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A KAISER WINDOW APPROACH FOR DESIGNING NONUNIFORM OVERSAMPLED *M*-CHANNEL FILTERBANKS

Robert Bregović,[1] Bogdan Dumitrescu,[1] Tapio Saramäki,[1] and Riitta Niemistö [2]

[1] Institute of Signal Processing, Tampere University of Technology
P. O. Box 553, FIN-33101 Tampere, Finland, {bregovic, bogdand, ts}@cs.tut.fi
[2] Nokia Research Center
P. O. Box 100, 33721 Tampere, Finland, riitta.niemisto@nokia.com

ABSTRACT

This paper shows how the windowing technique based on the use of the Kaiser window can be utilized for designing multiple prototype filters for generalized DFT (GDFT) modulated filterbanks (FBs). Some of the filters in these FBs are used for building a nonuniform M-channel FB. Using the proposed approach results in a simple and fast design algorithm for generating FBs having good overall properties, as is indicated by means of examples.

1. INTRODUCTION

Multirate filterbanks (FBs) are nowadays an important tool in many areas of digital signal processing [1]–[3]. Although, in most cases uniform FBs, that is, FBs having equal bandwidths (equal decimation factors) in all channels, are used, there are many applications in which it is beneficially to use the so-called nonuniform FBs [4], [5]. These FBs have subbands with different decimation factors, and correspondingly, different subband bandwidths. A typical example of such applications is speech processing where it is beneficially to have a FB with channels that approximate the Bark scale [6], a psychoacoustic scale with bands corresponding to the human hearing system.

There are two practical approaches for designing *M*-channel nonuniform FBs. The first approach is based on designing one or more suitable two-channel uniform or nonuniform FBs, and then, generating the required *M*-channel FB by using a tree structure [2] with the two-channel FBs as building blocks. The second approach is to design one or more prototype filters and generate the FB by using the cosine, GDFT, or some other modulation techniques. If only one prototype filter is used, then some channels in the FB are obtained by recombination (subband merging) of the neighboring channels [7], [8], [9]. Alternatively, two or more prototype filters can be used for designing different sections in the nonuniform FB [10]–[17]. In general, the first approach is easier to design, but results in systems having longer system delays than those obtainable when using the second approach. The second approach will be used in this paper and as such is described in more detail later on.

Moreover, the main emphasize is laid on oversampled FBs generated from one or more prototype filters by using the GDFT modulation [13]. Such oversampled FBs are of interest in various applications, for example, in adaptive filters [14]. Various design methods for the design of oversampled FBs are discussed in [13]–[17]. Those methods, although efficient in finding good solutions, are based on complex optimisation techniques and as such difficult to implement and use.

In order to simplify the design procedure, in this paper the Kaiser window approach for designing finite-impulse response (FIR) filters [18] is utilized for designing the prototype filters. In the design procedure, the number of parameters to be optimised is equal only to the number of different sections, with each section containing FB channels having equal channel bandwidth. This considerably simplifies the complexity of the FB design.

The outline of this paper is as follows: Section 2 gives the structure of the nonuniform FBs considered in this paper. The Kaiser window approach for designing FIR filters is reviewed in Section 3. The proposed design method is presented in Section 4 and two examples of the resulting FBs are given in Section 5.

2. GDFT-MODULATED FILTERBANKS

The block diagram of an *M*-channel FB is shown in Figure 1. This system consists of an analysis filterbank (analysis filters $H_k(z)$ followed by down-samplers by factors of R_k), a synthesis filterbank (up-samplers by factors of R_k followed by synthesis filters $F_k(z)$), and a

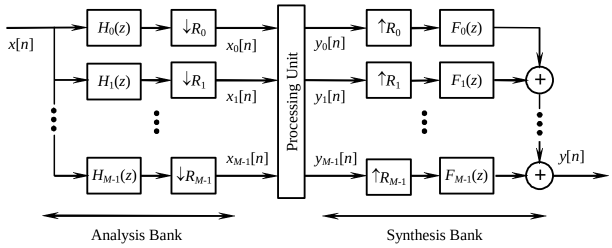

Figure 1. *M*-channel filterbank.

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processing unit between them. Depending on the choice of the decimation factors, and consequently, the bandwidths of the analysis and synthesis filters, the FB shown in Figure 1 is either uniform or nonuniform. Both of them will be discussed next.

2.1. Uniform Filterbanks

For a uniform FB, the decimation factors in all channels are identical, that is, $R_0 = R_1 = ... = R_{M-1} = R$ with R being smaller than or equal to M, the overall number of channels. The filters in the analysis and synthesis FBs are generated by using the following equations (GDFT modulation):

$$h_k[n] = h_p[n] e^{j\pi(k+1/2)(n-D/2)/M} \quad \text{for } n = 0, 1, \ldots, N_h \quad (1a)$$

$$f_k[n] = f_p[n] e^{j\pi(k+1/2)(n-D/2)/M} \quad \text{for } n = 0, 1, \ldots, N_f, \quad (1b)$$

for $k = 0, 1, \ldots, M-1$. In the above equations, $h_p[n]$ and $f_p[n]$ are real-valued coefficients of the two prototype filters and D is a positive integer corresponding to the FB delay.

In order to simplify the design, it is assumed that only one linear-phase FIR prototype filter is used for both the analysis and synthesis FBs. In this case, the FB delay is $D = N_h$ and the impulse response of the synthesis filters are related to those of the analysis filters as

$$f_k[n] = h_k^*[N_h - n] \quad (2)$$

for $k = 0, 1, \ldots, M-1$. Here, * stands for the complex conjugate.

Two major points should be emphasized here. First, although the prototype filters are filters with real-valued coefficients, due to (1a) and (1b), the channel filters are filters with complex-valued coefficients. Second, M in (1a) and (1b) corresponds to number of filters in the interval $[0, \pi]$. This is enough when processing real-valued signals.

2.2. Nonuniform Filterbanks

For nonuniform FBs, in turn, the decimation factors vary from channel to channel. In the most general case, each channel has a different decimation factor, that is, $R_0 \neq R_1 \neq ... \neq R_{M-1}$. In order to keep the flexibility provided by the nonuniform FB concept and in order to simplify the FB design, the main emphasize in this contribution is laid on nonuniform FBs that are constructed by using two or more uniform sections, each of them having a different down-sampling factor. In each of these sections only some filters in the corresponding uniform FBs are used. An example of a FB having two sections is given in Figure 2. The first section, denoted as S_1, consists of six filters belonging to a 12-channel uniform FB and the second section, denoted as S_2, consists of three filters belonging to a 6-channel uniform FB.

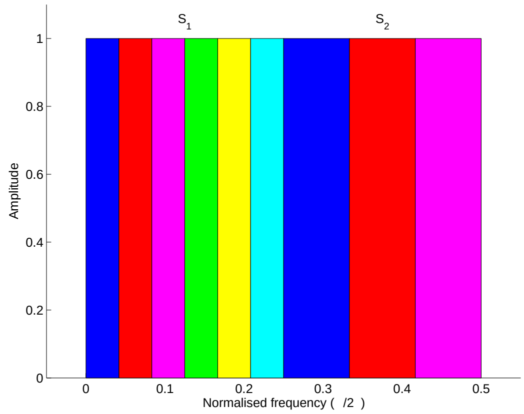

Figure 2. *M*-channel nonuniform filterbank for $M = 9$ containing two uniform sections.

Filters in each section are obtained by using the GDFT modulation, as given by (1a) and (1b). A different prototype filter is used for each section. For later use, the prototype filter used for designing the *i*th section will be denoted by $h_p^{(i)}$.

In order to cover the overall spectrum of a real-valued input signal, that is, the interval $[0, \pi]$, the following criterion must be satisfied in the most general case:

$$\sum_{i=1}^{S} \frac{m_i}{M_i} = 1 , \quad (3)$$

where S is the number of sections, whereas M_i and m_i are the total number of channels of the uniform FB in section S_i and the number of used channels in section S_i, respectively. In the example case of Figure 2, this corresponds to $S = 2$, $M_1 = 12$, $M_2 = 6$, $m_1 = 6$, and $m_2 = 3$. The overall number of channels in the FB is

$$M = \sum_{i=1}^{S} m_i . \quad (4)$$

Again, in the above example, this corresponds to $M = m_1 + m_2 = 9$.

An alternative method considered in [14], is to insert a separate transition filter between two uniform sections. This transition filter enables having different filter lengths in each section and/or different filter transition bandwidths and stopband attenuations. In this paper, this transition filter is omitted in order to prepare a fast design routine. However, a compromise is made, that is, the transition bandwidth of all filters in the filterbank is the same. Consequently, the filter order is dictated by the highest stopband attenuation.

2.3. Critically Sampled and Oversampled FBs

In the past, the main emphasize has been put on critically sampled (maximally decimated) FBs, that is, FBs with down-sampling factors given as $R_k = M_k$ [1], [2]. When using critically sampled FBs the smallest achievable number of samples in each channel is obtained without loosing information, and as such the fewest number of operations is required to process signals in the subbands. However, due to the critical sampling, signals from adjacent channel are aliasing into the neighboring channels. These alias errors are canceled in the synthesis side, but only in the case where the signals are not altered in the subbands (for example in the case of lossless coding). This assumption is not valid for most applications. Therefore, it is beneficially to use oversampled FBs, that is, FBs with $R_k < M_k$ [15].

2.4. Basic Relations

The input-output relation for the nonuniform FB in Figure 1 is

$$Y(z) = T_0(z)X(z) + \sum_{k=0}^{M-1}\frac{F_k(z)}{R_k}\sum_{l=1}^{R_k-1}H_k(zW_{R_k}^l)X(zW_{R_k}^l), \quad (5)$$

where $W_R = \mathrm{e}^{-\mathrm{j}2\pi/R}$ and

$$T_0(z) = \sum_{k=0}^{M-1}\frac{1}{R_k}H_k(z)F_k(z) \quad (6)$$

is the overall input-output distortion transfer function. When designing filters building the FB it is desired to obtain a system having good frequency selectivity in subbands as well as low distortion at the output, that is, $y[n] \approx x[n-D]$. When designing such nonuniform FBs the overall input-output transfer function can be reduced only to the amplitude distortion, that is,

$$Y(z) \approx T_0(z)X(z). \quad (7)$$

In order for this equation to be a good enough approximation of the system under consideration, it is required that the decimation factor R_k is chosen adequately, as illustrated in Figure 3, that is, π/R_k has to be larger than the channel bandwidth including both transition bands. In this case the alias error is on the level of the filter stopband attenuations, and as such can be ignored. Moreover, the aliasing errors are negligible even if the subband signals change.

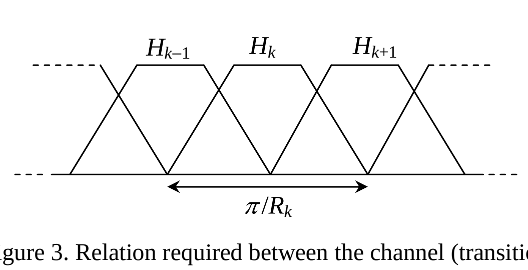

Figure 3. Relation required between the channel (transition) bandwidths and the decimation factor.

3. KAISER WINDOW TECHNIQUE FOR DESIGNING FIR FILTERS

The transfer function of an Nth-order FIR filter is given by:

$$H(z) = \sum_{n=0}^{N}h[n]z^{-n}. \quad (8)$$

When designing such a filter using the window technique, the filter impulse-response coefficients are derived as [18]

$$h[n] = h_{id}[n]w[n] \quad (9a)$$

for $n = 0, 1, ..., N$, where

$$h_{id}[n] = \begin{cases} \dfrac{\sin(\omega_0(n-N/2))}{\pi(n-N/2)} & 0 \le n \le N \\ 0 & \text{otherwise}, \end{cases} \quad (9b)$$

are the truncated (finite) impulse-response coefficients of an ideal lowpass filter with the cut-off frequency ω_0 and $w[n]$ is a window function defining the filter stopband attenuation as well as filter transition bandwidth. In order to achieve the required flexibility when choosing the filter attenuation, an adjustable window function, known as the Kaiser window, is used in this contribution. For the Kaiser window, the window function is given by [19]

$$w[n] = \frac{I_0\left(\beta\sqrt{1-\left(\dfrac{n}{N/2}-1\right)}\right)}{I_0(\beta)} \quad (10)$$

for $n = 0, 1, ..., N$, where $I_0(\beta)$ is the modified zeroth-order Bessel function expressed as

$$I_0(x) = 1 + \sum_{r=1}^{\infty}\left(\frac{(x/2)^r}{r!}\right)^2. \quad (11)$$

Here, β is an adjustable parameter that controls the minimum attenuation $A_{sb} = -20\log_{10}(\delta_s)$ in the filter stopband and can be very accurately estimated as

$$\beta = \begin{cases} 0.1102(A_{sb} - 8.7) & \text{for } A_s > 50 \\ 0.5842(A_{sb} - 21)^{0.4} + 0.07886(A_{sb} - 21) & \text{for } 21 \le A_s \le 50 \\ 0 & \text{for } A_s < 21. \end{cases} \quad (12)$$

The minimum filter order that guarantees the desired stopband attenuation can be estimated by using following formula:

$$N = \frac{2.056\,A_{sb} - 16.4}{2.285(\Delta\omega)} \quad (13)$$

with $\Delta\omega$ being the normalized transition bandwidth. This windowing technique is used next for designing FBs.

4. PROPOSED APPROACH

As shown in Section 2, nonuniform FBs discussed in this paper are designed by combining two or more uniform sections, each of them derived by using the GDFT modulation of one prototype filter. In order to simplify the design, the same prototype filter is used on the analysis and synthesis side. The goal now is to design the prototype filters for all uniform sections such that the overall FB has good properties, that is, high channel selectivity and low reconstruction error.

In the case of FBs with many channels and high stopband attenuation, the orders of the prototype filters become high. Therefore, some sophisticated optimization techniques are required for designing those filters. Such techniques are complicated to implement as well as time consuming for using. In order to speed up the design process, this paper presents the use of the Kaiser window method for designing the prototype filters.

4.1. Design Problem

This paper considers the following design problem for an M-channel nonuniform FB, as shown in Figure 1:
Given S, the desired number of sections, $1/M_i$, the widths of channels in each section, m_i, the number of channels in each section, and A_i for $i = 1, 2, ..., S$, the stopband attenuations for the sections as well as the desired decimation factors R_i,

$$\text{minimize } \delta \tag{14a}$$

such that

$$\left\| T_0(e^{j\omega}) \right| - 1 \right| \le \delta \tag{14b}$$

and for $i = 1, 2, ..., S$

$$\left| H_p^{(i)}(e^{j\omega}) \right| \le \delta_i \text{ for } \omega \in \left[\frac{\pi}{R_i}, \pi \right] \tag{14c}$$

with

$$T_0(e^{j\omega}) = \sum_{k=0}^{M-1} \frac{1}{R_k} H_k(e^{j\omega}) F_k(e^{j\omega}) , \tag{14d}$$

and

$$\delta_i = 10^{-A_i/20}. \tag{14e}$$

In the above problem, the goal is to minimize the FB distortion δ subject to the filter stopband constraints, that is, the stopband attenuation in each section has to be as good as or better than given by the specifications. How to select prototype filter orders and solve the above problem is discussed in the next section.

4.2. Proposed Design Method

In this paper, the problem stated in Section 4.1 is solved by properly applying the windowing technique that was discussed in Section 3. For this purpose, the following four-step procedure is used:

First, based on the design requirements, that is, R_i, M_i, m_i, and A_i for $i = 1, 2, ..., S$, the prototype filter order in section S_i is determined by using (13). [1] The transition width of the ith prototype filter can be evaluated as:

$$\Delta\omega_i = 2\left(\frac{\pi}{R_i} - \frac{\pi}{M_i} \right). \tag{15}$$

Second, the values of the Kaiser windows $w_i[n]$ for $n = 0, 1, ..., N_i$ and $i = 1, 2, ..., S$, (each section) are determined by using (9)–(12).

Third, the cut-off frequencies $\omega_i^{(0)}$ required for generating the ideal filters, as given by (9b), are calculated by solving the following simple optimization problem:
Minimize:

$$\delta \tag{16a}$$

such that

$$\left\| T_0(e^{j\omega}, \boldsymbol{\omega}_0) \right| - 1 \right| \le \delta \tag{16b}$$

with $T_0(e^{j\omega}, \boldsymbol{\omega}_0)$ given by (14d). Here,

$$\boldsymbol{\omega}_0 = \left[\omega_1^{(0)} \ \omega_2^{(0)} \ .... \ \omega_S^{(0)} \right]^T \tag{16c}$$

is the vector containing the unknown parameters, that is, the cut-off frequencies of the ideal filters for each section. For evaluating (16b), the distortion function $T_0(e^{j\omega}, \boldsymbol{\omega}_0)$ is evaluated on a grid of frequencies. It should be noted that the stopband attenuation and the filter transition width are fixed by selecting an appropriate Kaiser window as discussed at Steps 1 and 2. This optimization problem can be solved quickly by using any available unconstrained optimization routine, for example, the function `fminimax` included in the Optimization Toolbox [21] provided by MathWorks, Inc..

Fourth, the prototype filters are generated by using (9a) and (9b), whereas the filter in the FB are generated out of those prototype filters by using the GDFT modulation as given by (1a) and (1b).

If the resulting FB does not satisfy all criteria, then it is necessary to increase some or all filter orders and repeat the procedure.

4.3. Discussion about the Proposed Design Method

The following points regarding the proposed method should be emphasized. First, by utilizing the windowing technique, a design method has been derived that can be used to quickly design prototype filters for nonuniform FBs under consideration. This is mostly due to the fact that there are only S unknowns to be optimized.

Second, the implementation of the method is very easy and straightforward. For solving the optimization problem under consideration any basic optimization routine, for example `fminimax`, can be used.

Third, although a simple design method is used, the generated FBs possess low distortion errors and satisfactory stopband attenuations. This is achieved by using filters building the FB of higher order than it would be required in the optimum case. Nevertheless, this differ-

[1] It should be noted that (13) is just an approximation for the filter order. The required (best) filter order might be lower or higher.

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ence is negligible comparing to the easiness of the design.

Fourth, in the proposed method, linear-phase FIR filters are used. Therefore, the FB delay is equal to the order of the longest prototype filter. Low-delay FBs can not be designed in this way due to the restriction of the windowing technique. This might be a problem in applications where the FB delay is critical. In this case, another design method, like the one proposed in [13] has to be used.

5. Examples

In this section two different nonuniform FBs are designed by using the proposed method.

Example 1: It is desired to design a nonuniform FB having two sections and the following channel distribution: $m_1 = 24$, $M_1 = 48$, $m_2 = 8$, $M_2 = 16$ as shown in Figure 4. The required filter stopband attenuations are $A = 60$dB and the down-sampling factors are $R_1 = 32$ and $R_2 = 20$. The distortion transfer function given by (14d), has been evaluated in 1024 points on the interval $[0, \pi]$.

In this case, the required filter order, for both prototype filters as given by (13), are $N_1 = N_2 = 237$. The optimal values of the cut-off frequencies are $\omega_0 = [0.043875\ 0.109094]$. The amplitude responses of the prototype filters are shown in Figure 5, whereas the amplitude responses of the filters building the filterbank and the distortion of the overall bank are shown in Figures 6 and 7, respectively. The optimal value for δ can be seen from Figure 7. The overall design took 3.7 seconds on a PC with a 2.8 GHz Pentium 4 processor running Matlab 6.5 under Windows XP.

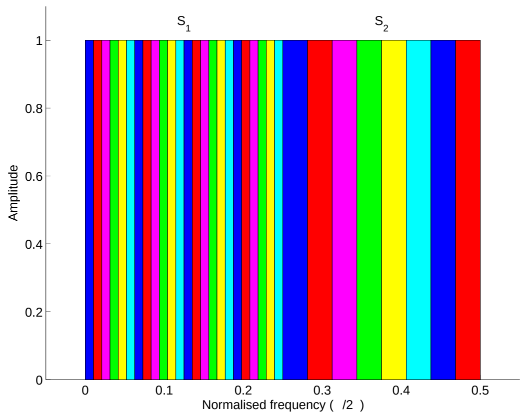

Figure 4. Example 1 – channel distribution.

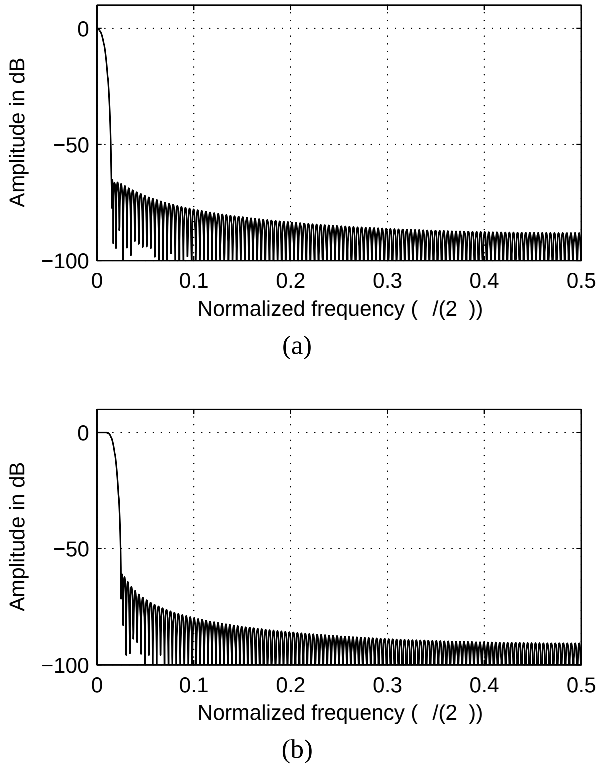

Figure 5. Example 1 – Amplitude response of the prototype filters. (a) Section 1. (b) Section 2.

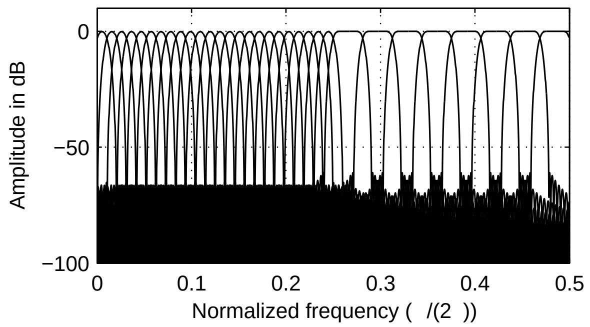

Figure 6. Example 1 – Amplitude response of analysis filters.

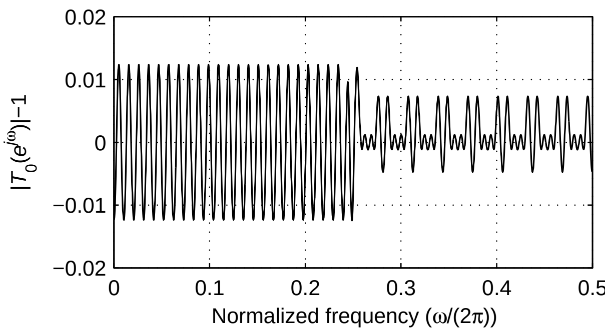

Figure 7. Example 1 – Amplitude distortion.

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Example 2: It is desired to design a nonuniform FB having three sections and the following channel distribution: $m_1 = 16$, $M_1 = 48$, $m_2 = 4$, $M_2 = 12$, $m_3 = 2$, $M_3 = 6$ as shown in Figure 8. The required filter stopband attenuations are $A = 80$dB and the down-sampling factors are $R_1 = 32$, $R_2 = 16$, and $R_3 = 9$. Again, 1024 points have been used for evaluating (14d).

In this case, the required filter order, for both prototype filters as given by (13), are $N_1 = N_2 = N_3 = 315$. The optimal values of the cut-off frequencies are $\omega_0 = [0.042583 \quad 0.140760 \quad 0.271657]$. The amplitude responses of the prototype filters are shown in Figure 10, whereas the amplitude responses of the filters building the filterbank and the distortion of the overall bank are shown in Figures 9 and 11, respectively. The optimal value for δ can be seen from Figure 9. In this case the overall design took 2.9 seconds.

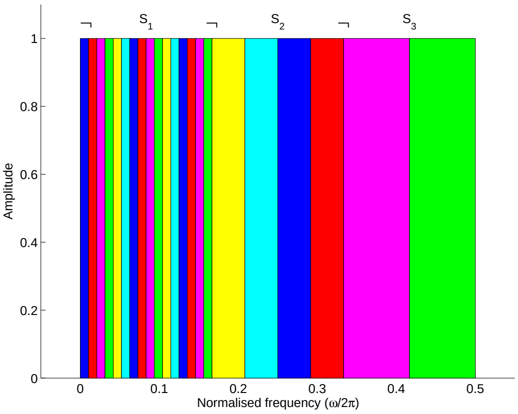

Figure 8. Example 2 – channel distribution.

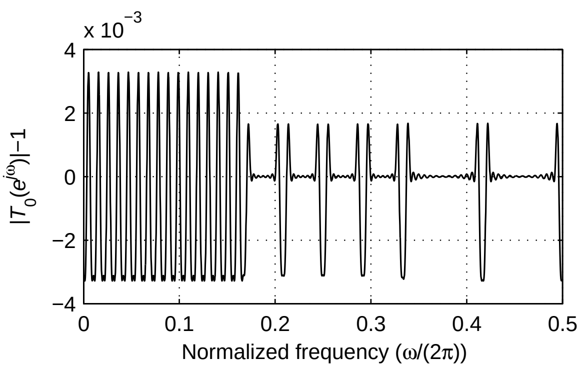

Figure 9. Example 2 – Amplitude distortion.

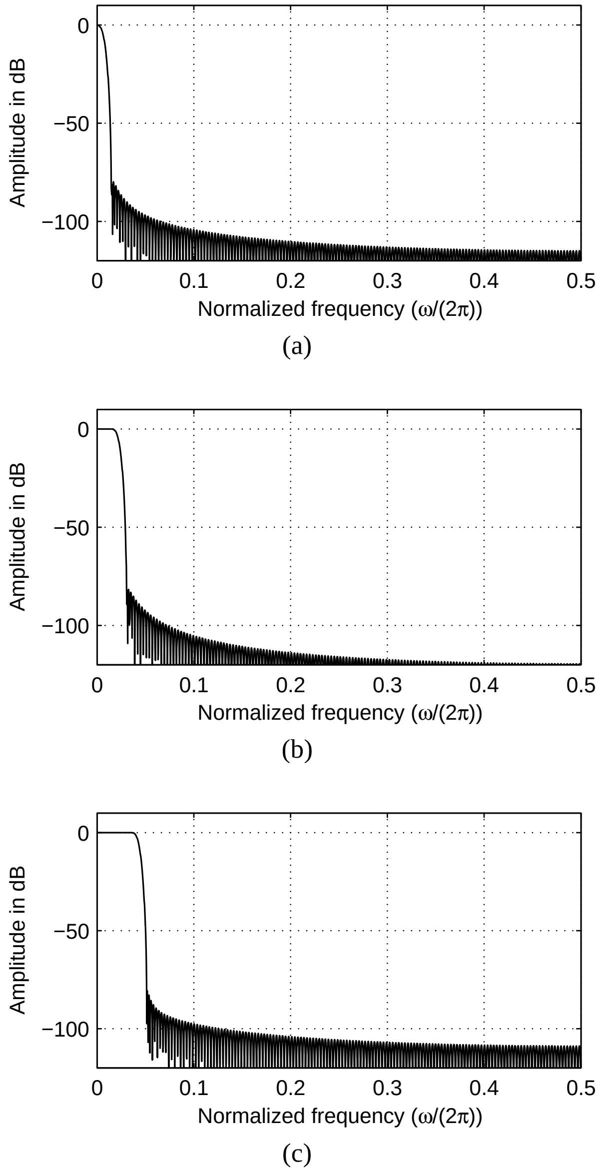

Figure 10. Example 2 – Amplitude response of the prototype filters. (a) Section 1. (b) Section 2. (c) Section 3.

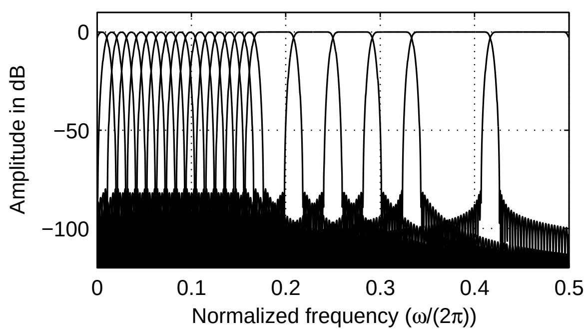

Figure 11. Example 2 – Amplitude response of analysis filters.

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