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INTERVAL DECOMPOSITION LATTICES ARE BALANCED

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Abstract. Intervals in binary or n -ary relations or other discrete structures generalize the concept of an interval in a linearly ordered set. They are defined abstractly as closed sets of a closure system on a set, satisfying certain axioms. Join-irreducible partitions into intervals are characterized in the lattice of all interval decompositions. This result is used to show that the lattice of interval decompositions is balanced, and the case when this lattice is distributive is also characterised.

1. Preliminaries

Decompositions into intervals were first studied by Hausdorff [12, 13], in the context of linearly ordered sets, then extended to partially ordered sets and graphs (see Sabidussi [20]), appearing in particular in the study of comparability graphs (Gallai [9]). The concept of decomposition was extended to hypergraphs and directed graphs by Dörfler and Imrich [4] and Imrich [3], and to higher arity relational structures by Fraïssé [6, 7, 8]. A general, abstract theory of decompositions was first presented by Möhring and Radermacher [17] and Möhring [16]. Under mild stipulations about what is to be considered an interval, interval decompositions constitute a complete lattice. In [17] it was proved that this lattice is semimodular whenever it is of a finite length. This result was extended in [5] to arbitrary interval decomposition lattices, and their meet-irreducible elements were also described. Another proof for semimodularity can be found in [14]. We note that in graph theory, interval decompositions are closely related to the transitive orientation problem (see [9] and [15]). In the present paper, we prove further properties of the lattice of interval decompositions. First, we characterize the join-irreducible elements in this lattice, and using this

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result we show that the lattice is balanced. As a consequence, several other properties of the lattice of interval decompositions are deduced, and the case when this lattice is distributive is characterized.

A closure system (V, \mathcal{Q}) , $\mathcal{Q} \subseteq \mathcal{P}(V)$ is called *algebraic* if the union of any nonempty chain of closed sets is closed. An *interval system* (V, \mathcal{I}) was defined in [5] as an algebraic closure system with the following properties:

- (I₀) $\{x\} \in \mathcal{I}$ for all $x \in V$ and $\emptyset \in \mathcal{I}$,
- (I₁) $A, B \in \mathcal{I}$ and $A \cap B \neq \emptyset$ imply $A \cup B \in \mathcal{I}$,
- (I₂) for any $A, B \in \mathcal{I}$ the relations $A \not\subseteq B$ and $B \not\subseteq A$ imply $A \setminus B \in \mathcal{I}$ (and $B \setminus A \in \mathcal{I}$).

Examples of interval systems given in [5] include modules of graphs and relational intervals. These latter include order intervals in linearly ordered sets.

In any closure system (V, \mathcal{Q}) , a set $A \in \mathcal{Q}$ is called a *strong set*, if for every $B \in \mathcal{Q}$, $A \cap B = \emptyset$ or $A \subseteq B$ or $B \subseteq A$. The empty set, V and the singletons $\{a\}$, $a \in V$ are called *improper strong sets*. Let \mathcal{S} stand for the set of strong sets in (V, \mathcal{Q}) ; then (V, \mathcal{S}) is a closure system satisfying conditions (I₁) and (I₂) trivially. Let (V, \mathcal{Q}) be algebraic and satisfy condition (I₀), then any singleton and \emptyset are strong sets and (V, \mathcal{S}) is an interval system.

We note that restricting a closure system (V, \mathcal{Q}) to a nonempty set $A \subseteq V$ we obtain again a closure system (A, \mathcal{Q}_A) with $\mathcal{Q}_A = \{Q \cap A \mid Q \in \mathcal{Q}\}$. Clearly, for any $A \in \mathcal{Q}$ we have $\mathcal{Q}_A \subseteq \mathcal{Q}$, and (A, \mathcal{Q}_A) is an interval system whenever (V, \mathcal{Q}) is an interval system.

DEFINITION 1.1. A *decomposition* in a closure system (V, \mathcal{Q}) is a partition $\pi = \{A_i \mid i \in I\}$ of the set V such that $A_i \in \mathcal{Q}$, for all $i \in I$. The decomposition π is said to be *proper*, if it has at least two distinct blocks A_i . If (V, \mathcal{Q}) is an interval system, then π is called an *interval decomposition*. The set of all decompositions in (V, \mathcal{Q}) is denoted by $\mathcal{D}(V, \mathcal{Q})$.

Let $\text{Part}(V)$ denote the lattice of all partitions of V . Since $\mathcal{D}(V, \mathcal{Q}) \subseteq \text{Part}(V)$, it is ordered by refinement, where for any $\pi_1, \pi_2 \in \mathcal{D}(V, \mathcal{Q})$, $\pi_1 \leq \pi_2$ holds if and only if every block of π_2 is the union of some blocks of π_1 . In [5] we proved the following.

PROPOSITION 1.2. *Let (V, \mathcal{Q}) be a closure system. Then $\mathcal{D}(V, \mathcal{Q})$ is a complete lattice with the greatest element $\nabla = \{V\}$. If (V, \mathcal{Q}) is algebraic and satisfies condition (I₀), then $\mathcal{D}(V, \mathcal{Q})$ is a complete sublattice of $\text{Part}(V)$ if and only if it satisfies condition (I₁).*

EXAMPLE 1.3. If $T = (V, E)$ is a finite tree, then the vertex sets of its subtrees form a closure system (V, \mathcal{Q}) which satisfies conditions (I₀) and (I₁). Then $\mathcal{D}(V, \mathcal{Q})$ is a finite sublattice of $\text{Part}(V)$, according to Proposition 1.2.

We prove that $\mathcal{D}(V, \mathcal{Q})$ is a Boolean lattice isomorphic to $(\mathcal{P}(E), \subseteq)$. Indeed, given $S \subseteq E$ define $\pi(S)$ as the equivalence relation on V in which two vertices are equivalent if they are connected in the tree T by a path containing only edges from S . Since the classes of $\pi(S)$ induce subtrees of T , $\pi(S)$ is a decomposition in (V, \mathcal{Q}) . Then the isomorphism of $\mathcal{P}(E)$ to the lattice $\mathcal{D}(V, \mathcal{Q})$ is given by the mapping

$$S \longmapsto \pi(S).$$

The following result was proved in full generality in [17] and [5]:

PROPOSITION 1.4. *If (V, \mathcal{Q}) is an algebraic closure system satisfying condition (I_1) , then $\mathcal{D}(V, \mathcal{Q})$ is an algebraic semimodular lattice.*

Therefore, for an interval system (V, \mathcal{I}) , the lattice $\mathcal{D}(V, \mathcal{I})$ is always an algebraic semimodular sublattice of $\text{Part}(V)$.

REMARK 1.5. Let (V, \mathcal{Q}) be a closure system satisfying (I_0) . Then clearly, $\Delta = \{\{x\} \mid x \in V\}$ is the least element of $\mathcal{D}(V, \mathcal{Q})$, and to any $A \in \mathcal{Q} \setminus \{\emptyset\}$ corresponds the decomposition

$$\pi_A = \{A\} \cup \{\{x\} \mid x \in V \setminus A\}.$$

Moreover, if $\pi = \{A_i \mid i \in I\} \in \mathcal{D}(V, \mathcal{Q})$, then

$$(1) \quad \pi = \bigvee \{\pi_{A_i} \mid i \in I\},$$

where \bigvee means the join in the complete lattice $\mathcal{D}(V, \mathcal{Q})$.

A decomposition $\pi = \{A_i \mid i \in I\}$ in a closure system (V, \mathcal{Q}) is called a *strong decomposition* if every A_i , $i \in I$, is a strong set in (V, \mathcal{Q}) . Since the strong decompositions in (V, \mathcal{Q}) can be considered also as decompositions in the closure system (V, \mathcal{S}) , they form the complete lattice $\mathcal{D}(V, \mathcal{S})$ whose greatest element is $\nabla = \{V\}$.

An element a of a lattice L is called *standard* (see Grätzer [10]), if

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y) \text{ holds, for all } x, y \in L.$$

The standard elements of L form a distributive sublattice of L denoted by $S(L)$. An element a is called a *neutral* element of L , if for any $x, y \in L$ the sublattice of L generated by the set $\{a, x, y\}$ is distributive (see e.g. [10]). Clearly, any neutral element is standard, too. The following result was proved also in [5]:

THEOREM 1.6. *Let (V, \mathcal{Q}) be a closure system and \mathcal{S} be the family of its strong sets. Then the strong decompositions in (V, \mathcal{Q}) are standard elements of $\mathcal{D}(V, \mathcal{Q})$, and $\mathcal{D}(V, \mathcal{S})$ is a distributive sublattice of $\mathcal{D}(V, \mathcal{Q})$ and of $\text{Part}(V)$.*

A set $A \in \mathcal{Q}$ of a closure system (V, \mathcal{Q}) is called *fragile* if it is the union of two disjoint nonempty members of \mathcal{Q} , otherwise A is called *nonfragile*. This

generalizes the concept of fragility studied by Habib and Maurer [11] in the context of module systems of graphs. In view of [5], if (V, \mathcal{Q}) is an interval system, then any nonfragile interval $A \in \mathcal{Q}$ is a strong set.

2. Completely join-irreducible elements in $\mathcal{D}(V, \mathcal{I})$

An element $p \in L \setminus \{0\}$ of a complete lattice L is called *completely join-irreducible* if for any system of elements $x_i \in L$, $i \in I$, the equality $p = \bigvee \{x_i \mid i \in I\}$ implies that $p = x_i$ for some $i \in I$. Let $J(L)$ stand for the set of completely join-irreducible elements of L . The completely meet-irreducible elements of L are defined dually, consisting a set $M(L)$. Let us now define $a_* := \bigvee \{x \in L \mid x < a\}$ for any $a \in L \setminus \{0\}$, and $a^* := \bigwedge \{x \in L \mid x > a\}$, for any $a \in L \setminus \{1\}$. Denoting by \prec the covering relation in a lattice L , we can observe that

$$p \in L \setminus \{0\} \text{ is completely join-irreducible} \Leftrightarrow p_* < p \Leftrightarrow p_* \prec p, \text{ and}$$

$$m \in L \setminus \{1\} \text{ is completely meet-irreducible} \Leftrightarrow m < m^* \Leftrightarrow m \prec m^*.$$

In this section, we characterize the completely join-irreducible elements in the lattice $\mathcal{D}(V, \mathcal{I})$ of interval decompositions, and we show that they are closely related to the strong sets of (V, \mathcal{I}) . The following result from [5] will be used.

LEMMA 2.1. *Let $\pi_1 = \{B_j \mid j \in J\}$ and $\pi_2 = \{A_i \mid i \in I\}$ be two decompositions in a closure system (V, \mathcal{Q}) . If $\pi_1 \prec \pi_2$ holds in $\mathcal{D}(V, \mathcal{Q})$ then there exists a unique $k \in I$ and $K \subseteq J$ with at least two elements such that $A_k = \bigcup_{j \in K} B_j$ and $A_i \in \pi_1$, for all $i \in I \setminus \{k\}$.*

We shall need also the following:

LEMMA 2.2. *Let (V, \mathcal{Q}) be a closure system satisfying condition (I_0) and $\pi \in \mathcal{D}(V, \mathcal{Q})$. Then π is completely join-irreducible in $\mathcal{D}(V, \mathcal{Q})$ if and only if there exists a nonempty closed set $A \in \mathcal{Q}$ with $\pi = \pi_A$ and such that A admits a greatest proper decomposition into closed sets.*

Proof. Assume that $\pi = \{A_i \mid i \in I\}$ is a completely join-irreducible element in $\mathcal{D}(V, \mathcal{Q})$. Since $\pi = \bigvee \{\pi_{A_i} \mid i \in I\}$ according to (1), we get that $\pi = \pi_{A_k}$, for some $k \in K$. Then, in view of Lemma 2.1, $\pi_* = \{B_j \mid j \in J\} \cup \{x \mid x \in V \setminus A_k\}$, where $B_j \in \mathcal{Q}$, $|J| \geq 2$, and A_k is the disjoint union $\bigcup_{j \in J} B_j$. Clearly, $\mu = \{B_j \mid j \in J\}$ is a proper decomposition of (A_k, \mathcal{Q}_{A_k}) . Let $\nu = \{C_t \mid t \in T\}$ be an arbitrary decomposition of (A_k, \mathcal{Q}_{A_k}) such that $\nu \neq \{A_k\}$. Then it is easy to see that $\nu^+ = \{C_t \mid t \in T\} \cup \{x \mid x \in V \setminus A_k\}$ is a decomposition in $\mathcal{D}(V, \mathcal{Q})$ and $\nu^+ < \pi_{A_k} = \pi$. Since π is completely join-irreducible, we get $\nu^+ < \pi_*$. Hence, the partition $\nu = \{C_t \mid t \in T\}$ of A is a refinement of the

partition $\mu = \{B_j \mid j \in J\}$ of A . Thus, $\nu \leq \mu$ holds in $\mathcal{D}(A_k, \mathcal{Q}_{A_k})$, and this means that A_k admits μ as a greatest proper decomposition.

Conversely, let $A \in \mathcal{Q} \setminus \{\emptyset\}$ be a closed set that admits a greatest proper decomposition $A = \{B_j \mid j \in J\}$, $|J| \geq 2$. We prove that π_A is a completely join-irreducible element in $\mathcal{D}(V, \mathcal{Q})$. Since $B_j \in \mathcal{Q}_A \subseteq \mathcal{Q}$ and $B_j \neq \emptyset$, for each $j \in J$, $\{B_j \mid j \in J\} \cup \{\{x\} \mid x \in V \setminus A\}$ is a decomposition in $\mathcal{D}(V, \mathcal{Q})$.

Now, let $\mu \in \mathcal{D}(V, \mathcal{Q})$, $\mu < \pi_A$ arbitrary. Since the partition μ is a refinement of π_A , it has the form $\mu = \{C_t \mid t \in T\} \cup \{\{x\} \mid x \in V \setminus A\}$, where $C_t \in \mathcal{Q}$, $|T| \geq 2$, and $A = \bigcup_{t \in T} C_t$. Then $\{C_t \mid t \in T\}$ is a proper decomposition in (A, \mathcal{Q}_A) , and hence $\{C_t \mid t \in T\} \leq \{B_j \mid j \in J\}$. Because this result yields $\mu \leq \{B_j \mid j \in J\} \cup \{\{x\} \mid x \in V \setminus A\}$, we deduce $(\pi_A)_* \leq \{B_j \mid j \in J\} \cup \{\{x\} \mid x \in V \setminus A\} < \pi_A$, and this implies that π_A is completely join-irreducible. ■

PROPOSITION 2.3. *Let (V, \mathcal{I}) be an interval system. Then π is a completely join-irreducible element in $\mathcal{D}(V, \mathcal{I})$ if and only if $\pi = \pi_A$, where A is an interval admitting a greatest proper decomposition $\{B_j \mid j \in J\}$, such that each B_j , $j \in J$ is a strong set in (V, \mathcal{I}) . If this decomposition of A has at least three blocks, then A is strong.*

Proof. Let $A \in \mathcal{I} \setminus \{\emptyset\}$ be an interval and $\mu = \{B_j \mid j \in J\}$, $|J| \geq 2$, the greatest proper decomposition of A . Then $B_j \in \mathcal{I}$, for all $j \in J$. If $|J| \geq 3$, then A cannot be the union of two disjoint nonempty members of \mathcal{I} . Therefore, A is nonfragile, and hence it is a strong set according to [5]. In view of Lemma 2.2, to prove our statement it is enough to show that each B_j , $j \in J$ is strong.

First, assume that $|J| \geq 3$. Then A is a strong set. Let $C \in \mathcal{I}$ such that $C \cap B_j \neq \emptyset$, for some $j \in J$. Since A is strong, now $C \cap A \neq \emptyset$ implies that either $A \subseteq C$ or $C \subsetneq A$ holds. In the first case $B_j \subseteq C$. If $C \subsetneq A$, then $\nu = \{C\} \cup \{\{x\} \mid x \in A \setminus C\}$ is a decomposition in (A, \mathcal{I}_A) and $\nu \neq \{A\}$. Hence $\nu \leq \mu$, and this implies $C \subseteq B_j$, because $C \cap B_j \neq \emptyset$. Therefore, B_j is a strong set of (V, \mathcal{I}) .

Let $|J| = 2$. Then $A = B_1 \cup B_2$ and $\mu = \{B_1, B_2\}$ is the greatest proper decomposition in (A, \mathcal{I}_A) .

Suppose that $C \cap B_1 \neq \emptyset$, for some $C \in \mathcal{I}$. Then also $C \cap A \neq \emptyset$. If $A \subseteq C$ then $B_1 \subseteq C$ holds. Let $A \not\subseteq C$. If $C \subseteq A$, then using the same argument as in the above case $|J| \geq 3$, we obtain $C \subseteq B_1$. If $C \not\subseteq A$ then $A \setminus C \in \mathcal{I} \setminus \{\emptyset\}$. Because $C \cap A$ and $A \setminus C$ are nonempty intervals, $\rho = \{C \cap A, A \setminus C\}$ is a proper decomposition in (A, \mathcal{I}_A) . Hence, $\rho \leq \mu$. Since ρ is a maximal proper partition of the set A , we get $\rho = \mu$. Then $B_1 = C \cap A$ or $B_1 = A \setminus C$. The second case is excluded, because $C \cap B_1 \neq \emptyset$. Thus, we deduce $B_1 \subseteq C$.

Therefore, $C \cap B_1 = \emptyset$ or $B_1 \subseteq C$ or $C \subseteq B_1$ must hold for any $C \in \mathcal{I}$, and this means that B_1 is a strong set. The fact that B_2 is strong is proved similarly. ■

As an immediate consequence of Proposition 2.3 we obtain:

COROLLARY 2.4. *Let (V, \mathcal{I}) be an interval system. If π is a completely join-irreducible element in $\mathcal{D}(V, \mathcal{I})$, then π_* is a strong decomposition.*

A lattice L is called *geometric*, if it is atomistic, semimodular and algebraic. Geometric lattices are also dually atomistic. This follows e.g. from [10, Lemma 391] and semimodularity.

COROLLARY 2.5. *Let (V, \mathcal{I}) be an interval system such that $\mathcal{D}(V, \mathcal{I})$ is of finite length. Then the following assertions are equivalent:*

- (i) $\mathcal{D}(V, \mathcal{I})$ is an atomistic lattice;
- (ii) $\mathcal{D}(V, \mathcal{I})$ is a geometric lattice;
- (iii) $\mathcal{D}(V, \mathcal{I})$ is a dually atomistic lattice;
- (iv) The only strong intervals in \mathcal{I} are V, \emptyset , and the singletons.

Proof. (i) \Rightarrow (ii) is clear, because $\mathcal{D}(V, \mathcal{I})$ is an algebraic semimodular lattice, according to Proposition 1.5. The implication (ii) \Rightarrow (iii) is obvious, and (iii) \Rightarrow (iv) follows from [5, Corollary 3.7].

(iv) \Rightarrow (i). Since $\mathcal{D}(V, \mathcal{I})$ is a lattice of finite length, any element of it is a join of some completely join-irreducible elements (see e.g. [1]). Now, assume that (V, \mathcal{I}) has no proper strong intervals, and let π be a completely join-irreducible element of $\mathcal{D}(V, \mathcal{I})$. Since $\pi > \Delta$, we get $\pi_* \geq \Delta$. Because by Corollary 2.4 $\pi_* \neq \{V\}$ is a strong decomposition, in view of Proposition 2.3, we get that any block of π_* is of the form $\{a\}$, $a \in V$. Then $\pi_* = \Delta$. Since Δ is the 0-element of $\mathcal{D}(V, \mathcal{I})$, it follows that π is an atom. Hence $\mathcal{D}(V, \mathcal{I})$ is an atomistic lattice. ■

3. Interval systems with distributive decomposition lattices

In this section, we are going to characterize interval systems (V, \mathcal{I}) having a distributive decomposition lattice $\mathcal{D}(V, \mathcal{I})$. (The question was raised by R. H. Möhring.) First, we introduce some notations and recall some known results.

Let L be a lattice, and for each element $a \in L$ define the set $J(a) = \{j \in J(L) \mid j \leq a\}$. Then $a = \bigvee J(a)$, whenever L is of finite length.

REMARK 3.1. In [18] it was shown that for all algebraic lattices in which $a = \bigvee J(a)$, for all $a \in L$ the following assertions are equivalent:

- (a) L is distributive.
- (b) $J(a \vee b) = J(a) \vee J(b)$, for all $a, b \in L$.
- (c) $j \leq a \vee b \Leftrightarrow (j \leq a \text{ or } j \leq b)$, for any $a, b \in L$ and all $j \in J(L)$.

A family of sets $A_i \subseteq V$, $1 \leq i \leq n$, is called *connected*, if $A_i \cap A_{i+1} \neq \emptyset$, for all $1 \leq i \leq n-1$. In view of [5], if (V, \mathcal{I}) is an interval system and $\{A_i \mid 1 \leq i \leq n\} \subseteq \mathcal{I}$ is a connected family, then $\bigcup_{i=1}^n A_i \in \mathcal{I}$. For any sets A, B let $A\Delta B = (A \setminus B) \cup (B \setminus A)$. We say that A and B are *overlapping* sets if all the relations $A \cap B \neq \emptyset$, $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$ hold.

LEMMA 3.2. *Let (V, \mathcal{I}) be an interval system and $A, B \in \mathcal{I}$ two overlapping sets. Then the following assertions are equivalent:*

- (i) $A\Delta B \in \mathcal{I}$.
- (ii) *There exist some nonempty intervals $C, D \in \mathcal{I}$ with $C \subseteq A \setminus B$, $D \subseteq B \setminus A$, such that $C \cup D \in \mathcal{I}$ holds.*

Proof. (i) \Rightarrow (ii). Take $C = A \setminus B$ and $D = B \setminus A$. Then (i) yields $C \cup D \in \mathcal{I}$.

(ii) \Rightarrow (i). Since A, B are overlapping sets $A \setminus B \neq \emptyset$, $B \setminus A \neq \emptyset$, and the sets $A \setminus B$, $C \cup D$ and $B \setminus A$ form a connected family. Hence $A\Delta B = (A \setminus B) \cup (B \setminus A) = (A \setminus B) \cup (C \cup D) \cup (B \setminus A) \in \mathcal{I}$. ■

THEOREM 3.3. *Let (V, \mathcal{I}) be an interval system such that $\mathcal{D}(V, \mathcal{I})$ has finite length. Then the following are equivalent:*

- (i) *The lattice $\mathcal{D}(V, \mathcal{I})$ is distributive.*
- (ii) *If A, B are overlapping intervals, then $A\Delta B \notin \mathcal{I}$.*

Proof. (i) \Rightarrow (ii). Let $\mathcal{D}(V, \mathcal{I})$ be of finite length, and suppose that for some overlapping intervals $A, B \in \mathcal{I}$ we have $A\Delta B \in \mathcal{I}$. Since \mathcal{I} is an interval system,

$$\begin{aligned}\pi_1 &= \{A, B \setminus A\} \cup \{\{x\} \mid x \in V \setminus (A \cup B)\}, \\ \pi_2 &= \{A \setminus B, B\} \cup \{\{x\} \mid x \in V \setminus (A \cup B)\} \quad \text{and} \\ \pi_3 &= \{A \cap B, A\Delta B\} \cup \{\{x\} \mid x \in V \setminus (A \cup B)\}\end{aligned}$$

are decompositions, and $\pi_2 \vee \pi_3 = \{A \cup B\} \cup \{\{x\} \mid x \in V \setminus (A \cup B)\}$, hence $\pi_1 \wedge (\pi_2 \vee \pi_3) = \pi_1$. On the other hand, $\pi_1 \wedge \pi_2 = \pi_1 \wedge \pi_3 = \{A \setminus B, A \cap B, B \setminus A\} \cup \{\{x\} \mid x \in V \setminus (A \cup B)\}$ implies

$$(\pi_1 \wedge \pi_2) \vee (\pi_1 \wedge \pi_3) = \{A \setminus B, A \cap B, B \setminus A\} \cup \{\{x\} \mid x \in V \setminus (A \cup B)\}.$$

Thus we get

$$\pi_1 \wedge (\pi_2 \vee \pi_3) \neq (\pi_1 \wedge \pi_2) \vee (\pi_1 \wedge \pi_3),$$

a contradiction with (i).

(ii) \Rightarrow (i). Conversely, assume that (ii) holds. Let $\pi_1, \pi_2, \nu \in \mathcal{D}(V, \mathcal{I})$ such that ν is completely join-irreducible and $\nu \leq \pi_1 \vee \pi_2$. We prove that $\nu \leq \pi_1$ or $\nu \leq \pi_2$ holds, and in view of Remark 3.1 this will imply that $\mathcal{D}(V, \mathcal{I})$ is distributive, i.e. (i).

In view of Proposition 2.3, $\nu = \pi_A$, for some $A \in \mathcal{I}$ having a proper decomposition $A = \bigcup\{B_j \mid j \in J\}$, $|J| \geq 2$ such that each B_j , $j \in J$ is a strong set in (V, \mathcal{I}) .

First, suppose $|J| \geq 3$. Then by Proposition 2.3 A is a strong set, and consequently $\nu = \pi_A$ is a strong decomposition. Thus ν is a standard element of $\mathcal{D}(V, \mathcal{I})$, according to Theorem 1.6. Since $\mathcal{D}(V, \mathcal{I})$ is a semimodular lattice of finite length, in view of Stern [21], ν is also a neutral element, and hence the sublattice generated by π_1, π_2, ν in $\mathcal{D}(V, \mathcal{I})$ is distributive. If neither $\nu \leq \pi_1$ nor $\nu \leq \pi_2$ holds, then $\pi_1 \wedge \nu, \pi_2 \wedge \nu < \nu$ imply $\pi_1 \wedge \nu, \pi_2 \wedge \nu \leq \nu_*$, because ν is completely join-irreducible. Thus we obtain $\nu = \nu \wedge (\pi_1 \vee \pi_2) = (\nu \wedge \pi_1) \vee (\nu \wedge \pi_2) \leq \nu_*$, a contradiction. This means that $\nu \leq \pi_1$ or $\nu \leq \pi_2$ holds necessarily, finishing the proof of this subcase.

Next, assume that $|J| = 2$. Then $A = B_1 \cup B_2$, where B_1, B_2 are disjoint and nonempty strong intervals of (V, \mathcal{I}) . Denote by ρ_1, ρ_2 and ρ_A the equivalence relations on V corresponding to the partitions π_1, π_2 and π_A , respectively. Then $\rho_A \leq \rho_1 \vee \rho_2$ implies that for any $x \in B_1$ and $y \in B_2$ there exists a sequence $z_0, z_1, \dots, z_n \in V$, with $x = z_0, y = z_n$ such that, for all $1 \leq i \leq n$ either $(z_{i-1}, z_i) \in \rho_1$ or $(z_{i-1}, z_i) \in \rho_2$ holds. Let us select the elements $x \in B_1$ and $y \in B_2$ so that the length n of the above sequence is as small as possible.

Assume that $n = 1$. Then $(x, y) \in \rho_1$ or $(x, y) \in \rho_2$ holds. This means $(x, y) \in C$, for some block $C \in \pi_1$, or $(x, y) \in D$, for some block $D \in \pi_2$. Since B_1, B_2 are disjoint, $C \subseteq B_1$ and $C \subseteq B_2$ can not hold simultaneously. If $(x, y) \in C$, then $C \cap B_1 \neq \emptyset, C \cap B_2 \neq \emptyset$ imply $B_1, B_2 \subseteq C$, because $C \in \mathcal{I}$ and B_1, B_2 are strong intervals. Hence $A = B_1 \cup B_2 \subseteq C$, and this yields $\nu = \pi_A \leq \pi_1$. Similarly, in the case $(x, y) \in D$ we obtain $A = B_1 \cup B_2 \subseteq D$ and $\nu = \pi_A \leq \pi_2$.

Finally, we prove that $n \geq 2$ is not possible. Let $n \geq 2$. Then there are intervals $X_1, X_2, \dots, X_n \in \pi_k, k \in \{1, 2\}$, such that $z_{i-1}, z_i \in X_i$, for all $1 \leq i \leq n$. Then $X_i \subseteq V, 1 \leq i \leq n$ is a connected family of sets, and we get $C = X_1 \cup \dots \cup X_{n-1} \in \mathcal{I}$ and $D = X_2 \cup \dots \cup X_n \in \mathcal{I}$. Since the length of the path z_0, z_1, \dots, z_n is as small as possible, $z_1 \notin B_1, z_{n-1} \notin B_2$, and we infer $B_1 \cap D = \emptyset$ and $B_2 \cap C = \emptyset$.

Indeed, $B_1 \cap D \neq \emptyset$ would imply $B_1 \cap X_l \neq \emptyset$, for some $l \in \{2, \dots, n\}$. Then replacing $x = z_0$ by $x' \in B_1 \cap X_l$, we would obtain a sequence x', z_1, \dots, z_n which connects $x' \in B_1$ to $y = z_n \in B_2$ and has length less than or equal to $n - 1$, contrary to our assumption. Similarly, we can prove $B_2 \cap C = \emptyset$. Because $z_1 \in C \setminus B_1$ and $z_{n-1} \in D \setminus B_2$, we have $C \not\subseteq B_1$ and $D \not\subseteq B_2$. Since $z_0 \in C \cap B_1$ and $z_n \in D \cap B_2$ and B_1, B_2 are strong intervals, we get $B_1 \subseteq C$ and $B_2 \subseteq D$. Hence $B_1 \subseteq C \setminus D, B_2 \subseteq D \setminus C$. Since $C \cap D \supseteq X_2 \neq \emptyset$ and $C \setminus D, D \setminus C$ are nonempty sets, C and D are overlapping intervals. As $B_1 \cup B_2 = A \in \mathcal{I}$, and $B_1 \cup B_2 \subseteq (C \setminus D) \cup (D \setminus C) = C \Delta D$ by Lemma 3.2 we obtain $C \Delta D \in \mathcal{I}$, contradicting our assumption. ■

4. Further properties of the lattice $\mathcal{D}(V, \mathcal{I})$

A lattice L of finite length is called a *strong lattice* if for any join-irreducible element $j \in J(L)$, and for all $x \in L$

$$j \leq j_* \vee x \text{ implies } j \leq x.$$

Obviously, any atomistic lattice L is strong. We say that a lattice L is *dually strong*, if its dual $L^{(d)}$ is strong. The lattice L is called *consistent* if, for any $j \in J(L)$ and each $x \in L$, the element $x \vee j$ is join-irreducible in the interval $[x, 1]$. If for any $j \in J(L)$ and $m \in M(L)$ with $j \not\leq m$

$$j \vee m = m^* \Leftrightarrow j \wedge m = j_*$$

holds true, then L is called a *balanced lattice*. We say that L *satisfies the Kurosh-Ore replacement property for join-decompositions* (\vee -KORP, for short), if for every $a \in L$, and any two irredundant join-decompositions

$$a = j_1 \vee \cdots \vee j_m \quad \text{and} \quad a = k_1 \vee \cdots \vee k_n,$$

with $j_1, \dots, j_m, k_1, \dots, k_n \in J(L)$, each j_i can be replaced by a k_p so that

$$a = j_1 \vee \cdots \vee j_{i-1} \vee k_p \vee j_{i+1} \vee \cdots \vee j_m.$$

REMARK 4.1. It is well-known that any semimodular lattice of finite length is dually strong (see e.g. Stern [21]). It belongs to the folklore that a lattice L of finite length is balanced if and only if both L and $L^{(d)}$ are strong. Crawley showed [1] that L satisfies \vee -KORP if and only if L is consistent. Let L be of finite length. As it is noted in [21], from the previous facts together with a result of Walendziak [22, Thm.1] the equivalence of the next assertions follows:

- (a) L is semimodular and has the \vee -KORP.
- (b) L is semimodular and balanced;
- (c) L is semimodular and consistent;
- (d) L is semimodular and strong.

THEOREM 4.2. *Let (V, \mathcal{I}) be an interval system. If the lattice $\mathcal{D}(V, \mathcal{I})$ has finite length, then it is a balanced lattice that satisfies \vee -KORP.*

Proof. Since $\mathcal{D}(V, \mathcal{I})$ is a semimodular lattice of finite length, in order to prove our theorem, in view of Remark 4.1, it suffices only to show that $\mathcal{D}(V, \mathcal{I})$ is strong. Take any $j \in J(\mathcal{D}(V, \mathcal{I}))$ and $x \in \mathcal{D}(V, \mathcal{I})$ with $j \leq j_* \vee x$. Because any join-irreducible element of a lattice of finite length is also completely join-irreducible, j_* is a standard element in $\mathcal{D}(V, \mathcal{I})$, according to Corollary 2.4. Thus we obtain

$$(2) \quad j = j \wedge (j_* \vee x) = (j \wedge j_*) \vee (j \wedge x) = j_* \vee (j \wedge x).$$

Since j is join-irreducible and $j_* < j$, (2) implies $j = j \wedge x$. Hence $j \leq x$, and this proves that $\mathcal{D}(V, \mathcal{I})$ is strong. ■

The above theorem has a further consequence for finite interval decomposition lattices.

A *tolerance* of a lattice L is a reflexive and symmetric relation $T \subseteq L^2$ compatible with the operations of L . A *block* of T is a maximal set $B \subseteq L$ satisfying $B^2 \subseteq T$. Suppose that L is of finite length. Then any block B of T has the form of an interval $B = [u, v]$, $u, v \in L$, $u \leq v$, and the compatibility property of T makes it possible to build a "factor lattice" L/T , whose elements are the blocks of T (see Czédli [2]). T is called a *glued tolerance*, if it contains all covering pairs of L . Since every intersection of glued tolerances of L is again a glued tolerance of L , there exists a least tolerance $\Sigma(L)$ comprising all pairs $x \prec y$ in L , called the *skeleton tolerance* of L . The lattice L is said to be *glued by geometric lattices*, if all blocks of $\Sigma(L)$ are geometric lattices.

Reuter [19] proved (see also [21; Thm. 4.6.8]) that for a finite lattice L , the assertions (a), (b), (c) and (d) of Remark 4.1 are equivalent to the following statement:

(e) L is glued by geometric lattices.

Therefore, by Theorem 4.2 we infer:

COROLLARY 4.3. *Let (V, \mathcal{I}) be an interval system such that the lattice $\mathcal{D}(V, \mathcal{I})$ of its decompositions is finite. Then $\mathcal{D}(V, \mathcal{I})$ is glued by geometric lattices.*

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