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POLYNOMIAL STABILITY OF ABSTRACT WAVE EQUATIONS

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ABSTRACT

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Abstract wave equations model a large class of linear dynamical systems, i.e., systems that evolve linearly over time. For example, this vast class of abstract wave equations encloses many important second-order partial differential equations that are frequent in applications. As with any partial differential equation, an important question arises whether a solution to an abstract wave equation exists for all possible initial conditions. By a solution we simply mean a function that satisfies both the abstract wave equation and the existing boundary and initial conditions. Answering this question leads us to the widely known criteria for well-posed problems by Hadamard. It turns out that all abstract wave equations are well-posed, which is implied by the theory of so-called strongly continuous semigroups.

Having established the existence of solutions for abstract wave equations, it is natural to ask how the solutions behave over time. In particular, we are interested in the limiting, that is, asymptotic behaviour of the solutions as time elapses. If the solutions corresponding to all initial conditions eventually converge to some equilibria, then we call the associated abstract wave equation asymptotically stable. In case the rate of convergence is also uniform for all solutions, the solutions actually converge to their equilibria at an exponential rate, yielding exponential stability. In general, a solution to an asymptotically stable abstract wave equation can approach its equilibrium arbitrarily slowly and thus preclude any uniform rate of convergence. However, with certain assumptions we obtain results for strongly continuous semigroups that guarantee both asymptotic stability and a uniform rate of convergence for a particular subset of solutions called classical solutions.

In this thesis we examine the polynomial stability of abstract wave equations. Put simply, all classical solutions to an abstract wave equation should converge to their equilibria at a polynomial rate. A polynomially stable system is always asymptotically stable but not necessarily exponentially stable. Although polynomial stability is a special case of a more general semi-uniform stability, for the time being counterparts to important results implying exponential stability only exist for polynomial stability. The key idea in these results is to investigate how the norm of a resolvent associated with the abstract wave equation grows on the imaginary axis. The slower this norm grows, the faster the classical solutions converge. At the end of this thesis we analyze a system from the literature and its two variants in great detail. We recast these systems as abstract wave equations and study their stability with the theory and tools we obtain along the way.

Keywords: polynomial stability, abstract wave equation, strongly continuous semigroup

TIIVISTELMÄ

Jasper Kosonen: Abstraktien aaltoyhtälöiden polynomiaalinen stabiilisuus
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Abstraktit aaltoyhtälöt muodostavat laajan luokan lineaarisia dynaamisia systeemejä eli systeemejä, jotka kehittyvät lineaarisesti ajan suhteen. Abstraktien aaltoyhtälöiden avulla voidaan mallintaa esimerkiksi tärkeissä sovelluksissa usein esiintyviä toisen kertaluvun osittaisdifferentiaaliyhtälöitä. Osittaisdifferentiaaliyhtälöiden tapaan ensimmäiseksi on oleellista selvittää, löytyykö abstraktille aaltoyhtälölle aina ratkaisu kutakin alkuarvoa kohden. Ratkaisulla tarkoitetaan yksinkertaisesti funktiota, joka toteuttaa sekä abstraktin aaltoyhtälön että siinä esiintyvät reuna- ja alkuarvot. Tämän kysymyksen kautta päästään käsiksi esimerkiksi Hadamardin hyvin asetetuille ongelmille määrittelemiin kriteereihin. Abstraktit aaltoyhtälöt voidaan näyttää hyvin asetetuiksi ongelmiksi niin sanottujen vahvasti jatkuvien puoliryhmien avulla.

Ratkaisujen olemassaolon jälkeen toinen mielenkiintoinen kysymys on selvittää, miten abstraktin aaltoyhtälön ratkaisut käyttäytyvät asymptoottisesti. Toisin sanoen on hyödyllistä tietää, mitä ratkaisuille tapahtuu ajan kuluessa äärettömiin. Mikäli kutakin alkuarvoa vastaava ratkaisu suppenee lopulta jotakin tasapainotilaa kohti, sanotaan abstraktia aaltoyhtälöä asymptoottisesti stabiiliksi. Jos lisäksi kaikki ratkaisut suppenevat tasapainotilaansa samalla nopeudella, suppenemisen on tapahduttava itse asiassa eksponentiaalisella nopeudella. Tällöin puhutaan eksponentiaalisesta stabiilisuudesta. Yleisessä tapauksessa asymptoottisesti stabiilin abstraktin aaltoyhtälön ratkaisu voi kuitenkin lähestyä tasapainotilaansa mielivaltaisen hitaasti, minkä vuoksi ratkaisut eivät aina suppene tasapainotiloihinsa yhtenäisellä nopeudella. Tietyin oletuksin vahvasti jatkuville puoliryhmille löytyy tuloksia, jotka takaavat sekä asymptoottisen stabiilisuuden että yhtenäisen suppenemisnopeuden kaikille niin sanotuille klassisille ratkaisuille.

Tässä diplomityössä tutustutaan abstraktien aaltoyhtälöiden polynomiaaliseen stabiilisuuteen, eli abstraktin aaltoyhtälön kaikki klassiset ratkaisut asettuvat tasapainotilaansa hitaimmillaan polynomiaalisella nopeudella. Polynomiaalisesti stabiili systeemi on aina asymptoottisesti stabiili mutta ei välttämättä eksponentiaalisesti stabiili. Vaikka polynomiaalinen stabiilisuus on itse asiassa erikoistapaus yleisemmästä osittain yhtenäisestä stabiilisuudesta (engl. semi-uniform stability), tällä hetkellä moni tunnetuista eksponentiaalisen stabiilisuuden takaavista tuloksista on pystytty yleistämään ainoastaan polynomiaaliselle stabiilisuudelle. Tulosten perustana on tutkia abstraktiin aaltoyhtälöön liittyvän resolventtioperaattorin normin kasvua imaginääriakselilla. Mitä hitaammin resolventtioperaattorin normi kasvaa, sitä nopeammin klassiset ratkaisut suppenevat. Työn lopussa analysoidaan kirjallisuudesta poimittua systeemiä yksityiskohtaisesti siten, että systeemi ja sen kaksi muunnelmaa muokataan abstrakteiksi aaltoyhtälöiksi. Kunkin systeemin stabiilisuutta tarkastellaan työn aikana rakentuvasta polynomiaalisen stabiilisuuden näkökulmasta.

Avainsanat: polynomiaalinen stabiilisuus, abstrakti aaltoyhtälö, vahvasti jatkuva puoliryhmä

PREFACE

It feels great to have finished this thesis. It has certainly been an interesting journey to the world of linear operator semigroups with a glimpse of how these semigroups are being applied even in present-day mathematical research. After taking a course on advanced functional analysis in the fall of 2022, I wanted to see what research in this field of mathematics would look like. Therefore, I approached Assoc. Prof. Lassi Paunonen and asked if he wanted to supervise my thesis. Fortunately he agreed and suggested a topic close to his own research, namely to study the polynomial stability of abstract wave equations. Although ambitious, this topic provided me with a suitable amount of challenge with many moments of both dread and triumph.

First and foremost, I want to thank both of my supervisors, Assoc. Prof. Lassi Paunonen and Dr. Nicolas Vanspranghe. Their guidance has improved the quality of this thesis immensely, and I could always rely on their help when trying to understand many of the challenging concepts within this thesis. I am also very grateful for the commitment that they demonstrated throughout the journey from start to finish – from the frequent meetings to giving valuable comments on the thesis very quickly after each submission. A special thank you goes to Dr. Vanspranghe for the many insightful conversations we had in the office and also for answering some of my questions while out in the wilderness of Norway! I also want to thank my colleague Dr. Thavamani for her part in the thesis.

Although the recent coronavirus pandemic and the unsettling situation in the East do not bode too well for the future, I am lucky to have a supporting network of family members and friends around me. Thank you all who have supported me during this journey. I especially want to thank my parents and siblings who have greatly shaped me into the person I am today. On one hand I feel happy to finally graduate but on the other hand the unpredictability of the future is honestly a bit daunting. Come what may, I am still eagerly looking forward to seeing what the future has in store for me.

Tampere, 9th September 2023

Jasper Kosonen

CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

A^*	adjoint of the operator A
$\partial\Omega$	boundary of the set Ω
$A \in \mathcal{L}(X, Y)$	bounded linear operator A with domain $\mathcal{D}(A) = X$ and normed vector spaces X and Y
$\overline{A}$	closure of the operator A
$\overline{S}$	closure of the set S
$\mathbb{C}$	complex numbers
$C(\Omega)$	continuous functions on Ω
$\frac{d}{dt}$	differentiation with respect to t
$\bigoplus_{i=1}^{n} S_i$	direct sum of the sets S_i
$\mathcal{D}(A)$	domain of the operator A
A^α	fractional power of the operator A
ω_0	growth bound of a strongly continuous semigroup
$i\mathbb{R}$	imaginary numbers
$\operatorname{Im} z$	imaginary part of the complex number z
$\inf S$	infimum of the set S
$\langle \cdot, \cdot \rangle_X$	inner product on the inner product space X
$\mathbb{Z}$	integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
A^{-1}	inverse operator of the injective operator A
$\mathcal{N}(A)$	kernel of the operator A
Δ	Laplacian operator
$\int_\Omega f(\xi)\, d\xi$	Lebesgue integral of f over domain Ω
$L^p(\Omega)$	Lebesgue space as an equivalence class
$A : \mathcal{D}(A) \subseteq X \to Y$	linear operator A with domain $\mathcal{D}(A)$ and vector spaces X and Y
$x \mapsto y$	mapping from x to y
$\mathbb{N}$	natural numbers $\{1, 2, 3, \ldots\}$
$\lVert\cdot\rVert_X$	norm on the normed space X

$\|A\|$	operator norm $\|A\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1}\|Ax\|_Y$
$\frac{\partial}{\partial t}$	partial differentiation with respect to t
$\sigma_p(A)$	point spectrum of the operator A
$\mathcal{R}(A)$	range of the operator A
$\mathbb{R}$	real numbers
$\operatorname{Re} z$	real part of the complex number z
$\rho(A)$	resolvent set of the operator A
$\lim_{t\to a^+} f(t)$	right-side limit of $f(t)$ at $t = a$
$(x_k)_{k\in\mathbb{N}} \subseteq X$	sequence $(x_1, x_2, \ldots)$ where $x_k \in X$ for all $k \in \mathbb{N}$
ℓ^p	sequence space $\{(x_k)_{k\in\mathbb{N}} \subseteq \mathbb{C} : \sum_{k=1}^{\infty} \|x_k\|^p < \infty\}$
$C_0^\infty(\Omega)$	smooth functions on Ω with compact support
$H^k(\Omega)$	Sobolev space $W^{k,2}(\Omega)$
$H_0^1(\Omega)$	Sobolev space $W_0^{1,2}(\Omega)$
$s(A)$	spectral bound of A
$\mathsf{WP}_{s,\delta(s)}(A)$	spectral subspace of the operator A
$\sigma(A)$	spectrum of the operator A
$(T(t))_{t\geq 0}$	strongly continuous semigroup
$\sup_{x\in S} f(x)$	supremum of the set $\{f(x) : x \in S\}$
x^T	transpose of the vector x
$D^\alpha f$	the α^{th}-weak derivative of f

1. INTRODUCTION

Differential equations model the world around us. In the vast field of physics alone, there is a plethora of laws and equations in the form of differential equations that dictate how objects move and how waves behave over time, for example. Although ordinary differential equations are useful in many applications, *partial differential equations* arise frequently when modeling linear dynamical systems [31, p. vii], i.e., systems that evolve linearly over time from an initial state [21, Ch. I]. Given an initial state of such a system, it is natural to ask if we can describe how the initial state evolves when time elapses. In theory, solving the partial differential equation analytically yields a perfect description of the evolution of an initial state. With more and more complicated systems, however, analytical solutions are often beyond our reach and instead we need numerical solutions.

To approach a numerical solution of a linear dynamical system, we first need to be certain that a solution even exists. Should a solution exist, we cannot necessarily approach it with numerical methods if the system is not *asymptotically stable*. By asymptotic stability we simply mean that all initial states of the system eventually converge to an equilibrium as time elapses. It turns out that if all solutions converge to their equilibria at a uniform rate, the rate of convergence is bounded by a decaying exponential [21, Prop. V.1.2]. In this case we say that the system is exponentially stable. However, there are many asymptotically stable systems with solutions that converge at a strictly slower rate [17, 16]. This poses the question whether all so-called *classical* solutions converge at a uniform rate. In particular, if the convergence of all classical solutions happens at a polynomial rate then we arrive at *polynomially stable* systems.

In this thesis, we study the polynomial stability of a particular class of dynamical systems called *abstract wave equations*. Due to their abstract nature, these equations model various interesting and important real-life systems. In addition to the classical wave equation in physics, abstract wave equations can be used to model how different waves behave when exposed to damping, how certain coupled systems interact with each other and how vibrations affect structural beams, for example. Our goal in this thesis is to conduct a literary survey on the topics and tools with which we can investigate the polynomial stability of abstract wave equations. We also apply these tools to analyze concrete examples of dynamical systems. As prerequisites, the reader should possess a solid understanding on introductory-level functional analysis as well as matrix analysis.

The structure of this thesis is as follows. In Chapter 2, we go through the theoretical background of our survey. We define important concepts like *Sobolev spaces* and *closed operators*, and we obtain relevant results concerning the spectral theory of closed operators. Afterwards, we move on to Chapter 3 and study the basic properties of *strongly continuous semigroups*. We learn that with strongly continuous semigroups we can prove the well-posedness of so-called *abstract Cauchy problems*. In Chapter 3, we also define the stability of a dynamical system mathematically using strongly continuous semigroups. In particular, we define what we mean by polynomial stability. Our literary survey culminates in Chapter 4 where we define abstract wave equations and study conditions under which an abstract wave equation is polynomially stable. A key insight in this chapter is that all abstract wave equations are in fact abstract Cauchy problems. We also present a few examples of abstract wave equations from the literature. In Chapter 5, we combine the knowledge from the preceding chapters and study a concrete system of coupled wave equations. We incorporate three different dampings to the system and investigate how the damping affects the stability of the system in each case. Finally, we present the main conclusions and discussion in Chapter 6.

2. THEORETICAL BACKGROUND

In this chapter we start building the sufficient theoretical foundation for the rest of this thesis. Our ultimate objective is to gain the knowledge and tools with which we can study the polynomial stability of abstract wave equations. Such a literary survey is not a trivial task, and therefore we go through the relevant topics in the course of the next three chapters. To study the polynomial stability of an abstract wave equation in Chapter 4, we first need a solid understanding of strongly continuous semigroups and their connection with abstract Cauchy problems. Therefore, Chapter 3 introduces these new concepts in greater detail. The theory of strongly continuous semigroups, however, relies heavily on the spectral theory of *closed operators*. Closed operators and their spectra are part of advanced functional analysis, and we cover them in Section 2.2 of this chapter. In Section 2.1 we also introduce the concept of *Sobolev spaces* which we need particularly in Chapter 5 with practical applications.

2.1 Sobolev Spaces

In this section we define the concept of Sobolev spaces. We denote a general Sobolev space by $W^{k,p}(\Omega)$ where $k \in \mathbb{N}$, $1 \leq p \leq \infty$ and $\Omega \subseteq \mathbb{R}^n$ is an open and bounded set with $n \in \mathbb{N}$ [22, p. 260][31, Def. 6.63][14, p. 216]. We call k the *order* of the space. Conceptually, the Sobolev space $W^{k,p}(\Omega)$ contains the Lebesgue p-integrable functions whose particular *weak derivatives* belong to the same Lebesgue space $L^p(\Omega)$. This concept is very useful in the theory of partial differential equations [31, p. 205].

We define the weak derivative as in [22, pp. 257–258] using the following notation. Suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index of order $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n = k$ where each α_j is a non-negative integer. If we denote the space of infinitely differentiable functions with compact support by $C_0^\infty(\Omega)$, then we call any element $\phi \in C_0^\infty(\Omega)$ a *test function* [22, p. 258]. For a test function $\phi : \Omega \to \mathbb{R}$ we define

$$D^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial \xi_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial \xi_n^{\alpha_n}} \phi.$$

We say a function $f \in L^p(\Omega)$ has the α^{th}-*weak derivative* if there is a function $g \in L^p(\Omega)$

such that

$$\int_{\Omega} f(\xi)(D^{\alpha}\phi)(\xi)\, d\xi = (-1)^{|\alpha|} \int_{\Omega} g(\xi)\phi(\xi)\, d\xi$$

for all test functions $\phi \in C_0^{\infty}(\Omega)$. In this case, we denote $D^{\alpha} f = g$.

Although the general Sobolev space $W^{k,p}(\Omega)$ is in fact a Banach space, only the Sobolev space $W^{k,2}(\Omega)$ is actually a Hilbert space [22, Thm 5.2.2, p. 260][31, Thm. 6.65][14, Prop. 8.1, p. 217]. We denote this special Sobolev space by $H^k(\Omega)$ and call it *the* Sobolev space of order k. The following definition is an adaptation from [22, p. 260], [31, Def. 6.63] and [14, p. 216].

Definition 2.1 (Sobolev space)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded set with $n \in \mathbb{N}$ and let $k \in \mathbb{N}$. We define the *Sobolev space of order* k as the space

$$H^k(\Omega) = \{f \in L^2(\Omega) : D^{\alpha} f \in L^2(\Omega) \text{ for all } |\alpha| \leq k\}.$$

We define the norm on $H^k(\Omega)$ as

$$\|f\|_{H^k}^2 = \sum_{0 \leq |\alpha| \leq k} \|D^{\alpha} f\|_{L^2}^2, \qquad f \in H^k(\Omega).$$

Note that the space $H^0(\Omega)$ is actually the space $L^2(\Omega)$. It is also evident from the above definition that $H^j(\Omega)$ is a subset of $H^k(\Omega)$ whenever $j > k$. Therefore, we can focus on studying the properties of the Sobolev space $H^1(\Omega)$ as the other Sobolev spaces then inherit the same properties [14, p. 217]. A particularly important subset of $H^1(\Omega)$ is the space $H_0^1(\Omega)$ that is the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $\|\cdot\|_{H^1}$ [22, p. 261]. Recall that the Lebesgue spaces $L^p(\Omega)$ are defined as equivalence classes with respect to sets of measure zero [7, Def. 7.16]. Therefore, an arbitrary function in $H^1(\Omega)$ need not have well-defined values on the boundary $\partial\Omega$. By the Trace Theorem [22, Thm. 5.5.1][31, Thm. 6.103] there exists a bounded linear operator with which we can study the values of functions in $H^1(\Omega)$ on the boundary $\partial\Omega$ in a generalized sense known as the *trace*. As a consequence, the space $H_0^1(\Omega)$ contains the functions in $H^1(\Omega)$ the traces of which are equal to 0 on the boundary $\partial\Omega$ [22, Thm 5.5.2][31, Thm. 6.110].

With the help of the space $H_0^1(\Omega)$ we can compactly encode a zero Dirichlet boundary condition present in many partial differential equations [31, p. 15]. For example, the intersection $H^k(\Omega) \cap H_0^1(\Omega)$ contains all the functions in $H^k(\Omega)$ that also satisfy the zero Dirichlet boundary condition. We need such spaces later with concrete examples of partial differential equations. For further information on Sobolev spaces, see Chapter 5 in [22], Section 6.4 in [31] and Chapter 8 in [14].

2.2 Spectral Theory of Closed Operators

Closed operators form a large class of linear operators. For example, this particular class contains all bounded linear operators [35, Thm 5.1][25, Thm. 10.3-2 (c)][28, pp. 241–242]. As closed operators are ubiquitous in applications of partial differential equations [35, p. 209], they are also prevalent throughout this thesis. Therefore, we first define closed operators and then study their spectral properties in this section. Keeping our ultimate goal in mind, we focus especially on the spectral theory of *self-adjoint* and *skew-adjoint* operators and of self-adjoint operators *with compact resolvents*. Note that all the operators in this thesis are linear operators by default.

We start by defining the aforementioned class of linear operators. The following definition is equivalent to the definition in [35, p. 209].

Definition 2.2 (Closed operator)**.** Let $A : \mathcal{D}(A) \subseteq X \to Y$ be an operator between Banach spaces X and Y. We say A is a *closed* operator if for an arbitrary sequence $(x_k)_{k\in\mathbb{N}} \subseteq \mathcal{D}(A)$ satisfying

$$\lim_{k\to\infty} \|x_k - x\|_X = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \|Ax_k - y\|_Y = 0$$

for some $x \in X$ and $y \in Y$, we also have that $x \in \mathcal{D}(A)$ and $Ax = y$.

We already know that a bounded operator $A \in \mathcal{L}(X, Y)$ is also a closed operator, but not all closed operators are bounded [35, Thm 5.1][25, Thm. 10.3-2 (c)][28, pp. 241–242]. In fact, many operators defined in terms of ordinary or partial differentiation turn out to be unbounded but closed operators [35, p. 209]. In the next example we define a closed operator that we need frequently in Chapter 5. For other examples of closed operators, see Examples IV.5.1 and IV.5.2 in [35].

Example 2.3 (Positive Dirichlet Laplacian)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded set with $n \in \mathbb{N}$. Moreover, assume the boundary $\partial\Omega$ is sufficiently smooth. We define the *positive Dirichlet Laplacian* as the operator

$$Lf = -\Delta f = -\left(\frac{\partial^2 f}{\partial \xi_1^2} + \frac{\partial^2 f}{\partial \xi_2^2} + \ldots + \frac{\partial^2 f}{\partial \xi_n^2}\right), \qquad f \in \mathcal{D}(L) = H^2(\Omega) \cap H_0^1(\Omega).$$

The space $H^2(\Omega) \cap H_0^1(\Omega)$ is a particular subspace of the Sobolev space $H^2(\Omega)$ defined in Section 2.1. The reason why L is a closed operator follows from a quintessential result called *elliptic regularity* [14, Thm. 9.25][22, Thm. 6.3.1]. Due to elliptic regularity, the inverse L^{-1} exists and is bounded, implying that for all $g \in L^2(\Omega)$ the equation $Lf = g$ has a solution $f_g \in H^2(\Omega)$. Therefore, L is a closed operator by [35, Thm. IV.5.8].

Having defined closed operators and seen an example, we can now move on to defining basic concepts in spectral theory. The following definition is an adaptation from its counterparts in [35, p. 264], [28, Def. 6.5.2] and [25, Def. 7.2-1].

Definition 2.4 (Resolvent set, spectrum)**.** Let $A : \mathcal{D}(A) \subseteq X \to X$ be a closed operator on a Banach space X. We define the *resolvent set* $\rho(A)$ of A as

$$\rho(A) = \{\lambda \in \mathbb{C} : \text{The operator } \lambda - A \text{ has a bounded inverse } (\lambda - A)^{-1} \in \mathcal{L}(X)\}.$$

If $\lambda \in \rho(A)$, then we call the bounded operator $(\lambda - A)^{-1}$ the *resolvent* of $\lambda - A$. Furthermore, we call the set $\sigma(A) = \mathbb{C} \setminus \rho(A)$ the *spectrum* of A. We call the subset

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ is not injective}\} \subseteq \sigma(A)$$

the *point spectrum* of A.

Like the above definition suggests, one major aspect of spectral theory deals with the study of particular inverse operators and their properties. Spectral theory arises quite naturally in various situations that involve solving equations, and studying the spectrum of an operator can also give useful insight to the operator itself [25, p. 363]. Recall that closed operators on Banach spaces have bounded inverses if and only if they are bijective operators [35, Thm. IV.5.5]. Therefore, it is easy to see from Definition 2.4 that the point spectrum $\sigma_p(A)$ is indeed a subset of $\sigma(A)$.

Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is a closed operator on a Banach space X and $\lambda \in \sigma_p(A)$. By Definition 2.4 the operator $\lambda - A$ is not injective and there must exist some non-zero element $\varphi \in \mathcal{D}(A)$ for which

$$(\lambda - A)\varphi = 0 \qquad \text{or equivalently} \qquad A\varphi = \lambda\varphi. \tag{2.1}$$

We call $\lambda \in \sigma_p(A)$ an *eigenvalue* of A [35, p. 265][28, p. 412][25, Def. 7.2-1]. We call the non-zero elements $\varphi \in \mathcal{D}(A)$ satisfying (2.1) the *eigenvectors* of A corresponding to the eigenvalue λ [28, p. 411]. Next we recall the definition of the kernel.

Definition 2.5 (Kernel)**.** Let $A : \mathcal{D}(A) \subseteq X \to Y$ be an operator between vector spaces X and Y. We call the space

$$\mathcal{N}(A) = \{x \in \mathcal{D}(A) : Ax = 0\}$$

the *kernel* of the operator A.

The kernel is also known as the *null space* and it is a subspace of the vector space X [35, p. 14][28, p. 166][25, Thm. 2.6-9 (c)]. Therefore, for an eigenvalue $\lambda \in \sigma_p(A)$ the kernel $\mathcal{N}(\lambda - A)$ contains both the linearly independent eigenvectors of A corresponding to λ and their linear combinations. We call the dimension of $\mathcal{N}(\lambda - A)$ the *geometric multiplicity* of λ [28, p. 411].

2.2.1 Self-Adjoint and Skew-Adjoint Operators

Now we embark on studying so called *self-adjoint* and *skew-adjoint* operators. We leave the more general Banach space setting behind and focus mainly on operators defined between Hilbert spaces for the rest of this chapter. With regard to this thesis, self-adjoint and skew-adjoint operators form the two most relevant subclasses of closed operators as we shall see. In this subsection we define these subclasses and lay out many of the useful properties that self-adjoint and skew-adjoint operators possess. However, we first have to define the *adjoint* of an operator. In the following definition a *densely defined* operator $A : \mathcal{D}(A) \subseteq X \to X$ simply means that the domain $\mathcal{D}(A)$ is dense in X.

Definition 2.6 (Adjoint)**.** Let $A : \mathcal{D}(A) \subseteq X \to Y$ be a densely defined closed operator between Hilbert spaces X and Y. The *adjoint* of A is an operator $A^* : \mathcal{D}(A^*) \subseteq Y \to X$ defined in the following way. An element $y \in Y$ is in $\mathcal{D}(A^*)$ if and only if there exists some $z_y \in X$ (necessarily unique, cf. infra) satisfying

$$\langle Ax, y\rangle_Y = \langle x, z_y\rangle_X \qquad \text{for all } x \in \mathcal{D}(A).$$

For all such elements $y \in \mathcal{D}(A^*)$ we define $A^*y = z_y$.

The above definition can be found in [35, p. 242], for example. Assuming A to be densely defined is the key reason why the adjoint A^* is well-defined, i.e., for every $y \in \mathcal{D}(A^*)$ there exists one and only one $z_y \in X$ [35, p. 242]. Note that for a bounded operator $A \in \mathcal{L}(X, Y)$, the existence of $z_y \in X$ follows from the famous Riez Representation Theorem [28, p. 527][25, Thm. 3.8-1]. The following proposition shows that adjoints are also closed operators.

Proposition 2.1. *Suppose $A : \mathcal{D}(A) \subseteq X \to Y$ is a densely defined closed operator between Hilbert spaces X and Y. Then the adjoint A^* of A is a closed operator.*

Proof. Let $(y_k)_{k\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{D}(A^*)$ such that $y_k \to y$ and $A^*y_k \to x$ for some $y \in Y$ and $x \in X$ as $k \to \infty$. We need to show that $y \in \mathcal{D}(A^*)$ and $A^*y = x$. To this end, let $z \in \mathcal{D}(A)$ be arbitrary. As the inner products $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$ are continuous [35, Thm. 6.3][25, Lemma 3.2-2], we obtain

$$\begin{aligned}\langle Az, y\rangle_Y &= \langle Az, \lim_{k\to\infty} y_k\rangle_Y = \lim_{k\to\infty}\langle Az, y_k\rangle_Y \\ &= \lim_{k\to\infty}\langle z, A^*y_k\rangle_X = \langle z, \lim_{k\to\infty} A^*y_k\rangle_X = \langle z, x\rangle_X.\end{aligned}$$

We can now deduce by Definition 2.6 that $y \in \mathcal{D}(A^*)$ and $A^*y = x$. Therefore, A^* is a closed operator by Definition 2.2. □

The adjoint of a densely defined operator has plenty of interesting properties. Whereas

Section IV.11 in [35] studies adjoints of general densely defined operators, Section 5.22 in [28] and Section 3.9 in [25] cover adjoints of bounded operators. For our purposes, the results in the following lemma are sufficient as we shall see in Chapters 4 and 5.

Lemma 2.2. *Suppose* $A : \mathcal{D}(A) \subseteq X \to Y$ *is a densely defined operator between Hilbert spaces* X *and* Y *and* $B \in \mathcal{L}(X,Y)$ *is a bounded operator. Then*

(i) The adjoint B^* *of* B *exists and is a bounded operator with* $\|B^*\| = \|B\|$*, and*

(ii) $(A+B)^*y = A^*y + B^*y$ *for all* $y \in \mathcal{D}((A+B)^*) = \mathcal{D}(A^*)$*.*

Proof. The first property is proven in [35, Eq. IV.11-3], [28, Thm 3.9-2] and [25, Thm. 5.22.2]. To prove the second property, let $x \in \mathcal{D}(A)$ and $y \in \mathcal{D}(A^*)$ be arbitrary. Note that bounded perturbations do not affect the domains, i.e., $\mathcal{D}(A) = \mathcal{D}(A+B)$. As $\mathcal{D}(A^*)$ is a subset of $Y = \mathcal{D}(B^*)$, we have

$$\begin{aligned}\langle (A+B)x, y\rangle_Y &= \langle Ax, y\rangle_Y + \langle Bx, y\rangle_Y \\ &= \langle x, A^*y\rangle_X + \langle x, B^*y\rangle_X = \langle x, A^*y + B^*y\rangle_X.\end{aligned}$$

This implies by Definition 2.6 that $\mathcal{D}(A^*) \subseteq \mathcal{D}((A+B)^*)$ and $(A+B)^*y = A^*y + B^*y$ for all $y \in \mathcal{D}(A^*)$. We still need to show that $\mathcal{D}((A+B)^*) \subseteq \mathcal{D}(A^*)$. To this end, let $y \in \mathcal{D}((A+B)^*)$ be arbitrary. Now by Definition 2.6

$$\langle (A+B)x, y\rangle_Y = \langle x, (A+B)^*y\rangle_X \quad \Longleftrightarrow \quad \langle Ax, y\rangle_Y = \langle x, ((A+B)^* - B^*)y\rangle_X.$$

The above implies that $y \in \mathcal{D}(A^*)$ and therefore $\mathcal{D}((A+B)^*) \subseteq \mathcal{D}(A^*)$. □

We are now ready to define what we mean by self-adjoint and skew-adjoint operators. As the names suggest, a self-adjoint operator is simply its own adjoint and a skew-adjoint operator is the opposite of its adjoint. The following definition is an adaptation from its counterparts in [35, p. 380], [28, Def. 5.23-2] and [25, Def. 10.2-5].

Definition 2.7 (Self-adjoint, skew-adjoint)**.** Let $A : \mathcal{D}(A) \subseteq X \to X$ be a densely defined operator on a Hilbert space X. We say A is *self-adjoint* if

$$Ax = A^*x \tag{2.2}$$

for all $x \in \mathcal{D}(A) = \mathcal{D}(A^*)$ If instead $Ax = -A^*x$ for all $x \in \mathcal{D}(A) = \mathcal{D}(A^*)$, then we say A is a *skew-adjoint* operator.

A concise way to write that an operator A is self-adjoint or skew-adjoint is simply $A = A^*$ or $A = -A^*$, respectively. If we have a self-adjoint operator A, then $-iA$ is a skew-adjoint operator. Vice versa, we obtain a self-adjoint operator from a skew-adjoint operator by multiplying it by i. As a consequence, the spectra $\sigma(A)$ and $\sigma(-iA)$ for a self-adjoint operator A are rotationally symmetrical with respect to the origin. In other words, we obtain the one from the other by rotating the complex plane by 90 degrees clockwise or anti-clockwise. The same applies also for the spectra $\sigma(A)$ and $\sigma(iA)$ if A is a skew-adjoint operator. For this reason, we will only focus on self-adjoint operators for the rest of this section. Results for skew-adjoint operators are often relatively easy to deduce from their counterparts concerning self-adjoint operators.

Sometimes we can show that an operator A satisfies (2.2) for all $x \in \mathcal{D}(A) \subseteq \mathcal{D}(A^*)$. In this case we call A a *symmetric* operator [35, p. 345][28, p. 493][25, Def. 10.2-3]. Note that *skew-symmetric* operators behave similarly. However, being a symmetric operator is not sufficient to being a self-adjoint operator as Definition 2.7 requires that the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ coincide. It is actually sufficient to only show that $\mathcal{D}(A^*)$ is a subset of $\mathcal{D}(A)$ because for a symmetric operator the other inclusion follows immediately from Definition 2.6. The former inclusion can often be difficult to show. Fortunately, there are many useful results to bypass this difficulty. For example, we can use Lemma 2.2 (ii) to show that $A + B$ is a self-adjoint operator if the operators A and B are self-adjoint for a bounded operator B. Another useful result following from [35, Thm. VI.8.1] states that a symmetric operator A is actually self-adjoint if parts of the real axis are contained in the resolvent set of A.

Lemma 2.3. *If $A : \mathcal{D}(A) \subseteq X \to X$ is a symmetric operator in a Hilbert space X and the intersection $\rho(A) \cap \mathbb{R}$ is not empty, then A is self-adjoint.*

In fact, it turns out that the entire spectrum of a self-adjoint operator is always located on the real axis [35, pp. 380–381][24, p. 271]. In addition to this useful fact, we also obtain an estimate for the operator norm of the resolvent. We conclude this subsection with the following proposition stating the aforementioned properties.

Proposition 2.4 ([24, Thm. V.3.16])**.** *Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is a self-adjoint operator on a Hilbert space X. Then $\sigma(A) \subset \mathbb{R}$ and*

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{|\operatorname{Im} \lambda|} \qquad \text{for all } \lambda \notin \mathbb{R}.$$

For a skew-adjoint operator, the above proposition implies that the spectrum is located on the imaginary axis instead. We also need to change the imaginary part of λ to the real part in the resolvent estimate. For the most curious of readers, theory of self-adjoint operators can be found in [35, Sec. VI.8] and [24, Ch. 3 & 4], for example.

2.2.2 Self-Adjoint Operators with Compact Resolvents

As our last endeavour in this chapter, we turn our attention to self-adjoint operators with a very special property. Namely, we assume that the resolvents of an operator are *compact* operators. These type of operators arise frequently enough in problems involving differential equations [35, p. 362], and therefore we dedicate this subsection for the study of self-adjoint operators with compact resolvents. We first recall the definition of a compact operator in a more general setting involving normed vector spaces.

Definition 2.8 (Compact operator)**.** Let X and Y be normed vector spaces and assume $K \in \mathcal{L}(X, Y)$ is a bounded operator. We say K is a *compact* operator if for every bounded sequence $(x_k)_{k\in\mathbb{N}} \subseteq X$ the sequence $(Kx_k)_{k\in\mathbb{N}} \subseteq Y$ has a convergent subsequence.

The above definition is an adaptation from [35, pp. 293–294], [28, Thm. 5.24.5 (a) & (d)] and [25, Thm. 8.1-3]. Compact operators have many nice algebraic and spectral properties. For example, compact operators form a subspace of bounded operators and the composition of two bounded operators is a compact operator if either of the operators is a compact operator [35, Thm. V.7.1 & V.7.2]. Furthermore, the spectrum of a compact operator contains at most a countable set of discrete points that can only accumulate at the origin, and all the non-zero points in the spectrum are eigenvalues [35, Thm. V.7.10].

Definition 2.9 (Compact resolvents)**.** Let $A : \mathcal{D}(A) \subseteq X \to X$ be an operator on a Banach space X with a non-empty resolvent set $\rho(A)$. We say A has *compact resolvents* if $(\lambda - A)^{-1} \in \mathcal{L}(X)$ is a compact operator for one $\lambda \in \rho(A)$.

A similar definition can be found in [21, Def. II.4.24]. The reason why a single compact resolvent implies that all resolvents are compact follows quite nicely from the *resolvent equation* [35, Eq. (V.2-3)][25, Eq. (7.4.1)][21, Eq. (IV.1.2)]. Suppose $\mu \in \rho(A)$ such that the resolvent $(\mu - A)^{-1}$ is a compact operator. Now for any $\lambda \in \rho(A)$ we have

$$(\lambda - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1} + (\mu - A)^{-1}.$$

Therefore, $(\lambda - A)^{-1}$ is a compact operator by the aforementioned algebraic properties of compact operators.

We are now ready for the most important result of this subsection. If we assume that a self-adjoint operator has compact resolvents, then we can express the operator as its spectral representation as is implied by [35, Thm. VI.5.1].

Theorem 2.5. *Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is a self-adjoint operator with compact resolvents on a Hilbert space X. Then, there exists an orthonormal basis $\{\varphi_k\}_{k\in\mathbb{N}}$ of X and a set of real numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ such that we can express A in the form*

$$Ax = \sum_{k=1}^{\infty} \lambda_k \langle x, \varphi_k \rangle_X \varphi_k, \qquad x \in \mathcal{D}(A) \tag{2.3}$$

where

$$\mathcal{D}(A) = \left\{ x \in X : \sum_{k=1}^{\infty} |\lambda_k|^2 |\langle x, \varphi_k \rangle_X|^2 < \infty \right\}.$$

In the above theorem the real numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ have the property that $|\lambda_k| \to \infty$ as $k \to \infty$ and φ_k is the normalized eigenvector of A corresponding to the eigenvalue λ_k for all $k \in \mathbb{N}$ [35, pp. 361–362]. Indeed, we have that $\lambda \in \sigma(A)$ if and only if $\lambda \in \{\lambda_k\}_{k\in\mathbb{N}}$.

Theorem 2.5 becomes really useful for us as we need the concept of *the square root* of an operator in Chapter 4. More generally, we can define an arbitrary *fractional power* of an operator [27, Ch. 5]. However, these concepts only make sense if the operator is *positive*, yielding the following definition. Note that for a self-adjoint operator $A : \mathcal{D}(A) \subseteq X \to X$ we have for all $x \in \mathcal{D}(A)$ that

$$\langle Ax, x \rangle_X = \langle x, Ax \rangle_X = \overline{\langle Ax, x \rangle_X}.$$

In particular, this implies that $\langle Ax, x \rangle_X \in \mathbb{R}$ for all $x \in \mathcal{D}(A)$.

Definition 2.10 (Positive operator)**.** Let $A : \mathcal{D}(A) \subseteq X \to X$ be a self-adjoint operator on a Hilbert space X. If for all $x \in \mathcal{D}(A)$

$$\langle Ax, x \rangle_X \geq 0,$$

then we say that A is a *positive* operator.

Positive operators are defined similarly in [35, p. 348], [28, Def. 5.23.7] and [25, p. 470]. As a consequence, the eigenvalues of a positive operator A are non-negative. To see this, let $\lambda \in \sigma_p(A)$ be an arbitrary eigenvalue of A with the corresponding normalized eigenvector $\varphi \in \mathcal{D}(A)$. Now

$$0 \leq \langle A\varphi, \varphi \rangle_X = \langle \lambda\varphi, \varphi \rangle_X = \lambda.$$

Therefore, we can utilize the results of Theorem 2.5 and obtain the following definition.

Definition 2.11 (Fractional power)**.** Let $A : \mathcal{D}(A) \subseteq X \to X$ be a positive operator with compact resolvents on a Hilbert space X expressed in the form (2.3). For a fixed real number $\alpha > 0$, we define the *fractional power* A^α as the operator $A^\alpha : \mathcal{D}(A^\alpha) \subseteq X \to X$ such that

$$A^\alpha x = \sum_{k=1}^{\infty} \lambda_k^\alpha \langle x, \varphi_k \rangle_X \varphi_k, \qquad x \in \mathcal{D}(A^\alpha) \tag{2.4}$$

where

$$\mathcal{D}(A^\alpha) = \left\{ x \in X : \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |\langle x, \varphi_k \rangle_X|^2 < \infty \right\}. \tag{2.5}$$

It follows from results in [27, Ch. 5] that the fractional powers are also positive operators. Note that we can obtain the the square root of A, i.e., the operator $A^{1/2}$ simply by substituting $\alpha = 1/2$ into (2.4). We conclude this subsection by showing that the composition of fractional powers behaves as expected.

Lemma 2.6. *Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is a positive operator and $\alpha > 0$ and $\beta > 0$. Then*

$$A^\alpha A^\beta x = A^{\alpha+\beta} x \qquad \textit{for all } x \in \mathcal{D}(A^{\alpha+\beta}).$$

Proof. Let $x \in \mathcal{D}(A^{\alpha+\beta})$ be arbitrary. We first show that $x \in \mathcal{D}(A^\beta)$. Recall that the values λ_k diverge to infinity [35, p. 361]. In particular, there exists some $N \in \mathbb{N}$ such that $\lambda_k > 1$ whenever $k \geq N$. As $x \in \mathcal{D}(A^{\alpha+\beta})$, we have that

$$\sum_{k=1}^{\infty} \lambda_k^{2\beta} |\langle x, \varphi_k \rangle_X|^2 \leq \sum_{k=1}^{N-1} \lambda_k^{2\beta} |\langle x, \varphi_k \rangle_X|^2 + \sum_{k=N}^{\infty} \lambda_k^{2(\beta+\alpha)} |\langle x, \varphi_k \rangle_X|^2 < \infty$$

and thus $x \in \mathcal{D}(A^\beta)$ by Definition 2.11. As $\{\lambda_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of X, we observe that

$$\begin{aligned}
\sum_{k=1}^{\infty} \lambda_k^{2\alpha} \left|\langle A^\beta x, \varphi_k \rangle_X\right|^2 &= \sum_{k=1}^{\infty} \lambda_k^{2\alpha} \left| \left\langle \sum_{j=1}^{\infty} \lambda_j^\beta \langle x, \varphi_j \rangle_X \varphi_j, \varphi_k \right\rangle_X \right|^2 \\
&= \sum_{k=1}^{\infty} \lambda_k^{2\alpha} \left| \sum_{j=1}^{\infty} \lambda_j^\beta \langle x, \varphi_j \rangle_X \langle \varphi_j, \varphi_k \rangle_X \right|^2 \\
&= \sum_{k=1}^{\infty} \lambda_k^{2\alpha} \left| \lambda_k^\beta \langle x, \varphi_k \rangle_X \right|^2 = \sum_{k=1}^{\infty} \lambda_k^{2(\alpha+\beta)} |\langle x, \varphi_k \rangle_X|^2.
\end{aligned}$$

The above expression is finite because $x \in \mathcal{D}(A^{\alpha+\beta})$. Therefore, $A^\beta x \in \mathcal{D}(A^\alpha)$ by (2.5). Now a direct computation yields

$$A^\alpha A^\beta x = \sum_{k=1}^{\infty} \lambda_k^\alpha \left\langle \sum_{j=1}^{\infty} \lambda_j^\beta \langle x, \varphi_j \rangle_X \varphi_j, \varphi_k \right\rangle_X \varphi_k = \sum_{k=1}^{\infty} \lambda_k^{\alpha+\beta} \langle x, \varphi_k \rangle_X \varphi_k = A^{\alpha+\beta} x$$

for all $x \in \mathcal{D}(A^{\alpha+\beta})$. □

We have now obtained sufficient knowledge on closed operators and their spectral properties. In the next chapter we focus on the main theoretical tool for analyzing the polynomial stability of abstract wave equations later in Chapter 4. As we shall see, closed operators are at the heart of this concept.

3. STRONGLY CONTINUOUS SEMIGROUPS

In this chapter we introduce a very important and versatile theoretical tool called a *strongly continuous semigroup*. Strongly continuous semigroups form the backbone of the analysis in the subsequent chapters as they provide us with a way to study the existence and stability of the solutions to so-called *abstract Cauchy problems*. After defining strongly continuous semigroups, we study how we can generate them in Section 3.1. We learn that for each strongly continuous semigroup there is a unique closed operator called the *generator*. We also examine under which assumptions a given closed operator generates a strongly continuous semigroup. In Section 3.2 we deepen our knowledge by focusing on so-called strongly continuous *contraction* semigroups. We shall see that strongly continuous contraction semiroups are prevalent in Chapters 4 and 5. In Section 3.3 we define abstract Cauchy problems and discuss the aforementioned connection between them and strongly continuous semigroups. Lastly, we study the stability of strongly continuous semigroups in Section 3.4.

We start by defining a strongly continuous semigroup. The following definition is a combination of [21, Def. I.5.1] and [21, Prop. I.5.3].

Definition 3.1 (Strongly continuous semigroup). Let $(T(t))_{t\geq 0}$ be a family of bounded operators on a Banach space X. We say $(T(t))_{t\geq 0}$ is a *strongly continuous semigroup* if it satisfies the *semigroup properties*

$$T(t+s) = T(t)T(s) \quad \text{for all } t,s\geq 0 \qquad \text{and} \qquad T(0) = I, \tag{3.1}$$

and for all $x \in X$ the mapping $t \mapsto T(t)$ is *strongly continuous* at $t = 0$, i.e.,

$$\lim_{t\to 0^+} \|T(t)x - x\|_X = 0 \qquad \text{for all } x \in X. \tag{3.2}$$

Note that if $(T(t))_{t\geq 0}$ is a strongly continuous semigroup then we have $T(t) \in \mathcal{L}(X)$ for all $t \geq 0$ by assumption. Furthermore, the semigroup properties in (3.1) immediately imply that the operators $T(t)$ and $T(s)$ commute for all $t, s \geq 0$. The book [21] by Engel & Nagel contains multiple insightful examples of strongly continuous semigroups. For our purposes, the most important examples of strongly continuous semigroups are abstract wave equations, and we study them later in Chapter 4. Therefore, we leave out any examples of strongly continuous semigroups for the time being. For concrete examples of strongly continuous semigroups, see Chapter VI in [21].

The semigroup properties in (3.1) also allow us to interpret $T(t)x$ as the state of some dynamical system that has evolved for t time units from an initial state $x \in X$. This interpretation is depicted in Figure 3.1. Suppose we have an initial state $x \in X$ of a dynamical system and $t, s \geq 0$. Starting from the initial state $T(0)x = x$, after $t + s$ time units the system is in the state $T(t + s)x$. Had we observed this system at time t, we would have observed the system in the state $T(t)x$. After letting the system evolve for s more time units, from our perspective the system would be in the state $T(s)T(t)x$. Our intuition about dynamical systems insists that the states $T(t+s)x$ and $T(s)T(t)x$ should be equal for all initial states $x \in X$.

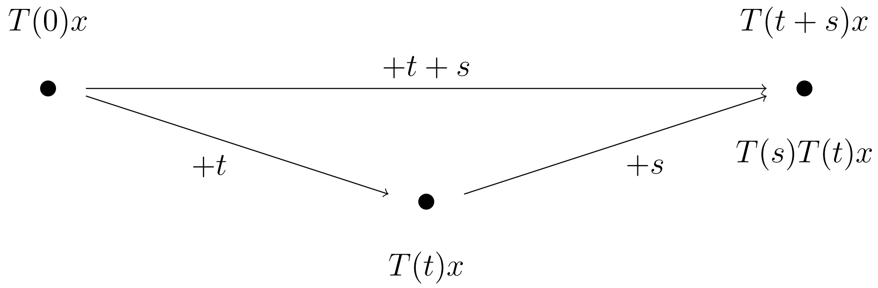

Figure 3.1. *The semigroup properties in* (3.1) *model the evolution of a dynamical system over time. Any initial state $x \in X$ of the system should evolve regardless of when we observe the system, implying $T(t + s)x = T(s)T(t)x$ for all $x \in X$.*

Despite modeling the evolution of a dynamical system, the semigroup properties in (3.1) alone are not that useful for us. If we combine these properties with the property of strong continuity in (3.2), a rich theory of strongly continuous semigroups arises [21, p. 37]. In the following sections we study only the main aspects of this theory. The theory of strongly continuous semigroups is covered in greater detail in [21], [22, Sec 7.4] and [31, Ch. 11], for example.

3.1 Generating a Strongly Continuous Semigroup

Suppose $(T(t))_{t\geq 0}$ is a strongly continuous semigroup. Recall from Definition 3.1, the mapping $t \mapsto T(t)x$ is continuous at $t = 0$ for all $x \in X$. As a consequence of strong continuity, the mapping $t \mapsto T(t)x$ is continuous for all $t \geq 0$ [21, Prop. I.5.3]. Furthermore, if the mapping $t \mapsto T(t)x$ is right-differentiable at $t = 0$ then it is actually differentiable for all $t \geq 0$ [21, Lem. II.1.1]. These properties motivate us to define the following operator which is at the very core of every strongly continuous semigroup.

Definition 3.2 (Inifinitesimal generator)**.** Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup. The *(infinitesimal) generator* of $(T(t))_{t\geq 0}$ is the operator $A : \mathcal{D}(A) \subseteq X \to X$

defined by

$$Ax = \lim_{h \to 0^+} \tfrac{1}{h}(T(h)x - x), \qquad x \in \mathcal{D}(A)$$

where

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{h \to 0^+} \tfrac{1}{h}(T(h)x - x) \text{ exists} \right\}.$$

The above definition is used in [22, p. 437] and [31, Def. 11.10]. Definition II.1.2 in [21] characterizes the domain $\mathcal{D}(A)$ equivalently as all the elements $x \in X$ for which the mapping $t \mapsto T(t)x$ is differentiable. Here differentiability means that for all $t \geq 0$ there exists some $y_t \in X$ such that

$$\lim_{h \to 0} \left\| \frac{T(t+h)x - T(t)x}{h} - y_t \right\|_X = 0.$$

We denote this limit simply by $\frac{d}{dt}T(t)x = y_t$.

Given a strongly continuous semigroup, we can now construct its generator using Definition 3.2. However, we ultimately want to be able to deduce whether a given operator generates a strongly continuous semigroup. Therefore, we first study many useful properties of generators in the following subsection. We can then refer to these properties as necessary conditions later in Subsection 3.1.2 where we investigate which closed operators indeed generate strongly continuous semigroups.

3.1.1 Properties of the Generator

We start with the most immediate properties and then work our way toward some useful spectral properties of generators. The following lemma gathers important properties of generators, implied by Definition 3.2.

Lemma 3.1 ([21, Lem. II.1.3 (i)–(iii)][22, Thm. 7.4.1][31, Lem. 11.11 1.–3.]). *Suppose the operator $A : \mathcal{D}(A) \subseteq X \to X$ is the generator of a strongly continuous semigroup $(T(t)_{t \geq 0})$. Then the following properties hold.*

(i) The operator A is a linear operator.

(ii) If $x \in \mathcal{D}(A)$, then for all $t \geq 0$ we have that $T(t)x \in \mathcal{D}(A)$ and

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x.$$

(iii) If $x \in X$, then for all $t \geq 0$ we have that

$$\int_0^t T(s)x \, ds \in \mathcal{D}(A). \tag{3.3}$$

The integral in (3.3) needs to be understood as a Bochner integral [6, pp. 6–15]. Its definition is similar to Lebesgue integrals but it includes also functions with values from arbitrary Banach spaces. We need Lemma 3.1 especially when we discuss the connection between strongly continuous semigroups and the solutions to abstract Cauchy problems in Section 3.3.

Note that it is not obvious whether the domain of the generator A in Definition 3.2 contains anything else but the zero element. However, the generator of a strongly continuous semigroup would not be a particularly useful concept if the domain was not sufficiently large. Fortunately, the next theorem reveals a fundamental truth about generators.

Theorem 3.2 ([21, Thm II.1.4]). *The generator of a strongly continuous semigroup is a unique, closed and densely defined operator. Moreover, the generator determines the semigroup uniquely.*

By determining the semigroup uniquely, we mean that if two strongly continuous semigroups have the same generator then these semigroups must be identical [21, pp. 51–52]. The generator itself is unique because the limit in Definition 3.2 is unique.

Next we focus on some fundamental spectral properties of generators. We begin with the following proposition which states that all strongly continuous semigroups are bounded by an exponential term. We shall see shortly that this bounding term sets a certain boundary for the spectrum of generators.

Proposition 3.3 ([21, Prop. I.5.5]). *Suppose $(T(t))_{t\geq 0}$ is a strongly continuous semigroup. Then there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \tag{3.4}$$

for all $t \geq 0$.

The above proposition raises an interesting question. Namely, for how small $\omega \in \mathbb{R}$ can we still find $M \geq 1$ such that (3.4) holds? The answer to this question leads us to the *growth bound* of a strongly continuous semigroup $(T(t))_{\geq 0}$. Suppose the infimum of all possible $\omega \in \mathbb{R}$ satisfying (3.4) exists and denote it by ω_0, i.e.,

$$\omega_0 = \inf\{\omega \in \mathbb{R} : \omega \text{ satisfies (3.4) for some } M_\omega \geq 1\}. \tag{3.5}$$

We call ω_0 the *growth bound* of $(T(t))_{t\geq 0}$ [21, Def I.5.6]. We are now ready to formulate the next proposition which states the spectral properties of any generator. The result is a combination of [21, Thm. II.1.10 (ii)–(iii)] and [21, Cor. II.1.11].

Proposition 3.4. *Suppose $(T(t))_{t\geq 0}$ is a strongly continuous semigroup with the generator $A : \mathcal{D}(A) \subseteq X \to X$. Moreover, suppose $\omega \in \mathbb{R}$ and $M \geq 1$ are such that* (3.4) *holds for all $t \geq 0$. If $\lambda \in \mathbb{C}$ satisfies $\operatorname{Re}\lambda > \omega$, then $\lambda \in \rho(A)$ and*

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n}$$

for all $n \in \mathbb{N}$.

As an immediate consequence of the above proposition, the spectrum of any generator is always contained in a left half-plane of $\mathbb{C}$. Therefore, we can define the *spectral bound* of a generator A as [21, Def. II.1.12]

$$s(A) = \sup_{\lambda \in \sigma(A)} \operatorname{Re}\lambda.$$

In particular, any $\lambda \in \mathbb{C}$ belongs to the resolvent set of A if $\operatorname{Re}\lambda > \omega_0$ where ω_0 is the growth bound (3.5). This implies immediately that $s(A) \leq \omega_0$. In fact, the spectral bound of a generator and the growth bound satisfy [21, Cor. II.1.13]

$$-\infty \leq s(A) \leq \omega_0 < \infty.$$

Note that in (3.4) we implicitly allow the infimum to be formally equal to $-\infty$. If $\omega_0 = -\infty$, then we find $M_\omega \geq 1$ for all $\omega \in \mathbb{R}$ such that (3.4) holds.

3.1.2 Generation Theorems

We have now a good overview of the most fundamental properties which any generator of a strongly continuous semigroup must possess. Indeed, if an operator lacks any of the properties mentioned in Subsection 3.1.1 then the operator does not generate a strongly continuous semigroup. The next theorem is a well-known theoretical result regarding the generation of a strongly continuous semigroup.

Theorem 3.5 ([21, Thm. II.3.8 (a), (c)][31, Thm. 11.17]). *Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is a linear operator on a Banach space X and suppose $\omega \in \mathbb{R}$ and $M \geq 1$ are constants. Then the operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ satisfying*

$$\|T(t)\| \leq Me^{\omega t}, \qquad \text{for all } t \geq 0$$

if and only if A is a closed and densely defined operator such that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ we have $\lambda \in \rho(A)$ and

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n} \tag{3.6}$$

for all $n \in \mathbb{N}$.

The above theorem is known in the literature as the Hille–Yosida Theorem [31, p. 401]. Despite its theoretical importance, Theorem 3.5 rarely gets used in practical situations because showing the required resolvent estimate in (3.6) can often be too difficult if not even impossible [21, p. 78][31, p. 405]. Therefore, we wish for results the premises of which are easier to verify. Fortunately, standard perturbation theory can help us if we already know that an operator generates a strongly continuous semigroup.

Theorem 3.6 ([21, Thm. III.1.3]). *Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ satisfying*

$$\|T(t)\| \leq Me^{\omega t} \qquad \text{for all } t \geq 0$$

where $\omega \in \mathbb{R}$ and $M \geq 1$ are some constants implied by Proposition 3.3. If $B \in \mathcal{L}(X)$ is a bounded operator, then the operator $C : \mathcal{D}(C) \subseteq X \to X$ defined by

$$Cx = (A + B)x = Ax + Bx, \qquad x \in \mathcal{D}(C) = \mathcal{D}(A)$$

generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ satisfying

$$\|S(t)\| \leq Me^{(\omega+M\|B\|)t}$$

for all $t \geq 0$.

The above theorem implies that any bounded perturbation of a generator results in an operator that also generates a strongly continuous semigroup. Theorem 3.6 becomes especially useful in situations where we can express a given operator as a bounded perturbation of a simpler operator. By simpler we mean that we either already know that the operator generates a strongly continuous semigroup or it is easier to show that the operator is the generator of a strongly continuous semigroup. For more perturbation theory of strongly continuous semigroups, see Chapter III in [21] and Chapter 9 in [24].

3.2 Strongly Continuous Contraction Semigroups

So far we have studied the generation of strongly continuous semigroups in a very general sense. In this section we focus on a particular class of strongly continuous semigroups called strongly continuous *contraction* semigroups. It turns out that the generators of such semigroups have an additional property of being *maximally dissipative* operators. As we shall see, maximally dissipative generators arise naturally in Chapter 4 where we study abstract wave equations.

Recall from Proposition 3.3 that for every strongly continuous semigroup $(T(t))_{t\geq 0}$ there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t}$$

holds for all $t \geq 0$. If the exponential bound in (3.4) holds for some $\omega \leq 0$, then $(T(t))_{t\geq 0}$ belongs to the following special class of strongly continuous semigroups.

Definition 3.3 (Bounded, contraction)**.** Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup. We call $(T(t))_{t\geq 0}$ *bounded* if there exists some $M \geq 1$ such that

$$\|T(t)\| \leq M \qquad \text{for all } t \geq 0. \tag{3.7}$$

If (3.7) holds with $M = 1$, then we say $(T(t))_{t\geq 0}$ is a *contraction* semigroup.

The above definition is used in [21, Def. I.5.6], [22, p. 437] and [31, p. 405]. If $(T(t))_{t\geq 0}$ is a strongly continuous contractive semigroup with the generator $A : \mathcal{D}(A) \subseteq X \to X$ then by Definition 3.3 for an arbitrary $x \in X$ we have that

$$\|T(t)x\|_X \leq \|x\|_X, \qquad \text{for all } t \geq 0.$$

Moreover, the resolvent estimate (3.6) in the Hille–Yosida Theorem implies that if $\lambda > 0$ then $\lambda \in \rho(A)$ and the generator A satisfies

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda}.$$

The above inequality implies particularly that $\|(\lambda - A)x\|_X \geq \lambda\|x\|_X$ for all $x \in \mathcal{D}(A)$. Operators with these properties are called *maximally dissipative* operators [21, Def. II.3.13].

Definition 3.4 (Maximally dissipative operator)**.** Let $A : \mathcal{D}(A) \subseteq X \to X$ be an operator on a Banach space X. We say A is a *dissipative* operator if

$$\|(\lambda - A)x\|_X \geq \lambda\|x\|_X$$

for all $\lambda > 0$ and $x \in \mathcal{D}(A)$. If in addition $\mathcal{R}(\lambda - A) = X$ for all $\lambda > 0$, then we say A is a *maximally dissipative* operator.

We are now ready for the second well-known theoretical result regarding the generation of a strongly continuous contraction semigroup.

Theorem 3.7 ([21, Thm. II.3.15])**.** *Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is a densely defined and dissipative operator on a Banach space X. Then the closure $\overline{A}$ of A generates a contraction semigroup if and only if the range $\mathcal{R}(\lambda - A)$ is dense in X for some $\lambda > 0$.*

Theorem 3.7 is known in the literature as the Lumer–Phillips Theorem [31, p. 405]. It is a reformulation of the Hille–Yosida Theorem and it emphasizes the dense range condition, ensuring that the closure of a densely defined and dissipative operator generates a strongly continuous contraction semigroup [21, p. 83]. Note in Theorem 3.5 that in the case of a strongly continuous semigroup with the generator $A : \mathcal{D}(A) \subseteq X \to X$ we only have to investigate the resolvent estimate (3.6) for $n = 1$ which holds if and only if A is a dissipative operator. Proposition 3.4 then implies that $\mathcal{R}(\lambda - A) = X$ for all $\lambda > 0$. Therefore, the closure $\overline{A}$ of the operator A generates a strongly continuous contraction semigroup by Theorem 3.7.

We conclude this section by considering strongly continuous contraction semigroups on Hilbert spaces instead of general Banach spaces. In this setting, one particularly neat consequence of the Lumer–Phillips Theorem is that all skew-adjoint operators generate strongly continuous contraction semigroups [21, Thm. II.3.24][31, Ex. 11.23]. To prove this useful fact, we first look at the following lemma which characterizes all dissipative operators on Hilbert spaces.

Lemma 3.8. *Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is an operator on a Hilbert space X. Then the operator A is dissipative if and only if*

$$\operatorname{Re}\langle Ax, x\rangle_X \leq 0$$

for all elements $x \in \mathcal{D}(A)$.

Proof. Suppose first that A is a dissipative operator, and let $\lambda > 0$ and $x \in \mathcal{D}(A)$ be arbitrary. By Definition 3.4, we have that

$$(\lambda \|x\|_X)^2 \leq \|(\lambda - A)x\|_X^2 = \lambda^2 \|x\|_X^2 + \|Ax\|_X^2 - 2\lambda \operatorname{Re}\langle Ax, x\rangle_X.$$

The above inequality implies $\operatorname{Re}\langle Ax, x\rangle_X \leq \frac{\|Ax\|_X}{2\lambda}$. As this holds for all $\lambda > 0$, we can deduce that $\operatorname{Re}\langle Ax, x\rangle_X \leq 0$.

Then suppose $\operatorname{Re}\langle Ax, x\rangle_X \leq 0$ for all $x \in \mathcal{D}(A)$. Now for all $\lambda > 0$ we have that

$$\|(\lambda - A)x\|_X^2 = \lambda^2 \|x\|_X^2 + \|Ax\|_X^2 \underbrace{-2\lambda \operatorname{Re}\langle Ax, x\rangle_X}_{\geq 0} \geq \lambda^2 \|x\|_X^2.$$

This implies $\|(\lambda - A)x\|_X \geq \lambda \|x\|_X$ for all $x \in \mathcal{D}(A)$. Therefore, we can conclude by Definition 3.4 that A is a dissipative operator. ☐

The next useful corollary of the Lumer–Phillips Theorem follows quite nicely from the results we have obtained in Subsection 2.2.1.

Corollary 3.9. *Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is a skew-adjoint operator on a Hilbert space X. Then A generates a strongly continuous contraction semigroup.*

Proof. The results in Subsection 2.2.1 imply that A is a closed and densely defined operator. Therefore, the closure $\overline{A}$ of A is simply the operator A itself. We can also deduce from Proposition 2.4 that $\sigma(A) \subset i\mathbb{R}$.

Let $x \in \mathcal{D}(A)$ be arbitrary. As A is a skew-adjoint operator, we have by Definition 2.7 that

$$\langle Ax, x\rangle_X = \langle x, -Ax\rangle_X,$$

and we immediately obtain $\mathrm{Re}\langle Ax, x\rangle_X = 0 \leq 0$. Therefore, A is a dissipative operator by Lemma 3.8. Note that for any $\lambda > 0$ we have that $\lambda \in \rho(A)$ and consequently $\mathcal{R}(\lambda - A) = X$ by Definition 2.4. In particular, the range $\mathcal{R}(\lambda - A)$ is dense in X, and the claim follows from Theorem 3.7. $\square$

The above corollary is particularly useful when paired with Theorem 3.6. In other words, every bounded perturbation of a skew-adjoint operator generates a strongly continuous semigroup. However, Theorem 3.6 does not guarantee that the boundedly perturbed generator generates another strongly continuous contraction semigroup. Fortunately, there are results which guarantee the generation of a strongly continuous contraction semigroup under relatively mild assumptions on the perturbing operator [21, Sec. III.2]. Instead of pursuing a complete picture on this matter here, we only present the next practical corollary of Theorem 3.6.

Corollary 3.10. *Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is the generator of a strongly continuous contraction semigroup on a Hilbert space X and $B \in \mathcal{L}(U, X)$ is a bounded operator where U is another Hilbert space. Then, the operator $A - BB^* : \mathcal{D}(A) \subseteq X \to X$ generates a strongly continuous contraction semigroup.*

Proof. As $B \in \mathcal{L}(U, X)$ is a bounded operator, Theorem 3.6 implies that $A - BB^*$ generates a strongly continuous semigroup. By Proposition 3.4 the range must now satisfy $\mathcal{R}(\lambda - (A - BB^*)) = X$ for some $\lambda > 0$. The claim follows from Theorem 3.7 if $A - BB^*$ is a dissipative operator.

Recall from Theorem 3.7 that A is a dissipative and densely defined operator as it generates a strongly continuous contraction semigroup. By Lemma 3.8, now $\mathrm{Re}\langle Ax, x\rangle_X \leq 0$ for all $x \in \mathcal{D}(A)$. Therefore, for an arbitrary $x \in \mathcal{D}(A)$ we obtain

$$\begin{aligned}
\mathrm{Re}\langle (A - BB^*)x, x\rangle_X &= \mathrm{Re}\langle Ax, x\rangle_X - \mathrm{Re}\langle BB^*x, x\rangle_X \\
&\leq -\mathrm{Re}\langle B^*x, B^*x\rangle_U \\
&= -\|B^*x\|_U^2 \leq 0,
\end{aligned}$$

implying that $A - BB^*$ is also a dissipative operator. $\square$

The premises of the above corollary are somewhat specific for a good reason. We shall see in Chapter 4 that operators of the form $A - BB^*$, where $A : \mathcal{D}(A) \subseteq X \to X$ is a skew-adjoint operator and $B \in \mathcal{L}(U, X)$ is a bounded operator between two Hilbert spaces U and X, are intrinsic to abstract wave equations.

3.3 Abstract Cauchy Problems

In this section we discuss the important connection between strongly continuous semigroups and the existence of solutions to a ubiquitous class of differential equations called *abstract Cauchy problems*. We emphasize that this connection is the main reason why strongly continuous semigroups are so versatile tools in a plethora of applications involving partial differential equations [21, p. 151][22, p. 436].

We start by defining the abstract Cauchy problem and its associated state space.

Definition 3.5 (Abstract Cauchy problem)**.** Let $A : \mathcal{D}(A) \subseteq X \to X$ be a linear operator on a Banach space X. We call the differential equation

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0 \in X \tag{3.8}$$

defined for all $t > 0$ an *abstract Cauchy problem*. We call the space X the *state space*.

For clarity, differentiating with respect to t in the above theorem simply refers to the familiar limit of the difference quotient

$$\frac{d}{dt}x(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

provided that the above limit exists with respect to the norm on X. As we study solving the abstract Cauchy problem in (3.8), we first define the notion of *well-posedness* in the sense of Hadamard [31, p. 8][21, Thm. II.6.7 (d)].

Definition 3.6 (Well-posedness)**.** We say the abstract Cauchy problem in (3.8) is *well-posed* if for all initial conditions $x_0 \in X$

(i) a solution $x(t)$ exists,

(ii) the solution $x(t)$ is unique and

(iii) for every $\tau > 0$ there exists some $M > 0$ such that

$$\|x(t)\|_X \leq M\|x_0\|_X, \qquad t \in [0, \tau].$$

Our main goal in this section is to examine under which assumptions the abstract Cauchy problem in (3.8) is well-posed in the sense of the above definition. To answer this important question, we first need a definition of a solution to (3.8). In the following definition we define which functions $x(t)$ we classify as the *classical solutions* to (3.8).

Definition 3.7 (Classical solution)**.** Let $x : [0, \infty) \to X$ be a continuously differentiable function in the state space X such that $x(t) \in \mathcal{D}(A)$ and $x(t)$ satisfies the abstract Cauchy problem in (3.8) for all $t \geq 0$. We call x a *classical solution* of (3.8).

The classical solutions of (3.8) are defined similarly in [21, Def. II.6.1 (ii)] and [31, p. 399]. We are now ready for the important connection between abstract Cauchy problems and strongly continuous semigroups. Namely, if the operator A in the abstract Cauchy problem in (3.8) generates a strongly continuous semigroup and the initial condition x_0 is in the domain of A then we obtain the classical solutions to (3.8) with help of the strongly continuous semigroup [21, Prop. II.6.2]. We state this result in the following theorem.

Theorem 3.11. *Suppose the operator $A : \mathcal{D}(A) \subseteq X \to X$ in an abstract Cauchy problem* (3.8) *generates a strongly continuous semigroup $(T(t))_{t\geq 0}$. If the initial condition satisfies $x_0 \in \mathcal{D}(A)$, then the function $x : (0, \infty) \to X$ defined by*

$$x(t) = T(t)x_0$$

is the unique classical solution to (3.8)*.*

Proof. By Definition 3.1 we observe immediately that

$$x(0) = T(0)x_0 = Ix_0 = x_0.$$

As $x_0 \in \mathcal{D}(A)$, Lemma 3.1 (i) implies that

$$\frac{d}{dt}x(t) = \frac{d}{dt}T(t)x_0 = AT(t)x_0 = Ax(t).$$

Combining these results, we have shown that the function x is a classical solution to (3.8) by Definition 3.7. To show the uniqueness of the solution, suppose $y(t)$ is another solution to (3.8). We obtain the following abstract Cauchy problem

$$\frac{d}{dt}(y(t) - T(t)x_0) = A(y(t) - T(t)x_0), \qquad y(0) - T(0)x_0 = 0.$$

Evidently, the zero function is a solution to the above equation. As the zero function is actually the unique solution [21, p. 146], we have that $y(t) = T(t)x_0$. □

Note that strongly continuous semigroups yield the classical solutions if the initial condition x_0 is in the domain of the operator A in (3.8). However, we already know from (iii) in Lemma 3.1 that if $x_0 \in X$ then $\int_0^t T(s)x_0 \, ds \in \mathcal{D}(A)$ for all $t \geq 0$. Therefore, modifying Definition 3.7 suitably allows us to still obtain solutions, albeit more relaxed than classical solutions, to (3.8) with strongly continuous semigroups. We can especially relax the requirement for continuously differentiable solutions which leads us to the concept of *mild solutions* [21, Def. II.6.3].

Definition 3.8 (Mild solution)**.** Let $x : [0, \infty) \to X$ be a continuous function such that for all $t \geq 0$ we have $\int_0^t x(s)\, ds \in \mathcal{D}(A)$ and

$$x(t) = A \int_0^t x(s)\, ds + x_0$$

where x_0 is the initial condition in (3.8). We call x a *mild solution* of (3.8).

It is not too difficult to see that all classical solutions to (3.8) are also mild solutions. If the operator A in an abstract Cauchy problem (3.8) generates a strongly continuous semigroup $(T(t))_{t\geq 0}$, then for all initial conditions $x_0 \in X$ the function $x : [0, \infty) \to X$ defined by $x(t) = T(t)x_0$ is the unique mild solution to (3.8) [21, Prop. II.6.4]. Recall from Proposition 3.3 there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)x_0\|_X \leq \|T(t)\| \|x_0\|_X \leq Me^{\omega t}\|x_0\|_X$$

for all $t \geq 0$. Now for all $\tau > 0$ the term $Me^{\omega t}$ has an upper bound on the closed and bounded interval $[0, \tau]$. Therefore, the abstract Cauchy problem in (3.8) is well-posed in the sense of Definition 3.6 if the operator A generates a strongly continuous semigroup.

3.4 Stability Analysis

We know now that we can express the mild solutions $x(t)$ of particular abstract Cauchy problems with the help of strongly continuous semigroups. Knowing that the mild solutions exist is one thing, but knowing how they behave *asymptotically*, i.e., for large $t > 0$ is also very relevant information [21, p. 295]. Therefore, in this section we turn our attention to the *strong stability* of strongly continuous semigroups. Conceptually, the strong stability of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X means that for all $x \in X$ the mild solutions $T(t)x$ decay to zero as t grows indefinitely [21, p. 296]. Based on the way the mild solutions $T(t)x$ decay, we can distinguish different stability properties of the strongly continuous semigroup. In this section we focus particularly on so-called *exponential* and *polynomial* stability.

We start by defining the concept of strong stability.

Definition 3.9 (Strong stability)**.** Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup. We say $(T(t))_{t\geq 0}$ is *strongly stable* if

$$\lim_{t\to\infty} \|T(t)x\|_X = 0$$

for all $x \in X$.

The above definition is adapted from Definition V.1.1 (c) in [21]. The same definition also lists other similar notions of stability with different operator topologies. However, these notions are not equivalent as there are strongly continuous semigroups that are stable in one operator topology but unstable in another [21, p. 297].

In the following two subsections we introduce the concepts of exponential and polynomial stability of strongly continuous semigroups. Both exponentially and polynomially stable strongly continuous semigroups are strongly stable, as we shall see. The two aforementioned stability types differ in the way the mild solutions of the corresponding abstract Cauchy problem decay.

3.4.1 Exponential Stability

Exponential stability of strongly continuous semigroups is studied in Chapter V of [21], for example. In this subsection we first define what we mean by exponential stability and then investigate conditions which guarantee that a strongly continuous semigroup is exponentially stable. We start with the following definition.

Definition 3.10 (Exponential stability)**.** Let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup. We say $(T(t)_{t\geq 0}$ is *exponentially stable* if there exist some constants $M \geq 1$ and $\omega > 0$ such that

$$\|T(t)x\|_X \leq Me^{-\omega t}\|x\|_X, \qquad x \in X \tag{3.9}$$

for all $t \geq 0$.

Note that (3.9) in the above definition implies immediately that $\|T(t)\| \leq Me^{-\omega t}$, and as a consequence $\|T(t)\| \to 0$ as $t \to \infty$. On the other hand, the operator norm $\|T(t)\|$ decays if and only if it decays at an exponential rate [21, Prop. V.1.2]. It is also evident that an exponentially stable strongly continuous semigroup is strongly stable.

Therefore, a strongly continuous semigroup is exponentially stable if and only if all of its mild solutions decay at a *uniform* rate. We can also deduce from Proposition 3.3 that we obtain exponential stability if and only if the growth bound in (3.5) satisfies $\omega_0 < 0$. We are now ready for the first important result regarding exponential stability. The following theorem draws the connection between exponential stability of a strongly continuous semigroup and spectral properties of its generator.

Theorem 3.12 ([21, Thm. V.1.11])**.** *Suppose $(T(t))_{t\geq 0}$ is a strongly continuous semigroup with the generator $A : \mathcal{D}(A) \subseteq X \to X$ on a Hilbert space X. Then $(T(t))_{t\geq 0}$ is exponentially stable if and only if $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \subset \rho(A)$ and the resolvent satisfies*

$$\sup_{\operatorname{Re}\lambda > 0} \|(\lambda - A)^{-1}\| < \infty.$$

The above result is known in the literature as the Gearhart–Prüss Theorem. If we are working with a bounded strongly continuous semigroup, then the following useful corollary of Theorem 3.12 arises.

Corollary 3.13 ([20, Thm 1.1]). *Suppose $(T(t))_{t\geq 0}$ is a bounded strongly continuous semigroup with the generator $A : \mathcal{D}(A) \subseteq X \to X$ on a Hilbert space X. Then, $(T(t))_{t\geq 0}$ is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and the resolvent satisfies*

$$\sup_{s\in\mathbb{R}} \|(is - A)^{-1}\| < \infty.$$

For bounded strongly continuous semigroups, being exponentially stable is therefore equivalent to having bounded resolvents of the generator on the imaginary axis. As a final note before moving on to polynomial stability, Corollary 3.13 gives us a convenient way to show that a bounded strongly continuous semigroup is not exponentially stable. Indeed, if we find a sequence of real numbers $(s_k)_{k\in\mathbb{N}}$ and a corresponding sequence of elements $(x_k)_{k\in\mathbb{N}} \subseteq X$ such that $\|x_k\|_X = 1$ for all $k \in \mathbb{N}$ and

$$\|(is_k - A)x_k\|_X \to 0 \qquad \text{as} \qquad k \to \infty,$$

then the resolvent is not bounded on the imaginary axis. To see this, we define a new sequence $(y_k)_{k\in\mathbb{N}} \subseteq X$ such that $y_k = \frac{(is_k - A)x_k}{\|(is_k - A)x_k\|}$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be arbitrary. Now $y_k \in \mathcal{R}(is_k - A)$ and $\|y_k\| = 1$. However, the resolvent satisfies

$$\|(is_k - A)^{-1}y_k\| = \left\| \frac{(is_k - A)^{-1}(is_k - A)x_k}{\|(is_k - A)x_k\|} \right\| = \frac{1}{\|(is_k - A)x_k\|}.$$

The right side of the above equation diverges to infinity as k grows indefinitely.

3.4.2 Polynomial Stability

Recall from the preceding subsection that exponential stability arises in situations where all mild solutions decay at a uniform and therefore exponential rate. However, there are strongly stable strongly continuous semigroups with mild solutions that decay at a strictly slower rate [17, 16]. The asymptotic behaviour of such strongly continuous semigroups has been an active topic of research since the turn of the millennium [11, 16]. One particularly interesting aspect of this research introduces different notions of stability that fall between strong stability and exponential stability. In this subsection we take a brief look at *polynomial* stability and present a fairly recent result that links polynomial stability to the rate at which the resolvent of the generator grows on the imaginary axis.

Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is the generator of a strongly continuous semigroup

$(T(t))_{t\geq 0}$ on a Banach space X and assume $0 \in \rho(A)$. If $x \in \mathcal{D}(A)$, then

$$\|T(t)x\|_X = \|T(t)A^{-1}Ax\|_X \leq \|T(t)A^{-1}\|\|Ax\|_X. \tag{3.10}$$

Therefore, the decay rate of all classical solutions is bounded by the decay rate of the operator norm $\|T(t)A^{-1}\|$. Although (3.10) holds only for classical solutions, it turns out that if $(T(t))_{t\geq 0}$ is a bounded strongly continuous semigroup then the decay of $\|T(t)A^{-1}\|$ actually implies strong stability. We prove this important fact in the following lemma.

Lemma 3.14. *Suppose $A : \mathcal{D}(A) \subseteq X \to X$ is the generator of a bounded strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X and assume $0 \in \rho(A)$. If*

$$\lim_{t\to\infty} \|T(t)A^{-1}\| = 0,$$

then $(T(t))_{t\geq 0}$ is strongly stable.

Proof. We need to show that $\lim_{t\to\infty}\|T(t)x\|_X = 0$ for all $x \in X$. To this end, let $x \in X$ and $\varepsilon > 0$ be arbitrary. Let $M \geq 1$ be a constant such that $\|T(t)\| \leq M$ for all $t \geq 0$. Now for all $x' \in \mathcal{D}(A)$

$$\begin{aligned}\|T(t)x\|_X &\leq \|T(t)x - T(t)x'\|_X + \|T(t)x'\|_X \\ &\leq \|T(t)\|\|x - x'\|_X + \|T(t)x'\|_X \\ &\leq M\|x - x'\|_X + \|T(t)x'\|_X.\end{aligned}$$

As $\mathcal{D}(A)$ is dense in X by Theorem 3.2, there exists $x' \in \mathcal{D}(A)$ such that $\|x - x'\| < \frac{\varepsilon}{2M}$. In addition, as $\lim_{t\to\infty}\|T(t)A^{-1}\| = 0$ by assumption the inequality in (3.10) implies that we can make $\|T(t)x'\|_X$ arbitrarily small for sufficiently large t. In particular, there exists some $t_\varepsilon > 0$ such that $\|T(t)x'\| < \frac{\varepsilon}{2}$ whenever $t \geq t_\varepsilon$. Combining these results, we obtain

$$\|T(t)x\|_X \leq M\|x - x'\|_X + \|T(t)x'\|_X < M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $t \geq t_\varepsilon$. Therefore, $(T(t))_{t\geq 0}$ is strongly stable by Definition 3.9. □

The above lemma motivates the investigation of the conditions for a strongly continuous semigroup $(T(t))_{t\geq 0}$ and its generator $A : \mathcal{D}(A) \subseteq X \to X$ that guarantee the decay of the norm $\|T(t)A^{-1}\|$ in (3.10). Fortunately, we know by [10, Prop. 7.3] that if $(T(t))_{t\geq 0}$ is a bounded strongly continuous semigroup on a Hilbert space and $i\mathbb{R} \subset \rho(A)$ then indeed $\lim_{t\to\infty}\|T(t)A^{-1}\| = 0$. As pointed out in [13], the decay rate of $\|T(t)A^{-1}\|$ in (3.10) can be arbitrarily slow and it is related to the growth of the resolvent of A on the imaginary axis. With this in mind, we obtain the following definition.

Definition 3.11 (Polynomial stability)**.** Let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup with the generator $A : \mathcal{D}(A) \subseteq X \to X$ on a Hilbert space X, and assume the generator A satisfies $i\mathbb{R} \subset \rho(A)$. We say $(T(t))_{t\geq 0}$ is *polynomially stable* if there exist some constants $M > 0$, $\alpha > 0$ and $t_0 > 0$ such that

$$\|T(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}} \qquad \text{for all } t \geq t_0.$$

Note that by (3.10) polynomial stability implies a uniform polynomial decay rate only for all classical solutions. Although Lemma 3.14 guarantees that polynomially stable strongly continuous semigroups are strongly stable, without any additional assumptions we cannot deduce a uniform decay rate for all mild solutions. This is the key difference between exponential and polynomial stability. As a uniform decay rate for all mild solutions is equivalent to having exponential stability [21, Thm. V.1.2], polynomial stability indeed falls between strong stability and exponential stability.

Next we present a quintessential result from the 2010 paper by Borichev & Tomilov [13]. The following theorem is an adaptation of [13, Thm. 2.4], and we can see its resemblance with Corollary 3.13, i.e., the reformulation of the Gearhart–Prüss Theorem for a bounded strongly continuous semigroup on a Hilbert space.

Theorem 3.15 ([13, Thm. 2.4 (i) & (iv)]). *Suppose $(T(t))_{t\geq 0}$ is a bounded strongly continuous semigroup with the generator $A : \mathcal{D}(A) \subseteq X \to X$ on a Hilbert space X, and assume the generator A satisfies $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\alpha > 0$ there exist constants $M_1 > 0$ and $t_0 > 0$ such that*

$$\|T(t)A^{-1}\| \leq \frac{M_1}{t^{1/\alpha}}, \qquad t \geq t_0$$

if and only if there exist constants $M_2 > 0$ and $s_0 > 0$ such that

$$\|(is - A)^{-1}\| \leq M_2|s|^{\alpha}, \qquad |s| \geq s_0.$$

Again, the growth rate of the resolvents of the generator on the imaginary axis alone determines the stability properties of the associated strongly continuous semigroup. We observe from Corollary 3.13 and Theorem 3.15 that the slower the resolvent grows on the imaginary axis, the faster the classical solutions must decay. In particular, we obtain Corollary 3.13 formally from Theorem 3.15 by taking the limit as $\alpha \to 0$. If $\alpha \to \infty$, then we no longer have polynomial stability.

We conclude this subsection by noting that polynomial stability actually belongs to a broader stability class called *semi-uniform* stability [16]. We can define semi-uniform stability for a bounded strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X with the generator $A : \mathcal{D}(A) \subseteq X \to X$ that has no spectrum on the imaginary axis by simply requiring that

$$\lim_{t\to\infty} \|T(t)A^{-1}\| = 0$$

holds [16][10, Prop. 7.3]. Although fascinating, semi-uniform stability is beyond the scope of this thesis. For more information on semi-uniform stability, see the survey [16] by Chill et al. and the references therein.

4. ABSTRACT WAVE EQUATIONS

In this chapter we study a large class of linear dynamical systems called *abstract wave equations*. These type of systems arise in many useful modeling applications, such as modeling the propagation of waves through different media [18, 19, 32], the vibrations of structural beams [9, 30] and many types of coupled systems [33, 1, 3], for example. As with all dynamical systems, we are interested in the well-posedness of abstract wave equations. We learn in Section 4.1 that for every abstract wave equation there exists a corresponding strongly continuous semigroup. Combining this fact with the knowledge from Chapter 3, we move on to Section 4.2 and study the stability of abstract wave equations. Ultimately, we wish to obtain sufficient conditions under which a given abstract wave equation is polynomially stable. To conclude this chapter, we dedicate Section 4.3 for showcasing a few examples of abstract wave equations from the literature.

We start by defining the aforementioned class of dynamical systems. The next definition is an adaption from [8, Eq. (1.1)] fit for our purposes.

Definition 4.1 (Abstract wave equation)**.** Let $L : \mathcal{D}(L) \subseteq H \to H$ be a positive operator with compact resolvents on a Hilbert space H and let $D \in \mathcal{L}(U, H)$ be a bounded operator from another Hilbert space U to H. We call a differential equation of the form

$$\begin{cases} \frac{d^2}{dt^2}w(t) + Lw(t) + DD^* \frac{d}{dt}w(t) = 0, & t > 0, \\ w(0) = w_0 \in \mathcal{D}(L^{1/2}) \quad \text{and} \quad \frac{d}{dt}w(0) = w_1 \in H \end{cases} \tag{4.1}$$

an *abstract wave equation*.

The operator DD^* in the above definition models the *damping* of the system [21, p. 367]. If the system has no damping, i.e., $D = 0$, we refer to the system simply as being undamped. Note that we take the possible boundary conditions, such as Dirichlet boundary conditions, into consideration by incorporating them in the domain of the operator L. For concrete examples of abstract wave equations, see Section 4.3.

4.1 Well-Posedness of the System

Our goal in this section is to show that all abstract wave equations are well-posed in the sense of Definition 3.6. As discussed in Section 3.3, an abstract Cauchy problem is well-posed if the operator A in Definition 3.5 generates a strongly continuous semigroup. We shall see that we can express the abstract wave equation in (4.1) as an abstract Cauchy problem (3.8) and construct the generator of a strongly continuous semigroup in the process. Therefore, we can characterize the mild solutions to (4.1) with the help of strongly continuous semigroups.

Note that the abstract wave equation in (4.1) is a second-order differential equation, but the abstract Cauchy problem (3.8) is a first-order differential equation. Fortunately, we can reduce (4.1) to a first-order problem by expressing it formally as

$$\frac{d}{dt}\begin{pmatrix} w(t) \\ \frac{d}{dt}w(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -L & -DD^* \end{pmatrix} \begin{pmatrix} w(t) \\ \frac{d}{dt}w(t) \end{pmatrix} \tag{4.2}$$

where $(w(t), \frac{d}{dt}w(t))^T \in H \times H$ for all $t \geq 0$ and $(w(0), \frac{d}{dt}w(0))^T = (w_0, w_1)^T$. As for the state space X, choosing it suitably we can relate the norm of the mild solutions with the total energy of the system. If we choose the state space $X = \mathcal{D}(L^{1/2}) \times H$ and endow it with the inner product defined by

$$\langle x, y\rangle_X = \langle L^{1/2}x_1, L^{1/2}y_1\rangle_H + \langle x_2, y_2\rangle_H$$

for all $x = (x_1, x_2)^T \in X$ and $y = (y_1, y_2)^T \in X$, then X is actually a Hilbert space and we can define the total energy of the system in terms of the norm on X induced by the inner product [21, pp. 374–375][36, p. 225].

Definition 4.2 (Total energy). Let $L : \mathcal{D}(L) \subseteq H \to H$ be a positive operator with compact resolvents on a Hilbert space H and let $w : [0, \infty) \to H$ be a mild solution to the abstract wave equation in (4.1) for all $t \geq 0$. We call the quantity

$$\mathcal{E}_w(t) = \frac{1}{2}\left\| \begin{pmatrix} w(t) \\ \frac{d}{dt}w(t) \end{pmatrix} \right\|_X^2 = \frac{1}{2}\|L^{1/2}w(t)\|_H^2 + \frac{1}{2}\|\tfrac{d}{dt}w(t)\|_H^2, \qquad t \geq 0 \tag{4.3}$$

the *total energy* of the mild solution w at time t.

The total energy in (4.3) is actually a constant if the system has no damping. We can obtain this result from (4.1) by standard multiplier techniques and setting $D = 0$.

Section 2.1 in [8] and Section 3.2 in [15] suggest that we can express the formal operator on the right side of (4.2) as the operator $A - BB^*$ with suitable choices for A and B. If we define the operator $A : \mathcal{D}(A) \subseteq X \to X$ where $\mathcal{D}(A) = \mathcal{D}(L) \times \mathcal{D}(L^{1/2})$ by

$$Ax = \begin{pmatrix} x_2 \\ -Lx_1 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{D}(A) \tag{4.4}$$

and $B \in \mathcal{L}(U, X)$ by

$$Bu = \begin{pmatrix} 0 \\ Du \end{pmatrix}, \qquad u \in U, \tag{4.5}$$

then we can reformulate the abstract Cauchy problem in (4.2) as

$$\frac{d}{dt} \begin{pmatrix} w(t) \\ \frac{d}{dt} w(t) \end{pmatrix} = (A - BB^*) \begin{pmatrix} w(t) \\ \frac{d}{dt} w(t) \end{pmatrix}. \tag{4.6}$$

In the following lemma we prove that the operator B defined in (4.5) is a bounded operator. We also obtain an explicit formula for the adjoint of B.

Lemma 4.1. *Suppose $D \in \mathcal{L}(U, H)$ is a bounded operator. Then the operator B defined in* (4.5) *is a bounded operator and its adjoint $B^* \in \mathcal{L}(X, U)$ is of the form*

$$B^* x = D^* x_2, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X. \tag{4.7}$$

Proof. Let $u \in U$ be arbitrary. As $D \in \mathcal{L}(U, H)$ is a bounded operator, we obtain

$$\|Bu\|_H^2 = \|L^{1/2} 0\|_H^2 + \|Du\|_H^2 \leq \|D\|^2 \|u\|_U^2.$$

The above inequality implies that $B \in \mathcal{L}(U, X)$ is a bounded operator. By Lemma 2.2 (i), the adjoint $B^* \in \mathcal{L}(X, U)$ is also a bounded operator. To show that (4.7) holds, let $u \in U$ and $x = (x_1, x_2)^T \in X$ be arbitrary. As $x_2 \in H$, we have

$$\langle Bu, x \rangle_X = \left\langle \begin{pmatrix} 0 \\ Du \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle_X = \langle L^{1/2} 0, L^{1/2} x_1 \rangle_H + \langle Du, x_2 \rangle_H = \langle u, D^* x_2 \rangle_U.$$

Therefore, the operator defined in (4.7) is the adjoint of B by Definition 2.6. □

We are now ready to prove the main result of this section. Namely, for all abstract wave equations the operator $A - BB^*$ in (4.6) actually generates a strongly continuous contraction semigroup. Note that introducing the operators A and B allows us to prove this result with a simple yet effective perturbation argument from Corollary 3.10.

Theorem 4.2. *Suppose A and B are the operators defined in* (4.4) *and* (4.5)*, respectively. Then, the operator $A - BB^* : \mathcal{D}(A) \subseteq X \to X$ generates a strongly continuous contraction semigroup.*

Proof. By Corollary 3.10, it is sufficient to show that A generates a strongly continuous contraction semigroup. We first observe that for arbitrary $x = (x_1, x_2)^T \in \mathcal{D}(A)$ and

$y = (y_1, y_2)^T \in \mathcal{D}(A)$ the self-adjointness of $L^{1/2}$ implies that

$$\begin{aligned}\langle Ax, y\rangle_X &= \left\langle \begin{pmatrix} x_2 \\ -Lx_1 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_X \\ &= \langle L^{1/2}x_2, L^{1/2}y_1\rangle_H + \langle -Lx_1, y_2\rangle_H \\ &= \langle x_2, Ly_1\rangle_H + \langle L^{1/2}x_1, -L^{1/2}y_2\rangle_H \\ &= \langle L^{1/2}x_1, L^{1/2}(-y_2)\rangle_H + \langle x_2, -(-Ly_1)\rangle_H \\ &= \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, -\begin{pmatrix} y_2 \\ -Ly_1 \end{pmatrix} \right\rangle_X = \langle x, -Ay\rangle_X.\end{aligned}$$

Therefore, A is a skew-symmetric operator by Definition 2.7.

We also know that A is invertible [21, p. 375]. In particular, the intersection $\rho(A) \cap i\mathbb{R}$ is not empty. As discussed in Subsection 2.2.1, we can now deduce that A is in fact a skew-adjoint operator, and the claim follows from Corollary 3.9. □

As a consequence of Theorem 4.2, all abstract wave equations are well-posed in the sense of Definition 3.6. In particular, suppose $(T(t))_{t\geq 0}$ is the strongly continuous contraction semigroup generated by the operator $A - BB^*$ in (4.6). Then for a given initial condition $x = (w_0, w_1)^T$ where $w_0 \in \mathcal{D}(L^{1/2})$ and $w_1 \in H$ we have that $x \in X$ and thus we obtain the corresponding mild solution $w(t)$ to the abstract wave equation in (4.1) from the first component of $T(t)x$.

Before moving on to the next section where we study the polynomial stability of abstract wave equations more closely, we prove a result that we need later in Subsection 5.3.3. The following lemma draws a connection between the point spectra $\sigma_p(A)$ and $\sigma_p(L)$.

Lemma 4.3. *Suppose L is the operator in the abstract wave equation* (4.1) *and suppose operator A is defined as in* (4.4)*. Then*

$$\sigma_p(A) = \{\pm i\sqrt{\lambda} : \lambda \in \sigma_p(L)\}.$$

Proof. First, let $\mu \in \sigma_p(A)$ be an arbitrary eigenvalue of A corresponding to an eigenvector $\psi = (\psi_1, \psi_2)^T \in X$. Now

$$A\psi = \mu\psi \qquad \Longleftrightarrow \qquad \begin{pmatrix} \psi_2 \\ -L\psi_1 \end{pmatrix} = \begin{pmatrix} \mu\psi_1 \\ \mu\psi_2 \end{pmatrix} \qquad \Longleftrightarrow \qquad L\psi_1 = -\mu^2\psi_1,$$

implying that $-\mu^2$ is an eigenvalue of L when $\psi_1 \neq 0$. If $\psi_1 = 0$, then also $\psi_2 = 0$ which contradicts the fact that ψ must be non-zero as an eigenvector. Therefore, $\mu = \pm i\sqrt{\lambda}$ where $\lambda \in \sigma_p(L)$ is an eigenvalue of L and $\mu \in \{\pm i\sqrt{\lambda} : \lambda \in \sigma_p(L)\}$.

Then, let $\lambda \in \sigma_p(L)$ be an arbitrary eigenvalue of L corresponding to a non-zero eigenvector $\varphi \in H$. Now

$$A \begin{pmatrix} \varphi \\ \pm i\sqrt{\lambda}\varphi \end{pmatrix} = \begin{pmatrix} \pm i\sqrt{\lambda}\varphi \\ -L\varphi \end{pmatrix} = \begin{pmatrix} \pm i\sqrt{\lambda}\varphi \\ -\lambda\varphi \end{pmatrix} = \pm i\sqrt{\lambda} \begin{pmatrix} \varphi \\ \pm i\sqrt{\lambda}\varphi \end{pmatrix},$$

implying that $\pm i\sqrt{\lambda}$ is an eigenvalue of A. Therefore, $\pm i\sqrt{\lambda} \in \sigma_p(A)$. □

Having established the well-posedness, we are ready to investigate the polynomial stability of abstract wave equations.

4.2 Sufficient Conditions for Polynomial Stability

We have now reached the pinnacle of our literary survey. In this section we obtain sufficient conditions for the operators L and D in an abstract wave equation (4.1) so that the abstract wave equation is polynomially stable. The main result of both this section and this thesis is Theorem 4.5. This theorem is at the very core of Chapter 5 where we investigate the stability of a concrete example of an abstract wave equation.

We start by defining the polynomial stability of a solution to the abstract wave equation in (4.1). Let A and B be the operators defined as in (4.4) and (4.5), respectively. We know by Theorem 4.2 that for every abstract wave equation there is a strongly continuous contraction semigroup $(T(t))_{t\geq 0}$ with the generator $A - BB^*$. Recall from Definition 3.11 that if $i\mathbb{R} \subset \rho(A - BB^*)$ then $(T(t))_{t\geq 0}$ is polynomially stable if and only if there exist some constants $M > 0$, $\alpha > 0$ and $t_0 > 0$ such that

$$\|T(t)x\|_X \leq \frac{M}{t^{1/\alpha}} \|(A - BB^*)x\|_X \tag{4.8}$$

for all $x \in \mathcal{D}(A)$ and $t \geq t_0$. Note that the damping of the abstract wave equation (4.1) is implicitly present in the above expression in the operator B. However, we wish to express the right side of (4.8) in a more general form by using only the skew-adjoint operator A. The result in the following lemma makes this reformulation possible.

Lemma 4.4. *Suppose A and B are the operators defined as in* (4.4) *and* (4.5)*, respectively, and assume that $i\mathbb{R} \subset \rho(A - BB^*)$. Then for all $x \in \mathcal{D}(A)$ we have*

$$\frac{\|Ax\|_X}{1 + \|BB^*(A - BB^*)^{-1}\|} \leq \|(A - BB^*)x\|_X \leq (1 + \|BB^*A^{-1}\|)\|Ax\|_X.$$

Proof. Let $x \in \mathcal{D}(A)$ be arbitrary. We first observe that

$$\begin{aligned}\|Ax\|_X &= \|Ax - BB^*x + BB^*x\|_X \\ &\le \|(A-BB^*)x\|_X + \|BB^*(A-BB^*)^{-1}(A-BB^*)x\|_X \\ &\le \|(A-BB^*)x\|_X + \|BB^*(A-BB^*)^{-1}\|\|(A-BB^*)x\|_X \\ &= (1+\|BB^*(A-BB^*)^{-1}\|)\|(A-BB^*)x\|_X.\end{aligned}$$

Then we observe that

$$\begin{aligned}\|(A-BB^*)x\|_X &\le \|Ax\|_X + \|BB^*A^{-1}Ax\|_X \\ &\le \|Ax\|_X + \|BB^*A^{-1}\|\|Ax\|_X \\ &= (1+\|BB^*A^{-1}\|)\|Ax\|_X.\end{aligned}$$

The claim follows directly from these two inequalities. □

By Lemma 4.4, we can express the inequality in (4.8) equivalently as

$$\|T(t)x\|_X^2 \le \frac{M}{t^{2/\alpha}}\|Ax\|_X^2, \qquad t \ge t_0$$

where $x \in \mathcal{D}(A)$, $M > 0$, $\alpha > 0$ and $t_0 > 0$. Now (4.3) suggests that we can express the polynomial stability of abstract wave equations naturally in terms of the total energy $\mathcal{E}_w(t)$. Indeed, if $i\mathbb{R} \subset \rho(A - BB^*)$ then we say that the abstract wave equation in (4.1) is polynomially stable if there exist constants $M > 0$, $\alpha > 0$ and $t_0 > 0$ such that for all initial conditions $w_0 \in \mathcal{D}(L)$ and $w_1 \in \mathcal{D}(L^{1/2})$ the total energy $\mathcal{E}_w(t)$ in (4.3) satisfies

$$\mathcal{E}_w(t) \le \frac{M}{t^{2/\alpha}}(\|Lw_0\|_H^2 + \|L^{1/2}w_1\|_H^2), \qquad t \ge t_0.$$

The above characterization for polynomial stability of (4.1) is consistent with its counterparts in [8, (1.2)] and [15, (3.1)].

Next we define the *wavepackets* of a self-adjoint operator with compact resolvents. The wavepackets play a key role in determining the polynomial stability as we shall see. The following definition is an adaptation from [15, Def. 3.4].

Definition 4.3 (Wavepacket)**.** Let $L : \mathcal{D}(L) \subseteq X \to X$ be a self-adjoint operator on a Hilbert space X with compact resolvents and let $I_s = (s - \delta(s), s + \delta(s))$ for some $s \in \mathbb{R}$ and $\delta(s) > 0$. If we denote $\{\lambda_n\}_{n=1}^{n_{I_s}} = I_s \cap \sigma(L)$, then we define $\mathrm{WP}_{s,\delta(s)}(L)$ as the spectral subspace

$$\mathrm{WP}_{s,\delta(s)}(L) = \{0\} \cup \bigoplus_{i=1}^{n_{I_s}} \mathcal{N}(\lambda_i - L). \tag{4.9}$$

We call $x \in \mathrm{WP}_{s,\delta(s)}(L)$ a *wavepacket* of L.

If we drop the assumption of L having compact resolvents in the above definition, it becomes much more difficult to define the spectral subspace $\mathsf{WP}_{s,\delta(s)}(L)$. Recall from Theorem 2.5 that if L is a self-adjoint operator with compact resolvents then the spectrum of L consists of only discrete eigenvalues with finite multiplicity on the real axis. Therefore, the wavepackets in (4.9) are always finite linear combinations of the eigenvectors associated with the eigenvalues of L in the interval I_s. As a special case, it is possible to have that $I_s \cap \sigma(L) = \emptyset$, and this corresponds to having $\mathsf{WP}_{s,\delta(s)}(L) = \{0\}$.

We are finally ready to state the main result of this section and of this entire thesis. The next theorem is an adaptation from [15, Thm. 3.9] combined with [15, Rem. 3.7].

Theorem 4.5. *Suppose L and D are the operators in an abstract wave equation* (4.1) *with the property that the operators A and B defined as in* (4.4) *and* (4.5)*, respectively, satisfy $i\mathbb{R} \subset \rho(A - BB^*)$. If there exist bounded functions $\gamma, \delta : (0,\infty) \to (0,\infty)$ and some constants $\alpha > 0$ and $K > 0$ such that*

$$\|D^*\varphi\|_U \geq \gamma(s)\|\varphi\|_H, \qquad \varphi \in \mathsf{WP}_{s,\delta(s)}(L^{1/2}), \quad s \geq s_0 > 0 \tag{4.10}$$

and $\gamma(|s|)^2\delta(|s|)^2 \geq K|s|^{-\alpha}$ for all $|s| \geq s_0$, then the abstract wave equation in (4.1) *is polynomially stable. That is, there exists some $M > 0$ and $t_0 > 0$ such that for all initial conditions $w_0 \in \mathcal{D}(L)$ and $w_1 \in \mathcal{D}(L^{1/2})$ the total energy $\mathcal{E}_w(t)$ in* (4.3) *satisfies*

$$\mathcal{E}_w(t) \leq \frac{M}{t^{2/\alpha}}(\|Lw_0\|_H^2 + \|L^{1/2}w_1\|_H^2) \tag{4.11}$$

for all $t \geq t_0$.

Proof. We know immediately by [15, Thm. 3.9] combined with [15, Rem. 3.7] that under the above assumptions we have

$$\|(is - (A - BB^*))^{-1}\| \leq \frac{C_1}{\gamma(|s|)^2\delta(|s|)^2}, \qquad |s| \geq s_0$$

for some $C_1 > 0$. The assumption $\gamma(|s|)^2\delta(|s|)^2 \geq K|s|^{-\alpha}$ for all $|s| \geq s_0$ implies that there exist some $C_2 > 0$ such that

$$\|(is - (A - BB^*))^{-1}\| \leq C_2|s|^\alpha$$

for all $|s| \geq s_0$. Now Theorem 3.15 combined with Definition 3.11 implies that there exists $C_3 > 0$ and $t_0 > 0$ such that the strongly continuous contraction semigroup $(T(t))_{t\geq 0}$ generated by $A - BB^*$ satisfies

$$\|T(t)x\| \leq \frac{C_3}{t^{1/\alpha}}\|(A - BB^*)x\|_X, \qquad t \geq t_0$$

for all $x \in \mathcal{D}(A)$. We obtain by Lemma 4.4 that if $M \geq \frac{1}{2}C_3^2(1 + \|BB^*A^{-1}\|)^2 > 0$ then for all initial conditions of the form $x = (w_0, w_1)^T \in \mathcal{D}(A)$ the total energy $\mathcal{E}_w(t)$ in (4.3) satisfies

$$\begin{aligned}\mathcal{E}_w(t) = \frac{1}{2}\|T(t)x\|_X^2 &\leq \frac{C_3^2}{2t^{2/\alpha}}\|(A - BB^*)x\|_X^2 \\ &\leq \frac{C_3^2}{2t^{2/\alpha}}(1 + \|BB^*A^{-1}\|)^2\|Ax\|_X^2 \\ &\leq \frac{M}{t^{2/\alpha}}(\|Lw_0\|_H^2 + \|L^{1/2}w_1\|_H^2)\end{aligned}$$

for all $t \geq t_0$. □

Similarly to Section 3.4, we obtain the exponential stability for abstract wave equations from Theorem 4.5 as the limiting case when $\alpha \to 0$. This corresponds to the special case where the product of the bounded functions $\gamma(s)$ and $\delta(s)$ is bounded below by a positive constant. As a consequence, we get the following useful corollary.

Corollary 4.6. *Suppose the assumptions of Theorem 4.5 hold and we have the bounded functions $\gamma, \delta : (0, \infty) \to (0, \infty)$ satisfying the inequality in* (4.10) *for some $s_0 > 0$. If for all $|s| \geq s_0$ the product $\gamma(|s|)\delta(|s|)$ is bounded below by a positive constant, then the abstract wave equation in* (4.1) *is exponentially stable. That is, there exist some $M > 0$, $\varepsilon > 0$ and $t_0 > 0$ such that for all initial conditions $w_0 \in \mathcal{D}(L)$ and $w_1 \in \mathcal{D}(L^{1/2})$ the total energy $\mathcal{E}_w(t)$ in* (4.3) *satisfies*

$$\mathcal{E}_w(t) \leq Me^{-\varepsilon t}(\|Lw_0\|_H^2 + \|L^{1/2}w_1\|_H^2) \tag{4.12}$$

for all $t \geq t_0$.

Proof. Let A and B be the operators defined as in (4.4) and (4.5), respectively. We can follow a similar reasoning as in the proof of Theorem 4.5. In doing so we obtain that there exists $K > 0$ such that

$$\|(is - (A - BB^*))^{-1}\| \leq K$$

for all $|s| \geq s_0$. Therefore, Corollary 3.13 implies that there is $C_1 > 0$, $\varepsilon_0 > 0$ and $t_0 > 0$ such that the strongly continuous contraction semigroup $(T(t)_{t\geq 0})$ generated by $A - BB^*$ satisfies

$$\|T(t)x\|_X \leq C_1 e^{-\varepsilon_0 t}\|(A - BB^*)x\|_X$$

for all $x \in \mathcal{D}(A)$. We obtain by Lemma 4.4 that if $M \geq \frac{1}{2}C_1^2(1 + \|BB^*A^{-1}\|)^2 > 0$ and $\varepsilon = 2\varepsilon_0 > 0$ then for all initial conditions of the form $x = (w_0, w_1)^T \in \mathcal{D}(A)$ the total energy $\mathcal{E}_w(t)$ in (4.3) satisfies

$$
\begin{aligned}
\mathcal{E}_w(t) = \frac{1}{2}\|T(t)x\|_X^2 &\leq \frac{1}{2}C_1^2 e^{-2\varepsilon_0 t}\|(A - BB^*)x\|_X^2 \\
&\leq \frac{1}{2}C_1^2(1 + \|BB^*A^{-1}\|)^2 e^{-\varepsilon t}\|Ax\|_X^2 \\
&\leq Me^{-\varepsilon t}(\|Lw_0\|_H^2 + \|L^{1/2}w_1\|_H^2)
\end{aligned}
$$

for all $t \geq t_0$. □

We conclude this section by discussing the optimality of the decay rates we obtain in Theorem 4.5. Note that the polynomial decay rate in (4.11) only tells us how fast the total energy of all classical solutions must decay. However, there are always classical solutions the total energy of which decays at a strictly faster rate [13, Thm. 2.4 (v)]. Therefore, an interesting question arises whether we can make the parameter $\alpha > 0$ in (4.11) any smaller such that (4.11) still holds for all classical solutions.

If the parameter $\alpha > 0$ is as small as it can possibly be, then we say the decay rate obtained from Theorem 4.5 is *sharp*. The polynomial decay rate in Theorem 4.5 follows from finding a polynomial growth bound for the resolvent on the imaginary axis. As a consequence, we can show that a given decay rate $t^{-1/\alpha}$ for $\alpha > 0$ and $t \geq t_0$ is sharp if the growth of the resolvent on the imaginary axis is also bounded below by a multiple of $|s|^\alpha$ for all $|s| \geq s_0$ [15, Prop. 5.3]. The sharpness of the decay rates is studied with more care in [15, Sec. 5], for example. In particular, Theorem 5.5 in [15] gives us a concrete way to deduce if the polynomial decay rate in (4.11) is sharp. However, the polynomial decay rates obtained in Theorem 4.5 are sufficient for our purposes even if the decay rates are not necessarily sharp.

4.3 Examples of Abstract Wave Equations

In this final section before Chapter 5, we turn our attention to a few examples of the abstract wave equation in (4.1). As mentioned at the beginning of this chapter, abstract wave equations turn out to be a useful and versatile class of dynamical systems with many applications. In the following examples we get to see three different dynamical systems from the literature and briefly discuss how we can interpret them as abstract wave equations. We then analyze the last system more carefully in Chapter 5.

Our first example is from the paper by Cox & Zuazua [18]. In their paper, Cox & Zuazua study the decay rate of energy in a damped string.

Example 4.4 ([18, Eq. (1.1)]). Let $\Omega = (0, 1)$ and let $a \in L^\infty(\Omega)$ be a non-negative function such that $a > 0$ on a subinterval of Ω. Consider the displacement u of a string that is fixed at its ends and experiences a viscous damping $2a$. If the string is of unit length, then we can model the aforementioned system with the following initial boundary value problem

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(\xi,t) - \frac{\partial^2}{\partial \xi^2}u(\xi,t) + 2a(\xi)\frac{\partial}{\partial t}u(\xi,t), & \text{in } \Omega \times (0,\infty) \\ u(0,t) = u(1,t) = 0 & \text{for } t \in (0,\infty) \end{cases}$$

with the initial conditions

$$u(\xi,0) = u_0(\xi) \qquad \text{and} \qquad \frac{\partial}{\partial t}u(\xi,0) = u_1(\xi)$$

defined in the domain Ω.

We can express the system in Example 4.4 as an abstract wave equation (4.1). It is not too difficult to see that we can simply choose $H = U = L^2(\Omega)$ with the canonical inner product on $L^2(\Omega)$ and define the operator $L : \mathcal{D}(L) \subseteq H \to H$ as

$$Lf = -\Delta f, \qquad f \in \mathcal{D}(L) = H^2(\Omega) \cap H_0^1(\Omega)$$

and the operator $D \in \mathcal{L}(U,H)$ as

$$(Du)(\xi) = \sqrt{2a(\xi)}u, \qquad \text{for a.e. } \xi \in \Omega.$$

Now the initial conditions should satisfy $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$.

We know that if $a \in L^\infty(\Omega)$ is non-negative on an open interval then the system in Example 4.4 is exponentially stable [18, Sec. 1]. The stability of similar damped wave equations on one-dimensional domains is studied more in [15, Sec. 6.2], for example. If we replace the viscous damping $2a(\xi)$ with a weaker damping, then we can obtain polynomial stability for the system in Example 4.4 [15, Sec. 6.2.2].

As our second example we have the transverse vibrations of a Timoshenko type beam from the paper by Raposo et al. [30]. Timoshenko beams typically model thick beams that can experience shear and rotary inertia [34, p. 5]. Shear forces cause the cross sections of the beam to rotate, which is a feature that distinguishes Timoshenko beams from the classical Euler–Bernoulli beams [34, pp. 10, 5]. For more information on Timoshenko type beams, see the book [34] by Stojanović & Kozić.

Example 4.5 ([30, Eq. (1.5)–(1.7)]). Let $\Omega = (0,L)$ where $L > 0$ is the length of a Timoshenko type beam. We denote by ρ, I_ρ, E, I and K the mass per unit length, the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section and the shear modulus, respectively. Consider the transverse displacement u of the beam and the rotation angle v of the beam. We can now model vibrating beams subjected to two frictional mechanisms with the following system of partial differential equations

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(\xi,t) - \frac{K}{\rho}\frac{\partial}{\partial \xi}(\frac{\partial}{\partial \xi}u(\xi,t) - v(\xi,t)) + \frac{1}{\rho}\frac{\partial}{\partial t}u(\xi,t) = 0, & \text{in } \Omega \times (0,\infty) \\ \frac{\partial^2}{\partial t^2}v(\xi,t) - \frac{EI}{I_\rho}\frac{\partial^2}{\partial \xi^2}v(\xi,t) + \frac{K}{I_\rho}(\frac{\partial}{\partial \xi}u(\xi,t) - v(\xi,t)) + \frac{1}{I_\rho}\frac{\partial}{\partial t}v(\xi,t) = 0, & \text{in } \Omega \times (0,\infty) \\ u(0,t) = u(L,t) = v(0,t) = v(L,t) = 0, & \text{for } t \in (0,\infty) \end{cases}$$

with the initial conditions

$$\begin{cases} u(\xi,0) = u_0(\xi), \\ v(\xi,0) = v_0(\xi), \\ \frac{\partial}{\partial t}u(\xi,0) = u_1(\xi), \text{ and} \\ \frac{\partial}{\partial t}v(\xi,0) = v_1(\xi) \end{cases}$$

defined in the domain Ω.

We can express the system in Example 4.5 as an abstract wave equation (4.1). This time we choose $H = U = L^2(\Omega) \times L^2(\Omega)$ endowed with their canonical inner products. We then define the operator $L : \mathcal{D}(L) \subseteq H \to H$ as

$$Lf = \begin{pmatrix} -\frac{K}{\rho}\frac{\partial}{\partial \xi}(\frac{\partial}{\partial \xi}f_1 - f_2) \\ -\frac{EI}{I_\rho}\frac{\partial^2}{\partial \xi^2}f_2 + \frac{K}{I_\rho}(\frac{\partial}{\partial \xi}f_1 - f_2) \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{D}(L) = (H_2(\Omega) \cap H_0^1(\Omega))^2.$$

and the operator $D \in \mathcal{L}(U,H)$ as

$$Du = \begin{pmatrix} \frac{1}{\sqrt{\rho}}u_1 \\ \frac{1}{\sqrt{I_\rho}}u_2 \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in U.$$

Now the initial conditions should satisfy $u_0, v_0 \in H^2(\Omega) \times H_0^1(\Omega)$ and $u_1, v_1 \in H_0^1(\Omega)$.

We know that the system in Example 4.5 is exponentially stable [30, Thm. 3.1]. An interesting variation of this system is studied in [9]. In their paper, Bassam et al. [9] study the polynomial stability of a Timoshenko beam that is damped only partially on its boundary. However, this type of damping that affects the system on the boundary $\partial\Omega$ does not result in a bounded damping operator D.

Our third and final example comes from the paper by Santos et al. [33] where the authors prove that a particular coupled system of wave equations is polynomially stable.

Example 4.6 ([33, Eq. (1.1)–(1.5)]). Let $\Omega \subseteq \mathbb{R}^n$ with $n \in \mathbb{N}$ be an open and bounded set with sufficiently smooth boundary $\partial\Omega$. Consider the displacements u and v of two elastic membranes subject to an elastic force attracting one membrane to the other with a coefficient $\kappa > 0$. If the membranes are made of the same material and only one membrane experiences a viscous damping, then denoting $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T$ we can model the aforementioned system with the following system of partial differential equations

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(\xi,t) - \Delta u(\xi,t) + \kappa v(\xi,t) + \frac{\partial}{\partial t}u(\xi,t) = 0 & \text{in } \Omega \times (0,\infty) \\ \frac{\partial^2}{\partial t^2}v(\xi,t) - \Delta v(\xi,t) + \kappa u(\xi,t) = 0 & \text{in } \Omega \times (0,\infty) \\ u(\xi,t) = v(\xi,t) = 0 & \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

with initial conditions

$$\begin{cases} u(\xi,0) = u_0(\xi), \\ v(\xi,0) = v_0(\xi), \\ \frac{\partial}{\partial t}u(\xi,0) = u_1(\xi) \quad \text{and} \\ \frac{\partial}{\partial t}v(\xi,0) = v_1(\xi) \end{cases}$$

defined in the domain Ω.

We can express the system in Example 4.6 as an abstract wave equation (4.1). Instead of examining the above system in this section, we dedicate the entirety of Chapter 5 to study the system and its variations in greater detail. We conclude this section by mentioning that with the partial viscous damping $\frac{\partial}{\partial t}u$ present in the system of Example 4.6 the system is only polynomially stable [33, Thm. 3.2]. We see in the following chapter that replacing the partial viscous damping in Example 4.6 with different types of damping can change the stability of the system.

5. A COUPLED SYSTEM OF WAVE EQUATIONS

In this chapter we analyze the stability of a coupled system of wave equations. Our goal here is to investigate the stability of the system defined in Example 4.6 using the tools we have obtained from the preceding chapters. In addition to the partial viscous damping present in Example 4.6, we analyze the system with two more types of damping. We first recast the systems as an abstract wave equation (4.1) in Section 5.1. After that we analyze the eigenvalues of the undamped system with their corresponding eigenvectors in Section 5.2. Finally in Section 5.3, we can incorporate the different dampings into the undamped system and use the results in Section 4.2.

We start by formulating the undamped version of the system in Example 4.6. Let $\Omega \subseteq \mathbb{R}^n$ with $n \in \mathbb{N}$ be an open and bounded set with a sufficiently smooth boundary $\partial\Omega$ and take some $\kappa > 0$. Denoting $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$, consider the following system of partial differential equations

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(\xi,t) - \Delta u(\xi,t) + \kappa v(\xi,t) = 0 & \text{in } \Omega \times (0,\infty) \\ \frac{\partial^2}{\partial t^2} v(\xi,t) - \Delta v(\xi,t) + \kappa u(\xi,t) = 0 & \text{in } \Omega \times (0,\infty) \\ u(\xi,t) = v(\xi,t) = 0 & \text{on } \partial\Omega \times (0,\infty) \end{cases} \tag{5.1a}$$

with initial conditions

$$\begin{cases} u(\xi,0) = u_0(\xi), \\ v(\xi,0) = v_0(\xi), \\ \frac{\partial}{\partial t} u(\xi,0) = u_1(\xi) \quad \text{and} \\ \frac{\partial}{\partial t} v(\xi,0) = v_1(\xi) \end{cases} \tag{5.1b}$$

in the domain Ω. We call the parameter κ the *coupling parameter* of the above system. Note that without the coupling parameter the two wave equations in (5.1a) would evolve independently of one another. From now on we omit the spatial variables.

The stability for damped variations of (5.1) has been studied in the literature, for example in [33], [1], [3], [26], [4], [5] and [23]. From these papers, [33], [1] and [3] study systems that we can express as an abstract wave equation (4.1). The systems in [26], [4], [5] and [23] experience a damping that affects the system on the boundary $\partial\Omega$ which does not produce a bounded damping operator D. Therefore, in this chapter we focus mainly on the papers by Santos et al. [33], Abdallah et al. [1] and Alabau et al. [3]. Although the stability of the systems in these papers has already been proven [33, Thm. 3.2][1, Thm. 3.1, Thm. 3.2][3, Thm. 4.2], the framework from Chapter 4 provides us with an alternative route to obtain the same conclusions.

5.1 Recasting the System as an Abstract Wave Equation

Let us recast the undamped system (5.1) as an abstract wave equation. We do this by defining the space H and the operator $L : \mathcal{D}(L) \subseteq H \to H$ in Definition 4.1 suitably and showing that these H and L indeed satisfy the definition. In particular, we need to show that H is a Hilbert space with a suitable inner product and L is a positive operator with compact resolvents. For the undamped system (5.1) the damping operator $D \in \mathcal{L}(U, H)$ is simply the zero operator. In Section 5.3 we define the damping operator D for three different types of damping.

Recall from Definition 4.1 that an abstract wave equation is of the form

$$\frac{d^2}{dt^2}w(t) + Lw(t) + DD^*\frac{d}{dt}w(t) = 0$$

with the initial conditions $w(0) = w_0 \in \mathcal{D}(L^{1/2})$ and $\frac{d}{dt}w(0) = w_1 \in H$. With this in mind we can choose $H = L^2(\Omega) \times L^2(\Omega)$ and define the operator $L : \mathcal{D}(L) \subseteq H \to H$ such that

$$Lf = \begin{pmatrix} -\Delta f_1 + \kappa f_2 \\ \kappa f_1 - \Delta f_2 \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{D}(L) = \mathcal{D}(L_0) \times \mathcal{D}(L_0) \tag{5.2}$$

where L_0 is the positive Dirichlet Laplacian

$$L_0 g = -\Delta g = -\left(\frac{\partial^2 g}{\partial \xi_1^2} + \frac{\partial^2 g}{\partial \xi_2^2} + \ldots + \frac{\partial^2 g}{\partial \xi_n^2}\right), \qquad g \in \mathcal{D}(L_0) = H^2(\Omega) \cap H_0^1(\Omega)$$

from Example 2.3. In addition to being a closed operator, we take it as given that L_0 is in fact a *positive* operator with compact resolvents. After a careful study, these facts about the operator L_0 follow from the results in [14, Ch. 9] and [22, Ch. 6]. Note that L_0 has compact resolvents as a consequence of the famous Rellich–Kondrachov Theorem by which the Sobolev space $H^1(\Omega)$ is *compactly embedded* in certain Lebesgue spaces [14, Thm. 9.16][22, Thm. 5.7.1].

Proposition 5.1. *The space $H = L^2(\Omega) \times L^2(\Omega)$ is a Hilbert space when endowed with the inner product defined by*

$$\langle f, g\rangle_H = \langle f_1, g_1\rangle_{L^2} + \langle f_2, g_2\rangle_{L^2} \tag{5.3}$$

for all elements $f = (f_1, f_2)^T \in H$ and $g = (g_1, g_2)^T \in H$.

Proof. It is easy to see that (5.3) indeed defines an inner product in H because it utilizes the inner product of the known Hilbert space $L^2(\Omega)$. To show that H is a Hilbert space with the inner product (5.3), let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in H where $f_n = (f_n^1, f_n^2)^T$ and $f_n^1, f_n^2 \in L^2(\Omega)$. Now for $i \in \{1,2\}$ we have that for all $m, n \in \mathbb{N}$

$$\|f_n - f_m\|_H^2 = \|f_n^1 - f_m^1\|_L^2 + \|f_n^2 - f_m^2\|_L^2 \geq \|f_n^i - f_m^i\|_L^2,$$

implying that $(f_n^i)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $L^2(\Omega)$. As $L^2(\Omega)$ is a Hilbert space, we have that $f_n^i \to f^i \in L^2(\Omega)$ as $n \to \infty$. Consequently, $f = (f^1, f^2)^T \in H$ and

$$\|f_n - f\|_H^2 = \|f_n^1 - f^1\|_L^2 + \|f_n^2 - f^2\|_L^2 \to 0$$

as $n \to \infty$. Therefore, the space H is a Hilbert space. $\square$

The inner product in (5.3) falls under the canonical way to define the inner product on spaces that are formed as Cartesian products of Hilbert spaces. Knowing that H is now a Hilbert space, we focus on proving the necessary properties of the operator L in the following propositions. We start by proving the next lemma which states that L is a self-adjoint operator.

Lemma 5.2. *The operator L defined in* (5.2) *is a self-adjoint operator.*

Proof. We first observe that we can decompose L into the operators $L_{-\Delta}$ and L_κ where we define $L_{-\Delta}$ by

$$L_{-\Delta} f = \begin{pmatrix} -\Delta f_1 \\ -\Delta f_2 \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{D}(L_{-\Delta}) = \mathcal{D}(L_0) \times \mathcal{D}(L_0) \tag{5.4}$$

and

$$L_\kappa f = \begin{pmatrix} \kappa f_2 \\ \kappa f_1 \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H. \tag{5.5}$$

The claim follows from Lemma 2.2 (ii) if we manage to show that both $L_{-\Delta}$ and L_κ are self-adjoint operators and L_κ is also a bounded operator. We can see quite directly that the operator L_κ is a bounded operator with $\|L_\kappa\| = \kappa$. The operator L_κ is also self-adjoint because for all $f = (f_1, f_2)^T \in \mathcal{D}(L)$

$$\begin{aligned} \langle L_\kappa f, g\rangle_H &= \langle \kappa f_2, g_1\rangle_{L^2} + \langle \kappa f_1, g_2\rangle_{L^2} \\ &= \langle f_1, \kappa g_2\rangle_{L^2} + \langle f_2, \kappa g_1\rangle_{L^2} = \langle f, L_\kappa g\rangle_H. \end{aligned}$$

The operator $L_{-\Delta}$ is a block diagonal operator of the form diag(L_0, L_0). Similar to block matrices, the spectrum $\sigma(L_{-\Delta})$ coincides with $\sigma(L_0)$. As now $0 \in \rho(L_0)$, we can deduce that $\rho(L_{-\Delta}) \cap \mathbb{R} \neq \emptyset$. By Lemma 2.3, it suffices to show that $L_{-\Delta}$ is symmetric. To this end, let $f = (f_1, f_2)^T \in \mathcal{D}(L_{-\Delta})$ and $g = (g_1, g_2)^T \in \mathcal{D}(L_{-\Delta})$ be arbitrary. We can use the Green's Formula [22, Thm. C.2] to obtain

$$\begin{aligned}\langle L_{-\Delta} f, g\rangle_H &= \langle -\Delta f_1, g_1\rangle_{L^2} + \langle -\Delta f_2, g_2\rangle_{L^2} \\ &= \langle f_1, -\Delta g_1\rangle_{L^2} + \langle f_2, -\Delta g_2\rangle_{L^2} = \langle f, L_{-\Delta} g\rangle_H,\end{aligned}$$

implying that $L_{-\Delta}$ is a symmetric operator. □

As our next endeavour we prove that L is a positive operator given that the coupling parameter κ is sufficiently small.

Proposition 5.3. *Suppose λ is the smallest eigenvalue of L_0, that is, the positive Dirichlet Laplacian defined in Example 2.3. If $\kappa < \lambda$, then the operator L defined in* (5.2) *is a positive operator.*

Proof. Let $\kappa < \lambda$. We already know by Lemma 5.2 that L is a self-adjoint operator. To prove that L is a positive operator, let $f = (f_1, f_2)^T \in \mathcal{D}(L)$ be arbitrary. We have

$$\begin{aligned}\langle Lf, f\rangle_H &= \langle -\Delta f_1 + \kappa f_2, f_1\rangle_{L^2} + \langle \kappa f_1 - \Delta f_2, f_2\rangle_{L^2} \\ &= \langle -\Delta f_1, f_1\rangle_{L^2} + \langle -\Delta f_2, f_2\rangle_{L^2} + \kappa(\langle f_1, f_2\rangle_{L^2} + \langle f_2, f_1\rangle_{L^2}).\end{aligned}$$

To simplify the inner product $\langle -\Delta f_i, f_i\rangle_{L^2}$ where $i \in \{1, 2\}$, we integrate by parts using the Green's Formula [22, Thm. C.2], yielding

$$\begin{aligned}\langle -\Delta f_i, f_i\rangle_{L^2} &= \int_\Omega (-\Delta f_i)(\xi)\overline{f_i(\xi)}\, d\xi \\ &= -\int_{\partial\Omega} (\nabla f_i)(\xi) \cdot \nu(\xi)\overline{f_i(\xi)}\, d\xi + \int_\Omega (\nabla f_i)(\xi) \cdot \overline{(\nabla f_i)(\xi)}\, d\xi\end{aligned}$$

where $\nu(\xi)$ denotes the unit outward normal of Ω at $\xi \in \partial\Omega$. As $f_i \in H^2(\Omega) \cap H_0^1(\Omega)$, the integral on the boundary vanishes and we are left with

$$\begin{aligned}\langle -\Delta f_i, f_i\rangle_{L^2} &= \int_\Omega (\nabla f_i)(\xi) \cdot \overline{(\nabla f_i)(\xi)}\, d\xi \\ &= \int_\Omega \left|\left(\frac{\partial}{\partial \xi_1} f_i\right)(\xi)\right|^2 + \left|\left(\frac{\partial}{\partial \xi_2} f_i\right)(\xi)\right|^2 + \ldots + \left|\left(\frac{\partial}{\partial \xi_n} f_i\right)(\xi)\right|^2 d\xi \\ &= \|\nabla f_i\|_{L^2}^2.\end{aligned}$$

The Poincaré Inequality implies that $\|f_i\|_{L^2} \leq C\|\nabla f_i\|_{L^2}$ where the constant $C > 0$ is called the *Poincaré constant* [31, Thm. 6.101][14, Prop. 8.13]. For our open and bounded domain Ω with a sufficiently smooth boundary $\partial\Omega$, we simply have $C^2 = 1/\lambda$.

In addition, we can estimate with the Cauchy–Schwarz Inequality that

$$\kappa(\langle f_1, f_2\rangle_{L^2} + \langle f_2, f_1\rangle_{L^2}) = 2\kappa \operatorname{Re}\langle f_1, f_2\rangle_{L^2} \geq -2\kappa\|f_1\|_{L^2}\|f_2\|_{L^2}.$$

Combining the above results with the fact that $2\|f_1\|_{L^2}\|f_2\|_{L^2} \leq \|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2$, we finally obtain

$$\langle Lf, f\rangle_H \geq \frac{1}{C^2}(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2) - \kappa(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2) = (\lambda - \kappa)\|f\|_H^2. \tag{5.6}$$

Now $\lambda - \kappa > 0$ by assumption, implying that the operator L is a positive operator by Definition 2.10. □

The estimate (5.6) shows that the operator L defined in (5.2) is actually a so-called *coercive* operator [31, Rem. 8.15][14, p. 138]. Coercivity is a stronger property than positivity, and due to L being a coercive operator we can deduce that $0 \in \rho(L)$, for example. Having proven that L is a positive operator, we conclude this section by showing that the operator L has also compact resolvents.

Proposition 5.4. *The operator L defined in* (5.2) *has compact resolvents.*

Proof. By Definition 2.9, we only need to show that $(\lambda - L)^{-1}$ is a compact operator for some $\lambda \in \mathbb{C}$. Similarly to the proof of Lemma 5.2, we can decompose L into the operators $L_{-\Delta}$ and L_κ defined as in (5.4) and (5.5), respectively. Hence, we can apply some fundamental techniques of bounded perturbations to analyze the resolvents of L. Recall that as $L_{-\Delta}$ is a self-adjoint operator the spectrum of $L_{-\Delta}$ lies entirely on the real axis by Proposition 2.4. Therefore, for all $\lambda \in \rho(L_{-\Delta}) \setminus \mathbb{R}$ we have that

$$\lambda - L = \lambda - L_{-\Delta} - L_\kappa = (I - L_\kappa(\lambda - L_{-\Delta})^{-1})(\lambda - L_{-\Delta}).$$

We can now estimate $\|(\lambda - L_{-\Delta})^{-1}\| \leq \frac{1}{\operatorname{Im}\lambda}$ for all $\lambda \in \rho(L_{-\Delta}) \setminus \mathbb{R}$ by Proposition 2.4. Note that because $\sigma(L_{-\Delta}) \subset \mathbb{R}$, we can make this resolvent norm arbitrarily small. In particular, if $\operatorname{Im}\lambda > \kappa$ then $\|(\lambda - L_{-\Delta})^{-1}\| < \frac{1}{\kappa} = \frac{1}{\|L_\kappa\|}$. Moreover,

$$\|L_\kappa(\lambda - L_{-\Delta})^{-1}\| \leq \|L_\kappa\|\|(\lambda - L_{-\Delta})^{-1}\| < 1,$$

which implies that the operator $I - L_\kappa(\lambda - L_{-\Delta})^{-1}$ has a bounded inverse by the Neumann Series [35, Thm. IV.1.4][28, pp. 253–254]. Suppose then that $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda > \kappa$. The operator $(\lambda - L_{-\Delta})^{-1}$ is a compact operator because $-\Delta$ has compact resolvents. Therefore, the operator

$$(\lambda - L)^{-1} = (\lambda - L_{-\Delta})^{-1}(I - L_\kappa(\lambda - L_{-\Delta})^{-1})^{-1}$$

is actually the composition of a compact operator and a bounded operator. As discussed in Subsection 2.2.2, such an operator is compact. □

We have now shown a way to express the undamped system (5.1) as an abstract wave equation (4.1). Recall that in this undamped case the damping operator $D \in \mathcal{L}(U, H)$ is simply the zero operator. In Section 5.3 we introduce three types of damping to the system, and each of these dampings results in a different damping operator $D \in \mathcal{L}(U, H)$ where U is some Hilbert space. Despite the differences between the damping operators, the results in this section imply that we can express each resulting system as an abstract wave equation. This is really important because we know by Section 4.1 that abstract wave equations are well-posed. In other words, for all initial conditions $(u_0, v_0)^T \in \mathcal{D}(L^{1/2})$ and $(u_1, v_1)^T \in H$ we obtain the unique mild solutions to both the system in (5.1) and the systems in Section 5.3. Furthermore, we can study the stability of these systems with the tools from Chapter 4.

Note that up until this point we have not been restricted to any particular domain Ω. However, we will next analyze the eigenvalues of the undamped system (5.1). As a reminder for the reader, the eigenvalues of the system depend highly on the domain Ω. For particular domains, the eigenvalue analysis can become very difficult if not impossible. In the following sections, we focus on the domain $\Omega = (0, 1)$.

5.2 Eigenvalues and Eigenvectors of the Undamped System

Our goal in this section is to find all the eigenvalues of the operator L defined in (5.2) and their corresponding eigenvectors. We are particularly interested in the eigenvalues of L because with them we can analyze the wavepackets of $L^{1/2}$ in Theorem 4.5. After all, we are eventually looking for the stability of the damped systems.

As discussed at the end of Section 5.1, let $\Omega = (0, 1)$ throughout this section. We have already shown that if the coupling parameter κ satisfies $\kappa < \lambda$ where λ is the smallest eigenvalue of L_0, then L is a positive operator with compact resolvents. Recall that as a consequence the spectrum of L consists of only discrete eigenvalues on the positive real axis. In fact, we obtain these eigenvalues by perturbing the eigenvalues of L_0 by the coupling parameter κ as we will see in the next proposition. Note that as $\Omega = (0, 1)$ we obtain $\sigma(L_0) = \{\pi^2 n^2 : n \in \mathbb{N}\}$.

Proposition 5.5. *Suppose the operator L is defined as in* (5.2) *with $0 < \kappa < \pi^2$. Then*

$$\sigma(L) = \sigma_p(L) = \{\pi^2 n^2 \pm \kappa : n \in \mathbb{N}\}$$

and for all $n \in \mathbb{N}$ the kernels satisfy $\mathcal{N}(\pi^2 n^2 \pm \kappa - L) = \operatorname{span}\{(\sin(\pi n \cdot), \pm \sin(\pi n \cdot))^T\}$.

Proof. We begin looking for the eigenvalues λ of L and their corresponding eigenvectors $\varphi = (\varphi_1, \varphi_2)^T \in \mathcal{D}(L)$ by considering first the general equation

$$L\varphi = \lambda\varphi \qquad \Longleftrightarrow \qquad \begin{cases} -\frac{\partial^2}{\partial\xi^2}\varphi_1 + \kappa\varphi_2 = \lambda\varphi_1 \\ \kappa\varphi_1 - \frac{\partial^2}{\partial\xi^2}\varphi_2 = \lambda\varphi_2 \end{cases} \tag{5.7}$$

without the boundary conditions presented in (5.1b). We can reduce this second order problem to a first order problem with standard techniques, expressing it in the form

$$\frac{\partial}{\partial\xi}\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \frac{\partial}{\partial\xi}\varphi_1 \\ \frac{\partial}{\partial\xi}\varphi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda & \kappa & 0 & 0 \\ \kappa & -\lambda & 0 & 0 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \frac{\partial}{\partial\xi}\varphi_1 \\ \frac{\partial}{\partial\xi}\varphi_2 \end{pmatrix}. \tag{5.8}$$

We denote by $\mathcal{A}$ the matrix in (5.8) multiplying the vector $(\varphi_1, \varphi_2, \frac{\partial}{\partial\xi}\varphi_1, \frac{\partial}{\partial\xi}\varphi_2)^T$. It is straight-forward to check that the eigenvalues of $\mathcal{A}$ are

$$\mu_1 = -\sqrt{\kappa - \lambda}, \qquad \mu_2 = \sqrt{-\kappa - \lambda}, \qquad \mu_3 = -\mu_2 \qquad \text{and} \qquad \mu_4 = -\mu_1,$$

and their corresponding eigenvectors are of the form

$$\psi_1 = \begin{pmatrix} 1 \\ 1 \\ \mu_1 \\ \mu_1 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} 1 \\ -1 \\ \mu_2 \\ -\mu_2 \end{pmatrix}, \qquad \psi_3 = \begin{pmatrix} 1 \\ -1 \\ -\mu_2 \\ \mu_2 \end{pmatrix} \qquad \text{and} \qquad \psi_4 = \begin{pmatrix} 1 \\ 1 \\ -\mu_1 \\ -\mu_1 \end{pmatrix}.$$

With these, we can express the general solution of (5.8) as

$$\left(\varphi_1, \varphi_2, \frac{\partial}{\partial\xi}\varphi_1, \frac{\partial}{\partial\xi}\varphi_2\right)^T = \sum_{k=1}^{4} C_k e^{\mu_k\xi}\psi_k \tag{5.9}$$

where the C_k's are some complex constants [2, p. 133]. As we strive for solving the equation (5.7), we are actually only interested in the two first components of (5.9). Namely, the general form for the eigenvectors of L corresponding to an eigenvalue λ is

$$\varphi(\xi) = C_1\begin{pmatrix} e^{\mu_1\xi} \\ e^{\mu_1\xi} \end{pmatrix} + C_2\begin{pmatrix} e^{\mu_2\xi} \\ -e^{\mu_2\xi} \end{pmatrix} + C_3\begin{pmatrix} e^{-\mu_2\xi} \\ -e^{-\mu_2\xi} \end{pmatrix} + C_4\begin{pmatrix} e^{-\mu_1\xi} \\ e^{-\mu_1\xi} \end{pmatrix}.$$

Taking the boundary conditions in (5.1b) into account, we obtain the following system of equations for the constants C_k.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ e^{\mu_1} & e^{\mu_2} & e^{-\mu_2} & e^{-\mu_1} \\ e^{\mu_1} & -e^{\mu_2} & -e^{-\mu_1} & e^{-\mu_1} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.10}$$

We denote by $\mathcal{C}$ the matrix in (5.10) multiplying the vector $(C_1, C_2, C_3, C_4)^T$. We wish to find non-trivial solutions of this equation. This can only happen if the determinant of $\mathcal{C}$ is zero, generating necessary conditions for the eigenvalues of L. After some lengthy, but elementary, computations, we arrive at the equation

$$\det(\mathcal{C}) = 4e^{\mu_1-\mu_2}(e^{-2\mu_1} - 1)(e^{2\mu_2} - 1) = 0.$$

We can now substitute $\mu_1 = -\sqrt{\kappa - \lambda}$ and $\mu_2 = \sqrt{-\kappa - \lambda}$ back into the above equation. Consequently, the candidates for the eigenvalues of L are of the form

$$\lambda_n = \pi^2 n^2 + \kappa \qquad \text{or} \qquad \lambda_n = \pi^2 n^2 - \kappa \tag{5.11}$$

for all $n \in \mathbb{Z}$. Equivalently, we can only consider all non-negative integers $n \geq 0$. To see which of these candidates represent true eigenvalues of L, we substitute them back into (5.10) and solve for the constants C_k. Let us start by focusing on the first branch of eigenvalues where $\lambda_n = \pi^2 n^2 + \kappa$. After substitution we get $C_1 = -C_4$, $C_2 = C_3 = 0$ and C_4 is free. Therefore, the multiplicity of the eigenvalues λ_n is one and their corresponding eigenvectors are of the form

$$\varphi(\xi) = c \begin{pmatrix} e^{\mu_1 \xi} \\ e^{\mu_1 \xi} \end{pmatrix} - c \begin{pmatrix} e^{-\mu_1 \xi} \\ e^{-\mu_1 \xi} \end{pmatrix}$$

where $c \in \mathbb{C}$ and

$$\mu_1 = -\sqrt{\kappa - \lambda_n} = -\sqrt{\kappa - (\pi^2 n^2 + \kappa)} = -i\pi n.$$

We can simplify the eigenvectors further by using Euler's Identity. We obtain

$$\varphi_n(\xi) = c \begin{pmatrix} e^{-i\pi n\xi} - e^{i\pi n\xi} \\ e^{-i\pi n\xi} - e^{i\pi n\xi} \end{pmatrix} = -2ci \begin{pmatrix} \sin(\pi n\xi) \\ \sin(\pi n\xi) \end{pmatrix} = c_1 \begin{pmatrix} \sin(\pi n\xi) \\ \sin(\pi n\xi) \end{pmatrix}$$

for $c_1 \in \mathbb{C}$. We also see from the above equation that only the case $n = 0$ does not produce valid eigenvectors. Similarly, we can investigate the second branch of eigenvalues where $\lambda_n = \pi^2 n^2 - \kappa$ and obtain for $n \in \mathbb{N}$ the corresponding valid eigenvectors

$$\varphi_n(\xi) = c_2 \begin{pmatrix} \sin(\pi n \xi) \\ -\sin(\pi n \xi) \end{pmatrix}$$

where $c_2 \in \mathbb{C}$. In both cases the eigenvectors belong to the domain $\mathcal{D}(L)$. Therefore, the eigenvalues of L and the kernels $\mathcal{N}(\pi^2 n^2 \pm \kappa - L)$ are exactly as claimed. □

In Proposition 5.3, we obtain an upper bound for the coupling parameter κ. Note that Proposition 5.5 agrees well with this bound in terms of the operator L being a positive operator. Indeed, we proved that the eigenvalues of L are precisely the eigenvalues of the operator L_0 perturbed by the terms κ and $-\kappa$. where L_0 is the positive Dirichlet Laplacian. The eigenvalues of a positive operator must be positive, and so the upper bound obtained in Proposition 5.3 has a nice geometrical interpretation. We depict this result schematically in Figure 5.1.

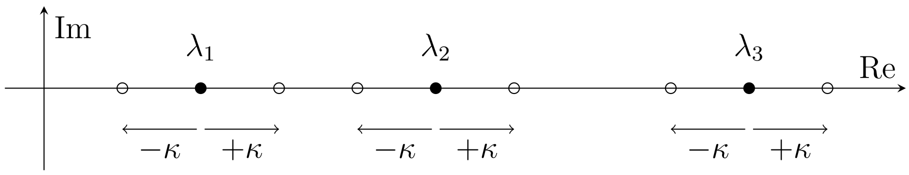

Figure 5.1. *We obtain the eigenvalues (white circles) of the operator L defined in* (5.2) *by perturbing the eigenvalues (black circles) of the positive Dirichlet Laplacian L_0 defined in Example 2.3 by the coupling parameter κ. Therefore, we must have $\kappa < \lambda_1$ for L to be a positive operator.*

We have now acquired a complete picture of the eigenvalues of L and their corresponding eigenvectors. We will next focus on the eigenvalues of $L^{1/2}$ which by Definition 2.11 are simply the square roots of the eigenvalues in (5.11). Whereas there is an obvious uniform gap of 2κ between the eigenvalues of L, the same is not true for the eigenvalues of $L^{1/2}$. This important result follows from the next lemma.

Lemma 5.6. *Suppose $\lambda_n^+ = \pi^2 n^2 + \kappa$ and $\lambda_n^- = \pi^2 n^2 - \kappa$ for all $n \in \mathbb{N}$ where $\kappa < \pi^2$. Then for all $n \in \mathbb{N}$*

$$\frac{\kappa}{\pi n} \leq \sqrt{\lambda_n^+} - \sqrt{\lambda_n^-} \leq \frac{\kappa}{\pi n} + \varepsilon(n)$$

where the function $\varepsilon : [1, \infty) \to (0, \infty)$ satisfies $\lim_{x \to \infty} x\varepsilon(x) = 0$.

Proof. We first observe that for all $n \in \mathbb{N}$ we have

$$\sqrt{\lambda_n^+} - \sqrt{\lambda_n^-} = \sqrt{\pi^2 n^2 + \kappa} - \sqrt{\pi^2 n^2 - \kappa} = \pi n \left(\sqrt{1 + \frac{\kappa}{\pi^2 n^2}} - \sqrt{1 - \frac{\kappa}{\pi^2 n^2}} \right).$$

To study the behaviour of the above expression, we apply Taylor's Theorem to express the square roots around 1 as the sum of their corresponding first-degree Taylor polynomials

P_1 and the Lagrange error terms R_1. Note that both radicals $1 + \frac{\kappa}{\pi^2 n^2}$ and $1 - \frac{\kappa}{\pi^2 n^2}$ converge to 1 as $n \to \infty$. Now for all $0 < h < 1$

$$\sqrt{1+h} = 1 + \frac{h}{2} - \frac{h^2}{8z_1^{3/2}} \qquad \text{and} \qquad \sqrt{1-h} = 1 - \frac{h}{2} - \frac{h^2}{8z_2^{3/2}}$$

where z_1 is between 1 and $1+h$ and z_2 is between 1 and $1-h$. We are now interested in finding bounds for $\sqrt{1+h} - \sqrt{1-h}$. As

$$\sqrt{1+h} - \sqrt{1-h} = h - \frac{h^2}{8}\left(\frac{1}{z_1^{3/2}} - \frac{1}{z_2^{3/2}}\right), \tag{5.12}$$

we immediately obtain a lower bound

$$h - \frac{h^2}{8}\left(\frac{1}{z_1^{3/2}} - \frac{1}{z_2^{3/2}}\right) \geq h - \frac{h^2}{8}\left(\frac{1}{1^{3/2}} - \frac{1}{1^{3/2}}\right) = h \tag{5.13}$$

and an upper bound

$$h - \frac{h^2}{8}\left(\frac{1}{z_1^{3/2}} - \frac{1}{z_2^{3/2}}\right) \leq h - \frac{h^2}{8}\left(\frac{1}{(1+h)^{3/2}} - \frac{1}{(1-h)^{3/2}}\right). \tag{5.14}$$

Let $f : (0,1) \to (0,\infty)$ be a function defined by

$$f(h) = -\frac{h^2}{8}\left(\frac{1}{(1+h)^{3/2}} - \frac{1}{(1-h)^{3/2}}\right).$$

Evidently, $\lim_{h\to 0} f(h) = 0$. Therefore, the bounds in (5.13) and (5.14) vanish as $h \to 0$. Let $h_x = \frac{\kappa}{\pi^2 x^2}$ for all $x \in [1,\infty)$. Clearly $0 < h_x < 1$ and $h_x \to 0$ as $x \to \infty$. We define the function $\varepsilon : [1,\infty) \to (0,\infty)$ by $\varepsilon(x) = \pi x f(h_x)$. Consequently, $\lim_{x\to\infty} x\varepsilon(x) = 0$. If we now combine (5.12), (5.13) and (5.14), then we obtain

$$\frac{\kappa}{\pi n} = \pi n h_n \leq \sqrt{\lambda_n^+} - \sqrt{\lambda_n^-} \leq \pi n(h_n + f(h_n)) = \frac{\kappa}{\pi n} + \varepsilon(n)$$

for all $n \in \mathbb{N}$. $\square$

The above result implies that the eigenvalues of $L^{1/2}$ are not uniformly separated. However, if we consider an eigenvalue $\sqrt{\lambda_n^\pm}$ then we know by Lemma 5.6 that the open interval $(\sqrt{\lambda_n^\pm} - \frac{\kappa}{\pi n}, \sqrt{\lambda_n^\pm} + \frac{\kappa}{\pi n})$ does not contain any other eigenvalues of $L^{1/2}$. We need to keep this fact in mind in especially the next section where we embark on studying the stability of (5.1) with different dampings.

5.3 Stability of the System with Different Dampings

In this section we study the stability of the system presented in (5.1) after we have incorporated different kinds of dampings into the system. We keep the boundary and initial conditions the same for the damped system. In the following subsections, we consider three damping terms of the form $DD^*(\frac{\partial}{\partial t}u, \frac{\partial}{\partial t}v)^T$ where $D \in \mathcal{L}(U, H)$ is a bounded operator from a Hilbert space U to the state space H. We study the system (5.1) with two viscous dampings, with partial viscous damping and with partial weak damping. All of these dampings produce polynomial stability for the system, but only for two viscous dampings we also obtain exponential stability.

We start by formulating the total energy of the upcoming systems. We already know by Section 5.1 that we can express the systems as abstract wave equations. Let L be the operator defined as in (5.2) and suppose $w(t) = (u(\cdot, t), v(\cdot, t))^T \in \mathcal{D}(L)$ for all $t > 0$. Now the total energy $\mathcal{E}_w(t) = \mathcal{E}_{u,v}(t)$ in (4.3) takes on the form

$$\begin{aligned}
\mathcal{E}_{u,v}(t) &= \frac{1}{2}\|L^{1/2}w(t)\|_H^2 + \frac{1}{2}\|\tfrac{d}{dt}w(t)\|_H^2 = \frac{1}{2}\langle Lw(t), w(t)\rangle_{L^2} + \frac{1}{2}\langle \tfrac{d}{dt}w(t), \tfrac{d}{dt}w(t)\rangle_{L^2} \\
&= \frac{1}{2}\int_\Omega \|(\nabla u)(\xi,t)\|^2 + \|(\nabla v)(\xi,t)\|^2 + 2\kappa \operatorname{Re}(u(\xi,t)\overline{v(\xi,t)})\, d\xi \\
&\quad + \frac{1}{2}\int_\Omega \left|\tfrac{\partial}{\partial t}u(\xi,t)\right|^2 + \left|\tfrac{\partial}{\partial t}v(\xi,t)\right|^2 d\xi. \qquad (5.15)
\end{aligned}$$

One of our goals in this section is to show that for a given system the total energy above is bounded by a decaying term. The type of stability follows from how fast the bounding term decays. In the next subsection we see that with two viscous dampings we obtain an exponential decay rate for the bounding term.

5.3.1 Two Viscous Dampings

Consider the following system of partial differential equations

$$\begin{cases} \frac{\partial^2}{\partial t^2}u - \Delta u + \kappa v + \frac{\partial}{\partial t}u = 0 & \text{in } \Omega \times (0,\infty) \\ \frac{\partial^2}{\partial t^2}v - \Delta v + \kappa u + \frac{\partial}{\partial t}v = 0 & \text{in } \Omega \times (0,\infty) \end{cases} \qquad (5.16)$$

where $\Omega = (0,1)$ and $0 < \kappa < \pi^2$. We call this type of damping *two viscous dampings*. A system similar to (5.16) has been proved to be exponentially stable with the help of Grönwall's Lemma in [12, Eq. (1.5)–(1.7)]. To use Grönwall's Lemma, we first have to multiply the equations in (5.16) by suitable test functions and then integrate the equations by parts. This leads us to terms that are related to the total energy of the system and its derivative with respect to time. Theorem 3.1 in [1] suggests that we could try a similar approach with the system in (5.16).

However, we prove the exponential stability of (5.16) using an alternative route. We already possess full knowledge of the eigenvalues of $L^{1/2}$ and their corresponding eigenvectors. As for the damping operator $D \in \mathcal{L}(U, H)$, the two viscous dampings in the system (5.16) imply that we must have

$$DD^* \begin{pmatrix} \frac{\partial}{\partial t} u \\ \frac{\partial}{\partial t} v \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial t} u \\ \frac{\partial}{\partial t} v \end{pmatrix}.$$

In particular, the damping operator $D = I$ satisfies the above equality. This of course implies immediately that $U = H$ and $D^* = I$. We are now ready to prove that the system is exponentially stable.

Theorem 5.7. *The system defined in* (5.16) *is exponentially stable. That is, there exist some $M > 0$ and $\varepsilon > 0$ such that for all initial conditions $(u_0, v_0)^T \in (H^2(\Omega) \cap H_0^1(\Omega))^2$ and $(u_1, v_1)^T \in H_0^1(\Omega) \times H_0^1(\Omega)$ the total energy $\mathcal{E}(t)$ in* (5.15) *of the corresponding solution of* (5.16) *satisfies*

$$\mathcal{E}(t) \leq Me^{-2\varepsilon t} \int_\Omega |L_0 u_0 + \kappa v_0|^2 + |\kappa u_0 + L_0 v_0|^2 \, d\xi$$
$$+ Me^{-2\varepsilon t} \int_\Omega \|(\nabla u_1)(\xi)\|^2 + \|(\nabla v_1)(\xi)\|^2 + 2\kappa \operatorname{Re}(u_1(\xi)\overline{v_1(\xi)}) \, d\xi$$

for all $t > 0$.

Proof. Let L be the operator defined in (5.2) and $D = I$. By Corollary 4.6, it suffices to show that there exist positive constants γ and δ such that the inequality in (4.10) holds. To this end, let $\delta = 1$. Now

$$\|D^*\varphi\|_U = \|I\varphi\|_U = \|\varphi\|_H = 1 \cdot \|\varphi\|_H, \qquad \varphi \in \mathsf{WP}_{s,\delta}(L^{1/2}), \qquad s > 0,$$

implying that we can choose $\gamma = 1$. Therefore, Corollary 4.6 implies exponential stability for the system (5.16). As for bounding the total energy $\mathcal{E}_{u,v}(t)$, we obtain directly by substituting $w_0 = (u_0, v_0)^T$ and $w_1 = (u_1, v_1)^T$ into (4.12) that

$$\mathcal{E}_{u,v}(t) \leq Me^{-2\varepsilon t} \int_\Omega |L_0 u_0 + \kappa v_0|^2 + |\kappa u_0 + L_0 v_0|^2 \, d\xi$$
$$+ Me^{-2\varepsilon t} \int_\Omega \|(\nabla u_1)(\xi)\|^2 + \|(\nabla v_1)(\xi)\|^2 + 2\kappa \operatorname{Re}(u_1(\xi)\overline{v_1(\xi)}) \, d\xi$$

for all $t > 0$. □

The exponential stability of the system (5.16) is not a surprise. The viscous damping term represents quite a strong damping in the domain Ω. Precisely like in Example 4.4, the viscous damping term would result in exponential stability in both components separately if we removed the coupling from the system. Although the stability of the coupled system depends heavily on the coupling, we can generally expect that coupling exponentially stable systems together should not affect the stability too drastically.

5.3.2 Partial Viscous Damping

Next we study perhaps a more interesting system where the viscous damping affects only one of the components. This way, the other component experiences the damping solely via the coupling of the components. We call this type of damping *partial viscous damping*.

Consider the following system of partial differential equations

$$\begin{cases} \frac{\partial^2}{\partial t^2}u - \Delta u + \kappa v + \frac{\partial}{\partial t}u = 0 & \text{in } \Omega \times (0,\infty) \\ \frac{\partial^2}{\partial t^2}v - \Delta v + \kappa u = 0 & \text{in } \Omega \times (0,\infty) \end{cases} \tag{5.17}$$

where $\Omega = (0,1)$ and $0 < \kappa < \pi^2$. The polynomial stability of this system has been proven in [33], for example. In this subsection we first prove that the system in (5.17) is not exponentially stable. Afterwards, we prove the polynomial stability of the system.

We start by defining the damping operator $D \in \mathcal{L}(U,H)$ such that the operator DD^* corresponds to the damping of the system (5.17). In particular, we must have

$$DD^* \begin{pmatrix} \frac{\partial}{\partial t}u \\ \frac{\partial}{\partial t}v \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial t}u \\ 0 \end{pmatrix}.$$

With this in mind, we can define the damping operator $D \in \mathcal{L}(U,H)$ simply by

$$Df = \begin{pmatrix} f \\ 0 \end{pmatrix}, \qquad f \in U = L^2(\Omega). \tag{5.18}$$

It is not too difficult to see that D is indeed a bounded operator. Therefore, the adjoint D^* exists. Moreover, the adjoint is also bounded by Lemma 2.2 (i), so it suffices to only find the formula for D^*. This we can do by using the definition of the adjoint.

Proposition 5.8. *The adjoint of D is the bounded operator $D^* \in \mathcal{L}(H,U)$ defined by*

$$D^*g = g_1, \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in H.$$

Proof. We observe that for an arbitrary element $g = (g_1, g_2)^T \in H$ and $f \in U$ we have

$$\langle Df, g\rangle_H = \left\langle \begin{pmatrix} f \\ 0 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle_H = \langle f, g_1\rangle_{L^2} + \langle 0, g_2\rangle_{L^2} = \langle f, g_1\rangle_{L^2}.$$

This implies by Definition 2.6 that $D^* g = g_1$. □

As we are dealing with only partial viscous damping instead of two viscous dampings, we have a reason to expect that the system in (5.17) is not exponentially stable. We prove the lack of exponential stability similarly to the proof of Proposition 2.2 in [26]. Essentially, we show that for the operators A and B defined as in (4.4) and (4.5), respectively, the resolvent $(\lambda - (A - BB^*))^{-1}$ is not bounded on the imaginary axis. As discussed in Subsection 3.4.1, we can do this by constructing a sequence of imaginary numbers $(\lambda_k)_{k\in\mathbb{N}}$ that correspond to normalized elements $\varphi_k \in \mathcal{D}(A)$. The resolvent is unbounded if the values $(\lambda_k - (A - BB^*))\varphi_k$ converge to zero as $k \to \infty$.

Theorem 5.9. *The system defined in* (5.17) *is not exponentially stable.*

Proof. Let A and B be the operators defined in (4.4) and (4.5), respectively, where L is the operator in (5.2) and D is the operator in (5.18). Let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of eigenvalues of the positive Dirichlet Laplacian L_0 and let their corresponding normalized eigenvectors form the sequence $(\psi_k)_{k\in\mathbb{N}}$. With these sequences, we can define a new sequence of elements $(\varphi_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(A)$ such that

$$\varphi_k = \frac{1}{\sqrt{2}} \left(0, \frac{\psi_k}{i\sqrt{\mu_k}}, 0, \psi_k\right)^T$$

for all $k \in \mathbb{N}$. We begin by showing that $\|\varphi_k\|_X = 1$ for all $k \in \mathbb{N}$. We obtain

$$\begin{aligned}
\|\sqrt{2}\varphi_k\|_X^2 &= \left\| \begin{pmatrix} 0 \\ \frac{\psi_k}{i\sqrt{\mu_k}} \\ 0 \\ \psi_k \end{pmatrix} \right\|_X^2 = \left\| L^{1/2} \begin{pmatrix} 0 \\ \frac{\psi_k}{i\sqrt{\mu_k}} \end{pmatrix} \right\|_H^2 + \left\| \begin{pmatrix} 0 \\ \psi_k \end{pmatrix} \right\|_H^2 \\
&= \frac{1}{\mu_k} \left\langle L \begin{pmatrix} 0 \\ \psi_k \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_k \end{pmatrix} \right\rangle_H + \|0\|_{L^2}^2 + \|\psi_k\|_{L^2}^2 \\
&= \frac{1}{\mu_k} \left\langle \begin{pmatrix} \kappa\psi_k \\ -\Delta\psi_k \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_k \end{pmatrix} \right\rangle_H + 1 \\
&= \frac{1}{\mu_k} \langle -\Delta\psi_k, \psi_k\rangle_{L^2} + 1 = \frac{1}{\mu_k} \langle \mu_k\psi_k, \psi_k\rangle_{L^2} + 1 = 2,
\end{aligned}$$

implying that $\|\varphi_k\|_X = 1$. Furthermore, we have

$$\begin{aligned}
&\|(i\sqrt{\mu_k} - (A - BB^*))\varphi_k\|_X^2 \\
&= \frac{1}{2}\left\| \begin{pmatrix} 0 \\ i\sqrt{\mu_k} \cdot \frac{\psi_k}{i\sqrt{\mu_k}} \\ 0 \\ i\sqrt{\mu_k}\psi_k \end{pmatrix} - \begin{pmatrix} 0 \\ \psi_k \\ \frac{\kappa\psi_k}{i\sqrt{\mu_k}} \\ -\Delta\frac{\psi_k}{i\sqrt{\mu_k}} \end{pmatrix} \right\|_X^2 = \frac{1}{2}\left\| \begin{pmatrix} 0 \\ 0 \\ -\frac{\kappa\psi_k}{i\sqrt{\mu_k}} \\ i\sqrt{\mu_k}\psi_k - i\sqrt{\mu_k}\psi_k \end{pmatrix} \right\|_X^2 \\
&= \frac{1}{2}\left\| \frac{\kappa\psi_k}{i\sqrt{\mu_k}} \right\|_{L^2}^2 = \frac{\kappa^2}{2\mu_k} \to 0 \qquad \text{as} \qquad k \to \infty.
\end{aligned}$$

Therefore, the resolvent of $A - BB^*$ is unbounded on the imaginary axis. Corollary 3.13 implies now that the system (5.17) is not exponentially stable. □

In Theorem 3.2 of [33], the system (5.17) with partial viscous damping is shown to be polynomially stable. Santos et al. [33] prove their result by multiplying the differential equations in (5.1) with suitable functions and applying standard techniques in manipulating the resulting expressions. We prove the polynomial stability of the system (5.17) using Theorem 4.5 combined with all the knowledge we have obtained so far.

Theorem 5.10. *The system defined in* (5.17) *is polynomially stable. That is, there exists some* $M > 0$ *such that for all initial conditions* $(u_0, v_0)^T \in (H^2(\Omega) \cap H_0^1(\Omega))^2$ *and* $(u_1, v_1)^T \in H_0^1(\Omega) \times H_0^1(\Omega)$ *the total energy* $\mathcal{E}_{u,v}(t)$ *in* (5.15) *satisfies*

$$\begin{aligned}
\mathcal{E}_{u,v}(t) \le \frac{M}{t} &\int_\Omega |L_0u_0 + \kappa v_0|^2 + |\kappa u_0 + L_0 v_0|^2 \, d\xi \\
&+ \frac{M}{t}\int_\Omega \|(\nabla u_1)(\xi)\|^2 + \|(\nabla v_1)(\xi)\|^2 + 2\kappa \operatorname{Re}(u_1(\xi)\overline{v_1(\xi)})\, d\xi
\end{aligned}$$

for all $t > 0$.

Proof. Let L and D be the operators defined in (5.2) and (5.18), respectively. We define the function $\delta(s) = \min\{\frac{\kappa}{2\pi s}, \frac{\kappa}{2\pi}\}$ for all $s > 0$. By Definition 4.3, the wavepackets of $L^{1/2}$ are now of the form

$$\mathsf{WP}_{s,\delta(s)}(L^{1/2}) = \operatorname{span}(\{\varphi_n^\pm : s - \delta(s) < \sqrt{\pi^2 n^2 \pm \kappa} < s + \delta(s)\}) \cup \{0\}$$

where $\varphi_n^\pm = (\sin(\pi n\cdot), \pm\sin(\pi n\cdot))^T$ is the normalized eigenvector of $L^{1/2}$ corresponding to the eigenvalue $\sqrt{\pi^2 n^2 \pm \kappa}$ for all $n \in \mathbb{N}$. We know by Proposition 5.5 and Lemma 5.6 that for all $s > 0$ there can be at most one eigenvector $\varphi_n^\pm$ corresponding to an eigenvalue within the interval $(s - \delta(s), s + \delta(s))$. Now let $s > 0$ and let $w \in \mathsf{WP}_{s,\delta(s)}(L^{1/2})$ be arbitrary. Note that if $\mathsf{WP}_{s,\delta(s)}(L^{1/2}) = \{0\}$, then any function $\gamma(s)$ satisfies the inequality in (4.10) trivially.

Otherwise we have $w = c\varphi_n^{\pm}$ where $c \in \mathbb{C}$ is arbitrary and n satisfies

$$s - \delta(s) < \sqrt{\pi^2 n^2 \pm \kappa} < s + \delta(s).$$

As $\varphi_n^{\pm} = (\sin(\pi n \cdot), \pm \sin(\pi n \cdot))^T$, in this case we also obtain

$$\begin{aligned}\|D^* w\|_U^2 &= \|D^*(c\varphi_n^{\pm})\|_U^2 = |c|^2 \|\sin(\pi n \cdot)\|_{L^2}^2 \\ &= \frac{|c|^2}{2}(\|\sin(\pi n \cdot)\|_{L^2}^2 + \|\pm \sin(\pi n \cdot)\|_{L^2}^2) \\ &= \frac{|c|^2}{2}\|\varphi_n^{\pm}\|_H^2 = \frac{1}{2}\|c\varphi_n^{\pm}\|_H^2 = \frac{1}{2}\|w\|_H^2,\end{aligned}$$

implying that $\|D^* w\|_U = \frac{1}{\sqrt{2}}\|w\|_H$. Thus, we can choose $\gamma(s) = \frac{1}{\sqrt{2}}$ for all $s > 0$, and the functions $\gamma(s)$ and $\delta(s)$ satisfy the inequality in (4.10). As $\gamma(|s|)^2 \delta(|s|)^2 = \frac{\kappa^2}{8\pi^2}|s|^{-2}$, the polynomial stability of the system (5.17) follows from Theorem 4.5. As for bounding the total energy $\mathcal{E}_{u,v}(t)$, we obtain directly by substituting $w_0 = (u_0, v_0)^T$ and $w_1 = (u_1, v_1)^T$ into (4.11) that

$$\begin{aligned}\mathcal{E}_{u,v}(t) \leq \frac{M}{t}\int_\Omega &|L_0 u_0 + \kappa v_0|^2 + |\kappa u_0 + L_0 v_0|^2 \, d\xi \\ &+ \frac{M}{t}\int_\Omega \|(\nabla u_1)(\xi)\|^2 + \|(\nabla v_1)(\xi)\|^2 + 2\kappa \operatorname{Re}(u_1(\xi)\overline{v_1(\xi)}) \, d\xi\end{aligned}$$

for all $t > 0$. □

The polynomial decay rate we obtain in Theorem 5.10 agrees well with the results of Theorem 3.2 in [33]. Note that Theorem 4.5 only gives us an upper bound for the polynomial decay rate. Therefore, an interesting question arises whether we can improve the bound for this particular system. As discussed in Section 4.2, there are methods for assessing the sharpness of the obtained decay rate. However, we do not pursue to answer this question in this thesis.

We conclude this subsection by discussing the key differences in the proofs for exponential stability and polynomial stability. In the proof of Theorem 5.7, i.e., when we proved that the system (5.16) was exponentially stable, we chose the functions γ and δ to be constants. This was easily justified due to the simplicity of the damping operator D. However, we did not make δ a constant in the proof of Theorem 5.10. In fact, if we try to study the inequality in (4.10) with a constant δ we have to be more careful due to the nature of the partial viscous damping.

Lemma 5.6 implies that for any constant $\delta > 0$, there is some $s > 0$ such that the interval $(s - \delta, s + \delta)$ contains more than one eigenvalue of $L^{1/2}$. To study the wavepackets with respect to such an interval, by Definition 4.3 we need to consider the linear combination of all the eigenvectors that have their corresponding eigenvalue within the interval. The simplest example of this is when there are two eigenvalues of $L^{1/2}$ in the interval, i.e., for some $s > 0$ and $\delta > 0$ we have

$$\sqrt{\pi^2(n-1)^2+\kappa} < s-\delta < \sqrt{\pi^2 n^2-\kappa} < \sqrt{\pi^2 n^2+\kappa} < s+\delta < \sqrt{\pi^2(n+1)^2-\kappa}$$

for some $n \in \mathbb{N}$. Then the wavepackets w are of the form

$$w = a\begin{pmatrix}\sin(\pi n\cdot)\\ \sin(\pi n\cdot)\end{pmatrix} + b\begin{pmatrix}\sin(\pi n\cdot)\\ -\sin(\pi n\cdot)\end{pmatrix}$$

for some $a \in \mathbb{C}$ and $b \in \mathbb{C}$. By definition, the partial viscous damping affects only the first component of the above wavepacket. If $a = -b$, then

$$\|D^* w\|_U = 0 < \|w\|_H,$$

and such a wavepacket does not satisfy the inequality in (4.10). Although not trivial, the above result implies that the system in (5.17) is not *exactly observable* [36, Thm. 6.9.3], yielding the lack of exponential stability in another way. This phenomenon is present in all the cases where the damping affects only one component of the system and the eigenvalues of the undamped system are not uniformly separated. In particular, we encounter the same phenomenon in the next subsection with partial weak damping.

5.3.3 Partial Weak Damping

Finally, we turn our attention to a system with *partial weak damping*. Similar to the system with partial viscous damping, this type of damping affects only one part of the system. The weak damping introduces a *damping function* to the system, and we can regulate the damping by varying this function.

Consider the following system of partial differential equations

$$\begin{cases} \frac{\partial^2}{\partial t^2}u - \Delta u + \kappa v + b\int_\Omega b(r)\frac{\partial}{\partial t}u(r,\cdot)\,dr = 0 & \text{in } \Omega\times(0,\infty) \\ \frac{\partial^2}{\partial t^2}v - \Delta v + \kappa u = 0 & \text{in } \Omega\times(0,\infty) \end{cases} \tag{5.19}$$

where $b \in L^2(\Omega)$ is the real-valued damping function, $\Omega = (0,1)$ and $0 < \kappa < \pi^2$. The stability of a similar system has been studied in [15, Sec 6.2.2], for example. Our first goal in this subsection is to prove that the system in (5.19) is not exponentially stable. As our second goal, we strive for characterizing the polynomial stability of the system (5.19) in terms of the damping function $b \in L^2(\Omega)$. The results in the latter part are similar to the results in Section 6.2.2 of [15].

We start by finding suitable damping operators $D \in \mathcal{L}(U,H)$ and $D^* \in \mathcal{L}(H,U)$. The weak damping in the system (5.19) implies that we must have

$$DD^* \begin{pmatrix} \frac{\partial}{\partial t} u \\ \frac{\partial}{\partial t} v \end{pmatrix} = \begin{pmatrix} b \int_\Omega b(r) \frac{\partial}{\partial t} u(r, \cdot)\, dr \\ 0 \end{pmatrix}.$$

Therefore, one simple choice for the operator $D \in \mathcal{L}(U, H)$ is to define it as

$$Dz = \begin{pmatrix} bz \\ 0 \end{pmatrix}, \qquad z \in U = \mathbb{C}. \tag{5.20}$$

Again, it is not too difficult to see that the operator D above is a bounded operator. We look for its adjoint D^* with the help of Definition 2.6. Here the formula of the adjoint falls nicely after manipulating the expressions in a suitable way. Lemma 2.2 (i) implies then that the adjoint is a bounded operator.

Proposition 5.11. *The adjoint of D is the bounded operator $D^* \in \mathcal{L}(H, U)$ defined by*

$$D^* g = \int_\Omega b(r) g_1(r)\, dr, \qquad g = (g_1, g_2)^T \in H.$$

Proof. We observe that for all elements $g = (g_1, g_2)^T \in H$ and $z \in U = \mathbb{C}$ we have

$$\begin{aligned} \langle Dz, g \rangle_H &= \left\langle \begin{pmatrix} bz \\ 0 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle_H = \langle bz, g_1 \rangle_{L^2} + \langle 0, g_2 \rangle_{L^2} \\ &= \int_\Omega b(r) z \overline{g_1(r)}\, dr = z \overline{\int_\Omega b(r) g_1(r)\, dr} = \langle z, \int_\Omega b(r) g_1(r)\, dr \rangle_U. \end{aligned}$$

In the penultimate simplification we use the fact that $z \in \mathbb{C}$ and b is a real valued function. Therefore, the adjoint satisfies $D^* g = \int_\Omega b(r) g_1(r)\, dr$ by Definition 2.6. □

Next we state that the system (5.19) is not exponentially stable. Instead of showing that the resolvent is unbounded on the imaginary axis, we simply refer to the results in [3]. Following [3], we can show that the growth bound of the strongly continuous semigroup associated with the system (5.19) satisfies $\omega_0 \geq 0$. Therefore, the system cannot be exponentially stable.

Theorem 5.12 ([3])**.** *The system defined in* (5.19) *is not exponentially stable.*

Having proven the lack of exponential stability, we embark on studying the polynomial stability of the system (5.19). The damping function $b \in L^2(\Omega)$ plays an important role in stabilizing the system. Therefore, our goal here is to investigate how the damping function affects the stability. The following analysis differs from the analysis in Section 6.2.2 of [15] because we lack the uniform gap between the eigenvalues of $L^{1/2}$. We state the main result of this subsection in the next theorem.

Theorem 5.13. *Suppose* $b \in L^2(\Omega)$ *is the damping function of the system in* (5.19). *If there exist some constants* $K > 0$ *and* $\beta > 0$ *such that for all* $n \in \mathbb{N}$ *we have*

$$|\langle b, \sin(\pi n\cdot)\rangle_{L^2}| \geq K n^{-\beta}, \tag{5.21}$$

then the system in (5.19) *is polynomially stable. That is, there exist some constants* $M > 0$ *and* $t_0 > 0$ *such that for all initial conditions* $(u_0, v_0)^T \in (H^2(\Omega) \cap H_0^1(\Omega))^2$ *and* $(u_1, v_1)^T \in H_0^1(\Omega) \times H_0^1(\Omega)$ *the total energy* $\mathcal{E}_{u,v}(t)$ *in* (5.15) *satisfies*

$$\begin{aligned}\mathcal{E}_{u,v}(t) \leq \frac{M}{t^{1/(1+\beta)}} &\int_\Omega |L_0 u_0 + \kappa v_0|^2 + |\kappa u_0 + L_0 v_0|^2 \, d\xi \\ &+ \frac{M}{t^{1/(1+\beta)}} \int_\Omega \|(\nabla u_1)(\xi)\|^2 + \|(\nabla v_1)(\xi)\|^2 + 2\kappa \operatorname{Re}(u_1(\xi)\overline{v_1(\xi)}) \, d\xi\end{aligned}$$

for all $t \geq t_0$.

Proof. Let $K > 0$ and $\beta > 0$ such that (5.21) holds for all $n \in \mathbb{N}$ and $s \geq s_0 > 0$. Then let L and D be the operators defined in (5.2) and (5.20), respectively. We define the function $\delta(s) = \min\{\frac{\kappa}{2\pi s}, \frac{\kappa}{2\pi}\}$ for all $s > 0$. Similar to the proof of Theorem 5.10, the wavepackets of $L^{1/2}$ are now of the form

$$\mathsf{WP}_{s,\delta(s)}(L^{1/2}) = \operatorname{span}(\{\varphi_n^\pm : s - \delta(s) < \sqrt{\pi^2 n^2 \pm \kappa} < s + \delta(s)\}) \cup \{0\}$$

where $\varphi_n^\pm = (\sin(\pi n\cdot), \pm\sin(\pi n\cdot))^T$ is the normalized eigenvector of $L^{1/2}$ corresponding to the eigenvalue $\sqrt{\pi^2 n^2 \pm \kappa}$ for all $n \in \mathbb{N}$. Now let $s > 0$ and let $w \in \mathsf{WP}_{s,\delta(s)}(L^{1/2})$ be arbitrary. As before, the case $\mathsf{WP}_{s,\delta(s)}(L^{1/2}) = \{0\}$ is trivial. Suppose then $w = c\varphi_n^\pm$ where $c \in \mathbb{C}$ is arbitrary and $n \in \mathbb{N}$ satisfies

$$s - \delta(s) < \sqrt{\pi^2 n^2 \pm \kappa} < s + \delta(s).$$

As $\varphi_n^\pm = (\sin(\pi n\cdot), \pm\sin(\pi n\cdot))^T$ is normalized, we obtain

$$\begin{aligned}\|D^* w\|_U = \|D^*(c\varphi_n^\pm)\|_U &= |c|\left|\int_\Omega b(r)\sin(\pi n r)\, dr\right| \\ &= \left|\int_\Omega b(r)\sin(\pi n r)\, dr\right| |c| \|\varphi_n^\pm\|_H \\ &= \left|\int_\Omega b(r)\overline{\sin(\pi n r)}\, dr\right| \|c\varphi_n^\pm\|_H \\ &= |\langle b, \sin(\pi n\cdot)\rangle_{L^2}| \|w\|_H \\ &\geq K n^{-\beta}\|w\|_H.\end{aligned}$$

Defining $\gamma(s) = K(1+s)^{-\beta}$ for all $s > 0$ now guarantees that $\|D^* w\|_U \geq \gamma(s)\|w\|_H$ for all wavepackets $w \in \mathsf{WP}_{s,\delta(s)}(L^{1/2})$ and $s > 0$. Furthermore, there exists some $s_0 > 0$ such that

$$\gamma(s) \geq \frac{K}{2} s^{-\beta}, \qquad \text{for all } s \geq s_0.$$

To see this, the above inequality is equivalent to having $2 \geq (1 + \frac{1}{s})^{\beta}$ which evidently holds for all sufficiently large values of s.

Now $\gamma(|s|)^2\delta(|s|)^2 \geq \frac{K^2\kappa^2}{16\pi^2}|s|^{-2(1+\beta)}$ for all $|s| \geq \max\{s_0, 1\}$ and Theorem 4.5 implies that the system (5.17) is polynomially stable. As for bounding the total energy $\mathcal{E}_{u,v}(t)$, we obtain directly by substituting $w_0 = (u_0, v_0)^T$ and $w_1 = (u_1, v_1)^T$ into (4.11) that

$$\begin{aligned}\mathcal{E}_{u,v}(t) \leq \frac{M}{t^{1/(1+\beta)}} \int_{\Omega} |L_0 u_0 + \kappa v_0|^2 + |\kappa u_0 + L_0 v_0|^2 \, d\xi \\ + \frac{M}{t^{1/(1+\beta)}} \int_{\Omega} \|(\nabla u_1)(\xi)\|^2 + \|(\nabla v_1)(\xi)\|^2 + 2\kappa \operatorname{Re}(u_1(\xi)\overline{v_1(\xi)}) \, d\xi\end{aligned}$$

for all $t \geq t_0$. □

The above theorem gives us a way to deduce if a given damping function $b \in L^2(\Omega)$ renders the system (5.19) polynomially stable. Note that the bound for the decay rate is worse in Theorem 5.13 than in Theorem 5.10. As $(\sqrt{2}\sin(\pi n\cdot))_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(0,1)$ [31, Thm. 6.37], we can interpret the terms $\langle b, \sin(\pi n\cdot)\rangle_{L^2}$ in the criterion (5.21) as the scaled *Fourier sine coefficients* [35, p. 88][28, p. 307][25, p. 157] of the damping function $b \in L^2(\Omega)$. We denote the coefficients $\langle b, \sin(\pi n\cdot)\rangle_{L^2}$ by b_n for brevity. For a large class of damping functions, we obtain explicit expressions for the scaled Fourier sine coefficients. Table 5.1 contains a few examples of them.

Table 5.1. *Explicit expressions for the scaled Fourier sine coefficients b_n with different damping functions $b \in L^2(\Omega)$.*

$b(\xi)$	1	ξ	$\xi(1-\xi)$	$\xi^2(1-\xi)$
b_n	$\frac{1+(-1)^n}{\pi n}$	$\frac{(-1)^n}{\pi n}$	$\frac{2((-1)^n+1)}{\pi^3 n^3}$	$\frac{2(2(-1)^n-1)}{\pi^3 n^3}$

Next we discuss an interesting consequence of Theorem 5.13. Theorem 5.13 grants us polynomial stability if the damping function $b \in L^2(\Omega)$ satisfies (5.21). However, the question remains, given a certain bound for the decay rate of the system in (5.19), whether we can find a suitable damping function $b \in L^2(\Omega)$ that matches the aforementioned bound for the decay rate. Fortunately, Theorem 5.13 allows us to construct such a damping function under some additional assumptions.

Corollary 5.14. *Suppose $0 < \beta < 2/3$. For a damping function $b \in L^2(\Omega)$ defined by*

$$b(\xi) = \sum_{n=1}^{\infty} n^{1-\frac{1}{\beta}} \sin(\pi n\xi), \qquad \text{for all } \xi \in \Omega, \tag{5.22}$$

the system in (5.19) *is polynomially stable. That is, there exist some constants* $M > 0$ *and* $t_0 > 0$ *such that for all initial conditions* $(u_0, v_0)^T \in (H^2(\Omega) \cap H_0^1(\Omega))^2$ *and* $(u_1, v_1)^T \in H_0^1(\Omega) \times H_0^1(\Omega)$ *the total energy* $\mathcal{E}_{u,v}(t)$ *in* (5.15) *satisfies*

$$\begin{aligned}\mathcal{E}_{u,v}(t) \leq \frac{M}{t^\beta} \int_\Omega |L_0 u_0 + \kappa v_0|^2 + |\kappa u_0 + L_0 v_0|^2 \, d\xi \\ + \frac{M}{t^\beta} \int_\Omega \|(\nabla u_1)(\xi)\|^2 + \|(\nabla v_1)(\xi)\|^2 + 2\kappa \operatorname{Re}(u_1(\xi)\overline{v_1(\xi)}) \, d\xi\end{aligned}$$

for all $t \geq t_0$.

Proof. Let $0 < \beta < 2/3$. We first justify that the damping function b defined in (5.22) is indeed in $L^2(\Omega)$. We know that the sequence $(\sqrt{2}\sin(\pi n \cdot))_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(0, 1)$, which implies that

$$\|b\|_{L^2}^2 = \sum_{n=1}^{\infty} \left| \frac{n^{1-\frac{1}{\beta}}}{\sqrt{2}} \right|^2 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{\beta}-2}}.$$

The above sum converges because now $\frac{2}{\beta} - 2 > \frac{2}{\frac{2}{3}} - 2 = 3 - 2 = 1$. In other words, we have that $b \in L^2(\Omega)$. Now $b_n = \frac{1}{2} n^{1-\frac{1}{\beta}}$ for all $n \in \mathbb{N}$ and therefore the claim follows directly from Theorem 5.13. □

As we have seen, the damping function $b \in L^2(\Omega)$ plays a key role in the stability of the system (5.19). For the rest of this subsection we focus on investigating the properties of the damping function. We begin with an interesting result on regularity. Suppose that we have achieved some polynomial rate of decay with a damping function $b \in L^2(\Omega)$. According to results in [29], we can actually achieve the same polynomial rate of decay with a more regular damping function $\tilde{b} \in C([0, 1])$.

Recall from Corollary 3.13 that any bounded strongly continuous semigroup is exponentially stable exactly when the resolvent of its generator is bounded on the imaginary axis. Therefore, a slower growth rate of the resolvent will result in a faster decay rate for the classical solutions of the system. The growth of the resolvent is linked to the decay rate of b_n [15, Thm. 3.9]. As a rule of thumb, the faster the terms b_n decay, the faster the resolvent grows. Therefore, a slow decay rate for the terms b_n implies a fast decay rate for the classical solutions. The following lemma sheds light on the slowest possible decay rate for the terms b_n.

Lemma 5.15. *Suppose* $b \in L^2(\Omega)$ *and* b_n *is the scaled Fourier sine coefficient in* (5.21) *for all* $n \in \mathbb{N}$. *Then the sequence* $(b_n)_{n\in\mathbb{N}}$ *is in the sequence space*

$$\ell^2 = \left\{ (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{C} : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}.$$

Proof. Note that $(\sqrt{2}\sin(\pi n\cdot))_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(0,1)$. Therefore,

$$\infty > \|b\|_{L^2}^2 = \sum_{n=1}^{\infty} \left|\langle b, \sqrt{2}\sin(\pi n\cdot)\rangle_{L^2}\right|^2 = 2\sum_{n=1}^{\infty} |b_n|^2,$$

implying that $(b_n)_{n\in\mathbb{N}} \in \ell^2$. □

In particular, the scaled Fourier sine coefficients b_n in (5.21) must decay faster than the terms $n^{-1/2}$. Note that in Theorem 5.13 the limiting case $\beta = 2/3$ corresponds to exactly this decay rate.

Remark 5.1. One way to control the decay rate of b_n is to control whether then damping function b is in the domain of the operator L_m defined for all $m \in \mathbb{N}$ by

$$L_m f = L_0^m f, \qquad f \in \mathcal{D}(L_m) = H^{2m}(\Omega) \cap H_0^1(\Omega).$$

Here the operator L_0 is again the positive Dirichlet Laplacian defined in Example 2.3. As discussed in Section 2.1, the subset property of Sobolev spaces imply that for $j, k \in \mathbb{N}$ we have $\mathcal{D}(L_j) \subseteq \mathcal{D}(L_k)$ whenever $j > k$. Definition 2.11 allows us to express the domain of L_m as

$$\mathcal{D}(L_m) = \left\{ f \in L^2(\Omega) : \sum_{n=1}^{\infty} |\lambda_n|^{2m} \left|\langle f, \sqrt{2}\sin(\pi n\cdot)\rangle_{L^2}\right|^2 < \infty \right\} \tag{5.23}$$

because L_m is a positive and self-adjoint operator with compact resolvents. In the above expression the eigenvalues $|\lambda_k|^{2m}$ diverge to infinity quite rapidly. This implies that if b is in the set (5.23) then the terms $\left|\langle b, \sqrt{2}\sin(\pi n\cdot)\rangle_{L^2}\right|^2$ need to counteract this divergence by decaying sufficiently fast. Thus, if we wish for a slow decay rate of b_n then it is necessary that $b \notin \mathcal{D}(L_m)$ for a small $m \in \mathbb{N}$.

Note that the damping function b cannot be in $\mathcal{D}(L_m)$ if it violates either the differentiability requirements or the boundary conditions. For example, in Table 5.1 all the functions are infinitely many times differentiable, but they or their derivatives fail the boundary conditions. For an arbitrary test function $\psi \in C_0^\infty(\Omega)$, things become more interesting as $\psi \in \mathcal{D}(L_m)$ for all $m \in \mathbb{N}$. In particular, this implies that the terms $\left|\langle \psi, \sqrt{2}\sin(\pi n\cdot)\rangle_{L^2}\right|^2$ need to decay faster than any polynomial. As a consequence, the system is neither exponentially stable nor polynomially stable, and we actually arrive at semi-uniform stability. As discussed at the end of Subsection 3.4.2, semi-uniform stability is beyond the scope of this thesis, but see [16] and the references therein for further information.

As our final conclusion, we notice that the system in (5.19) cannot be polynomially stable if $b_n = 0$ for some $n \in \mathbb{N}$. In this case, parts of the imaginary axis are contained in the spectrum of the resolvent, which is not consistent with Definition 3.11.

Proposition 5.16. *Suppose L and D are the operators defined in* (5.2) *and* (5.20)*, respectively, b_n is the scaled Fourier sine coefficient in* (5.21) *for some $n \in \mathbb{N}$ and $\lambda_n^\pm$ is the eigenvalue of L corresponding to the eigenvector $\varphi_n^\pm$ in b_n. Moreover, suppose that A and B are operators defined by* (4.4) *and* (4.5)*, respectively. If $b_n = 0$ for some $n \in \mathbb{N}$, then we have that $\pm i\sqrt{\lambda_n^\pm} \in \sigma(A - BB^*)$.*

Proof. As $b_n = 0$, we have that $D^*\varphi_n^\pm = 0$. Lemma 4.3 implies that $\pm i\sqrt{\lambda_n^\pm} \in \sigma(A)$. Now for an eigenvector $\psi_n = (\varphi_n^\pm, \pm i\sqrt{\lambda_n^\pm}\varphi_n^\pm)^T$ of A we have

$$\begin{aligned}
(\pm i\sqrt{\lambda_n^\pm} - (A - BB^*))\psi_n &= (\pm i\sqrt{\lambda_n^\pm} - A)\psi_n + BB^*\psi_n \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ DD^*(\pm i\sqrt{\lambda_n^\pm}\varphi_n^\pm) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \pm i\sqrt{\lambda_n^\pm} DD^*\varphi_n^\pm \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

The above implies that $\pm i\sqrt{\lambda_n^\pm} \in \sigma(A - BB^*)$ by Definition 2.4. $\square$

To obtain polynomial stability, we therefore require $b_n \neq 0$ for all $n \in \mathbb{N}$. We see in Table 5.1 that even some simple damping functions $b \in L^2(\Omega)$ fail this requirement. We observe that $\sin(\pi n\xi)$ is symmetric on the interval $(0, 1)$ for all odd n and anti-symmetric for all even n with respect to the midpoint of the interval. Therefore, any damping function $b \in L^2(\Omega)$ which is either symmetric or anti-symmetric on the interval $(0, 1)$ with respect to the midpoint will not produce polynomial stability. This is because the product of a symmetric and anti-symmetric function will be an anti-symmetric function. The integral of such a function is zero if the domain of integration is symmetric to the symmetry axis. In other words, for such $b \in L^2(\Omega)$ some of the terms b_n will be zero, and we will not obtain polynomial stability for the system (5.19).

6. CONCLUSIONS AND DISCUSSION

The abstract wave equation in (4.1) models a linear dynamical system that encapsulates various types of second order partial differential equations arising in practical applications. In particular, we can use abstract wave equations to model how waves behave when exposed to different dampings in the domain, how certain coupled systems interact and how structural beams vibrate, for example. Because every abstract wave equation has an associated strongly continuous semigroup by Theorem 4.2, we know that abstract wave equations are well-posed in the sense of Definition 3.6. In addition to well-posedness, Theorem 4.5 gives us sufficient conditions for polynomial stability of abstract wave equations. The polynomial decay rate of the total energy in Theorem 4.5 follows from a more general result [15, Thm. 3.9] and Theorem 3.15, stating the equivalence between polynomial stability and polynomial growth rate of the resolvent on the imaginary axis.

In Chapter 5 we study a coupled system of wave equations with three different types of damping. We notice that we can recast the systems in (5.16), (5.17) and (5.19) as abstract wave equations. Furthermore, we observe in Section 5.3 that for each type of damping the coupled system is strongly stable. However, only the system (5.16) with two viscous dampings is exponentially stable as stated in Theorem 5.7. By Theorem 5.10 and Theorem 5.13 we obtain polynomial stability for the systems (5.17) and (5.19) with partial viscous damping and weak damping, respectively. However, we point out that the decay rates of the total energy in Theorem 5.7, Theorem 5.10 and Theorem 5.13 are not necessarily sharp.

For further study, we have a lot of options. One natural way to generalize the systems in Chapter 5 is to advance from the one-dimensional domain $\Omega = (0, 1)$ into more general closed and bounded two-dimensional domains $\Omega \subseteq \mathbb{R}^2$ with suitable boundaries. Another option is to generalize the damping that the system experiences. Now the damping affects uniformly on the domain, but perhaps a non-uniform damping is equally interesting. Also, there is really no reason why the damping should remain inside the domain. We can modify the damping so that the system is only damped on the boundary. In doing so, we have to leave the framework presented in Chapter 4 behind and embrace the more advanced framework in [15]. Fortunately, this thesis provides us with a solid foundation to pursue the aforementioned systems and their stability.

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