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The optimal way to play the most difficult repeated two-player coordination games

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ABSTRACT

This paper investigates repeated win-lose coordination games (WLC-games). We analyze which protocols are optimal for these games, covering both the worst case and average case scenarios, i,e., optimizing the guaranteed and expected coordination times. We begin by analyzing Choice Matching Games (CM-games) which are a simple yet fundamental type of WLC-games, where the goal of the players is to pick the same choice from a finite set of initially indistinguishable choices. We give a fully complete classification of optimal expected and guaranteed coordination times in two-player CM-games and show that the corresponding optimal protocols are unique in every case—except in the CM-game with four choices, which we analyze separately.

Our results on CM-games are essential for proving a more general result on the difficulty of all WLC-games: we provide a complete analysis of least upper bounds for optimal expected coordination times in all two-player WLC-games as a function of game size. We also show that CM-games can be seen as the most difficult games among all two-player WLC-games, as they turn out to have the greatest optimal expected coordination times.

1. Introduction

Pure win-lose coordination games (WLC-games) are simple yet fundamental games where all players receive the same payoffs: 1 (win) or 0 (lose). This paper studies *repeated* WLC-games, where the players make simultaneous choices in discrete rounds until (if ever) succeeding to coordinate on a winning profile. *Choice matching games* (CM-games) can be seen as the simplest class of such games. The choice matching game CM_m^n has n players with the goal to choose the same choice among m different indistinguishable choices, with no communication during play. The players can use the history of the game (i.e., the players' choices in previous rounds) for their benefit as the game proceeds. For simplicity, we denote the two-player game CM_m^2 by CM_m.

A paradigmatic real-life scenario with a choice matching game relates to a phenomenon that has humorously been called "pavement tango" or "droitwich" in [1]. Here two people try to pass each other but may end up blocking each other by repeatedly moving sideways into the same direction. For another example of a choice matching game, consider CM_3, the coordination-based variant of the rock–paper–scissors game, pictured as a bipartite graph with three horizontal edges below. In the game, the two players (i.e., columns) coordinate if they succeed choosing nodes from the same row.

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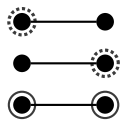
m	Optimal **expected** coordination time in CM_m	Unique optimal protocol for expected time	Optimal **guaranteed** coordination time in CM_m	Unique optimal protocol for guaranteed time
1	1	(any)	1	(any)
2	2	WM	∞	—
3	$1+\frac{2}{3}$	LA	2	LA
4	$2+\frac{1}{2}$	—	∞	—
5	$2+\frac{1}{3}$	LA	3	LA
6	$2+\frac{2}{3}$	WM	∞	—
7	$2+\frac{5}{7}$	WM	4	LA
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k$	$3-\frac{1}{k}$	WM	∞	—
$2k+1$	$3-\frac{2}{2k+1}$	WM	k	LA

Fig. 1. A complete analysis of two-player choice matching games.

The players first choose randomly; suppose they select the nodes in dotted circles. Next the players have three options: (1) to repeat their first choice, (2) to try to coordinate with the first choice of the other player, or (3) to select the choice which has not been picked yet. Supposing the players behave symmetrically, (3) is the only way to guarantee coordination in the second round.



A general n-player WLC-game is a generalization of CM_m^n where the players do not necessarily have to choose from the same row to coordinate, and it may not even suffice to choose from the same row. In classical matrix form representation, two-player choice matching games have ones on the diagonal and zeroes elsewhere, while general two-player WLC-games have general distributions of ones and zeroes. See Definition 2.1 for the full formal details.

In repeated WLC-games, it is natural to try to coordinate as quickly as possible. There are two main scenarios to be investigated: *guaranteeing coordination* (with certainty) in as few rounds as possible and minimizing the *expected number of rounds* for coordination. The former concerns the number of rounds it takes to coordinate in the *worst case* and is measured in terms of *guaranteed coordination times* (GCTs). The latter relates to the *average case* analysis measured in terms of *expected coordination times* (ECTs).

Our contributions. We provide a comprehensive study of upper bounds for coordination in *all* two-player repeated WLC-games, including a classification of related optimal strategies (called *protocols* in this work). WLC-games are represented in a novel way as *relational structures*, which is a key to many of the techniques used in the paper. The class of two-player WLC-games essentially coincides with the class of bipartite graphs, while n-player WLC-games correspond to n-partite hypergraphs, cf. Definition 2.1. CM-games are central to our work, being a fundamental class of games and also the most difficult games for coordination—in a sense made precise below.

Two protocols play a central role in our study. We first introduce the *loop avoidance* protocol LA (cf. Definition 4.1) that instructs the players to play such that the generated history of choices always reduces the symmetries of the game structure, which amounts to reducing the corresponding automorphism group. We also introduce the *wait-or-move* protocol WM (cf. Definition 4.5), essentially instructing the players to randomly alternate between two choices that both coordinate with at least one of the opponent's two choices. We show that WM leads to coordination in *all* two-player WLC-games *very fast*, the ECT being at most $3-2p$, where p is the probability of coordinating in the first round with random choices, that is, the probability of coordinating when both players choose based on the uniform distribution over their set of choices. We then provide a complete analysis of the optimal ECTs and GCTs in all choice matching games CM_m, also identifying the involved optimal protocols and showing their *uniqueness*, where possible. The table in Fig. 1 summarizes these results. The table also demonstrates that our analysis is fully *complete*; to see this, please note that we shall prove (1) that there exists a continuum of optimal protocols for CM_4 and (2) that for all even m, no protocol *guarantees* a win in CM_m.

In addition to studying CM-games, we provide the following complete characterization of upper bounds for the optimal ECTs in all two-player WLC-games *as a function of game size* (a game in standard matrix form is of size m when the maximum of the number of rows and columns is m):

Theorem. *For any m, the greatest optimal* ECT *among two-player* WLC*-games of size m is as follows:*

Game size	$m \in \mathbb{Z}_+ \setminus \{3, 5\}$	$m = 5$	$m = 3$
Greatest optimal ECT	$3 - \frac{2}{m}$	$2 + \frac{1}{3}$	$\frac{1+\sqrt{4+\sqrt{17}}}{2}$ ($\approx$1.925)

Furthermore, we establish that CM_m has the strictly greatest optimal ECT out of all two-player WLC-games of size $m \neq 3$, making CM-games the most difficult WLC-games to coordinating in. We give a separate complete analysis of the case $m = 3$.

Notes on applications. In genuine real-life scenarios, it can be highly inefficient – even practically impossible – to determine the absolutely optimal protocols that take into account the full game structure. Indeed, this generally requires an analysis that identifies, e.g., which choices are automorphic to each other. Computing automorphisms of (even bipartite) graphs is well known to be hard. However, our analysis gives an instant way of finding a fast protocol for any two-player WLC-game G, and furthermore, the above theorem guarantees that the ECT of that protocol in G is no worse than the upper bound of the optimal ECTs of the games of the same size.

It is also worth nothing here that our analysis concerns only two-player games. However, the arguments for this case are already rather involved, so the n-player case is expected to be highly complex and is thus left for the future. Furthermore, the two-player case is an especially important case covering, e.g., learning of communication protocols in distributed systems.

Related work. While the current paper is much more about algorithmic than classical game theory, coordination games (see, e.g., [2–4]) are a key topic also in the classical literature on games. The notions of *convention* and *focal point* play an important role in the related theory, also in the current paper. For some early foundations on conventions, see, inter alia, Schelling [19] and Lewis [16]. Repeated games are – likewise – a key topic, see for example [12,17,18]. For seminal work on *repeated coordination games*, see for example the articles [5,6,15]. Repeated coordination games have a wide range of applications, from learning and social choice theory to symmetry breaking in distributed systems.

WLC-games are a simple class of games that have not been extensively studied in the literature. In particular, choice matching games clearly constitute a *highly fundamental class of games*, and it is thus surprising that the analysis of the current paper has not been previously carried out. Thus the related analysis is well justified; it *closes an obvious gap* in the literature. Indeed, CM-games precisely capture the highly fundamental coordination problem of *picking the same choice from a set of choices*.

On the technical level, our study differs from the classical work on repeated games where the focus is on accumulated payoffs instead of the discrete outcomes of WLC-games. Indeed, repeated WLC-games are based on *reachability objectives*. Especially our worst case (but also the average case) analysis has only superficial overlap with classical work on repeated games. The most notable paper discussing a setting similar to the one we study in this paper is the seminal article [6] that studies a type of WLC-games. Also the papers [7–10] have a similar focus, as they also concentrate on coordination games with discrete win-lose outcomes. However, the papers [7–10] do not investigate optimality of protocols in repeated games, unlike the current paper. The seminal work [6] introduces (what is equivalent to) the two-player CM-games in the final section of the paper. They also essentially identify the optimal ways of playing CM_2 and CM_3 (discussed also in this article), although in the setting with accumulated payoffs. Furthermore, they observe that a protocol essentially equivalent to WM is the best way to play CM_6, an observation we also make in our framework. However, optimality of WM in CM_6 is not proved in [6]. This would require an extensive analysis proving that the players cannot make beneficial use of asymmetric histories created by non-coordinating choices. The main related technical difficulty is to show *uniqueness* of the optimal protocol, which we shall do in each case where a unique optimal protocol exist.

The framework of [6] also bears some conceptual similarities to ours, e.g., the authors also identify *structural protocols* (cf. Definition 3.5 below) as the natural notion of strategy for studying their framework. Furthermore, they make extensive use of focal points in analyzing how asymmetric histories can be used for coordination. Relating to uniqueness of protocols, [11] argues that individual rationality considerations not are not sufficient for "learning how to coordinate" in the setting of [6]. We agree with [11] that some conventions are needed if many protocols lead to the optimal result. However, as we can prove *uniqueness* of the optimal protocols for CM_m (for $m \neq 4$), then arguably rational players should adopt precisely these protocols in CM-games.

Concerning the work in [6], we note that some of the general background aims there differ from ours. In particular, while we focus on ECTs and GCTs of reachability objectives, the setting in [6] relates mainly to studying equilibrium properties of optimal attainable strategies in infinite plays.

We note that we have reported a subset of the results in the current article in the workshop abstract [14], with the proofs mainly omitted or sketched.

Techniques used. The core of our work relies on an original approach to games based on relational structures, as opposed to using the traditional matrix form representation. This enables us to use graph-theoretic ideas, which is essential in organizing our approach. Both in the worst-case and in average-case analysis, the main technical work relies heavily on analysis of symmetries—especially the way the groups of automorphisms of games evolve when playing coordination games. Theorem 5.2 – a central result in the worst-case analysis – is proved by reducing the cardinality of the automorphism groups of the WLC-games in a maximally fast fashion. In the average-case analysis, Theorems 6.2, 6.3, 6.5 are proved via a combination of (1) analyzing extrema, (2) keeping track of the automorphism groups of games, (3) applying graph-theoretic methods and notions, and (4) using focal points for breaking symmetry. An especially important part here is to show that we identify the *unique* optimal protocols for the games involved. Theorem 7.4 relies on earlier theorems and an extensive and exhaustive analysis of certain bipartite graphs. In principle, one could of course reduce our arguments into the setting with games presented in matrix form. However, we claim that this would lead to a rather awkward approach, as our arguments are based on natural graph-theoretic notions like degrees of nodes; special features such as cycles; general symmetries (automorphisms) et cetera. Such notions are key elements in our analysis.

Some of the our proofs require small technical results that are not central to the current paper. The detailed proofs of all these results can be found in the technical report [13]. We shall always give a clear reference when such a result is used, often also accompanied with an intuitive sketch of the proof.

2. Preliminaries

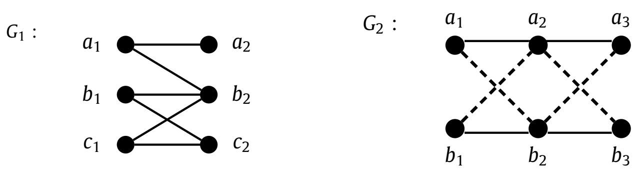
We define (pure) win-lose coordination games as *relational structures* as follows.

Definition 2.1. An n-player **win-lose coordination game (WLC-game)** is a relational structure $G = (A, C_1, \ldots, C_n, W_G)$ where A is a finite domain of **choices**, each C_i is a non-empty unary relation (representing the choices of player i) such that $C_1 \cup \cdots \cup C_n = A$, and $W_G \subseteq C_1 \times \cdots \times C_n$ is an n-ary **winning relation**. For technical convenience, we assume the players have pairwise disjoint choice sets, i.e., $C_i \cap C_j = \emptyset$ for all i and $j \neq i$. A tuple $\sigma \in C_1 \times \cdots \times C_n$ is a **choice profile** for G and the choice profiles in W_G are **winning (choice) profiles**. We assume there are no surely losing choices, i.e., choices $c \in A$ that do not belong to any winning choice profile, as rational players would never select such choices.

We will use the visual representation of WLC games as hypergraphs; two-player games become just bipartite graphs under this scheme. The choices of each player are displayed as columns of nodes, starting from the choices of player 1 on the left and ending with the column of choices of player n. The winning relation consists of lines that represent the winning choice profiles. Thus winning choice profiles are also called **edges**. See Example 2.2 below for an illustration.

Consider a WLC-game $G = (A, C_1, \ldots, C_n, W_G)$ with n players and m winning choice profiles that do not intersect, i.e., none of the m winning choice profiles share a choice $c \in A$. Such games form a simple yet fundamental class of games, where the goal of the players is simply to pick the same "choice", i.e., to simultaneously pick one of the m winning profiles. These games are called **choice matching games**. We let CM^n_m denote the choice matching game with n players and m choices for each player. In this article, we extensively make use of the two-player choice matching games, CM^2_m. For these games, we will omit the superscript "2" and simply denote them by CM_m. (Recall the example CM_3 pictured in the introduction.)

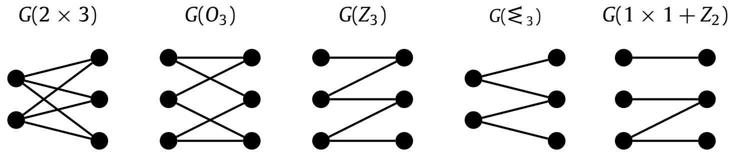
Example 2.2. Here we give two examples of drawings of WLC-games: a two-player game G_1 with 3 choices for both players and a total of 6 winning profiles represented as edges; and a three-player WLC-game G_2 with 2 choices for each player and 4 winning profiles, each represented as a triple of choices connected by (solid or dotted) lines.



We now specify some useful notational conventions for identifying some special WLC-games (see also the figure below for related examples).

- Let $m_1, \ldots, m_n \in \mathbb{Z}_+$. We write $G(m_1 \times \cdots \times m_n)$ for the n-player WLC-game where the player i has m_i choices and the winning relation is the *universal relation* $C_1 \times \cdots \times C_n$.
- Let $m \geq 2$. We write $G(O_m)$ for the two-player WLC-game in which both players have m choices and the winning relation W_G forms a $2m$-cycle through all the $2m$ choices. (Thus the game graph of this WLC-game corresponds to the cycle graph C_{2m}.) Similarly we write $G(Z_m)$ for the two-player WLC-game where both players have m choices and W_G forms a $(2m-1)$-edge path through all choices. Moreover, $G(\lesssim_m)$ denotes a WLC-game where the player 1 has $m-1$ choices, the player 2 has m choices and W_G forms a $(2m-2)$-edge path through all the choices; the game obtained by permuting the players in $G(\lesssim_m)$ is denoted by $G(\gtrsim_m)$.

- Suppose that $G(A)$ and $G(B)$ have been defined and both have the same number of players. Then $G(A+B)$ is the *disjoint union* of $G(A)$ and $G(B)$, i.e., the game obtained by assigning to each player a disjoint union of her/his choices in $G(A)$ and $G(B)$, with the winning relation for $G(A+B)$ being the union of the winning relations in $G(A)$ and $G(B)$.
- If $m \in \mathbb{Z}_+$, then $G(mA) := G(A + \cdots + A)$ (with A repeated m times). So, for example, $G(m(1 \times 1))$ is the 2-player choice matching game CM_m.



Interestingly, out of all n-player WLC-games where each of the n players has m choices, the game CM_m^n has the least probability of coordination when each player plays randomly. In this sense these games can be seen the most difficult for coordination. A fully compelling reason for the maximal difficulty of choice matching games is given later on by Theorem 7.5.

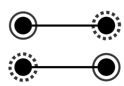
3. Repeated WLC-games

A **repeated play of a** WLC**-game** G consists of consecutive (one-step) plays of G. The repeated play is continued until the players successfully coordinate, i.e., select their choices from a winning choice profile. This may lead to infinite plays. We assume that each player can remember the full history of the repeated play and use this information when planning choices. The history of the play after k rounds is encoded in a sequence $\mathcal{H}_k$ defined as follows.

Definition 3.1. Let G be an n-player WLC-game. A pair $(G, \mathcal{H}_k)$ is called a **stage** k **(or** k**th stage) in a repeated play of** G, where the **history** $\mathcal{H}_k$ is a k-sequence of choice profiles in G. More precisely, $\mathcal{H}_k = \left(H_i\right)_{i\in\{1,\dots,k\}}$ where each H_i is an n-ary relation $H_i = \{(c_1, \dots, c_n)\}$ with a single tuple $(c_1, \dots, c_n) \in C_1 \times \cdots \times C_n$. In the case $k = 0$, we define $\mathcal{H}_0 = \emptyset$. The stage $(G, \mathcal{H}_0)$ is the **initial stage** (or the 0th stage). Like G, also $(G, \mathcal{H}_k)$ is a relational structure.

A stage k contains a history specifying precisely k choice profiles chosen in a repeated play. A winning profile of $(G, \mathcal{H}_k)$ is called a **touched edge** if it contains some choice c picked in some round $1, \dots, k$ leading to $(G, \mathcal{H}_k)$. As we assume that the players only need to coordinate once, we consider repeated plays only up to the first stage where some winning choice profile is selected. If coordination occurs in the kth round, the kth stage is called the **final stage** of the repeated play. (But a play can take infinitely long without coordination.)

Below is a drawing of the stage 2 in a repeated play of CM_2, the "coordination game variant" of the *matching pennies game* (or the "pavement tango" from the introduction). Here the players have failed to coordinate in round 1 (having picked the choices with dotted circles) and then failed again by both swapping their choices in round 2 (solid circles).



A protocol gives a *mixed strategy for all stages in all* WLC*-games and for all player roles* i:

Definition 3.2. A **protocol** π is a function outputting a probability distribution $f : C_i \to [0, 1]$ (so $\sum_{c\in C_i} f(c) = 1$) with the input of a player i and a stage $(G, \mathcal{H}_k)$ of a repeated WLC-game.

Since a protocol can depend on the full history of the current stage, it gives a mixed, memory-based strategy for any repeated WLC-game. Thus protocols can informally be regarded as global "behavior styles" of agents over the class of all repeated WLC-games. It is important to note that all players can see (and remember) the previous choices selected by all the other players—and also the order in which the choices have been made.

In the scenario that we study, it is obvious to require that the protocols should act *independently of the names of choices and the names (or ordering) of player roles i.*[1] In [6], this requirement follows from the "assumption of no common language" (for describing the game), and in [10], such protocols are called *structural*. We shall adopt the terminology of [10]. To extend this concept for repeated games, we first need to define the notion of a *renaming*. The intuitive idea of renamings is to extend *isomorphisms* between game graphs – including the history – to additionally enable *permuting the players* $1, \dots, n$ (see Example 3.4 for an illustration of the definition).

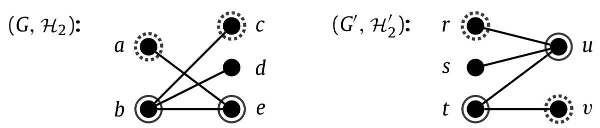
[1] Note that if this assumption is not made, coordination can trivially be guaranteed in a single round in any WLC-game by using a protocol which chooses some winning choice profile with probability 1.

Definition 3.3. A **renaming** between stages $(G, \mathcal{H}_k)$ and $(G', \mathcal{H}'_k)$ of n-player WLC-games G and G' is a pair (β, h) where β is a permutation of $\{1, \ldots, n\}$ and h a bijection from the domain of G to that of G' such that

- $c \in C_{\beta(i)} \Leftrightarrow h(c) \in C'_i$ for all $i \leq n$ and c in the domain of G,
- $(c_1, \ldots, c_n) \in W_G \Leftrightarrow (h(c_{\beta(1)}), \ldots, h(c_{\beta(n)})) \in W_{G'}$,
- $(c_1, \ldots, c_n) \in H_i \Leftrightarrow (h(c_{\beta(1)}), \ldots, h(c_{\beta(n)})) \in H'_i$ for all $i \leq k$.

If $(G, \mathcal{H}_k)$ and $(G', \mathcal{H}'_k)$ have the same domain A, we say that (β, h) is a **renaming of** $(G, \mathcal{H}_k)$. Choices $c \in C_i$ and $d \in C_j$ are **structurally equivalent**, denoted by $c \sim d$, if there is a renaming (β, h) of $(G, \mathcal{H}_k)$ such that $\beta(i) = j$ and $h(c) = d$. It is easy to see that $\sim$ is an equivalence relation on A. We denote the equivalence class of a choice c by $[c]$.

Example 3.4. Below we have two structurally equivalent stages $(G, \mathcal{H}_2)$ and $(G', \mathcal{H}'_2)$, where the players have selected the choices with dotted circles in round 1 and the choices with solid circles in round 2.



Definition 3.5. A protocol π is **structural** if it is indifferent with respect to renamings, i.e., if $(G, \mathcal{H}_k)$ and $(G', \mathcal{H}'_k)$ are stages with a renaming (β, h) between them, then for any i and any $c \in C_i$, we have $f(c) = f'(h(c))$, where $f = h((G, \mathcal{H}_k), i)$ and $f' = h((G', \mathcal{H}'_k), \beta(i))$.

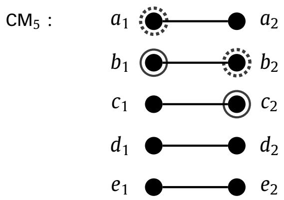
Note that a structural protocol may depend on the full history which records even the order in which the choices have been played, and the created history of course modifies the underlying structure. Hereafter we assume all protocols to be structural.

Definition 3.6. Let G be a WLC-game and let S and S' be stages of G. Let $\sim$ (respectively, $\sim'$) be the structural equivalence relation over S (respectively, S'). We say that S and S' are **automorphism-equivalent** if $\sim = \sim'$. The stages S and S' are **structurally similar** if one can be obtained from the other by a chain of renamings and automorphism-equivalences.

We will next discuss the notion of focal points. We note that in our work, focal points – to be defined formally below – can be identified only based on the structure of the game graph (or game stage, to be exact). Thus our notion differs from approaches where focal points are explicitly marked (say, via a unary relation designated specifically for labeling focal points).

Now, a choice c in a stage S is a **focal point** if it is not structurally equivalent to any other choice in that same stage S, with the possible exception of choices c' that *all belong to some single edge with* c. A focal point breaks symmetry and can be used for winning a repeated coordination game. This requires that the players have some (possibly prenegotiated) way to agree on which focal point to use. See the following example for an illustration.

Example 3.7. Consider the first two rounds of the game CM_5, pictured below, where the players fail to coordinate by first selecting the pair (a_1, b_2) and then fail again by selecting the pair (b_1, c_2).

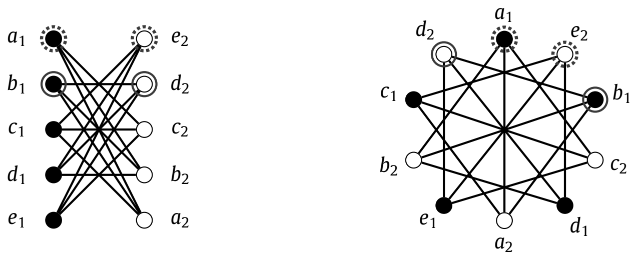


The structural equivalence classes become modified in this scenario as follows: (1) Initially all choices are structurally equivalent; (2) After the first round, the equivalence classes are $\{a_1, b_2\}$, $\{b_1, a_2\}$ and $\{c_1, d_1, e_1, c_2, d_2, e_2\}$; (3) After the second round, the equivalence classes are $\{a_1\}$, $\{a_2\}$, $\{b_1\}$, $\{b_2\}$, $\{c_1\}$, $\{c_2\}$ and $\{d_1, e_1, d_2, e_2\}$.

There are no focal points in the initial stage S_0 and the same is true for the next stage S_1. However, in the stage S_2, all the choices $a_1, b_1, c_1, a_2, b_2, c_2$ become focal points, and the players can thus immediately guarantee coordination in the third round by selecting any winning pair of focal points, i.e., any of the pairs (a_1, a_2), (b_1, b_2), (c_1, c_2). (We note that, from the point of view of the general study of rational choice, it may not be obvious which of these pairs should be selected, so a convention may be needed to fix which protocol to use.)

For another type of example on focal points, consider the game $\overline{G(O_5)}$ – displayed below – which can be described as the *complement game* of the cycle game $G(O_5)$. In the pictures below, we present $\overline{G(O_5)}$ also in the form where the choices are arranged in a cycle and we draw the choices of player 2 in white for clarity.

$\overline{G(O_5)}$:



Note that all choices are initially structurally equivalent in $\overline{G(O_5)}$. Suppose then that the players fail to coordinate in the first round. This can happen only if they select choices that are "adjacent in the cycle" (see the picture above). Hence, by symmetry, we may assume that the players choose the pair (a_1, e_2) in the first round. Then the equivalence classes after the first round are $\{a_1, e_2\}$, $\{b_1, d_2\}$, $\{c_1, c_2\}$, $\{d_1, b_2\}$ and $\{e_1, a_2\}$. Hence the players can guarantee coordination in the second round by selecting the winning pair (b_1, d_2) of focal points (or alternatively the pair (c_1, c_2) or (d_1, b_2)).

In repeated coordination games, it is natural to try to *coordinate as quickly as possible*. There are two principal scenarios related to optimizing coordination times: the *average case* and the *worst case*. The former concerns the expected number rounds for coordination and the latter the maximum number in which coordination can be guaranteed with certainty.

Definition 3.8. Let $(G, \mathcal{H}_k)$ be a stage and let π be a protocol. The **one-shot coordination probability** (OSCP) **from** $(G, \mathcal{H}_k)$ **with** π is the probability of coordinating in a single round from $(G, \mathcal{H}_k)$ when each player follows π. The **expected coordination time** (ECT) **from** $(G, \mathcal{H}_k)$ **with** π is the expected value for the number of rounds until coordination from $(G, \mathcal{H}_k)$ when all players follow π. The **guaranteed coordination time** (GCT) **from** $(G, \mathcal{H}_k)$ **with** π is the number n such that the players are *guaranteed* to coordinate from $(G, \mathcal{H}_k)$ in n rounds, but not in $n - 1$ rounds, when all players follow π, if such a number exists. Otherwise this value is ∞.

The OSCP, ECT and GCT from the initial stage $(G, \emptyset)$ with π are referred to as the OSCP, ECT and GCT in G with π. We say that π is ECT**-optimal** for G if π gives the minimum ECT in G, i.e., the ECT given by any protocol π' is at least as large as the one given by π. GCT**-optimality** of π for G is defined analogously.

A simple (and often quite efficient) way to play repeated WLC-games is to follow a "greedy protocol" which optimizes OSCP locally in every stage of the game. However, as we will see, such a protocol is not always ECT or GCT-optimal in a repeated game, which makes the optimization of ECTs and GCTs more difficult (cf. Remarks 6.4 and 7.3).

It is possible that there are several different protocols giving the optimal ECT (or GCT) for a given WLC-game. If two protocols π_1 and π_2 are both optimal, it may be that the optimal value is nevertheless *not* obtained when some of the players follow π_1 and the others π_2. This leads to a meta-coordination problem about choosing the same optimal protocol to follow. However, such a problem is avoided if there exists a unique optimal protocol.

Definition 3.9. Let π be a protocol and G a WLC-game. We say that π is **uniquely** ECT**-optimal** for G if π is ECT-optimal for G and the following holds for all other protocols π' that are ECT-optimal for G: for any stage S in G that is reachable with π, we have $\pi'(S) = \pi(S)$. **Unique** GCT**-optimality** of π for G is defined analogously.[2]

The next lemma states that two structurally similar stages are essentially the same stage with respect to different ECTs and GCTs. The proof is straightforward.

Lemma 3.10. *Assume stages S and S' of G are structurally similar. Now, for any protocol π, there exists a protocol π' which gives the same* ECT *and* GCT *from S' as π gives from S.*

4. Protocols for repeated WLC-games

In this section we introduce two special protocols, the *loop avoidance protocol* LA and the *wait-or-move protocol* WM. Informally, LA asserts that in every round, every player i should avoid – if possible – all choices c that could possibly make the resulting stage automorphism-equivalent (cf. Definition 3.6) to the current stage, i.e., the stage just before selecting c.

[2] Note that if two different protocols are uniquely ECT-optimal for G (and similarly for unique GCT-optimality), then their behavior on G can differ only on stages that are not reachable in the first place by the protocols. Also, their behavior can of course differ on games other than G.

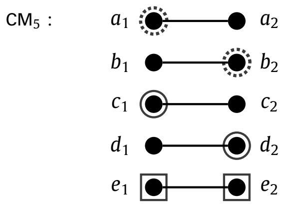
Definition 4.1. The **loop avoidance protocol** (LA) asserts that in every round, every player i should avoid – if possible – all choices c for which the following condition holds: if the player i selects c, then there exist choices for the other players so that the resulting stage is automorphism-equivalent to the current stage. If this condition holds for all choices of the player i, then i makes a random choice. Moreover, uniform probability is used among all the possible choices of i. Also, if in some stage S the players can guarantee a win in the next round, then the players are allowed to choose in the way that guarantees the win even if the resulting final stage is automorphism-equivalent to the stage S.

It is easy to see that LA avoids, when possible, all such stages that are structurally similar to *any* earlier stage in the repeated play. As structurally similar stages are essentially identical (cf. Lemma 3.10), repetition of such stages can be seen as a "loop" in the repeated play. When trying to *guarantee* coordination as quickly as possible, such loops should be avoided. In addition to this heuristic justification, Theorems 5.1 and 5.2 give a fully compelling justification for LA when considering guaranteed coordination in two-player CM-games. For now, we present the following easy results (for complete proofs, see the correspondingly numbered Propositions 4.2 and 4.3 in [13]); see also Example 4.4 below for an illustration of the use of LA.

Proposition 4.2. LA *is uniquely* ECT-*optimal and* GCT-*optimal in* CM_3.

Proposition 4.3. LA *guarantees coordination in* CM_m *in* $\lceil m/2 \rceil$ *rounds when m is odd, but* LA *does not guarantee coordination in* CM_m *for any even m.*

Example 4.4. We illustrate the use of the LA in CM_5. Suppose that coordination fails in the first round. By symmetry, we may assume that the players selected a_1 and b_2. Now, in the resulting stage S_1, the structural equivalence classes are $\{a_1, b_2\}$, $\{b_1, a_2\}$ and $\{c_1, d_1, e_1, c_2, d_2, e_2\}$. If the pair (b_1, a_2) is selected next, the equivalence classes do not change and thus the resulting next stage is automorphism-equivalent to S_1. Hence, by following LA, player 1 should avoid selecting b_1 and player 2 should avoid selecting a_2. For the same reason, they should also avoid selecting a_1 and b_2.



Hence, by following LA in S_1, the players will select among the set $\{c_1, d_1, e_1, c_2, d_2, e_2\}$. Supposing that they fail again in coordination, we may assume by symmetry that they selected the pair (c_1, d_2). The equivalence classes in the resulting stage S_2 are $\{a_1, b_2\}$, $\{b_1, a_2\}$, $\{c_1, d_2\}$, $\{d_1, c_2\}$ and $\{e_1, e_2\}$. Now, selecting any of the pairs (a_1, b_2), (b_1, a_2), (c_1, d_2) and (d_1, c_2) leads to a next stage which is automorphism-equivalent to S_2. Thus, by following LA in S_2, the players will select the pair (e_1, e_2). This leads to guaranteed coordination in the third round.

We next present the *wait-or-move protocol* WM, which naturally appears in numerous real-life two-player coordination scenarios. Informally, both players alternate (with equal probability) between two choices: the player's own initial choice and another choice that coordinates with the initial choice of the other player.

Definition 4.5. The **wait-or-move protocol** (WM) for repeated two-player WLC-games goes as follows: first randomly select any choice c, and thereafter choose, with equal probability, c or a choice c' that coordinates with the initial choice of the other player—thereby never picking other choices than c and c'. (Definition A.5 in [13] specifies WM in yet further detail.)

The following theorem shows that WM is very fast in relation to ECTs. This holds for *all* two-player WLC-games, not only choice matching games CM_m.

Theorem 4.6. *Let G be a* WLC-*game with one-shot coordination probability p when both players make their first choice randomly, i.e., select a choice according to the uniform distribution over their choice set. Then the expected coordination time by* WM *is at most* $3 - 2p$. *Thus the* ECT *with* WM *is strictly less than* 3 *in every two-player* WLC-*game.*

Proof (*Sketch*)**.** An upper bound for the ECT given by WM can be calculated as follows:

$$p + (1-p) \cdot \sum_{k \geq 2}^{\infty} \frac{2k}{2^k} = p + (1-p) \cdot 3 = 3 - 2p.$$

For all formal details, see the proof of Proposition 4.5 in [13]. □

It follows from the proof of Theorem 4.6 that the ECT with WM is *exactly* $3 - \frac{2}{m}$ in all choice matching games CM_m. Thus the last claim of the theorem *cannot be improved*, as the ECTs of the games CM_m grow asymptotically closer to the strict upper bound 3 when m is increased. In the particular case of CM_2, the ECT with WM is $3-\frac{2}{2} = 2$. Thus the following clearly holds.

Lemma 4.7. *When $S = (\mathsf{CM}_m, \mathcal{H}_k)$ is a non-final stage with exactly two touched edges, then the* ECT *from S with* WM *is 2. Moreover, in any* WLC*-game G, if $S' = (G, \mathcal{H}_k)$ is a non-final stage that is reachable by using* WM*, then the* ECT *from S' with* WM *is at most 2.*

WM eventually leads to coordination with asymptotic probability 1 in all two-player WLC-games. But it does not guarantee (with certainty) coordination in any number of rounds in WLC-games where the winning relation is not the total relation. In a typical real-life scenario, eternal non-coordination is of course impossible by WM, but it is conceivable, e.g., that two computing units using the very same pseudorandom number generator will never coordinate due to being synchronized to swap their choices in precisely the same rounds.

It is easy to show that WM is the unique protocol which gives the optimal ECT (namely, 2 rounds) in the "droitwich-scenario" of the game CM_2 (see the proof of Proposition 4.8 in [13]).

Proposition 4.8. WM *is uniquely* ECT*-optimal in* CM_2.

Next we compare the pros and cons of LA and WM in two-player CM-games. Recall that WM does not guarantee coordination in CM_m (when $m \neq 1$), while LA does guarantee coordination in CM_m if and only if m is *odd*. Concerning *expected* coordination times, it is easy to prove that WM gives a smaller ECT than LA in CM_m for all even m (except for the case $m = 2$, where WM and LA behave identically). Thus we now restrict attention to the games CM_m with odd m. Then, the probability of coordinating in the ℓth round of CM_m using LA, with $\ell \leq \lceil m/2 \rceil$, can relatively easily be seen to be calculable by the formula $P_{\ell,m}$ defined below (where the product is 1 when $\ell = 1$). And using the formula for $P_{\ell,m}$, we also get a formula for the expected coordination time E_m in CM_m with LA:

$$P_{\ell,m} = \frac{1}{m - 2\ell + 2} \prod_{k=0}^{k=\ell-2} \frac{m-2k-1}{m-2k}, \qquad E_m = \sum_{\ell=1}^{\ell=\lceil m/2 \rceil} \ell \cdot P_{\ell,m}.$$

Using this and Theorem 4.6, we can compare the ECTs in CM_m with LA and WM for odd m; see the following table.

m	ECT in CM_m with WM	ECT in CM_m with LA
1	1	1
3	$2 + \frac{1}{3}$	$1 + \frac{2}{3}$
5	$2 + \frac{3}{5}$	$2 + \frac{1}{3}$
7	$2 + \frac{5}{7}$	3
9	$2 + \frac{7}{9}$	$3 + \frac{2}{3}$

Especially the case $m = 7$ is interesting, as the ECT with LA is exactly 3 which is precisely the strict upper bound for the ECTs with WM for the class of all two-player choice matching games CM_m. Furthermore, $m = 7$ is the case where WM becomes faster than LA in relation to ECTs. Thus WM clearly stays faster than LA for all $m \geq 7$, including even values of m.

5. Optimizing guaranteed coordination times of CM-games

In this section we investigate when coordination can be guaranteed in two-player CM-games and which protocols give the optimal GCT for them. We first show that winning can never be guaranteed in choice matching games with an *even* number of choices.

Theorem 5.1. *For all even $m \geq 2$, there is no protocol guaranteeing coordination in* CM_m.

Proof. Let π be a protocol. As π is structural, it is possible that in each round of CM_m, the players pick a pair (c, c') of choices that are structurally equivalent. Suppose this indeed happens. Now, in each round, there are two scenarios relating to the way the players select their choices: (1) they both pick a choice from a touched edge; or (2) they both pick a choice from an untouched edge. As there is always an even number of untouched edges left in the game, the choice of type (2) will never guarantee coordination. And when the players have failed to coordinate so far, they will never succeed by making a choice of type (1) (due to structural equivalence of the choices). □

We then consider choice matching games CM_m with an odd m. Recall that the GCT with LA in these games is $\lceil m/2 \rceil$. The next theorem shows that this is the optimal GCT for CM_m, and moreover, LA is the unique protocol giving this GCT. The proof of Theorem 5.2 is an interesting exercise in the reducing of the automorphism group optimally fast.

Theorem 5.2. *For any odd $m \geq 1$,* LA *is uniquely* GCT*-optimal for* CM_m*. The obtained optimal* GCT *is* $\lceil m/2 \rceil$.

Proof. Let m be odd. Recall that, by Proposition 4.3, the GCT in CM_m with LA is $\lceil m/2 \rceil$ rounds. We assume, for contradiction, that there is some protocol $\pi \neq$ LA that guarantees coordination in CM_m in at most $\lceil m/2 \rceil$ of rounds, possibly less. As $\pi \neq$ LA, there exists some play of CM_m where both players follow π, and in some round, at least one of the players chooses a node on a touched edge; note that it is easy to see that LA never chooses from a touched edge in a CM-game with an odd number of edges (cf. Example 4.4). Now, let $S_\ell = (\mathsf{CM}_m, \mathcal{H}_\ell)$ be the first stage of that play when this happens—so if (c, c') is the most recently recorded pair of choices in S_ℓ, then at least one of c and c' is part of an edge that has already been touched in some earlier round. And furthermore, in all stages $S_{\ell'}$ with $\ell' < \ell$, the most recently chosen pair does not contain a choice belonging to an edge that was touched in some yet earlier round $\ell'' < \ell'$.

In the stage $S_{\ell-1}$ it therefore holds that for every choice profile (c_i, d_i), chosen in some round $i \leq (\ell - 1)$, the nodes c_i and d_i are structurally equivalent. Of course also the nodes of $S_{\ell-1}$ on so far untouched edges are structurally equivalent to each other. Furthermore, the number of already touched edges in $S_{\ell-1}$ is the *even* number $m' = 2(\ell - 1)$.

We next show that π does *not* guarantee a win in $\lceil m/2 \rceil - (\ell - 1)$ rounds when starting from the stage $S_{\ell-1}$. This completes the proof, contradicting the assumption that π guarantees a win in CM_m in at most $\lceil m/2 \rceil$ rounds.

Now, recall the stage S_ℓ from above where (c, c') contained a choice from an already touched edge. By symmetry, we may assume that c is such a choice. Starting from the stage $S_{\ell-1}$, consider a newly defined stage S'_ℓ where the first player again makes the choice c but the other player this time *makes a structurally equivalent choice* $c^* \sim c$. This is possible as π is a structural protocol. Now note that the choice profile (c, c^*) is not winning since c and c^* are structurally equivalent choices from already touched edges, and thus either (c, c^*) is a choice profile that has already been chosen in some earlier round $j < \ell$, or the nodes d, d^* adjacent in CM_m to c^*, c (respectively) form a choice profile (d, d^*) chosen in some earlier round $j < \ell$.

Therefore, in the freshly defined stage S'_ℓ, the players have in every stage (including the stage S'_ℓ itself) selected a choice profile that consists of two structurally equivalent choices. Both choices in the most recently selected choice profile in S'_ℓ have been picked from edges that have become touched even earlier. It now suffices to show that it can still take $\lceil m/2 \rceil - (\ell - 1)$ rounds to finish the game. To see that this is the case, we shall next consider a play from the stage S'_ℓ onwards where in each remaining round, the choice profile (e, e^*) picked by the players consists of structurally equivalent choices; such a play exists since π is structural.

Due to picking only structurally equivalent choices in the remaining play, when choosing a profile from the already touched part, the players will clearly never coordinate. And when choosing from the untouched part, immediate coordination is guaranteed if and only if there is only one untouched edge left. Therefore the players coordinate exactly when they ultimately select from the last untouched edge. As the stage S'_ℓ has precisely $m - 2(\ell - 1)$ untouched edges, winning in this play takes at least $\left\lceil \frac{m-2(\ell-1)}{2} \right\rceil = \lceil m/2 \rceil - (\ell - 1)$ rounds to win from S'_ℓ. □

6. Optimizing expected coordination times of CM-games

In this section we investigate which protocols give the best ECTs for two-player choice matching games. We also investigate when the best ECT is obtained by a unique protocol. We already know by Propositions 4.8 and 4.2 that the optimal ECTs for CM_2 and CM_3 are uniquely given by WM and LA, respectively. Thus it remains to consider the games CM_m with $m \geq 4$.

The main idea in the proofs is to investigate a formula (E), given below, for "locally optimizing" a probability distribution in certain stages $S = (\mathsf{CM}_m, \mathcal{H}_k)$. However, the formula (E) refers to ECTs (E_1 and E_2) from the subsequent stages $(\mathsf{CM}_m, \mathcal{H}_{k+1})$, and thus – in order to optimize the probability distribution in S – we should also know the optimal values for E_1 and E_2. However, finding out these values is problematic, especially because it turns out that E_1 (self-referentially) also denotes the ECT from S. But ultimately, we shall nevertheless "luckily" manage to optimize the probability distribution for S with (E) by giving good enough estimates for the values of E_1 and E_2.

The following auxiliary lemma gives a lower bound $\frac{3}{2}$ for nontrivial ECTs in CM-game stages. The claim of the lemma follows from the fact that in CM-games, the OSCPs below 1 clearly cannot be greater than $\frac{1}{2}$. Hence, if an optimal ECT from a stage S is greater than 1, then it must be at least $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 = \frac{3}{2}$. A complete proof is given in [13].

Lemma 6.1. *The* ECT *from* $(\mathsf{CM}_m, \mathcal{H}_k)$ *with no focal point is at least* $\frac{3}{2}$ *with any protocol.*

We first cover the case $m \geq 6$ and show that then WM is the unique protocol giving the best ECT. The remaining special cases $m = 4$ and $m = 5$ will then be examined.

Theorem 6.2. WM *is uniquely* ECT*-optimal for each* CM_m *with* $m \geq 6$.

Proof. We first present a formula for estimating the best ECTs in stages of CM_m (with any $m \geq 1$). Let $S := (\mathsf{CM}_m, \mathcal{H}_k)$ be a non-final stage with exactly two touched edges. Thus there are $n := m - 2$ untouched edges. Suppose the players use a protocol π behaving as follows in round $k+1$. Both players pick a choice from some touched edge with probability p and from an untouched edge with probability $(1 - p)$. A uniform distribution is used on choices in both classes: probability

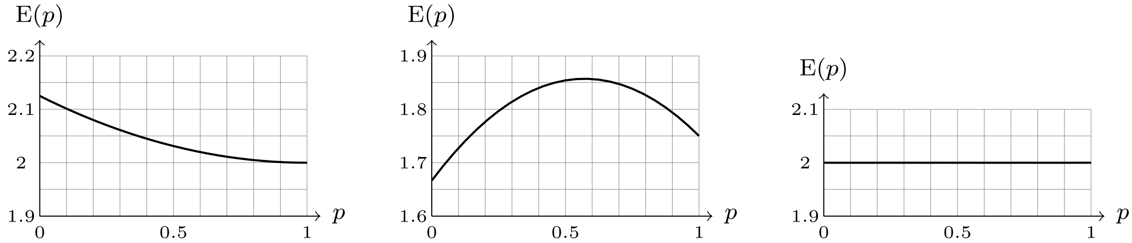


Fig. 2. Graph of (E) with (i) $n = 4$, $E_1 = 2$, $E_2 = \frac{3}{2}$; (ii) $n = 3$, $E_1 = \frac{3}{2}$, $E_2 = 1$; (iii) $n = 2$, $E_1 = E_2 = 2$.

$\frac{p}{2}$ for both choices on touched edges (which makes sense by Lemma B.2 of [13]) and probability $\frac{1-p}{n}$ for each choice on untouched edges (which is necessary with a structural protocol). If one player selects a choice c from a touched edge and the other one a choice c' from an untouched edge, the players win in the next round by choosing the edge with c'. Note that c' is a focal point, so the winning edge can be chosen by a structural protocol with probability 1. (Also other focal points arise which could alternatively be used; cf. Example 3.7):

Suppose then that E_1 is the ECT with π from a stage $(\mathsf{CM}_m, \mathcal{H}_{k+1})$ where both players have chosen a touched edge in round $k+1$ but failed to coordinate. Two different such stages $(\mathsf{CM}_m, \mathcal{H}_{k+1})$ exist, but they are automorphism-equivalent, so π can give the same ECT from both of them by Lemma 3.10. (Indeed, if π gave two different ECTs, it would make sense to adjust it to give the smaller one.) Similarly, suppose E_2 is the ECT with π from a stage $(\mathsf{CM}_m, \mathcal{H}'_{k+1})$ where both players have chosen an untouched edge in round $k+1$ but failed to coordinate. Note that all possible such stages $(\mathsf{CM}_m, \mathcal{H}'_{k+1})$ are renamings of each other, so π gives the same ECT from each one. We next establish that the expected coordination time from $(\mathsf{CM}_m, \mathcal{H}_k)$ with π is now given by the following formula (called formula (E) below):

$$p^2\Big(\frac{1}{2} + \frac{1}{2}(1 + E_1)\Big) \; + \; 2p(1-p)\cdot 2 \; + \; (1-p)^2\left(\frac{1}{n} + \frac{n-1}{n}(1 + E_2)\right) \tag{E}$$

Indeed, both players choose a touched edge in round $k+1$ with probability p^2. In that case the ECT from $(\mathsf{CM}_m, \mathcal{H}_k)$ is $\frac{1}{2} + \frac{1}{2}(1 + E_1)$, the first occurrence of $\frac{1}{2}$ corresponding to direct coordination and the remaining term covering the case where coordination fails at first. Both players choose an untouched edge in round $k+1$ with probability $(1-p)^2$, and then the ECT from $(\mathsf{CM}_m, \mathcal{H}_k)$ is $\frac{1}{n} + \frac{n-1}{n}(1 + E_2)$. The remaining term $2p(1-p)\cdot 2$ is the contribution of the case where one player chooses a touched edge and the other player an untouched one. The probability for this is $2p(1-p)$, and the remaining factor 2 indicates that coordination immediately happens in the subsequent round $k+2$ using the focal point created in round $k+1$.

We then present an informal argument sketch for the case CM_m with $m \geq 6$. We may assume that $E_1 \leq 2$ and $E_2 \geq \frac{3}{2}$ by Lemmas 4.7 and 6.1. Fig. 2 illustrates the graph of (E) with $E_1 = 2$, $E_2 = \frac{3}{2}$, $n = 4$, so then (E) has a unique minimum at $p = 1$ when $p \in [0, 1]$. This suggests that – under these parameter values – the players should always choose a touched edge in stages with exactly two touched edges. Clearly, lowering E_1, raising E_2 or raising n should make it even more beneficial to choose a touched edge. As we indeed can assume that $E_1 \leq 2$ and $E_2 \geq \frac{3}{2}$ in CM_m for $m \geq 6$, this informally justifies that WM is uniquely ECT-optimal in CM_m.

Next we formalize the argument above. Let $S := (\mathsf{CM}_m, \mathcal{H}_k)$, $m \geq 6$, be a non-final stage with precisely two touched edges and S' a stage extending S by one round where the players both choose an untouched edge but fail to coordinate. Let r_1 (respectively, r_2) be the infimum of all possible ECTs from S (respectively, S') with different protocols. Note that by Lemma 3.10, r_1 and r_2 are independent of which particular representative stages we choose, as long as the stages satisfy the given constraints. Let $\epsilon > 0$ and fix some numbers E_1 and E_2 such that $|E_1 - r_1| < \epsilon$ and $|E_2 - r_2| < \epsilon$. We assume $E_1 \leq 2$ and $E_2 \geq \frac{3}{2}$ by Lemmas 4.7 and 6.1. It is easy to show that with such E_1 and E_2, the minimum value of the formula (E) with $p \in [0, 1]$ is obtained at $p = 1$ (for any $n = m - 2 \geq 4$).

Thus, after the necessarily random choice in round one, the above reasoning shows the players should choose a touched edge with probability $p = 1$ in each round. Indeed, assume that the earliest occasion that a protocol π_k assigns $p \neq 1$ occurs in round k. Then, as shown above, the ECT of π_k can be strictly improved by letting $p = 1$ in that round. It is easy to see that the uniform probability over the touched choices should be used (formally this follows from the proof of Lemma B.2 given in [13]). □

We then cover the case for CM_5. The argument is similar to the case for CM_m with $m \geq 6$, but this time leads to the use of LA instead of WM.

Theorem 6.3. *For* CM_5, LA *is uniquely* ECT*-optimal.*

Proof. Recall the formula (E) from the proof of Theorem 6.2. Let $S := (\mathsf{CM}_5, \mathcal{H}_k)$ be a non-final stage with precisely two touched edges and S' a stage extending S by one round where the players both choose an untouched edge but fail to coordinate. The ECT-optimal protocol from S' chooses the unique winning pair of focal points in round $k+2$, so we have $E_2 = 1$. (cf. Example 4.4.) Let r_1 be the infimum of all possible ECTs from S with different protocols. Let $\epsilon > 0$ and fix some real number E_1 such that $|E_1 - r_1| < \epsilon$, assuming $E_1 \geq \frac{3}{2}$ (cf. Lemma 6.1). It is straightforward to show that with these values, and with $n = 3$, the minimum of (E) when $p \in [0, 1]$ is obtained at $p = 0$. (See also Fig. 2 for the graph of (E) when $E_1 = \frac{3}{2}$ for an illustration; even then the figure suggests to choose an untouched edge.)

Thus, after the necessarily random choice in round one, the above reasoning shows that the players should choose an untouched edge with probability 1 in the second round, thereby following LA. Coordination is guaranteed (latest) in the third round. □

Remark 6.4. It is interesting to note that in CM_5 it is not ECT-optimal to optimize OSCP in every round (like WM would do in CM_5). Indeed, in the second round LA gives the OSCP of $\frac{1}{3}$ while the optimal OSCP is $\frac{1}{2}$.

The only remaining case is CM_4. We will see that WM is ECT-optimal there, but not uniquely. Indeed, an equally good protocol is obtained as follows: first play randomly, and if coordination fails, switch to playing WM on the subgame defined by the *untouched* edges. Moreover, this leads to uncountably many protocols with equally good ECT as the players may arbitrarily alternate between the two subgames CM_2 formed by (1) the two edges touched in the first round and (2) the remaining two edges. Quite surprisingly, it also turns out that any probability distribution $p \in [0, 1]$ (for choosing an edge touched in the first round) can be used to achieve an optimal ECT of 2 rounds. We will sketch below a proof for this claim—also implying the ECT-optimality of WM in CM_4. A complete proof of this claim is given in the proof of Theorem 6.4 in [13].

Theorem 6.5. WM *is* ECT-*optimal for* CM_4, *but there are continuum many other protocols that are also* ECT-*optimal. In particular, any probability distribution* $p \in [0, 1]$ *in round* 2 *(for choosing an edge touched in the first round) can be used to achieve an optimal* ECT *of 2 rounds.*

Proof. (Sketch): Recall the formula (E) from the proof of Theorem 6.2. We may assume that $E_1, E_2 \leq 2$, because ECT $= 2$ can be obtained by applying WM either on the touched or untouched edges. We may also assume that $E_1 = E_2$; the justification of this assumption is based on structural resemblances of the scenarios of choosing a touched or untouched edge, see Lemma B.3 and Theorem 6.4 of [13] for a full argument.

We assume, for the sake of contradiction, that $E_1 = 2 - \epsilon$ (and thus $E_2 = 2 - \epsilon$) for some $\epsilon > 0$. With these values of E_1 and E_2, the least value of (E) (namely $2 - \epsilon$) is obtained when $p = 0$ or $p = 1$. Hence, the optimal ECT can be obtained by playing, say, among the touched edges only. However, by iterating this reasoning, the players should obtain the optimal ECT of $2 - \epsilon$ by always selecting among the touched edges. This is a contradiction as such a protocol is indeed WM which gives the ECT of 2 rounds.

Therefore we must have $E_1 = E_2 = 2$. Now (E) has the constant value 2 for all p (see Fig. 2). Thus any $p \in [0, 1]$ gives the least value 2 for (E). In particular, WM gives this optimal ECT for CM_4.

Note also that for any WLC-game, there exist at most continuum many protocols, for the following reasons. Firstly, for any fixed game G, since histories are finite, there exist countably many possibly histories in G. Secondly, for any history, there exits at most continuum many probability distributions for the choices of the next round.

See [13] for the full details of the current proof. □

Note that in the above proof we considered the round 2 as a special case. However, it is easy to see that the proof directly modifies to concern any reachable stage of the game.

Remark 6.6. Note in CM_4 that the values of p which are strictly between 0 and 1 give a smaller OSCP than WM, but they provide an opportunity of creating a focal point which can be used for immediate coordination in the next round. Indeed, if one player selects a touched edge and the other one an untouched edge, all of the nodes in CM_4 become focal points (cf. Example 3.7). However, as there is no unique pair of focal points, some convention may be needed to agree on which winning edge to select. It is interesting that this issue of multiple focal points does not arise in CM_m for any $m \neq 4$ when playing ECT-optimally.

We have thus given a *complete analysis* of optimal ECTs and GCTs in two-player CM-games, summarized in Fig. 1. Appendix D of [13] contains further reflections on the results. We note that the cases for CM_m with small m are exceptionally important from the point of view of applications, as such cases tend to occur more frequently in real-life scenarios.

7. The difficulty of WLC-games based on game size

In this section we give a complete characterization of the upper bounds of optimal ECTs in two-player WLC-games as a function of game size. For any $m \geq 1$, an m-**choice game** refers to any two-player WLC-game $G = (A, C_1, C_2, W_G)$ where $m = \max\{|C_1|, |C_2|\}$. Note that with the classical matrix representation of m-choice games, the parameter m corresponds to the greatest dimension of the matrix. In this section we will also show that CM_m can be regarded as the *uniquely most difficult* m-choice game for all $m \neq 3$, see Theorem 7.5.

Our first theorem shows that the wait-or-move protocol is reasonably "safe" to use in any m-choice game with $m \notin \{3, 5\}$ as it always guarantees an ECT which is at most the upper bound of the optimal ECTs of all m-choice games for the particular m.

Theorem 7.1. *Let $m \notin \{1, 3, 5\}$ and consider an m-choice game $G = (A, C_1, C_2, W_G) \neq \mathsf{CM}_m$. Then the* ECT *in G with* WM *is strictly smaller than the optimal* ECT *in* CM_m.

Proof. By Theorems 6.2, 6.5 and Proposition 4.8, the optimal ECT in CM_m is given by WM. We saw in Section 4 that the ECT with WM is $3 - \frac{2}{m}$ in CM_m and at most $3 - 2p$ in G, where p is the one-shot coordination probability when choosing randomly in G. Since G is an m-choice game, $|W_G| \geq m$. If $|W_G| > m$, then $p > \frac{m}{m^2} = \frac{1}{m}$. And if $|W_G| = m$, we have $p = \frac{m}{mn} = \frac{1}{n} > \frac{1}{m}$ where $n := \min\{|C_1|, |C_2|\} < m$ since $G \neq \mathsf{CM}_m$. In both cases, we have $3 - 2p < 3 - \frac{2}{m}$. □

The **greatest optimal** ECT among a class $\mathcal{G}$ of WLC-games is the value r such that, firstly, r is the optimal ECT for some $G \in \mathcal{G}$; and secondly, for every $G \in \mathcal{G}$, there is a protocol which gives it an ECT $\leq r$. By Theorem 7.1, the greatest optimal ECT among m-choice games is given by WM in CM_m for $m \notin \{1, 3, 5\}$. The case $m = 1$ is trivial, but for the special cases $m = 3$ and $m = 5$, we need to provide a systematic graph-theoretic analysis of all such m-choice games and their ECTs in order to identify the greatest optimal ECT among the class. This analysis is done in the following subsections.

7.1. Analysis of ECTs in 3-choice games

In this section we will show that, among all two-player 3-choice games, the greatest optimal ECT is uniquely realized by the game $G(1 \times 2 + 2 \times 1)$. We also show that the optimal ECT for this game is $\frac{1}{2}(1 + \sqrt{4 + \sqrt{17}})$ ($\approx 1,925$).

We first note that if either of the players has a choice of degree 3 in a 3-choice game G, then the optimal ECT in G is trivially 1. Thus we can restrict our analysis to the 3-choice games (up to structural equivalence) where the degree of each choice is at most 2. Note that the game graph of G must thus consist of components which are either *cycles* or *paths* (in particular, they are subgraphs of the form $G(O_n)$, $G(1 \times 1)$, $G(1 \times 2)$, $G(Z_n)$, $G(\lesssim_n)$; recall the notations from Example 2.2. We here systematically list all such 3-choice games G grouped by the number of edges in the winning relation W_G. (Note that we must have $3 \leq |W_G| \leq 6$ as G is a 3-choice game and the degree of each choice is at least 1 and at most 2.)

$\lvert W_G\rvert = 3$	$\lvert W_G\rvert = 4$	$\lvert W_G\rvert = 5$	$\lvert W_G\rvert = 6$
$G(1 \times 2 + 1 \times 1)$	$G(\lesssim_3)$	$G(O_2 + 1 \times 1)$	$G(O_3)$
$G(3(1 \times 1)) = \mathsf{CM}_3$	$G(Z_2 + 1 \times 1)$	$G(Z_3)$	
	$G(1 \times 2 + 2 \times 1)$		

Among these games, the only ones that do not have a focal point are the games CM_3, $G(O_3)$, $G(1 \times 2 + 2 \times 1)$ which we will analyze separately below.

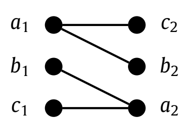
- CM_3: The optimal ECT here is $1 + \frac{2}{3}$ by Proposition 4.2 (see the table in Section 4).
- $G(O_3)$: The OSCP here is $\frac{2}{3}$. Suppose that the players simply make a random choice in every round (with uniform probability distribution). The obtained ECT can then be calculated as follows:

$$\sum_{k\geq 0}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^k (k+1) \;=\; 2 \cdot \sum_{k\geq 0}^{\infty} \frac{k+1}{3^{k+1}} \;=\; 2 \cdot \sum_{k\geq 1}^{\infty} \frac{k}{3^k} \;=\; 2 \cdot \frac{3}{4} \;=\; 1 + \frac{1}{2}.$$

  (It is relatively easy to show that this ECT will indeed be optimal for $G(O_3)$, but there is no need for us to prove it here.)
- $G(1 \times 2 + 2 \times 1)$: The optimal ECT for this game is proven in the proposition below.

Proposition 7.2. *The optimal* ECT *for $G(1 \times 2 + 2 \times 1)$ is $\frac{1}{2}(1 + \sqrt{4 + \sqrt{17}})$.*

Proof. We label the nodes in $G(1 \times 2 + 2 \times 1)$ as follows:



In the initial stage there are two structural equivalence classes:

(1) $\{a_1, a_2\}$ and $\{b_1, c_1, b_2, c_2\}$.

If players fail to coordinate by both selecting a node with degree 2, then the next stage will also be of type (1). However, if they fail to coordinate by selecting choices with degree 1, the equivalence class $\{b_1, c_1, b_2, c_2\}$ is split into two classes with two choices. We may assume by symmetry that the players chose b_1 and b_2, whence we have the following equivalence classes in the next stage:

(2) $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$.

If players fail to coordinate by selecting the pair (a_1, a_2), (b_1, b_2) or (c_1, c_2), then the next stage will also be of type (2). But if they fail to coordinate by one of them selecting from $\{b_1, b_2\}$ and the other one selecting from $\{c_1, c_2\}$, then all symmetries are broken and every choice turns into a focal point—and thus coordination can be guaranteed in the next round.

We first examine a stage S_2 of the type (2) and find the optimal probability distribution for it. The corresponding optimal ECT will be used later for finding the optimal ECT for a stage of type (1).

We first observe that in order to maximize the possibility of breaking symmetries and creating focal points, it is optimal for the players to use the uniform probability distribution for selecting between the sets $\{b_1, b_2\}$ and $\{c_1, c_2\}$. Thus, let p_2 denote the probability for selecting within $\{b_1, b_2, c_1, c_2\}$. Let E_2 denote the ECT for the remaining game if the players fail to coordinate and fail to create a focal point in S_2. (There are several ways how this can happen, but since all the resulting stages are of type (2), we may assume the same ECT for all of them by Lemma 3.10.)

Under the assumptions above, the ECT from S_2, with parameters p_2 and E_2, is given by the following function:

$$\begin{aligned} g(p_2, E_2) &= (1-p_2)^2(1+E_2) + 2p_2(1-p_2) + p_2^2\Big(\frac{1}{2}(1+E_2) + \frac{1}{2}\cdot 2\Big). \\ &= \frac{1}{2}(1+3E_2)p_2^2 - 2E_2p_2 + (1+E_2). \end{aligned}$$

The partial derivate $g_{p_2} = (1+3E_2) - 2E_2$ goes to zero when p_2 has the value $p_2^* := \frac{2E_2}{1+3E_2}$. Whenever $E_2 \geq 1$, the smallest value for $g(p_2, E_2)$ is obtained when $p_2 = p_2^*$. Because both $g(p, E_2)$ and E_2 refer to the ECT from a stage of type (2), E_2 obtains its smallest possible value when $E_2 = g(p_2^*, E_2)$. The only (positive) solution for this equation is $E_2 = \frac{3+\sqrt{17}}{4}$ (≈ 1.781). This is the optimal ECT from any stage of type (2).

Next we will use the value E_2 to determine the optimal ECT from a stage S_1 of type (1). Let E_1 denote the ECT for the remaining game if both players select within the set $\{a_1, a_2\}$. When p_1 denotes the probability of choosing within the set $\{b_1, c_1, b_2, c_2\}$, the ECT from S_1 is given by the following function:

$$\begin{aligned} f(p_1, E_1, E_2) &= (1-p_1)^2(1+E_1) + 2p_1(1-p_1) + p_1^2(1+E_2) \\ &= (E_1+E_2)p_1^2 - 2E_1p_1 + (1+E_1). \end{aligned}$$

The partial derivate $f_{p_1} = (2E_1+2E_2)p_1 - 2E_1$ goes to zero when p_1 has the value $p_1^* := \frac{E_1}{E_1+E_2}$. Whenever $E_1, E_2 \geq 1$, the smallest value for $f(p_1, E_1, E_2)$ is obtained when $p_1 = p_1^*$. Because both $f(p, E_1, E_2)$ and E_1 refer to the ECT from a stage of type (1), E_1 obtains its smallest possible value when $E_2 = \frac{3+\sqrt{17}}{4}$ and we have $E_1 = f(p_2^*, E_2)$. When $E_2 = \frac{3+\sqrt{17}}{4}$, the only (positive) solution for this equation is $E_1 = \frac{1}{2}(1+\sqrt{4+\sqrt{17}})$. This is the optimal ECT from any stage of type (1), and thus, in particular, it is the optimal ECT for the game $G(1\times 2+2\times 1)$. □

Remark 7.3. The optimal ECT for $G(1\times 2+2\times 1)$ is given by protocols that use the optimal values for E_1 and E_2 (given in the proof above) for calculating the probabilities p_1^* (≈ 0.5195) and p_2^* (≈ 0.5616), and the protocols use these probabilities for selecting a choice of degree one in stages of type (1) and (2), respectively. However, there is no unique protocol giving the optimal ECT since there are three winning pairs of focal points that are formed if the players break the symmetry in a stage of type (2).

Also note that, as in the case of CM_5 (c.f. Remark 6.4), it is not ECT-optimal to optimize OSCP in $G(1\times 2+2\times 1)$ in every round. Indeed, it is easy to see that the optimal OSCP for the first round is $\frac{1}{2}$ which is obtained by assigning the probability $\frac{1}{2}$ for playing a choice of degree one.

We conclude that the greatest optimal ECT among all 3-choice games (namely $\frac{1}{2}(1+\sqrt{4+\sqrt{17}})$) is uniquely realized by $G(1\times 2+2\times 1)$.

7.2. Analysis of ECT*s in 5-choice games*

In this section we will show that, among all two-player 5-choice games, the greatest optimal ECT is uniquely realized by the choice matching game CM_5. Recall that this ECT is obtained by the protocol LA by Theorem 6.3 and its value is $2+\frac{1}{3}$.

We first analyze 5-choice games G with $|W_G| > 8$. For such games, the one-shot coordination probability p, when the players make a random choice in the first round, is

$$p \;=\; \frac{|W_G|}{|C_1|\,|C_2|} \;\geq\; \frac{9}{25}.$$

Thus, by Theorem 4.6, the ECT for G by following WM is at most

$$3 - 2p \;\leq\; 3 - 2 \cdot \frac{9}{25} \;=\; 2 + \frac{7}{25} \;<\; 2 + \frac{1}{3}.$$

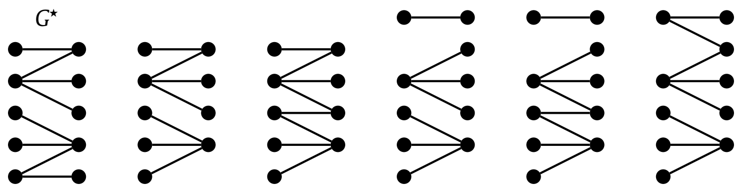
Thus G can be given a smaller ECT than the optimal ECT for CM_5.

Hence we can restrict our analysis to those 5-choice games G whose winning relation W_G has at most 8 edges. Moreover, we may also assume that neither of the players has a choice of degree 5 as otherwise the optimal ECT is trivially 1.

Suppose first that at least one of the players has a choice of degree 4. Since $|W_G| \leq 8$, neither of the players can have more than two such choices and it is impossible that both players have two such choices. If precisely one of the players has precisely one choice of degree 4 (and the other player zero or two such choices), then it is a focal point and the players can immediately coordinate. If one player has two choices, denoted by c and c', of degree 4 and the other player has no such choice, then there are (at least three) choices that are connected to both c and c'. Now the players can coordinate immediately by one of them selecting among $\{c, c'\}$ and the other one selecting among the choices which are connected to both c and c'. Finally, suppose that both players have exactly one choice of degree 4; we denote these by c_1 and c_2. If there is an edge between c_1 and c_2, then both of them are focal points. If there is no edge between c_1 and c_2, then we must have $G = G(1 \times 4 + 4 \times 1)$ as $|W_G| \leq 8$. The ECT for this game is analyzed later on below.

Suppose then that at least one of the players has a choice of degree 3 and none of the choices have a greater degree. As $|W_G| \leq 8$, both players have at most two choices of degree 3. We first show that it is impossible that both players have two choices of degree 3. If player 1 has two choices of degree 3, then (s)he can have at most 4 choices in total as the degree of every choice must be at least one. If also player 2 has two choices of degree 3, then (s)he also has at most 4 choices and thus G cannot be a 5-choice game.

We observe next that there clearly is a focal point in G if precisely one of the players has precisely one choice of degree 3 and the other player zero or two choices of degree 3. Suppose next that one player has two choices, c and c', of degree 3 and the other one has no such choices. Now there must be at least one choice which coordinates with both of c and c', and the players can guarantee coordination when one selects among $\{c, c'\}$ and the other one selects a choice which is connected to both c and c'. Finally, suppose that both players have exactly one choice of degree 3; these choices are denoted by c_1 and c_2. If there is an edge between c_1 and c_2, then they are focal points. If there is no edge between c_1 and c_2, then G must be one of the 5-choice games displayed below (note that these games have been obtained by adding 1 or 2 edges and 1 or 2 nodes to the 4-choice game $G(1 \times 3 + 3 \times 1)$).



All the other games above, except for the leftmost game $G^\star$, have a focal point. The game $G^\star$ is analyzed later on below.

We still need to analyze the case where all of the choices in G have a degree at most 2. The game graph of G must then consist of components which are either cycles or paths. We list here systematically all such 5-choice games G with $|W_G| \leq 8$.

$\lvert W_G\rvert = 5$	$\lvert W_G\rvert = 6$	$\lvert W_G\rvert = 7$	$\lvert W_G\rvert = 8$
$G(2(1 \times 2) + 1 \times 1)$	$G(\lesssim_3 + 1 \times 2)$	$G(O_2 + 1 \times 2 + 1 \times 1)$	$G(O_3 + 1 \times 2)$
$G(1 \times 2 + 3(1 \times 1))$	$G(\lesssim_3 + 2(1 \times 1))$	$G(O_2 + 3(1 \times 1))$	$G(O_3 + 2(1 \times 1))$
$G(5(1 \times 1)) = \mathsf{CM}_5$	$G(Z_2 + 1 \times 2 + 1 \times 1)$	$G(\lesssim_4 + 1 \times 1)$	$G(O_2 + \lesssim_3)$
	$G(Z_2 + 3(1 \times 1))$	$G(Z_3 + 1 \times 2)$	$G(O_2 + Z_2 + 1 \times 1)$
	$G(2(1 \times 2) + 2 \times 1)$	$G(Z_3 + 2(1 \times 1))$	$G(O_2 + 1 \times 2 + 2 \times 1)$
	$G(1 \times 2 + 2 \times 1 + 2(1 \times 1))$	$G(\lesssim_3 + Z_2)$	$G(\lesssim_5)$
		$G(\lesssim_3 + 2 \times 1 + 1 \times 1)$	$G(Z_4 + 1 \times 1)$
		$G(2Z_2 + 1 \times 1)$	$G(\lesssim_4 + 2 \times 1)$
		$G(Z_2 + 1 \times 2 + 2 \times 1)$	$G(Z_3 + Z_2)$
			$G(\lesssim_3 + \gtrsim_3)$

All of the games listed above have a focal point—except for the following four games: CM_5, $G(1 \times 2 + 2 \times 1 + 2(1 \times 1))$, $G(O_3 + 2(1 \times 1))$ and $G(\lesssim_3 + \gtrsim_3)$.

Next we analyze the ECTs for the above-identified 5-choice games G whose optimal ECT is greater than 1 and for which $|W_G| \leq 8$.

- $G(1 \times 4 + 4 \times 1)$: We obtain the ECT of 2 rounds with the following protocol: (1) in the first round, select the choice of degree 4 with probability $\frac{1}{2}$ and some of the choices of degree 1 with the total probability $\frac{1}{2}$; (2) if coordination does not succeed, then continue with WM. It is clear that this gives the same ECT as WM gives in the choice matching game CM_2, this ECT being 2.
- $G^\star$ (see the game graph given above): As in the previous case, we obtain the ECT of 2 rounds by first assigning the probability $\frac{1}{2}$ for selecting the choice with degree 3 and the probability $\frac{1}{2}$ for selecting the choice with degree 2, and by continuing with WM thereafter. Again it is clear that this gives the same ECT of 2 rounds as WM in CM_2.
- $G(\lesssim_3 + \gtrsim_3)$: Again – for practically the same reasons as above – we obtain the ECT 2 by first assigning the probability $\frac{1}{2}$ for selecting the choice which is "in the middle of a 5-choice path" and the total probability $\frac{1}{2}$ for selecting any other choice with degree 2, and by continuing with WM thereafter.
- $G(1 \times 2 + 2 \times 1 + 2(1 \times 1))$: Here the players can follow an optimal protocol for $G(1 \times 2 + 2 \times 1)$ in the corresponding subgame and thus obtain the ECT of less than 2 rounds (see Section 7.1).
- $G(O_3 + 2(1 \times 1))$: Here the players can keep selecting choices randomly within the subgame $G(O_3)$ to obtain the ECT of $1 + \frac{1}{2}$ rounds—as proven in Section 7.1.

Hence, we conclude that the greatest optimal expected coordination time among all 5-choice games (namely $2 + \frac{1}{3}$) is uniquely realized by CM_5.

7.3. *The most difficult two-player* WLC*-games*

By Theorem 7.1 and the analyses for the 3- and 5-choice cases in the previous subsections, we obtain the following results.

Theorem 7.4. *For any m, the greatest optimal* ECT *among m-choice games is given below:*

Game size	$m \in \mathbb{Z}_+ \setminus \{3, 5\}$	$m = 5$	$m = 3$
Greatest optimal ECT	$3 - \frac{2}{m}$	$2 + \frac{1}{3}$	$\frac{1+\sqrt{4+\sqrt{17}}}{2}$ ($\approx$1.925)

Theorem 7.5. *For $m \neq 3$, the greatest optimal* ECT *among m-choice games is uniquely realized by* CM_m.

Hence choice matching games can indeed be seen as *the most difficult* two-player WLC-games—excluding the interesting and important special case of 3-choice games. We also argue that the graph-theoretic analyses in the subsections above are interesting for their own sake and demonstrate the usefulness of representing two-player WLC-games as bipartite graphs.

8. Conclusion

In this paper we gave a complete analysis for two-player CM-games with respect to both GCTs and ECTs, including uniqueness proofs for the related protocols. We also found optimal upper bounds for optimal ECTs for all two-player WLC-games when determined according to game size only. Moreover, our arguments demonstrate the usefulness of representing WLC-games as hypergraphs. The current paper concentrated on the two-player case as this already turned out a challenging question. A complete characterization of the n-player case remains. This is expected to be a highly difficult task that is likely to require sophisticated arguments. Another natural further direction concerns questions relating to two-player common interest games which can be represented in our setting by bipartite graphs with weighted edges. Also, issues of computational complexity relating to WLC-games is an interesting direction.

Data availability

No data was used for the research described in the article

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