A Partial Internal Model for Approximate Robust Output Regulation of Boundary Control Systems*

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Abstract— Introduced for finite-dimensional systems by Francis and Wonham in the mid 70's, the *internal model principle* states that a stabilizing controller achieves asymptotic output tracking and disturbance rejection robustly if and only if it contains a p-copy of the exosystem frequencies, where p is the dimension of the output space of the plant. Later, the internal model principle has been extended, e.g., to boundary control systems on multidimensional spatial domains, and in this setting it follows from the principle that every robust output regulator is necessarily infinite-dimensional. However, it was recently established by the authors that robust *approximate* output tracking can be achieved with a finite-dimensional controller, and in the present paper, we formulate an internal model for this purpose. The efficiency of the method is numerically demonstrated using the heat equation on the unit square in $\mathbb{R}^2$ with boundary control and boundary observation.

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I. INTRODUCTION

The objective in *robust output regulation* is to design a control law for a given system in such a way that the output $y(t)$ of the system would converge asymptotically to a given reference signal $y_{ref}(t)$. Furthermore, a robust controller should also reject any external disturbance signal $d(t)$ and allow perturbations and uncertainties in the parameters of the system.

The *internal model principle* is the key to understanding how a controller can be robust. The principle indicates that a controller can solve the robust output regulation problem if (and only if) it contains a sufficient *internal model* of the dynamics of the exosystem that generates the disturbance and reference signals. The internal model principle was first introduced in the context of finite-dimensional systems by Francis and Wonham [3], [4]. Since then, the principle has been extended to infinite-dimensional systems, e.g., in [5], [8], [10], [13]. Most recently, the principle has been generalized to regular linear systems in [11] and to boundary control systems in [6], [7].

While the internal model principle does give the characterization of all robust regulating controllers for a large class of systems, in some cases the limitations set by practice may hinder the construction of such controllers. For example, if the output space of the system is infinite-dimensional, any robust controller for such a system is necessarily infinite-dimensional by the internal model principle. This has given rise to the novel concept of *approximate robust output regulation*, where, by relaxing the asymptotic accuracy of the output regulation, it is possible to construct finite-dimensional robust controllers even for systems with infinite-dimensional output spaces. Such approximate robust controllers have been constructed in recent papers [6], [9].

As the main contribution of this paper, by mimicking the internal model principle, we present sufficient conditions for a controller to solve the approximate robust output regulation problem. The conditions are stated as a modification of the $\mathcal{G}$-conditions [5, Def. 10] in Definition 3. In our main result (Theorem 4), we will derive an upper bound for the regulation error $e(t) := y(t) - y_{ref}(t)$, and show that it can be made arbitrarily small (Remark 5). Finally, we will present a controller structure for approximate robust output regulation by utilizing our main result. The results are presented for boundary control systems with infinite-dimensional output spaces, for which the proposed methodology is especially suited. For simplicity, we will make the additional assumption that the plant is initially exponentially stable.

The structure of the paper is as follows. In Section II, we present the plant, exosystem and controller. In Section III, we present the robust output regulation problem and the internal model principle as a background to their approximate counterparts which will be presented in Sections IV and IV-A. In Section IV-B, we will present a controller structure for approximate robust regulation, and in Section V, we will construct such a controller for the heat equation on a rectangular domain. Finally, in Section VI, the paper is concluded.

Here $\mathcal{L}(X, Y)$ denotes the set of bounded linear operators from a normed space X to a normed space Y. The domain, range, kernel, spectrum and resolvent of a linear operator A are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\sigma(A)$ and $\rho(A)$, respectively. The resolvent operator is defined for all $s \in \rho(A)$ by $R(s, A) := (s - A)^{-1}$.

II. THE PLANT, EXOSYSTEM AND CONTROLLER

The plant is given on a domain $\Omega \subset \mathbb{R}^n$ by the following equations:

$$\dot{x}(t) = \mathcal{A}x(t), \qquad x(0) = x_0 \qquad (1a)$$

$$\mathcal{B}x(t) = R_1 u(t) + R_2 d(t) \qquad (1b)$$

$$\mathcal{C}x(t) = y(t), \qquad (1c)$$

where the disturbance signal $d(t)$ is assumed to be generated by an exosystem that will be presented shortly. The operators $(\mathcal{A}, \mathcal{B})$ are defined such that they form a *boundary control system* [2, Def. 3.3.2]:

Definition 1: Let the state-space X and the input space U be Hilbert spaces and let $\mathcal{A} : X \supset \mathcal{D}(\mathcal{A}) \to X$ and $\mathcal{B} : X \supset \mathcal{D}(\mathcal{B}) \to U$ be linear operators such that $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{B})$. The system (1a)–(1b) is a *boundary control system* if the following hold:

1) The restriction $A := \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ of $\mathcal{A}$ to the kernel of $\mathcal{B}$ generates a C_0-semigroup on X.
2) There is an operator $B \subset \mathcal{L}(U, X)$ such that $\mathcal{R}(BR_1) \subset \mathcal{D}(\mathcal{A})$, $\mathcal{A}BR_1 \in \mathcal{L}(U, X)$ and $\mathcal{B}B = I_U$.

For the output, define the operator $\mathcal{C} : \mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{C}) \to Y$ for some (infinite-dimensional) Hilbert space Y with the properties that $\mathcal{C} \in \mathcal{L}(\mathcal{D}(A), Y)$, where $\mathcal{D}(A)$ is equipped with the graph norm of A, and $\mathcal{C}BR_1 \in \mathcal{L}(U, Y)$. Furthermore, we assume that $\mathcal{C}$ is an *admissible* [14, Def. 4.3.1] observation operator for A and for simplicity that A is the generator of an exponentially stable C_0-semigroup. Finally, R_1 and R_2 are arbitrary restrictions to parts of the boundary $\partial\Omega$ accessible via $\mathcal{B}$.

Let $W := \mathbb{C}^q$ and $S = \mathrm{diag}(i\omega_1, i\omega_2, \ldots, i\omega_q)$ for some $q \in \mathbb{N}$ such that $\omega_k \in \mathbb{R}$ and $\omega_k \neq \omega_j$ for $k \neq j$. Further let $E \in \mathcal{L}(W, U)$ and $F \in \mathcal{L}(W, Y)$. The exosystem that generates the disturbance signal $d(t)$ and the reference signal $y_{ref}(t)$ is given by

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \qquad (2a)$$

$$d(t) = Ev(t) \qquad (2b)$$

$$y_{ref}(t) = -Fv(t). \qquad (2c)$$

At this point, we present the transfer function of the plant (from u to y) by

$$P(s) = \mathcal{C}R(s, A)(\mathcal{A}B - sB)R_1 + \mathcal{C}BR_1, \quad s \in \rho(A).$$

For robust output regulation to be achievable, it is required that the transfer function $P(s)$ is surjective for every eigenvalue $i\omega_k$ of S. Thus, we make the rather standard assumption about the surjectivity of the transfer function, even though it is not necessarily required for achieving *approximate* robust output regulation.

The controller to be constructed is a linear system on a Banach space Z and of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0 \qquad (3a)$$

$$u(t) = Kz(t), \qquad (3b)$$

where $e(t) := y(t) - y_{ref}(t)$ denotes the regulation error, and $\mathcal{G}_1 \in \mathcal{L}(Z)$, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$ and $K \in \mathcal{L}(Z, U)$ are to be chosen later on.

When the plant, exosystem and controller are connected, they form a closed-loop system (see [6, Sec. 3] for the derivation and technical details). Define a new state variable by

$$x_e := \begin{bmatrix} 1 & -BR_1 K \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} - \begin{bmatrix} BR_2 Ev \\ 0 \end{bmatrix}.$$

The closed-loop system can be presented in the extended state-space $X_e := X \times Z$ as

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \qquad x_e(0) = (x_0, z_0) \qquad (4a)$$

$$e(t) = C_e x_e(t) + D_e v(t). \qquad (4b)$$

where

$$A_e = \begin{bmatrix} A - BR_1 K \mathcal{G}_2 \mathcal{C} & \mathcal{A}BR_1 K - BR_1 K \mathcal{G}_1' \\ \mathcal{G}_2 \mathcal{C} & \mathcal{G}_1' \end{bmatrix}$$

$$B_e = \begin{bmatrix} \mathcal{A}BR_2 E - BR_2 ES - BR_1 K \mathcal{G}_2 (\mathcal{C}BR_2 E + F) \\ \mathcal{G}_2 (\mathcal{C}BR_2 E + F) \end{bmatrix}$$

$$C_e = \begin{bmatrix} \mathcal{C} & \mathcal{C}BR_1 K \end{bmatrix}, \quad D_e = \mathcal{C}BR_2 E + F,$$

where we denote $\mathcal{G}_1' = \mathcal{G}_1 + \mathcal{G}_2 \mathcal{C}BR_1 K$ for brevity. The operator A_e has domain $\mathcal{D}(A_e) = \mathcal{D}(A) \times Z$. Due to the assumptions made on the parameters of the plant, exosystem and controller, the closed-loop system is a *regular linear system* (see [6, Thm. 3.1]).

III. THE ROBUST OUTPUT REGULATION PROBLEM AND THE INTERNAL MODEL PRINCIPLE

In order to present the robust output regulation problem, consider the perturbations $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ of the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ where the class $\mathcal{O}$ of admissible perturbations is such that

(i) the perturbed plant $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ is a boundary control system,
(ii) the operator $\tilde{\mathcal{C}}$ is admissible for $\tilde{A} := \tilde{\mathcal{A}}|_{\mathcal{N}(\tilde{\mathcal{B}})}$,
(iii) the eigenvalues of S are in the resolvent of $\tilde{A}$, i.e., $\{i\omega_k\}_{k=1}^q \subset \rho(\tilde{A})$,
(iv) the operators $\tilde{E}$ and $\tilde{F}$ are bounded.

These conditions are in particular satisfied for all sufficiently small bounded perturbations of the plant $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and for all bounded perturbations of E and F.

The Robust Output Regulation Problem. Choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that the following are satisfied:

1) The closed-loop semigroup generated by A_e is exponentially stable.
2) For all initial states $x_{e0} \in X_e$ and $v_0 \in W$, the regulation error satisfies $e^{\alpha \cdot} e(\cdot) \in L^2([0, \infty); Y)$ for some $\alpha > 0$ independent of x_{e0} and v_0.
3) If the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed to $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then for all initial states $x_{e0} \in X_e$ and $v_0 \in W$, the regulation error

satisfies $e^{\alpha'\cdot}e(\cdot) \in L^2([0,\infty); Y)$ for some $\alpha' > 0$ independent of x_{e0} and v_0.

We note that without the last item in the preceding list, i.e., the robustness requirement, the problem is called the *output regulation problem.*

The *internal model principle* can be expressed in the form of the $\mathcal{G}$-conditions [5, Def. 10]

$$\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \quad \forall k \in \{1, 2, \ldots, q\} \quad (5a)$$
$$\mathcal{N}(\mathcal{G}_2) = \{0\}, \quad (5b)$$

and hence, the $\mathcal{G}$-conditions can be used to characterize all robust regulating controllers. For boundary control systems, the result is given in [6, Thm. 4.7], [7, Thm. 4]:

Theorem 2: Assume that the closed-loop system is regular and exponentially stabilized by a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$. Then the controller solves the robust output regulation problem if and only if it satisfies the $\mathcal{G}$-conditions.

The rank-nullity theorem and the second $\mathcal{G}$-condition imply that $\dim Z \geq \dim \mathcal{R}(\mathcal{G}_2) = \dim Y$. Thus, for every system with an infinite-dimensional output space Y, every robust regulating controller is necessarily infinite-dimensional by Theorem 2. However, robust output regulation can still be achieved in an approximate sense by a finite-dimensional controller, which brings us to the concept of *approximate robust output regulation* considered in the following section.

IV. APPROXIMATE ROBUST OUTPUT REGULATION

Consider the following problem originally presented in [6, Sec. 4]:

The Approximate Robust Output Regulation Problem. For a given $\delta > 0$, choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that the following are satisfied:

1) The closed-loop system generated by A_e is exponentially stable.
2) For all initial states $x_{e0} \in X_e$, $v_0 \in W$, the regulation error satisfies

$$\int_t^{t+1} \|e(s)\|^2 ds \leq Me^{-\alpha t} \left(\|x_{e0}\|^2 + \|v_0\|^2\right) + \delta\|v_0\|^2$$

for some $M, \alpha > 0$ independent of $x_{e0} \in X_e, v_0 \in W$.
3) If the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed to $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then there exists a $\delta' > 0$ such that for all initial states $x_{e0} \in X_e, v_0 \in W$ the regulation error satisfies

$$\int_t^{t+1} \|e(s)\|^2 ds \leq M'e^{-\alpha't} \left(\|x_{e0}\|^2 + \|v_0\|^2\right) + \delta'\|v_0\|^2$$

for some $M', \alpha' > 0$ independent of x_{e0}, v_0.

Essentially, the problem formulation implies that for the unperturbed system, asymptotically the regulation error can be made smaller than $\delta\|v_0\|^2$ for any given $\delta > 0$. However, when the system is perturbed, there is no requirement that the asymptotic error is less than $\delta\|v_0\|^2$ but merely bounded.

We will essentially approach the concept of approximate robust output regulation by considering robust output regulation on a closed subspace of Y. Consider a partition $Y = Y_0 \oplus Y_1$ with $Y_0, Y_1 \neq \{0\}$. We will give sufficient conditions for a controller to solve the robust output regulation problem on Y_0. For systems with infinite-dimensional output spaces, the partition $Y = Y_0 \oplus Y_1$ can be chosen such that a controller that solves the robust output regulation problem on the finite-dimensional subspace Y_0 also solves the approximate robust output regulation problem. This will be discussed in more detail in Remark 5.

A. A Partial Internal Model

The concept of an internal model for approximate robust output regulation is presented in the following definition. Note that we omit the (classical) case $Y_1 = \{0\}$ as then the controller would contain the full internal model of the dynamics of the exosystem. Conversely, the case $Y_0 = \{0\}$ is omitted as then the controller would contain no internal model whatsoever.

Definition 3: Let $Y = Y_0 \oplus Y_1$ with $Y_0, Y_1 \neq \{0\}$. The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ contains an internal model on Y_0 of the dynamics of the exosystem if

$$\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \quad \forall k \in \{1, 2, \ldots, q\} \quad (6a)$$
$$\mathcal{N}(\mathcal{G}_2) \subset Y_1. \quad (6b)$$

In the following theorem, we will present an upper bound for the regulation error when a controller is constructed such that it satisfies Definition 3 for some partition $Y = Y_0 \oplus Y_1$. After presenting the result, we will comment on how it can be utilized in constructing a controller for approximate robust output regulation.

Theorem 4: Assume that a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ exponentially stabilizes the closed-loop system. If the controller satisfies Definition 3 for some partition $Y = Y_0 \oplus Y_1$, then there exist $M, \alpha > 0$, such that for all initial states $x_{e0} \in X_e, v_0 \in W$, the regulation error satisfies

$$\int_t^{t+1} \|e(s)\|^2 ds \leq Me^{-\alpha t} \left(\|x_{e0}\|^2 + \|v_0\|^2\right)$$
$$+ \|(I - Q)(C_e\Sigma + D_e)\|^2 \|v_0\|^2,$$

where Q a projection onto Y_0 along Y_1 and $\Sigma \in \mathcal{L}(W, X_e)$ is the unique solution of the Sylvester equation $\Sigma S = A_e\Sigma + B_e$. The corresponding estimate holds for all perturbations $\left(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}\right) \in \mathcal{O}$ that give rise to an exponentially stable closed-loop system.

Proof: Let $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ be such that the closed-loop semigroup $\tilde{T}_e$ generated by $\tilde{A}_e$ is exponentially stable. Due to the assumptions on the class $\mathcal{O}$ of perturbations and the regularity of the unperturbed closed-loop system, it follows that the perturbed closed-loop system is regular as well. Thus, as $\sigma(S) \subset i\mathbb{R}$, the Sylvester equation $\tilde{\Sigma}S = \tilde{A}_e\tilde{\Sigma} + \tilde{B}_e$ has a unique solution $\tilde{\Sigma} := \left(\tilde{\Pi}, \tilde{\Gamma}\right) \in \mathcal{L}(W, X_e)$ by [12], and by [11, Lem. 4.3], the regulation error can be written as

$$e(t) = \tilde{C}_e\tilde{T}_e(t)(x_{e0} - \tilde{\Sigma}v_0) + \left(\tilde{C}_e\tilde{\Sigma} + \tilde{D}_e\right)v(t). \quad (7)$$

Let $k \in \{1, 2, \dots, q\}$ be arbitrary and apply the Sylvester equation to the kth eigenvector ϕ_k of S associated with the eigenvalue $i\omega_k$. We obtain $\tilde{\Sigma} S \phi_k = \tilde{A}_e \tilde{\Sigma} \phi_k + \tilde{B}_e \phi_k$, i.e., $(i\omega_k - \tilde{A}_e) \tilde{\Sigma} \phi_k = \tilde{B}_e \phi_k$, which yields

$$
\begin{bmatrix}
(i\omega_k - \tilde{A} + \tilde{B}R_1 K \mathcal{G}_2 \tilde{\mathcal{C}}) \tilde{\Pi} \phi_k - \left(\tilde{\mathcal{A}} \tilde{B} R_1 K - \tilde{B} R_1 K \tilde{\mathcal{G}}_1' \right) \tilde{\Gamma} \phi_k \\
-\mathcal{G}_2 \tilde{\mathcal{C}} \tilde{\Pi} \phi_k + (i\omega_k - \tilde{\mathcal{G}}_1') \tilde{\Gamma} \phi_k
\end{bmatrix}
$$
$$
= \begin{bmatrix}
\tilde{\mathcal{A}} \tilde{B} R_2 \tilde{E} - \tilde{B} R_2 \tilde{E} S - \tilde{B} K \mathcal{G}_2 \left(\tilde{\mathcal{C}} \tilde{B} R_2 \tilde{E} + \tilde{F} \right) \\
\mathcal{G}_2 \left(\tilde{\mathcal{C}} \tilde{B} R_2 \tilde{E} + \tilde{F} \right)
\end{bmatrix} \phi_k,
$$

where $\tilde{\mathcal{G}}_1' = \mathcal{G}_1 + \mathcal{G}_2 \tilde{\mathcal{C}} \tilde{B} R_1 K$. The second line implies

$$
(i\omega_k - \mathcal{G}_1) \tilde{\Gamma} \phi_k
$$
$$
= \mathcal{G}_2 \left(\tilde{\mathcal{C}} \tilde{\Pi} \phi_k + \tilde{\mathcal{C}} \tilde{B} R_1 K \tilde{\Gamma} \phi_k + (\tilde{\mathcal{C}} \tilde{B} R_2 \tilde{E} + \tilde{F}) \phi_k \right),
$$

and now by Definition 3 we have

$$
Y_1 \ni \tilde{\mathcal{C}} \tilde{\Pi} \phi_k + \tilde{\mathcal{C}} \tilde{B} R_1 K \tilde{\Gamma} \phi_k + (\tilde{\mathcal{C}} \tilde{B} R_2 \tilde{E} + \tilde{F}) \phi_k
$$
$$
= \tilde{C}_e \tilde{\Sigma} \phi_k + \tilde{D}_e \phi_k.
$$

Since $\{\phi_k\}_{k=1}^q$ is the Euclidean basis for W, we obtain that

$$
Q \left(\tilde{C}_e \tilde{\Sigma} + \tilde{D}_e \right) = 0. \tag{8}
$$

Finally, we obtain by (7) that for all $t \geq 0$

$$
\int_t^{t+1} \|e(s)\|^2 ds
$$
$$
= \int_t^{t+1} \|\tilde{C}_e \tilde{T}_e(x_{e0} - \tilde{\Sigma} v_0) + (\tilde{C}_e \tilde{\Sigma} + \tilde{D}_e) v(t)\|^2 ds
$$
$$
\leq M' e^{-\alpha' t} (\|x_{e0}\|^2 + \|v_0\|^2) + \|\tilde{C}_e \tilde{\Sigma} + \tilde{D}_e\|^2 \|v_0\|^2
$$

for some $M', \alpha' > 0$ as $\tilde{\Sigma}$ is bounded, $\tilde{T}_e$ is exponentially stable, $\tilde{C}_e$ is admissible for $\tilde{A}_e$, and due to the structure of the signal generator $\|v(t)\| = \|e^{St} v_0\| = \|v_0\|$. Combining the preceding with (8), we obtain that

$$
\int_t^{t+1} \|e(s)\|^2 ds \leq M' e^{-\alpha' t} \left(\|x_{e0}\|^2 + \|v_0\|^2 \right)
$$
$$
+ \|(I - Q)(\tilde{C}_e \tilde{\Sigma} + \tilde{D}_e)\|^2 \|v_0\|^2,
$$

which concludes the proof. ∎

Remark 5: Considering the term $\|(I - Q)(C_e \Sigma + D_e)\|^2$ in the preceding theorem, since W is finite-dimensional there exists a unit vector $v_M \in W$ such that $\|C_e \Sigma + D_e\| = \|C_e \Sigma v_M + D_e v_M\|_Y$, where $C_e \Sigma v_M + D_e v_M := y_M$ is an element of Y. Now consider an orthonormal basis $\{\psi_k\}_{k=1}^\infty$ of Y so that we may write $y_M = \sum_{k=1}^\infty \langle y_M, \psi_k \rangle_Y \psi_k$. If the partition $Y = Y_0 \oplus Y_1$ is chosen such that $Y_0 = \text{span}\{\psi_k \,|\, k \in 1, 2, \dots, N_0\}$ for some $N_0 \in \mathbb{N}$, we have that $\|(I - Q) y_M\|_Y \to 0$ as $N_0 \to \infty$. Thus, for any given $\delta > 0$ it is possible to choose a sufficiently large N_0 such that $\|(I - Q)(C_e \Sigma + D_e)\|^2 \leq \delta$, which then provides a partition $Y = Y_0 \oplus Y_1$ with a finite-dimensional Y_0 such that

a controller satisfying Definition 3 for that partition solves the approximate robust output regulation problem to precision δ.

It should be noted that thus far we have merely *assumed* that the controller exponentially stabilizes the closed-loop system, as required when solving the approximate robust output regulation problem. In the next section, where we present a controller satisfying Definition 3, we will also show that it exponentially stabilizes the closed-loop system.

B. Construction of an Approximate Robust Controller

In this section, we will present a controller structure that solves the approximate robust output regulation problem. The controller structure is in fact the same as that presented in [6, Sect. 4.3], but here we will provide a simplified proof by utilizing Theorem 4 and Remark 5.

Let $Y = Y_0 \oplus Y_1$ such that Y_0 is a finite-dimensional closed subspace of Y and choose $Z = Y_0^q$. Choose the controller parameters as

$$
\mathcal{G}_1 = \text{diag} \left(i\omega_1 I_{Y_0}, i\omega_2 I_{Y_0}, \dots, i\omega_q I_{Y_0} \right), \tag{9a}
$$
$$
\mathcal{G}_2 = \left(\mathcal{G}_2^k \right)_{k=1}^q = (-Q)_{k=1}^q, \tag{9b}
$$
$$
K = \epsilon K_0 = \epsilon \left[K_0^1, K_0^2, \dots, K_0^q \right], \tag{9c}
$$

where Q is the orthogonal projection onto Y_0 and $\epsilon > 0$ is the tuning parameter. The parameters K_0^k can be chosen freely, albeit such that the controller also stabilizes the closed-loop system. We will utilize [6, Lem. 4.3] to achieve exponential stability, and thus, K_0^k must be chosen such that $\sigma(QP(i\omega_k) K_0^k) \subset \mathbb{C}_+$ for all $k \in \{1, 2, \dots, q\}$. We simply choose $K_0^k = (QP(i\omega_k))^\dagger$ (the Moore-Penrose pseudoinverse of $QP(i\omega_k)$), which is a valid choice due to the assumed surjectivity of $P(i\omega_k)$ for all $k \in \{1, 2, \dots, q\}$.

We note that in [6, Lem. 4.3], an extra feed-through term is required in the controller to exponentially stabilize the closed-loop system, but as we have assumed the plant to be exponentially stable, we can utilize [6, Lem. 4.3] without having the feed-through term in the controller.

Theorem 6: For all $\delta > 0$, there exists an $\epsilon^* > 0$ and a partition $Y = Y_0 \oplus Y_1$ such that for all $0 < \epsilon < \epsilon^*$, the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ with the parameter choices given in (9) solves the approximate robust output regulation problem.

Proof: The controller exponentially stabilizes the closed-loop system for all sufficiently small $\epsilon > 0$ by [6, Lem. 4.3], and for any $\delta > 0$ the partition $Y = Y_0 \oplus Y_1$ can be chosen such that $\|(I - Q)(C_e \Sigma + D_e)\|^2 \leq \delta$ by Remark 5. Thus, it remains to show that the controller satisfies Definition 3. By the choice of $\mathcal{G}_2$ it is clear that $\mathcal{N}(\mathcal{G}_2) \subset Y_1$ (in fact, $\mathcal{N}(\mathcal{G}_2) = Y_1$), so it remains to show that $\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ for all $k \in \{1, 2, \dots, q\}$.

Let $k \in \{1, 2, \dots, q\}$ and $w = (i\omega_k - \mathcal{G}_1) z = \mathcal{G}_2 y$ be arbitrary for some $z \in Z, y \in Y$. The diagonal structure of $\mathcal{G}_1$ implies that necessarily $0 = \mathcal{G}_2^k y = -Qy$, which is only possible when $y \in Y_1$. This further implies $w = \mathcal{G}_2 y = 0$, and since k and w were arbitrary, we have that $\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$. Thus, by Theorem 4 and Remark 5, the controller solves the approximate robust output

regulation problem to precision δ for all sufficiently small $\epsilon > 0$. ∎

V. EXAMPLE

In this section, we will use Theorem 6 to construct an approximate robust regulating controller for the 2D heat equation on a square $\Omega := (0,1) \times (0,1)$:

$$\dot{w}(\xi_1, \xi_2, t) = \frac{\partial^2 w(\xi_1, \xi_2, t)}{\partial \xi_1^2} + \frac{\partial^2 w(\xi_1, \xi_2, t)}{\partial \xi_2^2}, \ (\xi_1, \xi_2) \in \Omega$$

$$0 \equiv \frac{\partial w(\cdot, 1, t)}{\partial \xi_2} \equiv -\frac{\partial w(\cdot, 0, t)}{\partial \xi_2} \equiv w(0, \cdot, t)$$

$$u(t) = \frac{\partial w(1, \cdot, t)}{\partial \xi_1}$$

$$y(t) = w(1, \cdot, t). \tag{10}$$

In order to write the heat equation as a boundary control system, define the operator $\mathcal{A}$ such that $\mathcal{A}x = \nabla^2 x$ with domain

$$\mathcal{D}(\mathcal{A}) = \{x \in H^2(\Omega) \mid \partial_n x|_{\xi_2=0} = \partial_n x|_{\xi_2=1} = x|_{\xi_1=0} = 0\}.$$

Further define operators $\mathcal{B}$ and $\mathcal{C}$ such that $\mathcal{B}x(t) = \partial_n x(1, \cdot, t)$ and $\mathcal{C}x(t) = x(1, \cdot, t)$, so that (10) can be equivalently written as

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t), & x(0) = x_0, \\ \mathcal{B}x(t) &= u(t) + d(t), \\ \mathcal{C}x(t) &= y(t), \end{aligned} \tag{11}$$

where we added a disturbance signal $d(t)$ to the plant.

It is relatively easy to see that the operator $A := \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ generates an exponentially stable C_0-semigroup, and an operator B satisfying the criteria of Definition 1 is given by $(Bu)(\xi_1, \xi_2) = \xi_1 u$. Finally, define the input and output spaces as $U = Y = L^2([0,1])$, so that (11) indeed is an exponentially stable boundary control system (see [1] for a detailed consideration of the boundary controlled heat equation).

For the approximate robust output regulation problem, let $\delta = 10^{-4}$ be given. We choose the partition $Y = Y_0 \oplus Y_1$ such that

$$Y_0 := \text{span}\{\cos(k\pi\cdot) \mid k = 0, 1, \ldots, N_0 - 1\}$$

for some $N_0 \in \mathbb{N}$ which can essentially be considered an additional tuning parameter. The orthogonal projection Q onto Y_0 is then given by

$$Qy := \langle y, 1\rangle_{L^2([0,1])} + 2\sum_{k=1}^{N_0-1} \langle y, \cos(k\pi\cdot)\rangle_{L^2([0,1])} \cos(k\pi\cdot).$$

By standard Fourier analysis – and as already mentioned in Remark 5 – for every $y \in Y$, we have

$$\lim_{N_0 \to \infty} \|(I - Q)y\|_{L^2([0,1])} = 0,$$

and thus, for any given $\delta > 0$, we may choose N_0 such that the controller solves the approximate robust output regulation problem.

For the given $\delta = 10^{-4}$, we choose $N_0 = 15$. It should be noted that determining $\|(I-Q)(C_e\Sigma+D_e)\|$ is not that simple that it would be easy to state the smallest adequately large N_0, but since we only need to assure $\|(I-Q)(C_e\Sigma+D_e)\| \leq \delta$, a sufficiently large N_0 will do regardless. In practice, a suitable N_0 can be chosen by trial and error based on numerical simulations.

Let the reference and disturbance signals be given by

$$y_{ref}(\xi_2, t) = 2(\xi_2^2 - \frac{2}{3}\xi_2^3)\cos(\pi t) - \frac{1}{2}\sin(\pi\xi_2)$$

$$d(\xi_2, t) = (\xi_2 - 1)\sin(\pi t)$$

so that S can be chosen as $S = \text{diag}(-i\pi, 0, i\pi)$, and E and F are chosen such that $d = Ev$ and $y_{ref} = -Fv$. Thus, the controller parameters are chosen as

$$\mathcal{G}_1 = \text{diag}(-i\pi I_{Y_0}, 0_{Y_0}, i\pi I_{Y_0}) \tag{12a}$$

$$\mathcal{G}_2 = (-Q)_{k=1}^3 \tag{12b}$$

$$K = \epsilon \left[(QP(-i\pi))^\dagger, (QP(0))^\dagger, (QP(i\pi))^\dagger \right], \tag{12c}$$

where we choose $\epsilon = 0.6$ by trial and error to get the stability margin of the closed-loop system close to its maximal value.

For the simulation, the domain Ω is discretized by a square mesh of size $h = 2^{-5}$ and the transfer function P is approximated using the finite-difference discretized operators. The initial conditions are $x_0 = 0$, $z_0 = 0$ and $v_0 = 1$. The simulation results are presented in Figures 1 and 2 where the norm of the regulation error and the output profile for the controlled heat equation are displayed, respectively.
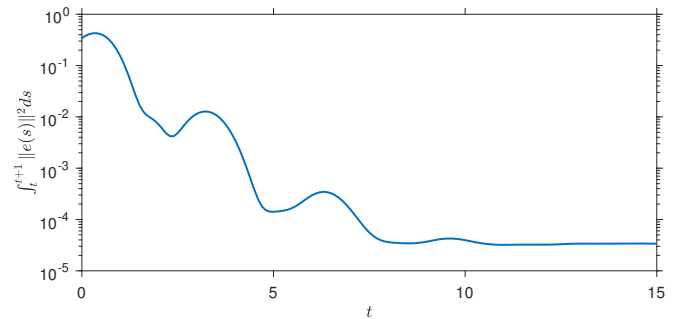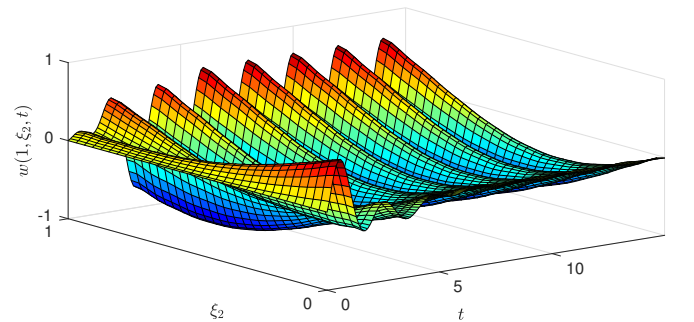

Fig. 1. The regulation error for $t \in [0, 15]$.

Fig. 2. Output profile for the controlled heat equation for $t \in [0, 15]$.

In Figure 1, the norm of the regulation error decreases to approximately $4 \cdot 10^{-5}$, implying that the chosen N_0 is indeed

sufficiently large. The rapid decay rate of the regulation error can also be observed in Figure 2, where the output profile of the controlled heat equation is displayed, as the output starts very rapidly to follow the periodic reference signal. In Figure 3, the state of the controlled heat equation is displayed at $t = 10$.
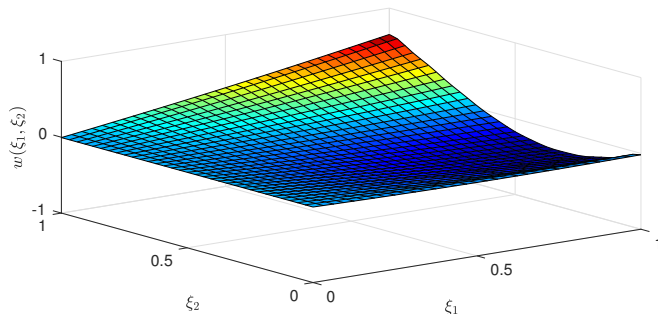

Fig. 3.   The state of the heat equation at $t = 10$.

VI. CONCLUSIONS

We considered a partial internal model for approximate robust output regulation and presented conditions which can be utilized in the construction of a controller. The considered system class was boundary control systems with infinite-dimensional output spaces, for which approximate robust output regulation is particularly relevant as a classical robust controller would necessarily be infinite-dimensional. Based on the presented results, a finite-dimensional approximate robust regulating controller was constructed for the heat equation on a square. The performance of the controller was demonstrated with numerical simulations.

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