

Generalizing Level Ranking Constraints for Monotone and Convex Aggregates

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In answer set programming (ASP), answer sets capture solutions to search problems of interest and thus the efficient computation of answer sets is of utmost importance. One viable implementation strategy is provided by translation-based ASP where logic programs are translated into other KR formalisms such as Boolean satisfiability (SAT), SAT modulo theories (SMT), and mixed-integer programming (MIP). Consequently, existing solvers can be harnessed for the computation of answer sets. Many of the existing translations rely on program completion and level rankings to capture the minimality of answer sets and default negation properly. In this work, we take level ranking constraints into reconsideration, aiming at their generalizations to cover aggregate-based extensions of ASP in more systematic way. By applying a number of program transformations, ranking constraints can be rewritten in a general form that preserves the structure of monotone and convex aggregates and thus offers a uniform basis for their incorporation into translation-based ASP. The results open up new possibilities for the implementation of translators and solver pipelines in practice.

1 Introduction

Answer set programming (ASP, [8]) offers rich rule-based languages for knowledge representation (KR) and reasoning. Given some search or optimization problem of interest, its *encoding* in ASP is a logic program whose answer sets capture solutions to the problem. Thus the efficient computation of answer sets is of utmost importance. One viable implementation strategy is provided by *translation-based* ASP where logic programs are translated into other KR formalisms such as Boolean satisfiability (SAT, [5]), SAT modulo theories (SMT, [4]), or mixed-integer programming (MIP, [32]). Consequently, existing solver technology can be harnessed for the computation of answer sets.

The semantics of answer set programming rests on *stable models* [14] that incorporate a notion of minimality and give a declarative semantics for default negation. Capturing these aspects in satisfaction-based formalisms such as pure SAT is non-trivial; see, e.g., [17, 22]. There are also various syntactic aggregations [2] that enable compact encodings but whose translation is potentially expensive if there is no respective primitive in the target formalism. A typical translation consists of several steps such as (i) *normalization* [7], (ii) *instrumentation* for loop prevention [6, 17, 23], and (iii) *completion* [10]. The first step concerns the removal of syntactic extensions that have been introduced to increase the expressive power of ASP in favor of *normal* rules. The second step either transforms the program or adds suitable constraints so that the difference between stable and *supported models* disappears. The third step captures supported models by transforming rules into equivalences. Ideally, the syntactic details of the target language are deferred during translation and incorporated at the very end; either after or while forming the completion. This strategy realizes a *cross-translation* approach [19] in analogy to modern compiler designs.

Many of the existing translations [20, 25, 26] essentially rely on *level ranking constraints* formalized by Niemelä [27] as formulas in difference logic [28]. Such constraints describe *level numbers* that order the atoms of a normal program in such a way that stable models can be distinguished among the supported ones [11]. Thus, level numbers are essential when it comes to capturing the minimality of stable models and the semantics of default negation properly. As shown in [17], level numbers can be made unique so that they match with the levels of atoms obtained by applying the *immediately true* operator \mathbf{T}_P iteratively. Uniqueness can also be enforced in terms of *strong* level ranking constraints [27]. Unique level numbers are also highly desirable when aiming at one-to-one correspondences with stable models, e.g., when counting solutions to problems or carrying out probabilistic inference [12].

In this work, we take level ranking constraints into reconsideration, aiming at generalizations that cover aggregate-based extensions of ASP in a more systematic way. So far, only normal programs are truly covered [17, 27] and the normalization of input programs is presumed. The generalization for weight constraint programs (WCPs), as sketched by Liu et al. [25], concerns only weak constraints and is confined to translations into MIP. However, the idea of avoiding or delaying normalization is interesting as such, opening up new possibilities for ordering the translation steps discussed above. For instance, if $\text{NORM}(\cdot)$ and $\text{LRC}(\cdot)$ stand for translations based on normalization and level ranking constraints, respectively, it would be highly interesting to compare $\text{LRC}(\text{NORM}(P))$ with potential generalizations $\text{LRC}(P)$ that express level ranking constraints at an aggregated level. Such representations are expected to be more compact and to favor level rankings with fewer variables. The resulting formulas can also be *Booleanized* afterwards [16], if translations toward SAT are desirable, or rewritten in some other form that complies with the intended back-end formalism. In the sequel, we use translations $\text{LRC}(\text{NORM}(P))$ of increasingly complex programs P as guidelines when lifting level ranking constraints for aggregates. The idea is to cover program classes involving standard aggregations subject to recursion. It turns out that the structure of monotone and convex aggregates can be preserved to a high degree, offering a uniform basis for their incorporation into translation-based ASP. On the one hand, the resulting generalizations exploit *ordered completion* [3] in the reformulation of *weak* level ranking constraints but, on the other hand, make novel contribution when imposing the uniqueness of level rankings with *strong* ones.

The rest of this article is structured as follows. We begin by recalling the basic notions of *logic programs* in Section 2, including the usual syntactic fragments, stable and supported model semantics, and other concepts relevant for this work. Then, in Section 3, we explain the details of ranking constraints in their standard form corresponding to *normal* logic programs. Actually, we present them in a slightly rewritten form in order to pave the way for their generalization for monotone aggregates in Section 4. Therein, we begin the analysis from the case of (positive) cardinality and weight rules and, eventually, incorporate negative conditions to ranking constraints. To illustrate the generality of the results even further, we investigate how certain convex aggregates get also covered via appropriate program transformations in Section 5. Finally, the conclusions of this work are presented in Section 6.

2 Preliminaries

In the sequel, we will consider *propositional logic programs* that are finite sets rules of the forms (1)–(5) below. In the rules, a , a_i 's, b_j 's, and c_k 's are *propositional atoms* (or *atoms* for short) and \sim denotes *default negation*. The rules in (1)–(5) are known as *normal*, *choice*, *cardinality*, *weight*, and *disjunctive* rules, respectively. Each rule has a *head* and a *body* separated by the \leftarrow sign, and the rough intuition is that if the *condition(s)* formed by the rule body are satisfied, then the respective head atom a in (1)–(4), or some of the head atoms a_1, \dots, a_h in (5) can be derived.

$$a \leftarrow b_1, \dots, b_n, \sim c_1, \dots, \sim c_m. \quad (1)$$

$$\{a\} \leftarrow b_1, \dots, b_n, \sim c_1, \dots, \sim c_m. \quad (2)$$

$$a \leftarrow l \leq \{b_1, \dots, b_n, \sim c_1, \dots, \sim c_m\}. \quad (3)$$

$$a \leftarrow w \leq \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}, \sim c_1 = w_{c_1}, \dots, \sim c_m = w_{c_m}\}. \quad (4)$$

$$a_1 \mid \dots \mid a_h \leftarrow b_1, \dots, b_n, \sim c_1, \dots, \sim c_m. \quad (5)$$

The choice regarding the head of (2) is optional while (5) insists on deriving at least one head atom a_i in a minimal way, as to be detailed in Definition 1. A positive body condition b_j holds if b_j can be derived by some other rules whereas $\sim c_k$ holds if c_k cannot be derived. A cardinality rule (3) demands that at least l of such conditions are met to activate the rule. Weight rules (4) are similar but body conditions b_j and $\sim c_k$ are valued by their respective non-negative integer weights w_{b_j} and w_{c_k} when it comes to reaching the bound w . In the sequel, we use shorthands $B^+(r) = \{b_1, \dots, b_n\}$, $B^-(r) = \{c_1, \dots, c_m\}$, and $B(r) = \{b_1, \dots, b_n\} \cup \{\sim c_1, \dots, \sim c_m\}$ when referring to the body conditions occurring in a rule r . The set of head atoms in r is denoted by $H(r)$ and for entire programs P , we define $H(P) = \bigcup_{r \in P} H(r)$.

Typical (syntactic) classes of logic programs are as follows: *normal* logic programs (NLPs) consist of normal rules (1) and the same can be stated about *disjunctive* logic programs (DLPs) and disjunctive rules (5) that are normal as a special case ($h = 1$). The class of *weight constraint programs* (WCPs) [30] is essentially based on normal rules (1) and the aggregated rule types in (2)–(4), out of which weight rules are expressive enough to represent the class of WCPs alone. Contemporary ASP systems—aligned with the ASP-core-2 language standard [9]—support these fragments so well that programmers can mix rule types freely in their encodings. When the fragment is not important, we may refer to *logic programs* or *programs* for short. Finally, we say that a rule is *positive*¹ if $m = 0$ and it is of the forms (1), or (3)–(5). An entire program is called positive if its rules are all positive.

The *signature* $\text{At}(P)$ of a logic program P is the set of atoms that occur in the rules of P . An *interpretation* $I \subseteq \text{At}(P)$ of P tells which atoms $a \in \text{At}(P)$ are *true* ($a \in I$, also denoted $I \models a$) whereas others are *false* ($a \in \text{At}(P) \setminus I$, denoted $I \not\models a$). Atoms are also called *positive literals*. Any *negative literal* $\sim c$, where c is an atom, is understood classically, i.e., $I \models \sim c$, iff $I \not\models c$. The relation \models extends for the bodies of normal/choice/disjunctive rules r as follows: $I \models B(r)$ iff $B^+(r) \subseteq I$ and $B^-(r) \cap I = \emptyset$. The body $l \leq B(r)$ of a cardinality rule r is satisfied in I iff $l \leq |B^+(r) \cap I| + |B^-(r) \setminus I|$. More generally, the body of a weight rule r in (4) is satisfied in I iff the *weight sum* $\text{WS}_I(b_1 = w_{b_1}, \dots, b_n = w_{b_n}, \sim c_1 = w_{c_1}, \dots, \sim c_m = w_{c_m}) = \sum_{b \in B^+(r) \cap I} w_b + \sum_{c \in B^-(r) \setminus I} w_c$ is at least w . For rules r , we have $I \models r$ iff the satisfaction of its body implies the satisfaction of its head, except that choice rules (2) are always satisfied. An interpretation $I \subseteq \text{At}(P)$ is a (*classical*) *model* of a program P , denoted $I \models P$, iff $I \models r$ for each $r \in P$. A model $M \models P$ is \subseteq -*minimal* iff there is no $M' \models P$ such that $M' \subset M$. The set of \subseteq -minimal models of P is denoted by $\text{MM}(P)$. If P is positive and non-disjunctive then $|\text{MM}(P)| = 1$ and the respective *least model* of P is denoted by $\text{LM}(P)$.

Definition 1 (Stable models [14, 15, 30]). *For a program P and an interpretation $I \subseteq \text{At}(P)$, the reduct P^I of P with respect to I contains*

1. *a rule $a \leftarrow B^+(r)$ for each normal rule (1) such that $B^-(r) \cap I = \emptyset$, and for each choice rule (2) such that $a \in I$ and $B^-(r) \cap I = \emptyset$;*
2. *a rule $a \leftarrow l' \leq B^+(r)$ for each cardinality rule (3) and the bound $l' = \max(0, l - |B^-(r) \setminus I|)$;*

¹Note that the head $\{a\}$ of choice rule embeds hidden (double) negation since it can be expressed as $a \leftarrow \sim \sim a$.

3. a rule $a \leftarrow w' \leq \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}$ for each weight rule (4) and the bound $w' = \max(0, w - \text{WS}_I(\sim c_1 = w_{c_1}, \dots, \sim c_m = w_{c_m}))$; and
4. a rule $a_1 \mid \dots \mid a_n \leftarrow \mathbf{B}^+(r)$ for each disjunctive rule (5) such that $\mathbf{B}^-(r) \cap I = \emptyset$.

An interpretation $M \subseteq \text{At}(P)$ is a stable model of the program P iff $M \in \text{MM}(P^M)$.

Example 1. Consider a cardinality rule $a \leftarrow 1 \leq \{b_1, \dots, b_n\}$ with choice rules $\{b_1\}, \dots, \{b_n\}$. Besides the empty stable model \emptyset , these rules induce $2^n - 1$ stable models $M = \{a\} \cup N$ with $\emptyset \subset N \subseteq \{b_1, \dots, b_n\}$: the head a is set true whenever at least one of b_1, \dots, b_n is chosen to be true. ■

In the sequel, we mostly concentrate on non-disjunctive programs P . Then, the stability of $M \subseteq \text{At}(P)$ can also be captured with the fixed point equation $M = \text{LM}(P^M)$. Moreover, the well-known \mathbf{T}_P operator, when applied to an interpretation $I \subseteq \text{At}(P)$, produces the set of atoms $a \in \text{At}(P)$ that are *immediately true* under I , i.e., for which there is a positive rule r having a as the head and whose body is satisfied by I . It follows that $M \models P$ holds for a non-disjunctive program P iff $\mathbf{T}_{P^M}(M) \subseteq M$ and M is a *supported model* of P iff $M = \mathbf{T}_{P^M}(M)$. Given a supported model M , the support is provided by the set of rules $\text{SuppR}(P, M) \subseteq P$ whose bodies are satisfied by M . Since $\text{LM}(P^M)$ is obtained as the least fixed point $\mathbf{T}_{P^M} \uparrow^\infty (\emptyset)$, each stable model of P is also supported. We write $\text{SM}(P)$ and $\text{SuppM}(P)$ for the sets of stable and supported models of P , respectively. Thus $\text{SM}(P) \subseteq \text{SuppM}(P)$ holds in general.

Next we recall some concepts related to modularity. First, given a WCP P , the set of *defining rules* for an atom $a \in \text{H}(P)$ is $\text{Def}_P(a) = \{r \in P \mid a \in \text{H}(r)\}$. Thus P can be partitioned as $\bigcup_{a \in \text{H}(P)} \text{Def}_P(a)$. Second, the *positive dependency graph* of P is $\text{DG}^+(P) = \langle \text{At}(P), \succeq_P \rangle$ where $a \succeq_P b$ holds for $a, b \in \text{At}(P)$, if $a \in \text{H}(r)$ and $b \in \mathbf{B}^+(r)$ for some rule $r \in \text{Def}_P(a)$. A *strongly connected component* (SCC) of $\text{DG}^+(P)$ is a maximal subset $S \subseteq \text{At}(P)$ such that all distinct atoms $a, b \in S$ depend on each other via directed paths in $\text{DG}^+(P)$. For an atom $a \in \text{H}(P)$, the SCC of a is denoted by $\text{SCC}(a)$. As shown in [29], each SCC S of a WCP P gives rise to a *program module* $P_S = \bigcup_{a \in S} \text{Def}_P(a)$ where pure body atoms $b \in \text{At}(P_S) \setminus S$ are treated as *input atoms* taking any truth value, intuitively defined by choice rules $\{b\}$. This yields a set of stable models $\text{SM}(P_S)$ for each module P_S based on Definition 1. Given two stable models $M \in \text{SM}(P)$ and $N \in \text{SM}(Q)$, we say that M and N are mutually *compatible*, if they agree on the truth values of atoms in $\text{At}(P) \cap \text{At}(Q)$, i.e., $M \cap \text{At}(Q) = N \cap \text{At}(P)$. The *module theorem* of [29] states that the stable models of P can be obtained as mutually compatible collections of stable models M_1, \dots, M_n for the program modules P_{S_1}, \dots, P_{S_n} induced by the SCCs S_1, \dots, S_n of P .

Finally, some notions of equivalence should be introduced. Logic programs P and Q are *weakly equivalent*, denoted $P \equiv Q$, iff $\text{SM}(P) = \text{SM}(Q)$. They are *strongly equivalent*, denoted $P \equiv_s Q$, iff $P \cup R \equiv Q \cup R$ for any other context program R [21]. Then $P \equiv_s Q$ implies $P \equiv Q$ but not vice versa. Strong equivalence can be characterized by using only contexts formed by *unary* positive rules $a \leftarrow b$, or semantically by using SE-models [31]. To address the correctness of various translations, however, more fine-grained relations become necessary. The signature $\text{At}(P)$ of a logic program P can be split into *visible* and *hidden* parts $\text{At}_v(P)$ and $\text{At}_h(P)$, respectively. Given a stable model $M \in \text{SM}(P)$, only its visible projection $M \cap \text{At}_v(P)$ is relevant when comparing P with other programs. Thus, P and Q are *visibly equivalent*, denoted $P \equiv_v Q$, iff $\text{At}_v(P) = \text{At}_v(Q)$ and $M \cap \text{At}_v(P) = N \cap \text{At}_v(Q)$ holds for each pair of models $M \in \text{SM}(P)$ and $N \in \text{SM}(Q)$ in a bijective correspondence [17]. There is a generalization of both \equiv_v and \equiv_s , viz. *visible strong equivalence* \equiv_{vs} , that incorporates context programs R that *respect the hidden atoms* of P and Q for comparisons [7]. The correctness of normalization has been addressed in this sense. E.g., for a weight rule r in (4), $\{r\} \equiv_{vs} \text{NORM}(\{r\})$, which means that r can be safely substituted by $\text{NORM}(\{r\})$ in contexts respecting the hidden atoms introduced by the normalization.

3 Level Rankings and Ranking Constraints

When a stable model $M \subseteq \text{At}(P)$ of a *non-disjunctive* logic program P is constructed using the reduct P^M and the \mathbf{T}_{P^M} operator, atoms true in the model M get divided into *levels* $M_i = (\mathbf{T}_{P^M} \uparrow^i(I)) \setminus (\mathbf{T}_{P^M} \uparrow^{i-1}(I))$ where $i > 0$ and $I \subseteq \text{At}(P) \setminus \text{H}(P)$ is a set of input atoms. By default $I = \emptyset$ and $M_0 = \emptyset$, but if $I \neq \emptyset$, then $M_0 = I$. For finite programs P , the index i is bounded from above by $|\text{At}(P)|$. Based on this division of atoms, it is possible to read off a *level ranking* $\# : \text{At}(P) \rightarrow \mathbb{N} \cup \{\infty\}$ for the atoms of the program [27]: the rank $\#a = i$, if $a \in M_i$, and $\#a = \infty$, if $a \notin M$. A *level numbering* $\#$ [17] extends any level ranking for the supporting rules $r \in \text{SuppR}(P, M)$ by the equality² $\#r = \max\{\#b \mid b \in \text{B}^+(r)\} + 1$. Intuitively, the level $\#r$ of a rule r indicates when r can be applied to derive its head and, consequently, $\#a = \min\{\#r \mid r \in \text{Def}_P(a) \cap \text{SuppR}(P, M)\}$. By these interconnections, we may use level rankings and numberings interchangeably in the sequel. If $r \notin \text{SuppR}(P, M)$, then $\#r = \infty$. The value ∞ emphasizes that an atom is never derived or a rule becomes never applicable. The other option is to restrict the domain of $\#$ to $M \cup \text{SuppR}(P, M)$ for which finite values exist, but some big value greater than any level rank is useful in practice. E.g., given an SCC S of the program P , the level ranks $\#a$ of atoms $a \in S$ can be effectively constrained by $0 < \#a < |S| + 1$; cf. (6) below.

Many existing translations of logic programs into SAT, SMT, and MIP rely on program completion [10]. The idea is to translate a (normal) logic program P into classical equivalences that capture the *supported models* of the program. The purpose of level ranking constraints [27], however, is to distinguish the stable ones among them by incorporating a requirement that there is a level ranking $\#$ for a model $M \in \text{SuppM}(P)$. These constraints can be expressed, e.g., as formulas in *difference logic* (DL). This SMT-style logic [28] enriches propositional formulas with difference constraints of the form $x - y \leq k$ where x, y are real/integer variables and k is a constant. The evaluation of a difference atom $x - y \leq k$ is based on an assignment $\tau : \mathcal{V} \rightarrow \mathbb{Z}$ on the set of variables \mathcal{V} in use. Given τ , the constraint $x - y \leq k$ is satisfied by τ , denoted by $\tau \models x - y \leq k$, iff $\tau(x) - \tau(y) \leq k$. A *DL-interpretation* is a pair $\langle I, \tau \rangle$ where I a standard propositional interpretation and τ an assignment. A formula ϕ of DL is satisfied by $\langle I, \tau \rangle$, denoted $\langle I, \tau \rangle \models \phi$, if ϕ evaluates to true under I by the usual propositional rules extended by the evaluation of difference atoms subject to τ .

A difference constraint $x_b - x_a \leq -1$ (i.e., $x_a > x_b$) can express that a is derived *after* b under the assumption that x_a and x_b store the level ranks of a and b , respectively. Based on this idea, we introduce formulas for the representation of level ranks. Their *scope* is specified in terms of a set of atoms $S \subseteq \text{At}(P)$ to be discussed in further detail below.

$$(1 \leq x_a \leq |S| + 1), \quad \neg a \rightarrow (x_a \geq |S| + 1), \quad (6)$$

$$\text{dep}(a, b) \leftrightarrow b \wedge (x_a > x_b), \quad (7)$$

$$\text{gap}(a, b) \leftrightarrow b \wedge (x_a > x_b + 1). \quad (8)$$

By the two formulas in (6), level ranks are positive and fixed to $|S| + 1$ if an atom a is false. In addition, we introduce two kinds of new atoms to help with the formulation of the actual level ranking constraints. First, the atom $\text{dep}(a, b)$ defined by (7), denotes an *active* dependency of a head atom a on a positive body atom b , i.e., b must be true. Such dependencies are deployed by Bomanson et al. [6], but we use a definition in terms of the difference constraint. Second, the atom $\text{gap}(a, b)$, as defined by (8), means a similar relationship except that b is derived so early that it is not critical for determining the exact level rank of a . Note that $\text{gap}(a, b)$ implies $\text{dep}(a, b)$ in general but not vice versa. In particular, if $\text{dep}(a, b)$ is true and $\text{gap}(a, b)$ is false, then b must be true and $x_a = x_b + 1$, indicating that a is derived right after b .

²This holds for rules whose body is essentially normal (1) while generalizations for more complex bodies follow.

Such body atoms b from the preceding level are relevant when a is derived by some rule $r \in \text{Def}_P(a)$ at level x_a .

In the following, we present a reformulation of level ranking constraints [27] by exploiting the dependency relations from (7) and (8). Our further goal is to incorporate the idea of *ordered completion* [3] for the sake of more compact representation. Given an atom $a \in \text{At}(P)$, its completion is based on the set $\text{Def}_P(a) = \{r_1, \dots, r_k\}$ of its defining rules. In the sequel, the *applicability* of a rule r_i is denoted by a new atom $\text{app}(r_i)$.

$$a \leftrightarrow \text{app}(r_1) \vee \dots \vee \text{app}(r_k), \quad (9)$$

$$\text{app}(r_i) \leftrightarrow \bigwedge_{b \in \mathbf{B}^+(r_i) \cap S} \text{dep}(a, b) \wedge (\mathbf{B}^+(r_i) \setminus S) \wedge \neg \mathbf{B}^-(r_i) \quad (1 \leq i \leq k), \quad (10)$$

$$\text{app}(r_i) \rightarrow \bigvee_{b \in \mathbf{B}^+(r_i) \cap S} \neg \text{gap}(a, b) \quad (1 \leq i \leq k, \mathbf{B}^+(r_i) \cap S \neq \emptyset), \quad (11)$$

$$\text{app}(r_i) \rightarrow (x_a \leq 1) \quad (1 \leq i \leq k, \mathbf{B}^+(r_i) \cap S = \emptyset). \quad (12)$$

Intuitively, the equivalence (9) sets the head atom a true if and only if at least one of its defining rules r_i is applicable. This, in turn, is defined by the equivalence (10) insisting that atoms in $\mathbf{B}^+(r_i) \cap S$ have been previously derived and all remaining positive and negative body conditions are satisfied. This formulation embeds both *weak* level ranking constraints [27] and ordered completion [3] but relative to the set S . The constraint (11) is the counterpart of *strong* level ranking constraints [27] enforcing the minimality of level ranks assigned to atoms. Besides this, the formula (12) resets the level of the head atom a to 1 when a can be derived by applying an *externally supporting* rule r_i with $\mathbf{B}^+(r_i) \cap S = \emptyset$.

Regarding the scope S , it is natural to think that the head atom a is included usually. Also, few special cases deserve further attention. (i) If $S = \text{At}(P)$, then the completion becomes fully ordered, i.e., $\mathbf{B}^+(r_i) \setminus S$ becomes empty in (10) and the formula (12) is generated only for $r_i \in \text{Def}_P(a)$ with an empty $\mathbf{B}^+(r_i)$. Moreover, if all atoms of $\text{At}(P)$ are completed using (9)–(12), the resulting formulas capture stable models directly, including level ranks of atoms. (ii) If $S = \text{SCC}(a)$, then the ordering becomes local to the component S . Then, if all atoms of S are completed, the formulas capture stable models M for the *program module* P_S induced by the component S [29]. It should be emphasized that the input atoms in $\text{At}(P_S) \setminus S$ are not subject to completion and they may vary freely. Therefore, given a set of facts $I \subseteq \text{At}(P_S) \setminus S$ as an actual *input* for P_S , the stable models of P_S become solutions to $M = \text{LM}(P_S^M \cup I)$ whose levels i are determined by $\mathbf{T}_{P_S^M} \uparrow^i(I)$. (iii) Finally, if $S = \emptyset$ and $a \notin S$ as an exception, equations (9) and (10) capture the standard completion of a , the formula (11) becomes void, and the formula (12) ensures that $x_a = 1$ whenever a is true.

Example 2. As a minimal example, consider $a \leftarrow a$ as the only rule r_1 of a program P and the SCC $S = \{a\} = \text{SCC}(a)$. We obtain the following formulas: $(1 \leq x_a \leq 2)$, $\neg a \rightarrow (x_a \geq 2)$, $\text{dep}(a, a) \rightarrow a \wedge (x_a > x_a)$, $\text{gap}(a, a) \rightarrow a \wedge (x_a > x_a + 1)$, $a \leftrightarrow \text{app}(r_1)$, $\text{app}(r_1) \leftrightarrow \text{dep}(a, a)$, $\text{app}(r_1) \rightarrow \neg \text{gap}(a, a)$. They can be satisfied by falsifying a and all new atoms, as well as by setting $x_a = 2$, indicating that $M_1 = \emptyset$ is stable. On the other hand, $M_2 = \{a\}$ is not stable, which can be realized by an attempt to make a true in the formulas listed above. Thus $\text{app}(r_1)$ and $\text{dep}(a, a)$ must be true, too, and $\text{gap}(a, a)$ false. By further inspection of the formulas, it follows that $x_a > x_a$ is true and $x_a > x_a + 1$ is false, both indicating a contradiction. ■

The case $S = \text{SCC}(a)$ is the most general one and deserves justifications for correctness due to reformulations done in view of [27] and the limitations of ordered completion [3] with regard to (11).

Definition 2. Given a normal logic program P and a scope $S \subseteq \text{At}(P)$ of completion, the tight ordered completion (TOC) of P relative to S is the set of formulas (6) for $a \in S$, (7) and (8) for $a, b \in S$ whenever $a \succeq_P b$, and (9)–(12) for each $a \in \text{At}(P)$ and $r_i \in \text{Def}_P(a)$.

The TOC of P relative to S is denoted by $\text{TOC}^S(P)$ and we omit S from the notation $\text{TOC}^S(P)$, if $S = \text{At}(P)$. It is worth noting that the length $\|\text{TOC}^S(P)\|$ stays linear in $\|P_S\|$.

Theorem 1. Let P be a normal logic program, S an SCC of P , and P_S the module of P induced by S .

1. If $M \subseteq \text{At}(P_S)$ is a stable model of P_S for an input $I \subseteq \text{At}(P_S) \setminus S$ and $\# : M \cap S \rightarrow \mathbb{N}$ the respective level ranking, then there is a model $\langle N, \tau \rangle$ for $\text{TOC}^S(P)$ such that $M = N \cap \text{At}(P_S)$, $\tau(x_a) = \#a$ for each $a \in M \cap S$, and $\tau(x_a) = |S| + 1$ for each $a \in S \setminus M$.
2. If $\langle N, \tau \rangle$ is a model of $\text{TOC}^S(P)$, then $M = N \cap \text{At}(P_S)$ is a stable model of P_S for the input $I = N \cap (\text{At}(P_S) \setminus S)$ and for each $a \in M \cap S$, $\#a = \tau(x_a) - \tau(z)$ is the respective level rank.

As a preparatory step toward generalizations for aggregated rules, our final example in this section illustrates TOC^S in the context of a cardinality rule (3) that is normalized before completion.

Example 3. Let us assume that an atom a is defined by a single cardinality rule $a \leftarrow 1 \leq \{b_1, \dots, b_n\}$ as part of a larger program P having an SCC $S = \text{SCC}(a)$ such that $\{b_1, \dots, b_n\} \subseteq S$. The rule is compactly expressible even without auxiliary atoms in terms of n positive normal rules

$$a \leftarrow b_1. \dots a \leftarrow b_n.$$

The tight ordered completion produces the following formulas for the joint head atom $a \in S$:

$$a \leftrightarrow \text{app}^1(a) \vee \dots \vee \text{app}^n(a), \quad (13)$$

$$\text{app}^1(a) \leftrightarrow \text{dep}(a, b_1), \dots, \text{app}^n(a) \leftrightarrow \text{dep}(a, b_n), \quad (14)$$

$$\text{app}^1(a) \rightarrow \neg \text{gap}(a, b_1), \dots, \text{app}^n(a) \rightarrow \neg \text{gap}(a, b_n). \quad (15)$$

In the above, we adopt a convention that $\text{app}^i(a)$ denotes the application of $r_i \in \text{Def}_P(a)$. Since each $b_i \in S$ the respective rules $a \leftarrow b_i$ may not contribute to external support via (12). ■

4 The Case of Monotone Aggregates

Cardinality rules (3) and weight rules (4) with *lower bounds* are widely used examples of monotone aggregates and, in particular, if the (anti-monotone) effect of negative literals is disregarded in the sense of stable models (cf. Definition 1). The level number $\#a$ of an atom $a \in \text{At}(P)$ is generalized in a straightforward way when *positive* cardinality/weight rules are incorporated to the definition of the \mathbf{T}_P operator [30]. As before, $\#a$ is the least value $i \in \mathbb{N}$ such that $a \in \mathbf{T}_P \uparrow^i(\emptyset)$ for positive programs P . Default negation is analogously treated via the reduct, i.e., given a stable model $M \subseteq \text{At}(P)$, the operator \mathbf{T}_{PM} can be used to assign level ranks for $a \in \text{At}(P)$. The goal of this section is to generalize tight ordered completion for rules involving monotone aggregates. The resulting formulas can be used to enforce stability in various settings where the semantics is no longer based on stable models themselves.

The normalization [7] of cardinality rules is used to guide our intuitions about the intended generalization of tight ordered completion. Besides this, to enable compact representations of aggregates as propositional formulas, we extend the language of difference logic by pseudo-Boolean constraints of the form $c_1 a_1 + \dots + c_m a_m \geq b$ where a_1, \dots, a_m are atoms, c_1, \dots, c_m their respective integer coefficients, and b an integer bound. Obviously, given an interpretation $\langle I, \tau \rangle$ in DL, we define $\langle I, \tau \rangle \models c_1 a_1 + \dots + c_m a_m \geq b$, iff $\sum_{I \models a_i} c_i \geq b$, since the truth values of a_1, \dots, a_m are determined by I independently of τ . Let us begin with an example that concentrates on a corner case ($l = 1$ and $m = 0$) of (3) from Example 1.

Example 4. Recalling formulas (13)–(15) from Example 3, we pull them back to the setting of the original cardinality rule $a \leftarrow 1 \leq \{b_1, \dots, b_n\}$ where b_1, \dots, b_n depend recursively on the head a . Based on a connecting formula $\text{app}^1(a) \vee \dots \vee \text{app}^n(a) \leftrightarrow \text{app}(a)$ on the applicability of the n rules in the normalization versus the applicability of the original rule, we rewrite (13)–(15) as follows:

$$a \leftrightarrow \text{app}(a), \quad (16)$$

$$\text{app}(a) \leftrightarrow (\text{dep}(a, b_1) + \dots + \text{dep}(a, b_n) \geq 1), \quad (17)$$

$$\text{app}(a) \rightarrow (\text{gap}(a, b_1) + \dots + \text{gap}(a, b_n) < 1), \quad (18)$$

where the new atoms $\text{dep}(a, b_1), \dots, \text{dep}(a, b_n)$ and $\text{gap}(a, b_1), \dots, \text{gap}(a, b_n)$ are still to be interpreted subject to formulas (6)–(8) as in the context of Example 3. ■

Note that the formula (18) expresses a *dynamic* check, i.e., it works for any subset B of $\{b_1, \dots, b_n\}$ of atoms *true* and *derived earlier* than a . If the cardinality rule is applied (i.e., $|B| \geq 1$), $\text{gap}(a, b_i)$ must be false for each $b_i \in B$, amounting to the effect of the individual implications in (15). The formulas in Examples 3 and 4 are based on different auxiliary atoms denoting the applicability of rules. The connecting formula $\text{app}^1(a) \vee \dots \vee \text{app}^n(a) \leftrightarrow \text{app}(a)$ describes their intended semantic interconnection for propositional interpretations M and N , i.e., $M \models \text{app}^1(a) \vee \dots \vee \text{app}^n(a)$ iff $N \models \text{app}(a)$. Because (7) and (8) disconnect all integer variables from the formulas under consideration, the following proposition concentrates on the propositional parts of DL-interpretations that are intended to satisfy TOC formulas in the end.

Proposition 1. *The formulas (13)–(15) and (16)–(18) constrain the respective interpretations $\langle M, \tau \rangle$ and $\langle N, \tau \rangle$ equivalently, as conveyed by the satisfaction of the formula $\text{app}^1(a) \vee \dots \vee \text{app}^n(a) \leftrightarrow \text{app}(a)$.*

Proof. Assuming the connecting formula makes formulas (13) and (16) equivalent. Formulas (13) and (14) imply $a \leftrightarrow \text{dep}(a, b_1) \vee \dots \vee \text{dep}(a, b_n)$ that is equivalent to (17) under (16). Finally, $\text{gap}(a, b_1) + \dots + \text{gap}(a, b_n) < 1$ is the same as $\neg \text{gap}(a, b_1) \wedge \dots \wedge \neg \text{gap}(a, b_n)$. These conditions are equally enforced by (15) and (18) when the connecting formula is satisfied. □

Proposition 1 indicates that the formulas (13)–(15) introduced for the normalizing rules can be safely substituted by the formulas (16)–(18) for the original rule. In this way, the aggregated condition is restored as a subformula in (17) while its negation incarnates in (18). Recall that the truth values of atoms $\text{gap}(a, b_i)$ are determined by (8). If (18) were not satisfied by $\langle N, \tau \rangle$, at least one $\text{gap}(a, b_i)$ atom must be true, i.e., $N \models b_i$ and $\tau \models (x_a > x_{b_i} + 1)$ assuming the satisfaction of (8). Thus b_i would be derived so early that the derivation of a is feasible before and the value of x_a could be decreased. Consequently, the joint effect of the formulas (17) and (18) is that $x_a = \min\{x_{b_i} + 1 \mid N \models b_i\}$ holds which is in harmony with the characterization of [17] when applied to the normalizing rules $a \leftarrow b_1. \dots a \leftarrow b_n$. Before addressing arbitrary cardinality rules, we draw the reader's attention to the other extreme.

Example 5. *When $l = n$ and $m = 0$ in (3), the rule can be directly cast as a positive normal rule $a \leftarrow b_1, \dots, b_n$. Still assuming that $\{b_1, \dots, b_n\} \subseteq \text{SCC}(a)$, the TOC formulas resulting from (9)–(11) are $a \leftrightarrow \text{app}^1(a)$, $\text{app}^1(a) \leftrightarrow \text{dep}(a, b_1) \wedge \dots \wedge \text{dep}(a, b_n)$, and $\text{app}^1(a) \rightarrow \bigvee_{1 \leq i \leq n} \neg \text{gap}(a, b_i)$. The corresponding aggregated formulas can be seen in the formulas (17) and (18) if the bound 1 is substituted by n . The resulting strong level ranking constraint ensures that at least one body atom b_i is derived just before a and $x_a = \max\{x_{b_i} \mid 1 \leq i \leq n\} + 1$. ■*

The preceding example reveals our plan when it comes to covering more general lower bounds $1 < l < n$ in (3) still pertaining to the positive case $m = 0$ and $\{b_1, \dots, b_n\} \subseteq \text{SCC}(a)$. In the sequel, we

write \subseteq_l to denote the l -subset relation restricted to subsets of size l exactly. Due to monotonicity, the satisfaction of the rule body $l \leq \{b_1, \dots, b_n\}$ essentially depends on the l -subsets of $\{b_1, \dots, b_n\}$. Thus, the cardinality rule (3) with $m = 0$ can be normalized by introducing a positive rule $a \leftarrow B$ for each $B \subseteq_l \{b_1, \dots, b_n\}$. The number of such rules $\binom{n}{l}$ is at maximum when l is n halved.³ In spite of exponential growth, the resulting normalization serves the purpose of understanding the effect of l on the required TOC formulas. To update equations (13)–(15) for this setting, we need a new atom $\text{app}^B(a)$ for every $B \subseteq_l \{b_1, \dots, b_n\}$ to capture the individual applicabilities of the respective positive rules $a \leftarrow B$:

$$a \leftrightarrow \bigvee_{B \subseteq_l \{b_1, \dots, b_n\}} \text{app}^B(a), \quad (19)$$

$$\text{app}^B(a) \leftrightarrow \bigwedge_{b \in B} \text{dep}(a, b) \quad (\text{for } B \subseteq_l \{b_1, \dots, b_n\}), \quad (20)$$

$$\text{app}^B(a) \rightarrow \bigvee_{b \in B} \neg \text{gap}(a, b) \quad (\text{for } B \subseteq_l \{b_1, \dots, b_n\}). \quad (21)$$

The connecting formula $\bigvee_{B \subseteq_l \{b_1, \dots, b_n\}} \text{app}^B(a) \leftrightarrow \text{app}(a)$ links the above back to the original rule $a \leftarrow l \leq \{b_1, \dots, b_n\}$ suggesting the revisions of (16)–(18) for any lower bound $1 \leq l \leq n$:

$$a \leftrightarrow \text{app}(a), \quad (22)$$

$$\text{app}(a) \leftrightarrow (\text{dep}(a, b_1) + \dots + \text{dep}(a, b_n) \geq l), \quad (23)$$

$$\text{app}(a) \rightarrow (\text{gap}(a, b_1) + \dots + \text{gap}(a, b_n) < l). \quad (24)$$

Most importantly, the length of the formulas (22)–(24) stays linear in n in contrast with their alternatives (19)–(21) based on l -subsets. The aggregate-based formulation covers all l -subsets of $\{b_1, \dots, b_n\}$ and their supersets that also satisfy the body of (3) with $m = 0$ by monotonicity.

Proposition 2. *The formulas (19)–(21) and (16)–(24) constrain the respective interpretations $\langle M, \tau \rangle$ and $\langle N, \tau \rangle$ equivalently, as conveyed by the satisfaction of the formula $\bigvee_{B \subseteq_l \{b_1, \dots, b_n\}} \text{app}^B(a) \leftrightarrow \text{app}(a)$.*

The proof is similar to that of Proposition 1 and amounts to showing that the (big) disjunctive formula $\bigvee_{B \subseteq_l \{b_1, \dots, b_n\}} \bigwedge_{b \in B} \text{dep}(a, b)$ is expressible as $(\text{dep}(a, b_1) + \dots + \text{dep}(a, b_n) \geq l)$. Similar aggregation is achieved in (24) in terms of $\text{gap}(a, b_1), \dots, \text{gap}(a, b_n)$. An important observation is that $\langle N, \tau \rangle \not\models (24)$ if and only if $N \models \text{app}(a)$ and $\exists B \subseteq_l \{b_1, \dots, b_n\}$ such that for each $b \in B$, $N \models b$, $N \models \text{gap}(a, b)$, and $\tau \models (x_a > x_b + 1)$. Since B reaches the bound l the value of x_a could be decreased to $\max\{\tau(x_b) \mid b \in B\} + 1$ or even further if $|B| > l$. Thus the satisfaction of (24) means that no $B \subseteq_l (\{b_1, \dots, b_n\} \cap N)$ of true atoms could be used to decrease the value of x_a . The net effect is that x_a has the critical minimum value. Since (23) is also satisfied there are at least l true atoms derived before a but sufficiently many of them are derived *just before* a . For those atoms b we have $x_a = x_b + 1$, $\text{dep}(a, b)$ true, but $\text{gap}(a, b)$ false!

Example 6. *Consider the rule $a \leftarrow 2 \leq \{b_1, b_2, b_3, b_4\}$ in the context of a model $\langle N, \tau \rangle$ where $N = \{a, b_1, b_3, b_4, \text{app}(a)\}$, $\tau(x_{b_1}) = \tau(x_{b_4}) = 2$, $\tau(x_{b_3}) = 1$, and $\tau(x_{b_2}) = 6$ by default as $N \not\models b_2$, see (6). Now the rule body is satisfied by three 2-subsets $B_1 = \{b_1, b_3\}$, $B_2 = \{b_1, b_4\}$, and $B_3 = \{b_3, b_4\}$ justifying the level rank $\tau(x_a) = 3$, since $\max\{\tau(x_b) \mid b \in B_i\} = 2$ for each $1 \leq i \leq 3$. We have $\text{gap}(b_i, a)$ true only for $i = 3$ and thus (24) is respected for $l = 2$. But, if $\tau(x_a) = 4$ had been alternatively set, the count of such b_i 's would be 3, falsifying (24). ■*

³Note that $\binom{n}{\lfloor n/2 \rfloor} \geq 3^{\lfloor n/2 \rfloor}$ from $n > 4$ onward.

4.1 Weights

So far, we have established relatively general form of TOC formulas (22)–(24) that cover cardinality rules (3) when $m = 0$. Before addressing negative body conditions ($m > 0$) and settings where SCCs play a major role, we take the weights of literals into consideration as already present in weight rules (4) when $m = 0$. Consequently, we have to substitute l -subsets used in (19)–(21) by *weighted* subsets $B \subseteq_w \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}$. Such a subset B can be formally defined in terms of the condition $\text{WS}_B(\{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}) \geq w$ from Section 2. It is clear by monotonicity that if $B \subseteq_w \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}$, then $B' \subseteq_w \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}$ for every B' with $B \subseteq B' \subseteq \{b_1, \dots, b_n\}$. A weighted set $B \subseteq_w \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}$ is defined to be \subseteq -minimal with respect to w , if for no $B' \subset B$, $B' \subseteq_w \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}$. We use \subseteq_w^{\min} to indicate such \subseteq -minimal weighted subsets of $\{b_1, \dots, b_n\}$. Assuming orthogonal generalizations of (19)–(21) for a *positive* weight rule (4) and the weighted subsets $B \subseteq_w^{\min} \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}$ of its body, we rather incorporate weights into the formulas (22)–(24) as follows:

$$a \leftrightarrow \text{app}(a), \quad (25)$$

$$\text{app}(a) \leftrightarrow (w_{b_1} \times \text{dep}(a, b_1) + \dots + w_{b_n} \times \text{dep}(a, b_n) \geq w), \quad (26)$$

$$\text{app}(a) \rightarrow (w_{b_1} \times \text{gap}(a, b_1) + \dots + w_{b_n} \times \text{gap}(a, b_n) < w). \quad (27)$$

Proposition 3. *The formulas (19)–(21) revised for weighted subsets B subject to the bound w and the formulas (25)–(27) constrain the respective interpretations $\langle M, \tau \rangle$ and $\langle N, \tau \rangle$ equivalently, as conveyed by the satisfaction of the equivalence $\bigvee_{B \subseteq_w^{\min} \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}} \text{app}^B(a) \leftrightarrow \text{app}(a)$.*

Proof. Due to high similarity with respect to Proposition 2, we just point out the equivalence of the formulas $\bigvee_{B \subseteq_w^{\min} \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}} \bigwedge_{b \in B} \text{dep}(a, b)$ and $(w_{b_1} \times \text{dep}(a, b_1) + \dots + w_{b_n} \times \text{dep}(a, b_n) \geq w)$. The equivalence involving $\text{gap}(a, b_1), \dots, \text{gap}(a, b_n)$ is analogous but negated. \square

Again, $\langle N, \tau \rangle \not\models (27)$ implies $N \models \text{app}(a)$ and for some $B \subseteq_w^{\min} \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}$, $N \models B$ and for every $b \in B$, $\tau \models (x_a > x_b + 1)$. Then, the value of x_a could be decreased to $\max\{\tau(x_b) \mid b \in B\} + 1$. Thus the formula (27) makes $\tau(x_a)$ minimal as before.

Example 7. *Let us consider a positive weight rule $a \leftarrow 7 \leq \{b_1 = 7, b_2 = 5, b_3 = 3, b_4 = 2, b_5 = 1\}$ in the context of a model $\langle N, \tau \rangle$ where N sets all body atoms b_1, \dots, b_5 true and τ the level numbers*

$$\tau(x_a) = 5, \quad \tau(x_{b_1}) = 5, \quad \tau(x_{b_2}) = 4, \quad \tau(x_{b_3}) = 3, \quad \tau(x_{b_4}) = 2, \quad \tau(x_{b_5}) = 1.$$

The \subseteq -minimal satisfiers of the body are $B_1 = \{b_1\}$, $B_2 = \{b_2, b_3\}$, and $B_3 = \{b_2, b_4\}$. The only atom in B_1 has a weight that reaches the bound 7 alone, but it is derived too late to affect the derivation of a . Both B_2 and B_3 yield the same value $\max\{\tau(x_b) \mid b \in B_i\} = 4$ and hence justify the one higher value 5 assigned to x_a . Interestingly, there is also an atom b_5 that is derived early, but whose weight is irrelevant for satisfying the rule body nor deriving a any earlier. In fact, this weighted atom could be safely deleted from the rule (under strong equivalence).

As regards the satisfaction of (27), the relevant body atoms are b_3 , b_4 , and b_5 , for which the atom $\text{gap}(a, b_i)$ is made true by (8). The sum of the respective weights $3 + 2 + 1$ is less than 7.

Also, note that the level numbers assigned by τ to b_1, \dots, b_5 can be easily arranged with positive rules, e.g., by using the chain of rules: $b_1 \leftarrow b_2$. $b_2 \leftarrow b_3$. $b_3 \leftarrow b_4$. $b_4 \leftarrow b_5$. b_5 . Given the respective program P , the operator \mathbf{T}_P should be applied 5 times to make a true. \blacksquare

4.2 Negative Conditions

Negative body conditions form the missing pieces when it comes to fully covering WCPs with level ranking constraints as embedded in tight ordered completion. To this end, our strategy is based on rewriting and ideas used in [7] where the correctness of normalization is first shown for positive programs and then generalized for programs with negation. In a nutshell, negative literals in (4) can be replaced by new atoms $\overline{c}_1, \dots, \overline{c}_m$ that respectively denote that c_1, \dots, c_m cannot be derived. These atoms are subsequently defined by (atomic) normal rules $\overline{c}_1 \leftarrow \sim c_1. \dots \overline{c}_m \leftarrow \sim c_m$. The outcome is a set of rules that is visibly strongly equivalent with the original weight rule (4). The completions of $\overline{c}_1, \dots, \overline{c}_m$ are $\overline{c}_1 \leftrightarrow \neg c_1, \dots, \overline{c}_m \leftrightarrow \neg c_m$, enabling the substitution of $\overline{c}_1, \dots, \overline{c}_m$ by $\neg c_1, \dots, \neg c_m$ in any formulas of interest. In this way, $\overline{c}_1, \dots, \overline{c}_m$ can be readily forgotten under classical semantics. The transformation described above leaves the SCCs of the program intact, because positive dependencies are not affected. Thus, besides taking care of negative body conditions, our next rewriting step recalls the scope $S \subseteq \text{At}(P)$ from (10)–(12): we present TOC formulas to cover WCPs split into modules based on SCCs. We say that a WCP is *pure* if it contains weight rules (4) only.

Definition 3. *Let P be a pure WCP and $S \subseteq \text{At}(P)$ an SCC of P used as the scope of completion. The tight ordered completion of P relative to S , denoted $\text{TOC}^S(P)$, consists of the formulas listed below:*

- If $|S| > 1$, then for each $a \in S$:
 1. the formulas (6);
 2. the formulas (7) and (8) for each $b \in S$ such that $a \succeq_P b$; plus
 3. the following formulas based on the definition $\text{Def}_P(a) = \{r_1, \dots, r_k\}$ in the program P :

$$a \leftrightarrow \text{app}^1(a) \vee \dots \vee \text{app}^k(a), \quad (28)$$

$$\text{app}^i(a) \leftrightarrow \text{int}^i(a) \vee \text{ext}^i(a), \quad (29)$$

$$\begin{aligned} \text{int}^i(a) \leftrightarrow & (\sum_{b \in \mathbb{B}^+(r_i) \cap S} (w_b \times \text{dep}(a, b)) + \sum_{b \in \mathbb{B}^+(r_i) \setminus S} (w_b \times b) \\ & - \sum_{c \in \mathbb{B}^-(r_i)} (w_c \times c) \geq w_{r_i} - w_i), \end{aligned} \quad (30)$$

$$\begin{aligned} \text{int}^i(a) \rightarrow & (\sum_{b \in \mathbb{B}^+(r_i) \cap S} (w_b \times \text{gap}(a, b)) + \sum_{b \in \mathbb{B}^+(r_i) \setminus S} (w_b \times b) \\ & - \sum_{c \in \mathbb{B}^-(r_i)} (w_c \times c) < w_{r_i} - w_i) \vee \text{ext}^i(a), \end{aligned} \quad (31)$$

$$\begin{aligned} \text{ext}^i(a) \leftrightarrow & (\sum_{b \in \mathbb{B}^+(r_i) \setminus S} (w_b \times b) - \sum_{c \in \mathbb{B}^-(r_i)} (w_c \times c) \\ & \geq w_{r_i} - w_i), \end{aligned} \quad (32)$$

$$\text{ext}^i(a) \rightarrow (x_a \leq 1), \quad (33)$$

where $w_i = \sum_{c \in \mathbb{B}^-(r_i)} w_c$ is the adjustment to the bound w_{r_i} of r_i .

- If $|S| = 1$ and $S = \{a\} = \text{SCC}(a)$, then (28)–(32) are replaced by the standard completion

$$\text{app}^i(a) \leftrightarrow \left(\sum_{b \in \mathbb{B}^+(r_i)} (w_b \times b) - \sum_{c \in \mathbb{B}^-(r_i)} (w_c \times c) \geq w_{r_i} - w_i \right). \quad (34)$$

In the equations of Definition 3, the treatment of negative literals occurring in a defining rule r_i is justified by their contribution $\sum_{c \in \mathbb{B}^+(r_i)} (w_c \times (1 - c)) = w_i - \sum_{c \in \mathbb{B}^+(r_i)} (w_c \times c)$ toward the bound w_{r_i} . Weight rules can also create external support in more flexible ways, i.e., if the bound w_{r_i} can be reached by satisfying any positive body conditions outside the SCC S in question or any negative body conditions. Moreover, a single weight rule r_i may justify the head a either internally or externally as formalized by (29), which is different from the case of normal rules. Formulas (30) and (32) capture this distinction.

Note that $\text{ext}^i(a)$ implies $\text{int}^i(a)$. The constraint (31) generalizes (11) while (33) is the analog of (12). The consequent of (31) is weakened by the condition $\text{ext}^i(a)$, since the other disjunct is falsified if r_i provides external support, $x_a = 1$ holds, and all atoms $\text{dep}(a, \cdot)$ and $\text{gap}(a, \cdot)$ associated with a are falsified. The reader may have noticed that the formulas concerning $\text{int}^i(a)$ and $\text{ext}^i(a)$ share a potentially large subexpression $s_i = \sum_{b \in B^+(r_i) \setminus S} (w_b \times b) - \sum_{c \in B^-(r_i)} (w_c \times c)$. Certain back-end formalisms, such as DL and MIP, enable the representation of this expression only once using an integer variable.

The following theorem states the correctness of $\text{TOC}(P)$ obtained as the union of $\text{TOC}^S(P)$ for the SCCs S of P . The claimed one-to-one correspondence, however, must take into account the fact that a satisfying assignment $\langle M, \tau \rangle$ in DL can be replicated into infinitely many copies by substituting τ in $\langle M, \tau \rangle$ by a function $\tau'(x) = \tau(x) + k$ for any $k \in \mathbb{Z}$. The remedy is to introduce a special variable z which is assumed to hold 0 as its value and the values of the other variables are set relative to the value of z . The current difference constraints mentioning only one variable must be rewritten using z . For instance, $1 \leq x_a \leq |S| + 1$ in (6) is expressed by the conjunction of $z - x_a \leq -1$ and $x_a - z \leq |S| + 1$, and this is how the variable z gets introduced.

Theorem 2. *Let P be a WCP with SCCs S_1, \dots, S_q and P_{S_1}, \dots, P_{S_q} the respective modules of P . Then P and the set of formulas $F = \bigcup_{j=1}^q \text{TOC}^{S_j}(P)$ are visibly equivalent (up to assigning $z = 0$).*

5 Generalizations Toward Convex Aggregates

Weight rules (4) can be generalized by introducing upper bounds u besides lower bounds l :

$$a \leftarrow l \leq \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}, \sim c_1 = w_{c_1}, \dots, \sim c_m = w_{c_m}\} \leq u. \quad (35)$$

This gives a rise to a *convex* condition which is easier to explain for positive rules ($m = 0$). If the condition can be satisfied by setting the atoms of $B_1 \subseteq \{b_1, \dots, b_n\}$ true and the same holds for a superset $B_2 \subseteq \{b_1, \dots, b_n\}$ of B_1 , then every intermediate set B' such that $B_1 \subseteq B' \subseteq B_2$ satisfies the condition, too. It is easy to check that this is a property of (35), since $B_1 \subseteq B_2$ implies $\text{WS}_{B_1}(\{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\}) \leq \text{WS}_{B_2}(\{b_1 = w_{b_1}, \dots, b_n = w_{b_n}\})$ in general. However, the bounds play a role here: upper bounds jeopardize monotonicity in general but convexity is still guaranteed. Thus, we use WCPs based on (35) to understand the role of level ranking constraints in the context of convex aggregates. The effect of negative literals is anti-monotonic, but their semantics is determined by the reduct as usual. Simons et al. [30] present a transformation that checks the upper bound of (35) with another weight rule. The rules below adopt this idea but using a constraint and new atoms that are in line with (34) and $r_i \in \text{Def}_P(a)$:

$$\text{app}^i(a) \leftarrow l \leq \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}, \sim c_1 = w_{c_1}, \dots, \sim c_m = w_{c_m}\}. \quad (36)$$

$$\text{vub}^i(a) \leftarrow u + 1 \leq \{b_1 = w_{b_1}, \dots, b_n = w_{b_n}, \sim c_1 = w_{c_1}, \dots, \sim c_m = w_{c_m}\}. \quad (37)$$

$$\leftarrow \text{app}^i(a), \text{vub}^i(a). \quad (38)$$

For the moment, this relaxes the notion of applicability for r_i but the constraint (38) makes sure that the upper bound is not *violated*. Since no atom depends positively on $\text{vub}^i(a)$, the resulting TOC formula is analogous to (34) with $\text{vub}^i(a)$ as its head. The constraint (38) can be expressed as $\neg(\text{app}^i(a) \wedge \text{vub}^i(a))$. The net effect is that if $\text{app}^i(a)$ is true, then $(\sum_{b \in B^+(r_i)} (w_b \times b) - \sum_{c \in B^-(r_i)} (w_c \times c)) \leq u - w_i$ must hold. Thus (30)–(32) can be revised for (35) by replacing the previous lower bound w_{r_i} with l and by incorporating $u - w_i$ into (30) and (32) as upper bounds; either as two pseudo-Boolean constraints or a combined one with two bounds. The respective upper bound does not play a role in (31) that concerns the criticality of the lower bound and this check does not interfere with the satisfaction of the upper bound due to convexity.

5.1 Abstract Constraint Atoms

Based on the preceding analysis of weight rules, we will rephrase our approach for an arbitrary convex aggregate $\text{Aggr}(B)$ that takes a set of (body) atoms B as input and accepts a certain subset \mathcal{S} of the powerset 2^B by evaluating to true. This set must satisfy the convexity condition, i.e., if $S_1, S_2 \in \mathcal{S}$ then $S \in \mathcal{S}$ for each intermediate set $S_1 \subseteq S \subseteq S_2$, too. Moreover, we let $\text{Aggr}^*(B)$ stand for the *monotonic* (upward) *closure* of $\text{Aggr}(B)$ based on the signature B . The set of satisfiers $\mathcal{S} \subseteq 2^B$ of the latter is extended to a set of satisfiers $\mathcal{S}^* = \{S \mid \exists S' \in \mathcal{S}, S' \subseteq S \subseteq B\}$ for the former. The syntax of logic programs can be extended by introducing aggregated conditions as rule bodies in analogy to (3) and (4):

$$a \leftarrow \text{Aggr}(b_1, \dots, b_n, \sim c_1, \dots, \sim c_m). \quad (39)$$

Let P be a logic program consisting of rules of the form (39) and $M \subseteq \text{At}(P)$ a *model* of P that satisfies all rules (39) in the standard sense, i.e., if the body is satisfied by M , then the head $a \in M$. The reduct P^M can be formed by including a positive rule $a \leftarrow \text{Aggr}^*(b_1, \dots, b_m, c_1 \in M? \perp : \top, \dots, c_m \in M? \perp : \top)$ for each (39) such that $M \models \text{Aggr}(b_1, \dots, b_n, \sim c_1, \dots, \sim c_m)$. In the above, we exploit *conditional substitutions* $c?v:u$ by the value v if the condition c is true and the value u otherwise. Thus, given a set of input atoms $I \subseteq \text{At}(P) \setminus \text{H}(P)$, we can calculate the least model of P^M and assign level ranks $\#a = i$ of atoms $a \in \text{H}(P)$ based on the membership $a \in \mathbf{T}_{P^M} \uparrow^i(I) \setminus \mathbf{T}_{P^M} \uparrow^{i-1}(I)$ as earlier. Assuming a defining rule $r_i \in \text{Def}_P(a)$ for an atom $a \in \text{H}(P)$ and the scope $S = \text{SCC}(a)$, we generalize (30)–(32) as follows:

$$\text{int}^i(a) \leftrightarrow \text{Aggr}(b_1 \in S? \text{dep}(a, b_1) : b_1, \dots, b_n \in S? \text{dep}(a, b_n) : b_n, \neg c_1, \dots, \neg c_m), \quad (40)$$

$$\begin{aligned} \text{int}^i(a) \rightarrow & \neg \text{Aggr}^*(b_1 \in S? \text{gap}(a, b_1) : b_1, \dots, b_n \in S? \text{gap}(a, b_n) : b_n, \neg c_1, \dots, \neg c_m) \\ & \vee \text{ext}^i(a), \end{aligned} \quad (41)$$

$$\text{ext}^i(a) \leftrightarrow \text{Aggr}(b_1 \in S? \perp : b_1, \dots, b_n \in S? \perp : b_n, \neg c_1, \dots, \neg c_m). \quad (42)$$

The formulas above assume that the aggregate $\text{Aggr}(\cdot)$ can be expressed as a propositional formula, or the target language has sufficient syntactic primitives available such as pseudo-Boolean constraints. Conditional substitutions are necessary, because we have not made any assumptions about ordering the arguments to $\text{Aggr}(\cdot)$. For instance, if the weight constraint from Example 7 were abstracted as $\text{Aggr}(b_1, b_2, b_3, b_4, b_5)$, the meaning of $\text{Aggr}(b_5, b_4, b_3, b_2, b_1)$ would become different due to asymmetric weights. As in the case of pure WCPs, the TOC formula (40) takes care of the (potential) internal support delivered by r_i and the support is ordered by the conjoined atoms $\text{dep}(a, b_i)$ so that the stability of models can be guaranteed. The formula (42) for external support is analogous except that the contribution of positive body conditions from the same component is hindered by substituting them by \perp . By the side of these formulas, the strong ranking constraint (41) ensures that $\text{Aggr}(\cdot)$ could not be satisfied with arguments derived way before a .

6 Conclusions

In this work, we investigate potential generalizations of *level ranking constraints* [27] for extended rule types involving aggregates. It turns out that the structure of aggregates can be preserved if the back-end language offers analogous syntactic primitives for expressing such conditions. As a consequence, it is not necessary to normalize rules before the introduction of ranking constraints in a form or another, depending on the target formalism and the back-end solver to be used for actual computations. The particular novelty of our approach lies in the generalization of *strong* level ranking constraints for WCPs and other

monotone/convex aggregates. Using them level rankings can be made unique which is desirable, e.g., when counting answer sets. A further by-product is that any WCP can be translated into a *tight* WCP if the formulas in $\text{TOC}(P)$ are expressed with rules rather than formulas, also justifying “*tight*” as part of TOC.

Although the results of this article are theoretical by nature, they enable new kinds of strategies when it comes to implementing the search of stable models using existing solver technology for SAT, SMT, and MIP. E.g., the presented TOC formulas offer a common ground for the translators in the LP2* family [19]. We leave *non-convex* aggregates as future work for two main reasons. First, there is no consensus about their semantics when recursive definitions are enabled [2]. The ASP-core-2 language standard assumes the *stratified* setting only whereas the Clingo system implements one particular semantics for recursive non-convex aggregates [13]. Second, there is also evidence [1] that the removal of non-convex aggregates tends to produce disjunctive rules which go beyond level rankings in the first place. One potential solution is provided by the *decomposition* of non-convex aggregates into their maximal convex regions, cf. [18, 24].

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