COMPLETE ADDITIVITY, COMPLETE MULTIPLICATIVITY, AND LEIBNIZ-ADDITIVITY ON RATIONALS

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Abstract

Completely additive (c-additive in short) functions and completely multiplicative (c-multiplicative in short) functions are ordinarily defined for positive integers but sometimes on larger domains. We survey this matter by extending these functions first to nonzero integers and thereafter to nonzero rationals. Then we can similarly extend Leibniz-additive (L-additive in short) functions. (A function is L-additive if it is a product of a c-additive and a c-multiplicative function.) We study some properties of these functions. The role of an L-additive function as a generalized arithmetic derivative is our special interest.

1. Introduction

We let $\mathbb{P}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}_+$, and $\mathbb{Q}$ denote the set of primes, positive integers, nonnegative integers, integers, positive rationals, and rationals, respectively. We also write

$$\mathbb{Z}_{\neq 0} = \mathbb{Z} \setminus \{0\}, \quad \mathbb{Q}_{\neq 0} = \mathbb{Q} \setminus \{0\}.$$

The arithmetic derivative, originally defined [1] on $\mathbb{N}$, can easily [12] be extended to $\mathbb{Z}$, and further to $\mathbb{Q}$. Also, the arithmetic partial derivative, defined [9] on $\mathbb{Z}_+$,

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can easily [4] be extended to $\mathbb{Q}$. More generally, this holds for the arithmetic subderivative, too. Its original definition [10] extends easily [6] to $\mathbb{Q}$.

Arithmetic functions have mostly been used for studying $\mathbb{Z}_+$, but sometimes they have been considered also in other domains. Let us take three examples. A completely multiplicative function is defined [7] on $\mathbb{Q}_+$. A completely additive and a completely multiplicative function are defined [8] on an arithmetic semigroup. These functions are defined [11] on an integral domain and extended to its field of fractions.

We are mainly concerned with a function $f = gh$, where g is completely additive and h completely multiplicative. We need certain properties of g and h for this purpose. Having studied g in Section 2 and h in Section 3, we focus on f in Section 4. The role of f as a generalized arithmetic derivative, considered in Section 5, is our special interest. As an application, we study in Section 6 an arithmetic differential equation and its generalization. A summarizing discussion in Section 7 completes our paper.

2. Complete additivity

Let A be a set satisfying

$$\emptyset \neq A \subseteq \mathbb{Q}, \quad 0 \notin A, \quad x, y \in A \Rightarrow xy \in A. \tag{1}$$

All functions we study in this paper are rational-valued. (In fact, they can be real-valued or even complex-valued, which, however, does not give us an additional benefit.) A function g on A is *completely additive* (in short, *c-additive*) if

$$g(xy) = g(x) + g(y) \tag{2}$$

for all $x, y \in A$. The reason for exluding 0 is that, if $0 \in A$, then

$$g(0) = g(0x) = g(0) + g(x)$$

for all $x \in A$, and the only c-additive function is therefore the zero function $\theta(x) = 0$. (If we accept ∞, we can invalidate this conclusion by defining $g(0) = \infty$.) If $1 \in A$, then substituting $x = y = 1$ in (2) yields

$$g(1) = 0.$$

Let $p \in \mathbb{P}$. The *p-adic ordinal* of $n \in \mathbb{Z}_+$,

$$\nu_p(n) = \max\{r \in \mathbb{N} : p^r \mid n\},$$

is c-additive on $\mathbb{Z}_+$. We have

$$n = \prod_{p \in \mathbb{P}} p^{\nu_p(n)}. \tag{3}$$

For $n > 1$, let

$$\{p_1, \ldots, p_k\} = \{p \in \mathbb{P} : \nu_p(n) \neq 0\} = \{p \in \mathbb{P} : \nu_p(n) > 0\}$$

and

$$n_i = p_i^{\nu_{p_i}(n)}, \quad i = 1, \ldots, k.$$

Then (3) reads

$$n = n_1 \cdots n_k. \tag{4}$$

In the following proposition, we recall [10, Theorem 1] that a c-additive function on $\mathbb{Z}_+$ is totally defined by its values at primes. We also see that in proving c-additivity, we do not need to show that

$$g(mn) = g(m) + g(n) \tag{5}$$

for all $m, n \in \mathbb{Z}_+$; to show it for all $m \in \mathbb{P}$ and $n \in \mathbb{Z}_+$ is enough.

Proposition 1. *Let g be a function on $\mathbb{Z}_+$. The following conditions are equivalent.*

(a) *g is completely additive.*

(b) *For all $n \in \mathbb{Z}_+$,*

$$g(n) = \sum_{p \in \mathbb{P}} \nu_p(n) g(p).$$

(c) *For all $p \in \mathbb{P}$, $n \in \mathbb{Z}_+$,*

$$g(pn) = g(p) + g(n).$$

Proof. (a)⇒(b). If $n = 1$, then (b) holds clearly. So, let $n > 1$ be as in (4). By induction on $\nu_{p_i}(n)$,

$$g(n_i) = \nu_{p_i}(n) g(p_i).$$

Therefore (b) follows by induction on k (defined in (4)).

(b)⇒(c). Let $p \in \mathbb{P}$ and $n \in \mathbb{Z}_+$. Since

$$\nu_p(pn) = \nu_p(n) + 1, \quad \nu_q(pn) = \nu_q(n), \quad p \neq q \in \mathbb{P},$$

we have

$$\begin{aligned} g(pn) &= \sum_{q \in \mathbb{P}} \nu_q(pn) g(q) = \sum_{\substack{q \in \mathbb{P} \\ q \neq p}} \nu_q(pn) g(q) + \nu_p(pn) g(p) \\ &= \sum_{\substack{q \in \mathbb{P} \\ q \neq p}} \nu_q(n) g(q) + (\nu_p(n) + 1) g(p) = \sum_{q \in \mathbb{P}} \nu_q(n) g(q) + g(p) = g(n) + g(p). \end{aligned}$$

(c)$\Rightarrow$(a). Substituting $n = 1$ in (c), we get $g(1) = 0$, and (5) therefore holds for $m = 1$. For $m > 1$, let

$$m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \quad p_1, \dots, p_k \in \mathbb{P},\, \alpha_1, \dots, \alpha_k \in \mathbb{Z}_+.$$

Since

$$\begin{aligned} g(p_i^{\alpha_i} n) &= g(p_i p_i^{\alpha_i - 1} n) = g(p_i) + g(p_i^{\alpha_i - 1} n) = g(p_i) + g(p_i p_i^{\alpha_i - 2} n) \\ &= g(p_i) + g(p_i) + g(p_i^{\alpha_i - 2} n) = g(p_i^2) + g(p_i^{\alpha_i - 2} n) = \cdots = g(p_i^{\alpha_i}) + g(n), \end{aligned}$$

we have

$$\begin{aligned} g(mn) &= g(p_1^{\alpha_1}(p_2^{\alpha_2} \cdots p_k^{\alpha_k} n)) = g(p_1^{\alpha_1}) + g(p_2^{\alpha_2}(p_3^{\alpha_3} \cdots p_k^{\alpha_k} n)) \\ &= g(p_1^{\alpha_1}) + g(p_2^{\alpha_2}) + g(p_3^{\alpha_3} \cdots p_k^{\alpha_k} n) = \dots \\ &= g(p_1^{\alpha_1}) + \cdots + g(p_k^{\alpha_k}) + g(n). \end{aligned} \tag{6}$$

In particular,

$$g(m) = g(p_1^{\alpha_1}) + \cdots + g(p_k^{\alpha_k}) + g(1) = g(p_1^{\alpha_1}) + \cdots + g(p_k^{\alpha_k}). \tag{7}$$

Now, (6) and (7) give $g(mn) = g(m) + g(n)$. □

Remark 1. The condition

(d) $g(pq) = g(p) + g(q) \quad$ for all $p, q \in \mathbb{P}$

does not imply (a). For a counterexample, define on $\mathbb{Z}_+$ that $g(n) = \sum_{p \in \mathbb{P}} \nu_p(n)$ if n has at most two prime factors, and $g(n) = 1$ otherwise. Then g satisfies (d) but does not satisfy (a), since $g(pqr) = 1$ but $g(p) + g(q) + g(r) = 3$ for all $p, q, r \in \mathbb{P}$ (all inequal).

Remark 2. If g is a c-additive function on $\mathbb{Z}_+$, then

$$g(cm) - g(cn) = g(c) + g(m) - (g(c) + g(n)) = g(m) - g(n) \tag{8}$$

for all $c, m, n \in \mathbb{Z}_+$. The converse is not true. For a counterexample, the function $g(n) = 1$ satisfies (8) but is not c-additive.

We extend a c-additive function on $\mathbb{Z}_+$ to that on $\mathbb{Z}_{\neq 0}$.

Theorem 1. *Let g be a completely additive function on $\mathbb{Z}_+$, and let $\tilde{g}$ be a function on $\mathbb{Z}_{\neq 0}$. The following conditions are equivalent.*

(a) (a_1) *$\tilde{g}$ is completely additive and* (a_2) *the restriction $\tilde{g}\,|_{\mathbb{Z}_+} = g$.*

(b) *$\tilde{g}(n) = g(|n|)$ for all $n \in \mathbb{Z}_{\neq 0}$.*

(c) (c_1) *$\tilde{g}(n) = \tilde{g}(-n)$ for all $n \in \mathbb{Z}_{\neq 0}$ and* (c_2) $\tilde{g}\mid_{\mathbb{Z}_+} = g$.

Proof. (a)⇒(b). Since

$$0 = \tilde{g}(1) = \tilde{g}((-1)^2) = 2\tilde{g}(-1),$$

it follows that

$$\tilde{g}(-1) = 0.$$

If $n > 0$, then (b) is just (a_2). If $n < 0$, then

$$\tilde{g}(n) = \tilde{g}((-1)|n|) = \tilde{g}(-1) + \tilde{g}(|n|) = 0 + g(|n|) = g(|n|).$$

(b)⇒(c). Trivial.

(c)⇒(a). Because (c_2) is the same as (a_2), our claim is that (c_1)⇒(a_1) under (c_2). If $m, n > 0$, then

$$\tilde{g}(mn) = g(mn) = g(m) + g(n) = \tilde{g}(m) + \tilde{g}(n).$$

If m and n have opposite signs, say $m > 0$ and $n < 0$, then

$$\tilde{g}(mn) = \tilde{g}(m(-n)) = g(m(-n)) = g(m) + g(-n) = \tilde{g}(m) + \tilde{g}(-n) = \tilde{g}(m) + \tilde{g}(n).$$

If $m, n < 0$, then

$$\begin{aligned}\tilde{g}(mn) &= \tilde{g}((-m)(-n)) = g((-m)(-n)) = g(-m) + g(-n)\\ &= \tilde{g}(-m) + \tilde{g}(-n) = \tilde{g}(m) + \tilde{g}(n),\end{aligned}$$

completing the proof. □

Applying Theorem 1 to $g = \nu_p$, we have

$$\tilde{\nu}_p(n) = \nu_p(|n|)$$

for all $n \in \mathbb{Z}_{\neq 0}$, $p \in \mathbb{P}$. Writing ν_p instead of $\tilde{\nu}_p$ above, (3) extends to $n \in \mathbb{Z}_{\neq 0}$:

$$n = (\operatorname{sgn} n) \prod_{p \in \mathbb{P}} p^{\nu_p(n)}, \tag{9}$$

where $\operatorname{sgn} n = n/|n|$. Consequently, we can replace $\mathbb{Z}_+$ with $\mathbb{Z}_{\neq 0}$ in Proposition 1.

Next, we extend a c-additive function on $\mathbb{Z}_{\neq 0}$ to that on $\mathbb{Q}_{\neq 0}$.

Theorem 2. *Let g be a completely additive function on $\mathbb{Z}_{\neq 0}$, and let $\tilde{g}$ be a function on $\mathbb{Q}_{\neq 0}$. The following conditions are equivalent.*

(a) (a_1) *$\tilde{g}$ is completely additive and* (a_2) $\tilde{g}\mid_{\mathbb{Z}_{\neq 0}} = g$.

(b) $\tilde{g}(m/n) = g(m) - g(n)$ *for all* $m, n \in \mathbb{Z}_{\neq 0}$.

(c) (c_1) $\tilde{g}(x/y) = \tilde{g}(x) - \tilde{g}(y)$ *for all* $x, y \in \mathbb{Q}_{\neq 0}$ *and* (c_2) $\tilde{g}\,|_{\mathbb{Z}_{\neq 0}} = g$.

Proof. We begin by noticing that the equations (b) and (c_1) are well-defined by Remark 2.

(a)⇒(b). Let $m, n \in \mathbb{Z}_{\neq 0}$. Since

$$0 = \tilde{g}(1) = \tilde{g}(n\,\frac{1}{n}) = \tilde{g}(n) + \tilde{g}(\frac{1}{n}) = g(n) + \tilde{g}(\frac{1}{n}),$$

we have

$$\tilde{g}(\frac{1}{n}) = -g(n);$$

and further,

$$\tilde{g}(\frac{m}{n}) = \tilde{g}(m\,\frac{1}{n}) = \tilde{g}(m) + \tilde{g}(\frac{1}{n}) = g(m) - g(n).$$

(b)⇒(c_2). If $m \in \mathbb{Z}_{\neq 0}$, then

$$\tilde{g}(m) = g(m) - g(1) = g(m) - 0 = g(m).$$

(b)⇒(c_1). Let

$$x = \frac{m}{n},\ y = \frac{r}{s}, \quad m, n, r, s \in \mathbb{Z}_{\neq 0}. \tag{10}$$

Then

$$\begin{aligned}\tilde{g}(\frac{x}{y}) &= \tilde{g}(\frac{ms}{nr}) = g(ms) - g(nr) = g(m) + g(s) - (g(n) + g(r))\\ &= \tilde{g}(\frac{m}{n}) - \tilde{g}(\frac{r}{s}) = \tilde{g}(x) - \tilde{g}(y).\end{aligned}$$

(c)⇒(a). Because (c_2) is just (a_2), we again claim that (c_1)⇒(a_1) under (c_2). Let x and y be as in (10). Then

$$\begin{aligned}\tilde{g}(xy) &= \tilde{g}(\frac{mr}{ns}) = \tilde{g}(mr) - \tilde{g}(ns) = g(mr) - g(ns)\\ &= g(m) + g(r) - (g(n) + g(s)) = \tilde{g}(\frac{m}{n}) + \tilde{g}(\frac{r}{s}) = \tilde{g}(x) + \tilde{g}(y),\end{aligned}$$

completing the proof. □

Applying Theorem 2 to $g = \nu_p$, we have

$$\tilde{\nu}_p(\frac{m}{n}) = \nu_p(m) - \nu_p(n)$$

for all $m, n \in \mathbb{Z}_{\neq 0}$, $p \in \mathbb{P}$. Writing ν_p instead of $\tilde{\nu}_p$, (9) extends to $x \in \mathbb{Q}_{\neq 0}$:

$$x = (\operatorname{sgn} x) \prod_{p \in \mathbb{P}} p^{\nu_p(x)}. \tag{11}$$

We can now replace $\mathbb{Z}_+$ with $\mathbb{Q}_{\neq 0}$ in Proposition 1.

3. Complete multiplicativity

Let a set A satisfy (1). A function h on A is *completely multiplicative* (*c-multiplicative* in short) if

$$h(x) \neq 0 \quad \text{and} \quad h(xy) = h(x)h(y) \tag{12}$$

for all $x, y \in A$. Ordinarily $A = \mathbb{Z}_+$ and the definition is weaker. It suffices that there is $x \in \mathbb{Z}_+$ satisfying $h(x) \neq 0$. Substituting $y = 1$ in (12) then yields

$$h(1) = 1.$$

Again, it is reasonable to exclude zero from A. Namely, if $0 \in A$, then

$$h(0) = h(0x) = h(0)h(x),$$

which implies that the only c-multiplicative function is the unit function $E(x) = 1$. (Accepting ∞ does not help now, because $\infty \cdot h(x) \neq \infty$ if $h(x) < 0$. Accepting $h(0) = 0$, which is justified by the ordinary definition, does not benefit us.)

The next proposition is analogous to Proposition 1.

Proposition 2. *Let h be a function on $\mathbb{Z}_+$. The following conditions are equivalent.*

(a) *h is completely multiplicative.*

(b) *For all $n \in \mathbb{Z}_+$,*
$$h(n) = \prod_{p \in \mathbb{P}} h(p)^{\nu_p(n)}.$$

(c) *For all $p \in \mathbb{P}$, $n \in \mathbb{Z}_+$,*
$$h(pn) = h(p)h(n).$$

Proof. A straightforward modification of the proof of Proposition 1. □

Remark 3. The condition

(d) $h(pq) = h(p)h(q)$ for all $p, q \in \mathbb{P}$

does not imply (a). For a counterexample, define on $\mathbb{Z}_+$ that $h(n) = n$ if n has at most two prime factors, and $h(n) = 1$ otherwise. Then h satisfies (d) but does not satisfy (a), since $h(pqr) = 1$ but $h(p)h(q)h(r) = pqr$ for all $p, q, r \in \mathbb{P}$ (all inequal).

Remark 4. Let h be completely multiplicative on $\mathbb{Z}_+$. Analogously to Remark 2,

$$\frac{h(cm)}{h(cn)} = \frac{h(c)h(m)}{h(c)h(n)} = \frac{h(m)}{h(n)} \tag{13}$$

for all $c, m, n \in \mathbb{Z}_+$. The converse is not true. For a counterexample, the function $h(x) = 2$ satisfies (13) but is not c-multiplicative.

Similarly to the c-additive case, we extend a c-multiplicative function on $\mathbb{Z}_+$ to that on $\mathbb{Z}_{\neq 0}$.

Theorem 3. *Let h be a completely multiplicative function on $\mathbb{Z}_+$, and let $\tilde{h}$ be a function on $\mathbb{Z}_{\neq 0}$. The following conditions are equivalent.*

(a) (a_1) $\tilde{h}$ *is completely multiplicative and* (a_2) $\tilde{h}\,|_{\mathbb{Z}_+} = h$.

(b) *Either*

(b_1) $\tilde{h}(n) = h(|n|)$ *for all* $n \in \mathbb{Z}_{\neq 0}$

or

(b_2) $\tilde{h}(n) = (\operatorname{sgn} n)h(|n|)$ *for all* $n \in \mathbb{Z}_{\neq 0}$.

(c) *Either*

(c_1) $\tilde{h}(n) = \tilde{h}(-n)$ *for all* $n \in \mathbb{Z}_{\neq 0}$

or

(c_2) $\tilde{h}(n) = -\tilde{h}(-n)$ *for all* $n \in \mathbb{Z}_{\neq 0}$,

and

(c_3) $\tilde{h}\,|_{\mathbb{Z}_+} = h$.

Proof. (a)$\Rightarrow$(b). Since

$$1 = \tilde{h}(1) = \tilde{h}((-1)^2) = \tilde{h}(-1)^2,$$

we get

$$\tilde{h}(-1) = \pm 1.$$

If $n > 0$, then $\tilde{h}(n) = h(n) = h(|n|)$. If $n < 0$, then

$$\tilde{h}(n) = \tilde{h}((-1)|n|) = \tilde{h}(-1)\tilde{h}(|n|) = \pm\tilde{h}(|n|) = \pm h(|n|).$$

(b)$\Rightarrow(\mathrm{c}_3)$, $(\mathrm{b}_1)\Rightarrow(\mathrm{c}_1)$, and $(\mathrm{b}_2)\Rightarrow(\mathrm{c}_2)$. Trivial.

$(\mathrm{c}_1), (\mathrm{c}_3)\Rightarrow$(a). Because (c_3) is just (a_2), we show that $(\mathrm{c}_1)\Rightarrow(\mathrm{a}_1)$ under (c_3). If $m, n > 0$, then

$$\tilde{h}(mn) = h(mn) = h(m)h(n) = \tilde{h}(m)\tilde{h}(n).$$

If m and n have opposite signs, say $m > 0$ and $n < 0$, then

$$\begin{aligned}\tilde{h}(mn) &= \tilde{h}(-m(-n)) = \tilde{h}(m(-n)) = h(m(-n)) \\ &= h(m)h(-n) = \tilde{h}(m)\tilde{h}(-n) = \tilde{h}(m)\tilde{h}(n).\end{aligned}$$

If $m, n < 0$, then

$$\begin{aligned}\tilde{h}(mn) &= \tilde{h}((-m)(-n)) = h((-m)(-n)) \\ &= h(-m)h(-n) = \tilde{h}(-m)\tilde{h}(-n) = \tilde{h}(m)\tilde{h}(n).\end{aligned}$$

$(\mathrm{c}_2), (\mathrm{c}_3)\Rightarrow$(a). A simple variant of the above. □

So, there are two possible extensions from $\mathbb{Z}_+$ to $\mathbb{Z}_{\neq 0}$. We choose one of them and extend it further to $\mathbb{Q}_{\neq 0}$.

Theorem 4. *Let h be a completely multiplicative function on $\mathbb{Z}_{\neq 0}$, and let $\tilde{h}$ be a function on $\mathbb{Q}_{\neq 0}$. The following conditions are equivalent.*

(a) (a_1) *$\tilde{h}$ is completely multiplicative and* (a_2) $\tilde{h}\,|_{\mathbb{Z}_{\neq 0}} = h$.

(b) $\tilde{h}(m/n) = h(m)/h(n)$ *for all* $m, n \in \mathbb{Z}_{\neq 0}$.

(c) (c_1) $\tilde{h}(x/y) = \tilde{h}(x)/\tilde{h}(y)$ *for all* $x, y \in \mathbb{Q}_{\neq 0}$ *and* (c_2) $\tilde{h}\,|_{\mathbb{Z}_{\neq 0}} = h$.

Proof. Proceed similarly to the proof of Theorem 2. The equations (b) and (c_1) are well-defined by Remark 4. □

We can now replace $\mathbb{Z}_+$ with $\mathbb{Q}_{\neq 0}$ in Proposition 2.

4. Leibniz-additivity

Let a set A satisfy (1). A function f on A is *Leibniz-additive* (*L-additive* in short), cf. [5, 10], if

$$f = gh, \tag{14}$$

where g is a completely additive and h completely multiplicative function on A. Then

$$f(xy) = g(xy)h(xy) = (g(x) + g(y))h(x)h(y) = f(x)h(y) + f(y)h(x) \tag{15}$$

for all $x, y \in A$. This may be considered a generalized Leibniz rule. In particular, if h is the identical function $N(n) = n$, then (15) gives the Leibniz rule

$$f(xy) = f(x)y + f(y)x.$$

If $1 \in A$, then

$$f(1) = g(1)h(1) = 0 \cdot 1 = 0.$$

We saw in Section 2 that a c-additive function on $\mathbb{Z}_+$ has a unique c-additive extension to $\mathbb{Z}_{\neq 0}$ and further to $\mathbb{Q}_{\neq 0}$. We also saw in Section 3 that a c-multiplicative function on $\mathbb{Z}_+$ has two c-multiplicative extensions to $\mathbb{Z}_{\neq 0}$, each having a unique c-multiplicative extension to $\mathbb{Q}_{\neq 0}$. Because the latter observation concerns also L-additive functions, we can directly go to the case $A = \mathbb{Q}_{\neq 0}$.

Theorem 5. *Let f be a function on $\mathbb{Q}_{\neq 0}$. The following conditions are equivalent.*

(a) *f is Leibniz-additive.*

(b) *There is a completely multiplicative function* h *on* $\mathbb{Q}_{\neq 0}$ *such that*

$$f(xy) = f(x)h(y) + f(y)h(x)$$

for all $x, y \in \mathbb{Q}_{\neq 0}$.

(c) *There is a completely multiplicative function* h *on* $\mathbb{Q}_{\neq 0}$ *such that*

$$f(\frac{x}{y}) = \frac{h(y)f(x) - h(x)f(y)}{h(y)^2}$$

for all $x, y \in \mathbb{Q}_{\neq 0}$.

(d) *There are functions* g *and* h *on* $\mathbb{P}$ *such that*

$$f(x) = \Big(\sum_{p\in\mathbb{P}} \nu_p(x)g(p)\Big) \prod_{p\in\mathbb{P}} h(p)^{\nu_p(x)}.$$

(e) *There is a completely multiplicative function* h *on* $\mathbb{Q}_{\neq 0}$ *such that*

$$f(py) = f(p)h(y) + f(y)h(p)$$

for all $p \in \mathbb{P}$, $y \in \mathbb{Q}_{\neq 0}$.

Proof. (a)⇒(b). This is already proved in (15).

(b)⇒(a). The function $g = f/h$ is c-additive, since

$$g(xy) = \frac{f(xy)}{h(xy)} = \frac{f(x)h(y) + f(y)h(x)}{h(x)h(y)} = \frac{f(x)}{h(x)} + \frac{f(y)}{h(y)} = g(x) + g(y)$$

for all $x, y \in \mathbb{Q}_{\neq 0}$.

(b)⇒(c). Since

$$f(1) = f(1 \cdot 1) = f(1)h(1) + f(1)h(1) = 2f(1) \cdot 1 = 2f(1),$$

we have

$$f(1) = 0;$$

and further,

$$f(y)h(\frac{1}{y}) + f(\frac{1}{y})h(y) = f(y\,\frac{1}{y}) = f(1) = 0$$

for all $y \in \mathbb{Q}_{\neq 0}$. Therefore

$$f(\frac{1}{y}) = -\frac{f(y)}{h(y)}h(\frac{1}{y}) = -\frac{f(y)}{h(y)}\frac{1}{h(y)} = -\frac{f(y)}{h(y)^2}.$$

Finally, for all $x, y \in \mathbb{Q}_{\neq 0}$,

$$f(\frac{x}{y}) = f(x\,\frac{1}{y}) = f(x)h(\frac{1}{y}) + f(\frac{1}{y})h(x) = \frac{f(x)}{h(y)} - \frac{h(x)f(y)}{h(y)^2} = \frac{h(y)f(x) - h(x)f(y)}{h(y)^2}.$$

(c)⇒(b). Let $x, y \in \mathbb{Q}_{\neq 0}$. Since

$$f(1) = \frac{h(1)f(1) - h(1)f(1)}{h(1)^2} = 0$$

and

$$f(\frac{1}{y}) = \frac{h(y)f(1) - h(1)f(y)}{h(y)^2} = \frac{h(y)\cdot 0 - 1\cdot f(y)}{h(y)^2} = -\frac{f(y)}{h(y)^2},$$

we have

$$f(xy) = f(\frac{x}{\frac{1}{y}}) = \frac{h(\frac{1}{y})f(x) - h(x)f(\frac{1}{y})}{h(\frac{1}{y})^2} = \frac{\frac{f(x)}{h(y)} + \frac{h(x)f(y)}{h(y)^2}}{\frac{1}{h(y)^2}} = f(x)h(y) + h(x)f(y).$$

(a)⇔(d). Apply Propositions 1 and 2, where $\mathbb{Z}_+$ is replaced with $\mathbb{Q}_{\neq 0}$.

(b)⇒(e). Trivial.

(e)⇒(a). The function $g = f/h$ is c-additive, because it satisfies condition (c) of Proposition 1, where $\mathbb{Z}_+$ is replaced with $\mathbb{Q}_{\neq 0}$. □

5. Arithmetic *gh*-derivative

Let f be as in (14) with $A = \mathbb{Q}_{\neq 0}$. If $h(x) = x$ and

$$g(x) = \sum_{p\in\mathbb{P}} \frac{\nu_p(x)}{p} \quad \text{or} \quad g(x) = \frac{\nu_p(x)}{p},$$

then f is the *arithmetic derivative* D [1, 12] in the first case, and the *arithmetic partial derivative* D_p [9] in the second. More generally, if $h(x) = x$ and

$$g(x) = \sum_{p\in S} \frac{\nu_p(x)}{p}, \tag{16}$$

where $\emptyset \neq S \subseteq \mathbb{P}$, then f is the *arithmetic subderivative* D_S [10] (or a general arithmetic derivative [2]). In particular, $D_{\mathbb{P}} = D$ and $D_{\{p\}} = D_p$. Usually $D(0)$, $D_p(0)$, and $D_S(0)$ are defined to be zero, but we leave them undefined.

An equivalent definition of D (and D_S, respectively) is the Leibniz rule together with $D(p) = 1$ for all $p \in \mathbb{P}$ ($D_S(p) = 1$ for all $p \in S$ and $D_S(p) = 0$ for all $p \in \mathbb{P} \setminus S$, respectively). If we assume only the Leibniz rule, then D_S generalizes to an L-additive function gN, where g is an arbitrary c-additive function on $\mathbb{Q}_{\neq 0}$. So, we must in this case choose $h = N$ (i.e., $h(x) = x$).

Ufnarovski and Åhlander [12, p. 18] generalized D (using some different notations) to

$$\tilde{D}(x) = x \sum_{p \in \mathbb{P}} \frac{\nu_p(x)\tilde{D}(p)}{p}$$

with prescribed $\tilde{D}(p)$ for all $p \in \mathbb{P}$. If each $\tilde{D}(p) = 1$, then $\tilde{D} = D$. Thus $\tilde{D} = gN$, where g is c-additive, satisfying

$$g(p) = \frac{\tilde{D}(p)}{p} \quad \text{for all } p \in \mathbb{P}.$$

We obtain a further generalization by replacing N with a c-multiplicative function h. In other words, we take any L-additive function and call it a generalized arithmetic derivative.

More precisely, we define that a function f on $\mathbb{Q}_{\neq 0}$ is an *arithmetic gh-derivative* if there are functions g and h on $\mathbb{Q}_{\neq 0}$ such that g is completely additive, h is completely multiplicative, and $f = gh$.

Although an arithmetic gh-derivative is nothing but an L-additive function, we find it reasonable to introduce this term and the notation

$$f = \Delta_{g,h}.$$

Then the arithmetic subderivative can be written as

$$D_S = \Delta_{g,N}, \quad g(p) = \frac{\chi_S(p)}{p} \quad \text{for all } p \in \mathbb{P},$$

where χ_S is the characteristic function of S (i.e., $\chi_S(p) = 1$ for $p \in S$ and $\chi_S(p) = 0$ for $p \in \mathbb{P} \setminus S$). In particular, we have for the arithmetic derivative

$$D = \Delta_{g,N}, \quad g(p) = \frac{1}{p} \quad \text{for all } p \in \mathbb{P},$$

and for the arithmetic partial derivative

$$D_p = \Delta_{g,N}, \quad g(q) = \frac{\delta_{pq}}{p} \quad \text{for all } q \in \mathbb{P},$$

where δ_{pq} is the Kronecker delta (i.e., $\delta_{pq} = 1$ if $p = q$, and $\delta_{pq} = 0$ if $p \neq q$).

For a generalization of $\Delta_{g,N}$ on $\mathbb{Z}_{\neq 0}$, cf. [3], let

$$\phi = \{\phi_p : p \in S\}$$

be a set of functions on $\mathbb{Z}_+$ such that the set

$$S_n = \{p \in S : \phi_p(n) \neq 0\}$$

is finite for all $n \in \mathbb{Z}_+$. Define the *generalized arithmetic subderivative* of $n \in \mathbb{Z}_+$ by
$$D_{S,\phi}(n) = n \sum_{p\in S} \frac{\phi_p(n)}{p}$$
and extend it to $\mathbb{Z}_{\neq 0}$ by $D_{S,\phi}(n) = (\operatorname{sgn} n) D_{S,\phi}(|n|)$ for all $n \in \mathbb{Z}_{\neq 0}$. The gN-derivative on $\mathbb{Z}_{\neq 0}$ is its special case. Namely, if
$$\phi_p(n) = p\nu_p(n)g(p) \quad \text{for all } p \in \mathbb{P},$$
then
$$\Delta_{g,N}(n) = ng(n) = n\sum_{p\in\mathbb{P}} \nu_p(n)g(p) = D_{\mathbb{P},\phi}(n).$$

The function
$$g(n) = \sum_{p\in S} \frac{\phi_p(n)}{p}$$
is not c-additive in general. However, if each ϕ_p is c-additive, then g is, too. Otherwise, for a trivial counterexample, let $S = \{2\}$ and $\phi_2 = E$. Then $g(mn) = \frac{1}{2}$ but $g(m) + g(n) = 1$ for all $m, n \in \mathbb{Z}_+$. Therefore g cannot be extended to $\mathbb{Q}_{\neq 0}$ as it was done in Theorem 2. Consequently, the same holds for $D_{S,\phi}$, too.

6. Equation $D_S(x) = ax$ and its generalization

We begin by recalling the already solved case of $S = \{p\}$ but formulate it slightly differently, because $D_p(0)$ is now undefined.

Theorem 6 ([4], Theorem 3)**.** *Let $p \in \mathbb{P}$ and $a \in \mathbb{Q}$. The equation*
$$D_p(x) = ax$$
has a solution $x \in \mathbb{Q}_{\neq 0}$ if and only if $ap \in \mathbb{Z}$. Then the set of its all solutions is
$$\left\{cp^{ap} : c \in \mathbb{Q}_{\neq 0},\, \nu_p(c) = 0\right\}.$$

We gave previously [6] a necessary and sufficient condition for a, under which the equation $D_S(x) = ax$ has a solution $x \in \mathbb{Q}_{\neq 0}$. We did not present the proof but committed to do so in a forthcoming paper. We do it now.

Theorem 7. *Let $\emptyset \neq S \subseteq \mathbb{P}$ and $a \in \mathbb{Q}$. The following conditions are equivalent.*

(a) *The equation*
$$D_S(x) = ax \tag{17}$$
has a solution $x \in \mathbb{Q}_{\neq 0}$.

(b) *The equation* (17) *has infinitely many solutions.*

(c) *There are* $p_1, \dots, p_k \in S$ *such that* $ap_1 \cdots p_k \in \mathbb{Z}$.

Proof. If $a = 0$, then all these conditions are trivially satisfied. So, we assume that $a \neq 0$.

(b)⇒(a). The proof of this implication is trivial.

(a)⇒(c). Let $x \in \mathbb{Q}_{\neq 0}$ satisfy (17). Since $-x$ also satisfies it, we can assume that $x > 0$. By (11),

$$x = \prod_{p \in \mathbb{P}} p^{\nu_p(x)}.$$

Let

$$S_x = \{p \in S : \nu_p(x) \neq 0\} = \{p_1, \dots, p_k\}$$

and write

$$\xi_i = \nu_{p_i}(x), \quad i = 1, \dots, k.$$

If $S_x = \emptyset$, then $x = 1$, implying by (17) that $ax = 0$. This is a contradiction, because $a, x \neq 0$. Hence $S_x \neq \emptyset$.

Now,

$$D_S(x) = x \sum_{i=1}^{k} \frac{\xi_i}{p_i} = \frac{x}{P} \sum_{i=1}^{k} \xi_i P_i,$$

where

$$P_i = \prod_{\substack{j=1 \\ j \neq i}}^{k} p_j, \quad P = p_1 \cdots p_k. \tag{18}$$

Since x satisfies (17), i.e.,

$$\frac{x}{P} \sum_{i=1}^{k} \xi_i P_i = ax,$$

we have

$$aP = \sum_{i=1}^{k} P_i \xi_i. \tag{19}$$

Because the right-hand side is an integer, so is the left-hand side, and (c) follows.

(c)⇒(b). Let $p_1, \dots, p_k$ be as in (c), and let $P_1, \dots, P_k, P$ as in (18). Because $\gcd(P_1, \dots, P_k) = 1$, the Diophantine equation (19) has infinitely many solutions $\xi_1, \dots, \xi_k \in \mathbb{Z}$. The equivalence of (19) and (17) completes the proof. □

We generalize this theorem.

Theorem 8. *Let $\emptyset \neq S \subseteq \mathbb{P}$, and let g and h be functions on $\mathbb{Q}_{\neq 0}$ such that h is completely multiplicative and g is completely additive with*

$$g(p) = \frac{\chi_S(p)}{p^{\alpha_p}}, \quad \alpha_p \in \mathbb{Z}_+,$$

for all $p \in \mathbb{P}$. Let $a \in \mathbb{Q}_{\neq 0}$. The following conditions are equivalent.

(a) *The equation*
$$\Delta_{g,h}(x) = ah(x) \tag{20}$$
has a solution $x \in \mathbb{Q}_{\neq 0}$.

(b) *The equation (20) has infinitely many solutions.*

(c) *There are $p_1, \ldots, p_k \in S$ such that $ap_1^{\alpha_{p_1}} \cdots p_k^{\alpha_{p_k}} \in \mathbb{Z}$.*

Proof. A straightforward generalization of the proof of Theorem 7. □

For $S = \{p\}$, we obtain a generalization of Theorem 6.

Corollary 1. *Let $\alpha \in \mathbb{Z}_+$, $p \in \mathbb{P}$, $a \in \mathbb{Q}$, and let g and h be functions on $\mathbb{Q}_{\neq 0}$ such that h is completely multiplicative and g is completely additive with*

$$g(q) = \frac{\delta_{pq}}{p^{\alpha}} \quad \textit{for all } q \in \mathbb{P}.$$

The equation
$$\Delta_{g,h}(x) = ah(x)$$
has a solution $x \in \mathbb{Q}_{\neq 0}$ if and only if $ap^{\alpha} \in \mathbb{Z}$. Then the set of its all solutions is

$$\left\{cp^{ap^{\alpha}} : c \in \mathbb{Q}_{\neq 0},\, \nu_p(c) = 0\right\}.$$

7. Discussion

We saw that a c-additive function on $\mathbb{Z}_+$ has a unique c-additive extension to $\mathbb{Z}_{\neq 0}$, and further to $\mathbb{Q}_{\neq 0}$. We also saw that a c-multiplicative function (and, consequently, a Leibniz-additive function, too) on $\mathbb{Z}_+$ has two c-multiplicative extensions to $\mathbb{Z}_{\neq 0}$, and each has a unique c-multiplicative extension to $\mathbb{Q}_{\neq 0}$.

We have previously [5, 10] considered an L-additive function $f = gh$ as a generalization of an arithmetic subderivative (in particular, as that of the arithmetic derivative). While studying generalized arithmetic differential equations, we found it reasonable to call f the gh-derivative and to write $f = \Delta_{g,h}$.

Solving an arithmetic differential equation – actually, already studying solvability of such an equation – is difficult in general. Two such equations relate to famous number-theoretic conjectures. Ufnarovski and Åhlander [12, Conjecture 5] conjectured that if $b \in \mathbb{Z}_+$, $b \geq 2$, then the equation $D(n) = 2b$ has a solution $n \in \mathbb{Z}_+$. This follows if the Goldbach conjecture is true. They also [12, Conjecture 9] conjectured that the equation $D^2(n) = 1$ has infinitely many solutions $n \in \mathbb{Z}_+$. This follows from the twin prime conjecture by [12, Theorem 10].

We studied the equation $\Delta_{g,h}(x) = ah(x)$. In the case of $h = N$, we solved it for $g(q) = \delta_{pq}/p$ (then the equation is $D_p(x) = ax$) in Theorem 6. For $g(p) = \chi_S(p)/p$ (then the equation is $D_S(x) = ax$), we found in Theorem 7 its solvability conditions but could not find the solutions. We also presented in Theorem 8 its slight generalization.

The equation $\Delta_{g,h}(x) = 1$ may look easy, but its solution is an open question even for $g(p) = 1/p$, $h = N$ (then the equation is $D(x) = 1$), see [12, Conjecture 12]. Instead, several equations of type $D_p(x) = \dots$ have been solved [4]. Some of these results can be extended to equations of type $D_S(x) = \dots$ (for example, Theorem 6 to Theorem 7).

The primes have useful properties in multiplication but not in addition. Perhaps, the most important such property is the fundamental theorem of arithmetic: every positive integer, greater than one, can be uniquely expressed as a product of prime factors (up to their ordering). However, primes have no special role in expressing a positive integer as a sum. (To be more precise, such a role is not currently known.)

Therefore, equations $D_S(x) = \dots$ are difficult. To illustrate why equations $\Delta_{g,h}(x) = \dots$ are even more difficult, let us look at the proof of Theorem 7. Then $h = N$ and g is as in (16). The problem is how to manage with

$$g(\sum_{i=1}^{k} \frac{\xi_i}{p_i})$$

without any knowledge about $g(x + y)$. Due to the fortunate interplay between g and h, we obtained (19).

In extending this theorem to $\Delta_{g,h}(x) = ax$, g and h have no connection, and it seems that we therefore can do nothing. Instead, in extending it to $\Delta_{g,h}(x) = ah(x)$, the interplay remains, and we obtain (under an assumption about g) Theorem 8.

So, there is much work to be done in this field. Another topic for further research is the study of gh-derivatives in abstract structures. As already noted in the introduction, c-additivity and c-multiplicativity can be defined in an integral domain, for example. Murashka, Goncharenko, and Goncharenko [11] have taken steps in this direction. One step has also been taken by Ufnarovski and Åhlander [12, Theorem 21]. We recall it (with different notation and slightly different formulation).

Let R be an integral domain with the corresponding additive group R^+ and multiplicative monoid R^*, and let $g : R^* \to R^+$ be a homomorphism. Then the function

$$f : R \to R : f(x) = xg(x),\ f(0) = 0,$$

satisfies the Leibniz rule. Conversely, if a function $f : R \to R$ satisfies the Leibniz rule, then the function

$$g : R^* \to R^+ : g(x) = \frac{f(x)}{x}$$

is a homomorphism. If R is a field, then g is a group homomorphism and

$$f(\frac{x}{y}) = \frac{yf(x) - xf(y)}{y^2}$$

for all $x, y \in R_{\neq 0} (= R \setminus \{0\})$.

The homomorphism g is just a c-additive function. Hence, this theorem is a special case of our results extended to R (and, according to our custom, with $f(0)$ undefined). These results are the following. Let R and g be as above, and let $h : R^* \to R^*$ be a homomorphism. Then the function

$$f = gh$$

is L-additive. Conversely, if a function $f : R_{\neq 0} \to R$ is L-additive, then the function

$$g : R^* \to R^+ : g(x) = \frac{f(x)}{h(x)}$$

is a homomorphism. If R is a field, then g is a group homomorphism and

$$f(\frac{x}{y}) = \frac{h(y)f(x) - h(x)f(y)}{h(y)^2}$$

for all $x, y \in R_{\neq 0}$.

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