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ORIGINAL ARTICLE

Robust output regulation of the linearized Boussinesq equations with boundary control and observation

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Abstract
We study a temperature and velocity output tracking problem for a two-dimensional room model with the fluid dynamics governed by the linearized translated Boussinesq equations. Additionally, the room model includes finite-dimensional models for actuation and sensing dynamics; thus, the complete model dynamics are governed by an ODE–PDE–ODE cascade. As the main contribution, we design a low-dimensional internal model-based controller for robust output tracking of the room model. The controller's performance is demonstrated through a numerical example.

Keywords Partial differential equations · Output regulation · Linear systems · Fluid flows · Coupled systems

1 Introduction

We consider fluid temperature and velocity control for a two-dimensional room model. In the model, behavior of the fluid within the room is described by the linearized Boussinesq equations. The Boussinesq equations are a system of partial differential equations coupling the fluid flow dynamics given by the incompressible Navier–Stokes

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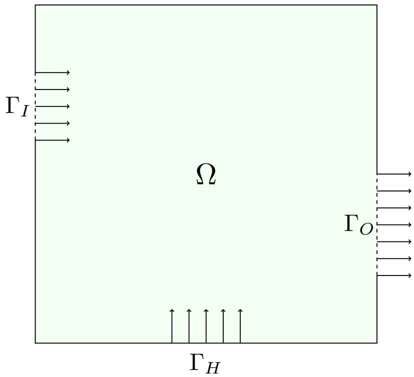


Fig. 1 A room with the boundary regions of interest highlighted

equations with the fluid temperature dynamics governed by the advection–diffusion equation, and they are commonly used for modeling non-isothermal flows, see, e.g., [1,10,12]. In this paper, we consider the linearized Boussinesq equations. As the main control problem, we study output tracking for the room model, where a mix of observations on the fluid temperature and velocity must converge to a desired reference trajectory over time, i.e.,

$$\|y(t) - y_{\text{ref}}(t)\| \to 0 \quad \text{as} \quad t \to \infty, \tag{1}$$

where $y(t) \in \mathbb{R}^{p_Y}$ is the observation and $y_{\text{ref}}(t) \in \mathbb{R}^{p_Y}$ is the reference output. The considered reference outputs are of the form

$$y_{\text{ref}}(t) = a_0(t) + \sum_{i=1}^{q_s} a_i(t)\cos(\omega_i t) + b_i(t)\sin(\omega_i t), \tag{2}$$

where $0 = \omega_0 < \omega_1 < \cdots < \omega_{q_s}$ are known frequencies and $a_i(t), b_i(t) \in \mathbb{R}^{p_Y}$ are polynomial vectors with possibly unknown coefficients but known maximal degrees. As the main contribution of this paper, we design a finite-dimensional controller for output tracking of the room model with the room geometry depicted in Fig. 1.

Fluid dynamics within the room are governed by the linearized translated Boussinesq equations. We focus on a control setup typical for rooms, where the physical control inputs act on the fluid near some parts of the boundary of the room, i.e., the walls, the floor or the roof, cf. [10,12]. In the model, the fluid flows into and out of the room through the boundary regions Γ_I and Γ_O, which represent an inlet and an outlet, respectively. Both the fluid velocity and the fluid temperature are controlled within Γ_I, cf. [12]. Additionally, the fluid temperature is controlled within Γ_H by a radiating heater, but no velocity control is applied within Γ_H and there is no fluid flow through this boundary section. Observations on the fluid are performed both within the boundary regions and inside the spatial domain. In addition to the fluid dynamics, the room model includes finite-dimensional dynamical models for the *actuators* and the *sensors* related to the fluid control and observation, respectively. Dynamic actuator modeling has been reported to increase model accuracy for an acoustic model in [46], and has been argued to be a more realistic approach to system modeling in general [13]. The complete room model is thus a coupled ODE–PDE–ODE model. Compared to a model with PDE dynamics only, the room model is more complex in the sense of having extended dynamics, but the control and observation operations are bounded. From now on, we will refer to the full ODE–PDE–ODE room model as the *cascade system*.

We achieve the output convergence (1) for the room model by implementing a controller introduced in [34]. The controller is based on the *internal model principle*, see [17,19,35], and has several desirable properties. It can be used in control of unstable systems, which is essential for this paper due to the fact that the linearized Boussinesq equations may be unstable [12] (depending on the room geometry and physical parameters). The controller does not require complete state information of the system but rather only uses the observation $y(t)$, and since the controller is based on a finite-dimensional approximation of the room model combined with model reduction, it is of low-order for fast computations. Finally, the controller is *robust* in that it tolerates small system uncertainties and rejects disturbance signals of the form

$$u_d(t) = c_0(t) + \sum_{i=1}^{q_s} c_i(t)\cos(\omega_i t) + d_i(t)\sin(\omega_i t), \tag{3}$$

which can be applied either within the boundary or inside the spatial domain of the room. Here, ω_i are the same frequencies as in (2) and $c_i(t), d_i(t) \in \mathbb{R}^d$ are polynomial vectors with possibly unknown coefficients but known maximal degrees. For linear systems, also several alternative output tracking controllers have been designed. However, these control solutions typically lack the robustness property of fault tolerance and disturbance rejection, cf. [14,18,45], are designed for stable systems only, cf. [22,38], or are infinite-dimensional, cf. [23,33].

Most of the previous work regarding control of the Boussinesq equations focuses on stabilization [12,26,37,42]. Examples of output tracking for both nonlinear and linear thermal fluid flows based on state feedback have been considered in [1] and references therein, and solution methods for the related regulator equations have been further developed in [2,3]. Additionally, robust output tracking for a simplified room model with only temperature dynamics and in-domain control and observation has been studied in [27]. Finally, addition of the actuator dynamics for classes of linear systems has previously been considered in [11,13,31].

In this work, we utilize the concept of *abstract boundary control systems*, see [16, Ch. 3.3], [15], [40, Ch. 10], to formulate the abstract system presentation for the cascade system with temperature, velocity and ODE dynamics. The boundary control system framework appears to not have been used in the analysis of incompressible Navier–Stokes-type fluid flows previously, yet it is a natural presentation choice for systems with boundary inputs and can be translated to the more familiar *abstract state space formulation*, see [40, Ch. 10]. We also use the boundary control system framework to study effects of the additional ODE dynamics on exponential stabilizability and exponential detectability of the room model. Furthermore, the boundary control system formulation could be used to justify alternative controller designs, namely those introduced in [28].

The paper is organized as follows. In Sect. 2, we first present the complete room model. We formulate the cascade system as an abstract linear control system in Sect. 2.2. Section 2.3 is devoted for stabilizability and detectability analysis of the cascade system in terms of us presenting sufficient conditions for these properties. In Sect. 3, we couple the room model with an error feedback controller to guarantee the output convergence (1). The controller is based on [34] and we present the design process for the cascade system. In Sect. 4, we present a numerical example of robust output tracking for the boundary controlled linearized Boussinesq equations with a mix of boundary and in-domain observations and including actuator and sensor dynamics. Finally, the paper is concluded in Sect. 5.

We use the following notation. For a linear operator A, $D(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote its domain, range and kernel, respectively. The spectrum of A is denoted by $\sigma(A)$ and the resolvent set by $\rho(A)$. The set of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. Finally, $\langle \cdot, \cdot \rangle_\Omega$ and $\langle \cdot, \cdot \rangle_\Gamma$ denote the L^2-inner product or duality pairing on the two-dimensional domain Ω and on the one-dimensional domain Γ, respectively.

2 The room model

We consider a two-dimensional model of a room depicted in Fig. 1 with the interior Ω and the boundary Γ. The room has two disjoint vents; an inlet Γ_I and an outlet Γ_O. We denote the walls of the room by $\Gamma_W = \Gamma \setminus (\Gamma_I \cup \Gamma_O)$ and assume "no-slip" velocity condition at the walls. Regarding temperature, we assume that there is a radiative heater located within $\Gamma_H \subset \Gamma_W$ and on $\Gamma_W \setminus \Gamma_H$ the temperature is fixed. In addition to the radiator, the fluid flow and the fluid temperature within the room are affected by Robin boundary control within the inlet. Finally, the fluid is assumed to be stress-free with unforced heat flux within the outlet.

We next formulate the linearized Boussinesq equations around a steady-state solution of the Boussinesq equations. The linearized Boussinesq equations are used to describe the flow and temperature evolution of the fluid within the room. The system of PDEs is coupled with abstract ODE systems governing the actuation and sensing dynamics, and we present the cascade system in an abstract form. Finally, we consider stabilizability and detectability properties of the cascade system

2.1 The linearized translated Boussinesq equations with actuation and sensing

The Boussinesq equations for $\xi \in \Omega$ and $t \geq 0$ are given by

$$\begin{aligned} \dot{w}(\xi, t) = {} & \frac{1}{Re} \Delta w(\xi, t) - (w(\xi, t) \cdot \nabla) w(\xi, t) - \nabla q(\xi, t) \\ & + \hat{e}_2 \frac{Gr}{Re^2} T(\xi, t) + f_w(\xi), \end{aligned} \tag{4a}$$

$$\dot{T}(\xi,t) = \frac{1}{RePr}\Delta T(\xi,t) - w(\xi,t)\cdot\nabla T(\xi,t) + f_T(\xi), \tag{4b}$$

$$0 = \nabla\cdot w(\xi,t), \quad w(\xi,0) = w_0(\xi), \quad T(\xi,0) = T_0(\xi), \tag{4c}$$

where T is the temperature, q is the pressure, and $w = [w_1,\ w_2]^{\mathrm{T}}$ is the velocity of the fluid. The functions f_w and f_T represent a body force and a heat source, respectively, and $\hat{e}_2 = [0,\ 1]^{\mathrm{T}}$ indicates the direction of buoyancy. Finally, the condition $0 = \nabla\cdot w(\xi,t)$ describes incompressibility of the fluid, Re is the Reynolds number, Gr is the Grashof number and Pr is the Prandtl number. By linearizing the above equations around a steady-state solution (w_{ss}, q_{ss}, T_{ss}) of (4) using the change of variables $v(\xi,t) = w(\xi,t) - w_{ss}(\xi)$, $\theta(\xi,t) = T(\xi,t) - T_{ss}(\xi)$, $p(\xi,t) = q(\xi,t) - q_{ss}(\xi)$, we arrive at the linearized translated Boussinesq equations

$$\begin{aligned}\dot{v}(\xi,t) = {} & \frac{1}{Re}\Delta v(\xi,t) - (w_{ss}(\xi)\cdot\nabla)v(\xi,t) - (v(\xi,t)\cdot\nabla)w_{ss}(\xi) \\ & - \nabla p(\xi,t) + \hat{e}_2\frac{Gr}{Re^2}\theta(\xi,t),\end{aligned} \tag{5a}$$

$$\dot{\theta}(\xi,t) = \frac{1}{RePr}\Delta\theta(\xi,t) - w_{ss}(\xi)\cdot\nabla\theta(\xi,t) - v(\xi,t)\cdot\nabla T_{ss}(\xi), \tag{5b}$$

$$0 = \nabla\cdot v(\xi,t), \quad v(\xi,0) = v_0(\xi), \quad \theta(\xi,0) = \theta_0(\xi). \tag{5c}$$

We consider the linearized translated Boussinesq equations subject to the boundary conditions

$$\big(\mathcal{T}(v(\xi,t),\, p(\xi,t))\cdot n + \alpha_v v(\xi,t)\big)\big|_{\Gamma_I} = \big[b_{v_I}(\xi)\ b_{dv_I}(\xi)\big]\begin{bmatrix} u_{bv_I}(t) \\ u_{dv_I}(t)\end{bmatrix}, \tag{5d}$$

$$\left(\frac{1}{RePr}\frac{\partial\theta}{\partial n}(\xi,t) + \alpha_\theta\theta(\xi,t)\right)\bigg|_{\Gamma_I} = \big[b_{\theta_I}(\xi)\ b_{d\theta_I}(\xi)\big]\begin{bmatrix} u_{b\theta_I}(t) \\ u_{d\theta_I}(t)\end{bmatrix}, \tag{5e}$$

$$\left(\frac{1}{RePr}\frac{\partial\theta}{\partial n}(\xi,t)\right)\bigg|_{\Gamma_H} = \big[b_{\theta_H}(\xi)\ b_{d\theta_H}(\xi)\big]\begin{bmatrix} u_{b\theta_H}(t) \\ u_{d\theta_H}(t)\end{bmatrix}, \tag{5f}$$

$$\big(\mathcal{T}(v(\xi,t),\, p(\xi,t))\cdot n\big)|_{\Gamma_O} = 0, \quad v(\xi,t)|_{\Gamma_W} = 0, \tag{5g}$$

$$\frac{\partial\theta}{\partial n}(\xi,t)|_{\Gamma_O} = 0, \quad \theta(\xi,t)|_{(\Gamma_W\setminus\Gamma_H)} = 0, \tag{5h}$$

where n denotes the unit outward normal vector of Γ, $\mathcal{T}$ is the fluid Cauchy stress tensor and $\alpha_v, \alpha_\theta \geq 0$ are constants, $u_b = [u_{bv_I},\ u_{b\theta_I},\ u_{b\theta_H}]^{\mathrm{T}}$ are control inputs, $u_d = [u_{dv_I},\ u_{d\theta_I},\ u_{d\theta_H}]^{\mathrm{T}}$ are disturbance inputs, and the control and the disturbance inputs are applied via the shape functions $b_{v_I}, b_{\theta_I}, b_{\theta_H}, b_{dv_I}, b_{d\theta_I}$ and $b_{d\theta_H}$. The inputs $u_b(t)$ are not directly generated by the controller, but are rather given as the output of the finite-dimensional actuator

$$\dot{x}_a(t) = A_a x_a(t) + B_a u(t), \quad x_a(0) = x_{a0} \in \mathbb{R}^{n_a}, \tag{6a}$$

$$u_b(t) = C_a x_a(t), \tag{6b}$$

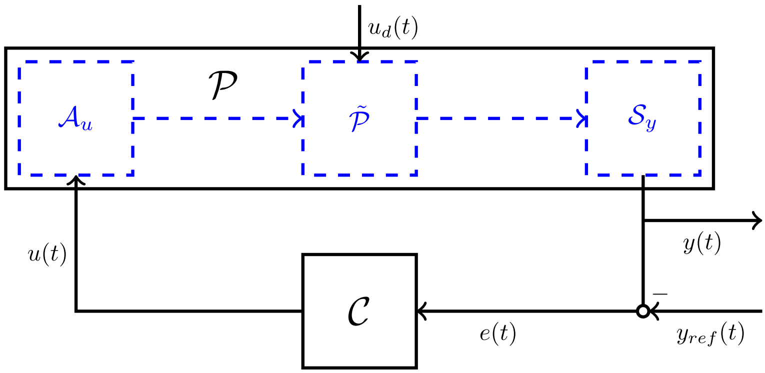


Fig. 2 A closed-loop control scheme with an actuator $\mathcal{A}_u$, a plant $\tilde{\mathcal{P}}$, a sensor $\mathcal{S}_y$, the cascade system $\mathcal{P}$ and a controller $\mathcal{C}$

which takes as its input the control signal $u(t)$ generated by the controller.

We are mainly interested in two types of observations. These are weighted temperature or velocity averages either over a two-dimensional domain $\Omega_C \subset \Omega$, given by

$$y_\Omega(t) = \left\langle c_\Omega(\xi), \left[v(\xi,t)\ \theta(\xi,t)\right]^{\mathrm{T}} \right\rangle_{\Omega_C}, \tag{7a}$$

or over a one-dimensional domain $\Gamma_C \subset \Gamma$, given by

$$y_\Gamma(t) = \left\langle c_\Gamma(\xi), \left[v(\xi,t)\ \theta(\xi,t)\right]^{\mathrm{T}} \right\rangle_{\Gamma_C}, \tag{7b}$$

and we denote by y_b the observation of interest consisting of a combination of the two types. Note that one may include several observations of one type with the restriction that one needs to increase the number of inputs $u(t)$ accordingly to at least match the number of observations, cf. Assumption 3.1 in Sect. 3. These additional inputs are included in (5d)–(5f) by considering vector valued u_{bv_I}, $u_{b\theta_I}$, $u_{b_{\theta_H}}$, b_{v_I}, b_{θ_I} and b_{θ_H}. Just as in the case of the fluid input, the fluid output is also processed by a finite-dimensional system, the sensor

$$\dot{x}_s(t) = A_s x_s(t) + B_s y_b(t), \quad x_s(0) = x_{s0} \in \mathbb{R}^{n_s}, \tag{8a}$$

$$y(t) = C_s x_s(t) \tag{8b}$$

with the observation $y(t)$. The complete plant thus consists of the linearized translated Boussinesq equations (5) coupled with the actuator (6) via the input u_b and with the sensor (8) via the output y_b. Recall that we refer to the system (5)–(8) as the cascade system. Figure 2 depicts the control scheme consisting of the cascade system and an *error feedback controller*.

The control goal is considered for the observation $y(t)$ of the sensor, which we want to converge exponentially to a prescribed reference trajectory $y_{\mathrm{ref}}(t)$ of the form

(2) despite the disturbance signal $u_d(t)$ given by (3). That is, for some $M_r, \omega_r > 0$ it should hold that

$$\|y(t) - y_{\mathrm{ref}}(t)\| \leq M_r \mathrm{e}^{-\omega_r t} P_0, \tag{9}$$

where P_0 is determined by the initial states of the linearized translated Boussinesq equations, the actuator, the sensor and the controller and the coefficients $a_i(t)$, $b_i(t)$, $c_i(t)$ and $d_i(t)$ of the reference signal (2) and the disturbance signal (3).

2.2 Abstract formulation of the control system

The controller to be implemented achieves output tracking for a class of abstract linear systems, which motivates us to present the cascade system as one. We define the system dynamics operator based on a weak formulation of the cascade system. We then follow up with formulation of the operators related to the abstract boundary control system representation of the cascade system, a formulation choice natural in the presence of boundary inputs such as (5d)–(5f). Finally, we connect the abstract boundary control system framework to the abstract state space formulation of the cascade system.

To prepare for the formulations, we define the spaces

$$\begin{aligned}
X_v &= \left\{ v \in (L^2(\Omega))^2 \middle| \nabla \cdot v = 0,\ (v \cdot n)|_{\Gamma_W} = 0 \right\},\\
X_b &= X_v \times L^2(\Omega),\\
X_\Gamma &= (L^2(\Gamma_I))^2 \times L^2(\Gamma_I) \times L^2(\Gamma_H),\\
H_v &= \left\{ v \in (H^1(\Omega))^2 \middle| \nabla \cdot v = 0,\ v|_{\Gamma_W} = 0 \right\},\\
H_\theta &= \left\{ \theta \in H^1(\Omega) \middle| \theta|_{(\Gamma_W \setminus \Gamma_H)} = 0 \right\},\\
H_b &= H_v \times H_\theta
\end{aligned}$$

concerning the Boussinesq equations and the spaces

$$X = X_b \times \mathbb{R}^{n_a} \times \mathbb{R}^{n_s}, \tag{10a}$$
$$H = H_b \times \mathbb{R}^{n_a} \times \mathbb{R}^{n_s} \tag{10b}$$

concerning the cascade system. Furthermore, for all $x = [x_b,\ x_a,\ x_s]^{\mathrm{T}} \in X$, where $x_b = [v,\ \theta]^{\mathrm{T}}$, we define the norms

$$\|x\|_X^2 = \|x_b\|_{X_b}^2 + \|x_a\|_{\mathbb{R}^{n_a}}^2 + \|x_s\|_{\mathbb{R}^{n_s}}^2, \tag{11a}$$
$$\|x\|_H^2 = \|x_b\|_{H_b}^2 + \|x_a\|_{\mathbb{R}^{n_a}}^2 + \|x_s\|_{\mathbb{R}^{n_s}}^2, \tag{11b}$$

and denote the input space $U = \mathbb{R}^m$, the output space $Y = \mathbb{R}^{p_Y}$ and the disturbance space $U_d = \mathbb{R}^d$.

The presented observations (7) are not the only possible choices, and before focusing on the system as a whole we present an assumption characterizing the class of suitable observations

$$y_b(t) = C_b x_b(t) \tag{12}$$

on the linearized Boussinesq equations.

Assumption 2.1 The observation operator satisfies $C_b \in \mathcal{L}(H_b, Y_b)$ for some $Y_b := \mathbb{R}^{p_b}$.

Lemma 2.2 *For system (5), both the observation $y_\Omega = \langle c_\Omega, x_b \rangle_{\Omega_C}$ and $y_\Gamma = \langle c_\Gamma, x_b \rangle_{\Gamma_C}$ in (7) with $c_\Omega \in (L^2(\Omega_C))^2 \times L^2(\Omega_C)$ and $c_\Gamma \in (L^2(\Gamma_C))^2 \times L^2(\Gamma_C)$ satisfy Assumption 2.1.*

Proof Clearly $\langle c_\Omega, \cdot \rangle_{\Omega_C} \in \mathcal{L}(X_b, \mathbb{R})$. Due to properties of the trace operator, we have for a constant $k > 0$

$$\langle c_\Gamma, x_b \rangle_{\Gamma_C} \leq \|c_\Gamma\|_{L^2(\Gamma_C)} \|x_b\|_{L^2(\Gamma_C)} \leq k \|x_b\|_{H^1(\Omega)},$$

thus $\langle c_\Gamma, \cdot \rangle_{\Gamma_C} \in \mathcal{L}(H_b, \mathbb{R})$. □

Existence and uniqueness of steady-state solutions for the Boussinesq equations are outside the scope of this work. For the following analysis of the cascade system, we assume that a weak steady-state solution

$$(w_{ss}, q_{ss}, T_{ss}) \in H_v \times L^2(\Omega) \times H_\theta$$

for the Boussinesq equations (4) exists. Discussion on existence and uniqueness of steady-state solutions for the Boussinesq equations can be found in, e.g., [29].

As the first step toward abstract formulation of the cascade system, we construct the system dynamics operator A via a weak formulation of the cascade system and verify that it generates a strongly continuous semigroup on X. To that end, we define the bilinear and trilinear forms

$$a_v(v, \psi) = \frac{2}{Re} \langle \epsilon(v), \epsilon(\psi) \rangle_\Omega + \alpha_v \langle v, \psi \rangle_{\Gamma_I} \quad \forall v, \psi \in H_v, \tag{13a}$$

$$a_\theta(\theta, \phi) = \frac{1}{Re\,Pr} \langle \nabla\theta, \nabla\phi \rangle_\Omega + \alpha_\theta \langle \theta, \phi \rangle_{\Gamma_I} \quad \forall \theta, \phi \in H_\theta, \tag{13b}$$

$$b_v(v_1, v_2, \psi) = \langle (v_1 \cdot \nabla) v_2, \psi \rangle_\Omega \quad \forall v_1, v_2, \psi \in H_v, \tag{13c}$$

$$b_\theta(v, \theta, \phi) = \langle v \cdot \nabla\theta, \phi \rangle_\Omega \quad \forall v \in X_v, \ \forall \theta, \phi \in H_\theta, \tag{13d}$$

$$b_0(\theta, \psi) = \left\langle \hat{e}_2 \frac{Gr}{Re^2} \theta, \psi \right\rangle_\Omega \quad \forall \theta \in L^2(\Omega), \ \forall \psi \in (L^2(\Omega))^2. \tag{13e}$$

Here,

$$\epsilon(v) = \frac{1}{2} \big(\nabla v + (\nabla v)^{\mathrm{T}} \big) \quad \forall v \in (H^1(\Omega))^2,$$

thus the Cauchy stress tensor is given by

$$\mathcal{T}(v, p) = \frac{2}{Re} \epsilon(v) - pI.$$

Note that $0 = -a_\theta(\theta, \phi)$ corresponds to a weak formulation of the stationary diffusion equation for the temperature subject to (5e), (5f) and (5h) with zero control and disturbance, i.e., when

$$\left(\frac{1}{RePr}\frac{\partial\theta}{\partial n} + \alpha_\theta\theta\right)\Big|_{\Gamma_I} = 0, \quad \left(\frac{1}{RePr}\frac{\partial\theta}{\partial n}\right)\Big|_{\Gamma_H} = 0.$$

Similarly, due to Stokes formula and incompressibility, $0 = -a_v(v, \psi)$ corresponds to a weak formulation of the stationary Stokes equation subject to (5d) and (5g) with zero control and disturbance, i.e., when

$$\big(\mathcal{T}(v, p)\cdot n + a_v v\big)\big|_{\Gamma_I} = 0.$$

Consider the cascade system (5)–(8) subject to a constant boundary disturbance signal $u'_d = [u'_{dv_I},\ u'_{d\theta_I},\ u'_{d\theta_H}]^{\mathrm{T}}$ and denote $g_{dv_I} = b_{v_I}u'_{bv_I}$, $g_{d\theta_I} = b_{\theta_I}u'_{d\theta_I}$, $g_{d\theta_H} = b_{\theta_H}u'_{d\theta_H}$. Now the boundary conditions (5d)–(5f) for the cascade system are

$$\begin{aligned}
\big(\mathcal{T}(v, p)\cdot n + \alpha_v v\big)\big|_{\Gamma_I} &= b_v C_{a_v} x_a + g_{dv_I},\\
\left(\frac{1}{RePr}\frac{\partial\theta}{\partial n} + \alpha_\theta\theta\right)\Big|_{\Gamma_I} &= b_{\theta_I} C_{a_{\theta_I}} x_a + g_{d\theta_I},\\
\left(\frac{1}{RePr}\frac{\partial\theta}{\partial n}\right)\Big|_{\Gamma_H} &= b_{\theta_H} C_{a_{\theta_H}} x_a + g_{d\theta_H},
\end{aligned}$$

where C_{a_v}, $C_{a_{\theta_I}}$ and $C_{a_{\theta_H}}$ are obtained from $C_a = [C_{a_v},\ C_{a_{\theta_I}},\ C_{a_{\theta_H}}]^{\mathrm{T}}$. A weak formulation for a steady-state solution of the cascade system subject to u'_d and a constant control input u' is now given by

$$\begin{aligned}
0 &= -a_v(v,\psi) - b_v(v, w_{ss}, \psi) - b_v(w_{ss}, v, \psi) + b_0(\theta,\psi) + \langle \mathcal{T}(v,p)\cdot n, \psi\rangle_{\Gamma_I}\\
&\quad + \alpha_v\langle v, \psi\rangle_{\Gamma_I}\\
&= -a_v(v,\psi) - b_v(v, w_{ss}, \psi) - b_v(w_{ss}, v, \psi) + b_0(\theta,\psi)\\
&\quad + \langle b_v C_{a_v} x_a + g_{dv_I}, \psi\rangle_{\Gamma_I} \quad \forall\psi \in H_v,\\
0 &= -a_\theta(\theta,\phi) - b_\theta(w_{ss},\theta,\phi) - b_\theta(v, T_{ss}, \phi) + \frac{1}{RePr}\Big\langle\frac{\partial\theta}{\partial n}, \phi\Big\rangle_{\Gamma_I\cup\Gamma_H} + \alpha_\theta\langle\theta,\phi\rangle_{\Gamma_I}\\
&= -a_\theta(\theta,\phi) - b_\theta(w_{ss},\theta,\phi) - b_\theta(v, T_{ss}, \phi) + \langle b_{\theta_I} C_{\theta_I} x_a + g_{d\theta_I}, \phi\rangle_{\Gamma_I}\\
&\quad + \langle b_{\theta_H} C_{\theta_H} x_a + g_{d\theta_H}, \phi\rangle_{\Gamma_H} \quad \forall\phi\in H_\theta,\\
0 &= \langle A_a x_a, \psi_a\rangle_{\mathbb{R}^{n_a}} + \langle B_a u', \psi_a\rangle_{\mathbb{R}^{n_a}} \quad \forall\psi_a \in \mathbb{R}^{n_a},\\
0 &= \langle A_s x_s, \psi_s\rangle_{\mathbb{R}^{n_s}} + \Big\langle B_s C_b \begin{bmatrix} v\\ \theta\end{bmatrix}, \psi_s\Big\rangle_{\mathbb{R}^{n_s}} \quad \forall\psi_s\in\mathbb{R}^{n_s}.
\end{aligned}$$

Motivated by the weak formulation, we define for $u'_d = 0$ and $u' = 0$ the bilinear form

$$\begin{aligned}
a_0(\Phi,\Psi) &= a_0((v,\theta,x_a,x_s),(\psi,\phi,\psi_a,\psi_s))\\
&= a_v(v,\psi) + a_\theta(\theta,\phi) + b_v(v, w_{ss},\psi) + b_v(w_{ss}, v, \psi) + b_\theta(w_{ss},\theta,\phi)
\end{aligned}$$

$$+ b_\theta(v, T_{ss}, \phi) - b_0(\theta, \psi) - \langle b_{v_I} C_{a_v} x_a, \psi\rangle_{\Gamma_I} - \langle b_{\theta_I} C_{a_{\theta_I}} x_a, \phi\rangle_{\Gamma_I}$$
$$- \langle b_{\theta_H} C_{a_{\theta_H}} x_a, \phi\rangle_{\Gamma_H} - \left\langle B_s C_b \left[v\ \theta\right]^{\mathrm{T}}, \psi_s\right\rangle_{\mathbb{R}^{n_s}} - \langle A_a x_a, \psi_a\rangle_{\mathbb{R}^{n_a}}$$
$$- \langle A_s x_s, \psi_s\rangle_{\mathbb{R}^{n_s}} \quad \forall \Phi, \Psi \in H, \tag{14}$$

and more generally for $u' = 0$ and some $u'_d \in \mathbb{R}^d$ the bilinear form

$$a_g(\Phi, \Psi) = a_0((v, \theta, x_a, x_s), (\psi, \phi, \psi_a, \psi_s)) - \langle g_{dv_I}, \psi\rangle_{\Gamma_I} - \langle g_{d\theta_I}, \phi\rangle_{\Gamma_I}$$
$$- \langle g_{d\theta_H}, \phi\rangle_{\Gamma_H} \quad \forall \Phi, \Psi \in H. \tag{15}$$

Using the bilinear form $a_0(\cdot, \cdot)$, we define the linear operator A by

$$\langle Ax, \Psi\rangle_X = -a_0(x, \Psi),$$
$$D(A) = \left\{x \in H \,\middle|\, \forall \Psi \in H,\ \Psi \to a_0(x, \Psi) \text{ is } X\text{-continuous}\right\}. \tag{16}$$

Note that the geometry of Ω and presence of the mixed boundary conditions reduce regularity of the solutions of (5) so that $D(A) \not\subset (H^2(\Omega))^2 \times H^2(\Omega) \times \mathbb{R}^{n_a} \times \mathbb{R}^{n_s}$, cf. [12,24,30].

The following semigroup generation result is not only needed for the abstract system formulation but also the coercivity and boundedness results for the bilinear form will be utilized for the controller implementation to achieve output tracking, cf. [34]. Similar results focusing mainly on semigroup generation instead of coercivity and boundedness of the bilinear forms for both the linearized Boussinesq equations and the linearized incompressible Navier–Stokes equations without additional ODE dynamics have been presented in multiple papers, see, e.g., [12,24,32].

Theorem 2.3 *Operator A is the generator of an analytic semigroup on X and the bilinear form $a_0(\cdot, \cdot)$ is H-bounded and H-coercive, i.e., H is continuously and densely embedded in X and there exist $c, \lambda, \gamma > 0$ such that for all $\Phi, \Psi \in H$*

$$|a_0(\Phi, \Psi)| \le c\|\Phi\|_H \|\Psi\|_H,$$
$$a_0(\Phi, \Phi) \ge \gamma\|\Phi\|_H^2 - \lambda\|\Phi\|_X^2.$$

Proof Throughout the proof, we denote by c a generic positive constant which may have a different value for each occurrence. We start by considering the terms $a_\theta(\cdot, \cdot)$ and $a_v(\cdot, \cdot)$. Now properties of the trace operator imply

$$0 \le \alpha_\theta \langle \theta, \theta\rangle_{\Gamma_I} \le c\|\theta\|_{H^1}^2,$$

thus using Poincare's inequality we get for $\theta \in H_\theta$ and a constant $c_\theta > 0$

$$a_\theta(\theta, \theta) = \frac{1}{RePr}\langle \nabla\theta, \nabla\theta\rangle_\Omega + \alpha_\theta\langle \theta, \theta\rangle_{\Gamma_I} \ge c_\theta\|\theta\|_{H^1}^2, \tag{17}$$

i.e., $a_\theta(\cdot,\cdot)$ is H_θ-coercive. Since

$$\begin{aligned}|a_\theta(\theta,\phi)| &\le \left|\frac{1}{RePr}\langle\nabla\theta,\nabla\phi\rangle_\Omega\right| + |\alpha_\theta\langle\theta,\phi\rangle_{\Gamma_I}| \\ &\le \frac{1}{RePr}\|\theta\|_{H^1}\|\phi\|_{H^1} + c\|\theta\|_{H^1}\|\phi\|_{H^1}, \end{aligned} \tag{18}$$

$a_\theta(\cdot,\cdot)$ is also H_θ-bounded.

Regarding $a_v(\cdot,\cdot)$, it similarly holds that

$$0 \le \alpha_v\langle v,v\rangle_{\Gamma_I} \le c\|v\|^2_{H^1},$$

and the norm $\|\epsilon(\cdot)\|_{L^2}$ is equivalent to the norm $\|\cdot\|_{H^1}$ through Korn's and Poincare's inequalities. Now for $v\in H_v$ and a constant $c_v>0$

$$a_v(v,v) = \frac{2}{Re}\langle\epsilon(v),\epsilon(v)\rangle_\Omega + \alpha_v\langle v,v\rangle_{\Gamma_I} \ge \frac{2}{Re}\|\epsilon(v)\|^2_{L^2} \ge c_v\|v\|^2_{H^1}, \tag{19}$$

thus $a_v(\cdot,\cdot)$ is H_v-coercive. Since additionally

$$\begin{aligned}|a_v(v,\psi)| &\le \left|\frac{2}{Re}\langle\epsilon(v),\epsilon(\psi)\rangle_\Omega\right| + |\alpha_v\langle v,\psi\rangle_{\Gamma_I}| \\ &\le c\big(\|v\|_{H^1}\|\psi\|_{H^1} + \|v\|_{H^1}\|\psi\|_{H^1}\big), \end{aligned} \tag{20}$$

$a_v(\cdot,\cdot)$ is also H_v-bounded. Combining (17)–(20), we have that there exist constants $c_1,\gamma_1>0$ such that for all $\phi_b,\psi_b\in H_b$ the bilinear form

$$a_1(\psi_b,\phi_b) = a_1((v,\theta),(\psi,\phi)) := a_v(v,\psi) + a_\theta(\theta,\phi)$$

satisfies

$$|a_1(\phi_b,\psi_b)| \le c_1\|\phi_b\|_{H_b}\|\psi_b\|_{H_b}, \tag{21a}$$

$$a_1(\phi_b,\phi_b) \ge \gamma_1\|\phi_b\|^2_{H_b}. \tag{21b}$$

The rest of the proof now consists of presenting estimates for the norms of the remaining terms of $a_0(\cdot,\cdot)$.

We immediately have that

$$|b_0(\theta,\psi)| \le c\|\theta\|_{L^2}\|\psi\|_{X_v}, \tag{22}$$

$$|\langle A_a x_a,\psi_a\rangle_{\mathbb{R}^{n_a}}| \le c\|x_a\|_{\mathbb{R}^{n_a}}\|\psi_a\|_{\mathbb{R}^{n_a}}, \tag{23}$$

$$|\langle A_s x_s,\psi_s\rangle_{\mathbb{R}^{n_s}}| \le c\|x_s\|_{\mathbb{R}^{n_s}}\|\psi_s\|_{\mathbb{R}^{n_s}}. \tag{24}$$

Regarding the form $b_\theta(\cdot,\cdot,\cdot)$, by Sobolev embeddings, L^2-duality of $H^{1/2}$ and $H^{-1/2}$ and Ladyzhenskaya's inequality

$$\begin{aligned}
&|b_\theta(v, T_{ss}, \theta)| \\
&= |\langle v \cdot \nabla T_{ss}, \theta\rangle_\Omega| \\
&\le |\langle v T_{ss}, \nabla\theta\rangle_\Omega| + |\langle v\cdot n, T_{ss}\theta\rangle_\Gamma| \\
&\le \|vT_{ss}\|_{L^2}\|\nabla\theta\|_{L^2} + c\|v\|_{H^1}\|T_{ss}\theta\|_{L^2} \\
&\le \|v\|_{L^4}\|T_{ss}\|_{L^4}\|\nabla\theta\|_{L^2} + c\|v\|_{H^1}\|T_{ss}\|_{L^4}\|\theta\|_{L^4} \\
&\le c\|T_{ss}\|_{H^1}\big(\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2}\|\nabla\theta\|_{L^2} + \|v\|_{H^1}\|\theta\|_{L^2}^{1/2}\|\nabla\theta\|_{L^2}^{1/2}\big). \qquad (25)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|b_\theta(w_{ss}, \theta, \theta)| &= \frac{1}{2}\langle w_{ss}, \nabla\theta^2\rangle_\Omega \\
&= \frac{1}{2}\langle w_{ss}\cdot n, \theta^2\rangle_\Gamma \\
&\le c\|w_{ss}\|_{H^1}\|\theta^2\|_{L^2} \\
&= c\|w_{ss}\|_{H^1}\|\theta\|_{L^4}^2 \\
&\le c\|\theta\|_{L^2}\|\nabla\theta\|_{L^2}. \qquad (26)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|b_\theta(v, \theta, \phi)| &\le |\langle v, \nabla(\theta\phi)\rangle_\Omega| + |\langle v\theta, \nabla\phi\rangle_\Omega| \\
&= |\langle v\cdot n, \theta\phi\rangle_\Gamma| + |\langle v\theta, \nabla\phi\rangle_\Omega| \\
&\le c\|v\|_{H^1}\|\theta\|_{H^1}\|\phi\|_{H^1}. \qquad (27)
\end{aligned}$$

Regarding the form $b_v(\cdot,\cdot,\cdot)$, we again use L^2-duality of $H^{1/2}$ and $H^{-1/2}$, Sobolev embeddings and Ladyzhenskaya's inequality to form the estimates. Now

$$\begin{aligned}
|b_v(w_{ss}, v, v)| &= |\langle (w_{ss}\cdot\nabla)v, v\rangle_\Omega| \\
&\le |\langle w_{ss}, (v\cdot\nabla)v\rangle_\Omega| + |\langle w_{ss}\cdot n, v\cdot v\rangle_\Gamma| \\
&\le c\big(\|w_{ss}\|_{L^4}\|v\|_{L^4}\|\nabla v\|_{L^2} + \|w_{ss}\|_{H^1}\|v\|_{L^4}^2\big) \\
&\le c\big(\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{3/2} + \|v\|_{L^2}\|\nabla v\|_{L^2}\big) \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
|b_v(v, w_{ss}, v)| &= |\langle (v\cdot\nabla)w_{ss}, v\rangle_\Omega| \\
&\le \|v\|_{L^4}^2\|\nabla w_{ss}\|_{L^2} \\
&\le c\|w_{ss}\|_{H^1}\|v\|_{L^2}\|\nabla v\|_{L^2}. \qquad (29)
\end{aligned}$$

Additionally,

$$\begin{aligned}
|b_v(v_1, v_2, \psi)| &\le |\langle v_1, (v_2\cdot\nabla)\psi\rangle_\Omega| + |\langle v_1\cdot n, v_2\cdot\psi\rangle_\Gamma| \\
&\le c\|v_1\|_{L^4}\|v_2\|_{L^4}\|\nabla\psi\|_{L^2} + c\|v_1\|_{H^1}\|v_2\cdot\psi\|_{L^2} \\
&\le c\big(\|v_1\|_{H^1}\|v_2\|_{H^1}\|\psi\|_{H^1} + \|v_1\|_{H^1}\|v_2\|_{L^4}\|\psi\|_{L^4}\big) \\
&\le c\|v_1\|_{H^1}\|v_2\|_{H^1}\|\psi\|_{H^1}. \qquad (30)
\end{aligned}$$

Finally, properties of the trace operator together with Assumption 2.1 and duality imply

$$|\langle b_t C_a \psi_a, \psi_b \rangle_\Gamma| \le c\|\psi_a\|_{\mathbb{R}^{n_a}} \|\psi_b\|_{H_b}, \tag{31}$$
$$|\langle B_s C_b \psi_b, \psi_s \rangle_{\mathbb{R}^{n_s}}| \le c\|\phi_s\|_{\mathbb{R}^{n_s}} \|\psi_b\|_{H_b}, \tag{32}$$

where $b_t := [b_{v_I},\ b_{\theta_I},\ b_{\theta_H}]$. Recalling the norm definitions (11), equations (21a), (22)–(24), (27) and (30)–(32) together imply H-boundedness of $a_0(\cdot,\cdot)$. H-coercivity of $a_0(\cdot,\cdot)$ follows from (21b) after applying Young's inequality to (25), (26), (28), (29), (31) and (32). Finally, H-coercivity and H-boundedness of $a_0(\cdot,\cdot)$ imply generation of an analytic semigroup on X, see, e.g., [6]. □

To formulate the cascade system as an abstract boundary control system in the sense of [16, Ch. 3.3], we next define the related operators. In what follows, $\mathbb{P}$ denotes the Leray projector as defined in [32, Lemma 2.2], which is used to eliminate the pressure term while imposing incompressibility. Define the operators

$$\mathcal{A}\begin{bmatrix} v \\ \theta \\ x_a \\ x_s \end{bmatrix} = \begin{bmatrix} \mathbb{P}\left(\frac{1}{Re}\Delta v - (w_{ss}\cdot\nabla)v - (v\cdot\nabla)w_{ss} + \frac{Gr}{Re^2}\hat{e}_2\theta\right) \\ \frac{1}{RePr}\Delta\theta - w_{ss}\cdot\nabla\theta - v\cdot\nabla T_{ss} \\ A_a x_a \\ A_s x_s + B_s C_b [v,\ \theta]^{\mathrm{T}} \end{bmatrix} : D(\mathcal{A}) \to X,$$

$$B_{\Gamma_u} = \begin{bmatrix} b_{v_I} & 0 & 0 \\ 0 & b_{\theta_I} & 0 \\ 0 & 0 & b_{\theta_H} \end{bmatrix} : U_b \to X_\Gamma,$$

$$B_{\Gamma_{u_d}} = \begin{bmatrix} b_{dv_I} & 0 & 0 \\ 0 & b_{d\theta_I} & 0 \\ 0 & 0 & b_{d\theta_H} \end{bmatrix} : U_d \to X_\Gamma,$$

$$B_\Gamma = \left[B_{\Gamma_u}\ B_{\Gamma_{u_d}}\right],$$

where $U_b := \mathbb{R}^{m_b}$. Operator $\mathcal{A}$ coincides with A on $D(A)$ but has a larger domain due to relaxed boundary conditions within the boundary parts Γ_I and Γ_H affected by disturbance inputs. That is, noting that the Neumann trace of H_b functions is in $(H^{-1/2}(\Gamma))^2 \times H^{-1/2}(\Gamma)$ and recalling the definition of $a_g(\cdot,\cdot)$ in (15), the domain is given by

$$D(\mathcal{A}) = \big\{x \in H \big| \, \exists g_{dv_I} \in (H^{-1/2}(\Gamma_I))^2,\ \exists g_{d\theta_I} \in H^{-1/2}(\Gamma_I),$$
$$\exists g_{d\theta_H} \in H^{-1/2}(\Gamma_H) : \forall \Psi \in H, \Psi \to a_g(x,\Psi) \text{ is } X\text{ -continuous}\big\}.$$

Corresponding to the control and disturbance boundary conditions (5d)–(5f), we define the operator

$$\tilde{\mathcal{B}}_b \begin{bmatrix} v \\ \theta \\ p \end{bmatrix} = \begin{bmatrix} \left(\mathcal{T}(v,p)\cdot n + \alpha_v v\right)\big|_{\Gamma_I} \\ \left(\frac{1}{RePr}\frac{\partial\theta}{\partial n} + \alpha_\theta\theta\right)\big|_{\Gamma_I} \\ \left(\frac{1}{RePr}\frac{\partial\theta}{\partial n}\right)\big|_{\Gamma_H} \end{bmatrix} : D(\tilde{\mathcal{B}}_b) \subset X_b \times L^2(\Omega) \to X_\Gamma.$$

The pressure p is uniquely defined by the velocity v, see [4], thus there exists an operator $\mathcal{B}_b$ such that

$$\mathcal{B}_b \begin{bmatrix} v \\ \theta \end{bmatrix} = \tilde{\mathcal{B}}_b \begin{bmatrix} v \\ \theta \\ p \end{bmatrix} \quad \forall \begin{bmatrix} v \\ \theta \\ p \end{bmatrix} \in D(\tilde{\mathcal{B}}_b), \quad D(\mathcal{B}_b) = \left\{(v, \theta) \in D(\tilde{\mathcal{B}}_b)\right\}.$$

To match the state variable $x \in X$ of the cascade system, we finally define the operator

$$\mathcal{B} = \left[\mathcal{B}_b \; 0 \; 0\right] : D(\mathcal{B}) = D(\mathcal{B}_b) \times \mathbb{R}^{n_a} \times \mathbb{R}^{n_s} \subset X \to X_\Gamma.$$

Since $A = \mathcal{A}|_{\mathcal{N}(\mathcal{B}-[0,\; B_{\Gamma_u} C_a,\; 0])}$ generates an analytic semigroup on X by Theorem 2.3 and $\mathcal{B}$ is onto X_Γ, cf. [12], by defining

$$B = \left[0_{X_b} \;\; B_a \;\; 0_{\mathbb{R}^{n_s}}\right]^{\mathrm{T}} \in \mathcal{L}(U, X) \tag{33}$$

we have that the cascade system (5)–(8) corresponds to the abstract boundary control system

$$\dot{x}(t) = \mathcal{A}x(t) + Bu(t), \tag{34a}$$

$$\mathcal{B}x(t) = B_\Gamma \begin{bmatrix} C_a x_a(t) \\ u_d(t) \end{bmatrix} \tag{34b}$$

on X with the (boundary) input space X_Γ in the sense of [16, Ch. 3.3]. The system observation is given by

$$y(t) = Cx(t), \quad C = [0_{X_b},\; 0_{\mathbb{R}^{n_a}},\; C_s] \in \mathcal{L}(X, Y). \tag{34c}$$

Note that the control and observation operators of the boundary control system are bounded and the disturbance $B_{\Gamma_{u_d}} u_d$ is the only boundary input of the boundary control system. The boundary control system formulation of the cascade system also has the following equivalent state space formulation.

Proposition 2.4 *The cascade system (5)–(8) can be formulated as*

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d u_d(t), \quad x(0) = x_0 \in X, \tag{35a}$$

$$y(t) = Cx(t) + D_d u_d(t), \tag{35b}$$

where the dynamics operator A defined as in (16) generates an analytic semigroup on the state space X defined in (10a) and the control operator B together with the observation operator C defined in (33) and (34c) are bounded. Additionally, a change of the state variable x can be applied such that also the resulting disturbance operator B_d is bounded.

Proof Existence of the state space formulation (35) follows from the boundary control system formulation as presented in [40, Ch. 10] and boundedness of B and C is apparent from their definitions. The change of variable $\tilde{x}_b = x_b - B_I B_{\Gamma_{u_d}}$, where $B_I \in \mathcal{L}(X_\Gamma, D(\mathcal{A}_b))$ is a right inverse of $\mathcal{B}_b$ used to homogenize the boundary conditions and obtain a bounded operator B_d, is presented in [16, Ch. 3.3]. Note that the change of variable utilizes smoothness of the disturbance signal u_d given by (3) and introduces a bounded feedthrough operator D_d into the system. □

The state space formulation together with the fact that the operators B, B_d, C and D_d are bounded will later in this paper be utilized for implementing the output tracking controller.

Remark 2.5 The controller to be implemented uses no information on the disturbance related operators B_d and D_d, thus we do not formulate the cascade system using the state variable $\tilde{x}_b$. However, one needs to verify that a representation using bounded disturbance operators B_d and D_d exists.

2.3 Stabilizability and detectability of the system

We will be using a controller including an observer-like structure, which means that we need to address both stabilizability and detectability properties of the cascade system (5)–(8). Here, we focus on deriving sufficient conditions for exponential stabilizability and exponential detectability of the cascade system based on properties of the linearized translated Boussinesq equations, the actuator and the sensor.

To begin with we note that, in addition to the cascade system, also the linearized translated Boussinesq equations (5) form an abstract boundary control system

$$\dot{x}_b(t) = \mathcal{A}_b x_b(t), \tag{36a}$$

$$\mathcal{B}_b x_b(t) = B_\Gamma \begin{bmatrix} u_b(t) \\ u_d(t) \end{bmatrix}, \tag{36b}$$

which can be verified by repeating the steps of Sect. 2.2 without the actuator and sensor dynamics. Here

$$\begin{aligned} \mathcal{A}_b &= \mathcal{A}|_{D(\mathcal{A}_b)}, \\ D(\mathcal{A}_b) &= \big\{ x_b \in H_b \big| \, \exists g_{v_I} \in (H^{-1/2}(\Gamma_I))^2, \ \exists g_{\theta_I} \in H^{-1/2}(\Gamma_I), \\ &\qquad \exists g_{\theta_H} \in H^{-1/2}(\Gamma_H) : \forall \varphi \in H_b, \ \varphi \to a_b(x_b, \varphi) \text{ is } X_b \text{ -continuous} \big\} \end{aligned}$$

with the bilinear form $a_b(\cdot, \cdot)$ defined by

$$\begin{aligned} & a_b((v, \theta), (\psi, \phi)) \\ &= a_v(v, \psi) + a_\theta(\theta, \phi) + b_v(v, w_{ss}, \psi) + b_v(w_{ss}, v, \psi) + b_\theta(w_{ss}, \theta, \phi) \\ &\quad + b_\theta(v, T_{ss}, \phi) - b_0(\theta, \psi) - \langle g_{v_I}, \psi \rangle_{\Gamma_I} - \langle g_{\theta_I}, \phi \rangle_{\Gamma_I} - \langle g_{\theta_H}, \phi \rangle_{\Gamma_H}. \end{aligned}$$

The associated generator of a strongly continuous semigroup is $A_b = \mathcal{A}_b|_{\mathcal{N}(\mathcal{B}_b)}$. Now recalling that $A = \mathcal{A}|_{\mathcal{N}(\mathcal{B}-[0,\ B_{\Gamma_u}C_a,\ 0])}$, A is described by

$$A\begin{bmatrix} x_b \\ x_a \\ x_s \end{bmatrix} = \begin{bmatrix} \mathcal{A}_b & 0 & 0 \\ 0 & A_a & 0 \\ -B_s C_b & 0 & A_s \end{bmatrix}\begin{bmatrix} x_b \\ x_a \\ x_s \end{bmatrix}, \tag{37a}$$

$$D(A) = \left\{(x_b, x_a, x_s) \in D(\mathcal{A}) \middle| \mathcal{B}_b x_b = B_{\Gamma_u} C_a x_a\right\}, \tag{37b}$$

which is the expression that we will use for the stabilizability and detectability analysis.

Recall that by Theorem 2.3 A generates an analytic semigroup on X. Additionally, by Theorem 2.3, Lax–Milgram theorem and compactness of the embedding H onto X, the resolvent of A is compact on X, cf. [37]. Thus, A has a finite number of isolated eigenvalues on the closed right half plane $\overline{\mathbb{C}_0^+}$, each with finite multiplicity. As such, stabilizability and detectability considerations of the cascade system with the bounded control operator B and the bounded observation operator C can be treated as controllability and observability problems of the finite-dimensional unstable part, see [16, Ch. 5.2]. That is, the pair (C, A) is exponentially detectable if and only if

$$\mathcal{N}(sI - A) \cap \mathcal{N}(C) = \{0\} \quad \text{for all } s \in \overline{\mathbb{C}_0^+}, \tag{38}$$

and the pair (A, B) is exponentially stabilizable if and only if

$$\mathcal{R}(sI - A) + \mathcal{R}(B) = X \quad \text{for all } s \in \overline{\mathbb{C}_0^+}. \tag{39}$$

Recalling Assumption 2.1, the observation (12) satisfies $C_b \in \mathcal{L}(D(\mathcal{A}_b), Y_b)$, thus it can be included into the abstract boundary control system framework as defined in [15]. For any $s \in \rho(A_b)$, the transfer function $P_b(s)$ of the triple $(\mathcal{A}_b, \mathcal{B}_b, C_b)$ is then defined by

$$P_b(s)u_b = C_b z(s), \tag{40}$$

where $z(s) \in D(\mathcal{A}_b)$ is the unique solution of the abstract elliptic problem

$$\mathcal{A}_b z(s) = s z(s), \tag{41a}$$

$$\mathcal{B}_b z(s) = B_{\Gamma_u} u_b, \tag{41b}$$

see [15]. The transfer functions for the actuator (6) and the sensor (8) are defined as $P_a(s) = C_a(sI - A_a)^{-1}B_a$ and $P_s(s) = C_s(sI - A_s)^{-1}B_s$, respectively.

For ease of notation, we define the operator B_b as

$$B_b = (\mathcal{A}_b - A_b)B_I B_{\Gamma_u} \in \mathcal{L}(U_b, X_{b_{-1}}), \tag{42}$$

where we recall that $B_I \in \mathcal{L}(X_\Gamma, D(\mathcal{A}_b))$ is a right inverse of $\mathcal{B}_b$. Here, A_b is regarded as an operator from X_b to $X_{b_{-1}}$, which is the completion of X_b with respect to the norm $\|\cdot\|_{X_{b_{-1}}} = \|(sI - A_b)^{-1}(\cdot)\|_{X_b}$ with $s \in \rho(A_b)$.

Lemma 2.6 *The boundary control system* $(\mathcal{A}_b, \mathcal{B}_b, C_b)$ *corresponds to a state space formulation* (A_b, B_b, C_b)*, which is a well-posed triple in the sense of* [41, Def. 4.8]. *Furthermore, the triple* (A_b, B_b, C_b) *is regular in the sense of* [43, Def. 4.1].

Proof The proof is a combination of existing results. We start by presenting results of [12, Sec. 2.2] relevant to the proof. Let $\lambda_0 \in \rho(A_b)$ have a large enough real part. Then, the fractional powers of $\lambda_0 I - A_b$ are well defined, $D((\lambda_0 I - A_b)^{1/2}) = H_b$ and

$$(\lambda_0 I - A_b)^{-1/4-\epsilon} B_b \in \mathcal{L}(U_b, X_b) \tag{43}$$

for any $\epsilon > 0$. Additionally, the operator A_b has a decomposition $A_b = A_{b2} + A_{b10}$, where A_{b2} with the domain $D(A_{b2}) = D(A_b)$ is self-adjoint and the generator of an exponentially stable analytic semigroup on X_b, and $A_{b10} \in \mathcal{L}(H_b, X_b)$.

Now by [41, Prop. 6.5] both (A_{b2}, B_b, C_b) and (A_{b2}, B_b, A_{b10}) are well-posed triples and even regular with zero feedthrough. We then use the feedback results of [21], which originate from [44], by considering the operator A_{b10} in the expression $A_b = A_{b2} + A_{b10}$ as an output feedback for A_{b2}. By [21, Lemma 12] the triple (A_b, B_b, C_b) is well-posed. Finally, the triple (A_b, B_b, C_b) is regular by [41, Prop. 5.13], since the degree of unboundedness is at most $1/2$ for C_b and strictly less than $1/2$ for B_b, cf. Assumption 2.1 and (43). □

Since the transfer function P_b is well defined, we can make the following assumptions.

Assumption 2.7 Assume that the following hold:

(i) The spectra $\sigma(A_b)$, $\sigma(A_a)$ and $\sigma(A_s)$ are pairwise disjoint on $\overline{\mathbb{C}_0^+}$.
(ii) The pair (C_s, A_s) is detectable.
(iii) For every $\lambda \in \sigma(A_b) \cap \overline{\mathbb{C}_0^+}$, $\mathcal{N}(P_s(\lambda)C_b) \cap \mathcal{N}(\lambda I - A_b) = \{0\}$.
(iv) For every $\lambda \in \sigma(A_a) \cap \overline{\mathbb{C}_0^+}$, $\mathcal{N}\big(P_s(\lambda)P_b(\lambda)C_a\big) \cap \mathcal{N}(\lambda I - A_a) = \{0\}$.

In particular, the assumption requires that also the pairs (C_b, A_b) and (C_a, A_a) are exponentially detectable even if C_b is unbounded [5].

Assumption 2.8 Assume that the following hold:

(i) The spectra $\sigma(A_b)$, $\sigma(A_a)$ and $\sigma(A_s)$ are pairwise disjoint on $\overline{\mathbb{C}_0^+}$.
(ii) The pair (A_a, B_a) is stabilizable and $P_a(\lambda)$ is surjective for every $\lambda \in \sigma(A_b) \cap \overline{\mathbb{C}_0^+}$.
(iii) For every $\lambda \in \sigma(A_s) \cap \overline{\mathbb{C}_0^+}$, $\mathcal{R}(\lambda I - A_s) + \mathcal{R}(B_s P_b(\lambda) P_a(\lambda)) = X_s$.
(iv) The pair $(\mathcal{A}_b, \mathcal{B}_b)$ is exponentially stabilizable, i.e., there exists $K_b \in \mathcal{L}(X_b, U_b)$ such that $\mathcal{A}_b|_{\mathcal{N}(\mathcal{B}_b - K_b)}$ generates an exponentially stable strongly continuous semigroup on X_b.

Note that due to the results in [5] we can limit our attention to considering bounded operators K_b in Assumption 2.8(iv).

Lemma 2.9 *If Assumption* 2.7 *holds, then the pair* (C, A) *is exponentially detectable.*

Proof To show that (38) holds for the pair (C, A), let $(x_b, x_a, x_s) \in \mathcal{N}(\lambda I - A) \cap \mathcal{N}(C)$, where $\lambda \in \overline{\mathbb{C}_0^+}$. Since $(x_b, x_a, x_s) \in D(A)$, using (34c) and (37) we have

$$\begin{cases} C_s x_s = 0, & \text{(44a)} \\ (\lambda I - A_a) x_a = 0, & \text{(44b)} \\ (\lambda I - \mathcal{A}_b) x_b = 0, & \text{(44c)} \\ \mathcal{B}_b x_b = B_{\Gamma_u} C_a x_a, & \text{(44d)} \\ (\lambda I - A_s) x_s - B_s C_b x_b = 0. & \text{(44e)} \end{cases}$$

If $\lambda \in \rho(A_a)$, (44b) implies $x_a = 0$, thus (44d) implies $x_b \in D(A_b)$. If $\lambda \in \rho(A_a) \cap \rho(A_b)$, then $x_b = 0$ by (44c) and furthermore (44a), (44e) and Assumption 2.7(ii) imply $x_s = 0$. If $\lambda \in \rho(A_a) \cap \sigma(A_b)$, then $x_s = (\lambda I - A_s)^{-1} B_s C_b x_b$ by (44e) and Assumption 2.7(i). By (44a), (44b) and Assumption 2.7(iii) we have $x_b = 0$, which then implies $x_s = 0$.

If $\lambda \in \sigma(A_a)$, then $\lambda \in \rho(A_b)$ by Assumption 2.7(i). Now combining (41)–(40) with (44c), (44d) and (44e) and using Assumption 2.7(i) again yields $x_s = (\lambda I - A_s)^{-1} B_s P_b(\lambda) C_a x_a$. Then, $x_a = 0$ by (44a), (44b) and Assumption 2.7(iv). By (44d) it now holds that $x_b \in D(A_b)$, thus $x_b = 0$ by (44c). Finally, $x_s = 0$ by (44e). Since $\lambda \in \overline{\mathbb{C}_0^+}$ was arbitrary, (38) holds and (C, A) is exponentially detectable. □

Lemma 2.10 *If Assumption 2.8 holds, then the pair (A, B) is exponentially stabilizable.*

Proof We show that (39) holds for the pair (A, B). Using (33) and (37), for arbitrary $z = (z_b, z_a, z_s) \in X$ and for any $\lambda \in \overline{\mathbb{C}_0^+}$ we need to find $x = (x_b, x_a, x_s) \in D(A)$ and $u \in U$ such that

$$\begin{cases} (\lambda I - \mathcal{A}_b) x_b = z_b, & \text{(45a)} \\ (\lambda I - A_a) x_a + B_a u = z_a, & \text{(45b)} \\ (\lambda I - A_s) x_s - B_s C_b x_b = z_s, & \text{(45c)} \\ \mathcal{B}_b x_b = B_{\Gamma_u} C_a x_a. & \text{(45d)} \end{cases}$$

If $\lambda \in \rho(A_b)$, there exists $\tilde{x}_b \in D(A_b)$ such that $z_b = (\lambda I - A_b)\tilde{x}_b = (\lambda I - \mathcal{A}_b)\tilde{x}_b$. Since $\mathcal{B}_b \tilde{x}_b = 0$, (45a) and (45d) form an abstract elliptic problem

$$(\lambda I - \mathcal{A}_b)(x_b - \tilde{x}_b) = 0, \tag{46a}$$
$$\mathcal{B}_b(x_b - \tilde{x}_b) = B_{\Gamma_u} C_a x_a, \tag{46b}$$

which has a unique solution $x_b - \tilde{x}_b = x_b - (\lambda I - A_b)^{-1} z_b \in D(\mathcal{A}_b)$ depending on x_a [40, Rem. 10.1.5]. If in addition $\lambda \in \rho(A_a) \cap \rho(A_s)$, (45b) and (45c) yield

$$x_a = (\lambda I - A_a)^{-1}(z_a - B_a u), \tag{47}$$
$$x_s = (\lambda I - A_s)^{-1}(z_s + B_s C_b x_b), \tag{48}$$

respectively, thus x and u for the arbitrary z can be solved from (46)–(48). If $\lambda \in \rho(A_b) \cap \rho(A_s) \cap \sigma(A_a)$, Assumption 2.8(ii) implies that there exist x_a and u such that (45b) holds. Then, x_b and x_s can be solved from (46) and (48). If $\lambda \in \rho(A_b) \cap \rho(A_a) \cap \sigma(A_s)$, we can use (47), (46) and (40) to rewrite (45c) as

$$\begin{aligned} z_s &= (\lambda I - A_s)x_s - B_s P_b(\lambda) C_a x_a + B_s C_b (\lambda I - A_b)^{-1} z_b \\ &= (\lambda I - A_s)x_s + B_s P_b(\lambda) P_a(\lambda) u - B_s P_b(\lambda) C_a (\lambda I - A_a)^{-1} z_a \\ &\quad + B_s C_b (\lambda I - A_b)^{-1} z_b. \end{aligned} \tag{49}$$

Now Assumption 2.8(iii), (46), (47) and (49) guarantee that there exist x and u such that (45a) holds.

By Assumption 2.8(i), the single case left to consider is $\lambda \in \sigma(A_b) \cap \rho(A_a) \cap \rho(A_s)$. In that case x_a and x_s can be solved from (47), (48) with the latter depending on the choice of x_b. By [40, Ch. 10] we have $\mathcal{A}_b = A_b + B_b \mathcal{B}_b$ on $D(\mathcal{A})$, where A_b again denotes the extension from X_b to $X_{b_{-1}}$ and B_b is defined in (42). Now using (45a) and (45d) yields

$$z_b = (\lambda I - \mathcal{A}_b)x_b = (\lambda I - A_b)x_b - B_b C_a x_a,$$

thus by (47) we have

$$z_b = (\lambda I - A_b)x_b + B_b P_a(\lambda) u - B_b C_a (\lambda I - A_a)^{-1} z_a.$$

Based on Lemma 2.6 and [39, Lemma 8.2.8], Assumption 2.8(ii),(iv) imply

$$X_b \subset R(\lambda I - A_b) + \mathcal{R}(B_b P_a(\lambda)),$$

thus there exists x_b, hence x and u, such that (45a) holds. Now (39) holds, i.e., the pair (A, B) is exponentially stabilizable. □

3 Robust output regulation

The output tracking goal (9) is in the case of abstract linear systems covered by the *robust output regulation problem*. We start by coupling an error feedback controller with the cascade system (35). The resulting system is called the *closed-loop system*. We then present the robust output regulation problem, which describes requirements for choosing the controller operators. Finally, we design an error feedback controller, introduced in [34], to solve the robust output regulation problem for the room model.

An error feedback controller on a Hilbert space Z is given by

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \quad z(0) = z_0 \in Z, \tag{50a}$$
$$u(t) = K z(t), \tag{50b}$$

where $\mathcal{G}_1$ generates a strongly continuous semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$, $K \in \mathcal{L}(Z, U)$ and $e(t) = y(t) - y_{\text{ref}}(t)$ is the regulation error. Coupling the controller with the cascade system (35) yields the closed-loop system, see [23,35],

$$\begin{aligned}\dot{x}_e(t) &= A_e x_e(t) + B_e w_{\text{ext}}(t), \quad x_e(0) = x_{e0},\\ e(t) &= C_e x_e(t) + D_e w_{\text{ext}}(t)\end{aligned}$$

on the Hilbert space $X_e := X \times Z$ with the state $x_e = [x,\ z]^{\mathrm{T}}$. Here, $w_{\text{ext}} = [u_d,\ y_{\text{ref}}]^{\mathrm{T}}$,

$$\begin{aligned}A_e &= \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 \end{bmatrix}, &\qquad B_e &= \begin{bmatrix} B_d & 0 \\ \mathcal{G}_2 D_d & -\mathcal{G}_2 \end{bmatrix},\\ C_e &= \begin{bmatrix} C, & 0 \end{bmatrix}, & D_e &= \begin{bmatrix} D_d, & -I \end{bmatrix}.\end{aligned}$$

The robust output regulation problem Design a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the following hold:

(I) The closed-loop system is exponentially stable.

(II) There exist $M_r, \omega_r > 0$ such that for all initial states $x_{e0} \in X_e$ of the closed-loop system and for all reference signals y_{ref} in (2) and disturbance signals u_d in (3)

$$\|y(t) - y_{\text{ref}}(t)\| \le M_r \mathrm{e}^{-\omega_r t}(\|x_{e0}\| + \|\Lambda\|), \tag{51}$$

where Λ is a vector consisting of the coefficients of the polynomials $a_i(t)$, $b_i(t)$, $c_i(t)$ and $d_i(t)$ of y_{ref} and u_d.

(III) If A, B, B_d, C, D_d in (35) are perturbed to $\tilde{A}, \tilde{B}, \tilde{B}_d, \tilde{C}, \tilde{D}_d$ in such a way that the closed-loop system remains exponentially stable, then for all $x_{e0} \in X_e$ and for all signals of the form (2), (3) the regulation error satisfies (51) for some $\tilde{M}_r, \tilde{\omega}_r > 0$.

By the internal model principle, an error feedback controller solves the robust output regulation problem precisely when the closed-loop system is exponentially stable and the controller incorporates a suitable *internal model* of the reference and disturbance signals [35]. That is, the controller must include the dynamics of the reference signal (2) and the disturbance signal (3) reduplicated according to the dimension p_Y of the output space Y, cf. the first step of the controller design algorithm. The following lemma, i.e., (A, B, C) having no transmission zeros at the relevant frequencies, is a standard necessary property for solvability of the robust output regulation problem.

Assumption 3.1 None of the systems (A_b, B_b, C_b), (A_a, B_a, C_a) and (A_s, B_s, C_s) has transmission zeros at the frequencies $\{i\omega_k\}_{k=0}^{q_s}$, i.e., for bounded stabilizing feedback operators K_b, K_a and K_s the transfer functions $P_{b,K_b}(i\omega_k) = C_b(i\omega_k I - A_b - B_b K_b)^{-1} B_b$, $P_{a,K_a}(i\omega_k) = C_a(i\omega_k I - A_a - B_a K_a)^{-1} B_a$ and $P_{s,K_s}(i\omega_k) = C_s(i\omega_k I - A_s - B_s K_s)^{-1} B_s$ are surjective for $k = 0, 1, \ldots, q_s$.

Lemma 3.2 *Given Assumption 3.1, the cascade system (A, B, C) has no transmission zeros at the frequencies $\{i\omega_k\}_{k=0}^{q_s}$.*

The proof follows immediately since the transfer function of the cascade system is product of the transfer functions of the three subsystems.

The particular controller design we propose for the cascade system is the one in [34, Sec. III.A], see also [36] for its boundary control system implementation.

I The observer-based finite-dimensional controller is given by

$$\dot{z}_1(t) = G_1 z_1(t) + G_2 e(t), \tag{52a}$$
$$\dot{z}_2(t) = (A_L^r + B_L^r K_2^r) z_2(t) + B_L^r K_1^N z_1(t) - L^r e(t), \tag{52b}$$
$$u(t) = K_1^N z_1(t) + K_2^r z_2(t), \tag{52c}$$

and is of the form (50) with $z(t) := [z_1(t),\ z_2(t)]^{\mathrm{T}} \in Z := Z_{im} \times \mathbb{C}^r$,

$$\mathcal{G}_1 = \begin{bmatrix} G_1 & 0 \\ B_L^r K_1^N & A_L^r + B_L^r K_2^r \end{bmatrix}, \quad \mathcal{G}_2 = \begin{bmatrix} G_2 \\ -L^r \end{bmatrix}, \quad K = \begin{bmatrix} K_1^N & K_2^r \end{bmatrix}.$$

For the cascade system (5)–(8), the operators in (52) are chosen according to the following algorithm.

I The Internal Model:

Choose $Z_{im} = Y^{n_0} \times Y^{2n_1} \times \cdots \times Y^{2n_{q_s}}$, where $n_i - 1$, $i \in \{0, 1, ..., q_s\}$ is the highest-order polynomial coefficient of the corresponding frequency ω_i in (2). Set $G_1 = \operatorname{diag}\big(J_0^Y, \dots J_{q_s}^Y\big) \in \mathcal{L}(Z_{im})$ and $G_2 = \big(G_2^k\big)_{k=0}^{q_s} \in \mathcal{L}(Y, Z_{im})$, where

$$J_0^Y = \begin{bmatrix} 0_p & I_p & & \\ & 0_p & \ddots & \\ & & \ddots & I_p \\ & & & 0_p \end{bmatrix}, \quad G_2^0 = \begin{bmatrix} 0_p \\ \vdots \\ 0_p \\ I_p \end{bmatrix}$$

and for $k = 1 \dots q_s$

$$J_k^Y = \begin{bmatrix} \Omega_k & I_{2p} & & \\ & \Omega_k & \ddots & \\ & & \ddots & I_{2p} \\ & & & \Omega_k \end{bmatrix}, \quad G_2^k = \begin{bmatrix} 0_{2p} \\ \vdots \\ 0_{2p} \\ I_p \\ 0_p \end{bmatrix}, \quad \Omega_k = \begin{bmatrix} 0_p & \omega_k I_p \\ -\omega_k I_p & 0_p \end{bmatrix}.$$

II Plant approximation and stabilization

For a sufficiently large $N \in \mathbb{N}$, discretize the operators A_b, B_b and C_b to obtain the finite-dimensional approximative system (A_b^N, B_b^N, C_b^N) on H_b^N. The chosen approximation method should satisfy the following assumption.

Assumption 3.3 The finite-dimensional approximating subspaces H_b^N of H_b have the following property: For any $x_b \in H_b$ there exists a sequence $x_b^N \in H_b^N$ such that

$$\|x_b^N - x_b\|_{H_b} \to 0 \text{ as } N \to \infty. \tag{53}$$

More specifically, the approximation of the cascade system should have the property equivalent to (53), and Assumption 3.3 implies the approximation property for the cascade system.

Lemma 3.4 *Let $H^N := H_b^N \times \mathbb{R}^{n_a} \times \mathbb{R}^{n_s}$ denote the finite-dimensional approximating subspaces of H. Given Assumption 3.3, there exists a sequence $x^N \in H^N$ such that $\|x^N - x\|_H \to 0$ as $N \to \infty$.*

The proof follows immediately by choosing $x^N = [x_b^N,\ x_a,\ x_s]^{\mathrm{T}}$, where x_b^N is such that Assumption 3.3 holds.

Denote the cascade system approximation on H^N by (A^N, B^N, C^N). Let $\alpha_1, \alpha_2 \geq 0$ and choose $Q_1 \in \mathcal{L}(U_0, X)$, $Q_2 \in \mathcal{L}(X, Y_0)$ such that $(A + \alpha_1 I, Q_1, C)$ and $(A + \alpha_2 I, B, Q_2)$ are exponentially stabilizable and detectable, where U_0 and Y_0 are Hilbert spaces. Denote by Q_1^N and Q_2^N the approximations of Q_1 and Q_2 on H^N. Choose $0 < R_1 \in \mathcal{L}(Y)$ and $0 < R_2 \in \mathcal{L}(U)$, and choose $Q_0 \in \mathcal{L}(Z_{im}, \mathbb{C}^{p_0})$ such that (Q_0, G_1) is observable. Let

$$A_c^N = \begin{bmatrix} G_1 & G_2 C^N \\ 0 & A^N \end{bmatrix}, \quad B_c^N = \begin{bmatrix} 0 \\ B^N \end{bmatrix}, \quad Q_c^N = \begin{bmatrix} Q_0 & 0 \\ 0 & Q_2^N \end{bmatrix}$$

and solve the finite-dimensional Riccati equations

$$(A^N + \alpha_1 I)\Sigma_N + \Sigma_N (A^N + \alpha_1 I)^* - \Sigma_N (C^N)^* R_1^{-1} C^N \Sigma_N = -Q_1^N (Q_1^N)^*,$$
$$(A_c^N + \alpha_2 I)^* \Pi_N + \Pi_N (A_c^N + \alpha_2 I) - \Pi_N B_c^N R_2^{-1} (B_c^N)^* \Pi_N = -(Q_c^N)^* Q_c^N.$$

Finally, define $L^N = -\Sigma_N C^N R_1^{-1} \in \mathcal{L}(Y, H^N)$ and $K^N := [K_1^N,\ K_2^N] = -R_2^{-1}(B_c^N)^* \Pi_N \in \mathcal{L}(Z_{im} \times H^N, U)$.

III Model reduction

For a sufficiently large $r \leq N$, apply balanced truncation, see [34, Sec. II-B] and the references therein, on the stable system

$$(A^N + L^N C^N, [B^N, L^N], K_2^N)$$

to obtain the reduced-order system

$$(A_L^r, [B_L^r,\ L^r], K_2^r).$$

By [34, Thm. III.1], the observer-based finite-dimensional controller solves the robust output regulation problem for a class of systems including the infinite-dimensional cascade system (5)–(8). Thus, the following holds for robust output tracking of the linearized translated Boussinesq equations (5) with the observation (12), the actuator dynamics (6) and the sensor dynamics (8).

Theorem 3.5 *Assume that the control and disturbance shape functions $[b_{v_I},\ b_{dv_I}] \in (L^2(\Gamma_I))^i$, $[b_{\theta_I},\ b_{d\theta_I}] \in (L^2(\Gamma_I))^j$ and $[b_{\theta_H},\ b_{d\theta_H}] \in (L^2(\Gamma_H))^k$, where i, j and k equal the sum of the number of control and disturbance inputs for* *the inlet velocity, the inlet temperature and the heating strip temperature, respectively. Assume that the observation on the fluid satisfies Assumption 2.1 and the linearized translated Boussinesq equations (5) with the observation (12), the actuator (6) and the sensor (8) satisfy Assumptions 2.7, 2.8 and 3.1. Then, the finite-dimensional low-order controller (52) solves the robust output regulation problem for the cascade system (5)–(8), thus achieving the output convergence (9), provided that Assumption 3.3 holds for the approximation method used in the controller design and the orders of approximation N and $r \leq N$ are large enough.*

4 Output tracking example for the room model

We consider robust output tracking of the linearized translated Boussinesq equations (5) with the actuator dynamics (6) and the sensor dynamics (8) in the rectangular room $\Omega = [0, 1] \times [0, 1]$ with the inlet, the outlet and the heating strip given by

$$\Gamma_I = \left\{ \xi_1 = 0,\ \frac{5}{8} \leq \xi_2 \leq \frac{7}{8} \right\}, \quad \Gamma_O = \left\{ \xi_1 = 1,\ \frac{1}{8} \leq \xi_2 \leq \frac{1}{2} \right\},$$
$$\Gamma_H = \left\{ \frac{3}{8} \leq \xi_1 \leq \frac{5}{8},\ \xi_2 = 0 \right\},$$

which roughly corresponds to Fig. 1. Physical parameters for the Boussinesq equations are chosen as $Re = 100$, $Gr = \frac{Re^2}{0.9}$ and $Pr = 0.7$. There are three control inputs on the fluid; one on each of v_{ξ_1} and θ within the inlet and one on θ within the heating strip, with coefficients $\alpha_v = \alpha_\theta = 1$ indicating Robin boundary conditions within the inlet. Additionally, we assume that a single disturbance signal acts within the inlet on v_{ξ_2}. Now $u_b(t) = [u_v(t),\ u_{\theta_I}(t),\ u_{\theta_H}(t)]^{\mathrm{T}} \in \mathbb{R}^3$ and $u_d(t) = u_{dv_I}(t) \in \mathbb{R}$. The control and disturbance shapes are given by

$$b_{v_I}(\xi_2) = \left[\exp\left(\frac{-0.00004}{((5/8-\xi_2)(7/8-\xi_2))^2} \right)\Big|_{\Gamma_I},\ 0 \right]^{\mathrm{T}},$$
$$b_{\theta_I}(\xi_2) = \exp\left(\frac{-0.00002}{((5/8-\xi_2)(7/8-\xi_2))^2} \right)\Big|_{\Gamma_I},$$
$$b_{\theta_H}(\xi_1) = \exp\left(\frac{-0.00001}{((3/8-\xi_1)(5/8-\xi_1))^2} \right)\Big|_{\Gamma_H},$$
$$b_{dv_I}(\xi_2) = \left[0,\ \exp\left(\frac{-0.0003}{((5/8-\xi_2)(7/8-\xi_2))^2} \right)\Big|_{\Gamma_I} \right]^{\mathrm{T}}$$

with the nonzero components depicted in Fig. 3.

We consider three observations on the linearized Boussinesq equations (5). These are given by

$$y_{b1}(t) = \frac{1}{|\Omega_\theta|} \int_{\Omega_\theta} \theta(\xi, t) d\xi, \quad y_{b2}(t) = \frac{1}{|\Gamma_O|} \int_{\Gamma_O} \theta(\xi, t) d\xi_2,$$

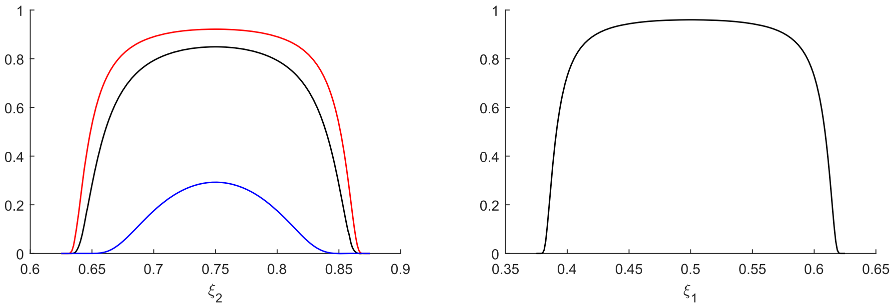


Fig. 3 Control and disturbance shape functions. On the left b_v (black), b_{θ_I} (red) and b_{dv} (blue), and on the right b_{θ_H}

$$y_{b3}(t) = \frac{1}{|\Omega_v|} \int_{\Omega_v} v_{\xi_1}(\xi, t) d\xi,$$

where $\Omega_\theta = [\frac{1}{8}, \ \frac{2}{8}] \times [\frac{5}{8}, \ \frac{6}{8}]$ and $\Omega_v = [\frac{3}{8}, \ \frac{4}{8}] \times [\frac{2}{8}, \ \frac{3}{8}]$, and we set $y_b = [y_{b1}, \ y_{b2}, \ y_{b3}]^{\mathrm{T}}$. The actuator (6) and the sensor (8) are characterized by the simple choices

$$A_a = A_s = -I_3, \quad B_a = C_a = B_s = C_s = I_3, \tag{54}$$

and the reference signals to be tracked are

$$y_{\mathrm{ref}1}(t) = -1 + \sin(t) + 0.3\cos(2t), \quad y_{\mathrm{ref}2}(t) = 0.5\cos(0.5t),$$
$$y_{\mathrm{ref}3}(t) = 1 + 0.5\sin(2t),$$

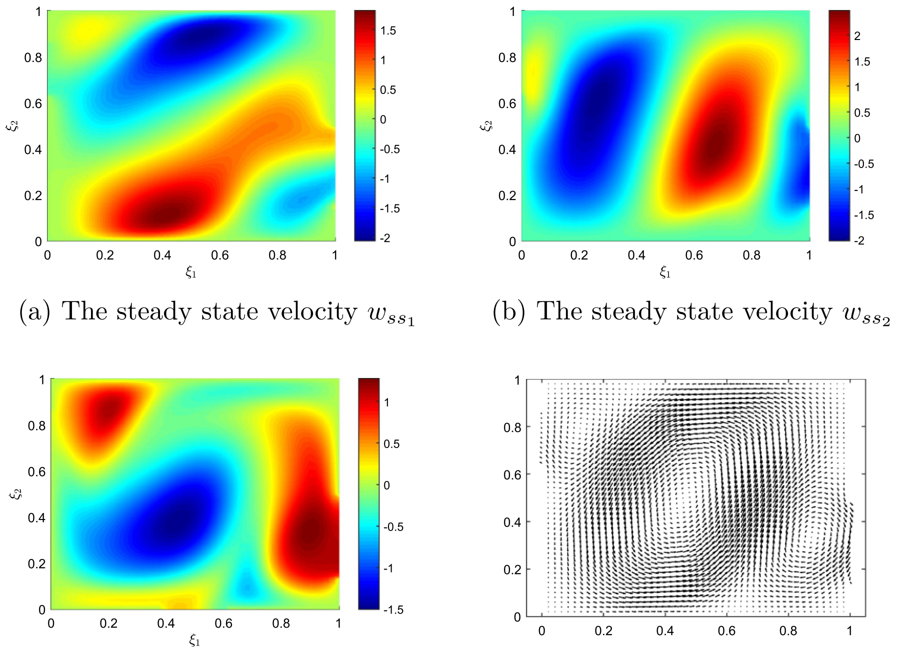
respectively. Finally, the disturbance signal is given by $u_d(t) = 2\sin(0.5t)$.

For the simulations, we use a uniform triangulation of Ω together with the Taylor–Hood finite element spatial discretization for the Navier–Stokes part of the Boussinesq equations and quadratic elements with the same triangulation for the advection–diffusion part. The incompressibility condition $\nabla \cdot v = 0$ is relaxed by using a penalty method to decouple the pressure term from the velocity, see, e.g., [20, Ch. 5.2] or [24], with the penalty parameter $\epsilon_p = 10^{-5}$. To approximate the infinite-dimensional system (A_b, B_b, C_b), we use the mesh size $h_{inf} = 1/24$, which results in approximation order $N_{inf} = 6728$ for the system (A, B, C) after accounting for the boundary conditions.

We use Newton's method to calculate a steady-state solution for the Boussinesq equations (4) subject to

$$f_T(\xi) = 5\sin(2\pi\xi_1)\cos(2\pi\xi_2),$$
$$f_w(\xi) = 4\left[\sin(2\pi\xi_1)\cos(2\pi\xi_2), \ -\cos(2\pi\xi_1)\sin(2\pi\xi_2)\right]^{\mathrm{T}}.$$

The steady-state solution may be, and according to numerical tests is, non-unique, and we choose the steady state (w_{ss}, T_{ss}) corresponding to the initial guess given by the



(a) The steady state velocity w_{ss_1} (b) The steady state velocity w_{ss_2}

(c) The steady state temperature T_{ss} (d) The steady state velocity field w_{ss}

Fig. 4 A steady-state solution $(w_{ss_1}, w_{ss_2}, T_{ss})$ of the Boussinesq equations (4)

steady-state solution (w_i, T_i) of (4) subject to

$$f_{T_i}(\xi) = 4\sin(2\pi\xi_1)\cos(2\pi\xi_2),$$
$$f_{w_i}(\xi) = 2\left[\sin(2\pi\xi_1)\cos(2\pi\xi_2),\ -\cos(2\pi\xi_1)\sin(2\pi\xi_2)\right]^{\mathrm{T}}.$$

The steady-state solution is depicted in Fig. 4.

We observe numerically that for the calculated steady-state solution A_b, thus also A due to the block triangular structure if rearranged according to the state (x_a, x_b, x_s), has a single pair of unstable eigenvalues

$$\lambda_{\pm} \approx 0.0621 \pm 0.4908i.$$

Exponential stabilizability, exponential detectability and Assumption 3.1 are checked numerically for the system (A_b, B_b, C_b), and for the choice (54) they are transferred to the system (A, B, C) by Lemmata 2.9, 2.10 and 3.2.

For the controller construction, we use a coarser linearized Boussinesq equations approximation with the mesh size $h = 1/16$. Using the penalty method would introduce additional modeling error, so we base the plant approximation (A^N, B^N, C^N) on the discretized Leray projector instead, see [7,8,25]. The discretized Leray projector is used merely as a theoretical tool, and the Riccati equations together with the

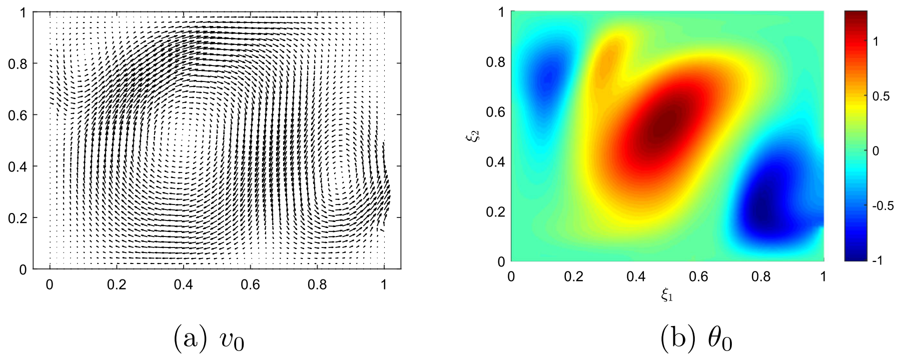


Fig. 5 The initial state (v_0, θ_0)

model-reduced operators A_L^r, B_L^r, K_2^r, L^r are instead solved using the differential-algebraic equation form of the Taylor–Hood-quadratic discretized cascade system. More specifically, we obtain solutions of the Riccati equations using the Generalized Low-Rank Cholesky Factor Newton Method, see Algorithm 2 in [7], with the initial stabilizing feedback solved by the Matlab's `icare` function using the penalized Taylor–Hood-quadratic discretization and the ADI shift parameters solved using the `LYAPACK` toolbox for Matlab. Finally, `balred` function of Matlab is utilized for the order reduction. Parameters of the Riccati equations are chosen as

$$\alpha_1 = 0.3, \quad \alpha_2 = 0.2, \quad R_1 = R_2 = I_3, \quad Q_1 Q_1^* = I_X, \quad Q_c^* Q_c = I_{Z_{im} \times X}.$$

To track $y_{\mathrm{ref}} = [y_{\mathrm{ref1}}, \ y_{\mathrm{ref2}}, \ y_{\mathrm{ref3}}]^{\mathrm{T}}$ while rejecting u_d, the internal model of the controller has 4 frequencies. Due to y_{ref} and u_d having constant coefficients, $\dim Z_{im} = 3 + 3 \cdot 3 \cdot 2 = 21$, thus us using the reduction order $r = 20$ results in the controller order $\dim Z = \dim Z_{im} + r = 41$.

Remark 4.1 Assumption 3.3 is not satisfied by the mixed finite element methods such as the Taylor–Hood discretization implemented in this work. Convergence properties of the approximation-based controllers for mixed finite elements are studied in [9], but the presented results are incomplete for our use. As such, finding suitable approximation methods or verifying convergence properties of commonly used approximation methods, such as mixed finite elements, requires further work.

For the simulations, we choose as the initial state of the closed-loop system

$$\begin{aligned} &\left[v_0, \ \theta_0, \ x_{a0}, \ x_{s0}, \ z_0\right]^{\mathrm{T}} \\ &= \left[w_i - w_{ss}, \ T_i - T_{ss}, \ 0, \ 0, \ 0\right]^{\mathrm{T}} \in X_v \times L^2(\Omega) \times \mathbb{R}^3 \times \mathbb{R}^3 \times Z \end{aligned}$$

and the state components v_0 and θ_0 are shown in Fig. 5.

Output tracking performance of the controller for $t \in [0, 50]$ is illustrated in Fig. 6.

The system output converges to the reference output with accurate tracking for $t > 30$. Initial oscillations of the observation are reasonable, but the state (v, θ) of the

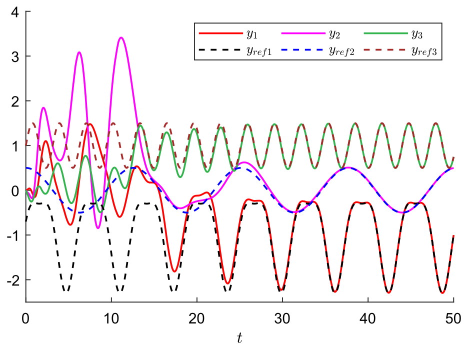
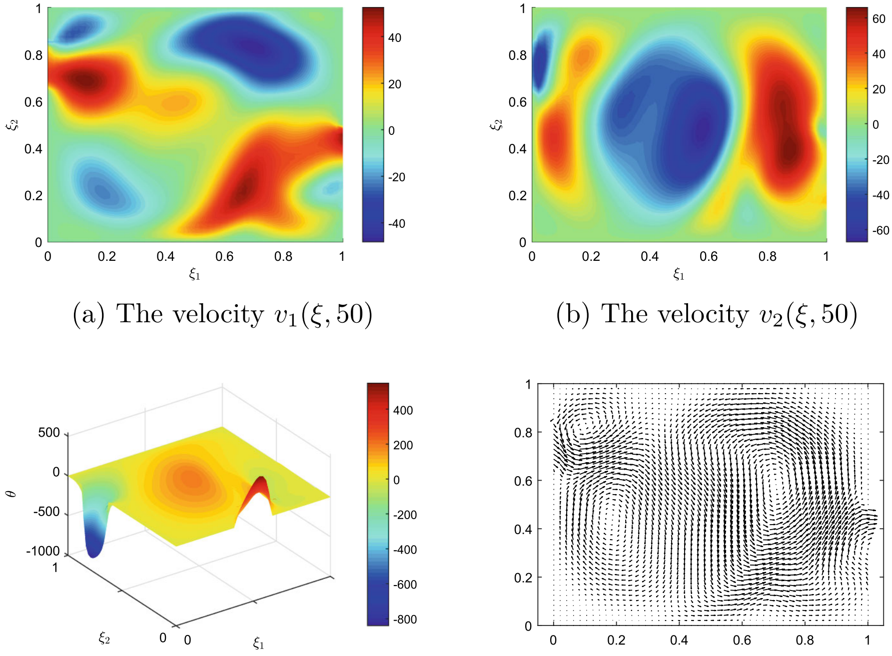


Fig. 6 The system output $y = [y_1,\ y_2,\ y_3]^{\mathrm{T}}$ and the reference output $y_{\mathrm{ref}} = [y_{\mathrm{ref}1},\ y_{\mathrm{ref}2},\ y_{\mathrm{ref}3}]^{\mathrm{T}}$



(a) The velocity $v_1(\xi, 50)$

(b) The velocity $v_2(\xi, 50)$

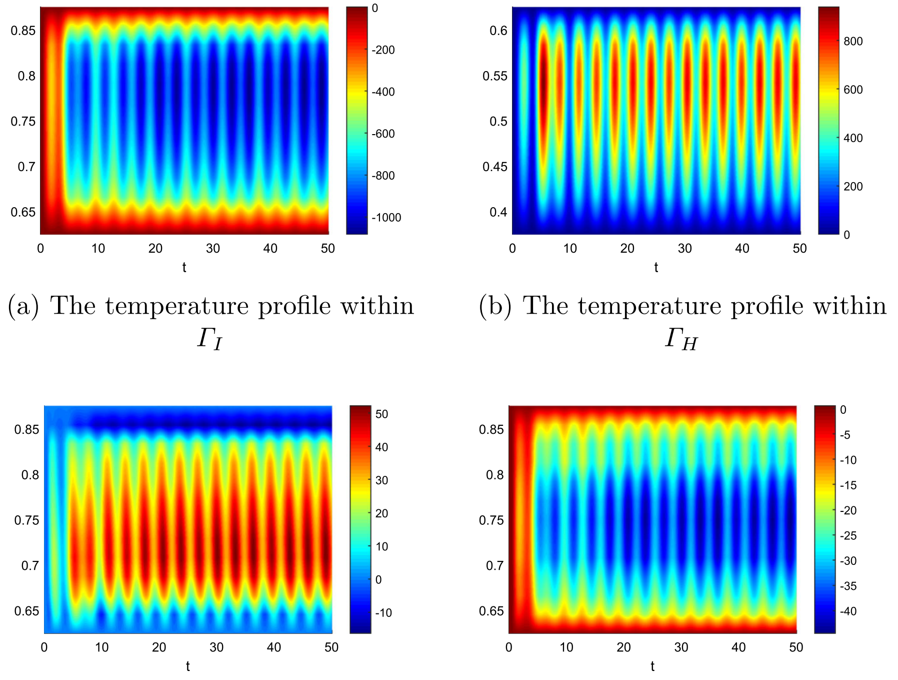
(c) The temperature $\theta(\xi, 50)$

(d) The velocity field $v(\xi, 50)$

Fig. 7 State (v_1, v_2, θ) of the linearized translated Boussinesq equations at the time $t = 50$

linearized translated Boussinesq equations experiences significant oscillations. The temperature values near the controlled strips Γ_I and Γ_H are a prime example of this behavior, as Fig. 7 suggests, while incompressibility leads to "less localized" velocity states.

The controlled strips actually maintain large amplitudes for both the velocity v and the temperature θ throughout the simulation disregarding the initial transient, as evident from Fig. 8.



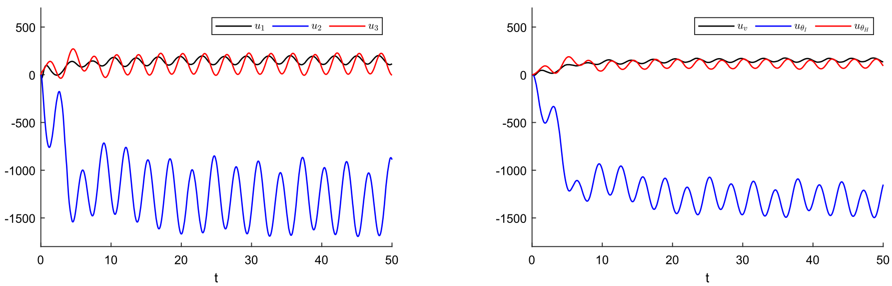
(a) The temperature profile within Γ_I

(b) The temperature profile within Γ_H

(c) The ξ_1-velocity profile within Γ_I

(d) The ξ_2-velocity profile within Γ_I

Fig. 8 State components within the controlled strips for $t \in [0, 50]$



(a) The plant input $u = [u_1,\ u_2,\ u_3]^T$

(b) The boundary input $u_b = [u_v,\ u_{\theta_I},\ u_{\theta_H}]^T$

Fig. 9 Plant input u and the corresponding boundary input u_b generated by the actuator

Similarly, the boundary inputs to the room do not change sign after the transient, see Fig. 9, although the plant input component u_3 just barely does.

5 Conclusion

We studied robust temperature and velocity output tracking of a two-dimensional room model with the fluid dynamics governed by the linearized Boussinesq equations. For the room model, we considered the natural case of the control applied on the fluid via the boundary and the observations performed on the fluid at the boundary. Related to the control and the observation operations, we modeled actuator and sensor dynamics of the system, thus the complete room model was an ODE–PDE–ODE cascade. We examined effects of the added actuator and sensor dynamics on the system properties such as semigroup generation, exponential stabilizability and exponential detectability, and implemented an internal model-based error feedback controller design for robust temperature and velocity output tracking for the room model. In addition to being robust, the controller is suitable for unstable systems, requires only output information instead of full state information and is of low order for efficient applicability.

As an example, we illustrated robust output tracking of the linearized Boussinesq equations with actuator and sensor dynamics in the case of three boundary controls, a mix of one boundary and two in-domain observations and a boundary disturbance, each affecting either a velocity or the temperature component of the system. The controller achieved accurate tracking with relatively small transient observation oscillation, although the system state reaches large absolute values locally. Analogous approach of actuator and sensor modeling can be applied for robust output tracking of a class of linear systems with boundary control and observation.

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