \text{ is MRP } - \text{ planned for } i = 1 \text{ to } n.
\end{aligned} \tag{1}$$

Definition 2.3 *(Production lead time)*. The production lead time $\ell_{A,production}$ of a MRP-planned item A is the maximum possible value $pl(B, A) + \ell_B$, along all paths from B to A, with exclusively MRP-planned item:

$$\ell_{A,production} = \max\left(pl(B, A) + \ell_B\right) \qquad \begin{aligned} &\text{if } B \text{ is connected to A,} \\ &\text{all items on the path from B to} \\ &\text{A are MRP-planned.} \end{aligned} \tag{2}$$

The production lead time denotes how long does it take to replenish an MRP-planned item along the MRP-planned path If A is an MRP-planned end item (finished good), the production lead time should be less than the customer required delivery lead time to ensure on-time delivery and a high service level for a MRP-planned sequence in a DDMRP environment.

$$\ell_{A,production} \leq \ell_{A,customer} \qquad \begin{aligned} &\text{where } \ell_{A,customer} \text{ customer required} \\ &\text{delivery lead time for end item } A \end{aligned} \tag{3}$$

A strategic inventory position is useful for items with a longer resulting production lead time than the customer required delivery lead time, see [7]. The strategic inventory position is managed by the reorder point method in a DDMRP environment, see [47]. This leads to the next definitions.

Definition 2.4 *(Customer distance)*. The customer distance $cd(A)$ of an item A is the maximum possible BOM-path length from item A to a connected end item. The customer distance of an end item equals its planned lead time.

In general, there is more than one connected end item, see item J with two connected end items A and K in Fig. 2. According to Fig. 2, we get $cd(J) = \max(pl(J, A), pl(J, K)) = \max(9, 10) = 10$

Definition 2.5 *(Customer buffer)*. The customer buffer $cb(A)$ of an item A is the minimum possible value *customer required delivery lead time of the connected end item minus the BOM-path length from item A to the connected end item*. The customer buffer of an end item is equals to its customer required delivery lead time minus its planned lead time.

The customer buffer can be equivalently defined by the minimum possible value *customer required delivery lead time minus the customer distance*.

Definition 2.6 *(Customer decoupling point)*. An item A is called the customer decoupling point if, and only if, item A is reorder planned and on all possible paths from item A to an end item, all items on the path are MRP-planned.

It follows directly from the definitions that the customer buffer of a customer decoupling point must be non-negative. We suggest,

$$cb(A) \geq 0 \text{ is a customer decoupling point.} \tag{4}$$

A negative customer buffer of a customer decoupling point means that there is a MRP-planned path to an end item with a longer BOM-path length than the customer required delivery lead time. An end item A should be reorder point planned if the planned lead time of the end item ℓ_A is longer than the customer required delivery lead time $\ell_{A,customer}$ to ensure on-time delivery. Sub item B should be reorder point planned (or downstream items) if the customer buffer is negative to ensure on-time delivery. This corresponds to the restriction formulated in the optimization model by [51] and describes the actively synchronized replenishment lead time for an end item A according [47,23]. MRP-planned items, when combined with lot-for-lot planning and zero safety stock, result in no inventory. The longer the sequence of MRP-planned items, the lower the total inventory across all items on the MRP-planned BOM path. Consequently, customer decoupling points tend to have the greatest possible customer distance and the smallest possible customer buffer. This leads us to the next definition.

Definition 2.7 *(Minimum customer buffer characteristic).* An item A fulfills the minimum customer buffer characteristic if and only if

$$cb(A) \geq 0 \text{ and } cb(A) - \ell_A < 0. \tag{5}$$

Next, we discuss an extension of the model of Sun et al. (2008), intermediates and purchased items which are subassemblies of reorder point planned items. Intermediate parts which are reorder point planned have, in general, MRP-planned sub parts as well as MRP-planned parent parts. For further discussion, we need the next definition.

Definition 2.8 *(Efficiency decoupling point).* An item A is called an efficiency decoupling point if, and only if, item A is reorder planned and not a customer decoupling point.

Definition 2.9 *(Replenishment lead time).* The replenishment lead time $\ell_{A,\text{replenishment}}$ of a reorder point planned item A is defined by the longest production lead time of all direct child MRP-planned sub items plus its own planned lead time. That means,

$$\ell_{A,\text{replenishment}} = \ell_A + \max \ell_{B,\text{production}} \text{ where } B \text{ is the direct child item of } A. \tag{6}$$

The reorder point must cover the maximum expected demand during the replenishment lead time to ensure an acceptable service rate. The longer the replenishment lead time, the longer the necessary reorder point and the higher the average inventory of the reorder planned item. The BOM-paths of MRP-planned parent parts of a reorder point planned item A combined with the external demand of an end item or the internal demand of a reorder planned parent item determine the demand of item A. There is a BOM-transition vector V_A with

$$\begin{aligned} x_A(k) = \;& V_A^t X_A(k) \text{ where} \\ x_A(k) \;& \text{is the demand of reorder point planned item A with respect to time } k. \\ V_A \;& \text{is the BOM transition column vector.} \\ X_A \;& (k) \text{is the demand column vector of parent parts (reorder point planned or end items).} \\ ()^t \;& \text{is the transpose operation.} \end{aligned} \tag{7}$$

The two vectors V_A and X_A depend heavily on the BOM-structure. All parent parts of A which are end items or reorder point planned items with the lowest lower-level code along the BOM-path must be taken into account in the construction of the two vectors. Furthermore, the time-phasing of the MRP algorithm must be considered.

For item D of Fig. 2, we get

$$\begin{aligned} \ell x_D(k) &= \alpha_{DB}\alpha_{BA} x_A(k+5), \\ V_D &= \alpha_{DB}\alpha_{BA}, \\ X_D(k) &= x_A(k+5), \end{aligned} \tag{8}$$

where α_{AB} is the BOM coefficient (how many items A are nedeed to produce one unit of item B). For item F, it is a little bit more complex. There are two paths with different planned lead times to the final product A to be considered for item F of Fig. 2.

$$\begin{aligned} x_F(k) &= \alpha_{FC}\alpha_{CA} x_A(k+5) + \alpha_{FA} x_A(k+1), \\ V_D &= \begin{pmatrix} \alpha_{FC}\alpha_{CA} \\ \alpha_{FA} \end{pmatrix}, \end{aligned} \tag{9}$$

$$X_D(k) = \begin{pmatrix} x_A(k+5) \\ x_A(k+1) \end{pmatrix}.$$

Assuming statistical independence of the demand as well as of their time shifted demand of all relevant parent items referenced in the BOM-transition vector, the average demand per time bucket and the demand variance per time bucket can be determined by

$$\mu_A = V_A^t \begin{pmatrix} \mu_{P_1} \\ \vdots \\ \mu_{P_m} \end{pmatrix} \text{where,} \tag{10}$$
μ_A is the average demand of reorder point planned item A,
μ_{P_i} is the average demand of relevant parent item P_i of A,

$$\sigma_A^2 = V_A^{2t} \begin{pmatrix} \sigma_{P_1}^2 \\ \vdots \\ \sigma_{P_m}^2 \end{pmatrix} \text{where,} \tag{11}$$
σ_A^2 is the demand variance of reorder point planned item A,
$\sigma_{P_i}^2$ is the demand variance of relevant parent item P_i of A,

and, finally,

$$V_A^2 = \begin{pmatrix} v_{A_1}^2 \\ \vdots \\ v_{A_m}^2 \end{pmatrix}. \tag{12}$$

There are two important details concerning the Equations (9)-(10). First, the structure of both vectors V_A and X_A has to be organized so that for one particular time, only one vector coefficient is used, otherwise the implicit assumption of statistical independence in the determination of the variance is no longer fulfilled. Second, the consumption of a reorder planned item is not equal to replenishment orders (neither in time nor in quantity) of the reorder planned item. The demand on intermediate child items is triggered by the replenishment orders. Thus, the demand of intermediate sub items is formed by the reorder point policy, especially by the applied lot size. This fact must be considered in the determination of the variance.

According [2], it is well known that the inventory position is uniformly distributed in the interval $[r, r+Q]$. If the inventory position is below the reorder point, a replenishment order is released by lot size equal to a multiple of Q. For the statistical distribution for the replenishment orders, we yield

$$\begin{aligned} &p(x=0) = \tfrac{1}{Q}\textstyle\int_0^Q F(\xi)\,d\xi, \\ &p(x=Q) = \tfrac{1}{Q}\textstyle\int_0^Q F(\xi+Q) - F(\xi)\,d\xi, \\ &\vdots \\ &p(x=iQ) = \tfrac{1}{Q}\textstyle\int_0^Q F(\xi+iQ) - F(\xi+(i-1)Q)\,d\xi, \\ &\text{whereby,} \end{aligned} \tag{13}$$
x is a random variable for replinishment order (upstream demand),
F is a statistical density function of the demand (downstream),
Q is the lot size .

The demand of upstream item is triggered by the replenishment orders. For Equation (10), we need to investigate the expectation value and the variance of replenishment order of reorder point planned items to determine the demand of upstream items. Before starting with these items, we need a Lemma.

Lemma 2.1. *For a statistical distribution with distribution function F and positive probabilities only for nonnegative values* ($F(x) = 0$ for $x < 0$) *the expectations value equals to*

$$\mu = \int_0^\infty 1 - F(x)\,dx. \tag{14}$$

Proof. By using partial integration, we conclude

$$\begin{aligned} \mu &= \int_0^\infty \xi f(\xi)\, d\xi = \lim_{x\to\infty} xF(x) - 0F(0) - \int_0^x F(\xi)\, d\xi \\ &\underset{\lim_{x\to\infty} F(x)=1}{=} \lim_{x\to\infty} \int_0^x 1 - F(\xi)\, d\xi = \int_0^\infty 1 - F(\xi)\, d\xi. \quad \square \end{aligned} \tag{15}$$

2.3. Analytical results

In this section, we prove analytical results in the context of the optimal decoupling point setting problem. We start by stating the following proposition.

Proposition 2.1. *For a reorder planned item with reorder point r, lot size Q, expectation value of the demand μ and demand variance σ^2 the expectation value of replenishment orders are equal to the expectation value of the demand*

$$\mu_x = \mu, \tag{16}$$

and the variance of the replenishment order is greater equal to the demand variance

$$\sigma_x^2 \geq \sigma^2. \tag{17}$$

Furthermore, the variance of the replenishment orders is an increasing function with respect to the lot-size Q

$$\sigma_x^2(Q_1) > \sigma_x^2(Q_2) \text{ if } Q_1 > Q_2. \tag{18}$$

For very small lot-sizes the variance of the replenishment orders converges to the variance of the demand,

$$\lim_{Q\to 0} \sigma_x^2(Q) = \sigma^2. \tag{19}$$

For high lot-sizes $Q > \mu + \sigma$ the variance is approximately equal to

$$\begin{aligned} &\sigma_x^2 \approx Q^2 - Q\int_0^Q F(\xi)\, d\xi - \mu^2, \\ &\sigma_x^2 \approx \mu(Q-\mu), \\ &v_x = \frac{\sigma_x}{\mu_x} \approx \sqrt{\frac{Q-\mu}{\mu}}. \end{aligned} \tag{20}$$

Proof. We start with the exploring expectation value

$$\begin{aligned} \mu_x &= \lim_{n\to\infty} \left(\sum_{i=1}^n \left(\frac{1}{Q} \int_0^Q F(\xi + iQ) - F(\xi + (i-1)Q)\, d\xi \right) Qi \right), \\ &= \lim_{n\to\infty} \left(\int_0^Q nF(\xi + nQ) - \sum_{i=1}^n F(\xi + (i-1)Q)\, d\xi \right), \\ &= \lim_{n\to\infty} \left(nQ - \int_0^{nQ} F(\xi)\, d\xi \right), \\ &= \int_0^\infty 1 - F(\xi)\, d\xi \underset{\text{Leema 2.1}}{=} \mu. \end{aligned} \tag{21}$$

The monotony of the variance can be proven by showing that the first derivative of the variance is strictly greater than zero. That is,

$$\begin{aligned} \sigma_x^2(Q) &= \lim_{n\to\infty} \left(\sum_{i=1}^n \left(\frac{1}{Q} \int_0^Q F(\xi + iQ) - F(\xi + (i-1)Q)\, d\xi \right) (Qi)^2 \right) - \mu_x^2, \\ &= Q \lim_{n\to\infty} \left(\sum_{i=1}^n \int_{iQ}^{(i+1)Q} F(\xi)\, d\xi - \int_{(i-1)Q}^{iQ} F(\xi)\, d\xi \right) - \mu_x^2, \\ &= Q\Psi - \mu_x^2, \end{aligned}$$

as it holds

$$\Psi(Q) = \lim_{n\to\infty} \left(\sum_{i=1}^n \int_{iQ}^{(i+1)Q} F(\xi)\, d\xi - \int_{(i-1)Q}^{iQ} F(\xi)\, d\xi \right) = \frac{\sigma_x^2 + \mu_x^2}{Q} > 0.$$

We also examine

$$\frac{\partial \sigma_x^2(Q)}{\partial Q} = \Psi(Q) + Q\frac{\partial \Psi(Q)}{\partial Q}, \tag{22}$$

and find that

$$
\begin{aligned}
&\tfrac{\partial\Psi(Q)}{\partial Q} = \lim_{n\to\infty}\left(\sum_{i=1}^{n}(i+1)F((i+1)Q) - iF(iQ) - iF(iQ) + (i-1)F((i-1)Q)\right)\\
&= \lim_{n\to\infty}\left(F(0) - F(Q) - nF(nQ) + (n+1)F((n+1)Q)\right)\\
&= F(0) - F(Q) + 1 \geq 0.
\end{aligned}
$$

Hence,

$$
\frac{\partial\sigma_x^2(Q)}{\partial Q} = \Psi(Q) + Q\frac{\partial\Psi(Q)}{\partial Q} > 0, \tag{23}
$$

and we conclude that

$$
\sigma_x^2(Q) \quad \text{is strictly monotone increasing.} \tag{24}
$$

For the variance we yield:

$$
\begin{aligned}
&\sigma_x^2(Q) = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\tfrac{1}{Q}\int_0^Q F(\xi+iQ) - F(\xi+(i-1)Q)\,d\xi\right)(Qi)^2\right) - \mu_x^2\\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\int_{(i-1)Q}^{iQ}\tfrac{F(\xi+Q)-F(\xi)}{Q}(Qi)^2\,d\xi\right)\right) - \mu_x^2\\
&\\
&\text{Finally, } \lim_{Q\to 0}\sigma_x^2(Q) = \int_0^\infty f(\xi)\xi^2 d\xi - \mu_x^2 = \sigma^2
\end{aligned} \tag{25}
$$

By using the Equations (23) - (24), we finally infer $\sigma_x^2(Q) \geq \sigma^2$ for all Q. For high lot sizes, we deduce

$$
\begin{aligned}
&\sigma_x^2(Q) = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\tfrac{1}{Q}\int_0^Q F(\xi+iQ) - F(\xi+(i-1)Q)\,d\xi\right)(Qi)^2\right) - \mu_x^2,\\
&\underset{Q>>\mu+\sigma}{\approx}\int_0^Q \tfrac{F(\xi+Q)-F(\xi)}{Q}Q^2 d\xi + \int_0^Q \tfrac{F(\xi+2Q)-F(\xi+Q)}{Q}(2Q)^2\,d\xi - \mu^2,\\
&= 4Q\int_0^Q F(\xi+2Q)\,d\xi - 3Q\int_0^Q F(\xi+Q)\,d\xi - Q\int_0^Q F(\xi)\,d\xi - \mu^2,\\
&\underset{Q>>\mu+\sigma}{\approx} Q\left(4Q - 3Q - \int_0^Q F(\xi)\,d\xi\right) - \mu^2 = Q^2 - Q\int_0^Q F(\xi)\,d\xi - \mu^2,\\
&= Q\left(Q - \int_0^Q F(\xi)\,d\xi\right) - \mu^2 \underset{Lemma\ 2.1}{\approx} Q\mu - \mu^2 = \mu(Q-\mu).
\end{aligned} \tag{26}
$$

Finally,

$$
v_x = \frac{\sigma_x}{\mu_x} = \frac{\sqrt{\mu(Q-\mu)}}{\mu} = \sqrt{\frac{Q-\mu}{\mu}}. \quad \square \tag{27}
$$

The demand coefficient of variation at a MRP-planned item with lot for lot as lot-sizing policy inherits the variation to the replenishment orders. For the demand $x(1), x(2), \ldots, x(T)$, the caused replenishment orders are $x(1+\ell_p), x(2+\ell_p), \ldots, x(T+\ell_p)$. Thus the statistical distribution especially the expectation value, variance and coefficient of variations of the demand and replenishment orders are the same.

The reorder point should be close to the maximum possible demand during the replenishment lead time $\ell_{A,replenishment}$ ensuring a high service level. For an end item A, a predefined customer required service level γ_A is given. Unlike Heydari [20], the service level for intermediates are not predefined, rather they are determined by ensuring the customer requested service levels. If a parent item has one exact child item, the required service level of the parent item inherits to the child item. The discussion of the service level for multiple child items is more sophisticated; a higher service level for multiple children items must be requested to ensure a particular service level of the parent item. Assuming statistically independence, the following equation holds true:

$$
\begin{aligned}
\gamma_p &= \gamma_1\gamma_2\cdots\gamma_n\\
\gamma_p &\in\,]0,1[\\
\gamma_i &\in\,]0,1[\text{ for } i = 1,\ldots,n \text{ where,}\\
&\gamma_p \text{ is the service level of the parent item,}\\
&\gamma_i \text{ is the service level of the i} - \text{th child item } (i = 1,\ldots,n),\\
&n \text{ is the number of children items.}
\end{aligned} \tag{28}
$$

The required service level remains if there is one child item. For multiple children items, the required service level of each child has to be greater than the required service level of the parent item, thus $\gamma_p < \gamma_i$ for all $i = 1,\ldots,n$. The required service levels of end items are predefined. An optimal propagation of the required service level upstream of the BOM-structure should take the cost structure into account. Thus, a child item with low inventory cost can be assigned by a higher required service level than child items with a high inventory cost to minimize the total inventory cost. The incurred cost by the target service level is equal to [40]:

$$\begin{aligned}&\sum_{i=1}^{n} c_i F^{-1}_{N(0,1)}\left(\gamma_i\right)\sqrt{\ell_{i,\text{replenishment}}}\sigma_i \underbrace{=}_{\sigma_i=\alpha_{ip}\sigma_p} \sigma_p \sum_{i=1}^{n} c_i F^{-1}_{N(0,1)}\left(\gamma_i\right)\alpha_{ip}\sqrt{\ell_{i,\text{replenishment}}} \text{ where}\\ &\gamma_p \text{ is the service level of the parent item,}\\ &\gamma_i \text{ is the service level of the i}-\text{th child item } (i=1,\dots,n),\\ &n \text{ is the number of children items,}\\ &c_i \text{ is the holdup cost.}\end{aligned} \tag{29}$$

For multiple children, an optimization problem can be formulated to distribute the optimal required service level cost:

$$\begin{aligned}&\min_{\gamma_1,\gamma_2,\dots,\gamma_n} \sum_{i=1}^{n} c_i F^{-1}_{N(0,1)}\left(\gamma_i\right)\alpha_{ip}\sqrt{\ell_{i,\text{replenishment}}}\\ &\text{subject to}\\ &\gamma_p=\gamma_1\gamma_2\dots\gamma_n.\end{aligned}$$

By using the substitution,

$$\begin{aligned}&g_i=c_i\alpha_{ip}\sqrt{\ell_{i,\text{replenishment}}}\\ &u_i=F^{-1}_{N(0,1)}\left(\gamma_i\right),\end{aligned}$$

the optimization problem yields to

$$\begin{aligned}&\min_{u_1,u_2,\dots,u_n} \sum_{i=1}^{n} g_i u_i\\ &\text{subject to}\\ &\gamma_p=F_{N(0,1)}(u_1)F_{N(0,1)}(u_2)\dots F_{N(0,1)}(u_n).\end{aligned} \tag{30}$$

We infer by utilizing Lagrangian relaxation,

$$\begin{aligned}&L(u_1,u_2,\dots,u_n,\lambda)=\sum_{i=1}^{n} g_i u_i + \lambda\left(\gamma_p-F_{N(0,1)}(u_1)F_{N(0,1)}(u_2)\dots F_{N(0,1)}(u_n)\right)\\ &\frac{\partial}{\partial u_i}L(u_1,u_2,\dots,u_n,\lambda)=g_i-\lambda\left(F_{N(0,1)}(u_1)\dots F_{N(0,1)}(u_{i-1})F_{N(0,1)}(u_{i+1})\dots F_{N(0,1)}(u_n)\right)f_{N(0,1)}\left(u_i\right)=0,\\ &\frac{g_i F_{N(0,1)}(u_i)}{f_{N(0,1)}(u_i)}=\lambda\gamma_p \text{ for } i=1,\dots,n.\end{aligned} \tag{31}$$

Also, we obtain $\frac{\partial}{\partial\lambda}L(u_1,u_2,\dots,u_n,\lambda)=\gamma_p-F_{N(0,1)}(u_1)F_{N(0,1)}(u_2)\dots F_{N(0,1)}(u_n)=0,$

$$\gamma_p=F_{N(0,1)}(u_1)F_{N(0,1)}(u_2)\dots F_{N(0,1)}(u_n).$$

Finally, the $n+1$ equations

$$\begin{aligned}&\frac{g_i F_{N(0,1)}(u_i)}{f_{N(0,1)}(u_i)}=\lambda\gamma_p \text{ for } i=1,\dots,n\\ &\gamma_p=F_{N(0,1)}(u_1)F_{N(0,1)}(u_2)\dots F_{N(0,1)}(u_n)\end{aligned}$$

have to be simultaneously solved for the $n+1$ variables $u_1,u_2,\dots,u_n,\lambda$. The required service levels of the children items are determined by the substitution

$$\gamma_i=F_{N(0,1)}\left(u_i\right).$$

Next, we model the inventory cost caused by the reorder point planned item A. For a service level of γ_A percent, the reorder point and average total inventory are determined by (see [40]):

$$\begin{aligned}
w_A = &\ \ell_{A,\text{replenishment}}\mu_A + F^{-1}_{N(0,1)}\left(\gamma_A\right)\sqrt{\ell_{A,\text{replenisment}}}\sigma_A \\
y_A = &\ c_A\left(w_A + \frac{Q_A}{2}\right) \text{ where,} \\
w_A &\ \text{is the reorder point.} \\
y_A &\ \text{is the average invenory cost.} \\
F^{-1}_{N(0,1)} &\ \text{is the quantile (inverse) of the standard normal distribution.} \\
\gamma_A &\ \text{is the required service level for item A.} \\
Q_A &\ \text{is the lot} - \text{size of item } A. \\
c_A &\ \text{is the unit holding cost for item A.}
\end{aligned} \tag{32}$$

The replenishment time increases if the reorder point planned stage is moved downstream (nearer to the customer). For the sake of completeness, we state that for a MRP-planned item, the inventory equals zero:

$$y_A = 0 \qquad \text{for MRP} - \text{planned items.} \tag{33}$$

The propagation of the demand coefficient of variation through the BOM-graph is essential for the performance of the production system. For exactly one parent item, the demand of the item is defined by the replenishment orders of the parent item, thus the demand of the item has the same statistical distribution as the replenishment orders of the parent item, especially if they have the same expectation value, variance and coefficient of variation. The next propositions discuss the situation with multiple parent items.

Proposition 2.2. *Item p has $n \geq 2$ parent items denoted by $p_1, \dots, p_n$ with expectation values $\mu_1, \mu_2, \dots, \mu_n > 0$, variances $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2 > 0$, coefficients of variations $v_1 = v_2 = \dots = v_n = v > 0$ and BOM-coefficients $\alpha_1, \alpha_2, \dots, \alpha_n > 0$. The demand coefficient of variation v_p of item p is strictly less than the coefficient of variation of the parent items replenishment orders v,*

$$v_p = Dv < v. \tag{34}$$

The dampening factor D is equal to

$$D = \frac{\sqrt{1 + \sum_{i=2}^{n}\left(\frac{\alpha_i\mu_i}{\alpha_1\mu_1}\right)^2}}{\left(1 + \sum_{i=2}^{n}\frac{\alpha_i\mu_i}{\alpha_1\mu_1}\right)} < 1. \tag{35}$$

Proof. We obtain

$$\begin{aligned}
v_p &= \frac{\sqrt{\sum_{i=1}^{n}\alpha_i^2\sigma_i^2}}{\sum_{i=1}^{n}\alpha_i\mu_i} = \frac{\alpha_1\sigma_1\sqrt{\sum_{i=1}^{n}\frac{\alpha_i^2\sigma_i^2}{\alpha_1^2\sigma_1^2}}}{\alpha_1\mu_1\sum_{i=1}^{n}\frac{\alpha_i\mu_i}{\alpha_1\mu_1}} = v\frac{\sqrt{1+\sum_{i=2}^{n}\frac{\alpha_i^2\sigma_i^2}{\alpha_1^2\sigma_1^2}}}{1+\sum_{i=2}^{n}\frac{\alpha_i\mu_i}{\alpha_1\mu_1}} = \\
&\underset{\substack{\frac{\sigma_1}{\mu_1} = \frac{\sigma_i}{\mu_i} \\ \frac{\mu_i}{\mu_1} = \frac{\sigma_i}{\sigma_1}}}{=} \underbrace{\frac{\sqrt{1^2 + \sum_{i=2}^{n}\left(\frac{\alpha_i\mu_i}{\alpha_1\mu_1}\right)^2}}{1 + \sum_{i=2}^{n}\frac{\alpha_i\mu_i}{\alpha_1\mu_1}}}_{<1} v < v.
\end{aligned}$$

For equal BOM-coefficients $\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha > 0$, the dampening factor simplifies to

$$D = \frac{1}{\sqrt{n}}. \text{ Thus } v_p = \frac{v}{\sqrt{n}} < v. \quad \square \tag{36}$$

Proposition 2.3. *Item p has $n \geq 2$ parent items $p_1, \dots, p_n$ with expectation values denoted by $\mu_1, \mu_2, \dots, \mu_n > 0$, variances $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2 > 0$, coefficients of variations $0 < v_1 \leq v_2 \leq \dots \leq v_n$ and BOM-coefficients $\alpha_1, \alpha_2, \dots, \alpha_n > 0$. The demand coefficient of variation v_p of item p is strictly less than the coefficient of the variation of the parent item p_n replenishment order v_n,*

$$v_p = D_n v_n < v_n. \tag{37}$$

The dampening factor D_n equals to

$$D_n = \frac{\sqrt{1 + \sum_{i=1}^{n-1}\left(\frac{\alpha_i\mu_i}{\alpha_n\mu_n}\right)^2}}{1 + \sum_{i=1}^{n-1}\frac{\alpha_i\mu_i}{\alpha_n\mu_n}} < 1. \tag{38}$$

Proof. By using the definition, we yield

$$v_p = \frac{\sqrt{\sum_{i=1}^{n} \alpha_i^2 \sigma_i^2}}{\sum_{i=1}^{n} \alpha_i \mu_i} = \frac{\alpha_n \sigma_n \sqrt{\sum_{i=1}^{n} \frac{\alpha_i^2 \sigma_i^2}{\alpha_n^2 \sigma_n^2}}}{\alpha_n \mu_n \sum_{i=1}^{n} \frac{\alpha_i \mu_i}{\alpha_n \mu_n}} = v_n \frac{\sqrt{1+\sum_{i=1}^{n-1} \frac{\alpha_i^2 \sigma_i^2}{\alpha_n^2 \sigma_n^2}}}{1+\sum_{i=1}^{n-1} \frac{\alpha_i \mu_i}{\alpha_n \mu_n}}$$

$$\underset{\substack{\frac{\sigma_n}{\mu_n} \geq \frac{\sigma_i}{\mu_i} \\ \frac{\mu_i}{\mu_n} \geq \frac{\sigma_i}{\sigma_n}}}{\leq} \underbrace{\frac{\sqrt{1^2 + \sum_{i=1}^{n-1} \left(\frac{\alpha_i \mu_i}{\alpha_n \mu_n} \right)^2}}{1 + \sum_{i=1}^{n-1} \frac{\alpha_i \mu_i}{\alpha_n \mu_n}}}_{=D_n<1} v_n < v_n. \quad \square$$

Proposition 2.4. *Item p has $n \geq 2$ parent items $p_1, \dots, p_n$ with expectation values denoted by $\mu_1, \mu_2, \dots, \mu_n > 0$, variances $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2 > 0$, coefficients of variations $0 < v_1 \leq v_2 \leq \dots \leq v_n$ and BOM-coefficients $\alpha_1, \alpha_2, \dots, \alpha_n > 0$. The demand coefficient of variation v_p of item p is strictly less than the coefficient of variation of the parent item p_k replenishment order v_k $(1 \leq k < n)$,*

$$v_p = D_k v_k < v_k, \tag{39}$$

if

$$v_n - v_k < S_k v_k, \tag{40}$$

is satisfied. The dampening factor D_k yields to

$$D_k = \frac{\sqrt{\sum_{i=1}^{n} \alpha_i^2 \mu_i^2 + \left(\frac{v_n}{v_k} - 1 \right) \sum_{i=k+1}^{n} \alpha_i^2 \mu_i^2}}{\sum_{i=1}^{n} \alpha_i \mu_i} < 1. \tag{41}$$

The distance factor S_k equals to

$$S_k = \frac{\left(\sum_{i=1}^{n} \alpha_i \mu_i \right)^2 - \sum_{i=1}^{n} \alpha_i^2 \mu_i^2}{\sum_{i=k+1}^{n} \alpha_i^2 \mu_i^2} > 1. \tag{42}$$

Proof. We conclude,

$$v_p = \frac{\sqrt{\sum_{i=1}^{n} \alpha_i^2 \sigma_i^2}}{\sum_{i=1}^{n} \alpha_i \mu_i} \underset{1 \leq k < n}{=} v_k v_n \frac{\sqrt{\sum_{i=1}^{n} \frac{\alpha_i^2 \sigma_i^2}{\alpha_k^2 \sigma_k^2 \alpha_n^2 \sigma_n^2}}}{\sum_{i=1}^{n} \frac{\alpha_i \mu_i}{\alpha_k \mu_k \alpha_n \mu_n}}$$

$$\underset{\substack{\frac{\sigma_i}{\sigma_k} \leq \frac{\mu_i}{\mu_k} \text{ for } i \leq k \\ \frac{\sigma_i}{\sigma n} \leq \frac{\mu_i}{\mu_n} \text{ for } i \leq n}}{\leq} v_k v_n \frac{\sqrt{\frac{1}{\alpha_k^2 \mu_k^2 \alpha_n^2 \sigma_n^2} \sum_{i=1}^{k} \alpha_i^2 \mu_i^2 + \frac{1}{\alpha_k^2 \sigma_k^2 \alpha_n^2 \mu_n^2} \sum_{i=k+1}^{n} \alpha_i^2 \mu_i^2}}{\frac{1}{\alpha_k \mu_k \alpha_n \mu_n} \sum_{i=1}^{k} \alpha_i \mu_i + \frac{1}{\alpha_k \mu_k \alpha_n \mu_n} \sum_{i=k+1}^{n} \alpha_i \mu_i}$$

$$= v_k v_n \frac{\mu_k \mu_n}{\mu_k \sigma_n} \frac{\sqrt{\sum_{i=1}^{k} \alpha_i^2 \mu_i^2 + \frac{\mu_k^2 \sigma_n^2}{\sigma_k^2 \mu_n^2} \sum_{i=k+1}^{n} \alpha_i^2 \mu_i^2}}{\sum_{i=1}^{k} \alpha_i \mu_i + \sum_{i=k+1}^{n} \alpha_i \mu_i}$$

$$= v_k \underbrace{\frac{\sqrt{\sum_{i=1}^{n} \alpha_i^2 \mu_i^2 + \left(\frac{v_n}{v_k} - 1 \right) \sum_{i=k+1}^{n} \alpha_i^2 \mu_i^2}}{\sum_{i=1}^{n} \alpha_i \mu_i}}_{=D_k}.$$

To continue, we must show that the dampening factor is less than one. Finally, we infer

$$D_k < 1 \Leftrightarrow \frac{\sqrt{\sum_{i=1}^n \alpha_i^2\mu_i^2 + \left(\frac{v_n}{v_k}-1\right)\sum_{i=k+1}^n \alpha_i^2\mu_i^2}}{\sum_{i=1}^n \alpha_i\mu_i} < 1$$
$$\Leftrightarrow \sum_{i=1}^n \alpha_i^2\mu_i^2 + \left(\frac{v_n}{v_k}-1\right)\sum_{i=k+1}^n \alpha_i^2\mu_i^2 < \left(\sum_{i=1}^n \alpha_i\mu_i\right)^2$$
$$\Leftrightarrow \frac{v_n}{v_k} < \underbrace{\underbrace{\frac{\left(\sum_{i=1}^n \alpha_i\mu_i\right)^2 - \sum_{i=1}^n \alpha_i^2\mu_i^2}{\sum_{i=k+1}^n \alpha_i^2\mu_i^2}}_{>0} + 1}_{=S_k>1}$$
$$\Leftrightarrow v_n - v_k < S_k v_k.$$

The above stated propositions express that if the difference of the coefficients of variation is not great, the demand is dampened going upstream of the BOM-structure with multiple parents. Multiple parents correspond to build variants or do use the same sub part for different (end) items. For small differences in the demand variation of the end items, it is useful in terms of reducing the demand variation if the strategic inventory position is upstream of the variant building point. Variants should be make to order, especially parent items with a higher coefficient of variation than v_{k^*} with

$$k^* = \max\left\{k = 0, 1, \dots, n \,\middle|\, v_n - v_k > S_k v_k\right\} \text{ whereby } v_0 = 0 \text{ and } S_0 = 1, \tag{43}$$

are dampened. The next proposition shows the situation for the special case of two parent items. □

Proposition 2.5. *Item p has $n = 2$ parent items p_1, p_2 with expectation values denoted by $\mu_1, \mu_2 > 0$, variances $\sigma_1^2, \sigma_2^2 > 0$, coefficients of variations $0 < v_1 < v_2$ and BOM-coefficients $\alpha_1, \alpha_2 > 0$. The demand coefficient of variation v_p of item p is strictly less than the coefficient of variation of the parent item p_1 replenishment order v_1*

$$v_p = D_1 v_1 < v_1, \tag{44}$$

if, and only if, the inequality

$$v_2^2 - v_1^2 < 2\frac{\alpha_1\sigma_1^2}{\alpha_2\mu_1\mu_2}, \tag{45}$$

is satisfied. The dampening factor D_1 is equal to

$$D_1 = \frac{\sqrt{1 + \frac{\alpha_2^2\sigma_2^2}{\alpha_1^2\sigma_1^2}}}{1 + \frac{\alpha_2\mu_2}{\alpha_1\mu_1}} < 1. \tag{46}$$

Proof.

We get straightforwardly $v_p = \frac{\sqrt{\alpha_1^2\sigma_1^2+\alpha_2^2\sigma_2^2}}{\alpha_1\mu_1+\alpha_2\mu_2} = v_1 \underbrace{\frac{\sqrt{1 + \frac{\alpha_2^2\sigma_2^2}{\alpha_1^2\sigma_1^2}}}{1 + \frac{\alpha_2\mu_2}{\alpha_1\mu_1}}}_{=D_1}$. Also

$$D_1 = \frac{\sqrt{1+\frac{\alpha_2^2\sigma_2^2}{\alpha_1^2\sigma_1^2}}}{1+\frac{\alpha_2\mu_2}{\alpha_1\mu_1}} < 1 \Leftrightarrow 1 + \frac{\alpha_2^2\sigma_2^2}{\alpha_1^2\sigma_1^2} < 1 + 2\frac{\alpha_2\mu_2}{\alpha_1\mu_1} + \frac{\alpha_2^2\mu_2^2}{\alpha_1^2\mu_1^2}$$

$$\underset{\text{multiply by } \frac{\alpha_1^2\sigma_1^2}{\alpha_2^2\mu_2^2}}{\Leftrightarrow} \frac{\sigma_2^2}{\mu_2^2} < 2\frac{\alpha_1\sigma_1^2}{\alpha_2\mu_1\mu_2} + \frac{\sigma_1^2}{\mu_1^2} \Leftrightarrow v_2^2 - v_1^2 < 2\frac{\alpha_1\sigma_1^2}{\alpha_2\mu_1\mu_2}. \quad \square$$

The demand coefficient of variation at a MRP-planned item with lot for lot as lot-sizing policy inherits the variation to the replenishment orders. For demand $x(1), x(2), \dots, x(T)$ the caused replenishment orders are $x(1+\ell_p), x(2+\ell_p), \dots, x(T+\ell_p)$. Thus, the statistical distribution, especially the variance and coefficient of variations of the demand and replenishment orders are the same. The following proposition addresses the situation for a reorder point planned item.

Proposition 2.6. *The coefficient of variation v_x of a replenishment order for a reorder point planned item with reorder point r, lot size Q, expectation value of the demand μ and demand variance σ^2 and with demand coefficient of variation $v = \frac{\mu}{\sigma}$ is dampened by*

$$v_x = D_{reorder} v < v, \tag{47}$$

if, and only if, the inequality

$$Q^2 Z^2 < 1 + v^2, \tag{48}$$

is satisfied. The dampening factor is equals to

$$D_{reorder} = \frac{\sqrt{Q^2 Z^2 - 1}}{v} < 1. \tag{49}$$

Proof. We infer

$$v_x = \frac{\sigma_x}{\mu_x} \underset{\text{Proposition 2.1}}{=} \frac{\sqrt{\mu^2 (Q^2 Z^2 - 1)}}{\mu}$$

$$= \sqrt{\frac{Q^2 Z^2 - 1}{v^2}} v = D_{reorder} v \tag{50}$$

$$D_{reorder} < 1 \Leftrightarrow Q^2 Z^2 - 1 < v^2$$

$$\Leftrightarrow Q^2 Z^2 < 1 + v^2 \quad \square$$

Proposition 2.6 expresses that for not very high lot sizes a reorder point dampens the variation upstream of the BOM-structure. It can be useful to use $W^2 = Z^2 \mu^2$ instead of Z^2. For W^2, we obtain

$$W^2 = \mu^2 Z^2 < \mu^2 \left(1 + v^2\right) < \mu^2 + \sigma^2. \tag{51}$$

Corollary 2.1. *The coefficient of variation v_x of a replenishment order for a reorder point planned item with reorder point r, lot size Q, expectation value of the demand μ and demand variance σ^2 and with demand coefficient of variation $v = \frac{\mu}{\sigma}$, is dampened*

$$v_x = D_{reorder} v < v, \tag{52}$$

if, and only if,

$$Q^2 W^2 < \mu^2 + \sigma^2, \tag{53}$$

holds true. The dampening factor is equal to

$$D_{\text{reorder}} = \frac{\sqrt{Q^2 W^2 - \mu^2}}{\sigma} < 1. \tag{54}$$

Proof. The proof follows directly from Proposition 2.1 and the definition of W as $W^2 = Z^2 \mu^2$. $\square$

For high lot sizes, we get the following approximation.

Proposition 2.7. *For the variance σ_x^2 and the coefficient of variation v_x of a replenishment order for a reorder point planned item with high lot sizes,*

$$Q^2 > \mu^2 + \sigma^2, \tag{55}$$

the following approximations hold:

$$\begin{aligned} &\sigma_x^2 \approx \mu (Q - \mu) \\ &v_x \approx \sqrt{\frac{Q-\mu}{\mu}} \end{aligned} \tag{56}$$

Proof.

$$\text{We start with } \sigma_x^2 \approx \underbrace{F_{N(\mu,\sigma^2)}(Q)}_{=1-\frac{\mu}{Q}} (0 - \mu)^2 + \underbrace{\left(1 - F_{N(\mu,\sigma^2)}(Q)\right)}_{=\frac{\mu}{Q}} (Q - \mu)^2$$

$$= \mu^2 - \frac{\mu^3}{Q} + \mu Q - 2\mu^2 + \frac{\mu^3}{Q} = \mu (Q - \mu). \tag{57}$$

$$\text{Finally, } v_x = \frac{\sigma_x}{\mu_x} = \frac{\sqrt{\mu(Q-\mu)}}{\mu} = \sqrt{\frac{Q-\mu}{\mu}}. \quad \square$$

After introducing a realistic cost model, we tackle the characteristics of optimal customer decoupling points and optimal efficiency decoupling point. Here, optimal relates to the minimum total inventory cost by ensuring customer requirements in terms of customer required delivery lead time and customer required service level.

Like in most multi echelon models [3], we assume increasing unit holing cost going downstream (nearer to the customer) in the BOM-structure based on value added processes, that is

$$\begin{aligned}&c_v \geq \textstyle\sum_{v_i \text{ is child item of } v} c_{v_i} + q o_{v_i}\\ &o_{v_i} \text{is the added value (proportionally cost for labor, energy, set} - \text{up, ...) for item } v_i\\ &q \text{ is the interest rate}\end{aligned} \tag{58}$$

This implies that it is better for the inventory cost to keep inventory in production systems in earlier stages (more upstream), rather than later (more downstream). For production lot sizes, we assume smaller lot sizes for lower low level items and greater lot sizes for items far away from end items [20] or multi-echelon models, see Hopp and Spearman [21],

$$Q_v \leq \min_{v_i \text{ is child item of } v} \left\{ Q_{v_i} \right\}. \tag{59}$$

The lot-sizing assumption reflects the higher customer orientation for items nearer to end items and the fact that in most production systems, changeover costs are higher in the first production steps. The set-up costs are indirectly considered by the operational unit cost.

Also, we state several reasons to position the customer decoupling point as upstream as possible (nearer to suppliers), namely,

1. Lower unit inventory cost at the reorder point planned item.
2. Lower demand coefficient of variation (for multiple parent structures) at the reorder point planned item (if the coefficients of demand variation are not too different).
3. Shorter replenishment lead time of the reorder point planned item.
4. Building variants make to order.

The replenishment lead time, as well as the consumption during the replenishment lead time, increases if the reorder point planned stage is moved downstream (nearer to the customer). This effect is confirmed in a simulation study by [19] for the customer order decoupling point. To meet the customer required delivery lead time, the most upstream position is limited by the claim of non-negative customer buffer. No strategic inventory position, especially no customer decoupling point, is necessary along all paths from an external sourced item A to a connected end item, if the external sourced item A fulfills the condition:

$$cb(A) - \ell_A \geq 0. \tag{60}$$

However, there are some reasons to position the customer decoupling point more downstream (nearer to customers):

1. Smaller needed service level at the reorder point planned item to fulfill the customer required service level for end items (for multiple child structures).
2. Lower average inventory caused by the lot sizes at the reorder point planned item.
3. Shorter customer distances.
4. Better demand dampening at the reorder point planned item because of smaller lot-sizes.

Shorter customer distances enable more revenue in some market environments. In our model, we only model that the customer distance must be shorter than the customer required delivery lead time. For planned lead times of an end item longer than the customer required lead time, the end item must be reorder planned to meet the customer requirement. The optimal number of customer decoupling points are a priori unknown. For efficiency decoupling point, the same characteristics for customer decoupling points are given, except for the conditions due to the customer required delivery lead time. Again, there are some reasons to position the efficiency decoupling point more upstream (nearer to suppliers):

1. Lower unit inventory cost at the reorder point planned item.
2. Lower demand coefficient of variation (for multiple parent structures) at the reorder point planned item (if the coefficients of demand variation are not too different).
3. Shorter replenishment lead time of the reorder point planned item (see Lemma 2.3).

There are some reasons to position the efficiency decoupling point more downstream (nearer to customers):

1. Smaller service level at the reorder point planned item to fulfill the customer required service level for end items (for multiple child structures).
2. Lower average inventory caused by the optimal lot-sizes at the reorder point planned item because of smaller set-up cost.
3. Shorter replenishment lead time of downstream reorder point planned items.
4. Better demand dampening at the reorder point planned item because of smaller lot-sizes.

The optimal number of efficiency decoupling points are a priori unknown. So, the following lemmas tackle the inventory cost change when moving the reorder point planned item up- or downstream.

Lemma 2.2. *For a directly linked parent item A with its child item B, the incurred cost of holding the reorder point inventory of the parent item is strictly greater than the incurred cost of holding the reorder point inventory of the child item*

$$c_A w_A > c_B w_B, \tag{61}$$

if there is only one child and the conditions of at least one of the Propositions 2.2, 2.3, 2.4 or 2.5 (demand variations are not too different) are fulfilled.

Proof.

We infer

$$\begin{aligned}
&E = c_A w_A - c_B w_B \text{ (inventory cost difference of holding the reorder points)}\\
&E = c_A\left(\ell_{A,rep}\mu_A + F^{-1}_{N(0,1)}(\gamma_A)\sqrt{\ell_{A,rep}}\sigma_A\right) - c_B\left(\ell_{B,rep}\mu_B + F^{-1}_{N(0,1)}(\gamma_B)\sqrt{\ell_{B,rep}}\sigma_B\right)\\
&\quad = c_A\mu_A\left(\ell_{A,rep} + F^{-1}_{N(0,1)}(\gamma_A)\sqrt{\ell_{A,rep}}v_A\right) - c_B\mu_B\left(\ell_{B,rep} + F^{-1}_{N(0,1)}(\gamma_B)\sqrt{\ell_{B,rep}}v_B\right)\\
&\quad = c_B\mu_B\left(\tfrac{c_A\mu_A}{c_B\mu_B}\left(\ell_{A,rep} + F^{-1}_{N(0,1)}(\gamma_A)\sqrt{\ell_{A,rep}}v_A\right) - \left(\ell_{B,rep} + F^{-1}_{N(0,1)}(\gamma_B)\sqrt{\ell_{B,rep}}v_B\right)\right)\\
&\quad = c_B\mu_B\left(\tfrac{c_A\mu_A}{c_B\mu_B}\ell_{A,rep} - \ell_{B,rep} + \tfrac{c_A\mu_A}{c_B\mu_B}F^{-1}_{N(0,1)}(\gamma_A)\sqrt{\ell_{A,rep}}v_A - F^{-1}_{N(0,1)}(\gamma_B)\sqrt{\ell_{B,rep}}v_B\right)\\
&c_A\mu_A > c_B\mu_B \Leftrightarrow \tfrac{c_A\mu_A}{c_B\mu_B} > 1 \text{ (because of the cost structre)}\\
&\ell_{A,rep} > \ell_{B,repl} \text{(because of } \ell_{A,rep} - \ell_A = \ell_{B,rep})\\
&\gamma_A = \gamma_B \text{ (because of single child condition)}\\
&v_A > v_B \text{(because of the demand variation dampening prepositions)}
\end{aligned}$$

This implies $E > 0$ and $c_A w_A > c_B w$. □

Lemma 2.3. *For a directly linked parent item A with its once child item B under lot-size Condition (59) the incurred demand variation upstream B depends on the positioning of the reorder point at A or B in the following manner*

$$\begin{aligned}
&\frac{v_{B,repl,A}}{v_{B,repl,B}} = \sqrt{\frac{\Psi(Q_A)Q_A-\mu^2}{\Psi(Q_B)Q_B-\mu^2}} \le 1,\\
&\text{whereby}\\
&v_{B,repl,A} \text{ is the coefficient of variation of replenishment order of item } B\\
&\qquad \text{if } A \text{ is reorder point planned and } B \text{ MRP} - \text{planned},\\
&v_{B,repl,B} \text{ is a coefficient of variation of replenishment order of item } A,\\
&\qquad \text{if } B \text{ is reorder point planned and } A \text{ MRP} - \text{planned},\\
&\Psi(Q) = \lim_{n\to\infty}\left(\sum_{i=1}^n \int_{iQ}^{(i+1)Q} F(\xi)\,d\xi - \int_{(i-1)Q}^{iQ} F(\xi)\,d\xi\right) = \frac{\sigma_x^2+\mu_x^2}{Q} > 0.
\end{aligned} \tag{62}$$

Proof. We yield

$$\begin{aligned}
&v_{B,repl,A} = \frac{\sigma_x^2(Q_A)}{\mu} = \sqrt{\Psi(Q_A)Q_A - \mu^2},\\
&v_{B,repl,B} = \frac{\sigma_x^2(Q_B)}{\mu} = \sqrt{\Psi(Q_B)Q_B - \mu^2},\\
&\frac{v_{B,repl,A}}{v_{B,repl,B}} = \sqrt{\frac{\Psi(Q_A)Q_A-\mu^2}{\Psi(Q_B)Q_B-\mu^2}} \underset{\substack{Q_B \ge Q_A \\ \text{Proposition 2.1}}}{\le} 1.
\end{aligned}$$

□

Lemma 2.3 says that the demand variation is smaller if the reorder point is nearer to the customer.

Proposition 2.8. *A customer decoupling point A fulfills the minimum customer buffer characteristic to ensure minimum total inventory if there is no multiple children structure downstream of the customer decoupling point, and the conditions of at least one of the Propositions 2.2, 2.3, 2.4 or 2.5 (demand variations are not too different) are fulfilled and the inequality,*

$$
\begin{aligned}
&Q_B < \underbrace{\frac{c_A}{c_B}}_{>1} Q_A + \frac{2}{c_B} \underbrace{E}_{>0} \quad \text{(lot size } Q_B \text{is not much bigger than lot size } Q_A\text{)},\\
&\text{where } E = c_A w_A - c_B w_B,
\end{aligned}
\tag{63}
$$

is satisfied and the conditions of Proposition 2.6 (lot sizes are not too high) is fulfilled and there is no reorder planned item upstream to B.

Proof. From the minimum customer buffer characteristic, it follows that only one more downstream positioning of the customer decoupling point A is possible. Let us assume the minimum customer buffer characteristic for the decoupling point A is not fulfilled, thus

$$
\begin{aligned}
&cb(A) < 0 \text{ (leads directly to a contradiction to A is a customer decoupling point)}\\
&\text{or}\\
&cb(A) - \ell_A \geq 0
\end{aligned}
\tag{64}
$$

is fulfilled. We must consider the case $cb(A) - \ell_A \geq 0$. No customer decoupling point on all pathes from A to an end item is needed, if A is an external sourced item. A must have a child item B which fulfills the minimum customer buffer characteristic $cb(B) \geq 0$, if A is not externally purchased. The child item B is a possible candidate for the customer decoupling point. We have only shown that decoupling point B leads to less total inventory costs than the customer decoupling point A.

$$
\begin{aligned}
&y_A > y_B\\
&\Leftrightarrow y_A - y_B \underbrace{=}_{E=c_A w_A - c_B w_B} E + c_A \frac{Q_A}{2} - c_B \frac{Q_B}{2} > 0\\
&\Leftrightarrow Q_B < \underbrace{\frac{c_A}{c_B}}_{>1} Q_A + \frac{2}{c_B} \underbrace{E}_{>0}
\end{aligned}
\tag{65}
$$

The influence of the customer decoupling point movement on the upstream reorder point planned items does not have to be considered because there is no upstream reorder point planned item. □

The interpretation of Proposition 2.8 is as follows: If there are no multiple children structure downstream of the customer decoupling point (between customer and decoupling point customer), demand variations are not to different, and upstream to the decoupling point lot sizes are not much bigger than lot sizes at the decoupling point, then an optimal customer decoupling point has the longest possible distance to the customers which ensures that the customer required deliver time is shorter that the planned lead time from the decoupling point to the customer. For practitioners the following rule of thumb can be derived: Move the customer decoupling point upstream as long as the customer required delivery time is shorter than the sum of the planed lead times from the customer decoupling point to the customers.

3. Summary and conclusion

In this study, we present a model centered on directed universal graphs for optimizing the position of decoupling points within a supply chain. Utilizing graph theory, we define key parameters such as production lead time, BOM-path length, customer distance, customer buffer, customer decoupling point, and replenishment lead time. A crucial distinction is made between customer decoupling points, where no other decoupling point exists downstream, and efficiency decoupling points, situated upstream to customer decoupling points, designed to minimize total inventory costs. We introduce the novel concept of minimum customer buffer characteristics as a means to typify an optimal customer decoupling point.

Our primary contributions involve formalizing the optimal customer decoupling point setting problem and deriving specific formulas to characterize these optimal points. The key scientific results comprise:

1. Demand Inheritance Rules: We establish rules governing the propagation of the demand coefficient of variation through the BOM-graph, providing a deeper understanding of the relationship between parent and children items (Proposition 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, and 2.7).
2. Characterization of Optimal Decoupling Point: We present a comprehensive characterization of an optimal decoupling point (Proposition 2.8).

Managerial insights are distilled into practical rules of thumb for practitioners. Decoupling points positioned more upstream (nearer to the suppliers) offer advantages such as lower holding costs due to reduced unit inventory costs, lower demand coefficient of variation, shorter replenishment lead times, and the ability to build more variants make-to-order at the decoupling point.

Conversely, downstream nearer to customers positions can reduce holding costs through smaller needed service levels and lot sizes at the decoupling point. Furthermore, shorter customer distances enable higher revenues for customer decoupling points and for efficiency decoupling points shorter replenishment lead time of downstream decoupling points facilitate lower holding costs of downstream decoupling points. In formalizing the optimal decoupling point problem and proving mathematical propositions, certain assumptions are made. Future research endeavors will focus on relaxing some of these assumptions. Additionally, we propose a numerical study to illustrate analytical results and analyze more general cases, providing a more comprehensive understanding of the optimal decoupling point setting problem.

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Data availability

No data was used for the research described in the article.

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