Approximate robust output regulation of boundary control systems

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***Abstract*—We extend the internal model principle for systems with boundary control and boundary observation, and construct a robust controller for this class of systems. However, as a consequence of the internal model principle, any robust controller for a plant with infinite-dimensional output space necessarily has infinite-dimensional state space. We proceed to formulate the approximate robust output regulation problem and present a finite-dimensional controller structure to solve it. Our main motivating example is a wave equation on a bounded multidimensional spatial domain with force control and velocity observation at the boundary. In order to illustrate the theoretical results, we construct an approximate robust controller for the wave equation on an annular domain and demonstrate its performance with numerical simulations.**

***Index Terms*—Robust control, Distributed parameter systems, Linear systems, Controlled wave equation**

I. Introduction

Intuitively speaking, the problem of output regulation of a given plant amounts to designing an output feedback controller which stabilizes the plant, and in addition the output of the plant tracks a given reference signal in spite of a given disturbance signal. If a single controller solves the output regulation problem for the plant and also for small perturbations of the plant, and for more or less arbitrary reference and disturbance signals, then the controller is said to solve the *robust* output regulation problem. See the beginning of §IV for exact definitions of these concepts.

Output tracking and disturbance rejection have been studied actively in the literature for distributed parameter systems with bounded control and observation operators [1], [2], [3], [4], [5] and robust controllers have been constructed for classes of systems with unbounded control and observation operators, such as well-posed [6] and regular [7] systems, in [8], [9], [10], [11]. The key in designing robust controllers is the *internal model principle* which in its classical form states that a controller can solve the robust output regulation problem only if it contains p copies of the dynamics of the exosystem, where p is the dimension of the output space of the plant. The internal model principle was first presented for finite-dimensional linear plants by Francis and Wonham [12] and Davison [13]. The principle was later generalized to infinite-dimensional linear systems in [11], [14], [15] under the assumption that the plant is regular.

In this paper, we focus on output regulation for boundary controlled systems with boundary observation. Our motivating example is a wave equation on a multidimensional spatial domain, with force control and velocity observation on a part of the boundary. This n-D wave system is challenging from the robust control point of view since it is neither regular nor well-posed. Moreover, the output space of the wave system is infinite-dimensional and then the *internal model principle* implies that any robust controller must also be infinite-dimensional. However, as the main contribution of this paper, we demonstrate that it is possible to achieve *approximate* tracking of the reference signal in the sense that the difference between the output and the reference signal becomes small as $t \to \infty$. More precisely, we introduce a new finite-dimensional controller that solves the robust output regulation problem in this approximate sense, hence extending the recent results of [16] to continuous time. At the same time, we extend the class of reference signals that can be tracked. As a part of the construction of this controller, we present an upper bound for the regulation error.

The second main result of this paper is a generalization of the internal model principle presented in [14], [15] to boundary control systems that are not necessarily regular linear systems. The sufficiency of the internal model for achieving robust control has been presented in [17], albeit here our formulation is more general in terms of boundary controls and disturbances. The necessity of the internal model is a new result for boundary control systems.

As our third main contribution we characterize and construct a minimal finite dimensional controller to solve the output regulation problem. Due to the reduced size of the controller, it does not have any guaranteed robustness properties. The controller concept was presented for regular linear systems in [11], and here we will generalize such controllers for boundary control systems.

In §II, we present the wave equation and show how it fits into the abstract framework of the later sections. In §III, we present the abstract plant, the exosystem and the controller (which is to be constructed), and reformulate the interconnection of these three systems as a regular input/state/output system. In §IV, we present the output regulation, the robust output regulation and the approximate robust output regulation problems, and present controller structures to solve them. A regulating controller without the robustness requirement is presented in §IV-A, and an approximate robust regulating controller is presented in §IV-C. In §IV-B, we present the

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internal model principle for boundary control systems, following which we present a precise robust regulating controller in §IV-D. In §V, we construct an approximate robust regulating controller for the wave equation on an annular domain and demonstrate its performance with numerical simulation. The paper is concluded in §VI.

Here $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from the normed space X to the normed space Y. The domain, range, kernel, spectrum and resolvent of a linear operator A are denoted by $\mathcal{D}(A), \mathcal{R}(A), \mathcal{N}(A), \sigma(A)$ and $\rho(A)$, respectively. The right pseudoinverse of a surjective operator P is denoted by $P^{[-1]}$.

II. The wave equation

In this section, we describe the example which motivates the robust output regulation theory in this paper, a wave equation (the plant) on a bounded domain $\Omega \subset \mathbb{R}^n$ with force control and velocity observation at a part of the boundary. We try to keep the exposition brief; more details can be found in [18], [19], [20].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (an open connected set) with a Lipschitz-continuous boundary $\partial\Omega$ split into two parts Γ_0, Γ_1 such that $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \partial\Omega$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and $\partial\Gamma_0, \partial\Gamma_1$ both have surface measure zero. We consider the wave equation

$$\begin{cases} \rho(\zeta)\dfrac{\partial^2 w}{\partial t^2}(\zeta,t) = \operatorname{div}\big(T(\zeta)\,\nabla w\,(\zeta,t)\big), \quad \zeta \in \Omega, \\ u(\zeta,t) = \nu \cdot T(\zeta)\,\nabla w(\zeta,t), \quad \zeta \in \Gamma_1, \\ y(\zeta,t) = \dfrac{\partial w}{\partial t}(\zeta,t), \qquad \zeta \in \Gamma_1, \\ 0 = \dfrac{\partial w}{\partial t}(\zeta,t), \qquad \zeta \in \Gamma_0,\ t>0 \\ w(\cdot,0) = w_0, \qquad \dfrac{\partial w}{\partial t}(\cdot,0) = w_1, \end{cases} \tag{II.1}$$

where $w(\zeta,t)$ is the displacement from the equilibrium at the point $\zeta \in \Omega$ and time $t \geq 0$, $\rho(\cdot)$ is the mass density, $T^*(\cdot) = T(\cdot) \in L^2(\Omega;\mathbb{R}^n)$ is Young's modulus and $\nu \in L^\infty(\partial\Omega;\mathbb{R}^n)$ is the unit outward normal at $\partial\Omega$. We require $\rho(\cdot)$ and $T(\cdot)$ to be essentially bounded from both above and below away from zero. Please note that the input u is the force perpendicular to Γ_1 and the output y is the velocity at Γ_1 while waves are reflected at the part Γ_0 of the boundary where the displacement is constant.

In order to solve the robust output regulation problem for the wave system, we shall need to stabilize (II.1) exponentially using a viscous damper on Γ_1, which corresponds to the output feedback

$$u(\zeta,t) = -b^2(\zeta)\,y(\zeta,t), \qquad \zeta \in \Gamma_1,\ t \geq 0.$$

This requires that we make some additional assumptions solely for the purpose of obtaining exponential stability (see §II-B below for more details). Additionally, to prove later on that the velocity observation on Γ_1 is admissible, we assume that

$$\delta := \inf_{\zeta\in\Gamma_1} b(\zeta)^2 > 0. \tag{II.2}$$

A. The wave equation as a formal boundary control system

Our first step is to show that the wave equation on a bounded domain in $\mathbb{R}^n$ can be written as a boundary control system (BCS) in the sense of [21]. To this end, we first write the wave equation

$$\rho(\zeta)\,\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \operatorname{div}\big(T(\zeta)\,\nabla w(\zeta,t)\big) \qquad \text{on} \quad \Omega \times \mathbb{R}_+$$

in the first-order form (as an equality in $L^2(\Omega)^{n+1}$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \rho(\cdot)\,\dot w(\cdot,t) \\ \nabla w(\cdot,t)\end{bmatrix} = \begin{bmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{bmatrix}\begin{bmatrix} 1/\rho(\cdot) & 0 \\ 0 & T(\cdot)\end{bmatrix}\begin{bmatrix} \rho(\cdot)\,\dot w(\cdot,t) \\ \nabla w(\cdot,t)\end{bmatrix}, \tag{II.3}$$

where div denotes the (distribution) divergence operator and ∇ is the (distribution) gradient. Hence, the state at any time is the pair of momentum and strain densities on Ω.

Under the standing assumptions on ρ and T, the operator of multiplication by $\mathcal{H} := \begin{bmatrix} 1/\rho(\cdot) & 0 \\ 0 & T(\cdot)\end{bmatrix}$ defines an inner product on $L^2(\Omega)^{n+1}$ via

$$\langle x,z\rangle_{\mathcal{H}} := \langle \mathcal{H}x, z\rangle_{L^2(\Omega)^{n+1}}$$

and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is equivalent to $\langle\cdot,\cdot\rangle_{L^2(\Omega)^{n+1}}$. The space $L^2(\Omega)^{n+1}$ equipped with this equivalent inner product is denoted by $X_{\mathcal{H}}$ and will be used as the state space of the plant.

We next introduce some function spaces for the wave equation. The notation $H^1(\Omega)$ stands for the Sobolev space of all elements of $L^2(\Omega)$ whose distribution gradient lies in $L^2(\Omega)^n$ and $H^1(\Omega)$ is equipped with the graph norm of the gradient. Similarly $H^{\operatorname{div}}(\Omega)$ is the space of all elements of $L^2(\Omega)^n$ whose distribution divergence lies in $L^2(\Omega)$, equipped with the graph norm of div. In order for (II.3) to make sense as an equation in $L^2(\Omega)^{n+1}$, we need for every fixed $t \geq 0$ that $\dot w(\cdot,t) \in H^1(\Omega)$, $\nabla w(\cdot,t) \in L^2(\Omega)$, and $T(\cdot)\,\nabla w(\cdot,t) \in H^{\operatorname{div}}(\Omega)$, or equivalently

$$\begin{bmatrix} \rho\,\dot w(t) \\ \nabla w(t)\end{bmatrix} \in \mathcal{H}^{-1}\begin{bmatrix} H^1(\Omega) \\ H^{\operatorname{div}}(\Omega)\end{bmatrix}, \qquad t \geq 0.$$

If $\Gamma_0 = \emptyset$, then the output y lives in the fractional-order space $H^{1/2}(\partial\Omega)$ on the boundary of Ω (see, e.g., [19, §13.5] or [20]). This space is important to us also when $\Gamma_0 \neq \emptyset$, because the *Dirichlet trace* γ_0 maps $H^1(\Omega)$ continuously onto $H^{1/2}(\partial\Omega)$. Indeed, we set

$$\mathcal{W} := \left\{ w \in H^{1/2}(\partial\Omega) \;\middle|\; w\big|_{\Gamma_0} = 0 \right\} \qquad \text{with}$$
$$\|w\|_{\mathcal{W}} := \left\| \gamma_0^{[-1]}\, w \right\|_{H^1(\Omega)},$$

where $|$ denotes the restriction to a given subdomain in the appropriate sense and

$$\gamma_0^{[-1]} := \gamma_0\big|_{\mathcal{N}(\gamma_0)^\perp}^{-1} \in \mathcal{L}\big(H^{1/2}(\partial\Omega); H^1(\Omega)\big).$$

Moreover, we introduce

$$H^1_{\Gamma_0}(\Omega) := \left\{ g \in H^1(\Omega) \;\middle|\; g\big|_{\Gamma_0} = 0 \right\},$$

with the norm inherited from $H^1(\Omega)$. This setup makes both $\mathcal{W}$ and $H^1_{\Gamma_0}(\Omega)$ Hilbert spaces; indeed, $H^{1/2}(\partial\Omega)$ is continuously embedded into $L^2(\partial\Omega)$ by [19, (13.5.3)], and so $H^1_{\Gamma_0}(\Omega)$ is the kernel of $P_{\Gamma_0}\gamma_0 \in \mathcal{L}\big(H^1(\Omega), L^2(\partial\Omega)\big)$, where

P_{Γ_0} is the orthogonal projection onto $L^2(\Gamma_0)$ in $L^2(\partial\Omega)$. This proves that $H^1_{\Gamma_0}(\Omega)$ is a Hilbert space, and moreover, γ_0 maps the Hilbert space $H^1_{\Gamma_0}(\Omega) \ominus \mathcal{N}(\gamma_0)$ unitarily onto $\mathcal{W}$ which is then also complete.

The embedding $\iota : \mathcal{W} \to L^2(\Gamma_1)$ is continuous, because $\iota = P_{\Gamma_1}\widetilde{\iota}\gamma_0\gamma_0^{[-1]}$, where $\widetilde{\iota}$ is the continuous embedding of $H^{1/2}(\partial\Omega)$ into $L^2(\partial\Omega)$. The embedding is also dense by [19, Thm 13.6.10], so that we may define $\mathcal{W}'$ as the dual of $\mathcal{W}$ with pivot space $L^2(\Gamma_1)$ (see [19, §2.9]). Then in particular

$$\langle \omega, w\rangle_{\mathcal{W}',\mathcal{W}} = \langle \omega, w\rangle_{L^2(\Gamma_1)}, \qquad \omega \in L^2(\Gamma_1),\, w \in \mathcal{W}.$$

Thm 1.8 in Appendix 1 of [18] states that the *restricted normal trace* $\gamma_\perp h := (\nu \cdot \gamma_0 h)\big|_{\Gamma_1}$, $h \in H^1(\Omega)^n$, has a unique extension to a continuous operator (still denoted by $\gamma_\perp$) that maps $H^{\mathrm{div}}(\Omega)$ *onto* $\mathcal{W}'$. Please note that $\gamma_\perp$ is *not* the Neumann trace γ_N: If $\Gamma_0 = \emptyset$, then $\mathcal{W} = H^{1/2}(\partial\Omega)$ and the relation between the two operators is $\gamma_N x = \gamma_\perp \nabla x$, for a sufficiently regular x, where the equality is in $H^{-1/2}(\partial\Omega)$. The space $H^{-1/2}(\partial\Omega)$ equals $\mathcal{W}'$ in the case where $\Gamma_0 = \emptyset$ (which is not the main case of interest to us, see (II.6) below).

Now we include the boundary condition at Γ_0 into the domain of $\left[\begin{smallmatrix} 0 & \mathrm{div} \\ \nabla & 0 \end{smallmatrix}\right]\mathcal{H}$, see (II.3), by requiring that $\dot w \in H^1_{\Gamma_0}(\Omega)$ instead of the weaker $\dot w \in H^1(\Omega)$ which we motivated above. We can then write (II.1) as

$$\begin{cases} \dot x(t) = \mathfrak{A}\mathcal{H}x(t), \\ u(t) = \mathfrak{B}\mathcal{H}x(t), \quad t \ge 0, \\ y(t) = \mathfrak{C}\mathcal{H}x(t), \end{cases} \qquad x(0) = \begin{bmatrix} \rho\, w_0' \\ \nabla w_0 \end{bmatrix}, \tag{II.4}$$

where $x(t) = \left[\begin{smallmatrix} \rho\, \dot w(t) \\ \nabla w(t) \end{smallmatrix}\right]$ is the state at time t, $\mathfrak{A} = \left[\begin{smallmatrix} 0 & \mathrm{div} \\ \nabla & 0 \end{smallmatrix}\right]$, $\mathfrak{B} = \begin{bmatrix} 0 & \gamma_\perp \end{bmatrix}$, and $\mathfrak{C} = \begin{bmatrix} \gamma_0 & 0 \end{bmatrix}$, with domains

$$\mathcal{D}(\mathfrak{A}) := \mathcal{D}(\mathfrak{B}) := \mathcal{D}(\mathfrak{C}) := \begin{bmatrix} H^1_{\Gamma_0}(\Omega) \\ H^{\mathrm{div}}(\Omega) \end{bmatrix} \subset X_{\mathcal{H}},$$

which is Hilbert when equipped with the graph norm of $\mathfrak{A}$. Here $X_{\mathcal{H}}$ is the state space, $U = \mathcal{W}'$ the input space, and $Y = \mathcal{W}$ the output space.

In [18, Thm 3.2] it was shown that (II.4) has the structure of a *boundary triplet* (or abstract *boundary space* in the original terminology of [22, §3.1.4]). This easily implies that the undamped wave equation is a boundary control system in the sense of Curtain and Zwart [21, Def. 3.3.2]:

Definition II.1. Let the *state space* X and *input space* U be Hilbert spaces, and let $\mathcal{A} : X \supset \mathcal{D}(\mathcal{A}) \to X$ and $\mathcal{B} : X \supset \mathcal{D}(\mathcal{B}) \to U$ be linear operators with $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B})$.

The control system $\dot x(t) = \mathcal{A}x(t)$, $\mathcal{B}x(t) = u(t)$, $t \ge 0$, $x(0) = x_0$, is called a *boundary control system (BCS)* if the following conditions are met:

1) The operator $A := \mathcal{A}\big|_{\mathcal{D}(A)}$ with domain
$$\mathcal{D}(A) := \mathcal{D}(\mathcal{A}) \cap \mathcal{N}(\mathcal{B})$$
generates a C_0-semigroup on X and
2) there exists a $B \in \mathcal{L}(U, X)$ such that $BU \subset \mathcal{D}(\mathcal{A})$, $\mathcal{A}B \in \mathcal{L}(U, X)$, and $\mathcal{B}B = I_U$.

An output equation may be added to the BCS by setting $y(t) = \mathcal{C}x(t)$, where $\mathcal{C}$ is a linear operator defined on $\mathcal{D}(\mathcal{C}) \supset \mathcal{D}(\mathcal{A})$ and mapping into some Hilbert *output space* Y, with the additional property that $\mathcal{C}B \in \mathcal{L}(U, Y)$. We shall briefly say that $(\mathcal{B}, \mathcal{A}, \mathcal{C})$ is a BCS on (U, X, Y) if all of the above conditions are met. Finally, we say that a BCS on (U, X, Y) is *(impedance) passive* if the input space U can be identified with the dual Y' of the output space and

$$\mathrm{Re}\, \langle \mathcal{A}x, x\rangle_X \le \mathrm{Re}\, \langle \mathcal{B}x, \mathcal{C}x\rangle_{Y',Y}, \qquad x \in \mathcal{D}(\mathcal{A}).$$

For more information on abstract passive BCS, we refer to [23], [24]. Unlike the setting of Malinen and Staffans, the original definition of Curtain and Zwart does not consider the observation operator $\mathcal{C}$ or passivity, and it is not assumed that $\mathcal{D}(\mathcal{A})$ is a Hilbert space. The robust output regulation theory presented in §IV below is formulated for the general, abstract systems in Definition II.1.

We now return to the particular case of the wave equation (II.4). However, later we shall need to use $L^2(\Gamma_1)$ as both input and output space rather than $\mathcal{W}'$ and $\mathcal{W}$. Fortunately, this can be achieved by restricting $(\mathfrak{B}\mathcal{H}, \mathfrak{A}\mathcal{H}, \mathfrak{C}\mathcal{H})$: Choose the new input space as $\mathcal{U} := L^2(\Gamma_1)$ and set

$$\mathcal{D}(\widetilde{\mathfrak{A}}) = \left\{ x \in \mathcal{H}^{-1}\mathcal{D}(\mathfrak{A}) \;\middle|\; \mathfrak{B}\mathcal{H}x \in L^2(\Gamma_1) \right\}$$

with the norm given by

$$\|x\|^2_{\mathcal{D}(\widetilde{\mathfrak{A}})} := \|\mathcal{H}x\|^2_{X_{\mathcal{H}}} + \|\mathcal{A}\mathcal{H}x\|^2_{X_{\mathcal{H}}} + \|\mathcal{B}\mathcal{H}x\|^2_{\mathcal{U}}.$$

Furthermore, we define the restrictions

$$\widetilde{\mathfrak{A}} := \mathfrak{A}\mathcal{H}\big|_{\mathcal{D}(\widetilde{\mathfrak{A}})},\ \widetilde{\mathfrak{B}} := \mathfrak{B}\mathcal{H}\big|_{\mathcal{D}(\widetilde{\mathfrak{A}})},\ \widetilde{\mathfrak{C}} := \iota\mathfrak{C}\mathcal{H}\big|_{\mathcal{D}(\widetilde{\mathfrak{A}})},$$

where $\iota : \mathcal{W} \to L^2(\Gamma_1)$ is again the (continuous) injection.

Theorem II.2. *The triple* $(\widetilde{\mathfrak{B}}, \widetilde{\mathfrak{A}}, \widetilde{\mathfrak{C}})$ *is a passive BCS on* $(\mathcal{U}, X_{\mathcal{H}}, \mathcal{U})$.

Proof. We first note that $\mathcal{N}(\widetilde{\mathfrak{B}}) = \mathcal{H}^{-1}\mathcal{N}(\mathfrak{B}) \subset \mathcal{D}(\widetilde{\mathfrak{A}})$ and then we prove that $\widetilde{\mathfrak{A}}\big|_{\mathcal{N}(\widetilde{\mathfrak{B}})}$ generates a unitary group on $X_{\mathcal{H}}$. It follows from [18, Cor. 3.4] that $\mathfrak{A}\big|_{\mathcal{N}(\mathfrak{B})}$ is a skew-adjoint, unbounded operator on $L^2(\Omega)^{n+1}$ and we will show that this implies that $\widetilde{\mathfrak{A}}\big|_{\mathcal{N}(\widetilde{\mathfrak{B}})}$ is skew-adjoint on $X_{\mathcal{H}}$. Indeed, for an arbitrary fixed $z \in X_{\mathcal{H}}$, there exists $w \in X_{\mathcal{H}}$ such that for all $x \in \mathcal{N}(\widetilde{\mathfrak{B}}) = \mathcal{H}^{-1}\mathcal{N}(\mathfrak{B})$ we have

$$\langle x, w\rangle_{X_{\mathcal{H}}} = \left\langle \widetilde{\mathfrak{A}}x, z\right\rangle_{X_{\mathcal{H}}} = \langle \mathfrak{A}\mathcal{H}x, \mathcal{H}z\rangle_{L^2(\Omega)^{n+1}} \tag{II.5}$$

if and only if $\mathcal{H}z \in \mathcal{D}(\mathfrak{A}\big|^*_{\mathcal{N}(\mathfrak{B})}) = \mathcal{N}(\mathfrak{B})$, where the adjoint is computed with respect to the inner product in $L^2(\Omega)^{n+1}$. Hence, $\widetilde{\mathfrak{A}}\big|_{\mathcal{N}(\widetilde{\mathfrak{B}})}$ has the same domain as its adjoint with respect to $X_{\mathcal{H}}$, and for every z in this common domain, (II.5) can be continued as

$$\left\langle \widetilde{\mathfrak{A}}x, z\right\rangle_{X_{\mathcal{H}}} = \langle \mathcal{H}x, -\mathfrak{A}\mathcal{H}z\rangle_{L^2(\Omega)^{n+1}} = \left\langle x, -\widetilde{\mathfrak{A}}z\right\rangle_{X_{\mathcal{H}}},$$

for all $x \in \mathcal{N}(\widetilde{\mathfrak{B}})$. By Stone's theorem, $\widetilde{\mathfrak{A}}$ generates a unitary group on $X_{\mathcal{H}}$.

As $\gamma_\perp$ maps $H^{\mathrm{div}}(\Omega)$ onto $\mathcal{W}'$, it is clear that $\widetilde{\mathfrak{B}}$ maps $\mathcal{D}(\widetilde{\mathfrak{A}})$ onto $\mathcal{U}$, and thus, $\widetilde{B} := \widetilde{\mathfrak{B}}^{[-1]} \in \mathcal{L}(\mathcal{U}, \mathcal{D}(\widetilde{\mathfrak{A}}))$ has the properties in Definition II.1.2. Moreover, the $\mathfrak{A}$-boundedness of $\mathfrak{C}$ and the fact that $\mathcal{H}\mathcal{D}(\widetilde{\mathfrak{A}})$ is continuously embedded in $\mathcal{D}(\mathfrak{A})$ imply that $\widetilde{\mathfrak{C}}\widetilde{B} \in \mathcal{L}(\mathcal{U}, \mathcal{W})$. Finally,

$$\mathrm{Re}\, \left\langle \widetilde{\mathfrak{A}}x, x\right\rangle_{X_{\mathcal{H}}} = \mathrm{Re}\, \left\langle \widetilde{\mathfrak{B}}x, \widetilde{\mathfrak{C}}x\right\rangle_{\mathcal{U}}, \qquad x \in \mathcal{D}\left(\widetilde{\mathfrak{A}}\right),$$

follows from the following integration by parts formula which was established in the appendix of [18], valid for all $h \in H^{\mathrm{div}}(\Omega)$ and $g \in H^1_{\Gamma_0}(\Omega)$:

$$\langle \operatorname{div} h, g\rangle_{L^2(\Omega)} + \langle h, \nabla g\rangle_{L^2(\Omega)^n} = \langle \gamma_\perp f, \gamma_0 g\rangle_{\mathcal{W}',\mathcal{W}};$$

recall that $\mathcal{W}'$ is the dual of $\mathcal{W}$ with pivot space $\mathcal{U}$ and that $\widetilde{\mathfrak{B}}x \in \mathcal{U}$ for $x \in \mathcal{D}(\widetilde{\mathfrak{A}})$. □

B. Exponential stabilization and admissible observation

The robust controller design in §IV involves exponential stabilization of the plant with output feedback, and in this section we will comment on this problem for the wave equation (II.1). We shall use a special case of a result by Guo and Yao [25] to obtain exponential stabilization using the so-called *multiplier method*. The case where all physical parameters are identity was covered also in [19], see [26], [27] for other related results.

In order to apply the multiplier method, we assume that the boundary $\partial\Omega$ is of class C^2 and that it is partitioned into the reflecting part Γ_0 and the input/output part Γ_1 in the following way (see [19, Chap. 7] for a longer discussion): Fix $\zeta^0 \in \mathbb{R}^n \setminus \overline{\Omega}$ arbitrarily and define $m(\zeta) := \zeta - \zeta^0$, $\zeta \in \mathbb{R}^n$. We assume that

$$\begin{aligned} &\Gamma_0 = \operatorname{int}\{\zeta \in \partial\Omega \mid m(\zeta)\cdot\nu(\zeta) \le 0\} \neq \emptyset \qquad \text{and} \\ &\Gamma_1 = \{\zeta \in \partial\Omega \mid m(\zeta)\cdot\nu(\zeta) > 0\} \neq \emptyset, \end{aligned} \tag{II.6}$$

and that the sets $\Gamma_0, \Gamma_1 \subset \partial\Omega$ form a partition of the boundary $\partial\Omega$ in the sense that $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \partial\Omega$. In our wave equation, we add a viscous damper $u = -b^2 y$ on Γ_1, where

$$b(\zeta)^2 := m(\zeta)\cdot\nu(\zeta), \qquad \zeta \in \Gamma_1. \tag{II.7}$$

This damper is rigorously interpreted as the following equation in $\mathcal{W}'$:

$$\gamma_\perp T\,\nabla w(t) = -b^2\,\gamma_0\,\dot w(t), \qquad t \ge 0.$$

In order to guarantee *exponential stability*, we do not need to explicitly make the common, but rather restrictive, assumption that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. However, combining the assumption that $\partial\Omega$ is of class C^2 with the assumption (II.2) that we need for the admissibility of velocity observation, we unfortunately seem to end up in a situation where necessarily $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

The total energy associated to a solution $x = \left[\begin{smallmatrix} g \\ h\end{smallmatrix}\right]$ of the wave equation in Thm II.2 at time t is

$$\frac{1}{2}\left\|\begin{bmatrix} g(t) \\ h(t)\end{bmatrix}\right\|^2_{X_\mathcal{H}} := \frac{1}{2}\int_\Omega \frac{1}{\rho(\zeta)}\,g(\zeta,t)^2 + h(\zeta,t)^* T(\zeta) h(\zeta,t)\,\mathrm{d}\zeta,$$

representing the sum of kinetic and potential energy.

Theorem II.3. *Assume that ρ and T are constant, that $\Omega \subset \mathbb{R}^n$ is a bounded C^2-domain with $n \le 3$, and that Γ_k satisfy* (II.6). *Then there exist $c > 1$ and $\omega > 0$, such that all $\left[\begin{smallmatrix} g \\ h\end{smallmatrix}\right] \in C^1(\mathbb{R}_+; X_\mathcal{H})$ with $\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{smallmatrix} g(t) \\ h(t)\end{smallmatrix}\right] = \mathcal{A}\mathcal{H}\left[\begin{smallmatrix} g(t) \\ h(t)\end{smallmatrix}\right]$ and $\gamma_\perp T\,h(t) = -(m\cdot\nu)\,\gamma_0\,g(t)$ for $t \ge 0$, and $h(0) \in \nabla H^1_{\Gamma_0}(\Omega)$, satisfy*

$$\left\|\begin{bmatrix} g(t) \\ h(t)\end{bmatrix}\right\|^2_{X_\mathcal{H}} \le c\,e^{-\omega t}\left\|\begin{bmatrix} g(0) \\ h(0)\end{bmatrix}\right\|^2_{X_\mathcal{H}}, \qquad t \ge 0. \tag{II.8}$$

Proof. Let $\left[\begin{smallmatrix} g \\ h\end{smallmatrix}\right]$ have the properties in the statement and let $\eta \in H^1_{\Gamma_0}(\Omega)$ be such that $\nabla\eta = h(0)$. Setting

$$w(t) := \eta + \frac{1}{\rho}\int_0^t g(s)\,\mathrm{d}s, \qquad t \ge 0, \tag{II.9}$$

we get that $\dot w(t) = g(t)/\rho$ and $\nabla w(t) = h(t)$ for all $t \ge 0$. Moreover, w is a classical solution of the wave equation since

$$\ddot w(t) = \operatorname{div}\left(\frac{T}{\rho}\,\nabla w\right)(t), \qquad t \ge 0, \tag{II.10}$$

with the left-hand side in $C\big(\mathbb{R}_+; L^2(\Omega)\big)$. Note that the constant matrix T/ρ is positive definite and hence invertible.

In [28, Ex. 3.1], a Riemannian manifold $(\mathbb{R}^n, g)$ is associated to (II.10), and it is concluded that the vector field $H := \sum_{k=1}^n (\zeta_k - \zeta_k^0)\,\partial/\partial\xi_k$ on this manifold satisfies the condition [25, (3.2)] with $a = 1$ (here ζ_k is coordinate number k of ζ). We further observe that $w(t) \in H^1_{\Gamma_0}(\Omega)$ and $\gamma_\perp T\nabla w(t) = -(m\cdot\nu)\,\gamma_0\,\dot w(t)$ for all $t \ge 0$, while $w(0) \in H^1_{\Gamma_0}(\Omega)$, and $\dot w(0) \in L^2(\Omega)$. By [25, Thm 1], we have (II.8). □

In general, a solution w of (II.3) is only required to be *constant* on Γ_0. The condition $h(0) \in \nabla H^1_{\Gamma_0}(\Omega)$ corresponds to the initial condition $w(0) \in H^1_{\Gamma_0}(\Omega)$ via (II.9), and this implies the stronger statement that w is constantly equal to *zero* on Γ_0. This is one way to guarantee that the potential energy decays to zero.

Returning to the case of the general BCS, we will replace the multiplication by $-m\cdot\nu$ on $L^2(\Gamma_1)$ by an admissible output feedback operator $Q \in \mathcal{L}(\mathcal{Y},\mathcal{U})$ which stabilizes the given BCS exponentially: Let $(\mathcal{B},\mathcal{A},\mathcal{C})$ be a BCS on (U,X,Y). We call $Q \in \mathcal{L}(Y,U)$ an *admissible (static output) feedback operator* for $(\mathcal{B},\mathcal{A},\mathcal{C})$ if $(\mathcal{B}+Q\mathcal{C},\mathcal{A},\mathcal{C})$ is a also a BCS. Moreover, let the Hilbert spaces Y and Y' be duals with some pivot Hilbert space $\widetilde{U}$, and let $Q \in \mathcal{L}(Y,Y')$. We say that Q is *uniformly accretive* if there exists some $\delta > 0$ such that

$$\operatorname{Re}\,\langle Qy, y\rangle_{Y',Y} \ge \delta\,\|y\|^2_{\widetilde{U}}, \qquad y \in Y.$$

By an *admissible observation operator* for a C_0-semigroup $\mathbb{T}$ on X with generator A, we mean a linear operator $C \in \mathcal{L}(\mathcal{D}(A),Y)$ for which there exist some $\tau > 0$ and $K_\tau \ge 0$ such that

$$\int_0^\tau \|C\,\mathbb{T}(t)\,x\|^2_Y\,\mathrm{d}t \le K_\tau^2\,\|x\|^2_X, \qquad \forall x \in \mathcal{D}(A). \tag{II.11}$$

If (II.11) holds for some $\tau > 0$ and $K_\tau \ge 0$, then for every $\tau > 0$ it is possible to choose a $K_\tau \ge 0$ such that (II.11) holds. The observation operator is *infinite-time admissible* if (II.11) holds for all $\tau > 0$ with K_τ replaced by some bound K which is independent of τ. In particular, if the semigroup $\mathbb{T}$ is exponentially stable, then every admissible observation operator is infinite-time admissible [19, Prop. 4.3.3].

Proposition II.4. *Let $(\mathcal{B},\mathcal{A},\mathcal{C})$ be a* passive *BCS on (Y',X,Y) and let $Q \in \mathcal{L}(Y,Y')$ be a uniformly accretive, admissible output feedback operator for $(\mathcal{B},\mathcal{A},\mathcal{C})$. The resulting BCS $(\mathcal{B}+Q\mathcal{C},\mathcal{A},\mathcal{C})$ is also passive and we denote its associated semigroup by $\mathbb{T}_Q$. The observation operator $\mathcal{C}$,* *interpreted as an operator mapping into the pivot space $\widetilde{Y}$ rather than into Y, is infinite-time admissible for $\mathbb{T}_Q$.*

Proof. By the definitions of admissible feedback operator and BCS, it follows that $(\mathcal{B}+Q\mathcal{C},\mathcal{A},\mathcal{C})$ is a BCS on $(Y',X,\widetilde{Y})$, and by definition the generator of $\mathbb{T}_Q$ is $A_Q := \mathcal{A}\big|_{\mathcal{N}(\mathcal{B}+Q\mathcal{C})}$.

For a fixed $x_0 \in \mathcal{D}(A_Q)$, the associated state trajectory $x(t) = \mathbb{T}_Q(t)\,x_0$ stays in $\mathcal{D}(A_Q)$, and by the assumed passivity, for all $t \geq 0$ we have

$$\begin{aligned}\mathrm{Re}\,\langle \mathcal{A}x(t), x(t)\rangle_X &\leq \mathrm{Re}\,\langle \mathcal{B}x(t), \mathcal{C}x(t)\rangle_{Y',Y}\\ &= -\mathrm{Re}\,\langle Q\mathcal{C}x(t), \mathcal{C}x(t)\rangle_{Y',Y}\,.\end{aligned}$$

Multiplying this by 2 and integrating over $[0,\tau]$, we get

$$\begin{aligned}\|x(\tau)\|_X^2 - \|x(0)\|_X^2 &= \int_0^\tau 2\mathrm{Re}\,\langle A_Q x(t), x(t)\rangle_X \,\mathrm{d}t\\ &\leq -2\delta\int_0^\tau \|\mathcal{C}x(t)\|_{\widetilde{Y}}^2\,\mathrm{d}t.\end{aligned}$$

Letting $\tau \to +\infty$, we obtain that $\mathcal{C}$ is infinite-time admissible, since

$$\int_0^\infty \|\mathcal{C}\mathbb{T}_Q(t)x_0\|_{\widetilde{Y}}^2\,\mathrm{d}t \leq \frac{1}{2\delta}\|x_0\|_X^2, \quad x_0 \in \mathcal{D}(A_Q).$$

□

We end the section by discussing the wave system as an example for the above abstract definitions. It is clear that the multiplication by $b^2 = m\cdot\nu$ in (II.7) is a bounded operator on $L^2(\Gamma_1)$, and hence it is also in $\mathcal{L}(\mathcal{W},\mathcal{W}')$ and it is uniformly accretive if (II.2) holds. Furthermore, multiplication by b^2 is an admissible feedback operator for the wave system in (II.4) and for its restriction in Thm II.2. Indeed, $\mathcal{N}(\mathfrak{B}\mathcal{H} + b^2\,\mathfrak{C}\mathcal{H}) = \mathcal{N}(\widetilde{\mathfrak{B}} + b^2\,\widetilde{\mathfrak{C}}) \subset \mathcal{D}(\widetilde{\mathfrak{A}})$, by [18, Thm 3.5] the operator $\mathfrak{A}\mathcal{H}\big|_{\mathcal{N}(\mathfrak{B}\mathcal{H}+b^2\,\mathfrak{C}\mathcal{H})} = \widetilde{\mathfrak{A}}\big|_{\mathcal{N}(\widetilde{\mathfrak{B}}+b^2\,\widetilde{\mathfrak{C}})}$ generates a contraction semigroup on $X_{\mathcal{H}}$, and the operators

$$\mathfrak{B}\mathcal{H} + b^2\mathfrak{C}\mathcal{H} = \begin{bmatrix} b^2\gamma_0 & \gamma_\perp\end{bmatrix}\mathcal{H} \qquad \text{and} \qquad \widetilde{\mathfrak{B}} + b^2\,\widetilde{\mathfrak{C}}$$

are continuous and surjective; hence they have right-inverses with the properties required in Definition II.1.

III. The plant, the controller, and the exosystem

In the next section, we solve the robust output regulation problem for a general BCS $(\mathcal{B},\mathcal{A},\mathcal{C})$ on the Hilbert spaces (U,X,Y); the system is not necessarily related to the wave equation. In the following we assume that the whole boundary $\partial\Omega$ is accessible via $\mathcal{B}$ and R_1, R_2 are arbitrary restrictions to certain parts of $\partial\Omega$. We first add an external disturbance w to the BCS, thus obtaining the plant

$$\begin{cases}\dot{x}(t) = \mathcal{A}x(t), \qquad x(0) = x_0,\\ \mathcal{B}x(t) = R_1u(t) + R_2w(t), \qquad t\geq 0,\\ \mathcal{C}x(t) = y(t),\end{cases} \tag{III.1}$$

where u and w may act on different parts of the boundary depending on R_1 and R_2.

In what follows, Q is such that R_1Q is an admissible static output feedback operator for (III.1) such that the semigroup $\mathbb{T}_s$ generated by $A_s := \mathcal{A}\big|_{\mathcal{D}(\mathcal{A})\cap\mathcal{N}(\mathcal{B}+R_1Q\mathcal{C})}$ is exponentially stable and $\mathcal{C}$ is an admissible observation operator for $\mathbb{T}_s$ (here the subscript 's' stands for "stabilized plant").

We will connect the plant to the dynamic controller

$$\begin{cases}\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2(y(t) - y_{ref}(t)), \quad z(0) = z_0\\ u(t) = Kz(t) - Q(y(t) - y_{ref}(t)), \qquad t\geq 0,\end{cases} \tag{III.2}$$

where y_{ref} is an external reference signal and the state space Z of the controller is a Hilbert space, but $\mathcal{G}_1 \in \mathcal{L}(Z)$ is bounded. Moreover, we assume that $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$, $K \in \mathcal{L}(Z,U)$ and $Q \in \mathcal{L}(Y,U)$. The disturbance signal w and the reference signal y_{ref} are assumed to be generated by an exosystem

$$\begin{cases}\dot{v}(t) = Sv(t), \quad v(0) = v_0,\\ w(t) = Ev(t), \qquad t\geq 0,\\ y_{ref}(t) = -Fv(t),\end{cases} \tag{III.3}$$

which is a linear system on a finite-dimensional space $W = \mathbb{C}^q$, $q\in\mathbb{N}$. We assume that $S = \mathrm{diag}(i\omega_1, i\omega_2, \ldots, i\omega_q)$ with $\omega_i \neq \omega_j$ for $i\neq j$, $E\in\mathcal{L}(W,U)$ and $F\in\mathcal{L}(W,Y)$.

Setting u and y equal in (III.1) and (III.2), and using (III.3), we obtain

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x\\ z\end{bmatrix} = \begin{bmatrix}\mathcal{A} & 0\\ \mathcal{G}_2\mathcal{C} & \mathcal{G}_1\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + \begin{bmatrix}0\\ \mathcal{G}_2F\end{bmatrix}v,\\ (R_2E - R_1QF)v = \begin{bmatrix}\mathcal{B}+R_1Q\mathcal{C} & -R_1K\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix},\\ e = \begin{bmatrix}\mathcal{C} & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + Fv,\end{cases} \tag{III.4}$$

where we chose the regulation error $e(t) =: y(t) - y_{ref}(t)$ as the output and the state-space is $X_e := X\times Z$. This system is no longer a BCS and we now proceed to write it in the standard input/state/output form. First we observe that we may interpret the feedthrough Q of the controller as a part of the plant without changing (III.4). This amounts to pre-stabilizing the plant via replacing the input equation of (III.1) by $(\mathcal{B}+R_1Q\mathcal{C})x(t) = R_1u(t) + (R_2E - R_1QF)v(t)$ and simultaneously removing the term $-Q(y(t)-y_{ref}(t))$ from the output equation of (III.2).

As R_1Q is assumed to be an admissible feedback operator, the pre-stabilized plant $(\mathcal{B}+R_1Q\mathcal{C},\mathcal{A},\mathcal{C})$ is a BCS and by Def. II.1.2, we can choose a right inverse $B_s\in\mathcal{L}(U,X)$ of $\mathcal{B}+R_1Q\mathcal{C}$ such that

$$B_sR_1U\subset\mathcal{D}(\mathcal{A}),\ \mathcal{A}B_sR_1\in\mathcal{L}(U,X),\ \mathcal{C}B_sR_1\in\mathcal{L}(U,Y). \tag{III.5}$$

In order to present the transfer function of $(\mathcal{B}+R_1Q\mathcal{C},\mathcal{A},\mathcal{C})$, consider the auxiliary function

$$P_0(\lambda) := \mathcal{C}(\lambda - A_s)^{-1}(\mathcal{A}B_s - \lambda B_s) + \mathcal{C}B_s, \quad \lambda\in\rho(A_s)\,.$$

Now, define the transfer function by

$$P_s(\lambda) := P_0(\lambda)R_1, \qquad \lambda\in\rho(A_s)\,. \tag{III.6}$$

The auxiliary function P_0 becomes useful later on in describing the mapping from v to y.

Now let $\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right]$ be a classical state trajectory of (III.4), i.e., $\left[\begin{smallmatrix}x\\ z\end{smallmatrix}\right]\in C^1(\mathbb{R}_+;X_e)$, $\mathcal{G}_2y_{ref}\in C(\mathbb{R}_+;Z)$, $(\mathcal{B}+R_1Q\mathcal{C})x\in C(\mathbb{R}_+;U)$, $w\in C^1(\mathbb{R}_+;U)$, and the first two lines of (III.4) hold for all $t \geq 0$. Next introduce a new state variable for (III.4) by

$$x_e := \begin{bmatrix} 1 & -B_s R_1 K \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} - \begin{bmatrix} B_s E_s v \\ 0 \end{bmatrix} \in C^1(\mathbb{R}_+; X_e),$$

where we denote $E_s := R_2 E - R_1 Q F$ for brevity. This transformation can be inverted as

$$\begin{bmatrix} x \\ z \end{bmatrix} := \begin{bmatrix} 1 & B_s R_1 K \\ 0 & 1 \end{bmatrix} x_e + \begin{bmatrix} B_s E_s v \\ 0 \end{bmatrix}. \quad \text{(III.7)}$$

Differentiating x_e and using the first line of (III.4), we get

$$\dot{x}_e = \begin{bmatrix} \mathcal{A} - B_s R_1 K \mathcal{G}_2 \mathcal{C} & \mathcal{A} B_s R_1 K - B_s R_1 K \tilde{\mathcal{G}}_1 \\ \mathcal{G}_2 \mathcal{C} & \tilde{\mathcal{G}}_1 \end{bmatrix} x_e + \begin{bmatrix} \mathcal{A} B_s E_s - B_s E_s S - B_s R_1 K \mathcal{G}_2 (\mathcal{C} B_s E_s + F) \\ \mathcal{G}_2 (\mathcal{C} B_s E_s + F) \end{bmatrix} v,$$

where we denote $\tilde{\mathcal{G}}_1 := \mathcal{G}_1 + \mathcal{G}_2 \mathcal{C} B_s R_1 K$ for brevity.

With the new state variable, the input equation of (III.4) becomes

$$E_s v = \begin{bmatrix} \mathcal{B} + R_1 Q \mathcal{C} & -K \end{bmatrix} \left(x_e + \begin{bmatrix} B_s \\ 0 \end{bmatrix} (R_1 K z + E_s v) \right)$$

which simplifies to $x_e \in \mathcal{N}\left(\begin{bmatrix} \mathcal{B} + R_1 Q \mathcal{C} & 0 \end{bmatrix}\right)$. Hence recalling that $A_s = \mathcal{A}|_{\mathcal{D}(\mathcal{A}) \cap \mathcal{N}(\mathcal{B} + R_1 Q \mathcal{C})}$ and defining

$$A_e := \begin{bmatrix} A_s - B_s R_1 K \mathcal{G}_2 \mathcal{C} & \mathcal{A} B_s R_1 K - B_s R_1 K \tilde{\mathcal{G}}_1 \\ \mathcal{G}_2 \mathcal{C} & \tilde{\mathcal{G}}_1 \end{bmatrix} \Big|_{\mathcal{D}(A_e)},$$
$$\mathcal{D}(A_e) := \mathcal{N}(\mathcal{B} + R_1 Q \mathcal{C}) \times Z, \quad \text{(III.8)}$$

we get that every classical solution of (III.4) satisfies $x_e(t) \in \mathcal{D}(A_e)$ for all $t \geq 0$ and $\dot{x}_e = A_e x_e + B_e v$, where the control operator $B_e \in \mathcal{L}(W, X_e)$ is

$$B_e := \begin{bmatrix} \mathcal{A} B_s E_s - B_s E_s S - B_s R_1 K \mathcal{G}_2 (\mathcal{C} B_s E_s + F) \\ \mathcal{G}_2 (\mathcal{C} B_s E_s + F) \end{bmatrix}.$$

Finally, using (III.7) the output for (III.4) becomes

$$e = \begin{bmatrix} \mathcal{C} & \mathcal{C} B_s R_1 K \end{bmatrix} x_e + (\mathcal{C} B_s E_s + F) v.$$

Thus, the closed-loop system is of the form

$$\begin{cases} \dot{x}_e = A_e x_e + B_e v, \\ e = C_e x_e + D_e v, \end{cases} \quad \text{(III.9)}$$

where

$$C_e := \begin{bmatrix} \mathcal{C} & \mathcal{C} B_s R_1 K \end{bmatrix}, \qquad \mathcal{D}(C_e) := \begin{bmatrix} \mathcal{D}(\mathcal{C}) \\ Z \end{bmatrix}, \qquad \text{and}$$
$$D_e := \mathcal{C} B_s E_s + F \in \mathcal{L}(W, Y).$$

We denote the transfer function of (III.9) from v to e with

$$P_e(\lambda) = C_e (\lambda - A_e)^{-1} B_e + D_e.$$

The above calculations show that every classical solution of (III.4) with $v \in C(\mathbb{R}_+; W)$ is also a classical solution of (III.9). Conversely, assume that $x_e \in C^1(\mathbb{R}_+; X_e)$ with $x_e(t) \in \mathcal{D}(A_e)$, $v \in C(\mathbb{R}_+; W)$ and (III.9) holds on $\mathbb{R}_+$. Then v, $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]$ in (III.7) and e satisfy (III.4). We conclude that (III.4) and (III.9) are equivalent systems in the sense that they have the same classical solutions.

The following result forms the basis for the output regulation theory in the next section. Note that we do not assume that the original plant (III.1) is well-posed or regular, but the closed-loop system (III.9) nevertheless has these properties.

Theorem III.1. *The operator A_e in* (III.8) *generates a C_0-semigroup $\mathbb{T}_e$ on X_e and C_e is an admissible observation operator for $\mathbb{T}_e$. The closed-loop system* (III.9) *is well-posed and regular such that $P_e(\lambda) \to D_e$ as $\operatorname{Re} \lambda \to \infty$.*

Proof. We begin by splitting $A_e = A_1 + A_2 + A_3$, where

$$A_1 = \begin{bmatrix} A_s & 0 \\ 0 & \mathcal{G}_1 \end{bmatrix}, \qquad \mathcal{D}(A_1) = \mathcal{D}(A_e),$$
$$A_2 = \begin{bmatrix} -B_s R_1 K \mathcal{G}_2 \mathcal{C} & 0 \\ \mathcal{G}_2 \mathcal{C} & 0 \end{bmatrix}, \qquad \mathcal{D}(A_2) = \mathcal{D}(A_e),$$
$$A_3 = \begin{bmatrix} 0 & \mathcal{A} B_s R_1 K - B_s R_1 K (\mathcal{G}_1 + \mathcal{G}_2 \mathcal{C} B_s R_1 K) \\ 0 & \mathcal{G}_2 \mathcal{C} B_s R_1 K \end{bmatrix},$$
$$\mathcal{D}(A_3) = X_e.$$

Here A_1 generates a C_0-semigroup $\mathbb{T}_1$ on X_e. The operator A_2 can be factored as

$$A_2 = \begin{bmatrix} -B_s R_1 K \mathcal{G}_2 \\ \mathcal{G}_2 \end{bmatrix} \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix},$$

where the first factor is bounded from Y into X_e. Our assumption that $\mathcal{C}$ is admissible for $\mathbb{T}_s$ implies that $\begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} : X_e \supset \mathcal{D}(A_e) \to Y$ is an admissible observation operator for $\mathbb{T}_1$, and by [19, Thm 5.4.2], $A_1 + A_2$ generates a C_0-semigroup $\mathbb{T}_2$ on X_e and $\begin{bmatrix} \mathcal{C} & 0 \end{bmatrix}$ is admissible for $\mathbb{T}_2$. Since A_3 is bounded, A_e generates a C_0-semigroup by [19, Thm 5.4.2] and due to the boundedness of $\mathcal{C} B_s R_1 K$, C_e is admissible for $\mathbb{T}_e$. As in addition B_e and D_e are bounded, the well-posedness and regularity of the closed-loop system follow immediately from [19, Thm 4.3.7] □

IV. Output Regulation

We begin this section by presenting the three output regulation problems considered in this paper. The structure for the remainder of this section will be presented after the problem definitions.

The Output Regulation Problem. For a given plant (III.1), choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ in (III.2) in such a way that the following are satisfied:

1) The closed-loop system generated by A_e is exponentially stable.
2) For all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\alpha \cdot} e(\cdot) \in L^2([0, \infty); Y)$ for some $\alpha > 0$ independent of $x_{e0} \in X_e$ and $v_0 \in W$.

Furthermore, if the controller solves the output regulation problem despite perturbations in the parameters of the plant or the exosystem, then we say that the controller solves the *robust* output regulation problem with respect to this class of perturbations. To make this precise, we first define the class of admissible perturbations:

Definition IV.1. A quintuple $(\mathcal{A}', \mathcal{B}', \mathcal{C}', E', F')$ of linear operators belongs to the *class $\mathcal{O}$ of admissible perturbations* if it has the following properties:

1) The triple $(\mathcal{B}' + R_1 Q\mathcal{C}', \mathcal{A}', \mathcal{C}')$ is a BCS on (U, X, Y).
2) The observation operator $\mathcal{C}'$ is admissible for the semigroup generated by $A'_s := \mathcal{A}'|_{\mathcal{N}(\mathcal{B}'+R_1Q\mathcal{C}')}$.
3) The eigenvalues of S are in the resolvent set of the perturbed pre-stabilized plant, i.e., $\{i\omega_k\}_{k=1}^q \subset \rho(A'_s)$.
4) $E' \in \mathcal{L}(W, U)$ and $F' \in \mathcal{L}(W, Y)$.

In the above definition it would appear that the class $\mathcal{O}$ of perturbations depends on Q. However, as Q only contributes to stabilizing the plant, we have much more freedom choosing Q than choosing the other controller parameters (as seen later on). For example, in the wave equation considered in Section II, any uniformly accretive operator can be chosen as Q. Therefore, in Definition IV.1, one could think of Q being chosen such that the class $\mathcal{O}$ is as large as possible. Moreover, if $(\mathcal{A}', \mathcal{B}', \mathcal{C}', E', F') \in \mathcal{O}$ then the transfer function (III.6) of the triple $(\mathcal{B}' + R_1 Q\mathcal{C}', \mathcal{A}', \mathcal{C}')$ is well-defined and bounded at the frequencies of the exosystem.

We make the natural assumption that the unperturbed system is in class $\mathcal{O}$ as well, that is, $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F) \in \mathcal{O}$. Note that this does not include the assumption that the semigroup generated by A_s is exponentially stable. Further note that even though $(\mathcal{B}, \mathcal{A}, \mathcal{C})$ is assumed to be a BCS, that is not required from $(\mathcal{B}', \mathcal{A}, \mathcal{C}')$ but only from $(\mathcal{B}' + R_1 Q\mathcal{C}', \mathcal{A}', \mathcal{C}')$.

From Definition IV.1 it follows that the perturbed closed-loop system is well-posed and regular. Please note that while no perturbations are allowed in the eigenvalues of the generator S of the exosystem or in the controller parameter $\mathcal{G}_1$, the parameters $(\mathcal{G}_2, K, Q)$ would in fact allow certain bounded perturbations. We will comment on this more thoroughly in Remark IV.9.

The Robust Output Regulation Problem. For a given plant, choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ in such a way that the following are satisfied:

1) The controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ solves the output regulation problem.
2) If the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed to $(\mathcal{A}', \mathcal{B}', \mathcal{C}', E', F') \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\alpha' \cdot} e(\cdot) \in L^2([0, \infty); Y)$ for some $\alpha' > 0$ independent of $x_{e0} \in X_e$ and $v_0 \in W$.

In Section IV-C, we will construct a controller that solves the robust output regulation problem *approximately*. That is, the regulation error does not decay asymptotically to zero but can be made small. For this purpose, we introduce the following new control problem:

The Approximate Robust Output Regulation Problem. Let $\delta > 0$ be given. Choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ in such a way that the following are satisfied:

1) The closed-loop system generated by A_e is exponentially stable.
2) For all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies
$$\int_t^{t+1} \|e(s)\|^2 \,\mathrm{d}s \le M e^{-\alpha t}(\|x_{e0}\|^2 + \|v_0\|^2) + \delta \|v_0\|^2$$
for some $M, \alpha > 0$ independent of $x_{e0} \in X_e, v_0 \in W$.
3) If the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed to $(\mathcal{A}', \mathcal{B}', \mathcal{C}', E', F') \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then there exists a $\delta' > 0$ such that for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies
$$\int_t^{t+1} \|e(s)\|^2 \,\mathrm{d}s \le M' e^{-\alpha' t}(\|x_{e0}\|^2 + \|v_0\|^2) + \delta' \|v_0\|^2$$
for some $M', \alpha' > 0$ independent of x_{e0}, v_0.

Remark IV.2. The approximate robust output regulation problem formulation implies that, in the absence of perturbations, the asymptotic regulation error must be smaller than $\delta\|v_0\|^2$ for any given (or in practice chosen) $\delta > 0$. However, when perturbations are present, the asymptotic regulation error is merely bounded by $\delta'\|v_0\|^2$. For details, see Theorem IV.11, (IV.14)–(IV.15) and the discussion therein.

Now that we have presented the different output regulation problems to be considered, the structure of the remaining section is as follows. Before proceeding to constructing the controllers, we will present two auxiliary results to be used throughout the remainder of this section. In §IV-A we present a regulating controller without the robustness requirement, in §IV-B we present the internal model principle for boundary control systems, in §IV-C we present an approximate robust controller, and finally in §IV-D we present a precise robust controller.

The following auxiliary result is a consequence of [15, Thm 4.1] under the assumption that the closed-loop system (III.9) is a regular linear system. The result states that the solvability of the *regulator equations*

$$\Sigma S = A_e \Sigma + B_e \tag{IV.1a}$$
$$0 = C_e \Sigma + D_e \tag{IV.1b}$$

is equivalent to the solvability of the output regulation problem. The result of [15, Thm 4.1] essentially follows from [15, Lem. 4.3] by which the regulation error can be written as

$$e(t) = C_e \mathbb{T}_e(t)(x_{e0} - \Sigma v_0) + (C_e \Sigma + D_e) v(t),$$

where the first part decays to zero at an exponential rate provided that $\mathbb{T}_e$ is exponentially stable, C_e is an admissible observation operator for $\mathbb{T}_e$ and Σ is the solution of (IV.1a).

Theorem IV.3. *Assume that the closed-loop system is regular and exponentially stabilized by a controller* $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$. *Then the controller solves the output regulation problem if and only if the regulator equations* (IV.1) *have a solution* $\Sigma \in \mathcal{L}(W, X_e)$. *The solution* Σ *is unique when it exists.*

Proof. We first note that the feedthrough term $-Qe(t)$ in the controller is not part of the controller in [15, Thm 4.1]. However, as in (III.4) we can interpret the feedthrough Q as a part of the plant (III.1) and simultaneously remove it from the controller (III.2), so that the input equation becomes $(\mathcal{B} + R_1 Q\mathcal{C})x(t) = R_1 u(t) + R_1 Q y_{ref}(t) + R_2 w(t)$. The closed-loop system is unaffected by this algebraic trick, and hence, we may continue with a pre-stabilized plant and the same controller structure as in [15, Thm 4.1].

Now the result follows from [15, Thm 4.1] as an exponentially stable semigroup is also strongly stable, and for A_e being the generator of an exponentially stable semigroup and $\sigma(S) \subset i\mathbb{R}$ the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ always has a unique solution $\Sigma \in \mathcal{L}(W, X_e)$ by [29, Cor. 8]. Furthermore, the exponential decay of the regulation error follows from the assumed exponential stability of the closed-loop system. □

Theorem IV.3 assumes that the controller exponentially stabilizes the closed-loop system. We will therefore need to show that the controllers we present in Proposition IV.6, Theorem IV.11 and Corollary IV.14 have this property. For this, we present the following tool which uses the notation of §III. Here we need to assume that there exists an operator Q as described in the following:

Lemma IV.4. *Let $Z = Y_N^q$, where Y_N is equal to $\mathbb{C}$ or a closed subspace of Y. Choose the controller parameter $Q \in \mathcal{L}(Y, U)$ such that the semigroup $\mathbb{T}_s$ generated by A_s is exponentially stable and $\mathcal{C}$ is an admissible observation operator for $\mathbb{T}_s$. Choose the remaining parameters as*

$$\mathcal{G}_1 = \operatorname{diag}(i\omega_1 I, i\omega_2 I, \dots, i\omega_q I) \in \mathcal{L}(Z),$$
$$K = \epsilon K_0 = \epsilon[K_0^1, K_0^2, \dots, K_0^q] \in \mathcal{L}(Z, U),$$
$$\mathcal{G}_2 = (\mathcal{G}_2^k P_N)_{k=1}^q \in \mathcal{L}(Y, Z),$$

where I is the identity in Y_N, and P_N is a projection onto Y_N in Y if $Y_N \subset Y$ or the identity on Y otherwise. Additionally, assume that $\mathcal{G}_2^k$ and K_0^k satisfy $\sigma(\mathcal{G}_2^k P_N P_s(i\omega_k) K_0^k) \subset \mathbb{C}_-$ for all $k \in \{1, 2, \dots, q\}$.

Then there exits an $\epsilon^* > 0$ such that the closed-loop system (III.9) *is exponentially stable for all $0 < \epsilon < \epsilon^*$.*

Proof. Define the operator $H = (H_1, H_2, \dots, H_q) \in \mathcal{L}(Z, X)$ by choosing

$$H_k := (i\omega_k - A_s)^{-1}(\mathcal{A}B_s - i\omega_k B_s)R_1 K_0^k$$

for all $k \in \{1, 2, \dots, q\}$. By the choice of H_k we have $(i\omega_k - A_s)H_k = \mathcal{A}B_s R_1 K_0^k - i\omega_k B_s R_1 K_0^k$, i.e., $H_k i\omega_k = A_s H_k + \mathcal{A}B_s R_1 K_0^k - B_s R_1 K_0^k i\omega_k$, and thus, $H\mathcal{G}_1 = A_s H + \mathcal{A}B_s R_1 K_0 - B_s R_1 K_0 \mathcal{G}_1$ due to the diagonal structure of $\mathcal{G}_1$. Define

$$R = \begin{bmatrix} -1 & \epsilon H \\ 0 & 1 \end{bmatrix} = R^{-1} \in \mathcal{L}(X_e)$$

and denote $\hat{A}_e = RA_eR^{-1}$. Note that as $\mathcal{R}(H) \subset \mathcal{N}(\mathcal{B} + R_1 Q\mathcal{C})$, it follows that $\mathcal{D}(\hat{A}_e) = \mathcal{D}(A_e)$. Using the above identity we can write $\hat{A}_e$ as

$$\hat{A}_e = \begin{bmatrix} A_s - \epsilon\tilde{H}\mathcal{G}_2\mathcal{C} & 0 \\ -\mathcal{G}_2\mathcal{C} & \mathcal{G}_1 + \epsilon\mathcal{G}_2\mathcal{C}\tilde{H} \end{bmatrix} + \epsilon^2 \begin{bmatrix} 0 & \tilde{H}\mathcal{G}_2\mathcal{C}\tilde{H} \\ 0 & 0 \end{bmatrix}.$$

where we denote $\tilde{H} := H + B_s R_1 K_0$ for brevity.

In the remaining part of the proof we apply the Gearhart-Prüss-Greiner Theorem in [30, Thm V.1.11]. More precisely, we will show that the resolvent of $\hat{A}_e$ is uniformly bounded on the closed right-half plane. We first note that since $\mathcal{C}$ is admissible for $\mathbb{T}_s$ which is exponentially stable, we have by [19, Thm 4.3.7] that $\mathcal{C}(\lambda - A_s)^{-1}$ is uniformly bounded for all $\lambda \in \overline{\mathbb{C}}_+$. Thus, as $\tilde{H}\mathcal{G}_2$ is bounded, there exists an $M_0 > 0$ such that $\|\tilde{H}\mathcal{G}_2\mathcal{C}(\lambda - A_s)^{-1}\| \le M_0$, and for $0 < \epsilon < M_0^{-1}$ a Neumann series expansion implies that $1 + \epsilon\tilde{H}\mathcal{G}_2\mathcal{C}(\lambda - A_s)^{-1}$ is invertible. Thus, we obtain that

$$(\lambda - A_s + \epsilon\tilde{H}\mathcal{G}_2\mathcal{C})^{-1} = (\lambda - A_s)^{-1}(1 + \epsilon\tilde{H}\mathcal{G}_2\mathcal{C}(\lambda - A_s)^{-1})^{-1}$$

is uniformly bounded in the right half plane. Hence, the semigroup generated by $A_s - \epsilon\tilde{H}\mathcal{G}_2\mathcal{C}$ is exponentially stable by [30, Thm V.1.11].

Note that by the choice of H_k we have

$$\begin{aligned} &\mathcal{C}(H_k + B_s R_1 K_0^k) \\ &= \mathcal{C}(i\omega_k - A_s)^{-1}(\mathcal{A}B_s - i\omega_k B_s)R_1 K_0^k + \mathcal{C}B_s R_1 K_0^k \\ &= P_s(i\omega_k)K_0^k, \end{aligned}$$

and thus $\sigma(\mathcal{G}_2^k P_N \mathcal{C}(H_k + B_s R_1 K_0^k)) \subset \mathbb{C}_-$ by the assumption made on $\mathcal{G}_2^k$ and K_0^k. Furthermore, since $\sigma(\mathcal{G}_1) = \{i\omega_k\}_{k=1}^q$, the operator $\mathcal{G}_1 + \epsilon\mathcal{G}_2\mathcal{C}\tilde{H}$ satisfies the stability conditions of the operator $A_c - \epsilon\tilde{P}K$ in [31, Appendix B]. Hence, by [31, Appendix B] there exist constants $M_1, \beta > 0$ such that for all $\epsilon > 0$ sufficiently small we have $\|\mathbb{T}_2(t)\| \le M_1 e^{-\epsilon\beta t}$ for $t \ge 0$, where $\mathbb{T}_2$ is the semigroup generated by $\mathcal{G}_1 + \epsilon\mathcal{G}_2\mathcal{C}\tilde{H}$. This further implies that

$$\|(\lambda - \mathcal{G}_1 + \epsilon\mathcal{G}_2\mathcal{C}\tilde{H})^{-1}\| \le \frac{M_1}{\epsilon\beta}, \qquad \lambda \in \overline{\mathbb{C}}_+.$$

Consider the operator $\hat{A}_e$ in the form $A_1 + \epsilon^2 A_2$. Since we have shown that the diagonal operators of A_1 generate exponentially stable semigroups and since $\mathcal{C}$ is admissible for A_s, it follows that A_1 is the generator of an exponentially stable semigroup. Furthermore, there exists an $M_2 > 0$ such that for all $\epsilon > 0$ sufficiently small, the estimate $\|(\lambda - A_1)^{-1}\| \le M_2/\epsilon$ holds for all $\lambda \in \overline{\mathbb{C}}_+$. Since A_2 is bounded, this implies that

$$\|\epsilon^2 A_2(\lambda - A_1)^{-1}\| \le \epsilon\|A_2\|M_2, \qquad \lambda \in \overline{\mathbb{C}}_+,$$

so that for $\epsilon < (\|A_2\|M_2)^{-1}$ we have $\|\epsilon^2 A_2(\lambda - A_1)^{-1}\| < 1$ on the closed right half plane. Using another Neumann series expansion, we obtain that

$$(\lambda - \hat{A}_e)^{-1} = (\lambda - A_1)^{-1}(1 - \epsilon^2 A_2(\lambda - A_1)^{-1})^{-1}$$

is uniformly bounded on $\overline{\mathbb{C}}_+$.

Thus, by the preceding argument there exists an $\epsilon^* > 0$ such that the resolvent of $\hat{A}_e$ is uniformly bounded on $\overline{\mathbb{C}}_+$ for all $0 < \epsilon < \epsilon^*$. By the Gearhart-Prüss-Greiner theorem, the semigroup $\hat{\mathbb{T}}_e$ generated by $\hat{A}_e$ is exponentially stable, and therefore, the semigroup $R\hat{\mathbb{T}}_e R^{-1}$ generated by A_e is exponentially stable as well, for all $0 < \epsilon < \epsilon^*$. □

A. A regulating controller

The following theorem gives necessary and sufficient conditions for a controller to achieve output regulation for the plant (III.1), i.e., a criterion equivalent to the solvability of the regulator equations. The result extends [15, Thm 5.1] to boundary control systems.

Theorem IV.5. *Assume that the closed-loop system is regular and exponentially stabilized by the controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$.*

Then the controller solves the output regulation problem if and only if the equations

$$P_s(i\omega_k)Kz_k = -P_0(i\omega_k)E_s\phi_k - F\phi_k \tag{IV.2a}$$
$$(i\omega_k - \mathcal{G}_1)z_k = 0 \tag{IV.2b}$$

have solutions $z_k \in Z$ for all $k \in \{1,2,\ldots,q\}$, where $\{\phi_k\}_{k=1}^q$ is the Euclidean basis of $\mathbb{C}^q$. Furthermore, the solutions z_k are unique when they exist.

Proof. Let us first assume that the controller solves the output regulation problem, i.e., by Theorem IV.3 the regulator equations have a solution $\Sigma = (\Pi,\Gamma)^T \in \mathcal{L}(W,X_e)$. Let $k \in \{1,2,\ldots,q\}$ be arbitrary. As ϕ_k is an eigenvector of S, applying the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ to ϕ_k yields $(i\omega_k - A_e)\Sigma\phi_k = B_e\phi_k$, i.e.,

$$\begin{bmatrix}(i\omega_k - A_s + B_sR_1K\mathcal{G}_2\mathcal{C})\Pi\phi_k - (\mathcal{A}B_sR_1K - B_sR_1K\tilde{\mathcal{G}}_1)\Gamma\phi_k \\ -\mathcal{G}_2\mathcal{C}\Pi\phi_k + (i\omega_k - \tilde{\mathcal{G}}_1)\Gamma\phi_k\end{bmatrix} = \begin{bmatrix}(\mathcal{A}B_sE_s - B_sE_sS - B_sR_1K\mathcal{G}_2(\mathcal{C}B_sE_s + F))\phi_k \\ \mathcal{G}_2(\mathcal{C}B_sE_s + F)\phi_k\end{bmatrix}.$$

where we again denote $\tilde{\mathcal{G}}_1 := \mathcal{G}_1 + \mathcal{G}_2\mathcal{C}B_sR_1K$. The second line implies

$$(i\omega_k - \mathcal{G}_1)\Gamma\phi_k = \mathcal{G}_2(\mathcal{C}\Pi + \mathcal{C}B_sR_1K\Gamma + (\mathcal{C}B_sE_s + F))\phi_k. \tag{IV.3}$$

Now, as applying the second regulator equation to ϕ_k yields

$$0 = C_e\Sigma\phi + D_e\phi_k = \mathcal{C}\Pi\phi_k + \mathcal{C}B_sR_1K\Gamma\phi_k + (\mathcal{C}B_sE_s + F)\phi_k, \tag{IV.4}$$

it follows from (IV.4) and (IV.3) that $(i\omega_k - \mathcal{G}_1)\Gamma\phi_k = 0$. If we choose $z_k = \Gamma\phi_k$, then (IV.2b) follows immediately. Furthermore, from (IV.4) we obtain

$$\mathcal{C}\Pi\phi_k = -\mathcal{C}B_sR_1K\Gamma\phi_k - (\mathcal{C}B_sE_s + F)\phi_k. \tag{IV.5}$$

Substituting $\mathcal{C}\Pi\phi_k$ for (IV.5) in the first line of the Sylvester equation yields

$$\begin{aligned}&(i\omega_k - A_s)\Pi\phi_k - \mathcal{A}B_sR_1K\Gamma\phi_k + B_sR_1K\mathcal{G}_1\Gamma\phi_k \\ &= (\mathcal{A}B_sE_s - B_sE_sS)\phi_k,\end{aligned} \tag{IV.6}$$

and utilizing $S\phi_k = i\omega_k\phi_k$ and $\mathcal{G}_1\Gamma\phi_k = i\omega\Gamma\phi_k$, we obtain from (IV.6) that

$$\Pi\phi_k = (i\omega_k - A_s)^{-1}(\mathcal{A}B_s - i\omega_kB_s)(R_1K\Gamma\phi_k + E_s\phi_k). \tag{IV.7}$$

Finally, substituting $\Pi\phi_k$ for (IV.7) in (IV.4) yields

$$0 = P_s(i\omega_k)K\Gamma\phi_k + P_0(i\omega_k)E_s\phi_k + F\phi_k,$$

from which (IV.2a) follows as we chose $z_k = \Gamma\phi_k$.

Now assume that equations (IV.2a)–(IV.2b) have solutions $z_k \in Z$. Define $\Pi \in \mathcal{L}(W,X), \Gamma \in \mathcal{L}(W,Z)$ and $\Sigma = (\Pi,\Gamma)^T$ by

$$\Gamma := \sum_{k=1}^q \langle\cdot,\phi_k\rangle z_k,$$
$$\Pi := \sum_{k=1}^q \langle\cdot,\phi_k\rangle(i\omega_k - A_s)^{-1}(\mathcal{A}B_s - i\omega_kB_s)(R_1Kz_k + E_s\phi_k). \tag{IV.8}$$

The definitions imply that $\mathcal{R}(\Sigma) \subset \mathcal{D}(A_e) \subset \mathcal{D}(C_e)$, and we will show that Σ is the solution of the regulator equations.

Let $k \in \{1,2,\ldots,q\}$ be arbitrary. Considering the first line of $(i\omega_k - A_e)\Sigma\phi_k - B_e\phi_k$, we obtain using (IV.2b), $S\phi_k = i\omega_k$, the definition of Π, and (IV.2a) that

$$\begin{aligned}&(i\omega_k - A_s)\Pi\phi_k + B_sR_1K\mathcal{G}_2\mathcal{C}\Pi\phi_k \\ &- (\mathcal{A}B_sR_1K - B_sR_1K(\mathcal{G}_1 + \mathcal{G}_2\mathcal{C}B_sR_1K))\Gamma\phi_k \\ &- (\mathcal{A}B_sE_s - B_sE_sS - B_sR_1K\mathcal{G}_2(\mathcal{C}B_sE_s + F))\phi_k \\ =\ & B_sR_1K\mathcal{G}_2(\mathcal{C}\Pi\phi_k + \mathcal{C}B_sR_1K\Gamma\phi_k + \mathcal{C}B_sE_s\phi_k + F\phi_k) \\ =\ & B_sR_1K\mathcal{G}_2(P_s(i\omega_k)K\Gamma\phi_k + P_0(i\omega_k)E_s\phi_k + F\phi_k) = 0.\end{aligned}$$

Note that by (IV.2a) we also have

$$\begin{aligned}C_e\Sigma\phi_k + D_e\phi_k &= \mathcal{C}\Pi\phi_k + \mathcal{C}B_sR_1K\Gamma\phi_k + \mathcal{C}B_sE_s\phi_k + F\phi_k \\ &= P_s(i\omega_k)K\Gamma\phi_k + P_0(i\omega_k)E_s\phi_k + F\phi_k = 0,\end{aligned}$$

i.e., Σ solves the second regulator equation. Finally, the second line of $(i\omega_k - A_e)\Sigma\phi_k - B_e\phi_k$ yields

$$\begin{aligned}&- \mathcal{G}_2\mathcal{C}\Pi\phi_k + (i\omega_k - \mathcal{G}_1)\Gamma\phi_k - \mathcal{G}_2\mathcal{C}B_sR_1K\Gamma\phi_k \\ &- \mathcal{G}_2(\mathcal{C}B_sE_s + F)\phi_k \\ =& - \mathcal{G}_2(\mathcal{C}\Pi\phi_k + \mathcal{C}B_sR_1K\Gamma\phi_k + \mathcal{C}B_sE_s\phi_k + F\phi_k) = 0.\end{aligned}$$

Thus, as $\{\phi_k\}_{k=1}^q$ is a basis of $\mathbb{C}^q$ and the choice of k was arbitrary, Σ is the solution of the regulator equations $\Sigma S = A_e\Sigma + B_e$ and $C_e\Sigma + D_e = 0$. Now, by Theorem IV.3, the controller solves the output regulation problem.

It yet remains to prove the uniqueness of the solutions z_k of (IV.2a)–(IV.2b). Let z_k and z_k' be two solutions of (IV.2a)–(IV.2b), and use (IV.8) to define $\Sigma = (\Pi,\Gamma)^T$ and $\Sigma' = (\Pi',\Gamma')^T$ corresponding to z_k and z_k', respectively. It now follows from the above proof that both Σ and Σ' satisfy the Sylvester equation, and by the uniqueness of the solution of the Sylvester equation we must have $\Sigma = \Sigma'$. In particular, $z_k = \Gamma\phi_k = \Gamma'\phi_k = z_k'$, i.e., the solutions z_k of (IV.2a)–(IV.2b) are unique. □

Based on Theorem IV.5, we can now construct a regulating controller for the plant (III.1). Choose $Z = W$ and choose the controller parameter $Q \in \mathcal{L}(Y,U)$ such that the semigroup $\mathbb{T}_s$ generated by A_s is exponentially stable and $\mathcal{C}$ is an admissible observation operator for A_s. Choose the remaining parameters as

$$\mathcal{G}_1 = S = \operatorname{diag}(i\omega_1, i\omega_2, \ldots, i\omega_q), \tag{IV.9a}$$
$$K = \epsilon K_0 = \epsilon\,[u_1, u_2, \ldots, u_q]\,, \tag{IV.9b}$$
$$\mathcal{G}_2 = (\mathcal{G}_2^k)_{k=1}^q = (-(P_s(i\omega_k)u_k)^*)_{k=1}^q, \tag{IV.9c}$$

where $\epsilon > 0$ is called the tuning parameter and $u_k \in U$ are chosen such that [32, Sec. 4.2]

$$\begin{cases} P_s(i\omega_k)u_k = y_k, & y_k \neq 0, \\ u_k \notin \mathcal{N}(P_s(i\omega_k)) \text{ arbitrary}, & y_k = 0, \end{cases} \tag{IV.10}$$

where we denote $y_k = -P_0(i\omega_k)E_s\phi_k - F\phi_k$. For this to be possible, we need to assume that $P_s(i\omega_k) \neq 0$ and $y_k \in \mathcal{R}(P_s(i\omega_k))$ for all $k \in \{1,2,\ldots,q\}$, so that there exist some $u_k \in U$ satisfying (IV.10). However, this assumption is also necessary for the solvability of the output regulation problem by Theorem IV.5.

Proposition IV.6. *There exists an $\epsilon^* > 0$ such that the controller with the parameter choices* (IV.9a)–(IV.9c) *solves the output regulation problem for all $0 < \epsilon < \epsilon^*$.*

Proof. First of all we note that the choices of $\mathcal{G}_1$ and K imply that the equations (IV.2a)–(IV.2b) have the solutions $z_k = \epsilon^{-1}\phi_k$ if $P_0(i\omega_k)E_s\phi_k + F\phi_k \neq 0$ or $z_k = 0$ otherwise. Now, as Q exponentially stabilizes the plant and $\mathcal{C}$ is admissible for A_s, $\sigma(\mathcal{G}_1) = \{i\omega_k\}_{k=1}^q$, and

$$\sigma(\mathcal{G}_2^k P_s(i\omega_k)K_0^k) = \sigma(-(P_s(i\omega_k)u_k)^* P_s(i\omega_k)u_k) \subset \mathbb{C}_-$$

as $P_s(i\omega_k)u_k \neq 0$ for $k \in \{1, 2, \ldots, q\}$, we have by Lemma IV.4 that there exists an $\epsilon^* > 0$ such that the closed-loop system is exponentially stable for all $0 < \epsilon < \epsilon^*$. Thus, by Theorem IV.5 the controller solves the output regulation problem. □

B. The Internal Model Principle

Before presenting an approximate robust controller in §IV-C and a robust controller in §IV-D, we will present a general result that characterizes robust controllers. That is, we will show that in order for a controller to achieve robust output regulation, it has to contain an internal model of the dynamics of the exosystem. We will express this using the following $\mathcal{G}$-conditions [33, Def. 10].

Definition IV.7. A quadruple of bounded operators $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ is said to satisfy the $\mathcal{G}$-conditions if

$$\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \quad \forall k \in \{1, 2, \ldots, q\} \quad \text{(IV.11a)}$$
$$\mathcal{N}(\mathcal{G}_2) = \{0\}. \quad \text{(IV.11b)}$$

Note that while the parameters K and Q are not present in the $\mathcal{G}$-conditions, they contribute to exponentially stabilizing the closed-loop system. The sufficiency part of the following result has been presented in the case $R_1 = R_2 = I$ in [17, Thm 4] and the necessity part extends the results of [14, Thm 5.2] and [11, Thm 7] to boundary control systems.

Theorem IV.8. *Assume that the closed-loop system is regular and exponentially stabilized by the controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$. Then the controller solves the robust output regulation problem if and only if it satisfies the $\mathcal{G}$-conditions.*

Proof. Let us assume that the controller solves the robust output regulation problem and show that (IV.11) hold starting with (IV.11a). Let $k \in \{1, 2, \ldots, q\}$ be arbitrary and $w \in \mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$. Then there exist $z \in Z$ and $y \in Y$ such that $w = (i\omega_k - \mathcal{G}_1)z = \mathcal{G}_2 y$. Let us leave the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ unperturbed and choose such perturbations from $\mathcal{O}$ that $E_s' = 0$ and $F' = \langle \cdot, \phi_k \rangle (y - P_s(i\omega_k)Kz)$. Choose $\Sigma = (\Gamma, \Pi)^T \in \mathcal{L}(W, X_e)$ such that

$$\Gamma = \langle \cdot, \phi_k \rangle z, \quad \Pi = \langle \cdot, \phi_k \rangle (i\omega_k - A_s)(\mathcal{A}B_s - i\omega_k B_s)R_1 K z,$$

which can be shown to be the solution of the Sylvester equation by a direct computation. As $C_e\Sigma\phi_k + D_e'\phi_k = 0$ by the controller solving the robust output regulation problem, we obtain

$$\begin{aligned} w = (i\omega_k - \mathcal{G}_1)z &= \mathcal{G}_2 y = \mathcal{G}_2(P_s(i\omega_k)Kz + F'\phi_k) \\ &= \mathcal{G}_2(\mathcal{C}\Pi\phi_k + \mathcal{C}B_s R_1 K\Gamma\phi_k + F'\phi_k) \\ &= \mathcal{G}_2(C_e\Sigma\phi_k + D_e'\phi_k) = 0, \end{aligned}$$

and thus $w = 0$, which concludes the first part of the necessity proof.

Let us now show that (IV.11b) holds. Let $y \in \mathcal{N}(\mathcal{G}_2)$ and let $\phi \in W$ be such that $\|\phi\| = 1$. Leave the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ unperturbed and choose $E' = 0$ and $F' = \langle \cdot, \phi \rangle y \in \mathcal{L}(W, Y)$. If we choose $\Sigma = 0 \in \mathcal{L}(W, X_e)$, for all $v \in W$ we have $\Sigma S v = 0$ and

$$\begin{aligned} A_e\Sigma v + B_e' v &= \begin{bmatrix} -B_s R_1 K \mathcal{G}_2 F' v \\ \mathcal{G}_2 F' v \end{bmatrix} = \begin{bmatrix} -\langle v, \phi \rangle B_s R_1 K \mathcal{G}_2 y \\ \langle v, \phi \rangle \mathcal{G}_2 y \end{bmatrix} \\ &= 0, \end{aligned}$$

and thus, $\Sigma = 0$ is the unique solution of the Sylvester equation. As the controller solves the robust output regulation problem, we have by Theorem IV.3 that

$$0 = C_e\Sigma\phi + D_e'\phi = F'\phi = \langle \phi, \phi \rangle y = y,$$

which concludes the necessity proof. The sufficiency part follows by simple modifications from [17, Thm. 4]. □

Remark IV.9. Theorem IV.8 states that any controller that stabilizes a regular closed-loop system exponentially and satisfies the $\mathcal{G}$-conditions solves the robust output regulation problem. In particular, this implies that if a robust regulating controller $(\mathcal{G}_1, \mathcal{G}_2, K, Q)$ is constructed, then every controller $(\mathcal{G}_1, \mathcal{G}_2', K', Q')$, where $(\mathcal{G}_2', K', Q')$ are boundedly perturbed $(\mathcal{G}_2, K, Q)$, solves the robust output regulation problem, provided that the closed-loop system remains exponentially stable and $(\mathcal{G}_1, \mathcal{G}_2')$ satisfy the $\mathcal{G}$-conditions. Note that only rather specific perturbations would be allowed in $\mathcal{G}_1$ as it has to include an exact internal model of the dynamics of the exosystem.

Note that the rank-nullity theorem and the second $\mathcal{G}$-condition imply that $\dim Z \geq \dim \mathcal{R}(\mathcal{G}_2) = \dim Y$. Thus, if the output space of the system is infinite-dimensional as, e.g., in the wave equation of §II, Theorem IV.8 implies that robust controllers for such systems are necessarily infinite-dimensional. However, we can construct a finite-dimensional controller that solves the robust output regulation problem *approximately*. We will construct such a controller in the next section. Finally, in §IV-D we will construct an infinite-dimensional controller that achieves exact robust output regulation. The following assumption is required for the remaining sections:

Assumption IV.10. The transfer function $P_s(\lambda)$ is surjective at all the eigenvalues $\{i\omega_k\}_{k=1}^q$ of S.

C. An approximate robust controller

In this section, we consider approximate robust output regulation on Y. We will solve the control problem by choosing a subspace Y_N of Y and constructing a controller that robustly tracks the reference signal projected onto Y_N. If Y_N is chosen to be finite-dimensional, we can construct a finite-dimensional robust regulating controller even if the output space of the system is infinite-dimensional. Furthermore, we derive an upper bound for the asymptotic regulation error. Our result generalizes the controller structure presented in [16, Thm. 3.5] where discrete-time systems with constant reference signals were considered.

Let Y_N be a closed subspace of Y and choose $Z := Y_N^q$. Choose the controller parameter $Q \in \mathcal{L}(Y, U)$ such that the semigroup $\mathbb{T}_s$ generated by A_s is exponentially stable and $\mathcal{C}$ is an admissible observation operator for $\mathbb{T}_s$. Choose the remaining parameters as

$$\mathcal{G}_1 = \operatorname{diag}(i\omega_1 I_{Y_N}, i\omega_2 I_{Y_N}, \ldots, i\omega_q I_{Y_N}), \tag{IV.12a}$$

$$K = \epsilon K_0 = \epsilon[K_0^1, K_0^2, \ldots, K_0^q] \in \mathcal{L}(Z, U), \tag{IV.12b}$$

$$\mathcal{G}_2 = (\mathcal{G}_{20}^k P_N)_{k=1}^q \in \mathcal{L}(Y, Z), \tag{IV.12c}$$

where P_N is a projection onto Y_N, and $\mathcal{G}_{20}^k$ and K_0^k are such that

$$\sigma(\mathcal{G}_{20}^k P_N P_s(i\omega_k) K_0^k) \subset \mathbb{C}_- \tag{IV.13}$$

for all $k \in \{1, 2, \ldots, q\}$. We can choose, e.g., $\mathcal{G}_{20}^k = -I_{Y_N}$ and $K_0^k = (P_N P_s(i\omega_k))^{[-1]}$ for $k \in \{1, 2, \ldots, q\}$, and conversely, the spectrum condition implies that $\mathcal{G}_{20}^k$ and $P_N P_s(i\omega_k) K_0^k$ are boundedly invertible.

In the following theorem, we show that a controller with the aforementioned structure solves the approximate robust output regulation problem. Furthermore, we will show that for some constants $M, \alpha > 0$ and all $t \geq 0$ the regulation error satisfies

$$\int_t^{t+1} \|e(s)\|^2 \,\mathrm{d}s \leq M e^{-\alpha t}(\|x_{e0}\|^2 + \|v_0\|^2) + \delta \|v_0\|^2, \tag{IV.14}$$

where x_{e0} and v_0 are the initial states of the closed-loop system and the exosystem, respectively, and δ is given by

$$\delta = \left\|(I - P_N) \textstyle\sum_{k=1}^q \left(P_s(i\omega_k) K z_k + P_0(i\omega_k) E_s v_k + F v_k\right)\right\|^2, \tag{IV.15}$$

where v_k are the components of the unit vector $v_{\max} \in W$ satisfying $\|C_e \Sigma + D_e\| = \|C_e \Sigma v_{\max} + D_e v_{\max}\|_Y$ and $z_k = \Gamma v_k$ where Γ is given in (IV.16). Note that since W is finite dimensional, $v_{\max}$ is well-defined. Further note that we cannot guarantee pointwise convergence for the regulation error, and therefore the upper bound is presented in the integral form. Finally, since $\sum_{k=1}^q \left(P_s(i\omega_k) K z_k + P_0(i\omega_k) E_s v_k + F v_k\right) \in Y$, the projection P_N (or rather the space Y_N) can be chosen such that δ becomes arbitrarily small. We will demonstrate this procedure in §V for the wave equation.

Theorem IV.11. *There exists an $\epsilon^* > 0$ such that for all $0 < \epsilon < \epsilon^*$ the controller with the parameter choices* (IV.12a)–(IV.12c) *solves the approximate robust output regulation problem and there exist some constants $M, \alpha > 0$ such that for all $t \geq 0$ the regulation error satisfies* (IV.14).

Furthermore, the controller is robust with respect to those perturbations of class $\mathcal{O}$ that give rise to an exponentially stable perturbed closed-loop system, and the regulation error behaves as in (IV.14) *for the perturbed parameters of the plant and the exosystem.*

Proof. By Lemma IV.4, the closed-loop system is exponentially stable for all sufficiently small $\epsilon > 0$. Thus, as $\sigma(S) \subset i\mathbb{R}$, the Sylvester equation has a unique solution $\Sigma = (\Pi, \Gamma)^T$, and a direct computation using (III.6) verifies that

$$\begin{aligned} \Gamma &= (\Gamma_k)_{k=1}^q \\ &= -\epsilon^{-1}\left(\langle \cdot, \phi_k\rangle (P_N P_s(i\omega_k) K_0^k)^{-1} P_N (P_0(i\omega_k) E_s + F)\phi_k\right)_{k=1}^q, \\ \Pi &= \sum_{k=1}^q \langle \cdot, \phi_k\rangle (i\omega_k - A_s)^{-1}(\mathcal{A} B_s - i\omega_k B_s)(R_1 K \Gamma + E_s)\phi_k \end{aligned} \tag{IV.16}$$

solves $\Sigma S \phi_k = A_e \Sigma \phi_k + B_e \phi_k$, i.e., $(i\omega_k - A_e)\Sigma\phi_k = B_e \phi_k$ for all $k \in \{1, 2, \ldots, q\}$. Here one also uses that our Γ satisfies

$$\begin{aligned} P_N P_s(i\omega_k) K \Gamma \phi_k &= \epsilon P_N P_s(i\omega_k) K_0^k \Gamma_k \phi_k \\ &= -P_N P_0(i\omega_k) E_s \phi_k - P_N F \phi_k. \end{aligned} \tag{IV.17}$$

Note that (IV.16) is well-defined and bounded since $P_N P_s(i\omega_k) K_0^k$ are boundedly invertible by (IV.13).

Let us now consider the behavior of the regulation error. By [15, Lem. 4.3], we may write

$$e(t) = C_e \mathbb{T}_e(t)(x_{e0} - \Sigma v_0) + (C_e \Sigma + D_e) v(t),$$

and we obtain that for all $t \geq 0$

$$\begin{aligned} &\int_t^{t+1} \|e(s)\|^2 \,\mathrm{d}s \\ &= \int_t^{t+1} \|C_e \mathbb{T}_e(s)(x_{e0} - \Sigma v_0) + (C_e \Sigma + D_e) v(s)\|^2 \,\mathrm{d}s \\ &\leq M e^{-\alpha t}(\|x_{e0}\|^2 + \|v_0\|^2) + \|C_e \Sigma + D_e\|^2 \|v_0\|^2 \end{aligned}$$

for some $M, \alpha > 0$ as Σ is bounded, $\mathbb{T}_e$ is exponentially stable, C_e is admissible for $\mathbb{T}_e$, and due to the structure of the signal generator $\|v(t)\| = \|e^{St} v_0\| = \|v_0\|$.

We will show that

$$\begin{aligned} &C_e \Sigma v_{\max} + D_e v_{\max} \\ &= (I - P_N) \sum_{k=1}^q \left(P_s(i\omega_k) K z_k + P_0(i\omega_k) E_s v_k + F v_k\right). \end{aligned}$$

A direct computation using (IV.16) shows that

$$\begin{aligned} &C_e \Sigma v_{\max} + D_e v_{\max} \\ &= \sum_{k=1}^q \left(P_s(i\omega_k) K \Gamma v_k + P_0(i\omega_k) E_s v_k + F v_k\right), \end{aligned} \tag{IV.18}$$

Denoting $z_k = \Gamma v_k$, we have by (IV.17) that

$$P_N P_s(i\omega_k) K z_k = -P_N P_0(i\omega_k) E_s v_k - P_N F v_k, \tag{IV.19}$$

and now, combining (IV.19) with (IV.18) yields

$$\begin{aligned} &C_e \Sigma v_{\max} + D_e v_{\max} \\ &= (I - P_N) \sum_{k=1}^q \left(P_s(i\omega_k) K z_k + P_0(i\omega_k) E_s v_k + F v_k\right), \end{aligned}$$

which implies (IV.15), and thus, (IV.14).

If the parameters $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed in such a way that the closed-loop system remains exponentially stable, then the regulation error asymptotically satisfies $\int_t^{t+1} \|e(s)\|^2 \,\mathrm{d}s \leq M' e^{-\alpha' t}(\|x_{e0}\|^2 + \|v_0\|^2) + \|C_e' \Sigma' +$

$D'_e\|^2\|v_0\|^2$ for all $t \geq 0$, where $M', \alpha' > 0$, and C'_e, D'_e and Σ' are related to the the perturbed closed-loop system. By repeating the above computations with the perturbed parameters we clearly obtain $C'_e\Sigma' v'_{\max} + D'_e v'_{\max} = (I - P_N)\sum_{k=1}^q P'_s(i\omega_k)Kz'_k + P'_0(i\omega_k)E'_s v'_k + F'v'_k)$, where z'_k is the unique solution of $P_N P'_s(i\omega_k)Kz'_k = -P_N P'_0(i\omega_k)E'_s v'_k - P_N F' v'_k$ in $\mathcal{N}(i\omega_k - \mathcal{G}_1)$. Thus, the controller approximately solves the robust output regulation problem. □

Remark IV.12. As an alternative to the error estimate given in (IV.14), one can make a coarser choice for δ that does not require $v_{\max}$:

$$\delta = \sum_{k=1}^q \left|\left|(I - P_N)\big(P_s(i\omega_k)Kz_k + P_0(i\omega_k)E_s\phi_k + F\phi_k\big)\right|\right|^2,$$

where $\{\phi_k\}_{k=1}^q$ is the Euclidean basis of W and $z_k = \Gamma\phi_k$.

Corollary IV.13. *In Theorem IV.11, the regulation error satisfies $e^{\beta\cdot}P_N e(\cdot) \in L^2([0,\infty); Y)$ for some $\beta > 0$ independent of $x_{e0} \in X_e$ and $v_0 \in W$. Under perturbations of class $\mathcal{O}$ that give rise to an exponentially stable closed-loop system, the regulation error satisfies $e^{\beta'\cdot}P_N e(\cdot) \in L^2([0,\infty); Y)$ for some $\beta' > 0$ independent of $x_{e0} \in X_e$ and $v_0 \in W$.*

Proof. Let us first show that $P_N C_e\Sigma + P_N D_e = 0$. A direct computation using (IV.16) together with (IV.17) shows that for all $k \in \{1, 2, \dots, q\}$:

$$\begin{aligned} &P_N C_e \Sigma\phi_k + P_N D_e\phi_k \\ &= P_N P_s(i\omega_k)K\Gamma\phi_k + P_N P_0(i\omega_k)E_s\phi_k + P_N F\phi_k = 0, \end{aligned}$$

and as $\{\phi_k\}_{k=1}^q$ form a basis of $\mathbb{C}^q$, we have that $P_N C_e\Sigma + P_N D_e = 0$. By the proof of Theorem IV.11 we now have for some $\beta > 0$ that

$$\begin{aligned} \int_t^{t+1} \|e^{\beta s}P_N e(s)\|^2\,\mathrm{d}s &\leq e^{\beta(t+1)}\int_t^{t+1}\|P_N e(s)\|^2\,\mathrm{d}s \\ &\leq e^{\beta}Me^{(\beta-\alpha)t}(\|x_{e0}\|^2 + \|v_0\|^2), \end{aligned}$$

so for any $0 < \beta < \alpha$ we obtain

$$\begin{aligned} \int_0^\infty \|e^{\beta s}P_N e(s)\|^2\,\mathrm{d}s &\leq e^{\beta}M(\|x_{e0}\|^2 + \|v_0\|^2)\sum_{t=0}^\infty e^{(\beta-\alpha)t} \\ &= \frac{e^{\beta}M(\|x_{e0}\|^2 + \|v_0\|^2)}{1 - e^{\beta-\alpha}}, \end{aligned}$$

by which $e^{\beta\cdot}P_N e(\cdot) \in L^2([0,\infty))$ for any $0 < \beta < \alpha$. By the robustness part of Theorem IV.11, the same holds for some $0 < \beta' < \alpha'$ under perturbations of class $\mathcal{O}$ that give rise to an exponentially stable closed-loop system. □

D. A robust controller

In this section, we utilize the approximate controller structure of the previous section to construct an exact robust controller which, however, necessarily has infinite-dimensional state space if the output space of the plant is infinite-dimensional. Thus, we choose $Z = Y^q$ and choose the controller parameter $Q \in \mathcal{L}(U, Y)$ such that the semigroup $\mathbb{T}_s$ generated by A_s is exponentially stable and $\mathcal{C}$ is an admissible observation operator for $\mathbb{T}_s$. Following [11, Sec. IV] or [17, Thm. 8], we choose the remaining parameters as

$$\mathcal{G}_1 = \operatorname{diag}(i\omega_1 I_Y, i\omega_2 I_Y, \dots, i\omega_q I_Y) \in \mathcal{L}(Z), \tag{IV.20a}$$

$$K = \epsilon K_0 = \epsilon\left[K_0^1, K_0^2, \dots, K_0^q\right] \in \mathcal{L}(Z, U), \tag{IV.20b}$$

$$\mathcal{G}_2 = (-(P_s(i\omega_k)K_0^k)^*)_{k=1}^q \in \mathcal{L}(Y, Z). \tag{IV.20c}$$

Above the components K_0^k can be chosen freely provided that $P_s(i\omega_k)K_0^k$ are invertible. If we choose $K_0^k = P_s(i\omega_k)^{[-1]}$, then $\mathcal{G}_2^k = -I_Y$ for all $k \in \{1, 2, \dots q\}$, then the controller is the same as the approximate controller for the choice $Y_N = Y$. The following result follows immediately from Corollary IV.13.

Corollary IV.14. *There exists an $\epsilon^* > 0$ such that a controller with the parameter choices given in* (IV.20a)–(IV.20c) *solves the robust output regulation problem for all $0 < \epsilon < \epsilon^*$.*

Remark IV.15. The above result also follows from Lemma IV.4 and Theorem IV.8 as the choice $K_0^k = P_s(i\omega_k)^{[-1]}$ yields $\sigma(\mathcal{G}_2^k P_s(i\omega_k)K_0^k) = \sigma(-I_Y) \subset \mathbb{C}_-$, which together with the choice of Q completes the assumptions of Lemma IV.4, by which the closed-loop system is exponentially stable. Furthermore, it has been shown in the proof of [11, Thm 8] that $\mathcal{G}_1$ and $\mathcal{G}_2$ in (IV.20a)–(IV.20c) satisfy the $\mathcal{G}$-conditions, and thus, the controller solves the robust output regulation problem by Theorem IV.8.

V. Approximate robust regulation of the wave equation

Consider the wave equation as given in (II.1) with the spatial domain $\Omega := \{\zeta \in \mathbb{R}^2 \mid 1 < \|\zeta\| < 2\}$. Choose the partition $\partial\Omega = \Gamma_0 \cup \Gamma_1$ where $\Gamma_0 = \{\zeta \in \partial\Omega \mid \|\zeta\| = 1\}$ and $\Gamma_1 = \{\zeta \in \partial\Omega \mid \|\zeta\| = 2\}$ which satisfies the assumption in (II.6), e.g., for $\zeta_0 = 0$, and thus the results presented in Section II-B are applicable.

For the approximate robust output regulation problem, let $\delta = 0.01$ be given. We choose the output space as $Y := L^2(\Gamma_1)$ which is equivalent to $L^2([0, 2\pi])$. Thus, for the finite-dimensional closed subspace Y_N we may choose, e.g.,

$$Y_N := \operatorname{span}\{1, \cos(k\cdot), \sin(k\cdot) \mid k = 1, \dots, N\},$$

and the projection P_N from Y onto Y_N is then given by

$$P_N y := \frac{1}{\sqrt{2\pi}}\langle y, 1\rangle + \frac{1}{\sqrt{\pi}}\sum_{k=1}^N\left(\langle y, \cos(k\cdot)\rangle + \langle y, \sin(k\cdot)\rangle\right). \tag{V.1}$$

By standard Fourier analysis, it holds that for all $f \in L^2([0, 2\pi])$, we have $\lim_{N\to\infty}\|(1 - P_N)f\| = 0$, and thus, by Theorem IV.11, for a given reference y_{ref}, we can choose N in (V.1) sufficiently large such that asymptotically the regulation error becomes smaller than $\delta\|v_0\|^2$ (in the L^2-sense).

Let the reference and disturbance signals be given by

$$\begin{aligned} y_{ref}(\theta, t) &= -\frac{1}{2\pi^2}(\pi - \theta)^2\sin(\pi t) - \frac{1}{2}\sin\left(\frac{\theta}{2}\right)\cos(2\pi t) \\ d(\theta, t) &= \cos(\theta)\sin(2\pi t) + \sin(\theta)\sin(\pi t) \end{aligned}$$

and the disturbance d acts on Γ_1. Thus, we choose $S = \operatorname{diag}(-2i\pi, -i\pi, i\pi, 2i\pi)$, and the operators E and F are chosen such that $y_{ref} = -Fv$ and $d = Ev$ for $v_0 = 1$. The controller parameter Q is chosen as $Q(\theta) = 3$, and according to §IV-C we choose

$$\mathcal{G}_1 = \operatorname{diag}(-2i\pi I_{Y_N}, -i\pi I_{Y_N}, i\pi I_{Y_N}, 2i\pi I_{Y_N}), \tag{V.2a}$$

$$K = \epsilon \left[K_0^1, K_0^2, K_0^3, K_0^4\right] \tag{V.2b}$$

$$\mathcal{G}_2 = (-P_N)_{k=1}^4 \tag{V.2c}$$

where $K_0^k = (P_N P_s(i\omega_k))^{[-1]}$, $N = 5$ and $\epsilon = 0.15$.

For simulation, the operators related to the wave equation are approximated by the orthonormal eigenfunctions of the Laplacian Δ with homogeneous boundary conditions. In polar coordinates, these are of the form

$$\phi_{n0}^1(r) = \frac{1}{\sqrt{2\pi}}\varphi_{n0}(r), \qquad n \in \mathbb{N}$$

$$\phi_{nm}^1(r,\theta) = \frac{1}{\sqrt{\pi}}\varphi_{nm}(r)\cos(m\theta), \qquad m, n \in \mathbb{N}$$

$$\phi_{nm}^2(r,\theta) = \frac{1}{\sqrt{\pi}}\varphi_{nm}(r)\sin(m\theta), \qquad m, n \in \mathbb{N},$$

where $\varphi_{nm}(r)$ are the appropriately normalized Bessel functions corresponding to the radial part of the Laplacian such that the functions $\{\phi_{nm}^{1,2}\}$ form an orthonormal basis of $L^2(\Omega)$. The eigenvalues are computed numerically and in the simulation we use $n = 8$ radial and $m + 1 = 12$ angular eigenfunctions corresponding to the eigenvalues. The transfer function P_s is computed using the approximated operators, and the initial conditions are given by $x_0 = 0$ and $z_0 = 0$.

In Figure 1, the output profile y of the controlled wave equation and the reference profile y_{ref} are displayed for $t \in [0, 10]$. It can be seen that the output starts to follow the reference signal rather soon, even though some undershooting can be observed throughout the simulation.

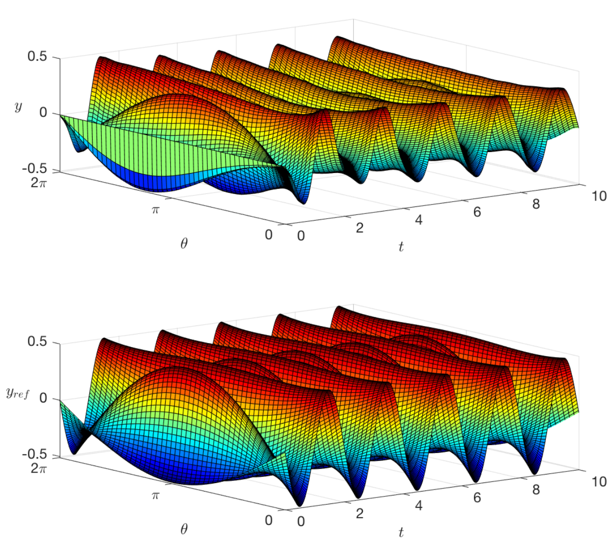

Fig. 1. The output profile y of the controlled wave equation and the reference profile y_{ref} for $t \in [0, 10]$ and in the same scales.

In Figure 2, the time average of the norm of the regulation error is displayed for $t \in [0, 20]$. Here it can be seen that, apart from the oscillations and initial errors, the regulation error decays at an exponential rate and that asymptotically it decays beyond the given $\delta\|v_0\|^2$. In Figure 3, the wave profile of the controlled system is displayed at time $t = 9$ and in Figure 4, the disturbance signal is displayed for $t \in [0, 6]$.

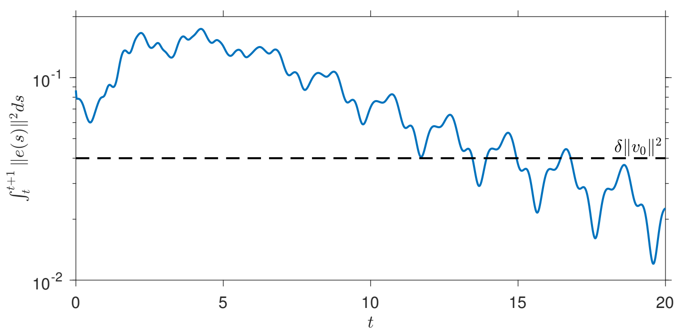

Fig. 2. The regulation error $\int_t^{t+1} \|y(s) - y_{ref}(s)\|^2 ds$ for $t \in [0, 20]$.

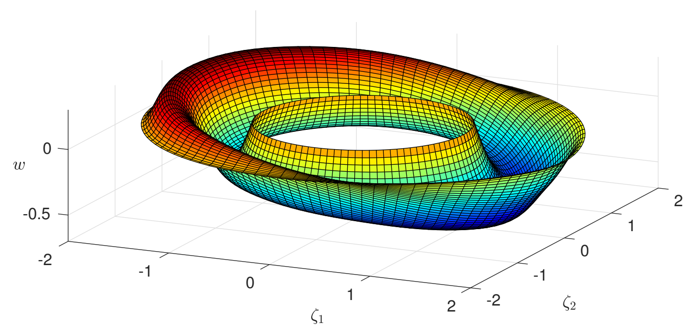

Fig. 3. The wave profile of the controlled system at $t = 9$.

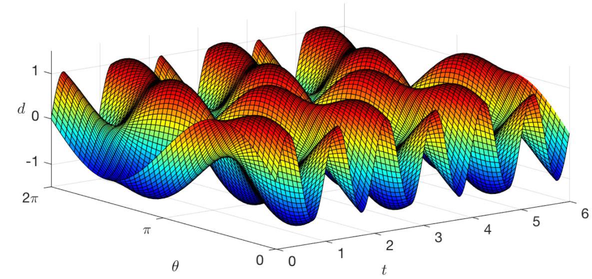

Fig. 4. The disturbance signal d for $t \in [0, 6]$.

VI. Conclusions

We developed output regulation for abstract boundary control systems, parametrizing all regulating and robust regulating controllers, and also suggesting some particular choices of such controllers. Since the internal model principle implies that the state space of any robust controller for a system with infinite-dimensional output space has infinite dimension, we extended the concept of approximate robust output regulation to boundary control systems. We demonstrated that approximate robust regulation can be achieved with a finite-dimensional controller by constructing such a controller for the two-dimensional wave equation and demonstrating its performance with numerical simulations.

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