Robust controllers for a heat equation using the Galerkin approximation

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***Abstract*— We consider the robust output tracking problem for an unstable one-dimensional heat equation. As the main contribution we propose a new way of designing finite-dimensional robust controllers based on Galerkin approximations of infinite-dimensional observer-based controllers. The results are illustrated with a concrete example where the finite-dimensional controllers are constructed using the Finite Element Method. The results are extendable for more general parabolic control systems.**

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I. Introduction

In this paper we consider robust output tracking for infinite-dimensional linear systems. Both the theoretical properties of the problem and the design of robust controllers has been studied extensively in the literature [1–9]. The fundamental *internal model principle* [6], [10], [11] provides a way of designing dynamic output feedback controllers that solve this so-called "robust output regulation problem". However, many controller design procedures presented in the recent articles [5], [9], [12] have the drawback that if the original system is unstable (and cannot be pre-stabilized with output or state feedback), then the controller requires a full order observer of the original system. Because of this, these controller designs lead to infinite-dimensional controllers for unstable systems.

The main purpose of this paper is to present new methods for the design of finite-dimensional robust controllers for a controlled heat equation with a finite-dimensional unstable part. The general approach is based on approximating the infinite-dimensional controller designed with existing methods with a suitable numerical approximation scheme. In the present paper we concentrate on studying a toy model — one-dimensional heat equation with a single unstable eigenvalue. However, the general approach approach is applicable for a more general class of parabolic systems, and these more general results are presented in a later paper.

Throughout the paper we consider the one-dimensional controlled heat equation on the domain $\Omega = (0, 1)$ with homogeneous Neumann boundary conditions [13, Examples 4.3.11, 5.2.8]

$$\frac{\partial x}{\partial t}(\xi,t) = \nu \frac{\partial^2 x}{\partial \xi^2}(\xi,t) + \frac{1}{b-a} 1_{[a,\,b]}(\xi) u(t), \tag{1a}$$

$$\frac{\partial x}{\partial \xi}(0,t) = 0 = \frac{\partial x}{\partial \xi}(1,t), \quad x(\xi,0) = x_0(\xi), \tag{1b}$$

$$y(t) = \frac{1}{d-c} \int_c^d x(\xi,t) d\xi. \tag{1c}$$

where $\nu > 0$, $0 \leq a < b \leq 1$, and $0 \leq c < d \leq 1$. In the robust output regulation problem our main goal is to design a controller in such a way that the output $y(t)$ of the system converges asymptotically to a given reference signal $y_{ref}(t)$

$$\|y(t) - y_{ref}(t)\| \to 0, \qquad \text{as} \quad t \to \infty$$

at an exponential rate. In addition, the control law is required to function even under small variations of the parameters of the system (1). We assume the reference signal is of the form

$$y_{ref}(t) = \sum_{k=1}^{q} y_{ref}^k e^{i\omega_k t} \tag{2}$$

for known distinct frequencies $(\omega_k)_{k=1}^q \subset \mathbb{R}$ and unknown amplitudes $(y_k)_{k=1}^q \subset \mathbb{C}$.

As our main result, we design a finite-dimensional dynamic error feedback controller that solves the robust output regulation problem. The controller is based on designing an infinite-dimensional internal model-based controller for the system and subsequently approximating the observer part using the Galerkin method. In particular, our approach uses the theory presented in [14] to study the preservation of the closed-loop stability under the approximations of the controller. Once the closed-loop system is stabilized, the internal model property of the controller will guarantee the robust tracking of the reference signal.

An alternative to the controller design approach used in this paper would be to first stabilize the system (1) using a finite-dimensional feedback controller (for example using the method presented in [13, Ch. 9]), and subsequently to design a finite-dimensional low-gain controller for the stabilized system. Compared to this procedure, the approximation approach used in this paper has the advantage that the controller does not require finding a suitable low-gain parameter for the controller. In addition, the reference [7] presents another alternative approach to design of finite-dimensional controllers using "dual observers".

The paper is organized as follows. In Section III we design an infinite-dimensional dynamic stabilizing controller for the heat equation. In Section IV we approximate the full controller using the Galerkin method and verify that for a sufficiently high order approximations the finite-dimensional controllers achieve robust output regulation. In Section V we demonstrate the results using a numerical simulation based on Finite Element approximations of the system (1).

Notation: $\mathbb{C}_0^+$ is the right-half plan containing all complex number s with Re $s > 0$. Through this paper we denote

$$\mathcal{H}_\infty = \{G : \mathbb{C}_0^+ \to \mathbb{C} \mid G \text{ analytic and } \sup_{s \in \mathbb{C}_0^+} |G(s)| < \infty\}.$$

Matrices whose entries are in $\mathcal{H}_\infty$ are indicated by $M(\mathcal{H}_\infty)$. Let $R(\mathcal{H}_\infty)$ indicate the transfer functions with both right and left coprime factorizations over $M(\mathcal{H}_\infty)$.

II. The Robust Output Regulation Problem

The system (1) can be rewritten in the abstract form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \in X\\ y(t) &= Cx(t)\end{aligned}$$

on the state space $X = L_2(0,1)$ with operators

$$\begin{aligned}A &= \frac{d^2}{d\xi^2},\\ D(A) &= \Big\{ f \in L^2(0,1) \mid f,\, \frac{df}{d\xi} \text{ are absolutely}\\ &\qquad \text{continuous },\, \frac{df^2}{d\xi^2} \in L^2(0,1),\\ &\qquad \text{and } \frac{df}{d\xi}(0) = 0 = \frac{df}{d\xi}(1) \Big\},\\ Bu &= gu \;\text{ with }\; g(\xi) = \frac{1}{b-a} 1_{[a,\,b]}(\xi), \;\text{ and}\\ Cx &= \langle x,\, h\rangle \;\text{ with }\; h(\xi) = \frac{1}{d-c} 1_{[c,\,d]}(\xi).\end{aligned}$$

The system operator A has the eigenvalues $\{-n^2\pi^2\}_{n=0}^\infty$ and the corresponding eigenvectors $\phi_0 \equiv 1$ and $\phi_n = \sqrt{2}\cos(n\pi\cdot)$ for $n \geq 1$. Here $B \in \mathcal{L}(\mathbb{C}, X)$ and $C \in \mathcal{L}(X, \mathbb{C})$.

Our goal is to design a dynamic error feedback controller of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t) \tag{3a}$$
$$u(t) = Kz(t) \tag{3b}$$

where $e(t) = y(t) - y_{ref}(t)$ is the *regulation error*, $\mathcal{G}_1 : D(\mathcal{G}_1) \subset Z \to Z$ generates a strongly continuous semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(\mathbb{C}, Z)$, and $K \in \mathcal{L}(Z, \mathbb{C})$. Letting $x_e(t) = (x(t), z(t))^\top$, the system and the controller can be written together as a *closed-loop system* on $X_e = X \times Z$ (see [5], [6] for details)

$$\begin{aligned}\dot{x}_e(t) &= A_e x_e(t) + B_e y_{ref}(t), \qquad x_e(0) = x_{e0} = (x_0, z_0)^\top\\ e(t) &= C_e x_e(t) + D_e y_{ref}(t)\end{aligned}$$

where

$$\begin{aligned}A_e &= \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 \end{bmatrix}, \qquad & B_e &= \begin{bmatrix} 0 \\ -\mathcal{G}_2 \end{bmatrix},\\ C_e &= \begin{bmatrix} C & 0 \end{bmatrix}, & D_e &= -I.\end{aligned}$$

The Robust Output Regulation Problem. *Choose $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that the following are satisfied:*

(a) *The semigroup $T_e(t)$ generated by A_e is exponentially stable.*

(b) *There exists $M_e, \omega_e > 0$ such that for all initial states $x_{e0} \in X_e$ and for all $y_{ref}(t)$ of the form* (2) *the regulation error satisfies*

$$\|y(t) - y_{ref}(t)\| \leq M_e e^{-\omega_e t}(\|x_{e0}\| + \|(y_{ref}^k)_k\|). \tag{4}$$

(c) *If (A, B, C) are perturbed to $(\tilde{A}, \tilde{B}, \tilde{C})$ in such a way that the perturbed closed-loop system is exponentially stable, then for all $x_{e0} \in X_e$ and for all $(y_{ref}^k)_k \in \mathbb{C}^q$ the regulation error satisfies* (4) *for some modified constants $\tilde{M}_e, \tilde{\omega}_e > 0$.*

The *internal model principle* [6, Thm. 6.9] implies that in order to achieve robust output tracking of the reference signal $y_{ref}(t)$, it is both necessary and sufficient that the following are satisfied.

- The controller (3) incorporates *an internal model* of the frequencies $(i\omega_k)_{k=1}^q$ of the signal $y_{ref}(t)$.
- The semigroup $T_e(t)$ generated by A_e is exponentially stable.

Since the plant is a single-input single-output system and $y_{ref}(t)$ is of the form (2), the internal model is defined as the property that [6, Sec. 6]

$$i\omega_k \in \sigma_p(\mathcal{G}_1) \qquad \text{for all} \quad k \in \{1, \ldots, q\}.$$

The system (1) is in general not stabilizable with static output feeback, and therefore the controller designs proposed in [5], [9] result in infinite-dimensional controllers. However, as discussed in the introduction, the fact that the plant has a finite-dimensional unstable part makes it reasonable to expect that the robust output regulation problem is solvable with a finite-dimensional controller. In this paper we show that the robust output regulation problem for the system (1) can be solved by constructing an infinite-dimensional controller using the procedures presented in [5], [9], and subsequently approximating the infinite-dimensional controller using the Galerkin method. The result is a finite-dimensional internal-model based controller $(\mathcal{G}_1^n, \mathcal{G}_2^n, K^n)$ that solves the robust output regulation problem for a sufficiently large $n \in \mathbb{N}$. In the construction we in particular use the results on preservation of stability under approximations of the controller presented by Morris [14], and Banks and Kunisch [15].

The main contribution of the paper is the construction of the finite-dimensional robust controller $(\mathcal{G}_1^n,\, \mathcal{G}_2^n,\, K^n)$ based on a Galerkin approximation. The construction will be completed in the following two sections. In particular, the controller incorporates a suitable internal model and is guaranteed to stabilize the closed-loop system provided that the order of the approximation is sufficiently high.

III. Designing an Infinite-Dimensional Controller for System (1)

In this section we design an infinite-dimensional robust controller for the system (1) based on the method presented in [9, Section 5]. This requires the following two properties.

- Assumption I1: The pair $(A,\, B)$ is exponentially stabilizable and the pair $(A,\, C)$ is exponentially detectable.
- Assumption I2: There exists $L_1 \in \mathcal{L}(\mathbb{C}, X)$ such that $A + L_1 C$ is exponentially stable and for every $k \in \{1, \ldots, q\}$ we have $P_L(iw_k) \neq 0$ where $P_L(\lambda) = CR(\lambda, A + L_1 C)B$.

The stabilizability and detectability of (A, B, C) of the system (1) hold for our system by [13, Ex. 5.2.8]. Moreover, the condition $P_L(i\omega_k) \neq 0$ for all $k \in \{1, \ldots, q\}$ can often be checked directly, or with using the transfer function $P(\lambda)$ of (A, B, C) and the relation $P_L(\lambda) = (I - CR(\lambda, A)L_1)^{-1}P(\lambda)$ for $\lambda \in \rho(A) \cap \rho(A + L_1C)$. Moreover, if $P_L(i\omega_k) \neq 0$ for some L_1 for which $A + L_1C$ is stable, then the same holds for all such L_1.

The following construction was presented in [9, Section 5], and it guarantees that the controller has an internal model and that the closed-loop system is exponentially stable.

- Step 1: We choose the state space of controller as $Z = Z_0 \times X$, and the general control structure of the operators $(\mathcal{G}_1, \mathcal{G}_2, K)$ as

$$\mathcal{G}_1 = \begin{bmatrix} G_1 & G_2C \\ 0 & A + BK_2 + LC \end{bmatrix},$$
$$\mathcal{G}_2 = \begin{bmatrix} G_2 \\ L \end{bmatrix}, \qquad K = \begin{bmatrix} K_1, & -K_2 \end{bmatrix}.$$

We choose $Z_0 = \mathbb{C}^q$ and $G_1 = \operatorname{diag}(i\omega_1, \ldots, i\omega_q) \in \mathbb{C}^{q\times q}$. Since $\sigma(G_1) = \sigma_p(G_1) = \{i\omega_k\}_{k=1}^q$, the triangular structure implies that $i\omega_k \in \sigma_p(\mathcal{G}_1)$ for all $k \in \{1, \ldots, q\}$, and thus the controller incorporates an internal model, as required. We choose the operator $K_1 = [K_1^1, \ldots, K_1^q] \in \mathcal{L}(Y^q, U)$ so that $K_1^k \neq 0$ for all $k \in \{1, \ldots, q\}$.
- Step 2: We choose $L_1 \in \mathcal{L}(\mathbb{C}, X)$ in such a way that $A + L_1C$ generates an exponentially stable semigroup. Then $P_L(i\omega_k) \neq 0$ for all k. We also choose $K_2 \in \mathcal{L}(X, \mathbb{C})$ in such a way that $A + BK_2$ generates an exponentially stable semigroup.
- Step 3: We define $H = [H_1, H_2, \ldots, H_q] \in \mathcal{L}(Z_0, X)$ where

$$H_k = R(iw_k, A + L_1C)BK_1^k.$$

Then, we define $C_1 = CH \in \mathcal{L}(Z_0, \mathbb{C})$.
- Step 4: We choose $G_2 \in \mathcal{L}(\mathbb{C}, Z_0)$ in such a way that $G_1 + G_2C_1 \in \mathbb{C}^{q\times q}$ is Hurwitz. Finally, we define $L = L_1 + HG_2$.

For our system, the following suitable choices of operators L_1 and K_2 are given in [13, Example 5.2.8].

Lemma 3.1: If we choose K_2 and L_1 so that $K_2z = -3\langle z, \phi_0\rangle = -3\langle z, 1\rangle$ and $L_1y = -3y\phi_0 = -3y$, then $A + BK_2$ and $A + L_1C$ generate exponentially stable semigroups.

IV. Approximating the Controller with the Galerkin Method

We use the results of Morris [14] based on convergence of infinite-dimensional systems to approximate the infinite-dimensional controller in the previous section with a finite-dimensional one. We begin by recalling general assumptions A1–A5 that guarantee the convergence of the approximate controllers to the original on in the graph topology. Subsequently, we collect more concrete sufficient conditions B1–B2 that are more easily checkable for parabolic systems.

A. The general assumptions

Consider a general single-input single-output system

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t),$$
$$y(t) = \mathcal{C}x(t)$$

on a Hilbert space $\mathcal{X}$. Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ be a sequence of finite-dimensional subspaces of $\mathcal{X}$ and define $P_n : \mathcal{X} \to \mathcal{X}_n$ to be the orthogonal projections onto $\mathcal{X}_n$. For each n, we define the *approximating system* $(\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n)$ where $\mathcal{B}_n := P_n\mathcal{B}$, and $\mathcal{C}_n$ is the restriction of $\mathcal{C}$ onto $\mathcal{X}_n$. The following assumptions were introduced in [14].

- Assumption A1: For all $x \in \mathcal{X}$, $\lim_{n\to\infty} \|P_nx - x\| = 0$.
- Assumption A2: For some $s \in \rho(\mathcal{A})$ and for all $x \in \mathcal{X}$,

$$\lim_{n\to\infty} \|P_nR(s, \mathcal{A})x - R(s, \mathcal{A}_n)P_nx\| = 0.$$

- Assumption A3: The semigroup $T_n(t)$ generated by $\mathcal{A}_n$ are uniformly bounded. That is, there exists $M, \omega \in \mathbb{R}$ and $N \in \mathbb{N}$ such that

$$\|T_n(t)\| \leq Me^{\omega t}, \quad \text{for all} \quad n \geq N.$$

- Assumption A4: If the original system is stabilizable, then the approximations are uniformly stabilizable. That is, there exists a sequence of operators K_n with $K_n \in \mathcal{L}(\mathcal{X}_n, \mathbb{C})$ and some $K \in \mathcal{L}(\mathcal{X}, \mathbb{C})$ such that for all $x \in \mathcal{X}$, $\lim_{n\to\infty} K_nP_nx = Kx$. Furthermore, for sufficiently large N the semigroups generated by $\mathcal{A}_n - \mathcal{B}_nK_n$ are uniformly bounded by $Me^{-\omega t}$ for some $M > 0$, $\omega > 0$, and all $n > N$.
- Assumption A5: If the original system is detectable, then the approximations are uniformly detectable. That is, there exists a sequence of operators L_n with $L_n \in \mathcal{L}(\mathbb{C}, \mathcal{X}_n)$ and some $L \in \mathcal{L}(\mathbb{C}, \mathcal{X})$ such that $\lim_{n\to\infty}\|L_n - P_nL\| = 0$. Furthermore, for sufficiently large N the semigroups generated by $\mathcal{A}_n - L_n\mathcal{C}_n$ are uniformly bounded by $Me^{-\omega t}$ for some $M > 0$, $\omega > 0$, and all $n > N$.

The following theorems from [14] play an important role later in showing that the closed-loop system is stabilized by an approximated controller.

Theorem 4.1 ([14, Thm. 4.2–4.3]): Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a stabilizable/detectable control system, and assume $(\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n)$ is a sequence of approximations satisfying assumptions A1–A3, and either A4 or A5. Then the approximating systems with transfer functions $G_n(s) = \mathcal{C}_nR(s, \mathcal{A}_n)\mathcal{B}_n$ converge to the original system in the graph topology on $M(\mathcal{H}_\infty)$.

B. Assumptions for parabolic systems

Banks and Kunisch [15] showed that assumptions A1–A4 hold for general Galerkin approximations of symmetric parabolic equations. These sufficient conditions were also generalized to a larger class of systems by Morris in [14, Section 5].

Let V be a Hilbert space, densely and continuously embedded in $\mathcal{X}$. We denote the the inner products on $\mathcal{X}$ and V with

$\langle\cdot,\,\cdot\rangle$ and $\langle\cdot,\,\cdot\rangle_V$, respectively. Analogously, denote by $\|\cdot\|$ and $\|\cdot\|_V$ the norms on $\mathcal{X}$ and V. Let $\mathcal{A}: D(\mathcal{A})\subset\mathcal{X}\to\mathcal{X}$ be a closed operator such that

$$\langle -\mathcal{A}\phi,\,\psi\rangle = a(\phi,\psi),\quad \forall\psi\in V$$

where $D(\mathcal{A})=\{\phi\in V\mid a(\phi,\cdot)\in\mathcal{X}\}$.

- Assumption B1: $a: V\times V\to\mathbb{C}$ is a continuous sesquilinear form, i.e. there exists $M>0$ such that

$$|a(\phi,\psi)|\leq M\|\phi\|_V\|\psi\|_V \tag{5}$$

for all $\phi,\psi\in V$.

- Assumption B2: $a(\cdot,\,\cdot)$ satisfies Garding's inequality, i.e. there exists $\gamma\geq 0$ and $\delta>0$ such that for all $\phi\in V$

$$\mathrm{Re}\; a(\phi,\,\phi)+\gamma\langle\phi,\,\phi\rangle\geq\delta\|\phi\|_V^2. \tag{6}$$

Let $\mathcal{X}_n\subset V$ be a sequence of finite-dimensional subspaces. When the operator $\mathcal{A}$ satisfies the two inequalities (5) and (6), it is only required that the subspaces $\mathcal{X}_n$ satisfy a *V-approximation property*. That is, for each $x\in V$, there exists a sequence $(x_n)_n$ with $x_n\in\mathcal{X}_n$ such that

$$\lim_{n\to\infty}\|x_n-x\|_V=0. \tag{7}$$

This condition is fulfilled, e.g., for several Finite Element approximation schemes (see [16, Sec. 3.2, Ch. 5] and [15]).

The finite-dimensional approximations $\mathcal{A}_n$ of the operator $\mathcal{A}$ are defined via

$$\langle -\mathcal{A}_n x_n,\,v_n\rangle = a(x_n,\,v_n),\quad\forall x_n,v_n\in\mathcal{X}_n. \tag{8}$$

The V-approximation property (7) and two assumptions of operator $\mathcal{A}$ imply all assumptions A1–A5. The proof can be found in [14, Section 5.2].

C. Approximation of controller

We will now approximate the infinite-dimensional controller by a finite-dimensional one in such a way that for a sufficiently large approximation order also the approximate controller stabilizes the closed-loop system. We consider an approximation of dynamic feedback controller

$$\dot{z}^n(t)=\mathcal{G}_1^n z^n(t)+\mathcal{G}_2^n e(t) \tag{9a}$$

$$u^n(t)=K^n z(t). \tag{9b}$$

For a fixed $n\in\mathbb{N}$ we choose the finite-dimensional state space as $Z_n=Z_0\times X_n$, and choose the operators as

$$\mathcal{G}_1^n=\begin{bmatrix}G_1 & G_2C^n\\ 0 & A^n+B^nK_2^n+L^nC^n\end{bmatrix},$$

$$\mathcal{G}_2^n=\begin{bmatrix}G_2\\ L^n\end{bmatrix},\qquad K^n=\begin{bmatrix}K_1, & -K_2^n\end{bmatrix},$$

where $(A^n,\,B^n,\,C^n)$ is an approximation of $(A,\,B,\,C)$ in X_n. The rest of the parameters are chosen using the following modified version of the algorithm in Section III.

- Step 1: G_1 and K_1 are chosen as in Section III. For these choices the controller incorporates an internal model.
- Step 2: We choose $L_1\in\mathcal{L}(\mathbb{C},X)$ and $K_2\in\mathcal{L}(X,\mathbb{C})$ in such a way that $A+L_1C$ and $A+BK_2$ generate exponentially stable semigroups, and define $L_1^n := P_nL_1\in\mathcal{L}(\mathbb{C},X_n)$ and $K_2^n:=K_2P_n\in\mathcal{L}(X_n,\mathbb{C})$.
- Step 3: The operator $H^n=(H_1^n,\,H_2^n,\,\ldots,\,H_q^n)\in\mathcal{L}(Z_0,X_n)$ is computed based on $(A^n,\,B^n,\,C^n)$ instead of $(A,\,B,\,C)$, and we define $C_1^n=C^nH_1^n\in\mathcal{L}(Z_0,\mathbb{C})$.
- Step 4: We choose $G_2\in\mathcal{L}(\mathbb{C},Z_0)$ such that $G_1+G_2C_1^n\in\mathbb{C}^{q\times q}$ is Hurwitz. Finally, we define $L^n=L_1^n+H^nG_2$.

Lemma 4.2: Suppose (A,B) is a stabilizable pair and A satisfies assumptions B1 and B2. Let $K_2\in\mathcal{L}(X,\mathbb{C})$ be such that $A+BK_2$ generates a stable semigroup. If V-approximation property (7) is true, there exists N_1 such that for all $n>N_1$, the semigroup $S_n(t)$ generated by $A_n+B_nK_2P_n$ are uniformly stable.

Lemma 4.3: Suppose (A,C) is a detectable pair and A satisfies assumptions B1 and B2. Let $L_1\in\mathcal{L}(\mathbb{C},X)$ be such that $A+L_1C$ generates a stable semigroup. If V-approximation property (7) is true, there exists N_2 such that for all $n>N_2$, the semigroup $S_n(t)$ generated by $A_n+P_nL_1C_n$ are uniformly stable.

In the following lemma we consider extended output operators

$$\mathcal{K}=\begin{bmatrix}K_1 & -K_2\\ 0 & C\end{bmatrix}\quad\text{and}\quad\mathcal{K}^n=\begin{bmatrix}K_1 & -K_2^n\\ 0 & C^n\end{bmatrix} \tag{10}$$

for the controller and its approximation, respectively. With these choices the extended systems $(\mathcal{G}_1,\,\mathcal{G}_2,\,\mathcal{K})$ and $(\mathcal{G}_1^n,\,\mathcal{G}_2^n,\,\mathcal{K}^n)$ are exponentially stabilizable and detectable.

Lemma 4.4: The approximating controllers with transfer functions $P_c^n(s)=\mathcal{K}^nR(s,\,\mathcal{G}_1^n)\mathcal{G}_2^n$ converge to the original system with $P_c(s)=\mathcal{K}R(s,\,\mathcal{G}_1)\mathcal{G}_2$ in the graph topology on $M(\mathcal{H}_\infty)$.

Proof: For brevity, we denote $A_1^n=A^n+B^nK_2^n+L^nC^n$. For $s\in\sigma(G_1)\cap\sigma(A_1^n)$ and for all $n\in\mathbb{N}$, we compute the transfer function $\mathcal{P}_c^n$ of the extended system $(\mathcal{G}_1^n,\,\mathcal{G}_2^n,\,\mathcal{K}^n)$ as follows

$$\begin{aligned}\mathcal{P}_c^n(s)&=\mathcal{K}^nR(s,\,\mathcal{G}_1^n)\mathcal{G}_2^n\\ &=\begin{bmatrix}K_1 & -K_2^n\\ 0 & C^n\end{bmatrix}\begin{bmatrix}sI-G_1 & -G_2C^n\\ 0 & sI-A_1^n\end{bmatrix}^{-1}\begin{bmatrix}G_2\\ L^n\end{bmatrix}\\ &=\begin{bmatrix}\mathcal{P}_1^n(s)\\ \mathcal{P}_2^n(s)\end{bmatrix}\end{aligned}$$

where

$$\begin{aligned}\mathcal{P}_1^n(s)&=K_1R(s,G_1)G_2\left(I+\mathcal{P}_2^n(s)\right)-K_2^nR(s,A_1^n)L^n,\\ \mathcal{P}_2^n(s)&=C^nR(s,A_1^n)L^n.\end{aligned}$$

By Theorem 4.1, Lemmas 4.2 and 4.3 we have that $C^nR(s,A_1^n)L^n$ converges to $CR(s,A+BK_2+LC)L$ and $K_2^nR(s,A_1^n)L^n$ also converges to $K_2R(s,A+BK_2+LC)L$. Because of this, also $\mathcal{P}_c^n(s)=\mathcal{K}^nR(s,\,\mathcal{G}_1^n)\mathcal{G}_2^n$ converges to $\mathcal{P}_c(s)=\mathcal{K}R(s,\,\mathcal{G}_1)\mathcal{G}_2$ in the graph topology on $M(\mathcal{H}_\infty)$.

■

We denote by $P(\lambda)=CR(\lambda,A)B$ the transfer function of the original system (1). We then recall a result in [14],

[17] concerning the convergence of closed-loop $\Delta(P_c^n, P)$ to $\Delta(P_c, P)$.

Theorem 4.5: Let $\{P_c^n\}$ be a sequence in $R(H_\infty)$, and $P_c,\, P \in R(H_\infty)$. Then $\Delta(P_c^n,\, P) \to \Delta(P_c,\, P)$ if and only if $P_c^n \to P_c$ in the graph topology.

The above convergence results allow us to prove that the approximated controller solves the robust output regulation problem provided that the accuracy of the approximation is sufficiently high. It should be noted that we are interested in closed-loop stability in the sense of the stability of the semigroup generated by A_e, but for achieving these we can use the analysis of the closed-loop transfer function and the well-known connection between internal and external stability, see e.g. [14, Theorem 2.1].

Theorem 4.6: There exists $N \in \mathbb{N}$ such that the finite-dimensional controller $(\mathcal{G}_1^n,\, \mathcal{G}_2^n,\, K^n)$ solves the Robust Output Regulation Problem for all $n > N$.

Proof: The block-triangular structure and the property $\sigma(G_1) = \sigma_p(G_1) = \{i\omega_k\}_{k=1}^q$ imply that $i\omega_k \in \sigma_p(\mathcal{G}_1^n)$ for all $k \in \{1, \ldots, q\}$, and thus the controller incorporates an internal model for every $n \in \mathbb{N}$. Thus it remains to show that the closed-loop system A_e^n is exponentially stable for all sufficiently large n.

To show closed-loop stability, we first consider a composite system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ with operators

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & \mathcal{G}_1 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B & 0 \\ 0 & \mathcal{G}_2 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C & 0 \\ 0 & K \\ 0 & [0, C] \end{bmatrix}$$

and we similarly define $(\mathcal{A}^n, \mathcal{B}^n, \mathcal{C}^n)$ using $(\mathcal{G}_1^n, \mathcal{G}_2^n, K^n)$. Since (A, B, C), $(\mathcal{G}_1, \mathcal{G}_2, K)$, and $(\mathcal{G}_1^n, \mathcal{G}_2^n, \mathcal{K}^n)$ are exponentially stabilizable and detectable, the same properties hold for the systems $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $(\mathcal{A}^n, \mathcal{B}^n, \mathcal{C}^n)$. In addition, by Theorem 4.1 and Lemma 4.4 we have that transfer function $\mathcal{P}^n(\lambda)$ of $(\mathcal{A}^n, \mathcal{B}^n, \mathcal{C}^n)$ converges to the transfer function $\mathcal{P}(\lambda)$ of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in the graph topology.

If we define $Q = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]$, then the semigroup associated to the closed-loop system $(\mathcal{A} + \mathcal{B}Q\mathcal{C}, \mathcal{B}, \mathcal{C})$ coincides with A_e, and therefore $(\mathcal{A} + \mathcal{B}Q\mathcal{C}, \mathcal{B}, \mathcal{C})$ is input-output stable. Moreover, the closed-loop systems $(\mathcal{A}^n + \mathcal{B}^n Q\mathcal{C}^n, \mathcal{B}^n, \mathcal{C}^n)$ converge to $(\mathcal{A} + \mathcal{B}Q\mathcal{C}, \mathcal{B}, \mathcal{C})$ in the graph topology (see [18, Section 7.2]). Because of this, for all large enough $n \in \mathbb{N}$ the systems $(\mathcal{A}^n + \mathcal{B}^n Q\mathcal{C}^n, \mathcal{B}^n, \mathcal{C}^n)$ are input-output stable (see [14], [19]), and since there systems are exponentially stablizable and detectable, also the semigroups generated by $\mathcal{A}^n + \mathcal{B}^n Q\mathcal{C}^n$ are exponentially stable. But since $\mathcal{A}^n + \mathcal{B}^n Q\mathcal{C}^n = A_e^n$, the proof is complete.

■

V. Discretization and a Numerical Example

Both the systems and the dynamic controllers must be discretized in simulation. To understand the concept that the approximating controllers also stabilizes the original system, we use two distinct discretizations of the heat equation (1). By using Finite Element Method, we firstly discretized the system with a finer mesh with N hat functions. The approximation of operators $(A,\, B,\, C)$ are $(A^N,\, B^N,\, C^N)$. We then define a coarse mesh with a considerably smaller number of hat functions $n \ll N$. In this mesh, the operators $(A,\, B,\, C)$ are approximated as $(A^n\, B^n,\, C^n)$. The controller (9) is computed based on operators $(A^n,\, B^n,\, C^n)$.

A. Discretization of operators $(A,\, B,\, C)$

To use a finite-element-based approach, we firstly introduce a uniform space mesh

$$\Omega_D^N = \left(\frac{1}{N},\, \frac{2}{N},\, \ldots,\, \frac{N-1}{N} \right)$$

containing interior points of Ω. We define the space step $\Delta\xi = \frac{1}{N}$ with $2 \le N \in \mathbb{N}$. We use the classical hat functions as basis functions $\phi_i(\xi) \in V$ with $i \in \{1, 2, \ldots, N-1\}$ defined for all $\xi \in \Omega$ as follows

$$\phi_i(\xi) = \begin{cases} 1 - i + \frac{\xi}{\Delta\xi} & \text{if } \xi \in [(i-1)\Delta\xi,\, i\Delta\xi], \\ 1 + i - \frac{\xi}{\Delta\xi} & \text{if } \xi \in [i\Delta\xi,\, (i+1)\Delta\xi], \\ 0 & \text{if } \xi \notin [(i-1)\Delta\xi,\, (i+1)\Delta\xi]. \end{cases}$$

A function $f \in V$ can be approximated by the values taking on Ω_D. Particularly, we approximate f by the function $\tilde{f}$ defined as

$$\tilde{f} \coloneqq \sum_{i=1}^{N-1} f(i\Delta\xi)\phi_i.$$

We define the evaluation vector at each points of the mesh Ω_D

$$\bar{f} = \begin{bmatrix} f(1\Delta\xi) & f(2\Delta\xi) & \ldots & f((N-1)\Delta\xi) \end{bmatrix}^\top$$

where $\mathbf{A}^\top$ stands for the transpose matrix of $\mathbf{A}$.

Two important matrices are so-called mass matrix $\mathbf{M} \coloneqq [\langle \phi_j,\, \phi_i \rangle]$ and stiffness matrix $\mathbf{S} \coloneqq [\langle \partial_x \phi_j,\, \partial_x \phi_i \rangle]$. We can compute both matrices explicitly as tridiagonal matrices

$$\mathbf{M}^N = \frac{\Delta\xi}{6} \begin{bmatrix} 4 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 4 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 4 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 & 4 & 1 \\ 0 & \ldots & 0 & 0 & 1 & 4 \end{bmatrix},$$

and

$$\mathbf{S}^N = \frac{1}{\Delta\xi} \begin{bmatrix} 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & -1 & 2 & -1 \\ 0 & \ldots & 0 & 0 & -1 & 2 \end{bmatrix},$$

Then we get the approximating operators $(A^N,\, B^N,\, C^N)$ as follows: $A^N = -\nu(\mathbf{M}^N)^{-1}\mathbf{S}^N$, $B^N = \bar{g}^N$, and $C^N = \Delta\xi(\bar{h}^N)^\top$ where $\bar{g}^N$ and $\bar{h}^N$ is the evaluation vectors of functions $g(\xi) = \frac{1}{b-a}1_{[a,\, b]}(\xi)$ and $h(\xi) = \frac{1}{d-c}1_{[c,\, d]}(\xi)$ respectively.

B. Discretization of operators ($\mathcal{G}_1$, $\mathcal{G}_2$, K) in controller

To design the feedback controller, firstly, we define another uniform coarser mesh with n interior points

$$\Omega_D^n = \left(\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right).$$

We then discretize operators (A, B, C) in this mesh as (A^n, B^n, C^n). Under the choice in Lemma 3.1 we can analogously approximate two operators K_2 and L_1. Following Section IV-C, we can construct a discretization version of operators ($\mathcal{G}_1$, $\mathcal{G}_2$, K) as ($\mathcal{G}_1^n$, $\mathcal{G}_2^n$, K^n).

C. Discretization in time

Finally, when we get the discretization of all operators, we will solve the coupled system of extended state $x_e = (x^N, z^n)^\top$

$$\dot{x}_e = \begin{bmatrix} \dot{x}^N \\ \dot{z}^n \end{bmatrix} = \begin{bmatrix} A^N & B^N K^n \\ \mathcal{G}_2^n C^N & \mathcal{G}_1^n \end{bmatrix} \begin{bmatrix} x^N \\ z^n \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathcal{G}_2^n \end{bmatrix} y_{ref}$$

using the function `ode23` in MATLAB.

D. A numerical example

We consider a particular example of the system (1) with $\nu = 1,\ a = \frac{1}{4},\ b = \frac{1}{2},\ c = \frac{1}{2},\ d = \frac{3}{4}$, and $x_0(\xi) = \cos(5\xi)$. To test the simulation, we choose a given reference signal $y_{ref}(t) = 3\cos t + \sin(2t) - 2\cos(3t)$, and the time interval $[0,\ 10]$.

We use two meshes: a fine mesh to discterize system with $N = 1000$ and a coarse one to design the dynamic feedback controller. We choose a sequence of $n \in \{5, 20, 100, 1000\}$. The controllers based on all coarse meshes stabilizes the output regulation problem even if in some cases we use an extremely small n.

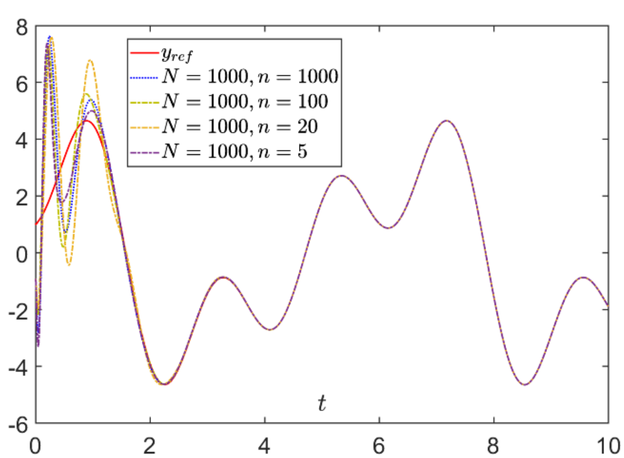

Fig. 1. Output under the finite-dimensional controllers.

VI. Conclusion

In this paper we have designed a finite-dimensional controller for robust output tracking of an unstable one-dimensional heat equation with bounded input and output operators. The controller design is based on constructing an infinite-dimensional observer-based controller and subsequently replacing it with a finite-dimensional Galerkin approximation. The approximation of controller here can also be completed for the type of controllers considered in [5]. The general approach is extendable for parabolic systems with a finite-dimensional unstable parts.

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