Tampere University

Elizaveta Ermakovich

RAMSEY NUMBERS

Bachelor thesis
Faculty of Engineering and Natural Sciences
April 2024

ABSTRACT

Elizaveta Ermakovich: Ramsey numbers
Bachelor thesis
Tampere University
Science and Engineering
April 2024

It is always guaranteed to find some order in chaos – we just need to take a large enough chaos. To explain why, we use the tools from the area of mathematics known as Ramsey theory.

Ramsey theory is a relatively new discipline that lies between the fields of combinatorics and graph theory. It deals with conditions under which some properties are guaranteed for large-scale systems. The measure of how large those systems need to be is characterized by Ramsey numbers. Almost a hundred years ago British mathematician Frank Ramsey proved that if we know the size of the structure that we are looking for, a large enough graph that guarantees the appearance of that structure in it always exists.

Today, we only know the first nine Ramsey numbers. The calculation of higher values remains an open challenge in mathematics. As the numbers increase, so does the uncertainty surrounding their exact values. For most of the Ramsey numbers, we only know the intervals for search, and the larger the numbers get, the wider those intervals are.

One extension of Ramsey theory is the induced Ramsey theory. Instead of looking for a complete subgraph, as in the classical Ramsey theory, induced Ramsey theory seeks to find an arbitrary subgraph in a larger graph. It is an even more complicated task, since we not only have to present a graph of a certain size but also of a certain structure.

In this thesis, we present a general overview of Ramsey theory. We define graph theoretical concepts related to Ramsey numbers. Then we prove the finiteness of Ramsey numbers and show some bounds for them. Lastly, we discuss a recent result in Ramsey theory and introduce induced Ramsey theory.

Keywords: graph theory, Ramsey numbers, Ramsey theory, induced Ramsey theory, combinatorics

PREFACE

This Bachelor thesis was completed as a part of my applied mathematics studies in Tampere University. My interest in graph theory first sparked when I took the 'Graph theory' course in my second year. University lecturer Riikka Kangaslampi, who was teaching it, became my thesis supervisor afterwards. Working on this thesis was an exciting process from start to finish, and the topic I went with has kept my brain occupied (in a good way) even in my daily life outside of studies. It was a valuable experience and I have definitely learned a lot.

I would like to thank Riikka for her kindness and wisdom, and especially for always being patient with me throughout the writing process. I truly could not wish for a better supervisor. Last but not least, I am grateful to everyone close to me for their support, and for always being ready to hear my endless chatters about Ramsey numbers.

Tampere, 22nd April 2024

Elizaveta Ermakovich

CONTENTS

1. Introduction 1
2. Fundamental concepts in graph theory 3
3. Ramsey theory and Ramsey numbers 11
   3.1 Ramsey theory 11
   3.2 Bounds on Ramsey numbers 16
   3.3 Induced Ramsey theory 19
4. Summary. 22

References 23

LIST OF SYMBOLS AND ABBREVIATIONS

$\{v_i, v_j\}$	edge between vertices v_i and v_j
$E(G)$	set of edges of graph G
G	graph
$\overline{G}$	complement of a graph G
K_n	complete graph on n vertices
$\mathbb{N}$	set of natural numbers $\{1, 2, 3, ...\}$
P_n	path on n vertices
$R_{ind}(H_i, H_j)$	induced Ramsey number
$R(k, l)$	2-color Ramsey number
$R(n_1, \ldots, n_k)$	multicolor Ramsey number
v	vertex of a graph
$V(G)$	set of vertices of graph G

1. INTRODUCTION

Graph theory started developing as a field of mathematics in 18th century. In 1735, Leonhard Euler, a Swiss mathematician, proposed a famous puzzle called "seven bridges of Königsberg". He asked whether one can cross each of the seven bridges of Königsberg city without crossing any of them twice. Soon Euler presented a proof that it was, in fact, impossible. That proof is often considered the first important result in graph theory. [1]

Ramsey theory, the focus of this thesis, was founded by a British mathematician Frank Ramsey. Despite of dying at the age of 26, he left a significant legacy in various areas of knowledge. Born in Cambridge, England, Ramsey was mainly working in the fields of mathematics, philosophy and economics [12]. However, his biggest impact was the theory that bears his name ever since – Ramsey theory.

Ramsey theory is a branch of combinatorics that is often presented with the means of graph theory. The most important result of Ramsey theory is Ramsey's theorem [22]. In simple words, it states that in any large enough graph, there will always be a specific arrangement present, regardless of how "disordered" that graph may appear. Since by the size of the graph we mean the number of vertices, or points, that it consists of, we want to determine how large *exactly* the graph should be to guarantee the presence of the above-mentioned properties. Such numbers of vertices are called Ramsey numbers, and they respectively depend on the size of the structures that we are looking for.

Unlike many concepts in mathematics, Ramsey numbers are easy to explain to anyone with enough mathematical literacy. The most popular illustration of them is the so-called "party problem" [7]. It states that at a party with 6 guests, there will always be either at least 3 people who know each other or at least 3 people who have never met. That is because the Ramsey number for the pair (3,3) is 6. This and many other problems in Ramsey theory are intuitive to visualize by coloring the edges of the graph – which is extensively used for explaining many of the related concepts.

Even though some of the small Ramsey numbers are known, the rest of them are not yet discovered. Today, the largest Ramsey number we know is 36 for the pair (3,9) [5], which does not seem impressive at first glance, but makes sense once the mechanism of computing such numbers is explained. The larger the graphs get, the more variants of them we have to search through [14]. At the present moment, our modern computational powers are far from being enough for that task.

For the majority of Ramsey numbers, we only know the interval where they lay, i.e. their lower and upper bounds. It is famously a hard task to find tighter bounds for Ramsey numbers, and different researchers have slowly been improving them since Ramsey introduced his theory [5]. A great contributor to that was Paul Erdős, a Hungarian mathematician who dedicated many of his works to Ramsey theory. However, since Erdős' 1935 paper [10], there has not been a significant improvement in determining the general formula for a non-trivial upper bound on Ramsey numbers.

In 2023, after 70 years of no major progress in optimizing the general formula for the upper bound, researchers from the Universities of Rio de Janeiro and Cambridge made a breakthrough. They improved the formula from 4^k to $(4-\epsilon)^k$, which is the first exponential improvement of the upper bound since 1935. [4]

Induced Ramsey theory is an extension of Ramsey theory that asks whether we can find a *specific* subgraph in a large graph. This changes the problem completely, since now we not only have to guarantee a certain size of the graph, but also construct it with a high precision. The fact that we can always find a graph with such properties is one of the most important results in Ramsey theory. [6]

Ramsey theory has important applications in the field of mathematics and beyond. Number theory, algebra, topology, set theory and many other areas of mathematics use its results. Other areas where Ramsey theory is applied are, for example, theoretical computer science, information theory, and communications. Finally, there are many interesting games that are based on Ramsey theory. [23][2]

The structure of of this thesis is as follows. Chapter 2 presents fundamental concepts in graph theory required to understand Ramsey theory and an example for developing an intuition about Ramsey numbers. The primary focus of the thesis revolves around Chapter 3, which explains the main results in Ramsey theory. Section 3.1 contains the statement and proof of Ramsey theorem, definition of a Ramsey graph and a generalized version of Ramsey theorem. Section 3.2 elaborates on upper and lower bounds on Ramsey numbers, as well as known Ramsey numbers and the recent improvement in determining diagonal Ramsey numbers bounds. Section 3.3 presents induced Ramsey theory, along with an upper bound for induced Ramsey numbers and an example of one of them. Finally, the main insights of the thesis are summarized in Chapter 4.

2. FUNDAMENTAL CONCEPTS IN GRAPH THEORY

In this chapter we define important concepts in graph theory that will be essential for understanding Ramsey's theory in Chapter 3.

A graph $G = (V, E)$ is a structure that consists of a non-empty set of vertices $V(G)$ and a set of edges $E(G)$. Edges connect pairs of distinct vertices or one vertex to itself.

The following is a formal definition of a graph:

Definition 2.1. *A graph G is an ordered pair $(V(G)), E(G))$ consisting of a non-empty set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$, of edges, together with an incidence function ψ_G that associates with each edge of G a pair of (not necessarily distinct) vertices of G. [3]*

Graphs can be either directed — with edges pointing from one vertex to another — or undirected — with edges connecting two vertices without specifying a direction for that connection. In this work we will only be focusing on undirected graphs.

Example 2.2. *In Figure 2.1 we present an example of an undirected graph with vertex set $V(G) = \{1, 2, 3, 4, 5\}$ and edge set $E(G) = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{4, 5\}\}$.*

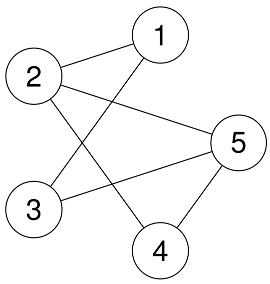


Figure 2.1. *A graph with 5 vertices and 6 edges.*

A graph that consists of n vertices is referred to as a *graph on n vertices*.

Sometimes a vertex in a graph can be connected to itself, or two vertices can be connected by multiple edges. An edge that connects a vertex to itself, that is, is of the form $\{v, v\}$, is called a *loop*. Edges, that connect the same two vertices, are called *parallel edges*.

Example 2.3. *In Figure 2.2, vertex 1 is connected to itself with a loop, and vertices 3 and 4 have two parallel edges between them.*

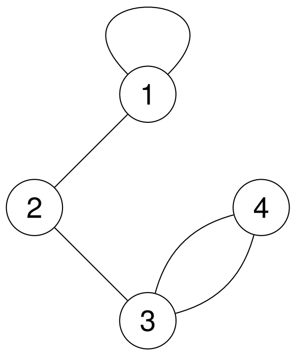


Figure 2.2. *A graph with a loop and parallel edges.*

Definition 2.4. *(Simple graph). A graph is called simple if it has no loops or parallel edges. [3]*

Definition 2.5. *(Adjacent vertices). Two vertices v, w of G are said to be adjacent, or neighbours, if $\{v, w\}$ is an edge of G. [6]*

For instance, vertices 1 and 2 in Figure 2.2 are adjacent, but vertices 1 and 4 are not. The number of adjacent vertices to a given vertex is referred to as the *degree* of that vertex. In this example, the degree of vertex 2 is equal to 2.

Later in Chapter 3, when talking about Ramsey numbers, we will need a special type of graphs where all vertices are adjacent to one another.

Definition 2.6. *(Complete graph). A complete graph is a simple graph in which any two vertices are adjacent. [3]*

Example 2.7. *The graph in Figure 2.3 is complete.*

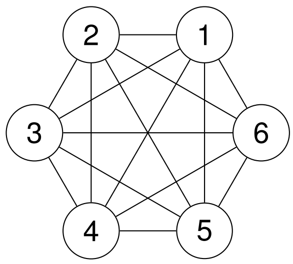


Figure 2.3. *A complete graph on 6 vertices.*

We call a complete graph on n vertices n*-complete* or K_n.

In Chapter 3, we will extensively use a technique called *graph coloring* (and specifically *edge coloring*). It is helpful especially for developing a visual intuition about Ramsey's theorem and Ramsey numbers. We briefly explain it next.

It is common to think of graph colorings as of ways to color the vertices or edges of a graph in such manner that no neighbouring elements have the same color. We introduce that way of interpreting it first and then follow to expand it to a more general definition of graph coloring. For simplicity, let us call the first version a "proper coloring", and the second version will then simply be referred to as "graph coloring".

First, we introduce vertex coloring, and then continue to explain edge coloring based on that.

Definition 2.8. *(Proper vertex coloring). A proper vertex coloring of a graph* $G = (V, E)$ *is a map* $c : V(G) \to S$ *such that* $c(v) \neq c(w)$ *whenever* v *and* w *are adjacent. The elements of* S *are called available* $colors$*. [6]*

Essentially, we are trying to color the vertices of a graph in such a way that no two neighbouring vertices have the same color.

Example 2.9. *Figure 2.4 shows how to "properly" color a given graph with 3 colors.*

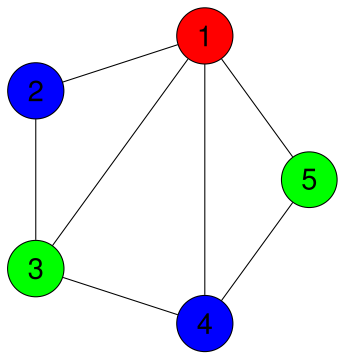


Figure 2.4. *Proper vertex coloring of a graph on 5 vertices.*

Definition 2.10. *(Proper edge coloring). A proper edge coloring of a graph* $G = (V, E)$ *is a map* $c : E(G) \to S$ *such that* $c(e) \neq c(f)$ *for any adjacent edges* e*,* f*.*

By adjacent edges we mean two edges that share a common vertex.

Example 2.11. *In Figure 2.5, the same graph as in Figure 2.4 is used, except now we color the edges instead of vertices. It can be easily seen that in this case we need not 3 but at least 4 colors, because vertex 1 has 4 incident edges.*

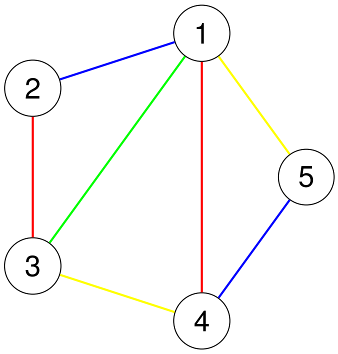


Figure 2.5. *Proper edge coloring of a graph on 5 vertices.*

However, we do not always need to color a graph in a way that no neighbours have the same color. Let us define the general case of edge coloring that we will use later in the end of this chapter and in Chapter 3.

Definition 2.12. *(Edge coloring). A coloring of the set of edges $E(G)$ of a graph G, with c colors, or c-coloring for short, is a partition (a way of splitting a set into smaller non-overlapping sets) of $E(G)$ into c classes (indexed by the "colors"). These colorings need not satisfy any non-adjacency requirements. [6]*

In other words, in an "improper" graph coloring we do not care whether adjacent edges have the same color or not. From now on referring to edge coloring, we will only mean that. For example, the edges of the graph in Figure 2.14 are colored "improperly".

Next, we introduce the concepts of a *subgraph*, an *induced subgraph* and a *clique*. To illustrate those structures, let us start with a graph G in Figure 2.6:

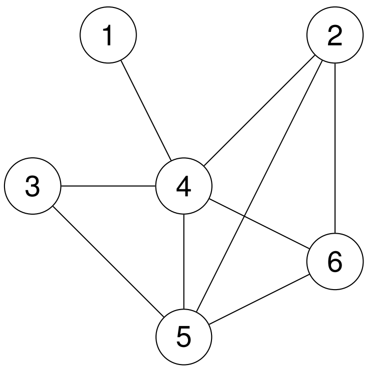


Figure 2.6. *A graph on 6 vertices.*

In simple words, a subgraph of a graph G is the subset of some vertices and edges of G, with the condition that if an edge is included in the subgraph, its incident vertices must also be included.

Example 2.13. *The graph in Figure 2.7 is one possible subgraph of G. We will call it G_1.*

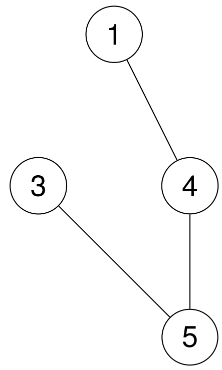


Figure 2.7. *G_1 is a subgraph of G on 4 vertices.*

Definition 2.14. *(Induced subgraph). A subgraph induced by Y is a subgraph of G whose vertex set is Y and whose edge set consists of all edges of G which have both ends in Y. [3]*

Thus, every induced subgraph is a subgraph, but not every subgraph is an induced subgraph. In Figure 2.8 we present G_2 – an induced subgraph of G. In this case the set Y consists of vertices 1, 3, 4, and 5.

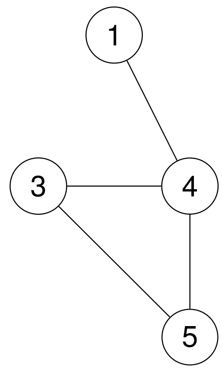
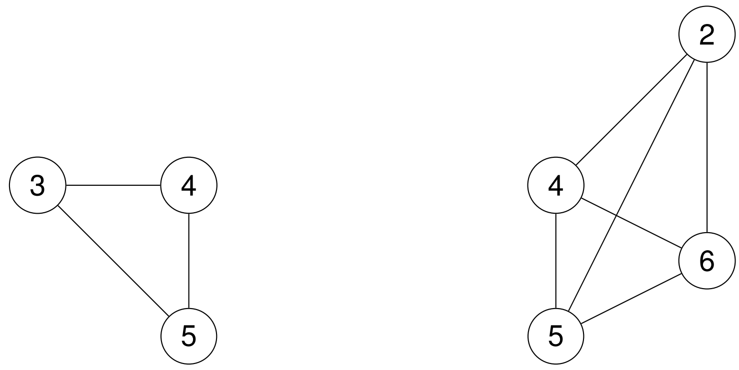


Figure 2.8. *G_2 is an induced subgraph of G on 4 vertices.*

The difference between G_1 (a subgraph) and G_2 (an induced subgraph) is only in the edge $\{3, 4\}$. While the inclusion of this edge is optional for a subgraph, it is necessary for an induced subgraph.

Finally, we call a complete induced subgraph of a graph G a *clique* on G. Two of the cliques in G are shown in Figure 2.9:



(a) *A 3-clique in G.* ***(b)*** *A 4-clique in G.*

Figure 2.9

Sometimes it is useful to construct a graph $\overline{G}$ that has the same vertices as the original graph G, but with the condition that all the edges that were not in G are included in $\overline{G}$ and the edges that were in G do not appear in $\overline{G}$. Such graph $\overline{G}$ is called the *complement* of G.

Definition 2.15. *(Complement of a graph). The complement $\overline{G}$ of $G(V, E)$ is the simple graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G. [3]*

Example 2.16. *The graph $\overline{G}$ in Figure 2.10 is the complement of G.*

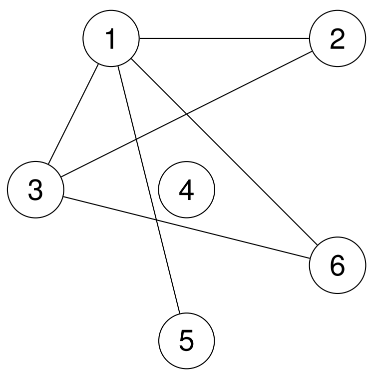


Figure 2.10. *$\overline{G}$, the complement of G.*

Lastly, we introduce a theorem that will be helpful in some proofs of this and the following chapter.

Theorem 2.17. *(Pigeonhole principle). Let f be a function from a finite set X into a finite set Y. If $n \geq 1$ and $|X| > n|Y|$, then there exists an element of Y that is the image under f of at least $n + 1$ elements of X. [8]*

Example 2.18. *A popular introduction to Ramsey theory is the so-called "party problem" (also known as "theorem on friends and strangers") [7]. Consider a party with 6 participants. Then each two of them either know or do not know each other. Is it then true that* *there are at least 3 people at the party who each know each other or otherwise at least 3 people who each have not met before? The answer to this question is affirmative, which can be shown with the following proof:*

Proof. We construct a graph that represents the party problem using edge coloring. Each vertex is a guest, and each edge is the relation between two guests: red edge in case if they know each other and blue edge otherwise. In other words, we want to prove that it is impossible to construct such a red-and-blue graph without either getting a red triangle or a blue triangle. Let us start with a graph with no edges in Figure 2.11:

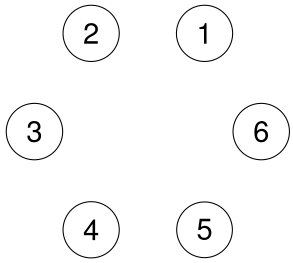


Figure 2.11. *A graph representing 6 guests at the party and no relations between them.*

Now, choose an arbitrary vertex – for example, vertex 1. We know that due to the pigeon-hole principle (Theorem 2.17), the person represented by that vertex will know or not know at least 3 other people at the party. We can take the first case as due to symmetry it does not change the essence of the proof. We now connect vertex 1 with any 3 other vertices (in this case, 2, 4 and 5) with red edges as shown in Figure 2.12:

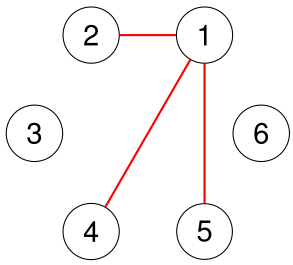


Figure 2.12. *Guest 1 knows guests 2, 4 and 5.*

Recalling that we are trying to avoid red triangles, we must connect vertices 2 and 4, 4 and 5, and 2 and 5 pairwise with blue edges (Figure 2.13).

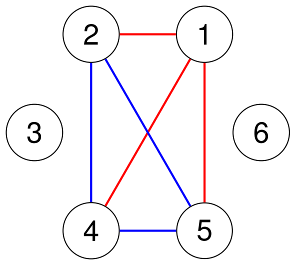


Figure 2.13. *Guests 2, 4 and 5 all do not know each other.*

As a result, we get a blue triangle, which shows that we will necessarily have a triangle of blue edges in any such scenario with 6 guests. The case with an "unavoidable" red triangle is proven identically. Thus, we have shown that at a party with 6 guests, there will either be at least 3 people that each know each other or at least 3 people that have never met.

□

It can also be shown (Figure 2.14) that a lower number of guests (namely, 5) will not necessarily have that property:

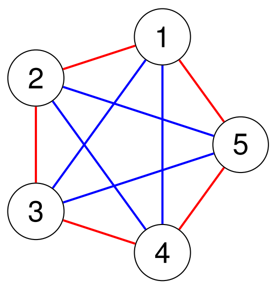


Figure 2.14. *A party with 5 guests, where no 3 people all know or do not know each other.*

We will later see that the graph above is the so-called *Ramsey graph* for the *Ramsey number* $R(3,3)$.

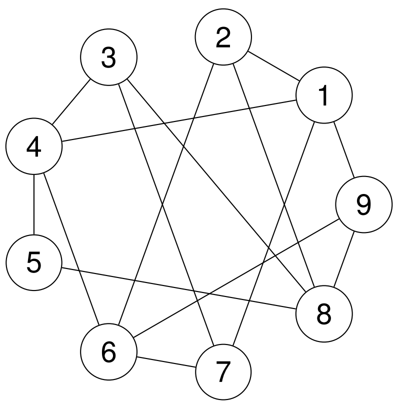
3. RAMSEY THEORY AND RAMSEY NUMBERS

Ramsey theory is generally concerned with finding certain substructures in large structures of a given size. Ramsey numbers, represented by $R(k, l)$, refer to the smallest natural number n where any graph of order at least n guarantees the existence of a k-clique, or the existence of a l-clique in its complement. Though established in 1930 by Frank Ramsey, understanding their precise values remains a challenge.

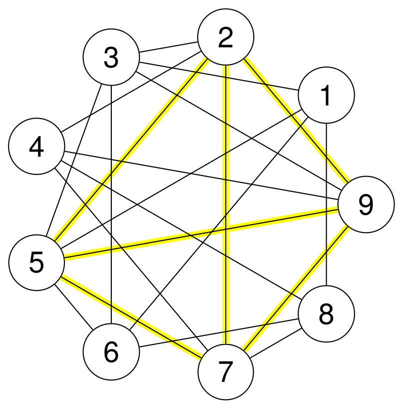
3.1 Ramsey theory

Theorem 3.1. *(Ramsey's theorem). For every $k, l \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that every graph of order at least n either contains a K_k, or its complement contains a K_l as an induced subgraph. [6]*

Example 3.2. *In Figure 3.1, a graph on 9 vertices (a) and its complement (b) are presented. 9 is known to be such n for $k = 3$ and $l = 4$, so we must expect either the original graph to have a K_3 or its complement to have a K_4. In this example, it is the complement that has a 4-complete graph as an induced subgraph.*



***(a)** A graph on 9 vertices containing no K_3.*



***(b)** The complement of the graph on the left with a K_4 induced by {2,5,7,9}.*

Figure 3.1

Definition 3.3. *(Ramsey number). The minimal integer $R(k, l)$ that depends on both k and l is called a Ramsey number. If $k = l$, we call it a diagonal Ramsey number.*

In other words, Ramsey's theorem states that for any positive integers k, l, a Ramsey number $R(k,l)$ is finite.

The two trivial cases are

$$\begin{aligned} R(1,l) &= R(k,1) = 1, \\ R(2,k) &= R(k,2) = k. \end{aligned} \tag{1}$$

In the first case, any 1-subgraph is monochromatic in every color as there are no edges in it. In the second case, if a simple graph has at least one edge, then that edge will be connecting two vertices. Therefore, either the graph or its complement will have an edge. If that graph is complete and monochromatic, we will find a K_k, and in any other case we will find a K_2 (essentially, just an edge).

The following inequality was presented by Erdős and Szekers in 1935 [10], and it is directly used to prove Ramsey's theorem. But first, let us prove the inequality itself.

Lemma 3.4.

$$R(k,l) \leq R(k-1,l) + R(k,l-1)$$

Proof. Assume that $R(k-1,l)$ and $R(k,l-1)$ are both finite (it is the only "interesting" case to prove, since if at least one of the components on the right is infinite, the inequality always holds). Now, consider a graph on exactly $R(k-1,l)+R(k,l-1)$ vertices whose edges are colored in red and blue. It can be thought of as if all the edges in the original (not necessarily complete) graph were red, and then we overlay the complement of that graph onto it and color its edges blue. The result is a complete red-and-blue graph.

Next, take an arbitrary vertex v in this graph. It is connected to every other vertex either by a red or a blue edge. Call the set of vertices that are connected to v with a red edge P and the set of vertices that are connected to v with a blue edge S. This is shown in Figure 3.2:

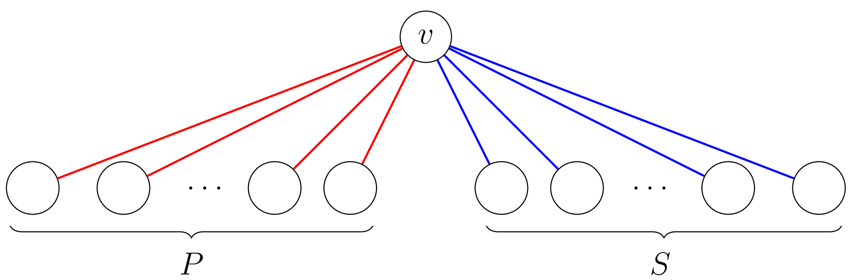


Figure 3.2. *Vertex v is connected to the rest of the vertices by red and blue edges.*

We may assume that either the cardinality (number of elements in the set X, denoted

$|X|$) of P is greater or equal to $R(k-1,l)$ or the cardinality of S is greater or equal to $R(k,l-1)$. In short, this is because the total number of vertices connected to v, $|P \cup S|$, is equal to $R(k-1,l)+R(k,l-1)-1$. However, let us show why either of the two cases must hold by negating the statement:

$$\overline{(|P| \geq R(k-1,l) \textit{ OR } |S| \geq R(k,l-1))}$$

$$\Longleftrightarrow$$

$$(|P| < R(k-1,l) \textit{ AND } |S| < R(k,l-1)).$$

But $|P|$, $|S|$, $R(k-1,l)$, and $R(k,l-1)$ are all integers, which means that

$$|P| \leq R(k-1,l)-1 \textit{ AND } |S| \leq R(k,l-1)-1.$$

Thus,

$$|P|+|S| \leq R(k-1,l)+R(k,l-1)-2,$$

which is a contradiction, since $|P|+|S| = |P \cup S| = R(k-1,l)+R(k,l-1)$

Now, we shall return to the proof of lemma, knowing that at least one of the cases must hold.

If the first case is true and $|P| \geq R(k-1,l)$, then it is guaranteed that the vertices in P form either a K_{k-1} or a K_l, i. e., a red $k-1$-clique or a blue l-clique. Since P contains all the vertices connected to v in red, including v into the K_{k-1} makes it a K_k, and we are done. The two above scenarios are illustrated in Figure 3.3:

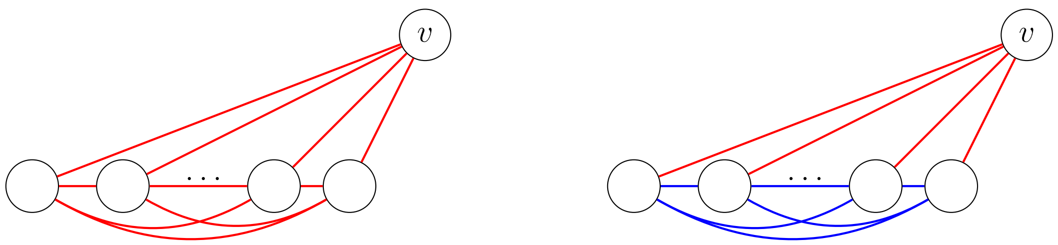


Figure 3.3. *Two possible complete graphs if* $|P| \geq R(k-1,l)$

On the other hand, if $|S| \geq R(k,l-1)$, the we have a K_k or a K_{l-1} in S respectively, and adding v, which is connected to vertices in S by blue edges, leads to the same result. The visual representation of that is similar to Figure 3.3 except the colors are swapped.

Since one of the two cases must hold, that concludes the proof of the inequality. The idea of the proof was taken from Bondy's and Murty's book "Graph Theory" [3]. □

Now, it is straightforward to show that Theorem 3.1 holds, i.e., $R(k,l) < \infty$.

Proof. We do that using induction. Assume that for some $n \in \mathbb{N}$, $R(k,l) < \infty$ when $k + l \leq n$.

We start with the base case $n = 2$ (see equations 1):

$$R(1,1) = 1.$$

Now, we use Lemma 3.4 to show the two cases when k or l is incremented by 1:

$$R(k+1,l) \leq R(k,l) + R(k+1,l-1),$$
$$R(k,l+1) \leq R(k-1,l+1) + R(k,l).$$

In both inequalities, we see that all indices on the right side sum up to $k + l = n$:

$$k + 1 + l - 1 = k + l = k - 1 + l + 1 = n.$$

That, together with the inductive hypothesis, leads us to the following:

$$R(k+1,l) < \infty,$$
$$R(k,l+1) < \infty.$$

Thus, the equation holds for both k and l incremented by 1, i.e., whenever their values sum up to $n + 1$. This completes the proof of Ramsey's theorem (3.1).

□

Definition 3.5. *(Ramsey graph). A Ramsey graph is a complete graph on $R(k,l) - 1$ vertices that does not contain either K_k or K_l. [3]*

Example 3.6. *A Ramsey graph for $k = 3$ and $l = 4$ is shown in Figure 3.4:*

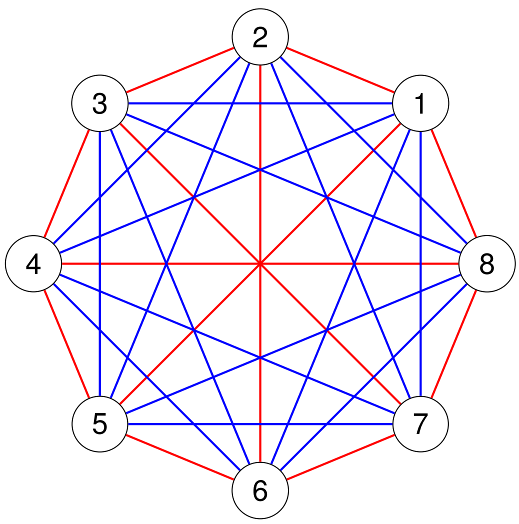


Figure 3.4. *A (3,4)-Ramsey graph.*

Brendan McKay, an Emeritus Professor in the School of Computing at the Australian National University, maintains a list of all known Ramsey graphs [19]. There, unlike in our definition (3.5), graphs of smaller size than $R(k,l)-1$ are also considered Ramsey graphs.

There is no general method for constructing Ramsey graphs, although, there are some useful techniques. If a known Ramsey number is of the form $p+1$, where p is a prime number (for example, $R(3,5)=14$ and $R(4,4)=18$), we can construct the Ramsey graph for it by regarding the vertices of it as a finite field and using residues modulo p to connect vertices with one of the colors. The rest of the edges will then have the second color. Graphs of this kind are also known as *Paley graphs* [20]. For larger Ramsey numbers, more complicated methods are used to construct their Ramsey graphs, such as probabilistic method [20] and boolean function representations [11].

Ramsey's theorem can also be extended beyond the 2-colorings into the multi-color case.

Definition 3.7. *(Multicolor Ramsey number). A multicolor Ramsey number $R(n_1,\ldots,n_k)$ is defined as the minimal number of vertices on a complete graph that, if colored in k colors, contains a monochromatic clique of size n_i for some $1\leq i\leq k$. [6]*

At the time of writing, only 2 multicolor Ramsey numbers are known: $R(3,3,3)=17$ and $R(3,3,4)=30$ [21]. Generalized Ramsey's theorem is almost identical to Ramsey's theorem for the 2-color case. 3.1

Theorem 3.8. *(Generalized Ramsey's theorem). [3]*

$$R(n_1,\ldots,n_k)<\infty,\quad \forall n_i\in\mathbb{N},\, n_i\geq 1.$$

Figure 3.5 shows an example of a Ramsey graph for $R(3,3,3)=17$ (thus, consisting of 16 vertices), colored in red, blue and green, that does not contain a monochromatic K_3.

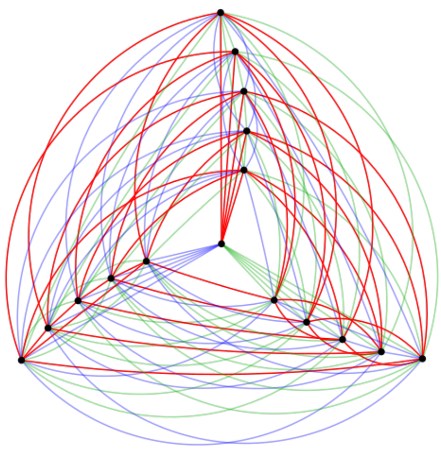


Figure 3.5. *A $R(3,3,3)$ Ramsey graph. [By User:Maproom - User:Maproom, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=52243290]*

3.2 Bounds on Ramsey numbers

Even though Ramsey theory has been around for almost a hundred years [22] the developments in determining the exact values of Ramsey numbers are fairly slow. This is because of a high computational complexity of such a task [14]. Even though there are much better algorithms for searching through large collections of graphs than simple "brute force", it is still unattainable with modern hardware. In the table 3.6 all $R(k,l)$ Ramsey numbers known to date are presented [24].

k	l	$R(k,l)$
3	3	6
3	4	9
3	5	14
3	6	18
3	7	23
3	8	28
3	9	36
4	4	18
4	5	25

Figure 3.6. *All known 2-color Ramsey numbers.*

As was mentioned earlier, it is famously a very complex problem to determine Ramsey numbers for greater values of k and l. For most of Ramsey numbers we do not know the exact value, but only the lower and upper bounds.

Using inequality 3.4, it is straightforward to prove that the following upper bound is true for any $R(k,l)$ with $k,l \geq 1$ [3]:

Theorem 3.9.

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

Proof. We prove Theorem 3.1 using induction.

The base case with $k = l = 1$ holds since $R(1,1) \leq \binom{1+1-2}{1-1} = \binom{0}{0} = 1$, which can also be seen from the trivial identities (1).

Now, assume that the inequality holds for some pair $k_1, l_1 \in \mathbb{N}$ such that $k_1+l_1 = k+l+1$. Then, using Lemma 3.4, we obtain

$$
\begin{aligned}
R(k_1, l_1) \leq R(k_1 - 1, l_1) + R(k_1, l_1 - 1) &\leq \binom{(k_1 - 1) + l_1 - 2}{(k_1 - 1) - 1} + \binom{k_1 + (l_1 - 1) - 2}{k_1 - 1} \\
&\leq \binom{k_1 + l_1 - 3}{k_1 - 2} + \binom{k_1 + l_1 - 3}{k_1 - 1} \\
&= \frac{(k_1 + l_1 - 3)!}{(k_1 - 2)!(l_1 - 1)!} + \frac{(k_1 + l_1 - 3)!}{(k_1 - 1)!(l_1 - 2)!} \\
&= \frac{(k_1 + l_1 - 3)!(k_1 + l_1 - 2)}{(k_1 - 1)!(l_1 - 1)!} \\
&= \frac{(k_1 + l_1 - 2)!}{(k_1 - 1)!(l_1 - 1)!} \\
&= \binom{k_1 + l_1 - 2}{k_1 - 1}.
\end{aligned}
$$

□

Although this upper bound is valid, it is comparatively weak. Better upper bounds can be determined for specific types of Ramsey numbers, such as diagonal Ramsey numbers, Ramsey numbers of the form $R(3, k)$, and so on. Next, we will briefly introduce the upper bounds for diagonal Ramsey numbers.

In the case of diagonal Ramsey numbers $R(k, k)$, the inequality in Theorem 3.1 can be simplified as follows:

$$
R(k, k) \leq \binom{k + k - 2}{k - 1} = \binom{2k - 2}{k - 1} = \frac{(2(k - 1))!}{((k - 1)!)^2},
$$

which, by Stirling's approximation [25] (outside the scope of this thesis) reduces inequality 3.1 to

$$
R(k, k) \leq \frac{4^k}{\sqrt{k}}. \tag{2}
$$

Arguably, the most well-known non-trivial upper bound for diagonal Ramsey numbers was proven by Erdős and Szekeres [13], and it states that

$$
R(k, k) \leq 4^k. \tag{3}
$$

A number of improvements was made to this formula by different mathematicians [5], lowering the limit by multiplying it by specific factors, as, for example, in formula 2. We will return to this in the end of this section.

As for the lower bounds, it was shown by Erdős [9] that the general formula for a lower bound on large diagonal Ramsey numbers can be derived using probabilistic method [18]. We present the said formula here without a proof:

$$R(r) > \frac{r2^{\frac{r}{2}}}{e\sqrt{2}}.$$

Bounds on Ramsey numbers are known to grow exponentially [5], and the higher they get, the wider the gaps are for the known lower and upper bounds. For instance, $R(5,5)$ is bounded by 43 and 48, $R(6,8)$ – by 134 and 495, and $R(9,10)$ lies between 581 and 12677.

In 2023, researchers from University of Rio de Janeiro and Cambridge made a significant breakthrough in defining the upper bounds for diagonal Ramsey numbers [4]. The famously known 4^k bound (3) was improved to $(4-\epsilon)^k$, in other words

$$R(k,k) \leq (4-\epsilon)^k \text{ for some constant } \epsilon > 0.$$

The values of ϵ the researchers used were 2^{-10} and 2^{-7}, and besides, they claim that those numbers can likely be optimized.

The method that Morris, Campos, Griffiths and Sahasrabudhe applied uses the concept of so-called "books". It is a way of partitioning the vertices of a complete 2-colored graph. In simple words, following the analogy of a real book, the spine of the book is a monochromatic clique, and the pages (vertices) of the book are connected to the spine by an edge of the same color as the clique. Roughly, a book can be visualized as the Figure 3.7 shows:

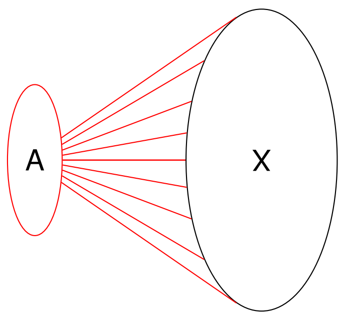


Figure 3.7. *A red "book", where A is a red clique and X contains the vertices connected to A with a red edge.*

The idea of the algorithm is to find multiple vertices forming a red clique (the spine of the book) in the beginning, so that the search for monochromatic cliques of either color is easier. After the book is found, the next step is to look for a clique in the pages of the book. While performing the algorithm, there is a danger of skipping too many vertices that could potentially be suitable for the growing clique. To avoid that, the group came up a method called “density-boost steps”, which helps to choose vertices more carefully in order to form the clique. [17]

The paper had a huge success in the community since it was the first major improvement for the upper bound on Ramsey numbers since 1935 [10]. The next step now is to optimize the value of ϵ, which, according to the authors of the paper, is rather straightforward and technical.

3.3 Induced Ramsey theory

Up until this point we have only been looking for cliques, or complete induced subgraphs, in larger graphs. Ramsey theory, however, does not stop there. The next question we ask is whether for any *arbitrary* graph H we can guarantee to find such a graph G that contains a monochromatic copy of H in any of its 2-colorings. Turns out, we can, and it is even true for two arbitrary graphs H_1 and H_2, i.e., one of them will always be induced by the red edges of G or the other one by its blue edges.

Theorem 3.10. *(Induced Ramsey's theorem). For any two graphs H_1, H_2 there exists a graph $G(H_1, H_2)$ such that every red-blue coloring of G yields either an induced $H_1 \in G$ with all its edges colored red or an induced $H_2 \in G$ with all its edges colored blue. [6]*

The minimum number of vertices in G is called the *induced Ramsey number* $R_{ind}(H_1, H_2)$.

Two extensive proofs of Theorem 3.10 are presented in Diestel's book "Graph theory" [6]. They are based on the idea of manually replacing vertices of (the already constructed) G with copies of H.

A non-trivial upper bound for induced Ramsey numbers was proven by Kohayakawa, Prömel, and Rödl in 1998, and it can be formulated in the following way: if H_1 is a graph on k vertices and H_2 is a graph on $t \geq k$ vertices, then the upper bound for the the induced Ramsey number $R_{ind}(H_1, H_2)$ is given by:

$$R_{ind}(H_1, H_2) \leq t^{ck \cdot log(\chi)}.$$

where χ is the edge chromatic number of H_2 (minimum number of colors needed for its proper edge coloring 2.10) and c is some absolute constant. [16]

Let us look at an example where $H_1 = H_2 = P_4$ (P_n is a so-called *path* on n vertices, which is a sequence of vertices, where each vertex is adjacent to the next one in the sequence [3]). It was shown by Harary, Nešetril and Rödl that $R_{ind}(P_4) \leq 8$ [15]. One of such graphs G on 8 vertices in shown in Figure 3.8.

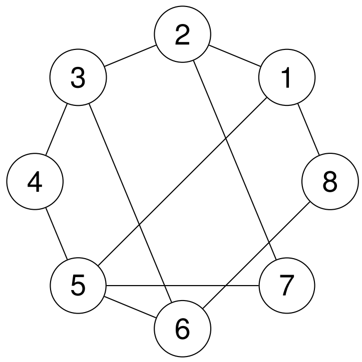
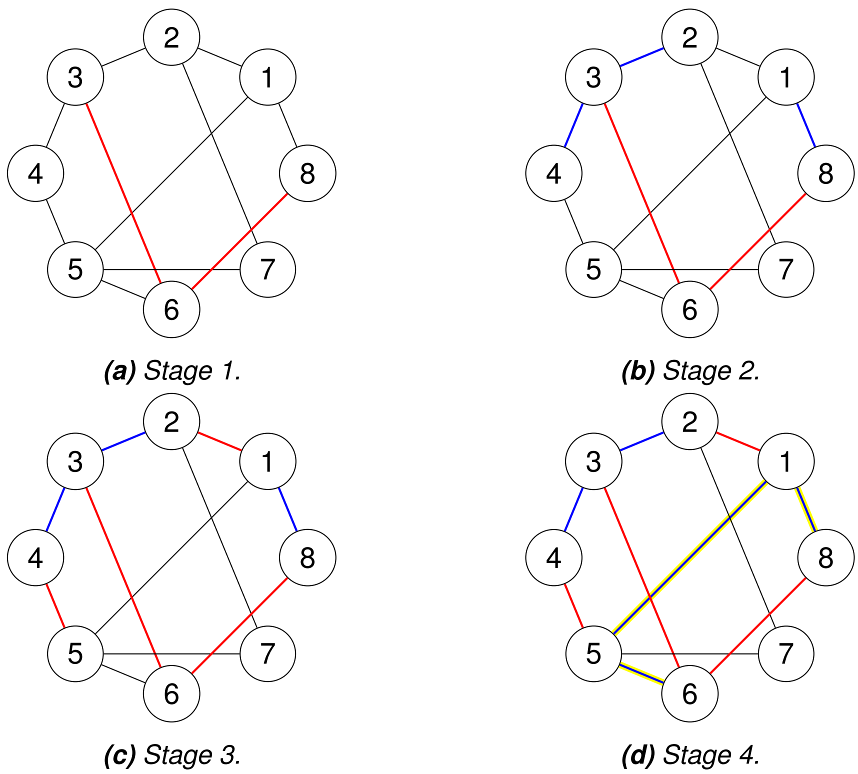


Figure 3.8. *Graph G containing a P_4 in every 2-coloring.*

The straightforward way to show that there will always be a monochromatic 4-path in G is by applying the pigeonhole principle (2.17) the same way as in Example 2.18, starting with any vertex that has 3 or more incident edges. Let us show that by starting with vertex 6. By pigeonhole principle, at least 2 of its 3 incident edges will have the same color - say, red.

We look at all 3 scenarios of that, starting by coloring edges $(6, 3)$ and $(6, 8)$ red (stage 1) in Figure 3.9. Next, in order to avoid a red P_4, we must color edges $(3, 4)$, $(1, 8)$, and $(2, 3)$ blue (stage 2). Now, at least edges $(4, 5)$ and $(1, 2)$ have to be red so there is no blue 4-path (stage 3). For the last step, we have to color $(1, 5)$ and $(5, 6)$ blue in order to avoid two potential red paths (stage 4), and that inevitably leads to a blue P_4 consisting of vertices 8, 1, 5, and 6, and at this point we are done.

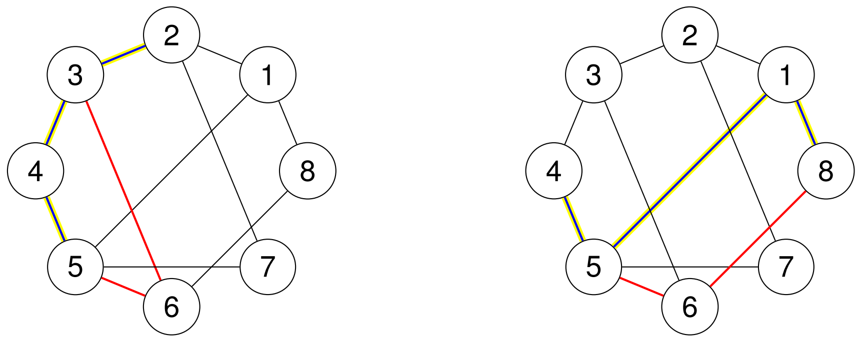


(a) *Stage 1.* ***(b)*** *Stage 2.*

(c) *Stage 3.* ***(d)*** *Stage 4.*

Figure 3.9. *Red $(6, 3)$ and $(6, 8)$.*

The other 2 cases are shown in Figure 3.10. When edges $(6,5)$ and $(6,3)$ are red, edges $(2,3)$, $(3,4)$, and $(4,5)$ have to be blue, thus forming a blue P_4. Finally, when edges $(6,5)$ and $(6,8)$ are red, edges $(4,5)$, $(5,1)$, and $(1,8)$ are blue and form a P_4 as well.



(a) *Red* $(6,5)$ *and* $(6,3)$.   ***(b)*** *Red* $(6,5)$ *and* $(6,8)$.

Figure 3.10

Starting with 2 blue edges instead of red would yield the same result due to symmetry. Thus, this is enough to show that in such a graph G, there will always be an induced monochromatic path on 4 vertices.

4. SUMMARY

The goal of this thesis was to introduce the reader to Ramsey theory and Ramsey numbers. The fundamentals of graph theory were given first, followed by the main chapter about the most important concepts in Ramsey theory. In that chapter, the proof of Ramsey's theorem was presented, as well as the proof of the upper bound for Ramsey numbers. Besides, concepts such as Ramsey graph, multicolor Ramsey number, and other (better) bounds for Ramsey numbers were shown. Additionally, a recent breakthrough in the search of Ramsey numbers was mentioned. To conclude the thesis and acquaint the reader with an extension of Ramsey theory, a brief overview of induced Ramsey theory was introduced in the end.

REFERENCES

[1] Alexanderson, G. Euler and Königsberg's bridges: a historical view. *Bulletin of the American Mathematical Society* 43.4 (2006).

[2] Beck, J. Ramsey games. *Discrete Mathematics* 249.1 (2002).

[3] Bondy, J. A. and Murty, U. S. R. *Graph Theory*. Springer, 2008.

[4] Campos, M., Griffiths, S., Morris, R. and Sahasrabudhe, J. An exponential improvement for diagonal Ramsey. *arXiv preprint* (2023). `https://arxiv.org/abs/2303.09521`(visited 2024-06-02).

[5] Conlon, D., Fox, J. and Sudakov, B. Recent developments in graph Ramsey theory. *Surveys in Combinatorics* 424 (2015).

[6] Diestel, R. *Graph Theory*. Springer, 2016.

[7] Do, N. Party problems and Ramsey theory. *Australian Mathematical Society Gazette* 33.5 (2006).

[8] Doerr, A. and Levasseur, K. *Applied Discrete Structures*. `https://discretemath.org/ads/index-ads.html`(visited 2024-10-01). 2023.

[9] Erdős, P. Some remarks on the theory of graphs. *Bulletin of the American Mathematical Society* 53 (1947).

[10] Erdős, P. and Szekeres, G. A Combinatorial Problem in Geometry. *Compositio Mathematica* 2 (1935).

[11] Gopalan, P. Constructing Ramsey graphs from Boolean function representations. *21st Annual IEEE Conference on Computational Complexity (CCC'06)*. 2006.

[12] Gottlieb, A. The Man Who Thought Too Fast. *The New Yorker* (2020). Accessed: 09.04.2024.

[13] Graham, R. L., Rothschild, B. L. and Spencer, J. H. *Ramsey theory*. Vol. 2. John Wiley & Sons, 1990.

[14] Haanpää, H. Computational Methods For Ramsey Numbers. *Helsinki University of Technology Laboratory for Theoretical Computer Science, Research Reports 65* (2000).

[15] Harary, F., Nešetril, J. and Rödl, V. Generalized Ramsey theory for graphs, XIV, Induced Ramsey numbers. *Graphs and other Combinatorial Topics (Prague, 1982)* (1982).

[16] Kohayakawa, Y., Prömel, H.-J. and Rödl, V. Induced Ramsey Numbers. *Combinatorica* 18 (1998).

[17] Matemática Pura e Aplicada, I. de. *(22/03/2023) - Doutorado: Ramsey Theory and Random Graphs - Rob Morris - Aula 01*. 2023. URL: https://www.youtube.com/watch?v=GA295heuHoY&list=PLo4jXE-LdDTTUDYMaYoWDOZ3ltOQ_XIoc.

[18] Matoušek, J. and Vondrák, J. *The Probabilistic Method*. Lecture notes. 2008.

[19] McKay, B. *Ramsey Graphs*. Accessed: 06.03.2024. URL: https://users.cecs.anu.edu.au/~bdm/data/ramsey.html.

[20] Naor, M. Constructing Ramsey graphs from small probability spaces. *IBM Research Report RJ 8810 (70940)* (1992).

[21] Radziszowski, S. P. Small Ramsey Numbers. *The Electronic Journal of Combinatorics* 1000 (2021). URL: http://www.combinatorics.org.

[22] Ramsey, F. P. On a Problem of Formal Logic. *Proceedings of the London Mathematical Society* s2-30.1 (1930).

[23] Rosta, V. Ramsey theory applications. *The Electronic Journal of Combinatorics [electronic only]* DS13 (2004).

[24] Weisstein, E. W. *Ramsey Number*. Accessed: 06.02.2024. URL: https://mathworld.wolfram.com/RamseyNumber.html.

[25] Weisstein, E. W. *Stirling's Approximation*. Accessed: 13.03.2024. URL: https://mathworld.wolfram.com/StirlingsApproximation.html.