Tampere University

Jetro Sihvonen

GLOBAL UNIFORM OUTPUT REGULATION OF NONLINEAR SYSTEMS

Master's thesis
Faculty of Engineering and Natural Sciences
Supervisors: Professor Lassi Paunonen
Postdoctoral Research Fellow Nicolas Vanspranghe
November 2024

ABSTRACT

Jetro Sihvonen: Global uniform output regulation of nonlinear systems
Master's thesis
Tampere University
Master's degree in science and engineering
November 2024

The output regulation problem is a control problem that encompasses tracking, disturbance rejection, synchronization and observer design problems. The goal in output regulation is for a given system to achieve asymptotic tracking of a reference signal. In this thesis the output regulation problem is considered for nonlinear systems. We give sufficient conditions for solvability of both the local and global output regulation problem. This requires considering the solvability of the so called regulator equations and proving the existence of a stabilizing controller such that the system is made uniformly convergent.

We first focus on presenting and discussing sufficient conditions for uniform convergence of systems with inputs. We discuss the Demidovich condition and extend it using contractivity to form a more general result guaranteeing exponential convergence. We consider two stabilizing controllers, one for quadratically stabilizable systems corresponding to the Demidovich condition and one for systems stabilizable using contractivity. Then we solve the regulator equations for a standard harmonic oscillator, a harmonic oscillator with nonlinear damping and some other simple systems. Also we present a known algorithm for solving the regulator equations for control affine systems and apply it to the Hopfield neural network model. Finally we apply the discussed control design methods to the harmonic oscillators and the Hopfield model to solve the uniform output regulation problem.

Keywords: Output regulation, uniform convergence, uniform contractivity, output tracking, disturbance rejection, nonlinear systems, regulator equations, harmonic oscillator, Hopfield model

TIIVISTELMÄ

Jetro Sihvonen: Epälineaaristen järjestelmien globaali tasainen lähdön regulaatio
Diplomityö
Tampereen yliopisto
Tekniikan ja luonnontieteiden maisteri tutkinto
Marraskuu 2024

Regulointi ongelma on säätöongelma, jota voidaan käyttää seuranta-, häiriönpoisto-, synkronointi- ja tarkkailijasuunnitteluongelmien tarkasteluun. Reguloinnin tavoitteena on saada jokin annetun järjestelmän mittaus seuraamaan asymptoottisesti referenssisignaalia siten että järjestelmä samalla hylkää häiriöt. Tässä opinnäytetyössä tarkastellaan regulointi ongelmaa epälineaarisille järjestelmille. Sekä paikallisen että globaalin tasaisen lähdön regulaatio ongelman riittävät ratkaistumisedellytykset esitetään ja tarkastellaan kahdessa pääosassa. Tätä varten tarkastellaan niin sanottujen regulaattoriyhtälöiden ratkaistavuutta, sekä sellaisen stabiloivan säätimen olemassaoloa, että järjestelmästä tulee tasaisesti konvergoituva.

Keskitymme ensin muodostamaan riittäviä ehtoja tasaiselle konvergenssille sisääntuloja sisältäville järjestelmille. Käsittelemme Demidovichin ehtoa ja laajennamme sitä käyttämällä kontraktiivisuuden käsitettä muodostaen yleisemmän tuloksen, joka takaa eksponentiaalisen konvergenssin. Käsittelemme kahta stabiloivaa säädintä, yhtä neliöllisesti stabiloitaville järjestelmille, joilla Demidovichin ehto täyttyy, ja toista järjestelmille, jotka voidaan stabiloida kontraktivuuden avulla. Toisekseen ratkaisemme regulaattoriyhtälöt yksinkertaiselle harmoniselle oskillaattorille, harmoniselle oskillaattorille, jossa on epälineaarinen vaimennus ja joillekin muille yksinkertaisille järjestelmille. Käsittelemme myös erästä tunnettua algoritmia regulaattoriyhtälöiden ratkaisemiseksi ohjausaffiineille järjestelmille ja sovellamme sitä Hopfield-hermoverkkomalliin. Lopuksi sovellamme käsiteltyjä ohjauksen suunnittelumenetelmiä harmonisiin oskillaattoreihin ja Hopfield-malliin tasaisen lähdön regulaatio ongelman ratkaisemiseksi.

Avainsanat: Regulointi, stabiilisuus, tasainen kontraktiivisuus, mittauksen seuranta, häiriönhylkääminen, epälineaariset järjestelmät, regulaattoriyhtälöt, harmoninen oskillaattori, Hopfieldin malli

PREFACE

This thesis project has been exciting and full of meaningful lessons. I am grateful for the guidance, suggestions and patience of my supervisors Lassi Paunonen and Nicolas Vanspranghe. It has been a delight working with you and getting to know the field of mathematical systems theory, while sharing the frustrations of complex technical details. I wish to also thank my family and friends for their support and encouragement. This thesis could not have been possible without you.

Finally thanks be to the Creator, for his marvelous design in which we always find things to ponder upon, to the Son through whom we are saved and through whom we find meaning and to the Holy Spirit who strengthens us and guides us. May all the glory be to the God in the highest.

In Tampere, 18th November 2024

Jetro Sihvonen

CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

δISS	incremental input-to-state stability
$(A)_{ij}$	The element of matrix A on row i and column j
$\mathbb{N}$	Natural numbers, i.e., $\{0, 1, 2, \dots\}$
C^1	Functions that are continuously differentiable
C^∞	Functions that are infinitely continuously differentiable
I	Identity matrix
ISS	Input-to-state stable
$J(x,t)$	Generalized Jacobian
$\|\| \cdot \|\|$	Euclidean vector norm
LMI	Linear matrix inequality
$L_g^n f(x)$	The n:th Lie derivative of f with respect to g
MIMO	Multiple input multiple output
$\overline{\mathbb{PC}}^n$	Class of bounded piecewise continuous n-dimensional vector functions
$\overline{\mathbb{PC}}(W)$	Class of bounded piecewise continuous vector functions that are, for all values, in the set W
PReLU	Parametric Rectified Linear Unit
$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Positive real numbers
$\mathbb{R}^n$	Real vectors of length n
SISO	Single input single output
TAU	Tampere University
A^T	Transpose of matrix A
TUNI	Tampere Universities
UBSS	Uniformly bounded steady state
URL	Uniform Resource Locator

1. INTRODUCTION

In this thesis we study system theory where the goal is to affect or control phenomena that can be modeled mathematically. We focus specifically on cases where a predefined behaviour is required and discuss the tools needed to achieve it.

A system models the behaviour of some chosen variable or set of variables in specific environment. More precisely a system is a mathematical model that describes how the state of the system changes over time. The state of a system consists of chosen parameters of interest and completely describes the system at a specific instance of time. In mathematical systems theory the model consists of one or more differential equations where the solution of the equations is the state of the system and the initial value is the state of the system at zero time, that is a known starting point.

Systems can have so called *inputs*, which model some external parameters that can be constant or varying with time and affect the system's behavior. Likewise systems can have *outputs*, which describe measurable parameters or effects of the system on specific parameters of interest. Different systems can be connected using their inputs and outputs. In this way we can form larger combined systems, the states of which are the states of the connected systems considered together.

To influence the behaviour of a system, inputs called *control inputs*, for which values can be chosen are considered. In order to control the system with these control inputs, we wish to design controllers which are systems that generate a control input as their output. This output can then be connected to the original systems control input to control the system. The controllers in most cases need information about the systems state to be able to generate a useful control input. The information we are able to measure and give to the controller can be thought as an output of the system. These outputs are called *measured outputs*.

Our goal is to study problems where we are given a required behaviour of the system in the form of a *reference signal*. More precisely the reference signal can vary with time or be constant and usually describes the wanted behaviour of some output of the system. In addition to a reference signal we consider a so called *disturbance signal*, which models possible external disturbances that affect the system. Naturally the goal is to control the system so that it is relatively unaffected by the disturbances and the chosen output

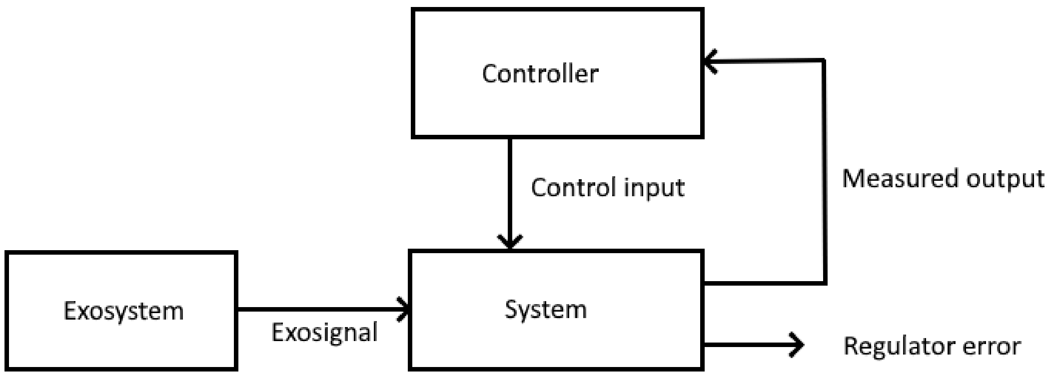


Figure 1.1. *The systems of the output regulation problem and their connections*

follows the reference signal.

The way the system is required to follow the reference signal can vary. In our case we want the output of the system to approach the reference signal with time, that is we want *asymptotic tracking*. To measure whether asymptotic tracking is achieved we define the regulator error, a measure between the reference signal and the output of the system. We then consider this error as an output of the system writing it as e and more precisely requiring that $e(t) \to 0$ as $t \to \infty$.

Also in this thesis we consider the cases where the reference signal and the disturbances can be modeled using another system, which outputs these signals. This external system that can be said to generate the reference and disturbances will be called the *exosystem* and the signal it generates will be called the *exosignal*. Using exosystems gives us a concrete way of describing exosignals and possible assumptions regarding them and is useful in analysis. Problems that consider asymptotic tracking of a reference signal under disturbances, when more formally defined are called *output regulation* problems. An illustration of the different systems of the output regulation problem is shown in Figure 1.1.

Depending on the form of the differential equation describing the system and possible restrictions on the initial conditions, different variants of the output regulation problem can be formulated. The system is called *linear* if the differential equations describing it are linear. Otherwise the system is called *nonlinear*. The problem is called a *global* problem if the goal is to solve it for all initial values, otherwise it is called a *local* problem. In this thesis we consider the global and the local nonlinear output regulation problem.

The output regulation problem is a useful problem formulation in many practical applications. For example systems with periodic and bounded reference signals can be considered through exosystems. Synchronization problems where the goal is to synchronize two different systems as well as observer design problems for autonomous (time independent) systems can be formed as output regulation problems. In this thesis we consider the output regulation of two different harmonic oscillators and a neural network model. [1]

The local output regulation problem has been solved in general for all nonlinear systems and so it is more interesting to focus on the global output regulation problem [1]. Such considerations are also of interest in semiglobal cases, where the problem is solved for a predefined set of initial conditions. Linear systems have also been studied extensively with applications also in approximating nonlinear systems [1]. However, especially when considering global output regulation it is not always possible to approximate real systems using linear theory as in general linear approximations hold only locally. Therefore discussion on nonlinear systems is of interest.

In this thesis we aim to discuss the nonlinear output regulation problem as defined in [1] and expand slightly on the existing theory. We will focus on the *global* problem, but will also give a *local* variant. In addition to formulating the output regulation problems, we discuss methods to show their *solvability* and some simple *controller design* methods.

The conditions for solvability that we will discuss can roughly be divided into three parts. First we need the problem to be well-posed. This will be achieved through continuity and boundedness assumptions discussed shortly in Chapter 2 for general systems. Second the system needs to be stable in the sense that solutions of the system's differential equations corresponding to different initial conditions approach each other. Stability will be the focus of Chapter 2. Thirdly there must exist a state of the system and a control input where the output corresponds to the trajectory to be tracked. This requirement will be discussed in Chapter 3, where these requirements are brought together in the problem formulation.

We will discuss some different methods for verifying the solvability conditions that will be formulated. Three different stability conditions are presented in Chapter 2. The first of which is a set of so called Lyapunov functions that guarantee stability, the second is the so called Demidovich condition and the third uses a stability notion called contractivity. The first two are discussed in [1], while the third we formulate using [2] and [3]. The condition using contractivity is more general than the Demidovich condition and we use it to extend the theory presented in [1]. Using contractivity is also simpler than the method using the Lyapunov functions directly and so of interest.

To guarantee the existence of a state and control that correspond to the reference signal a set of differential equations called the *regulator equations* of the system must be solvable. We will discuss an algorithm from [4] that can be used to solve them for a specific class of systems in Section 3.3. Other methods, including direct computation will also be discussed and demonstrated.

In Chapter 4 we use the solvability conditions discussed and theory from [1] to design controllers for three different types of systems. First we consider quadratically stabilizable systems, for which the Demidovich condition can be applied to solve the global output regulation problem. Then we discuss systems that can be made to be contractive to solve the global problem. Finally we present a simple controller for the local output regulation problem.

To showcase the controller design methods and the algorithm for solving the regulator equations we apply these results in Chapter 5. We study three cases two of which are harmonic oscillators, one standard one and one with a nonlinear damping term. The final case study is on Hopfield systems, which are used to model neural networks [5].

2. STABILITY OF NONLINEAR SYSTEMS

A central question in control problems is whether a system is stable in the sense that solutions of the system remain close to each other. Therefore stability is a key part in considering output regulation problems as well. In this chapter some common stability properties and methods we will utilize will be introduced. These include the concepts of uniform stability, uniform convergence, input-to-state stability and contractivity. The goal is to give conditions that guarantee *uniform convergence*, which we utilize to solve the output regulation problem. Three such conditions are formed using Lyapunov functions, the Demidovich condition and contractivity.

2.1 Uniform stability

The first concept introduced is uniform stability as it forms a basis for uniform convergence. For this consider a system

$$\dot{x} = f(x,t) \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state and $t \in \mathbb{R}$. A solution of the system will be denoted by $x(t)$, and the initial state by $x_0 = x(0)$. First when analyzing such systems, the existence and uniqueness of solutions corresponding to different initial conditions should be considered. For this we assume that f is piecewise continuous in t and locally Lipschitz continuous everywhere in x.

By system (2.1) being locally Lipschitz continuous everywhere in x we mean that on each closed interval $[a,b] \subset [0,\infty)$ the following condition applies. For each $x \in \mathbb{R}^n$, there exists a neighborhood D of x and a constant $L > 0$ such that for all $z, y \in D$ and all $t \in [a,b]$

$$\|f(z,t) - f(y,t)\| \leq L\|z - y\|. \tag{2.2}$$

These continuity assumptions on system (2.1) guarantee the existence of unique solutions locally. When these solutions are known to be bounded, there exists a unique solution that is defined for all $t \geq 0$, see eg. [6, Thm. 3.3]. Boundedness of solutions will be achieved later separately. To shorten notation when meaning that a function is locally Lipschitz continuous everywhere in x, we will write that the function is locally Lipschitz in x.

A system is said to be globally Lipschitz if there exists $L > 0$ such that (2.2) is satisfied for all $z, y \in \mathbb{R}^n$ and $t \in [a, b]$. The solutions of such systems are unique on $[a, b]$ [6, Thm. 3.2]. The global Lipschitz condition is stronger than the condition of locally Lipschitz in x so no additional knowledge of solutions is needed to establish uniqueness. However, we will use the local variant as it is sufficient.

Definition 2.1 ([1]). A solution of (2.1), $\bar{x}(t)$, defined for $t \in (0, \infty)$ is said to be

- uniformly Lyapunov stable if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|x_0 - \bar{x}_0\| < \delta$ implies $\|x(t) - \bar{x}(t)\| < \epsilon$.
- uniformly asymptotically stable if it is uniformly Lyapunov stable and there exists $\delta_* > 0$ such that for any $\epsilon > 0$ there exists $T = T(\epsilon) > 0$ such that $\|x_0 - \bar{x}_0\| < \delta_*$ implies $\|x(t) - \bar{x}(t)\| < \epsilon$ for all $t \geq T$.
- exponentially stable if there exist $\delta > 0$, $C > 0$, $\beta > 0$ such that $\|x_0 - \bar{x}_0\| < \delta$ implies
$$\|x(t) - \bar{x}(t)\| \leq Ce^{-\beta(t)}\|x_0 - \bar{x}_0\|, \quad \forall t \geq 0$$

It is noted that exponential stability implies uniform asymptotic stability. Also it is noted that these properties are *local* in the sense that they hold in some neighborhood of $\bar{x}_0$. To be useful in the global setting and wider predefined sets of initial conditions the following definitions are needed.

Definition 2.2 ([1]). A solution $\bar{x}(t)$ of system (2.1), which is defined for $t \in (0, \infty)$, is said to be

- asymptotically stable in a set $Z \subset \mathbb{R}^n$ if it is asymptotically stable and any solution of system (2.1) starting in $x_0 \in Z$ satisfies $\|x(t) - \bar{x}(t)\| \to 0$ as $t \to \infty$.
- uniformly asymptotically stable in $Z \subset \mathbb{R}^n$ if it is uniformly asymptotically stable and for any compact set $K \subset Z$ and any $\epsilon > 0$ there exists $T(\epsilon, K) > 0$ such that if $x_0 \in K$, then $\|x(t) - \bar{x}(t)\| < \epsilon$ for all $t \geq T(\epsilon, K)$.
- exponentially stable in $Z \subset \mathbb{R}^n$ if it is exponentially stable and there exist constants $C > 0$ and $\beta > 0$ such that any solution starting in $x_0 \in Z$ satisfies
$$\|x(t) - \bar{x}(t)\| \leq Ce^{-\beta(t)}\|x_0 - \bar{x}_0\|$$

If $Z = \mathbb{R}^n$ the solution is called globally (asymptotically, exponentially) stable.

Usually these definitions are given with respect to the origin, as systems can be transformed using the co-ordinate transformation $y := x - \bar{x}$ without loss of generality so that for the transformed system, $y(t) = 0$, $\forall t \in \mathbb{R}$ corresponds to $x(t) = \bar{x}(t)$, $\forall t \in \mathbb{R}$. However in analyzing convergence locally this transformation would make the region of attraction, that is the region where the stability properties hold, time dependent and thus analysis more complicated. Thus the definitions are given with respect to a solution $\bar{x}(t)$. Considering the global form no complications arise with the region of attraction and Definition 2.2 for $Z = \mathbb{R}^n$ is equivalent to standard forms. Regardless even then the co-ordinate transforms introduce additional complexity, so we consider this form for the global form as well. [1]

2.2 Stability with respect to inputs

Having introduced uniform stability, stability of systems when adding external inputs is considered next. In output regulation problems, the time dependence of a system is commonly due these inputs. Therefore systems of the form

$$\dot{x} = f(x, w), \tag{2.3}$$

where the state is $x \in \mathbb{R}^n$ and the input $w \in \mathbb{R}^m$, are considered. We assume that that $f \in C^1$. This then implies that $x, t \mapsto f(x, w(t))$ is locally Lipschitz everywhere in x for all piecewise continuous functions $w(\cdot)$ [6, Lemma 3.2].

The inputs must satisfy certain smoothness and boundedness properties. To condense notation of these assumptions input classes are introduced following [1, pg. 16].

Definition 2.3. Consider a set $W \subset \mathbb{R}^m$ and a function $w : \mathbb{R} \to W$.

- The class of inputs $\overline{\mathbb{PC}}(W)$ contains the function w if it is piecewise continuous and if there exists a compact set $K \subset W$ such that $w(t) \in K$ for all $t \in \mathbb{R}$.
- The second class of inputs is defined as $\overline{\mathbb{PC}}^m = \overline{\mathbb{PC}}(\mathbb{R}^m)$ and contains all bounded, piecewise continuous vector-valued functions.
- The class of inputs $I_s(W)$ contains functions that are generated by the system

  $$\dot{w} = s(w), \quad w \in \mathbb{R}^m, \tag{2.4}$$

  with a locally Lipschitz function $s : \mathbb{R}^m \to \mathbb{R}^m$ and initial values $w_0 \in W$. The set W is required to be invariant with respect to the system, that is $w_0 \in W \implies w(t) \in W$, $\forall t \in \mathbb{R}$.

While the first two classes of inputs just require smoothness and boundedness, the third class allows to consider inputs as signals generated by an external system, which will be referred to as the *exosystem*. This is not only useful in practice, where inputs may be generated by such systems, but also in analysis as the systems can be combined to consider a composite system without an input. The requirement of invariance guarantees that for $w(\cdot) \in I_s(W)$, we have $w(t) \in W$, $\forall t \in \mathbb{R}$. The functions in the class of inputs $I_s(W)$ are not necessarily bounded as W may be unbounded. Therefore the first two classes of inputs are necessary to describe instances where boundedness is required, as well as instances where the input does not necessarily need to be generated by a system of form (2.4).

As the input functions w are assumed to be piecewise continuous, $x, t \mapsto f(x, w(t))$ in (2.3) is piecewise continous with respect to time. We can then consider the system

$$\dot{x} = f(x, w(t)), \tag{2.5}$$

for some fixed piecewise continuous input function $w(\cdot)$. Since we also assumed that f is locally Lipschitz everywhere in x this system corresponds to system (2.1).

In the analysis of stability properties with inputs the notions of certain function classes are also highly useful in simplifying notation. We introduce these classes next, following [1, pg. 12].

Definition 2.4. A continuous function $\alpha : [0, a) \to [0, \infty)$ belongs to class $\mathcal{K}$ if

- it is strictly increasing, that is $b > c \implies \alpha(b) > \alpha(c)$
- it satisfies $\alpha(0) = 0$.

If in addition $a = \infty$ and $\lim_{r\to\infty} \alpha(r) \to \infty$, the function belongs to class $\mathcal{K}_\infty$.

Definition 2.5. A continuous function $\beta : [0, a) \times [0, \infty) = [0, \infty)$ belongs to class $\mathcal{KL}$ if

- for each fixed s, the mapping $\beta(\cdot, s)$ belongs to class $\mathcal{K}$
- for each fixed r the mapping $\beta(r, \cdot)$ is decreasing and $lim_{s\to\infty}\beta(r, s) = 0$.

Next we use these notions to introduce *input-to-state stability*, which describes the property that bounded inputs produce bounded states.

Definition 2.6 ([1])**.** Consider system (2.3) and suppose $f(0, 0) = 0$. The system is *locally input-to-state stable* if there exists a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ, and constants $k_x > 0$ and $k_w > 0$ such that for any initial state x_0 with $\|x_0\| \leq k_x$ and any input $w(\cdot)$ with $\|w(t)\| \leq k_w$, $\forall t \geq 0$, the solution $x(t)$ exists and satisfies

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\sup_{0\leq\tau\leq t} \|w(\tau)\|)$$

for all $t \geq 0$. It is said to be input-to-state-stable (ISS) if the inequality is satisfied for any initial state x_0 and input $w(\cdot) \in \overline{\mathbb{PC}}^m$.

Here the definition requires the origin to be an equilibrium. However later this will not be required as input-to-state convergence is defined, which is a combined property that requires a type of input-to-state stability. Input-to-state stability facilitates simple analysis of interconnected systems [1]. It is also equivalent to numerous other stability notions [7]. Thus it is a natural choice to describe stability with respect to inputs.

The property of ISS is connected with existence of bounded solutions for different inputs.
Lemma 2.7 ([1]). *Suppose that the system (2.3) is ISS. Then for every input $w(\cdot) \in \overline{\mathbb{PC}}^m$ there exists a solution $x_w(t)$ that is defined for all $t \in \mathbb{R}$ and satisfies*

$$\sup_{t\in\mathbb{R}} \|x_w(t)\| \leq \gamma\left(\sup_{t\in\mathbb{R}} \|w(t)\|\right)$$

for all $t \in \mathbb{R}$, where γ is the class $\mathcal{K}$ function corresponding to the definition of ISS.

Input-to-state stability of a system may be determined using an ISS-lyapunov function. We rewrite the result here for completeness.
Lemma 2.8 ([4]). *Consider system (2.3). Let $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^+$ be a C^1 function such that*

$$\alpha_1(\|x\|) \leq V(x,t) \leq \alpha_2(\|x\|),$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,w) \leq -\alpha_3(\|x\|) + \rho(\|w\|)$$

for all $(x,w) \in \mathbb{R}^n \times \mathbb{R}^m$, where α_1, α_2, α_3 are class $\mathcal{K}_\infty$ functions and ρ is a class $\mathcal{K}$ function. Then system (2.3) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1} \circ \rho$. The function V is called an ISS-Lyapunov function.

2.3 Uniform convergence of systems with inputs

In this section the definitions of different types of convergent systems are given following [1, pg. 15].
Definition 2.9. The system (2.1) is uniformly convergent in $Z \subset \mathbb{R}^n$ when

- there exists a solution $\bar{x}(t)$ that is defined and bounded for all $t \in \mathbb{R}$,
- $\bar{x}(t)$ is uniformly asymptotically stable in Z.

If $Z = \mathbb{R}^n$, then the system is globally uniformly convergent.

The solution $\bar{x}(t)$ is called a *steady-state solution* and thus distinguished from other solutions $x(t)$. While for other types of convergent systems this solution is not necessarily unique among trajectories with initial values $x_0 \in Z$, for uniformly convergent systems it is [1, Prop. 2.15]. Therefore we will call the solution $\bar{x}(t)$ *the* steady-state solution when considering uniformly convergent systems.

The set Z is referred to as the convergence region. If the steady-state solution of a uniformly convergent system in Z is exponentially stable, then the system is called exponentially convergent. Next we define convergence with respect to inputs using input classes.
Definition 2.10 ([1]). System (2.3) is uniformly (exponentially) convergent in a set $Z \subset \mathbb{R}^n$ for a class of inputs $\mathcal{N} \subset \overline{\mathbb{PC}}^m$ if system (2.5) is uniformly (exponentially)

convergent in Z for every input $w(\cdot) \in \mathcal{N}$.

To use the concept of input-to-state stability together with convergence input-to-state convergence is defined.

Definition 2.11 ([1]). System (2.3) is said to be input-to-state convergent if it is globally uniformly convergent for the class of inputs $\overline{\mathbb{PC}}^m$ and it is ISS with respect to each steady-state solution $\bar{x}_w(t)$ corresponding to some input $w(\cdot) \in \overline{\mathbb{PC}}^m$. By ISS with respect to the steady-state solution $x_w(t)$ we mean that there exists a $\mathcal{KL}$-function β and a $\mathcal{K}_\infty$-function γ such that any solution $\bar{x}(t)$ of system (2.3) corresponding to some input $w(t) + \Delta w(t)$ satisfies

$$\|x(t) - \bar{x}_w(t)\| \leq \beta(\|x_0 - \bar{x}_{w0}\|, t) + \gamma\left(\sup_{0 \leq \tau \leq t} \|\Delta w(\tau)\|\right). \tag{2.6}$$

In general, the functions β and γ in Definition 2.11 may depend on the particular input $w(\cdot)$. Here input-to-state stability is required with respect to the steady-states. This is similar to the more general condition, incremental input-to-state stability (δISS), where *any* two trajectories are required to satisfy (2.6) [8]. This extension allows us to consider systems where zero is not an equilibrium solution, that is systems where $f(0,0) \neq 0$, allowing for analysis of more general systems without a coordinate transform.

Input-to-state convergence is a global property. Thus another description of stability with respect to the inputs is needed, one that is applicable to local cases in the context of convergence. We use a somewhat more general property defined in [1, Def. 2.17].

Definition 2.12. The system (2.3) that is uniformly convergent in Z for a class of inputs $\mathcal{N}(W) \subset \overline{\mathbb{PC}}(W)$ is said to have the uniformly bounded steady state (UBSS) property if for any compact set $C \subset W$ there exists a compact set $D \subset \mathbb{R}^n$ such that for any input $w(\cdot) \in \mathcal{N}(W)$ the following implication holds:

$$w(t) \in C, \quad \forall t \in \mathbb{R} \implies x_w(t) \in D, \quad \forall t \in \mathbb{R}.$$

Input-to-state convergence implies global uniform convergence with the UBSS property [1, Prop. 2.19].

We introduce two different conditions that guarantee uniform convergence with the UBSS property or input-to-state convergence. The first one is a variant of the Demidovich condition generalized for inputs in [1]. The condition's application to stability is usually called *Krasovskii's method* [5].

Theorem 2.13 ([1]). *Consider system (2.3) and let the function* $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *be* C^1. *Suppose there exist matrices* $P = P^T > 0$ *and* $Q = Q^T > 0$ *such that*

$$P\frac{\partial f}{\partial x}(x, w) + \frac{\partial f^T}{\partial x}(x, w)P \leq -Q, \quad \forall x \in \mathbb{R}^n, \quad \forall w \in W \subset \mathbb{R}^m.$$

Then system (2.3) is globally exponentially convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}(W)$. If $W = \mathbb{R}^m$, then system (2.3) is input-to-state convergent.

The second condition uses Lyapunov-like functions and is more general than the Demidovich condition, which can be seen as a particular case corresponding to quadratic Lyapunov-like functions.

Theorem 2.14 ([1]). *Consider system (2.3). Suppose there exist C^1 functions $V_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ and $V_2 : \mathbb{R}^n \to \mathbb{R}^+$, $\mathcal{K}$-functions α_2, α_3, α_5, γ, and $\mathcal{K}_\infty$-functions α_1, α_4 satisfying the conditions*

$$\alpha_1(\|x_1 - x_2\|) \le V_1(x_1, x_2) \le \alpha_2(\|x_1 - x_2\|),$$
$$\frac{\partial V_1}{\partial x_1}(x_1, x_2)F(x_1, w) + \frac{\partial V_1}{\partial x_2}(x_1, x_2)F(x_2, w) \le -\alpha_3(\|x_1 - x_2\|),$$
$$\alpha_4(\|x\|) \le V_2(x) \le \alpha_5(\|z\|),$$
$$\frac{\partial V_2}{\partial x}(x)F(x, w) \le 0 \text{ for } \|x\| \ge \gamma(\|w\|),$$

for all x_1, x_2, $x \in \mathbb{R}^n$ and all $w \in \mathbb{R}^m$. Then system (2.3) is globally uniformly convergent and has the UBSS property for the class of inputs $\overline{\mathbb{PC}}^m$.

Finally we give a condition for *local* exponential convergence where there exists some neighborhood of the origin where the system in question is exponentially convergent.

Theorem 2.15 ([1]). *Consider system (2.3) with $f(0, 0) = 0$ and f being C^1 in some neighborhood of the origin, $(x, w) = (0, 0)$. Let $\mathcal{N} \subseteq \mathbb{PC}^m$ be some class of inputs containing the zero input, $w(t) = 0$. The following statements are equivalent:*

(i) System (2.3) is locally exponentially convergent for the class of inputs $\mathcal{N}$

(ii) The matrix Jacobian matrix at the origin, $(\partial F/\partial x)(0, 0)$ is Hurwitz.

2.4 Contractivity analysis

Contractivity is a stability notion that describes a set of initial conditions for which trajectories of a system converge to each other. In application it is similar to the concept of exponential convergence, but with the difference that contractivity does not require bounded steady states. However as will be seen, contractivity, when assuming bounded trajectories is in fact equivalent to exponential convergence. Thus it may be used to achieve another condition for uniform convergence.

Contractivity can be defined in numerous ways as demonstrated in [9]. Following the technique in [2], we consider the solutions of (2.1), $x(t)$, as trajectories and the analyze the relationship between two neighboring trajectories. To describe the distance between these neighboring trajectories we consider a virtual or infinitesimal displacement δx. The dynamics of virtual displacements are described by the first variation equation

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x,t)\delta x. \tag{2.7}$$

In using virtual displacements a quadratic Lyapunov function of the form $V(x, \delta x, t) = \delta x^T M(x,t)\delta x$, where $M : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is a matrix function can be considered. This Lyapunov function can be used to analyze the dynamics of the trajectories of system (2.1) with the goal of forming a condition using M that guarantees that neighboring trajectories approach each other. First the matrix function $M : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$ is required to be uniformly positive definite. Secondly it it must satisfy $M(x,t) = \Theta(x,t)^T\Theta(x,t)$, for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ for some invertible $\Theta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$.

By uniformly positive definite M it is meant that there exists $\alpha > 0$ such that for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$\frac{1}{2}(M(x,t)^T + M(x,t)) \leq \alpha I.$$

Since M is symmetric, this is equivalent to $M(x,t) \leq \alpha I$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. The matrix function M is formally a *time-dependent Riemannian metric*. By the notation $\dot{M}$ we mean the matrix function $\dot{M} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$ with entries

$$\dot{M}(x,t)_{ij} = \frac{\partial M_{ij}}{\partial t}(x,t) + \nabla_x M_{ij}(x,t) \cdot f(x,t) \tag{2.8}$$

for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $i, j \in [1, \ldots, n]$. [9]

Using the above assumptions and notations the Lyapunov function

$$V(x, \delta x, t) = \delta x^T M(x,t)\delta x \tag{2.9}$$

can be considered with the local coordinate transform $z = \Theta(x,t)x$, which leads to a coordinate change of the virtual displacements

$$\delta z = \Theta(x,t)\delta x. \tag{2.10}$$

Then analyzing the time derivative of this transformed virtual displacement along the trajectories $x(t)$ of system (2.1) leads to a sufficient condition guaranteeing that different trajectories converge to each other. Next we use this condition derived in [2] to define contractivity.

Definition 2.16 ([2]). Given a system $\dot{x} = f(x,t)$, a set of initial conditions $Z \subset \mathbb{R}^n$ is called a *contraction region* with respect to a uniformly positive definite metric $M(x,t) = \Theta(x,t)^T\Theta(x,t)$ if there exists $\beta_M > 0$ such that

$$J(x,t) := \left(\dot{\Theta}(x,t) + \Theta(x,t)\frac{\partial f}{\partial x}(x,t)\right)\Theta^{-1}(x,t) \leq -\beta_M I$$

or equivalently

$$\frac{\partial f^T}{\partial x}(x,t)M(x,t) + M(x,t)\frac{\partial f}{\partial x}(x,t) + \dot{M}(x,t) \leq -2\beta_M M(x,t) \tag{2.11}$$

for all $x \in Z$ and all $t \in \mathbb{R}$. In this case, J is called the *generalized Jacobian*. The constant β_M is the contraction rate.

Theorem 2.17 ([2]). *Consider the system (2.1) and a solution of that system $\bar{x}(t)$. Any solution $x(t)$ which starts in a ball, with respect to metric M centered around $\bar{x}(t)$ and contained at all times in a contraction region with respect to M, remains in that ball and converges exponentially to $\bar{x}(t)$.*
Furthermore global exponential convergence to the given solution $\bar{x}(t)$ is guaranteed if the whole state space is a contraction region with respect to the metric M.

Theorem 2.18 ([2]). *Consider the system (2.1) and assume that it is exponentially convergent. Then the system is contracting with respect to a uniformly positive definite metric M.*

These theorems connect contractivity to exponential convergence and vice versa. Thus, when there exists a bounded trajectory, the system is exponentially convergent if and only if there exists a metric M satisfying the conditions in Definition 2.16. This metric is called a contraction metric.

Compared to the Lyapunov functions in Theorem 2.14 guaranteeing convergence, the contractivity approach seems appealing. If we know that trajectories are bounded, we simply need to solve for a contraction metric M or show its existence. This can be simpler than the task of finding a Lyapunov function. Naturally this simplification leads to decreased generality as Theorem 2.14 guarantees uniform convergence, while Theorem 2.17 gives exponential convergence. However in situations where exponential convergence is required, the contractivity condition is necessary and no generality is lost.

On the other hand compared to the Demidovich condition, Theorem 2.13, the contractivity condition is clearly more general and complex. This can be seen by considering (2.11) with a metric that is constant with respect to both state and time. The condition becomes

$$\frac{\partial f^T}{\partial x}M + M\frac{\partial f}{\partial x} \leq -2\beta_M M, \tag{2.12}$$

because $\dot{M} = 0$. This can be seen to directly correspond to the Demidovich condition in Theorem 2.13. Contraction metrics can be guessed or in some simple cases computed directly. Also a numerical approach could be used, with one such presented for example in [10].

Another definition of contractivity using one sided Lipschitz constants, discussed in [5], allows for handling of other norms in addition to the Euclidean norm. However the discussion is on the cases corresponding to constant contraction metrics, and thus the

Euclidean norm case discussed in [5] corresponds to Theorem 2.13. Other variants of contractivity and historical connections to similar ideas are discussed extensively in both [5] and [9].

Further considering the relationship between convergence and contractivity the fact that they are not equivalent is demonstrated in [9]. However as shown, the concepts are complementary. In fact the use of contractivity to achieve convergence is a common method of analysis used especially with autonomous systems; first contractivity analysis is used to show that a system is incrementally stable and then convergent dynamics are constructed [9].

2.5 Universal contractivity of systems with inputs

In this section we consider contractivity in the context of systems with inputs. For this we consider system (2.3). The goal is to achieve input-to-state convergence, and since Theorem 2.17 guarantees uniform convergence, the focus will be on connecting contracting systems with inputs to ISS. Contractivity does not directly imply ISS, but with some simple assumptions, namely universal contractivity, there is a direct link to incremental input-to-state stability (δISS) [3], which as discussed in Section 2.3 is similar to the type of ISS needed for input-to-state convergence. The connection is formulated precisely here. First we define universal contractivity following [3].

Definition 2.19. The system (2.3) is *uniformly contracting* in a set of initial conditions $Z \subset \mathbb{R}^n$ and for a set of inputs $W_r \subset \mathbb{R}^m$ when $\frac{\partial f}{\partial w}$ is bounded and (2.3) is contracting in Z with respect to a metric $M = \Theta^T \Theta$ for all $w \in W_r$, that is there exists $\beta_M > 0$ such that

$$J(x, w, t) := \left(\dot{\Theta}(x, w, t) + \Theta(x, t) \frac{\partial f}{\partial x}(x, w, t) \right) \Theta^{-1}(x, t) \leq -\beta_M I,$$

for all $x \in Z$, $w \in W_r$ and $t \in \mathbb{R}$. The constant β_M is the contraction rate and J is the generalized Jacobian of the system. If $Z = \mathbb{R}^n$, the system is *globally uniformly contracting*.

When considering the system (2.3), we define the matrix function $\dot{M} : \mathbb{R}^{n+m} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ such that its entries are

$$\dot{M}_{ij}(x, w, t) = \frac{\partial M_{ij}}{\partial t}(x, t) + \nabla_x M_{ij}(x, t) \cdot f(x, w) \tag{2.13}$$

for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $t \in \mathbb{R}$ and $i, j \in [1, \ldots, n]$. This notation corresponds to (2.8) and explains the reason why we write $\dot{\Theta}$ with a dependency on w.

The Grönwall's Lemma is a highly useful result in contracting applications. The variant we use can be found for example in [11, Lemma 2.1].

Lemma 2.20. *Suppose that a C^1 function $v : \mathbb{R}_+ \to \mathbb{R}$ satisfies*

$$\dot{v}(t) \leq -\gamma v(t) + c, \quad v(0) = v_0, \quad \forall t \in \mathbb{R}_+,$$

where $0 < \gamma \in \mathbb{R}$, $c \in \mathbb{R}$ and $v_0 \in \mathbb{R}$. Then the inequality

$$v(t) \leq v_0 e^{-\gamma t} + \frac{c}{\gamma}(1 - e^{-\gamma t}),$$

holds $\forall t \in \mathbb{R}_+$

Another lemma that will be used is Young's inequality with ϵ.
Lemma 2.21. *Let $a, b \in \mathbb{R}^n$ then for arbitrary $\epsilon > 0$*

$$a^T b \leq \frac{a^T a}{2\epsilon} + \frac{\epsilon b^T b}{2}$$

Lemma 2.22. *Consider system (2.3), a class of inputs $\mathcal{N}$ and a set $W_r \subset \mathbb{R}^m$ such that for all $w(\cdot) \in \mathcal{N}$, $w(t) \in W_r$ for all $t \in \mathbb{R}$. Suppose there exists a metric $M : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times n}$ such that the system (2.3) is globally uniformly contracting for the set of inputs W_r at a rate λ with respect to M. Suppose also that M is bounded in the sense that there exists $\sigma_{min}, \sigma_{max} > 0$ such that*

$$\sigma_{min}^2 I \leq M(x,t) \leq \sigma_{max}^2 I, \quad \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}. \tag{2.14}$$

Finally assume that for an input $w(\cdot) \in \mathcal{N}$, there exists a corresponding bounded trajectory $x_w(\cdot)$ of (2.3). Then the system is input-to-state stable with respect to $x_w(\cdot)$.

Proof. The proof is based on the techniques in the proof of [12, Lemma 2.1]. We consider the steady state $x_w(\cdot)$ corresponding to input $w(\cdot)$ and a solution of the system (2.3), $x(\cdot)$, corresponding to input $w(\cdot) + \Delta w(\cdot)$ and with initial values $x_w(0) = x_{w0}$ and $x(0) = x_0$. First we form line segments s between x_0 and x_{w0}, and u between $w(\cdot)$ and $w(\cdot) + \Delta w(\cdot)$. These are defined by

$$s(\alpha) = \alpha x_0 + (1 - \alpha) x_{w0}, \quad \alpha \in [0, 1],$$

$$u(\beta, t) = w(t) + \beta \Delta w(t), \quad \beta \in [0, 1]$$

for all $t > 0$. The goal is to use the lengths of these line segments, $\|x_0 - x_{w0}\|$ and $\|\Delta w(t)\|$ to estimate $\|x(t) - x_w(t)\|$, $\forall t > 0$.

To do this we consider two curves (2.3), generated using $s(\cdot)$, $u(\cdot, \cdot)$ and solutions of (2.3) written here as $x(x_0, w(t), t)$, $\forall t > 0$. They are defined by $\phi_1(\alpha, t) = x(s(\alpha), u(0, t), t)$, $\alpha \in [0, 1]$ and $\phi_2(\beta, t) = x(s(1), u(\beta, t), t)$, $\beta \in [0, 1]$, for $t > 0$. These two curves combined defines a curve between $x(t)$ and $x_w(t)$ for $t > 0$ whose length satisfies

$$L(t) \le \int_0^1 \left\| \frac{\partial x(s(\alpha), u(0,t), t)}{\partial \alpha} \right\| d\alpha + \int_0^1 \left\| \frac{\partial x(s(1), u(\beta,t), t)}{\partial \beta} \right\| d\beta, \quad t > 0$$

This length then satisfies

$$\|x(t) - x_w(t)\| \le L(t), \quad t > 0. \tag{2.15}$$

We can further estimate the length $L(t)$ for $t > 0$ by defining v_1 and v_2 as

$$v_1(\alpha, t) := \frac{\partial x(s(\alpha), u(0,t), t)}{\partial \alpha} = \frac{\partial \phi_1}{\partial \alpha}(\alpha, t),$$

$$v_2(\beta, t) := \frac{\partial x(s(1), u(\beta,t), t)}{\partial \beta} = \frac{\partial \phi_2}{\partial \beta}(\beta, t).$$

To shorten notation when necessary we omit the dependency on α, β and t when notating ϕ_1, ϕ_2, v_1 and v_2. We transform v_1 using the local transformation $\Theta(\cdot,\cdot)$ corresponding to the contraction metric M and differentiate to get

$$\begin{aligned}
\frac{d}{dt}\Theta(\phi_1, t)v_1 &= \dot{\Theta}(\phi_1, u(0,t), t)v_1 + \Theta(\phi_1, t)\frac{d}{dt}\frac{\partial x(s(\alpha), u(0,t), t)}{\partial \alpha} \\
&= \dot{\Theta}(\phi_1, u(0,t), t)v_1 + \Theta(\phi_1, t)\frac{\partial}{\partial \alpha} f(x(s(\alpha), u(0,t), t), u(0,t), t) \\
&= \dot{\Theta}(\phi_1, u(0,t), t)v_1 + \Theta(\phi_1, t)\frac{\partial f(\phi_1, u(0,t), t)}{\partial x}\frac{\partial \phi_1}{\partial \alpha}(\alpha, t) \\
&= J(\phi_1, u(0,t), t)\Theta(\phi_1, t)v_1,
\end{aligned}$$

for $t > 0$, $\alpha \in [0,1]$, where J is the generalized Jacobian. This means that

$$\frac{d}{dt}(\Theta(\phi_1, t)v_1)^T \Theta(\phi_1, t)v_1 = 2(\Theta(\phi_1, t)v_1)^T J(\phi_1, u(0,t), t)\Theta(\phi_1, t)v_1.$$

Since the system is uniformly contracting with rate λ, $J(\phi_1(\alpha, t), u(0,t), t)$ must be uniformly negative definite for all $\alpha \in [0,1]$, $t > 0$ with

$$\frac{d}{dt}(\Theta(\phi_1, t)v_1)^T \Theta(\phi_1, t)v_1 \le -2\lambda(\Theta(\phi_1, t)v_1)^T \Theta(\phi_1, t)v_1,$$

which using the Lemma 2.20 gives

$$v_1^T \Theta(\phi_1, t)^T \Theta(\phi_1, t)v_1 \le v_{10}(\alpha)^T \Theta(0,t)^T \Theta(0,t)v_{10}(\alpha)e^{-2\lambda t}, \tag{2.16}$$

where $v_{10}(\alpha) = v_1(\alpha, 0)$. Like when writing v_1, to shorten notation, we notate v_{10} omitting the dependency on α when necessary. Since $\Theta(x,t)^T\Theta(x,t) = M(x,t)$ for all

$x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, estimating (2.16) using (2.14) yields

$$\sigma_{min}^2 v_1^T v_1 \leq v_1^T M(\phi_1, t) v_1 \leq v_{10}^T M(0,t) v_{10} e^{-2\lambda t} \leq \sigma_{max}^2 v_{10}^T v_{10} e^{-2\lambda t}.$$

Thus

$$v_1(\alpha,t)^T v_1(\alpha,t) \leq \frac{\sigma_{max}^2}{\sigma_{min}^2} v_{10}(\alpha)^T v_{10}(\alpha) e^{-2\lambda t},$$

holds, which further implies

$$\|v_1(\alpha,t)\| \leq \frac{\sigma_{max}}{\sigma_{min}} \|v_1(\alpha,0)\| e^{-\lambda t}, \tag{2.17}$$

for $t > 0$, $\alpha \in [0,1]$.

We will similarly transform v_2 using the local transform $\Theta(\cdot,\cdot)$. To shorten and clarify notation we will notate when necessary Θ, $\frac{\partial f}{\partial w}$ and $\frac{\partial f}{\partial x}$ without their dependencies on ϕ_2, α, t and u. Transforming v_2 and differentiating yields

$$\begin{aligned}
\frac{d}{dt}\Theta(\phi_2,t)v_2 &= \dot{\Theta}(\phi_2, u(\beta,t), t)v_2 + \Theta(\phi_2,t)\frac{d}{dt}\frac{\partial x(s(1), u(\beta,t), t)}{\partial \beta} \\
&= \dot{\Theta}(\phi_2, u(\beta,t), t)v_2 + \Theta(\phi_2,t)\frac{\partial}{\partial \beta} f(x(s(1), u(\beta,t), t), u(\beta,t), t) \\
&= \dot{\Theta}(\phi_2, u(\beta,t), t)v_2 + \Theta(\phi_2,t)\left(\frac{\partial f}{\partial x} v_2 + \frac{\partial f}{\partial w}\frac{\partial u}{\partial \beta}(\beta,t)\right) \\
&= J(\phi_2, u(\beta,t), t)\Theta(x,t)v_2 + \Theta(\phi_2,t)\frac{\partial f}{\partial w}\Delta w(t).
\end{aligned}$$

Thus

$$\frac{d}{dt}(\Theta v_2)^T \Theta v_2 = 2(\Theta v_2)^T J(\phi_2, u(\beta,t), t)\Theta v_2 + 2(\Theta v_2)^T \Theta \frac{\partial f}{\partial w}\Delta w(t), \tag{2.18}$$

holds. Using Young's inequality, Lemma 2.21 with $\epsilon = \frac{1}{\lambda}$ and the fact that Θ and $\frac{\partial f}{\partial w}$ are uniformly bounded we get

$$\begin{aligned}
2(\Theta v_2)^T \Theta \frac{\partial f}{\partial w}\Delta w(t) &\leq \lambda v_2^T \Theta^T \Theta v_2 + \frac{1}{\lambda}\Delta w(t)^T \frac{\partial f^T}{\partial w}\Theta^T \Theta \frac{\partial f}{\partial w}\Delta w(t). \\
&\leq \lambda(\Theta v_2)^T \Theta v_2 + c \sup_{0 \leq \tau \leq t} \|\Delta w(t)\|^2,
\end{aligned} \tag{2.19}$$

for some $c > 0$ dependent on λ and the upper bounds of Θ and $\frac{\partial f}{\partial w}$. Again since the system is globally contracting with rate $\lambda > 0$, $J(\phi_2, u(\beta,t), t)$ must be uniformly negative definite with $(\Theta v_2)^T J(\phi_2, u(\beta,t), t)\Theta v_2 \leq -\lambda(\Theta v_2)^T \Theta v_2$ for all $t > 0$ and

$\beta \in [0, 1]$. This with (2.18) and (2.19) implies that

$$\begin{aligned} \frac{d}{dt}(v_2\Theta)^T\Theta v_2 &= 2(v_2\Theta)J(\phi_2, u(\beta, t), t)\Theta v_2 + 2(v_2\Theta)\Theta\frac{\partial f}{\partial w}\Delta w(t) \\ &\leq -2\lambda(\Theta v_2)^T\Theta v_2 + \lambda(\Theta v_2)^T\Theta v_2 + c \sup_{0\leq\tau\leq t}\|\Delta w(t)\|^2 \\ &= -\lambda(\Theta v_2)^T\Theta v_2 + c \sup_{0\leq\tau\leq t}\|\Delta w(t)\|^2. \end{aligned}$$

Thus using Lemma 2.20,

$$v_2^T\Theta(\phi_2, t)^T\Theta(\phi_2, t)v_2 \leq v_{20}(\beta)^T\Theta(0, t)^T\Theta(0, t)v_{20}(\beta)e^{-\lambda t} + \gamma(\sup_{0\leq\tau\leq t}\|\Delta w(t)\|),$$

where $v_{20}(\beta) = v_2(\beta, 0)$ and $\gamma(r) = \frac{cr^2}{\lambda}$ for $r > 0$ implying that γ is a class $\mathcal{K}$ function. Since $\Theta(x, t)^T\Theta(x, t) = M(x, t)$ for all $x \in \mathbb{R}^n$ and $t > 0$, estimating this using (2.14) yields

$$\sigma_{min}^2 v_2(\beta, t)^T v_2(\beta, t) \leq \sigma_{max}^2 v_{20}(\beta)^T v_{20}(\beta)e^{-\lambda t} + \gamma(\sup_{0\leq\tau\leq t}\|\Delta w(t)\|).$$

Thus

$$v_2(\beta, t)^T v_2(\beta, t) \leq \frac{\sigma_{max}^2}{\sigma_{min}^2} v_{20}(\beta)^T v_{20}(\beta)e^{-\lambda t} + \frac{1}{\sigma_{min}^2}\gamma(\sup_{0\leq\tau\leq t}\|\Delta w(t)\|),$$

holds, which implies

$$\|v_2(\beta, t)\| \leq \frac{\sigma_{max}}{\sigma_{min}}\|v_2(\beta, 0)\|e^{-\frac{\lambda}{2}t} + \gamma_*(\sup_{0\leq\tau\leq t}\|\Delta w(t)\|), \tag{2.20}$$

where $\gamma_*(y) = \frac{1}{\sigma_{min}^2}|\sqrt{\gamma(y)}|$ for $y > 0$, implying that γ_* is a class $\mathcal{K}$ function. Finally taking (2.15), estimating with (2.17) and (2.20), and integrating with respect to α and β yields

$$\begin{aligned} \|x(t) - x_w(t)\| &\leq L_1(t) + L_2(t) \\ &\leq \frac{\sigma_{max}^2}{\sigma_{min}^2}L_1(0)e^{-\lambda t} + \frac{\sigma_{max}^2}{\sigma_{min}^2}L_2(0)e^{-\frac{\lambda}{2}t} + \gamma_*(\sup_{t_0\leq\tau\leq t}\|\Delta w(t)\|) \\ &= \frac{\sigma_{max}^2}{\sigma_{min}^2}\|x_0 - x_{w0}\|e^{-\lambda t} + \frac{\sigma_{max}^2}{\sigma_{min}^2}\|\Delta w(0)\|e^{-\frac{\lambda}{2}t} + \gamma_*(\sup_{t_0\leq\tau\leq t}\|\Delta w(t)\|) \\ &\leq \frac{\sigma_{max}^2}{\sigma_{min}^2}\|x_0 - x_{w0}\|e^{-\lambda t} + \frac{\sigma_{max}^2}{\sigma_{min}^2}\sup_{0\leq\tau\leq t}\|\Delta w(\tau)\| + \gamma_*(\sup_{0\leq\tau\leq t}\|\Delta w(t)\|), \end{aligned}$$

for all $t > 0$ Thus the system is input-to-state stable with respect to $x_w(\cdot)$ as defining $\gamma_0(y) = \frac{\sigma_{max}^2}{\sigma_{min}^2}y + \gamma_*(y)$ for $y > 0$ implies that γ_0 a class $\mathcal{K}$ function. □

Theorem 2.23. *Consider system (2.3), a class of inputs $\mathcal{N}$ and a set $W_r \subset \mathbb{R}^m$ such* *that for all* $w(\cdot) \in \mathcal{N}$*,* $w(t) \in W_r$ *for all* $t \in \mathbb{R}$*. Suppose the system (2.3), is globally uniformly contracting with respect to a bounded metric for the set of inputs* W_r *and that for some input* $w(\cdot) \in \mathcal{N}$*, there exists a corresponding bounded solution* $x_w(t)$*. Then the system is input-to-state convergent for the class of inputs* $\mathcal{N}$*.*

Proof. This is a direct consequence of Theorem 2.17 and Lemma 2.22. □

While the generalization of allowing *non-constant* contraction metrics, compared to Demidovich's condition in Theorem 2.13, allows for analysis of a wider class of systems the conditions of Theorem 2.23 require additional knowledge of the system. Of these while the boundedness of $\frac{\partial f}{\partial w}$ may be simple to determine, the existence of a bounded solution for some input may require computing solutions or the use of another Lyapunov function. Regardless this third applicable condition to achieve input-to-state convergence appears to have some potential in systems where these conditions are simple to verify.

It is possible to further generalize the results by allowing the metric M to depend on the input w as well. In this case $\dot{M}$ will also depend on the time derivative of $w(\cdot)$ and thus considering the class of inputs $I_s(W)$ with an exosystem would be of interest. However for simplicity we do not consider this generalization.

3. THE OUTPUT REGULATION PROBLEM

In this chapter we formulate the global and local variants of the *uniform output regulation problem* and give conditions for their solvability. The solvability conditions are based on uniform convergence of a combined system and the solvability of a set of differential equations called the *regulator equations*. In the latter part of this chapter we discuss methods of solving the regulator equations as uniform convergence was considered in the previous chapter.

3.1 The problem formulation

The output regulation problem is a control problem where the goal is track a reference signal for a given system with the presence of disturbances. In this section we formulate this problem precisely. However, to illustrate we first discuss a simple example.

3.1.1 Harmonic oscillator

We will consider as an example a harmonic oscillator which is described by a set of differential equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 + \frac{k}{m}u, \end{aligned}$$

where $k > 0$ is the spring constant, $m > 0$ is the mass, $u \in \mathbb{R}$ is the control input and $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ are *state variables* of the system that together describe the *state* of the system [13]. Physically x_1 is the position and x_2 the velocity of the harmonic oscillator. We notate the solution of the system at some initial point $x_0 = [x_1(0), x_2(0)] \in \mathbb{R}^2$ as $x(t) = [x_1(t), x_2(t)]$. Naturally we must also be able to control the system in some way. For this we consider exerting a force on the oscillator, which would translate to a control input affecting the acceleration $\dot{x}_2$. Thus the control input u affects $\dot{x}_2$.

For this oscillator, we wish the position to track a given trajectory $y_{ref}(\cdot)$. Thus we will form an error term to describe the difference between the given trajectory and the position,

$$e(t) = x_1(t) - y_{ref}(t), \quad \forall t > 0.$$

Finally to simplify the formulation of the control problem we introduce the *exosystem*, which is an external system that generates the reference signal. For the oscillator we consider a simple exosystem

$$\begin{aligned}\dot{w}_1 &= w_2\\ \dot{w}_2 &= -w_1.\end{aligned}$$

We will write the solution of this system, called the *exosignal* at initial point $w(0) = w_0 \in \mathbb{R}^2$ as $w(t) = [w_1(t), w_2(t)]$. This exosystem generates sinusoidal signals and thus is useful in many applications.

We do not define the exosystem to directly give the reference signal as we wish to generate more complex combinations of signals. In addition in formulating the output regulation problem we use the exosystem to describe disturbance as well. Thus we need to define the reference based on the exosignal. For example the relationship between the reference signal and the exosignal for the harmonic oscillator can be $y_{ref} = aw_1 + bw_2$, $a, b \in \mathbb{R}$.

Combining the above formulation into matrix form we have the following system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} u, \tag{3.1a}$$

$$e = Cx - Fw, \tag{3.1b}$$

$$\dot{w} = Sw, \tag{3.1c}$$

where $x(t), w(t), e(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, for all $t \in \mathbb{R}$, and

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} a & b \end{bmatrix},$$

for some $a, b \in \mathbb{R}$.

To form an output regulation problem for system (3.1), we must first consider how we wish the position x_1 to correspond to the reference signal y_{ref}. Since the error cannot be zero in all initial cases, we cannot directly require equality of the functions. It is possible to require that within some finite time the error becomes zero, however in the output regulation problem we require that the error converges zero. More precisely we

require that
$$\|x_1 - y_{ref}(t)\| = \|Cx(t) - Fw(t)\| = \|e(t)\| \to 0, \tag{3.2}$$
as $t \to \infty$. This is called *asymptotic output tracking* or asymptotic output zeroing and the error term is called the *regulator error*[1].

Two other conditions must also be satisfied for the output regulation problem to be solved. First the problem must satisfy *well-posedness* or regularity, that is the solutions of the system (3.1) must exist and be unique for $t \in \mathbb{R}$. Second the system must be made to forget its initial values in the sense that regardless of initial values it converges to the same steady state, which depends only on the exosignal. These two conditions are satisfied with the continuity assumptions discussed in Chapter 2 and when the system is made *uniformly convergent*. Other stability notions like exponential convergence could be used as well and so the problem will be called the *uniform output regulation problem*.

3.1.2 General problem formulation

In this subsection we formulate the problem more precisely in the general case following [1, pg. 32]. We consider a system of the form

$$\dot{x} = f(x, u, w), \tag{3.3a}$$
$$e = h(x, w), \tag{3.3b}$$
$$y = k(x, w), \tag{3.3c}$$

where the state is $x \in \mathbb{R}^n$, the control input is $u \in \mathbb{R}^d$, the exogenous signal is $w \in \mathbb{R}^m$, the regulated error is $e \in \mathbb{R}^p$ and the measured output is $y \in \mathbb{R}^l$. The *measured output* is the information that can be measured from the system and used for control. The exosystem is
$$\dot{w} = s(w), \tag{3.4}$$
with initial conditions in some set $W \subset \mathbb{R}^m$. The functions $f : \mathbb{R}^{n+d+m} \to \mathbb{R}^n$, $h : \mathbb{R}^{n+m} \to \mathbb{R}^p$, $k : \mathbb{R}^{n+m} \to \mathbb{R}^l$ and $s : \mathbb{R}^m \to \mathbb{R}^m$ are assumed to be continuous. In addition we assume that $f \in C^1$ and that s is locally Lipschitz with respect to w. As discussed in Section 2.1, when the solutions of the system are also bounded these assumptions guarantee *well-posedness*.

For the output regulation problem to be solved we must find a controller of the form

$$\dot{\xi} = \eta(\xi, y), \tag{3.5a}$$
$$u = \theta(\xi, y), \tag{3.5b}$$

where the state is $\xi \in \mathbb{R}^q$. This controller must be such that when connected with the system (3.3) and the exosystem (3.4) to form the combined system

$$\dot{x} = f(x, \theta(\xi, k(x, w)), w), \tag{3.6a}$$
$$\dot{\xi} = \eta(\xi, k(x, w)), \tag{3.6b}$$
$$\dot{w} = s(w), \tag{3.6c}$$

it satisfies the three conditions of asymptotic output tracking, uniform convergence and well-posedness. The system (3.6) is a *closed-loop* system as it does not depend on any external signals.

We will formulate both the local and the global uniform output regulation problem. For the *global* problem it is required that a controller satisfying the three conditions exists for all initial conditions of the system (3.3). The *local* problem needs only to be solved for a neigbourhood of the origin. Therefore, the two problems require some differing assumptions.

For the global uniform output regulation problem we will use the following assumption.
Assumption 3.1. *The set of initial conditions W of the exosystem (3.4) is invariant with respect to exosystem. Also for any compact set $K_0 \subset W$ there exists a compact set $K_w \subset W$ such that any solution $w(\cdot)$ starting in $w(0) \in K_0$ satisfies $w(t) \in K_w$, $\forall t \in \mathbb{R}$.*

This assumption guarantees that the solutions of the exosystem, $w(\cdot)$, are of classes $\overline{\mathbb{PC}}(W)$ and $I_s(W)$. Other assumptions are also possible for the global problem but require different formulations, so for simplicity this one is chosen. With this assumption we are ready to give the precise problem formulation.

The global uniform output regulation problem: Given an exosystem (3.4) satisfying Assumption 3.1, find a controller of the form (3.5) such that the closed-loop system (3.6) satisfies the following conditions:

- the functions on the right-hand side of the closed-loop system are C^1;
- the closed-loop system is globally uniformly convergent with the UBSS property;
- for all solutions of the closed-loop system and the exosystem corresponding to initial conditions $(x(0), \xi(0)) \in \mathbb{R}^{n+q}$ and $w(0) \in W$ the regulated output satisfies $\lim_{t\to\infty} e(t) = 0$.

The first condition is needed for well-posedness, the second condition guarantees uniform convergence with respect to inputs and the third condition is asymptotic output tracking.

The controller (3.5) is a general one that encompasses multiple, more specific types of controllers. If $y = k(x, w) = (x, w)$, the states of the system and the exosystem may be directly used for feedback and the controller is called a *state feedback* controller. If this is not the case the controller is called an *output feedback* controller. In some cases the controller can be chosen such that $\xi \equiv 0 \in \mathbb{R}^0$. Then the controller can be written as $u = \theta(y)$ and is called a *static* controller. Otherwise the controller is called a *dynamic* controller. A controller that is both a state feedback and a static controller is called a *static state feedback* controller.

We discussed the problem formulation with a focus on tracking. However the exosystem can be also used to model disturbances, which are external signals that we do not want affecting the system. Therefore the output regulation problem is a tracking problem with disturbances. If no reference signal is given the problem can be purely a disturbance rejection problem or on the other hand if no disturbances are present the problem is purely a tracking problem. The way the exosystem can be used to model both disturbances and reference signals is through describing them both in a single vector called the *exosignal* [4]. This kind of consideration simplifies the analysis of the problem.

The harmonic oscillator (3.1), which we considered as an example can be seen to satisfy the necessary assumptions on the system itself that are required in the problem formulation. The system is clearly C^1 and continuous with respect to the other variables. The exosystem (3.1c) satisfies Assumption 3.1 and is in fact a useful example of systems that do so. The justifications are discussed in Section 5.1, where the global uniform output regulation problem is also solved for the harmonic oscillator using tools we discuss in Chapter 4.

We will now formulate the *local uniform output regulation problem*. We now require the regulation problem to be solved only for initial values in a small neighborhood of the origin. Thus the exosystem (3.4) does not need to be stable globally, but also only for a neighborhood of the origin. Continuing to follow [1] we introduce a different assumption called the *neutral stability assumption*. Another similar notion of stability called Poisson stability is used in [4].

Assumption 3.2 ([1])**.** *The exosystem (3.4) with an equilibrium $\bar{w}(t) \equiv 0$ satisfies the* neutral stablility assumption *if the following two conditions hold:*

a) There exist a neighbourhood of the origin $W \subset \mathbb{R}^m$ such that any solution of the exosystem $w(t)$ starting at a point $w(0) = w_0 \in W$, there exists a sequence $\{t_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} w(t_k) = w_0$.

b) The equilibrium $\bar{w}(t)$ is such that for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|w_0\| < \delta$ implies $\|w(t) - \bar{w}(t)\| < \epsilon$ for all $t \in (-\infty, \infty)$.

The local uniform output regulation problem: Assume that for systems (3.3) and (3.4), $f(0,0,0) = 0$, $h(0,0) = 0$, $k(0,0) = 0$, $s(0) = 0$, and that the functions f and k are C^1. Given an exosystem (3.4) satisfying the neutral stability assumption, Assumption 3.2, find a controller of the form (3.5) with $\eta(0,0) = 0$, $\theta(0,0) = 0$ such that the closed-loop system (3.6) satisfies the following conditions:

- the functions on the right-hand side of the closed-loop system are C^1 with respect to (x, w, ξ);
- the closed-loop system is locally uniformly convergent for the class of inputs $I_s(W)$, where $W \subset \mathbb{R}^m$ is some neighborhood of the origin invariant with respect to (3.4);
- for all solutions of the closed-loop system and the exosystem corresponding to initial values close enough to the origin $(x, \xi, w) = (0, 0, 0)$ the regulated output satisfies $\lim_{t\to\infty} e(t) = 0$.

These requirements are similar to the global output regulation problem and similarly correspond to the three conditions discussed, well-posedness, uniform convergence and asymptotic output tracking. The second requirement is equivalent to the requirement that at the origin the Jacobian of the right hand side of the closed-loop system (3.6) has eigenvalues with strictly negative real parts.

3.2 Solvability of the output regulation problem using regulator equations

Having formulated the uniform output regulation problem, we give the solvability conditions in this section. The requirements of uniform convergence and continuously differentiable functions can be checked using results presented in Chapter 2. Therefore the solvability conditions we present focus on determining whether a system is able to achieve asymptotic output tracking. We analyze whether the closed-loop system can be made such that the steady-state is stable and follows the reference signal. In this section we follow [1].

To formulate the results precisely, the following definition describing behavior of the controller is needed.
Definition 3.3 ([1, Def. 4.14])**.** Consider controller (3.5). Let $y(t)$ and $u(t)$ be defined for all $t \in \mathbb{R}$ and $y(\cdot)$ and $u(\cdot)$ be bounded. We say that the input y *induces* the output u in controller (3.5), if for this y system (3.5) has a solution $\xi(t)$ defined for all $t \in \mathbb{R}$, with $\xi(\cdot)$ bounded and satisfying the equality $u(t) = \theta(\xi(t), y(t))$ for all $t \in \mathbb{R}$.

Another definition, describing the exosystem's limit points, from [1] is necessary as well.
Definition 3.4. A point $w_* \in \mathbb{R}^m$ is called an ω-limit point of the trajectory w of the exosystem (3.4) if for all $T > 0$ and all $\epsilon > 0$ there exists $t_* > T$ such that $\|w(t_*) - w_*\| < \epsilon$. The set of all ω-limit points of the trajectory w is called the ω-limit set and denoted by $\Omega(w_0)$. For trajectories starting in some set $W \subset \mathbb{R}^m$, we denote $\Omega(W) := \bigcup_{w_0 \in W} \Omega(w_0)$.

Using these definitions we give the necessary and sufficient conditions for solving the global uniform output regulation problem.

Theorem 3.5 ([1, Thm. 4.16]). *Consider the system (3.3) and the exosystem (3.4) satisfying Assumption 3.1. The global uniform output regulation problem is solvable if and only if the following conditions are satisfied:*

(i) There exist continuous functions $\pi : \mathbb{R}^m \to \mathbb{R}^n$ and $c : \mathbb{R}^m \to \mathbb{R}^d$ defined in some neighborhood of $\Omega(W)$ and satisfying the regulator equations

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w(t)), c(w(t)), w(t)), \tag{3.7a}$$

$$0 = h(\pi(w(t)), w(t)), \tag{3.7b}$$

for all $t \in \mathbb{R}$ and for all solutions $w(t)$ of exosystem (3.4) satisfying $w(t) \in \Omega(W)$ for all $t \in \mathbb{R}$.

(ii) There exists a controller of the form (3.5) such that for any solution of the exosystem $w(\cdot)$ lying in the set $\Omega(W)$ the input $y(w) := k(\pi(w), w)$ induces the output $u(w) = c(w)$, and the corresponding closed-loop system is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(W)$.

Solvability conditions for the robust and forward time variants of the uniform output regulation problem are also found in [1]. The *forward time* variant replaces Assumption 3.1 on the exosystem with an assumption that makes it possible to consider signals that are well defined only for positive time. The *robust* variant considers an additional system parameter that represents uncertainty in system behavior and measurement.

The next theorem gives a sufficient condition for the solvability of the local output regulation problem.

Theorem 3.6 ([1, Thm. 4.22]). *Consider system (3.3) and exosystem (3.4) that satisfies the Assumption 3.2. The local uniform output regulation problem is solvable if and only if the following conditions are satisfied:*

(i) There exist continuous mappings $\pi : \mathbb{R}^m \to \mathbb{R}^n$ and $c : \mathbb{R}^m \to \mathbb{R}^d$ defined in some neighborhood of the origin $W \subset \mathbb{R}^m$ invariant with respect to (3.4), such that $\pi(0) = 0$, $c(0) = 0$, and

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w(t)), c(w(t)), w(t)), \tag{3.8a}$$

$$0 = h(\pi(w(t)), w(t)), \tag{3.8b}$$

for all solutions of exosystem (3.4) satisfying $w(t) \in W$ for all $t \in \mathbb{R}$.

(ii) There exists a controller of the form (3.5) satisfying the following conditions:

a) There exists a continuous mapping $\sigma : W \to \mathbb{R}^q$ satisfying $\sigma(0) = 0$ and

$$\frac{d}{dt}\sigma(w(t)) = \theta(\sigma(w), k(\pi(w), w)), \tag{3.9a}$$

$$c(w(t)) = \theta(\sigma(w(t)), k(\pi(w(t)), w(t))) \tag{3.9b}$$

b) The closed-loop system corresponding to this controller is locally uniformly convergent for the class of inputs $I_s(W)$.

In the case of a static state feedback controller the mapping σ can be chosen to be zero and thus the regulator equation (3.9) can be written as

$$c(w(t)) = \theta(k(\pi(w(t)), w(t))). \tag{3.10}$$

Thus in this case we do not need to solve the equation (3.9a).

For both the local and the global uniform output regulation problem the solvability conditions consist of two central parts: the regulator equations and the existence of a controller that induces the required output and guarantees stability. Stability has been studied in Chapter 2 and designing the controllers will be considered in Chapter 4. Since the solutions of the regulator equations are usually used as a part in controller design, we will consider solving the regulator equations in the next section.

3.3 Methods for solving the regulator equations

The regulator equations may be solved directly in some cases, but in general it may be challenging [1]. For special classes of systems there are a number methods for solving the regulator equations of the local regulation problem (3.8). For example affine systems are studied in [14] and control affine systems in [15]. A numerical method for solving the regulator equations is presented in [16]. Various techniques are also collected in [4], one of which we follow to study control affine systems specifically.

The methods presented in [4, 14, 15, 16] are *local* in the sense that they only guarantee the existence of $\pi(\cdot)$ and $c(\cdot)$ in some neighborhood of the origin, and thus can not be applied to the global output regulation problem. However, in some cases the solutions may exist in some neighborhood of $\Omega(W)$ or they may be extended to this set and then applied directly to the global problem. Additionally local results are often given at the origin, thus requiring that $f(0, 0) = 0$. This is true for the result we discuss as well, allowing us to apply it only to specific cases when solving the global uniform output regulation problem. Nevertheless we consider them here with the note that similar methods can likely be extended to systems not satisfying $f(0, 0) = 0$ as well through a shift of co-ordinates.

An affine system is is a system that is linear with respect to some or all inputs. Restricting considerations to such systems simplifies methods significantly. A *control affine* system has the form

$$\dot{x} = f(x,w) + g(x,w)u, \tag{3.11a}$$
$$e = h(x,w), \tag{3.11b}$$
$$y = k(x,w) \tag{3.11c}$$

where the state is $x \in \mathbb{R}^n$, reference signal is $w \in \mathbb{R}^m$, control input is $u \in \mathbb{R}^d$, the measure output is $y \in \mathbb{R}^l$ and regulated output is $e \in \mathbb{R}^p$. This form of system will be analyzed in this section. It is assumed that $g : \mathbb{R}^{n+m} \to \mathbb{R}^{n\times d}$, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $h : \mathbb{R}^{n+m} \to \mathbb{R}^p$ are such that $f \in C^\infty$, $h \in C^\infty$, $g \in C^1$ and $k : \mathbb{R}^{n+m} \to \mathbb{R}^l$ is continuous. In addition we assume that $f(0,0) = 0$. The system (3.11) is a special case of systems of the form (3.3). The exosystem is defined as (3.4) with a set of initial conditions W.

The regulator equations (3.7) for system (3.11) have the form

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w(t)), w(t)) + g(\pi(w(t)), w(t))c(w(t)), \tag{3.12a}$$
$$0 = h(\pi(w(t)), w(t)). \tag{3.12b}$$

We consider the system (3.11) together with the exosystem (3.4) as a composite system,

$$\dot{x}_s = f_s(x_s) + g_s(x_s)u, \tag{3.13a}$$
$$e = h(x_s), \tag{3.13b}$$

where $x_s = [x,w]^T$,$f_s(x_s) = [f(x,w), s(w)]^T$ and $g_s(x_s) = [g(x,w), 0]^T$.

Lie derivatives are directional derivatives of a function along a vector field. Thus we can use them to simplify the notation of derivatives of the regulator error along the vector fields defined by f and g.

Definition 3.7 ([4, Def. 2.35])**.** The *Lie derivative* of a function $U : \mathbb{R}^d \to \mathbb{R}$ with $U \in C^1$ along the vector field $X : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is defined by

$$L_X U(x) := \frac{\partial U}{\partial x} X(x) = \sum_{i=1}^{d} \frac{\partial U}{\partial x_i} X_i(x),$$

For $k \in \mathbb{N}$ when $m = 1$, $U \in C^k$ and $X \in C^{k-1}$ we denote

$$L_X^k U(x) := \frac{\partial L_X^{k-1} U}{\partial x} X(x),$$
$$L_X^0 U(x) := U(x).$$

Another notion we need is the *vector relative degree*, which describes whether each output is connected to an input. The least number of state elements the output is dependent on to have a dependence on an input is its relative degree. The vector relative degree combines the relative degree of each output into a vector and it is required that each output is connected to a different input. We give the definition of the vector relative degree using a decoupling matrix for control affine systems.

Definition 3.8. [4, Def. 2.47] A system of the form (3.13) has a *vector relative degree* $\{r_1, r_2, \ldots, r_p\}$ at $(x, w) = 0$ when the decoupling matrix

$$D_s(x, w) := \begin{bmatrix} L_{g_s} L_{f_s}^{r_1 - 1} h_1(x, w) \\ L_{g_s} L_{f_s}^{r_2 - 1} h_2(x, w) \\ \vdots \\ L_{g_s} L_{f_s}^{r_p - 1} h_p(x, w) \end{bmatrix}$$

has full row rank at $(x, w) = 0$ and for $r_i \in \mathbb{N}$, $i \in [1, 2, \ldots, p]$

$$L_{g_s} L_{f_s}^k h_i(x, w) = 0$$

for all $0 \leq k < r_i - 1$ and for all x, w in an open neighbourhood of the origin.

The H-vector and E-vector of the system (3.11) are also necessary to simplify notation. The zero dynamics of the composite system (3.13) are defined as the dynamics when the H-vector is restricted to be zero. The E-vector on the other hand will be used to solve the state feedback control for the zero dynamics. These vectors are defined as

$$H_s(x,w) := \begin{bmatrix} h_1(x,w) \\ L_{f_s}h_1(x,w) \\ \vdots \\ L_{f_s}^{r_1-1}h_1(x,w) \\ \vdots \\ h_p(x,w) \\ L_{f_s}h_p(x,w) \\ \vdots \\ L_{f_s}^{r_p-1}h_p(x,w) \end{bmatrix}, \quad E_s(x,w) := \begin{bmatrix} L_{f_s}^{r_1}h_1(x,w) \\ L_{f_s}^{r_2}h_2(x,w) \\ \vdots \\ L_{f_s}^{r_p}h_p(x,w) \end{bmatrix}.$$

The zero dynamics correspond to the solutions of the regulator equations since $H_s(x,w) = 0$ defines the zero dynamics so that $e = h(x,w) = 0$. Now to further describe the zero dynamics a partition of the system's state is considered. The partition is made with respect to the sum $r := \sum_{i=1}^{p} r_i$, where r_i are components of a relative degree vector of the composite system (3.13). The partition is defined as $x^1 = [x_{j_1}, \ldots, x_{j_r}]^T$ and $x^2 = [x_{j_{r+1}}, \ldots, x_{j_n}]$ such that

$$\text{rank}\frac{\partial H_s}{\partial x^1}(0,0) = r.$$

Then the Implicit Function Theorem guarantees that there exists a function $\sigma : \mathbb{R}^{n-r+m} \to \mathbb{R}^r$ such that $\sigma(x^2,w) = x^1$, $\sigma(0) = 0$ and

$$0 = H_s(x,w)|_{x^1=\sigma(x^2,w)}$$

for some neighborhood of the origin. Using this partition we define

$$\delta(x^2,w) = \begin{bmatrix} (f_{j_{r+1}}(x,w) + g_{j_{r+1}}(x,w)u_e(x,w))|_{x^1=\sigma(x^2,w)} \\ \vdots \\ (f_{j_n}(x,w) + g_{j_n}(x,w)u_e(x,w))|_{x^1=\sigma(x^2,w)} \end{bmatrix}.$$

Having introduced these notations we will finally give a sufficient condition guaranteeing the existence of the regulator equations.

Theorem 3.9 ([4, Thm. 3.26]). *Suppose the composite system (3.13) at $(x,w) = 0$ has a relative degree $\{r_1, r_2, \ldots, r_p\}$ with $\sum_{i=1}^{p} r_i = r$. Assume that for some continuous* *function* $u_e : \mathbb{R}^{n+m} \to \mathbb{R}^m$ *satisfying*

$$E_s(x,w) + D_s(x,w)u_e(x,w) = 0, \quad \forall x \in \mathbb{R}^n, \forall w \in \Omega(W) \tag{3.14}$$

there exists a C^1 *function* $\pi_2 : \mathbb{R}^m \to \mathbb{R}^{n-r}$ *with* $\pi_2(0) = 0$ *and*

$$\frac{\partial \pi_2}{\partial w} s(w(t)) = \delta(\pi_2(w(t)), w(t)), \quad \forall t \in \mathbb{R}^m.$$

Then $\pi(w) = (\pi_1(w), \pi_2(w))$ *and* $c(w) = u_e(\pi(w), w)$ *where* $\pi_1(w) = \sigma(\pi_2(w), w)$ *are the solutions of the regulator equations (3.12).*

For $p < d$ the function u_e which is a feedback control is not unique. Because of this a partition can be made on the control similarly to the partition of the state and the extra components can be used to modify the zero dynamics of the system. This partition is defined as $u^1 \in \mathbb{R}^p, u^2 \in \mathbb{R}^{d-p}$ such that there exists a function $\bar{u}_e : \mathbb{R}^{n+m+d-p} \to \mathbb{R}^d$ satisfying

$$E_s(x,w) + D_s(x,w)\bar{u}_e(x,w,u^2) = 0, \quad \forall x \in \mathbb{R}^n, \forall w \in \mathbb{R}^m, \forall u^2 \in \mathbb{R}^{d-p}. \tag{3.15}$$

Naturally the extent the zero dynamics can be modified is limited to the effects of the control u^2, which contains the extra components. To determine the zero dynamics uniquely we must choose some feedback control $u^2 = \theta(x)$ where $\theta : \mathbb{R}^n \to \mathbb{R}^{d-p}$ is continuous and satisfies $\theta(0) = 0$.

Using Theorem 3.9 we can form an algorithm to solve the regulator equations. It is given below following the formulation in [4].

1) Calculate the vector relative degree, decoupling matrix, H-vector and E-vector of the composite system (3.13).

2) Solve the equation

$$H_s(x,w) = 0$$

for r components of x in terms of w and the remaining $n-r$ components of x. Form the partition fo state defined by some x^1 and x^2 and find the mapping $\sigma : \mathbb{R}^{n-r+m} \to \mathbb{R}^r$ satisfying $\sigma(x^2, w) = x^1$.

3) Solve $u_e(x,w)$ from equation (3.14) if it is possible to do uniquely. If necessary form the partition defined by some u^1 and u^2, solving the equation (3.15). Then choose a control θ, and set $u^2 = \theta(x)$ and $u_e(x,w) = \bar{u}_e(x,w,\theta(x))$.

4) Solve the differential equation

$$\dot{x}_{j_i} = (f_{j_i}(x,w) + g_{j_i}(x,w)u_e(x,w))|_{x^1=\sigma(x^2,w)}, \quad i = r+1, \dots, n,$$

denoting the solution by $\pi_2(w)$. Denote $\pi_1(w) = \sigma(\pi_2(w), w)$. Then the solutions of the regulator equations are $\pi(\cdot)$ and $c(\cdot)$ such that $\pi(w) = [\pi_1(w), \pi_2(w)]^T$ and $c(w) = u_e(\pi(w), w)$ for all $w \in \mathbb{R}^m$.

If the mappings σ and π_2 exist globally, the solutions of the regulator equations exist globally and can be applied to the global regulation problem when $f(0,0) = 0$. The next example demonstrates a simple system where this is the case.

Example 3.10. *Consider a simple system of the form*

$$\dot{x} = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \phi_2(x_2) \\ -\phi_1(x_1) \end{bmatrix} + u, \tag{3.16a}$$

$$y = x_1 - Fw, \tag{3.16b}$$

$$\dot{w} = s(w), \tag{3.16c}$$

where $x \in \mathbb{R}^2$, $u \in \mathbb{R}^2$, $w \in \mathbb{R}^q$, $q \in \mathbb{N}$, $F \in \mathbb{R}^{1\times q}$, $s : \mathbb{R}^q \to \mathbb{R}^q$ is a sufficiently smooth vector function, and ϕ_1 and ϕ_2 are sufficiently smooth scalar functions that satisfy $\phi_1(0) = 0$, $\phi_2(0) = 0$. To solve the regulator equations the algorithm presented above is followed.

1) The composite system has a relative degree of $r_1 = r = 1$. Direct computation shows that $D_a(x, w) = [1, 0]$ $H_s(x, w) = x_1 - Fw$ and $E_s(x, w) = -x_1 - \phi_2(x_2) - Fs(w)$.

2) The equality $H_s(x, w) = x_1 - Fw = 0$ implies $x_1 = Fw$. Thus the partition can be chosen as $x^1 = x_1$, $x^2 = x_2$ and $\sigma(x^2, w) = Fw$.

3) Since $p < m$, the control may also be partitioned as $u = [u_1, u_2]^T$, where only u_1 is significant. Then solving equation (3.14) corresponding to the system (3.16) gives

$$u_e(x, w) = \begin{bmatrix} Fs(w) + x_1 - \phi_2(x_2) \\ u_2 \end{bmatrix},$$

where u_2 can be chosen freely.

4) To determine the zero dynamics the system

$$\dot{x}_2 = -x_2 - \phi_1(x_1) + u_2|_{x_1=Fw},$$

needs to be solved. The solution depends on the choice of u_2, which for example may be chosen as $u_2 = \phi_1(w)$ for simplicity. Then $x_2 = 0$ solves the system. Thus solutions *of the regulator equations of the system (3.16) are given by*

$$\pi(w) = \begin{bmatrix} Fw \\ 0 \end{bmatrix}, \quad c(w) = \begin{bmatrix} Fs(w) + Fw - \phi_2(0) \\ \phi_1(Fw) \end{bmatrix}. \tag{3.17}$$

4. CONTROLLER DESIGN FOR OUTPUT REGULATION

In this chapter different approaches to controller design are discussed. The solutions of the regulator equations are assumed to be given and will be used together with stabilizing techniques to form controllers. The focus will be on two stabilization techniques for the global output regulation problem; one using *quadratic stabilizability* and the other *universal contractivity*. A simple technique related to the local output regulation problem will also be discussed.

4.1 Design strategy

In controller design a large variety of methods can be used. Commonly the solutions of the regulator equations are utilized with some appropriate stability notions. While it can be possible to formulate a full controller directly, usually to achieve the desired conditions, design is decomposed into two main parts. First the goal is to design a controller such that for any solution of exosystem (3.4) lying in the ω-limit set Ω, the input $y_w(t) = k(\pi(w(t)), w(t))$ induces the output $u_w(t) = c(w(t))$, where $\pi(\cdot)$ and $c(\cdot)$ are solutions of the regulator equations. Second the inducing controller is combined with the system and the resultant system extended to take an input from another controller. Then a second controller will be designed such that when it is in closed-loop with the extended system, the closed-loop system is input-to-state convergent. [1]

The two parts in design can be seen to correspond to the two main conditions required for the solvability of the output regulation problem given in Theorem 3.5 and Eq. (3.8). The first part essentially ensures that the control input corresponds to the solution $c(\cdot)$ of the regulator equations which describes the input required at the steady-state. The second part ensures that the closed-loop system is input-to-state convergent and so approaches the steady-state. Thus it is the stabilizing part and so systems for which the second controller can be formulated are called *stabilizable*.

This two part design is in different forms present in many methods of solving output regulation problems. For example in [15] and [14] the focus is on design related to the first controller that induces the output is focused and local stabilizability is simply assumed. The approach of using quadratic stabilizability in [1] concentrates on the second part of the controller design.

In the simplest case where the regulator equations can be solved and the output of the exosystem is measured, the first controller can be simply chosen as $u(w) = c(w(t))$. However in general solving the regulator equations is not easy and will require detailed techniques for specific systems and assumptions as seen in Chapter 3. Also, the exosystem cannot always be measured and in such a case the first controller should be a type of observer of the exosystem. In addition the choice of the first controller and its properties may affect the existence of a stabilizing controller and therefore the two parts of the overall problem are not in general disconnected. [1, pg. 75]

Regardless, for simplicity we will concentrate on systems with $y = (x, w)$, that is systems where the state and the output can be measured. In addition we will assume that the regulator equations are solvable and that the solutions are known. With these assumptions we will formulate a simple approach similar to the use of quadratic stabilizability in [1] to achieve stability through contractivity and then global output regulation. However first we will introduce quadratic stability and a controller design method for quadractically stabilizable systems.

While these approaches are rather restrictive as quadratic stability and contractivity both imply an exponential form of convergence, the methods are instructive and allow to achieve regulation relatively simply. For more general systems system specific Lyapunov-functions may be used to achieve uniform convergence using Theorem 2.14. Also global output regulation is already restrictive and many systems can only be regulated locally. In such cases controller design methods differ significantly as local convergence with the UBSS property is sufficient. Then it is sufficient to consider the linearization of the system. Still the global convergence requirements can be used to formulate estimates for the regions of admissible initial conditions as demonstrated in [1] using quadratic stability.

4.2 Quadratically stabilizable systems

In this section quadratic stability and quadratically stabilizable systems are introduced. Quadratic stability is practically correspondent to the Demidovich condition, so quadratically stabilizable systems present a group of systems that can be made uniformly convergent. This fact will be used for controller design. The methods discussed here are from [1] and easily incorporate inputs and numerical computation.

Assuming that $f : \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^n$ is C^1, to simplify notation it is defined for system (3.3) that

$$A(\zeta) = \frac{\partial f}{\partial x}(x, w, u),$$
$$B(\zeta) = \frac{\partial f}{\partial u}(x, w, u),$$

where $\zeta := (x, u, w) \in \mathbb{R}^{n+d+m}$.

Definition 4.1 ([1]). A matrix function $A : \mathbb{R}^{n+d+m} \to \mathbb{R}^{n\times n}$ is called quadratically stable over a set $\Xi \subset \mathbb{R}^{n+d+m}$ if for some $P = P^T > 0$ and $Q = Q^T > 0$

$$PA(\zeta) + A(\zeta)^T P \leq -Q, \quad \forall \zeta \in \Xi.$$

Definition 4.2 ([1]). A pair of matrix functions $A : \mathbb{R}^{n+d+m} \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R}^{n+d+m} \to \mathbb{R}^{n\times m}$ is said to be quadratically stabilizable over $\Xi \subset \mathbb{R}^{n+d+m}$ if there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that $A(\zeta) + B(\zeta)K$ is quadratically stable over Ξ.

To determine whether the pair (A, B) is quadratically stabilizable the matrix K needs to be solved. This in general requires solving a matrix inequality for all $\zeta \in \Xi$, which is computationally demanding. However, the inequality can be simplified to some linear matrix inequalities (LMIs) using a convex hull of matrices, making computations significantly simpler. [1] The convex hull of matrices $A_1, \ldots, A_k \in \mathbb{R}^{n\times n}$ is

$$\text{co}\{A_1, \ldots, A_k\} := \left\{ A \in \mathbb{R}^{n\times n} : A = \sum_{i=1}^{k} \lambda_i A_i, \quad \sum_{i=1}^{k} \lambda_i = 1, \quad \lambda_i \geq 0 \right\}.$$

Lemma 4.3 ([1]). *Consider a set $\Xi \subset \mathbb{R}^{n+d+m}$ and the matrix functions $A : \mathbb{R}^{n+d+m} \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R}^{n+d+m} \to \mathbb{R}^{n\times m}$.*
(i) Suppose there exist matrices $A_1, \ldots, A_p$ such that $A(\zeta) \in \text{co}\{A_1, \ldots, A_p\}$, $\forall \zeta \in \Xi$, and that there exists P such that the linear matrix inequality

$$PA_i + A_i^T P < 0, \quad i = 1, \ldots, p, \quad P = P^T > 0.$$

holds. Then A is quadratically stable over Ξ.
(ii) Suppose there exist matrices $A_1, \ldots, A_p$ and $B_1, \ldots, B_p$ such that

$$[A(\zeta)\, B(\zeta)] \in \text{co}\{[A_1\, B_1], \ldots, [A_p\, B_p]\}, \quad \forall \zeta \in \Xi,$$

and P such that the linear matrix inequality

$$A_i P + P A_i^T + B_i Y + Y^T B_i^T < 0, \quad i = 1, \ldots, p, \quad P = P^T > 0,$$

holds. Then the pair (A, B) is quadratically stabilizable over Ξ with the matrix $K = Y P^{-1}$.

As quadratic stabilizability can be determined through solving a linear matrix inequality, it is a practical stability condition in applications. This is because LMIs can be computed efficiently using computers. Also quadratic stability guarantees input-to-state convergence as systems that are quadratically stable can be shown to satisfy the Demidovich condition, Theorem 2.13. Thus quadratic stability is an excellent tool to use in controller design. However, as discussed in Section 4.1 quadratic stability is rather restrictive already in the sense that it corresponds to an exponential form of stability.

Next a controller that solves the global uniform output regulation problem for quadratically stabilizable systems is given.

Theorem 4.4 ([1]). *Consider system (3.3) with* $y = (x, w)$ *and exosystem (3.4). Suppose the regulator equations (3.7) are solvable and the corresponding continuous solutions* $\pi(w)$ *and* $c(w)$ *are globally defined. If the pair* $(A(\zeta), B(\zeta))$ *is quadratically stabilizable over* $\zeta \in \mathbb{R}^{n+d+m}$*, then the global uniform output regulation problem is solved by a controller of the form*

$$u = c(w) + K(x - \pi(w))$$

where the matrix K *is such that the matrix function* $A(\zeta) + B(\zeta)K$ *is quadratically stable over* $\zeta \in \mathbb{R}^{n+d+m}$*.*

Quadratic stability has useful properties for considering more general systems as well. Using the concept leads to good robustness properties and so similar conditions can be easily constructed for the robust regulation problem variants. Also in the case that the state is not measurable a similar control law can be constructed using an observer. [1]

The following example demonstrates the usage of Theorem 4.4 for a simple single input single output (SISO) system.

Example 4.5. *Consider a system with an arbitrary strictly decreasing function* $p : \mathbb{R} \to \mathbb{R}$ *that is* C^1*;*

$$\dot{x} = p(x) + u, \tag{4.1a}$$

$$y = x, \tag{4.1b}$$

$$e = x - Fw, \tag{4.1c}$$

$$\dot{w} = s(w), \tag{4.1d}$$

where $x, u, e \in \mathbb{R}$*,* $w \in \mathbb{R}^m$*,* $m \in \mathbb{N}$*,* $F \in \mathbb{R}^{1\times m}$ *and* $s : \mathbb{R}^m \to \mathbb{R}^m$ *is such that* $s(\cdot) \in I_s(W)$ *for some* $W \subset \mathbb{R}^m$*.*

The regulator equations of the system are

$$\frac{d}{dt}\pi(w(t)) = p(\pi(w(t))) + c(w(t)),$$
$$0 = \pi(w(t)) - Fw.$$

These yield the solutions

$$\pi(w) = Fw$$
$$c(w) = Fs(w) - p(Fw).$$

For an arbitrary $P > 0$ the quadratic stability condition holds:

$$PA(\zeta) + A(\zeta)^T P = 2P\frac{\partial p}{\partial x} < 0.$$

Thus the system is quadratically stable and by Theorem 4.4 the control can simply be chosen as

$$u = Fs(w) - p(Fw).$$

For example for $p(x) = -x - x^3$ the system is quadratically stable and the control applicable.

In this example it can be seen that if a SISO system can be transformed into the form (4.1) using state feedback, it is quadratically stabilizable. This occurs when the function p has a derivative that is upper bounded, that is $\sup_{x\in\mathbb{R}}(\frac{\partial p}{\partial x}) \leq a$ for some $a \in \mathbb{R}$. Then choosing a state feedback with $K \in \mathbb{R}$, such that $K = -q - a$ for some $q > 0$, stabilizes the system quadratically as for $P = 1$,

$$PA(\zeta) + A(\zeta)^T P + BY + Y^T B^T = 2\frac{\partial p}{\partial x} + 2K \leq -q. \tag{4.2}$$

Using Theorem 4.4 the global uniform output regulation problem for such a system is solved by $u = s(w) - p(w) + K(x - w)$.

4.3 Systems stabilizable using universal contractivity

The downside of quadratic stability is its conservative nature as stability is essentially achieved using a quadratic Lyapunov function with a constant matrix P. This is what motivates the discussion in this section on a slightly more general result using contractivity with a non-constant metric. For this systems that can be stabilized to be universally contracting will be considered. The controller design techniques here are similar to those used for quadratically stabilizable systems in [1] and discussed in Section 4.2.

Theorem 4.6. *Consider system (3.3) with $y = k(x, w) = [x, w]^T$ and exosystem (3.4). Suppose the regulator equations (3.7) are solvable and the corresponding continuous* *solutions $\pi(\cdot)$ and $c(\cdot)$ are globally defined and C^1. Suppose also that for $K \in \mathbb{R}^{d\times n}$ the system*

$$\dot{x} = f(x, Kx + c(w) - K\pi(w), w) := f_*(x, w), \tag{4.3}$$

where x is the state and w the external input, is globally contracting for the class of inputs $I_s(W)$ with a bounded metric. Finally suppose that the partial derivative $\frac{\partial f_*}{\partial w} = \frac{\partial f}{\partial w} + \frac{\partial f}{\partial u}(\frac{\partial c}{\partial w} - K\frac{\partial \pi}{\partial w})$ is bounded and for each input $w(\cdot) \in I_s(W)$ there exists a corresponding bounded solution $x_w(t)$. Then the controller

$$u = c(w) + K(x - \pi(w)), \tag{4.4}$$

solves the global exponential output regulation problem.

Proof. The regulator equations are assumed to be solved. The controller is such that for input $y_w(t) = [\pi(w(t), w(t))]^T$ it clearly induces the output $c(w(t))$. The closed loop system

$$\dot{x} = f(x, Kx + c(w) - K\pi(w), w)$$

is input-to-state convergent for the class of inputs $I_s(W)$ by Theorem 2.23 as

$$\frac{\partial}{\partial w} f(x, Kx + c(w) - K\pi(w), w) = \frac{\partial f_*}{\partial w} = \frac{\partial f}{\partial w} + \frac{\partial f}{\partial u}(\frac{\partial c}{\partial w} - K\frac{\partial \pi}{\partial w})$$

is bounded. Then by Theorem 3.5 the control u solves the global uniform output regulation problem. $\square$

Instead of a linear stabilizing control $K(x - \pi(w))$ a more general function $k(x, \pi(w))$ can also be fitted to achieve stability in a similar fashion. However such an approach is challenging without more specific assumptions on the system. Therefore we considered only the linear version.

In attempting to use contractivity as a tool for controller design in practice, an approach using LMI's similar to quadratic stability appears appealing. However since $\dot{M}$ is a part of the inequality (2.11), it must be bounded which creates challenges since for system (3.3) the entries of $\dot{M}$ are

$$\dot{M}_{ij}(x) = \nabla M_{ij}(x) \cdot f(x, u, w).$$

The dependence on u and w may prove especially challenging. Also with $\dot{M}$ the condition requires the solvability of a differential equation instead of a linear matrix inequality. This in turn means that results similar to Lemma 4.3 are not available. Thus computing varying metrics is again immediately more difficult. Therefore different methods are needed to utilize contractivity analysis with varying metrics. However there are simple cases where the metric can be directly computed.

While using contractivity directly to design a controller can be challenging, it can be used to verify the convergence of controllers designed using other methods. For example in [12] a contraction metric is used to verify that a controller for a robot manipulator is leads to a contracting system. On the other hand some simple cases where the system is not quadratically stabilizable, but is stabilizable such that the resulting system is contracting, do exist as demonstrated by the next example.

Example 4.7. *Consider the system*

$$\dot{x}_1 = x_1 + x_2 + u \tag{4.5a}$$

$$\dot{x}_2 = -x_2^3 + 1 \tag{4.5b}$$

$$e = x_1 - Fw \tag{4.5c}$$

$$\dot{w} = s(w) \tag{4.5d}$$

where $x_1(t), x_2(t), u(t), e(t) \in \mathbb{R}$, $w(t) \in \mathbb{R}^m$, *for all* $t \in \mathbb{R}$, $F \in \mathbb{R}^{1\times m}$ *and* $s : \mathbb{R}^m \to \mathbb{R}^m$ *is such that Assumption 3.1 holds for the set of initial conditions* $W \subset \mathbb{R}^m$ *and* $\frac{\partial s}{\partial w}$ *is bounded. The regulator equations of the system (4.5) are*

$$\begin{aligned}
\frac{d}{dt}\pi_1(w) &= \pi_1(w) + \pi_2(w) + c(w),\\
\frac{d}{dt}\pi_2(w) &= -\pi_2(w)^3 + 1,\\
0 &= \pi_1(w) - Fw.
\end{aligned}$$

They have a solution $\pi_1(w) = Fw$, $\pi_2(w) = 1$ *and* $c(w) = Fs(w) - Fw - 1$. *For system (4.5) we have*

$$A(\zeta) = \begin{bmatrix} 1 & 1 \\ 0 & -3x_2^2 \end{bmatrix}, \quad B(\zeta) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $\zeta = (x, w, u)$. *Such* $P, Y \in \mathbb{R}^{2\times 2}$, *do not exist that* $P = P^T > 0$ *and*

$$A(\zeta)P + PA(\zeta)^T + B(\zeta)Y + Y^T B(\zeta)^T < 0$$

when $\zeta = 0$. *This implies that the system (4.5) is not quadratically stabilizable. However the system (4.5) is stabilizable so that resultant system is contractive. If a linear control* $u = -2(x_1 - Fw) + c(w)$ *is chosen, of the form (4.4), the closed loop system of the* *form (4.3) is*

$$\dot{x}_1 = x_1 + x_2 - 2x_1 + Fs(w) - Fw - 1 + 2Fw \tag{4.6a}$$

$$= -x_1 + x_2 + Fs(w) + Fw - 1 \tag{4.6b}$$

$$\dot{x}_2 = -x_2^3 + 1. \tag{4.6c}$$

$$\dot{s} = s(w) \tag{4.6d}$$

Denote $f_*(x,w) = [-x_1 + x_2 + Fs(w) + Fw - 1, -x_2^3 + 1]^T$ *so that* $\dot{x} = f_*(x,w)$. *Thus we get*

$$A_*(\zeta) = \begin{bmatrix} -1 & 1 \\ 0 & -3x_2^2 \end{bmatrix}$$

where $\zeta = (x, w, u)$. *Choose a metric*

$$M(x) = M(x)^T = \begin{bmatrix} 1 & 0 \\ 0 & p(x_2) \end{bmatrix},$$

where

$$p(x_2) = \begin{cases} 1 & x_2 \leq -\frac{1}{2} \\ 1.5 - x_2 & -\frac{1}{2} < x_2 < \frac{1}{2} \\ 2 & x_2 \geq \frac{1}{2}. \end{cases}$$

Then the system $\dot{x} = f_*(x,w)$ *is globally contracting because*

$$A_*(x)M(x) + M(x)A_*(x)^T + \dot{M}(x) = \begin{bmatrix} -2 & 1 \\ 1 & (1 - x_2^3)\frac{\partial p}{\partial x_2} - 6x_2^2 p(x_2) \end{bmatrix} < M(x),$$

for all $x \in \mathbb{R}^2$, *where*

$$\frac{\partial p}{\partial x_2} = \begin{cases} -1 & -\frac{1}{2} < x_2 < \frac{1}{2} \\ 0 & \textit{otherwise.} \end{cases}$$

For each $w(\cdot)$ *starting in* W, *the corresponding trajectory of system (4.6) is* $x_w(t) = [Fw(t), 1]^T$. *Because* $w(\cdot)$ *is bounded,* $x_w(\cdot)$ *is bounded as well. The metric* $M(\cdot)$ *is also bounded because* $p(\cdot)$ *is bounded. Finally, the partial derivative* $\frac{\partial f_*}{\partial w}$ *is bounded because it was assumed that* $\frac{\partial s}{\partial w}$ *is bounded. Thus Theorem 4.6 guarantees that the control*

$$u = -2x_1 + Fs(w) - Fw - 1 + 2Fw,$$

solves the global uniform output regulation problem.

4.4 Locally stabilizable systems

The methods for designing controllers locally for the uniform output regulation problem are relatively similar to the global methods. The regulator equations are solved and then their solutions are utilized to find an applicable controller. However as the controller needs to work only in a neighborhood of the origin, the task is simpler. We will apply Theorem 3.6 in discussing a simple linear state feedback controller, following [4].

The controller needs to make the closed-loop system locally uniformly convergent. Because Theorem 2.15 guarantees that this is the case when the Jacobian of the system is Hurwitz, we consider the linearization of the system. The linearization approximates the system's dynamics near the origin and can be used to evaluate local stability properties. The linearization of the equation (3.3a) is

$$\dot{x} = \frac{\partial f}{\partial x}(0,0,0)x + \frac{\partial f}{\partial u}(0,0,0)u = A(0)x + B(0)u. \tag{4.7}$$

As the linearization describes the systems dynamics in a neighbourhood of the origin through a linear system we can use linear theory. Evaluating (4.7), the pair $(A(0), B(0))$ is stabilizable using a linear state feedback controller $u = Kx$ if $A(0)+B(0)K$ is Hurwitz for some $K \in \mathbb{R}^{d\times n}$ [4]. This fact is used to formulate the next result.

Theorem 4.8. *Consider system (3.3) with $f(0,0) = 0$, $y = (x, w)$ and exosystem (3.4) satisfying the neutral stability assumption. Suppose the regulator equations (3.8) are solvable and the corresponding continuous solutions $\pi(\cdot)$ and $c(\cdot)$ are defined in a some neighborhood of the origin invariant with respect to (3.4) and $\pi(0) = 0$, $c(0) = 0$. If there exists a matrix $K \in \mathbb{R}^{d\times n}$ such that $A(0) + B(0)K$ is Hurwitz, then the local uniform output regulation problem is solved by a controller of the form*

$$u = c(w) + Kx. \tag{4.8}$$

Proof. The controller $u = c(w) + Kx$ satisfies (3.9) with $\sigma(t) = 0$, $\forall t \in \mathbb{R}$ and $\theta(\xi, y) = c(w) + Kx$. The closed-loop system

$$\dot{x} = f(x, c(w) + Kx, w)$$

is locally exponentially convergent by Theorem 2.15 because

$$\frac{\partial}{\partial x} f(x, c(w) + Kx, w) = \frac{\partial f}{\partial x}(\xi) + \frac{\partial f}{\partial u}(\xi)K = A(\xi) + B(\xi)K,$$

is Hurwitz at $(x, w) = (0, 0)$. Thus by Theorem 3.6 the control $u = c(w) + Kx$ solves the local uniform output regulation problem. □

5. CASE STUDIES OF OUTPUT REGULATION

In this chapter we apply results on controller design from Chapter 4 to two variants of a harmonic oscillator and a neural network model. First we consider a standard model of the harmonic oscillator and secondly a similar model with a nonlinear damping term. Finally, the Hopfield model [5] representing a neural network is analyzed.

5.1 Harmonic oscillator

In this section we consider a simple harmonic oscillator and design a controller that achieves global output regulation. A harmonic oscillator can be modelled using a linear system [13]. As will be demonstrated, the system is quadratically stabilizable and the regulator equations are easily solvable. Knowing this we can use Theorem 4.4 to design a controller.

The system to be considered is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} u, \tag{5.1a}$$

$$e = Cx - Fw, \tag{5.1b}$$

$$\dot{w} = Sw, \tag{5.1c}$$

where $x(t), w(t), e(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, for all $t \in \mathbb{R}$, $k, m > 0$ and

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} a & b \end{bmatrix}, \tag{5.2}$$

for some $a, b \in \mathbb{R}$. Assume that $w_0 \in W_r = \{w \in \mathbb{R}^2 : \|w\| \le r\}$ for some $r > 0$. The solution of (5.1c) is

$$w(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} w_1(0) \\ w_2(0) \end{bmatrix}. \tag{5.3}$$

Thus $w(\cdot)$ is bounded and $w(t) \in W_r$ $\forall t \in \mathbb{R}$ implying that W_r is invariant with respect to the system (5.1c). Therefore $w(\cdot) \in I_s(W_r)$ and Assumption 3.1 is satisfied, because

W_r is compact.

The choices of C and F restrict the system to a case where the goal is to track the position of the oscillator to a linear combination of sine and cosine signals. This restriction is useful as it is not possible in general to control both the position and velocity of a harmonic oscillator separately due to the availability of a single control input.

The regulator equations of this system are

$$\begin{aligned}\frac{d}{dt}\pi_1(w(t)) &= \pi_2(w(t)),\\ \frac{d}{dt}\pi_2(w(t)) &= -\frac{k}{m}\pi_1(w(t)) + \frac{k}{m}c(w(t)),\\ 0 &= \pi_1(w(t)) - aw_1(t) - bw_2(t).\end{aligned}$$

Thus we get

$$\pi_1(w(t)) = aw_1(t) + bw_2(t),$$

which using (5.1) implies

$$\pi_2(w(t)) = \frac{d}{dt}(aw_1(t) + bw_2(t)) = aw_2(t) - bw_1(t).$$

Therefore we have

$$\begin{aligned}c(w(t)) &= \frac{m}{k}(\frac{d}{dt}\pi_2(w(t)) + \frac{k}{m}\pi_1(w(t)))\\ &= \frac{m}{k}(-aw_1(t) - bw_2(t)) + aw_1(t) + bw_2(t).\end{aligned}$$

Thus the solutions of regulator equations are

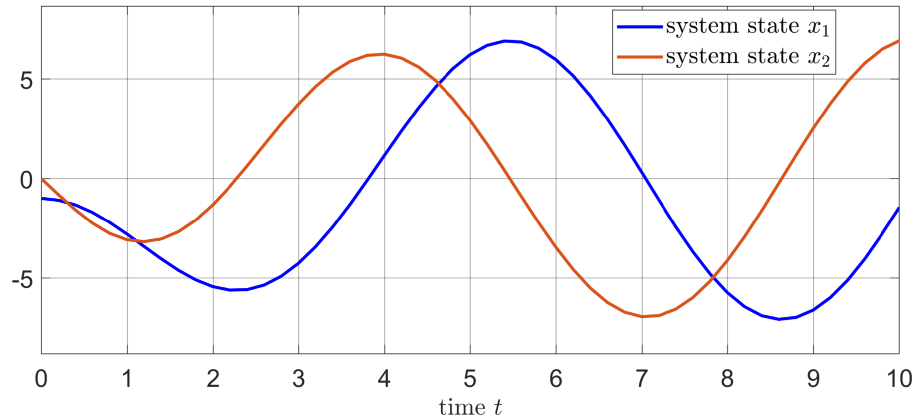
$$\begin{aligned}\pi_1(w(t)) &= aw_1(t) + bw_2(t),\\ \pi_2(w(t)) &= aw_2(t) - bw_1(t),\\ c(w(t)) &= (1 - \frac{m}{k})(aw_1(t) + bw_2(t)).\end{aligned}$$

The system is quadratically stabilizable with for example

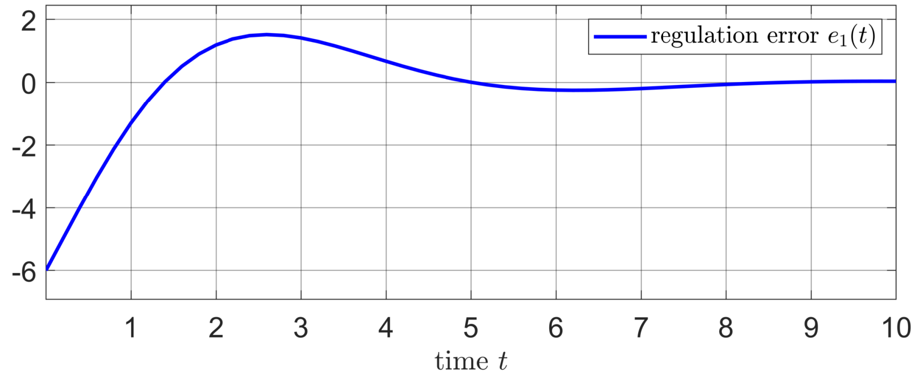
$$K = \begin{bmatrix} \frac{k-m}{k} & -\frac{m}{k}\end{bmatrix}, \quad P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \tag{5.4}$$

Thus by Theorem 4.4 the control can be chosen as

$$u = \frac{k-m}{k}Rw + K(x - Nw),$$



***(a)** State of the system*



***(b)** The regulated output of the system*

***Figure 5.1.** The simulated signals of system (5.1)*

where

$$R = \begin{bmatrix} a & b \end{bmatrix}, \quad Nw = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} w = \pi(w).$$

To verify the functionality of the controller Simulink™ is used to simulate the harmonic oscillator and its controller in a closed-loop system. For this purpose choose $k = 1$, $m = 2$, $a = 3$, $b = -1$, $w_0 = [1, -2]^T$ and $x_0 = [-1, 0]^T$. The simulation results are shown in Figure 5.1. The closed-loop system clearly achieves asymptotic output zeroing as the regulated output tends to zero and convergence is fast. Thus we have verified that the controller works.

5.2 Harmonic oscillator with nonlinear damping

In this section a nonlinear damping term is added to the oscillator considered in the previous section. As we will see, due to the damping term the system is no longer quadratically stabilizable. The regulator equations are solvable even with the damping term and thus we design a controller solving the local output regulation problem.

We consider the damped harmonic oscillator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -d(1+hx_2^2) \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} u \tag{5.5}$$

$$e = Cx - Fw \tag{5.6}$$

$$\dot{w} = Sw, \tag{5.7}$$

where $h, m, d, k > 0$, $x(t), u(t), w(t), e(t) \in \mathbb{R}^2$, for all $t \in \mathbb{R}$ and

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} a & b \end{bmatrix}, \tag{5.8}$$

for some $a, b \in \mathbb{R}$. Thus as in the previous example $w(\cdot) \in I_s(W_r)$ and Assumption 3.1 is satisfied. The regulator equations of the system are

$$\begin{aligned} \frac{d}{dt}\pi_1(w(t)) &= \pi_2(w(t)) \\ \frac{d}{dt}\pi_2(w(t)) &= -\frac{k}{m}\pi_1(w(t)) - d(\pi_2(w(t)) + h\pi_2(w(t))^3) + \frac{k}{m}c(w(t)) \\ 0 &= \pi_1(w(t)) - aw_1(t) - bw_2(t). \end{aligned}$$

Thus we get

$$\pi_1(w(t)) = aw_1(t) + bw_2(t),$$

which implies

$$\pi_2(w(t)) = \frac{d}{dt}(aw_1(t) + bw_2(t)) = aw_2(t) - bw_1(t).$$

Therefore we have

$$\begin{aligned} c(w(t)) =& \frac{m}{k}(\frac{d}{dt}\pi_2(w(t)) + \frac{k}{m}\pi_1(w(t)) + d(\pi_2(w(t)) + h\pi_2(w(t))^3)) \\ =& \frac{m}{k}(-aw_1(t) - bw_2(t)) + aw_1(t) + bw_2(t) \\ &+ \frac{md}{k}(aw_2(t) - bw_1(t) + h(aw_2(t) - bw_1(t))^3). \end{aligned}$$

Thus the solutions of the regulator equations are

$$\begin{aligned} \pi_1(w(t)) &= aw_1(t) + bw_2(t), \\ \pi_2(w(t)) &= aw_2(t) - bw_1(t) \\ c(w(t)) &= (1 - \frac{m}{k})(aw_1(t) + bw_2(t)) + \frac{md}{k}(aw_2(t) - bw_1(t) + h(aw_2(t) - bw_1(t))^3). \end{aligned}$$

For the system (5.5) we have

$$A(\zeta) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -d(1+3hx_2^2). \end{bmatrix}, \quad B(\zeta) = \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix},$$

where $\zeta = (x, w, u)$. For the system to be quadratically stable the inequality

$$PA(\zeta) + A(\zeta)^T P < 0, \quad \forall \zeta \in \mathbb{R}^6$$

must hold for some $P = P^T > 0$. Without loss of generality we can write,

$$P = \begin{bmatrix} 1 & p \\ p & p_2 \end{bmatrix}, \tag{5.9}$$

where $p, p_2 \in \mathbb{R}$. Then for quadratic stability the following inequalities must hold

$$\begin{aligned} -p\frac{k}{m} &< 0 \\ -p_2 d(1+hx_2^2) + p &< 0 \\ p\frac{k}{m}(p_2 d(1+hx_2^2) - p) - (1 - \frac{k}{m}p_2 - d(1+hx_2^2)p)^2 &> 0 \end{aligned}$$

for all $x_2 \in \mathbb{R}$. Thus $p > 0$ and since at large values of x_2, only the terms with x_2^2 are significant we have

$$p_2 p\frac{k}{m}d(1+hx_2^2) - d^2p^2(1+hx_2^2)^2 > 0,$$

implying

$$p_2\frac{k}{m} - dp(1+hx_2^2) > 0.$$

This however is not possible for an arbitrarily large x_2 with any choice of p_2 as $p > 0$. Therefore the system is not quadratically stable. Also the system is not quadratically stabilizable as it is not possible to suppress the growing term linearly.

The system is still *locally exponentially convergent* as

$$A(0) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -d, \end{bmatrix}$$

has eigenvalues $\lambda_1 = \frac{1}{2}(-d + \sqrt{d^2 - 4\frac{k}{m}})$, $\lambda_2 = \frac{1}{2}(-d - \sqrt{d^2 - 4\frac{k}{m}})$. For all values $d, k, m > 0$, the real parts are negative and so the matrix is Hurwitz. Thus by Theorem 4.8, the control $u = c(w)$ solves the local uniform output regulation problem.

5.3 Hopfield neural networks

Two models of neural circuits are analyzed in [5] using the notion of strong infinitesimal contractivity, which corresponds to the Demidovich condition, Theorem 2.13. In this section we solve the regulator equations for the *Hopfield model* and design a controller using quadratic stability to solve the output regulation problem. Output regulation of neural networks has not been considered widely. Examples can be found mainly for Boolean networks, which are considered for example in [17].

The Hopfield model is defined by

$$\dot{x} = -x + N\Psi(x) + Bu := f_H(x, u) \tag{5.10}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^d$ is an input, $N \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times d}$ and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a diagonal map $\Psi(x) = [\psi_1(x_1), \dots, \psi_n(x_n)]^T$ with component functions $\psi_i : \mathbb{R} \to \mathbb{R}$. We assume that these component functions, called *activation functions*, are differentiable and that $0 \leq \frac{\partial \psi_i}{\partial y}(y) < 1$ for all $y \in \mathbb{R}$.

In [5, Lemma 3.21] conditions for strong infinitesimal contraction of the other model introduced in [5], the firing rate model, are analyzed. Different norms, defined based on logarithmic functions and ℓ^1, ℓ^2 and ℓ^∞-norms, are used to formulate the conditions. Here we consider quadratic stability for the Hopfield model. This approach corresponds to strong infinitesimal contraction with an ℓ^2-norm and is chosen due to the similarity of the methods for different norms and models.

In the context of output tracking, we consider the system

$$\dot{x} = f_H(x, u) \tag{5.11a}$$

$$e = Cx - Fw \tag{5.11b}$$

$$y = (x, w) \tag{5.11c}$$

$$\dot{w} = s(w), \tag{5.11d}$$

where $x \in \mathbb{R}^n$ is the state, $e \in \mathbb{R}^p$ is the regulator error, $u \in \mathbb{R}^d$ is the control input, $w \in \mathbb{R}^m$ is the reference signal, $y \in \mathbb{R}^{n+m}$ is the measured output, $C \in \mathbb{R}^{p \times n}$, $F \in \mathbb{R}^{p \times m}$, $B \in \mathbb{R}^{n \times d}$ and $s : \mathbb{R}^m \to \mathbb{R}^m$ describes the exosystem.

First we consider solving the regulator equations. In general C is not invertible and in such cases direct computation is not feasible and different methods need to be utilized depending on the properties of C and A. We use the method outlined in Section 3.3 to solve the regulator equations for two cases, one where all nodes can be controlled, that is C has full row rank and B is invertible and another where the system models three nodes with a single input and a single output (SISO).

5.3.1 Hopfield system with control inputs on each node

First assume that C has full row rank and B is invertible, implying $n = d$. Also assume that $f(0,0) = 0$. We now follow the steps of the algorithm discussed in Section 3.3.

1) The composite system has a relative degree of $\{1,\ldots,1\}$ with $r = p$ because $D_a(x,w) = CB$ has full row rank. Then $H_s(x,w) = Cx - Fw$ and $E_s(x,w) = C(-x + N\Psi(x)) - Fs(w)$.

2) Since C has full row rank, the equation $Cx - Fw = 0$ can be solved for $r = p$ values of x, in terms of w and the remaining $n - r$ components of x such that the chosen r components of x define the partition x^1, x^2 satisfying rank$\frac{\partial H_s}{\partial x^1}(0) = r$. The solution of $Cx - Fw = 0$ and the partition define the mapping $\sigma : \mathbb{R}^{n-p+m} \to \mathbb{R}^p$ globally.

3) Solving $E_s(x,w) + D_s(x,w)u_e(x,w) = C(-x+N\Psi(x)) - Fs(w) - CBu_e(x,w) = 0$ gives $u_e(x,w) = B^{-1}(x - N\Psi(x) + C^+Fs(w))$ as a solution. The matrix $C^+ \in \mathbb{R}^{n\times p}$ is a left inverse of C that exists as C is of full row rank. Since u_e is not uniquely defined for $p < d$, it is possible to form a partition of u where only the elements u^1 are needed for output regulation. The function u_e can then depend on a control u^2. For simplicity we consider the case where u^2 is constant. Then we can choose any C^+, dependent on the wished effects on the zero dynamics of the system.

4) To determine the zero dynamics the differential equation

$$\begin{aligned}\dot{x}_{j_i} &= -x_{j_i} + \sum_{k=1}^{n} N_{j_ik}\psi_k(x_k) + x_{j_i} - \sum_{k=1}^{n} N_{j_ik}\psi_k(x_k) + \sum_{k=1}^{n}(C^+)_{j_ik}F_k s_k(w)\\ &= \sum_{k=1}^{n}(C^+)_{j_ik}F_k s_k(w),\end{aligned}$$

where $s_i(w)$ is the i:th element of the vector function $s(w)$ and F_i the i:th row of F, needs to be solved. Thus solutions of the zero dynamics depend on C^+, F and w, and are of the form $\pi^2(w) = \bar{C}^+Fw$, where $\bar{C}^+ \in \mathbb{R}^{(n-p)\times p}$ is dependent on C^+ and the partition defined. Then $\pi^1(w) = \sigma(\pi_2(w),w)$ and $c(w) = B^{-1}(x - N\Psi(x) + C^+Fs(w))$.

For example a system where the goal is to control a weighted sum of pairs of nodes may be modelled with

$$C = \begin{bmatrix} c_1 & c_2 & 0 & 0 & \ldots & 0 & 0\\ 0 & 0 & c_3 & c_4 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \ldots & c_{n-1} & c_n \end{bmatrix}.$$

with even n and $c_{2i-1} > 0$ for $i \in \{1,2,\ldots,\frac{n}{2}\}$. Then the matrix has full row rank and $r = m = \frac{n}{2}$. The equation $H_s(x,w) = Cx - Fw = 0$ can be solved for x_i,

$i \in \{1, 3, \ldots, n-1\}$ in terms of x_j, $j \in \{2, 4 \ldots, n\}$ and w. This yields

$$x_i = \frac{(Fw)_k}{c_i} - \frac{c_{i+1}x_{i+1}}{c_i},$$

where $k = \frac{i+1}{2}$, $i \in \{1, 3, \ldots, n-1\}$. Using this a partition $x^1 = \{x_1, x_3, \ldots, x_{n-1}\}$, $x^2 = \{x_2, x_4, \ldots, x_n\}$, can be formed. The mapping $\sigma : \mathbb{R}^n \to \mathbb{R}^{\frac{n}{2}}$ can be globally defined componentwise with $\sigma_i(x^2, w) = \frac{(Fw)_i}{c_{2i-1}} - \frac{c_{2i}x_{2i}}{c_{2i-1}}$, $i \in \{1, 2, \ldots \frac{n}{2}\}$. A left inverse of C is

$$C^+ = \begin{bmatrix} \frac{1}{c_1} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & \frac{1}{c_3} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{c_{n-1}} \\ 0 & 0 & \ldots & 0 \end{bmatrix},$$

which we choose for simplicity. The zero dynamics can be then solved from $\dot{x}_j = \sum_{k=1}^{n} c^+_{jk} F_k s_k(w) = 0$, $j \in \{2, 4 \ldots, n\}$ and so $x_j = x_j(0)$ are the solutions for this choice of C^+. Then we have $\pi^2(w) = 0$, $\pi^1_i(w) = \frac{(Fw)_i}{c_{2i-1}}$, implying that π and c are the solutions of the regulator equations where $\pi(w(t)) = C^+ Fw(t)$ and $c(w(t)) = B^{-1}(C^+ Fw(t) - N\Psi(C^+ Fw(t)) + C^* Fs(w(t)))$ for all $t \in \mathbb{R}$.

5.3.2 Three node Hopfield system with a single input and output

Next we consider a SISO example of the Hopfield model with 3 nodes. We choose

$$N = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T, \quad F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}. \tag{5.12}$$

We assume that the exosystem is $\dot{w} = Sw$, where

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{5.13}$$

so that as in the previous examples in Section 5.1 and Section 5.2, Assumption 3.1 is satisfied. Likewise we assume that $y = (x, w)$.

Thus the composite system (3.13) for these choices is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{w}_1\\ \dot{w}_2\end{bmatrix} = \begin{bmatrix}-x_1\\ -x_2+\psi_1(x_1)\\ -x_3+\psi_2(x_2)\\ w_2\\ -w_1\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\\ 0\\ 0\end{bmatrix} u, \tag{5.14a}$$

$$e = x_3 - F_1 w_1 - F_2 w_2, \tag{5.14b}$$

$$y = (x, w), \tag{5.14c}$$

$$\dot{w} = Sw. \tag{5.14d}$$

We further choose the activation functions ψ_1 as a hyperbolic tangent function and ψ_2 as a smoothened parametric rectified linear unit (PReLU) function to ensure solvability. These activation functions are used within the Hopfield model for example in [18]. The functions are defined as

$$\psi_1(x_1) = \begin{cases} x_1 & x_1 > 1 \\ mx_1 & x_1 < 1 \\ \psi_*(x_1) & \text{otherwise.} \end{cases}$$

$$\psi_2(x_2) = \tanh(x_2),$$

where $0 < m < 1$ and $\psi_*(\cdot)$ is a smooth approximation of the PreLU function on the interval $[-1, 1]$ constructed using mollifiers, which are smooth differentiable functions. Thus these activation functions satisfy the necessary assumptions as they are differentiable and clearly $0 \leq \frac{\partial \psi_i}{\partial x_i}(x_i) \leq 1$ as $\frac{\partial \psi_2}{\partial x_2}(x_2) = 1 - \tanh^2(x_2)$.

We follow the algorithm in Section 3.3 to solve the regulator equations of (5.14).

1) The system (5.14) has the Lie derivatives

$$
\begin{aligned}
L_{f_s}h(x,w) &= \begin{bmatrix} 0 & 0 & 1 & -F_1 & -F_2 \end{bmatrix} f_s(x,w) = -x_3 + \psi_2(x_2) - F_1 w_2 + F_2 w_1, \\
L_{f_s}^2 h(x,w) &= \begin{bmatrix} 0 & \frac{\partial \psi_2}{\partial x_2}(x_2) & -1 & F_2 & -F_1 \end{bmatrix} f_s(x,w) \\
&= \frac{\partial \psi_2}{\partial x_2}(x_2)(\psi_1(x_1) - x_2) - \psi_2(x_2) + x_3 + F_1 w_1 + F_2 w_2, \\
L_{f_s}^3 h(x,w) &= \begin{bmatrix} \frac{\partial \psi_1}{\partial x_1}(x_1)\frac{\partial \psi_2}{\partial x_2}(x_2) & (\psi_1(x_1) - x_2)\frac{\partial^2 \psi_2}{\partial x_2^2}(x_2) - 2\frac{\partial \psi_2}{\partial x_2}(x_2) & 1 & F_1 & F_2 \end{bmatrix} f_s \\
&= -x_1 \frac{\partial \psi_1}{\partial x_1}(x_1)\frac{\partial \psi_2}{\partial x_2}(x_2) + (\psi_1(x_1) - x_2)^2 \frac{\partial^2 \psi_2}{\partial x_2^2}(x_2) \\
&\quad - 2(\psi_1(x_1) - x_2)\frac{\partial \psi_2}{\partial x_2}(x_2) - x_3 + \psi_2(x_2) + F_1 w_2 - F_2 w_1.
\end{aligned}
$$

Likewise the Lie derivatives with respect to f_s and g_s are

$$
\begin{aligned}
L_{g_s}h(x,w) &= \begin{bmatrix} 0 & 0 & 1 & -F_1 & -F_2 \end{bmatrix} g_s(x,w) = 0, \\
L_{g_s}L_{f_s}h(x,w) &= \begin{bmatrix} 0 & \frac{\partial \psi_2}{\partial x_2}(x_2) & -1 & F_2 & -F_1 \end{bmatrix} g_s(x,w) = 0, \\
L_{g_s}L_{f_s}^2 h(x,w) &= \frac{\partial \psi_1}{\partial x_1}(x_1)\frac{\partial \psi_2}{\partial x_2}(x_2).
\end{aligned}
$$

Therefore, since $L_{g_s}L_{f_s}^2 h(x,w)|_{(x,w)=0} \neq 0$, the relative degree of the system is $r = 3$. We also get

$$
\begin{aligned}
D_s(x,w) &= \frac{\partial \psi_1}{\partial x_1}(x_1)\frac{\partial \psi_2}{\partial x_2}(x_2), \\
H_s(x,w) &= \begin{bmatrix} x_3 - F_1 w_1 - F_2 w_2 \\ -x_3 + \psi_2(x_2) - F_1 w_2 + F_2 w_1 \\ (\psi_1(x_1) - x_2)\frac{\partial \psi_2}{\partial x_2} - \psi_2(x_2) + x_3 + F_1 w_1 + F_2 w_2 \end{bmatrix} \\
E_s(x,w) &= -x_1 \frac{\partial \psi_1}{\partial x_1}(x_1)\frac{\partial \psi_2}{\partial x_2}(x_2) + (\psi_1(x_1) - x_2)^2 \frac{\partial^2 \psi_2}{\partial x_2^2}(x_2), \\
&\quad - 2(\psi_1(x_1) - x_2)\frac{\partial \psi_2}{\partial x_2}(x_2) - x_3 + \psi_2(x_2) + F_1 w_2 - F_2 w_1.
\end{aligned}
$$

2) Solving $H_s(x,w) = 0$ for x (since $r = 3 = n$), yields

$$
\begin{aligned}
x_3 &= F_1 w_1 + F_2 w_2 \\
\psi_2(x_2) &= F_1 w_1 + F_2 w_2 + F_1 w_2 - F_2 w_1 \\
\psi_1(x_1) &= \frac{1}{\frac{\partial \psi_2}{\partial x_2}(x_2)}(F_1 w_1 + F_2 w_2 - F_1 w_2 - F_2 w_1) + x_2
\end{aligned}
$$

To ensure that a solution x exists, we must restrict w as $|\psi_2(x_2)| = |\tanh(x_2)| \leq 1$

for all $x_2 \in \mathbb{R}$. We choose the set $W_r = \{w \in \mathbb{R}^2 : \|w\| < \frac{1}{\|\tilde{F}\|}\}$ where $\tilde{F} = \begin{bmatrix} F_1 - F_2 & F_1 + F_2 \end{bmatrix}$. Then for $w \in W_r$ we have

$$\begin{aligned} 1 > \|\tilde{F}\|\|w\| \geq & \left| \begin{bmatrix} F_1 - F_2 & F_1 + F_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right| \\ &= |F_1 w_1 + F_2 w_2 + F_1 w_2 - F_2 w_1|. \end{aligned}$$

With this choice for $w \in W_r$ we can write

$$x_2 = \tanh^{-1}(F_1 w_1 + F_2 w_2 + F_1 w_2 - F_2 w_1)$$

Finally since the range of $\psi_1(x_1)$ is $\mathbb{R}$ and it is clearly injective, we can write

$$x_1 = \psi_1^{-1} \left(\frac{F_1 w_1 + F_2 w_2 - F_1 w_2 - F_2 w_1}{1 - (F_1 w_1 + F_2 w_2 + F_1 w_2 - F_2 w_1)^2} + x_2 \right).$$

The partition for $r = n$ is trivial with $x = x^1$ and $x^2 \in \{0\}$, and the mapping σ is defined as the solution of $H_s(x, w) = 0$.

3) Solving $E_s(x, w) + D_s(x, w) u_e(x, w) = 0$ yields

$$u_e(x, w) = \frac{E_s(x, w)}{D_s(x, w)},$$

since $D_s(x, w), E_s(x, w) \in \mathbb{R}$.

4) Since the partition of the state is trivial with $x = x^1$, the zero dynamics are uniquely determined by x^1 and we do not need to solve them for x^2. Thus $\pi(w) = x^1(w)$ and $c(w) = u_e(\pi(w), w)$ for all $w \in W_r$. Since W_r is invariant with respect to the exosystem (5.14d) the functions π and c are the solutions of the regulator equations of (5.14) for all solutions of the exosystem satisfying $w(t) \in \Omega(W_r)$ for all $t \in \mathbb{R}$.

Having solved the regulator equations for two example systems, we consider stability next. The system (5.11) is quadratically stable when for some $P = P^T > 0$, $Q > 0$ we have

$$-2P^T + \frac{\partial \Psi}{\partial x}(x) N^T P + PN \frac{\partial \Psi}{\partial x}(x) \leq -Q, \quad \forall x \in \mathbb{R}^n$$

Using Lemma 4.3 this condition yields $n+1$ LMIs of the form

$$\begin{aligned} -2P &\leq -Q \\ -2P + nE_N B^T P + PNnE_i &\leq -Q, \quad i \in \{1, \ldots, n\} \end{aligned}$$

where $(E_i)_{ii} = 1$, $(E_i)_{jk} = 0$ for $j \neq i$ or $k \neq i$. This is as

$$\frac{\partial \Psi}{\partial x}(x)N \in \mathrm{co}\{0, nE_1N, nE_2N, \cdots, nE_nN\}, \quad \forall x \in \mathbb{R}^n$$

To see this consider $\frac{\partial \Psi}{\partial x}(x)N$ for an arbitrary fixed x and denote $\rho_i := (\frac{\partial \Psi}{\partial x}(x))_{ii}$. We have $0 \leq \rho_i \leq 1$, and thus $1 - \sum_{i=1}^{n} \frac{\rho_i}{n} \geq 0$. Then we can write $\lambda_i = \frac{\rho_i}{n}$ for $i \in \{1, \ldots, n\}$ and $\lambda_{n+1} = 1 - \sum_{i=1}^{n} \frac{\rho_i}{n}$. Using these we have

$$\frac{\partial \Psi}{\partial x}(x)N = \sum_{i=1}^{n} \frac{\rho_i}{n} nE_iN + (1 - \sum_{i=1}^{n} \frac{\rho_i}{n})0 = \sum_{i=1}^{n+1} \lambda_i nE_iN,$$

where $\sum_{i=1}^{n+1} \lambda_i = 1$, $\lambda_i \geq 0$ and so $\frac{\partial \Psi}{\partial x}(x)N \in \mathrm{co}\{0, nE_1A, nE_2N, \cdots, nE_nN\}$. The matrix E_i is called the matrix unit with row i and column i.

Similarly the system is quadratically stabilizable with $K = YP^{-1}$ if the LMIs

$$-2P \leq -Q,$$

$$-2P + nE_iN^TP + PNnE_i + Y + Y^T \leq -Q,$$

$i \in \{1, 2, \ldots, n\}$ are solvable.

Then using Theorem 4.4, the global uniform output regulation is solved by the controller $y = c(w) + K(x - \pi(w))$.

6. CONCLUSION

In this thesis we have investigated the output regulation problem in a broad manner. We have considered global and local variants of the problem together with sufficient solvability conditions. The solvability conditions in both cases consist of a stability requirement, a set of regulator equations and requirements for sufficient smoothness of functions. We also discussed the precise properties required and conditions for verifying them. Finally, we considered controller design in order to showcase methods for solving the output regulation problem for specific systems.

The central stability requirement investigated is uniform convergence with the uniformly bounded steady state property. In Chapter 2 three sufficient conditions for this property are presented. The first, Theorem 2.13, is the Demidovich condition implying an exponential form of convergence. The second, Theorem 2.14 uses Lyapunov-like functions directly implying uniform convergence with the UBSS property. The third, Theorem 2.23 is a combination of universal contractivity and existence of solutions, and also implies exponential convergence. These three stability conditions can be used in controller design and of them the first and the third are applied in Chapter 4.

The two approaches to controller design for global output regulation discussed in this thesis are rather similar, but with some key differences. Both approaches stabilize systems to an exponentially convergent form using linear state feedback control. The approach using contractivity is more complicated in computation and assumptions, requiring a contraction metric and boundedness of solutions to be known. Using quadratic stability on the other hand allows for simple computation, but is more restrictive. The contractivity approach can be seen as a slight generalization of the approach using quadratic stability.

In addition to the stability requirements, the regulator equations are also central in solving the output regulation problem. In fact, they could be said to be the defining factor in the solvability conditions of the output regulation problem as in most control problems some stability concepts are considered. The solutions of the regulator equations are used in Chapter 4 and so solving them is discussed in Section 3.3. In this section an algorithm to solve the equations for control affine systems is presented. The importance of the regulator equations cannot be understated especially as in some cases they can be solved rather easily as seen in Section 5.1 and Section 5.2.

The three case studies in Chapter 5 display not only solving the regulator equations, but also applications of the controller design methods discussed. The first case study involving a simple harmonic oscillator displays how for simple systems quadratic stability can be achieved with relatively low effort. The second case study shows how complexity leads to more difficult analysis and easily leads to a system that is not globally quadratically stabilizable even though it is locally stabilizable. The third case study focuses on applying the algorithm in Section 3.3 to solve the regulator equations of a neural network model. However it also displays the usage of quadratic stability.

The case studies in this thesis are limited to regulation of control affine quadratically stabilizable systems. Additional investigations could be conducted into systems that are not affine or systems that can be stabilized using universal contractivity. Other applications of contractivity could be studied as well, for example similarly to [12].

There are a number of assumptions made in the controller design methods discussed in this thesis which leave room for additional research. Only the simpler static state feedback controllers were considered, where as in practice controllers may have to be dynamic error feedback controllers. Similar methods could be used by constructing state observers. However approaches which directly design dynamic error feedback controllers could be also researched.

In addition to controller design, changes in the problem formulation can lead to interesting variants to consider. The assumptions on the exosystem can be varied for example to investigate the forward time variant of the global output regulation problem. Also the restrictive requirement of global convergence could be relaxed to a specified region or the convergence could be required to be uniform in a region and only convergent elsewhere.

Regarding computation, methods for finding contraction metrics could be investigated further. Direct computation appears difficult but investigating purely state dependent metrics or other simpler types of metrics could prove fruitful. In addition numerical methods could also be applicable in attempting to showing showing and using contractivity.

REFERENCES

[1] Pavlov, A. V., Wouw, N. van de and Nijmeijer, H. *Uniform Output Regulation of Nonlinear Systems A Convergent Dynamics Approach*. eng. 1st ed. 2006. Systems and Control: Foundations and Applications. Boston, MA: Birkhäuser Boston, 2006. ISBN: 0-8176-4465-2.

[2] Lohmiller, W. and Slotine, J.-J. E. On Contraction Analysis for Non-linear Systems. eng. *Automatica (Oxford)* 34.6 (1998), pp. 683–696. ISSN: 0005-1098.

[3] Jouffroy, J. A simple extension of contraction theory to study incremental stability properties. eng. *2003 European Control Conference (ECC)*. IEEE, 2003, pp. 1315–1321. ISBN: 3952417378.

[4] Huang, J. *Nonlinear output regulation: theory and applications*. eng. Society for Industrial and Applied Mathematics, 2004. ISBN: 0898715628.

[5] Bullo, F. *Contraction Theory for Dynamical Systems*. 1.1. Kindle Direct Publishing, 2023. ISBN: 979-8836646806. URL: https://fbullo.github.io/ctds.

[6] Khalil, H. *Nonlinear Systems*. Macmillan Publishing Company, 1992. ISBN: 9780023635410. URL: https://books.google.fi/books?id=RVHvAAAAMAAJ.

[7] Sontag, E. and Wang, Y. New characterizations of input-to-state stability. eng. *IEEE transactions on automatic control* 41.9 (1996), pp. 1283–1294. ISSN: 0018-9286.

[8] Angeli, D. A Lyapunov approach to incremental stability properties. eng. *IEEE transactions on automatic control* 47.3 (2002), pp. 410–421. ISSN: 0018-9286.

[9] Giesl, P., Hafstein, S. and Kawan, C. Review on Contraction Analysis and Computation of Contraction Metrics. eng. *Journal of computational dynamics* 10.1 (2023), pp. 1–47. ISSN: 2158-2505.

[10] Aylward, E. M., Parrilo, P. A. and Slotine, J.-J. E. Stability and robustness analysis of nonlinear systems via contraction metrics and SOS programming. eng. *Automatica (Oxford)* 44.8 (2008), pp. 2163–2170. ISSN: 0005-1098.

[11] Tsukamoto, H., Soon-Jo, C. and Slotine, J.-J. E. Contraction Theory for Nonlinear Stability Analysis and Learning-based Control: A Tutorial Overview. eng. *arXiv.org* (2021). ISSN: 2331-8422.

[12] Jouffroy, J. and Fossen, T. I. A Tutorial on incremental stability analysis using contraction theory. eng. *Modeling, identification and control* 31.3 (2010), pp. 93–106. ISSN: 0332-7353.

[13] Paunonen, L. *Mathematical Control Theory*. 2023. URL: https://homepages.tuni.fi/lassi.paunonen/course_material/Paunonen-MCT-2023.pdf.

[14] Cheng, D., Tarn, T. and Spurgeon, S. On the design of output regulators for nonlinear systems. eng. *Systems and control letters* 43.3 (2001), pp. 167–179. ISSN: 0167-6911.

[15] Huang, J. On the solvability of the regulator equations for a class of nonlinear systems. eng. *IEEE transactions on automatic control* 48.5 (2003), pp. 880–885. ISSN: 0018-9286.

[16] Numerical method for the solution of the regulator equation with application to nonlinear tracking. eng. *Automatica (Oxford)* 44.5 (2008), pp. 1358–1365. ISSN: 0005-1098.

[17] Li, H., Xie, L. and Wang, Y. Output Regulation of Boolean Control Networks. eng. *IEEE transactions on automatic control* 62.6 (2017), pp. 2993–2998. ISSN: 0018-9286.

[18] Wang, C., Liang, J. and Deng, Q. Dynamics of heterogeneous Hopfield neural network with adaptive activation function based on memristor. *Neural Networks* 178 (2024), p. 106408. ISSN: 0893-6080. DOI: https://doi.org/10.1016/j.neunet.2024.106408. URL: https://www.sciencedirect.com/science/article/pii/S0893608024003320.