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Contents

Abstract

In this report, the structural optimization of a three-bar truss is considered. We present the detailed solution of the minimum compliance problem as well as the stress-constrained minimum weight design. In both cases the truss is subject to two loading conditions. The compliance minimization problem is formulated as a multicriterion optimization problem, where the compliances of the different loading conditions are the conflicting criteria that are minimized simultaneously. We provide the analytical solution of this formulation as well as the global optimum of the stress-constrained minimum weight problem.

1 Introduction

In the structural optimization of trusses, three main problem categories can be identified. In *sizing optimization*, the optimal member profiles are determined for a fixed layout, where the coordinates of the joints as well as the connectivity of the members are given. In *geometry optimization*, some nodes are allowed to move. Finally, in topology optimization, members are allowed to vanish. The combination of geometry and topology optimization is sometimes called *layout optimization*. For a thorough review on topology optimization and the above terms, see [7, 10, 3].

In the present work, the topology optimization of a simple three-bar truss is considered. The purpose is to provide the research community a test case, where the analytical solution is known. This will serve as a reliable benchmark for testing numerical optimization algorithms, and for studying other phenomenon related to topology optimization of trusses.

In formulating the optimization problems, the *ground structure approach* [6] is adopted. In this approach, an initial truss with excessive number of nodes and members (the ground structure) is defined, and during the optimization process members are allowed to vanish, which leads to alterations of the topology. As the ground structure of the example problems possesses only three members, the number of topologies is small, and the problem can be solved analytically.

We consider the same ground structure for two sets of loading conditions. In the first case, the allowable stresses in tension and in compression are equal, while in the latter case, the allowable stress in compression is smaller than in tension. For both cases, we solve the minimum weight problem with stress constraints, and the compliance minimization problem. These are arguably the most studied problems in the literature on topology optimization. The motivation for considering these problems is to study their optimal topologies under multiple loading conditions. It is a well-known fact that under a single loading condition, the two problems have an identical optimum topology [6, 7]. It is also generally agreed that such a result cannot be stated in general, when multiple loading conditions are involved, but to the author's knowledge, a rigorous proof of this statement is still lacking. The computations in the present work will help to bridge this gap.

The paper is organized as follows. In Section 2, the general problem formulations are given. The three-bar truss with equal allowable stresses is solved in Section 3 and with unequal allowable stresses in Section 4. Finally, the computations are discussed in Section 5.

2 Problem formulations

In the following, we present the general problem formulations that are employed in this work. The stress-constrained minimum weight problem is one of the first problems ever studied problem in the structural optimization literature. We give both the general nonlinear formulation and a simplified linear formulation, where the compatibility conditions have been neglected.

For the compliance minimization problem, a multicriterion formulation is applied, where the conflicting criteria are the compliances of the different loading conditions. Albeit not new, this formulation has not been thoroughly studied in the literature.

2.1 Minimum weight problem

The general problem formulation for finding the minimum weight design with stress constraints is

$$\begin{aligned} \min_{A_i} \quad & V(\mathbf{x}) = \sum_{i=1}^{n} L_i A_i \\ \text{s.t} \quad & \mathbf{K}(\mathbf{x})\mathbf{u}_j = \mathbf{p}_j,\ j = 1,2,\ldots,n_L \\ & \underline{\sigma}_i \leq \sigma_{ij}(\mathbf{x}) \leq \overline{\sigma}_i, \quad \text{if } A_i > 0 \\ & \underline{A}_i \leq A_i \leq \overline{A}_i \end{aligned} \tag{P_V}$$

In the above, $\mathbf{x} = \{A_1\ A_2\ \cdots\ A_n\}$ is the design variable vector of cross-sectional areas of the ground structure members. $\mathbf{K}$ is the stiffness matrix, $\mathbf{u}_j$ and $\mathbf{p}_j$ are the displacement and load vectors of the j^{th} loading condition. The stress of member i in the j^{th} loading condition is denoted σ_{ij}. The member stresses are obtained from the equation

$$\sigma_j = \mathbf{S}\mathbf{u}_j \tag{1}$$

where $\mathbf{S}$ is the stress matrix. During the optimization, the structural analysis is performed separately for each given design variable vector. Thus, the stiffness equations are not considered as equality constraints. For formulating the stiffness and stress matrices, see for example [5].

In problem P_V, topology alterations are enabled by setting the lower bound for the member areas to zero, i.e. $\underline{A}_i = 0$ for all $i = 1,2,\ldots,n$. As the topology varies during the optimization process, the stiffness matrix may become singular or nearly singular, which causes numerical difficulties. However, this issue does not arise in the present work due to simplicity of the ground structure.

If the compatibility conditions are neglected, the minimum weight problem can be formulated as the following linear programming problem [6]:

$$\begin{aligned} \min_{A_i, N_{ij}} \quad & V(\mathbf{x}) = \sum_{i=1}^{n} L_i A_i \\ \text{s.t} \quad & \mathbf{B}\mathbf{N}_j = \mathbf{p}_j,\ j = 1,2,\ldots,n_L \\ & \underline{\sigma}_i A_i \leq N_{ij} \leq A_i \overline{\sigma}_i,\ i = 1,2,\ldots,n \\ & \underline{A}_i \leq A_i \leq \overline{A}_i,\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,n_L \end{aligned} \tag{P_{LP}}$$

where $\mathbf{B}$ is the statics matrix of the truss, and $\mathbf{N}_j$ is the vector of member forces and N_{ij} is the normal force of member i in the loading condition j, respectively.

For a single loading condition, it has been shown that the optimum topology is statically determinate, see [7]. It then follows that the solution of problem P_{LP} satisfies the compatibility conditions automatically, and its solution is also the global optimum of problem P_V. When several loading conditions are involved, the minimum weight structure is generally not statically determinate, and the solution of the LP formulation does not necessarily satisfy the compatibility conditions. In this case, problem P_{LP} yields a lower bound for the optimum of problem P_V.

2.2 Multicriterion compliance minimization

Minimizing the compliance under multiple loading conditions requires the choice of the objective function. Typically in the literature, either the weighted sum or the maximum of the compliances of the different loading conditions is minimized. Another approach, also adopted in the present work, is to consider the compliances of the different loading conditions as competing and conflicting criteria, and to minimize them simultaneously. This leads to the following multicriterion optimization formulation [8]:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \{\, c_1(\mathbf{x})\; c_2(\mathbf{x}) \;\cdots\; c_{n_L}(\mathbf{x}) \,\} \\ \text{s.e.} \quad & \mathbf{K}(\mathbf{x})\mathbf{u}_j = \mathbf{p}_j,\; j = 1,2,\ldots,n_L \\ & V(\mathbf{x}) \le V_{\max} \\ & \underline{A}_i \le A_i,\; i = 1,2,\ldots,n \end{aligned} \tag{P_C}$$

where $c_j(\mathbf{x}) = \mathbf{p}_j^{\mathrm{T}}\mathbf{u}_j(\mathbf{x}),\ j = 1,2,\ldots,n_L$, are the compliances of the loading conditions. Furthermore, the material volume upper bound can be written as $V_{\max} = kLA_0$, where $k > 0$.

To allow topology alterations, the lower bounds for member areas are set to $\underline{A}_i = 0$, $i = 1,2,\ldots,n$. Additionally, the cases where the structure is unable to satisfy the equilibrium equations due to removal of members, must be taken into account. We do this formally by defining the domain of the compliances as

$$D = \{\, \mathbf{x} \mid \exists \mathbf{u}_j :\ \forall j = 1,2,\ldots,n_L :\ \mathbf{K}(\mathbf{x})\mathbf{u}_j = \mathbf{p}_j \,\} \tag{2}$$

This set is the non-negative orthant of the design space, with certain parts of the boundary removed. It can be proven that D is a convex set [1]. Also, following the guidelines of [1], it can be shown that the compliances $c_j,\ j = 1,2,\ldots,n_L$, are convex functions in the set D.

Assuming that the member areas are chosen from the set D, the stiffness equation can be eliminated from Problem P_C by inverting the stiffness matrix and substituting the displacements into the expressions of the compliances. This reduces the problem size considerably, and the following form is obtained:

$$\begin{aligned} \min_{\mathbf{x}\in D} \quad & \{\, c_1(\mathbf{x})\; c_2(\mathbf{x}) \;\cdots\; c_{n_L}(\mathbf{x}) \,\} \\ \text{s.e.} \quad & V(\mathbf{x}) \le V_{\max} \\ & 0 \le A_i,\; i = 1,2,\ldots,n \end{aligned} \tag{P_C^0}$$

The feasible set is defined by linear constraints. It is easy to verify that problem P_C^0 satisfies the Kuhn-Tucker constraint qualification [9, pp.38]:

Lemma 1. Let the feasible set of a multicriterion problem $\Omega = \{\, \mathbf{x} \mid \mathbf{g}(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b} \le \mathbf{0} \,\}$. Then the problem satisfies the Kuhn-Tucker constraint qualification.

Proof. Let $\mathbf{x}^* \in \Omega$ and $J = \{\, j \mid g_j(\mathbf{x}^*) = 0 \,\}$. To show that the constraint qualification holds, we construct a vector-valued function $\mathbf{a} : [0,1] \to \mathbb{R}^n$, satisfying $\mathbf{a}(0) = \mathbf{x}^*$, $\mathbf{a}'(0) = \alpha\mathbf{d}$ and $\mathbf{g}(\mathbf{a}(t)) \le \mathbf{0}$ for all $t \in [0,1]$. Here $\alpha > 0$ is a constant and $\mathbf{d}$ is an arbitrary member of the set $\{\, \mathbf{v} \mid \nabla g_j^{\mathrm{T}}(\mathbf{x}^*)\mathbf{v} \le 0,\ \forall j \in J \,\}$.

A function satisfying the initial condition is $\mathbf{a}(t) = \mathbf{x}^* + \alpha t\mathbf{d}$. Next we define a constant $\alpha > 0$ such that $\mathbf{g}(\mathbf{a}(t)) \leq \mathbf{0}$ holds. Consider the function g_j, when $j \in J$ and when $j \notin J$:

$$j \in J: \qquad g_j(\mathbf{a}(t)) = (\mathbf{A}(\mathbf{x}^* + \alpha t\mathbf{d}))_j - b_j = \underbrace{(\mathbf{A}\mathbf{x}^*)_j}_{=b_j} + \alpha t \underbrace{(\mathbf{A}\mathbf{d})_j}_{\leq 0} - b_j \leq 0$$

$$j \notin J: \qquad g_j(\mathbf{a}(t)) = (\mathbf{A}\mathbf{x}^*)_j + \alpha t(\mathbf{A}\mathbf{d})_j - b_j$$
$$= b_j + \varepsilon_j + \alpha t(\mathbf{A}\mathbf{d})_j - b_j = \varepsilon_j + \alpha t(\mathbf{A}\mathbf{d})_j$$

where $\varepsilon_j = (\mathbf{A}(\mathbf{x}^* + \alpha t\mathbf{d}))_j - b_j < 0$.

Now $\varepsilon_j + \alpha t(\mathbf{A}\mathbf{d})_j \leq 0$ for all $t \in [0,1]$ at least, when $(\mathbf{A}\mathbf{d})_j \leq 0$. Assume then, that $(\mathbf{A}\mathbf{d})_j > 0$. Since g_j is an increasing function with respect to t, setting $t = 1$ yields an upper bound for α such that $g_j(\mathbf{a}(t)) \leq 0$ holds for all $t \in [0,1]$:

$$\alpha \leq \frac{-\varepsilon_j}{\nabla g_j^{\mathrm{T}}\mathbf{d}}$$

Taking into account all constraints, for which $\nabla g_j^{\mathrm{T}}\mathbf{d} > 0$, the following value for α is obtained

$$\alpha = \min_{j:\nabla g_j^{\mathrm{T}}\mathbf{d}>0} \left\{ \frac{-\varepsilon_j}{\nabla g_j^{\mathrm{T}}\mathbf{d}} \right\}$$

Thus, an appropriate function $\mathbf{a}$ has been constructed, and it can be deduced that the problem satisfies the constraint qualification.

□

In order to apply the KKT-conditions, we still need to verify that the compliance is continuously differentiable in the set D. Since in the set D, the stiffness matrix $\mathbf{K}$ is positive definite, we have $\det\mathbf{K}(\mathbf{x}) > 0$ for all $\mathbf{x} \in D$. The compliance of loading condition i can be written as

$$c_i(\mathbf{x}) = \mathbf{p}_i^{\mathrm{T}}\mathbf{K}(\mathbf{x})^{-1}\mathbf{p}_i = \sum_{s,t} K_{st}^{-1}(\mathbf{x}) p_{is} p_{it}$$

The inverse of the stiffness matrix can be written as

$$\mathbf{K}^{-1} = \frac{1}{\det\mathbf{K}} \operatorname{adj}\mathbf{K}$$

where $\operatorname{adj}\mathbf{K}$ is the adjugate. Since each element of $\mathbf{K}$ is a linear function of the member areas, $\det\mathbf{K}$ and the elements of $\operatorname{adj}\mathbf{K}$ are polynomial functions of the member areas. Then, as all elements, K_{ij}^{-1}, of the inverse matrix are rational functions, they are continuously differentiable ($\det\mathbf{K} > 0$) with respect to member areas. Thus, since the compliance is linear with respect to the K_{ij}^{-1}, it can be deduced that the compliance is continuously differentiable with respect to the member areas in the set D.

From the discussion above, it follows that the Pareto optimal solutions of the compliance minimization problem can be solved from the KKT-conditions. This is done in the following sections.

3 Equal allowable stresses in tension and in compression

In this section, we find the stress-constrained minimum weight design and the minimum compliance design for a three-bar truss, with equal allowable stresses in tension and in compression.

3.1 Ground structure and feasible topologies

We consider the structure shown in Fig. 1. The truss is subject to two loading conditions. In loading condition LC1, a horizontal load with magnitude $3F$ acting to the right and a vertical load with magnitude F, are applied. In loading condition LC2, only the horizontal load acting to the left with magnitude $3F$ is applied.

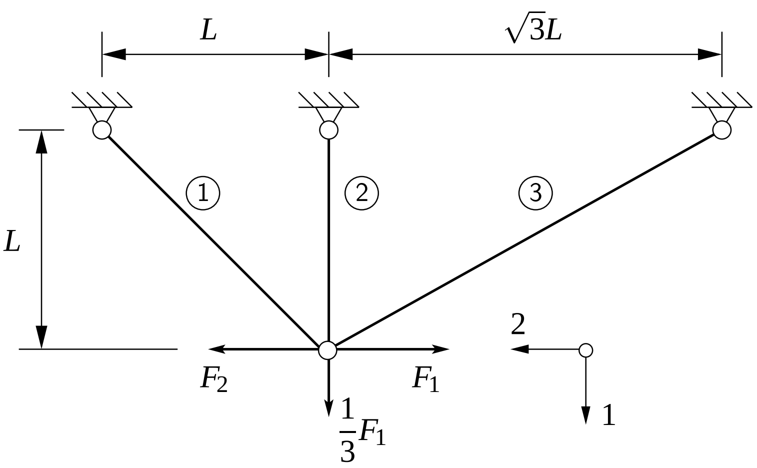

Figure 1: Three-bar truss. The loading conditions are: LC 1: $F_1 = 3F$, $F_2 = 0$; LC 2: $F_1 = 0$, $F_2 = 3F$.

The truss has four statically feasible topologies; three statically determinate that are obtained by removing one member from the ground structure, and one statically indeterminate, corresponding to the ground structure.

3.2 Minimum weight design

To solve the stress-constrained minimum weight design, Problem P_V is written for the three-bar truss. The stress bounds for all members are $\overline{\sigma}_i = -\underline{\sigma}_i = \frac{F}{A_0}$, where A_0 is a reference member area.

The stiffness matrix corresponding with the nodal displacements shown in Fig. 1 is

$$\mathbf{K}(\mathbf{x}) = \frac{E}{L}\begin{bmatrix} \frac{A_1}{2\sqrt{2}} + A_2 + \frac{A_3}{8} & -\frac{A_1}{2\sqrt{2}} + \frac{\sqrt{3}}{8}A_3 \\ -\frac{A_1}{2\sqrt{2}} + \frac{\sqrt{3}}{8}A_3 & \frac{A_1}{2\sqrt{2}} + \frac{3}{8}A_3 \end{bmatrix} \tag{3}$$

where E is the Young's modulus and L is the characteristic length appearing in Fig. 1.

The load vectors are

$$\mathbf{p}_1 = \begin{bmatrix} F \\ -3F \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 3F \end{bmatrix} \tag{4}$$

and the stress matrix is

$$\mathbf{S} = \frac{E}{L}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \\ \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix} \tag{5}$$

The nodal displacements in the two loading conditions are obtained by solving the stiffness equation. They have the following expressions:

$$\begin{aligned}
\text{LC1:} \qquad & u_{11}(\mathbf{x}) = \frac{-8\sqrt{2}A_1 + 6(1+\sqrt{3})A_3}{d(A_1,A_2,A_3)}\frac{FL}{E} \\
& u_{21}(\mathbf{x}) = \frac{-8\sqrt{2}A_1 - 48A_2 - 2(3+\sqrt{3})A_3}{d(A_1,A_2,A_3)}\frac{FL}{E} \\
\text{LC2:} \qquad & u_{12}(\mathbf{x}) = \frac{12\sqrt{2}A_1 - 6\sqrt{3}A_3}{d(A_1,A_2,A_3)}\frac{FL}{E} \\
& u_{22}(\mathbf{x}) = \frac{12\sqrt{2}A_1 + 48A_2 + 6A_3}{d(A_1,A_2,A_3)}\frac{FL}{E}
\end{aligned}$$

where

$$d(A_1,A_2,A_3) = 4\sqrt{2}A_1A_2 + (2\sqrt{2}+\sqrt{6})A_1A_3 + 6A_2A_3$$

Applying Eq. (1) gives the member stresses:

$$\begin{aligned}
\text{LC1:} \qquad & \sigma_{11}(\mathbf{x}) = \frac{24A_2 + 2(3+2\sqrt{3})A_3}{d(A_1,A_2,A_3)}F && \text{(tension)} \\
& \sigma_{21}(\mathbf{x}) = \frac{-8\sqrt{2}A_1 + 6(1+\sqrt{3})A_3}{d(A_1,A_2,A_3)}F && \text{(tension or compression)} \\
& \sigma_{31}(\mathbf{x}) = \frac{-2(\sqrt{2}+\sqrt{6})A_1 - 12\sqrt{3}A_2}{d(A_1,A_2,A_3)}F && \text{(compression)} \\
\text{LC2:} \qquad & \sigma_{12}(\mathbf{x}) = -\frac{24A_2 + 3(1+\sqrt{3})A_3}{d(A_1,A_2,A_3)}F && \text{(compression)} \\
& \sigma_{22}(\mathbf{x}) = \frac{12\sqrt{2}A_1 - 6\sqrt{3}A_3}{d(A_1,A_2,A_3)}F && \text{(tension or compression)} \\
& \sigma_{32}(\mathbf{x}) = \frac{3\sqrt{2}(1+\sqrt{3})A_1 + 12\sqrt{3}A_2}{d(A_1,A_2,A_3)}F && \text{(tension)}
\end{aligned}$$

The following observations can be made about the member stresses. Since $d > 0$ for all feasible cross-sections, in can be seen that in LC1, member 1 is always in tension and member 3 is always in compression. Member 2 can be in tension or in compression depending on the values of A_1 and A_3. In LC2, member 1 is in tension and member 3 is in compression, while member 2 can again be either in tension or in compression.

3.2.1 Statically determinate topologies

Problem P_V is non-convex, so it can have local minima. Furthermore, the feasible set contains degenerate parts that correspond to the removal of a member [4]. Gradient-based solution methods are unable to find solutions located in these parts. Thus, we consider each topology separately. When one of the members is removed, the remaining truss becomes statically determinate, and the minimum weight solution can be found by simple hand calculations.

Member 1 removed. When member 1 is removed from the ground structure, the normal forces in the remaining members 2 and 3 can be solved from the equilibrium equations of the loaded node. The normal forces are:

$$\begin{aligned} &\text{LC 1:} \quad N_{21} = (1+\sqrt{3})F \quad N_{31} = -2\sqrt{3}F \\ &\text{LC 2:} \quad N_{22} = -\sqrt{3}F \qquad N_{33} = 2\sqrt{3}F \end{aligned}$$

The cross-sectional area of member 2 is determined by LC1, whereas LC2 determines the cross-section of member 3. Setting the members to the stress limit gives the following optimal member areas and the minimum material volume of this topology:

$$A_1 = 0 \qquad A_2 = (1+\sqrt{3})A_0 \qquad A_3 = 2\sqrt{3}A_0 \qquad V = (5\sqrt{3}+1)LA_0 \approx 9.66025LA_0$$

Member 2 removed. When member 2 is removed from the ground structure, the normal forces of members 1 and 3 are

$$\begin{aligned} &\text{LC 1:} \quad N_{11} = \frac{\sqrt{2}(3+2\sqrt{3})}{2+\sqrt{3}}F \qquad N_{31} = \frac{-2(1+\sqrt{3})}{2+\sqrt{3}}F \\ &\text{LC 2:} \quad N_{12} = -\frac{3\sqrt{2}(1+\sqrt{3})}{2(2+\sqrt{3})}F \quad N_{32} = \frac{3(1+\sqrt{3})}{2+\sqrt{3}}F \end{aligned}$$

This time LC1 determines the optimum cross-sectional are of member 1 and LC2 determines the optimum value for A_3. The stress bounds are reached by the following member areas:

$$\begin{aligned} &A_1 = \frac{\sqrt{2}(3+2\sqrt{3})}{2+\sqrt{3}}A_0 \approx 2.4495A_0 \qquad A_2 = 0 \qquad A_3 = \frac{3(1+\sqrt{3})}{2+\sqrt{3}}A_0 \approx 2.1962A_0 \\ &V = \frac{2(6+5\sqrt{3})}{2+\sqrt{3}}LA_0 \approx 7.8564LA_0 \end{aligned}$$

It can be seen that this topology gives a better value for the material volume than the previous topology.

Member 3 removed. In this case, the normal forces of members 1 and 2 are

$$\begin{aligned} &\text{LC 1:} \quad N_{11} = 3\sqrt{2}F \qquad N_{21} = -2F \\ &\text{LC 2:} \quad N_{12} = -3\sqrt{2}F \quad N_{22} = 3F \end{aligned}$$

The optimum member areas and the minimum material volume are

$$A_1 = 3\sqrt{2}A_0 \approx 4.2426A_0 \qquad A_2 = 3A_0 \qquad A_3 = 0 \qquad V = 9LA_0$$

From the statically determinate topologies, the one where member 2 is removed gives the lowest material volume. We denote this value by

$$V_0 = \frac{2(6+5\sqrt{3})}{2+\sqrt{3}}LA_0 \approx 7.8564LA_0$$

3.2.2 Statically indeterminate topology

When the sign of the member stresses are taken into account, the material volume minimization problem can be written in the following standard form:

$$\begin{aligned}
\min_{\mathbf{x}} \quad & f(\mathbf{x}) = L(\sqrt{2}A_1 + A_2 + 2A_3) \\
& g_1(\mathbf{x}) = \sigma_{11}(\mathbf{x}) - \overline{\sigma} \leq 0 \\
& g_2(\mathbf{x}) = \sigma_{21}(\mathbf{x}) - \overline{\sigma} \leq 0 \\
& g_3(\mathbf{x}) = -\overline{\sigma} - \sigma_{21}(\mathbf{x}) \leq 0 \\
& g_4(\mathbf{x}) = -\overline{\sigma} - \sigma_{31}(\mathbf{x}) \leq 0 \\
& g_5(\mathbf{x}) = \sigma_{22}(\mathbf{x}) - \overline{\sigma} \leq 0 \\
& g_6(\mathbf{x}) = \sigma_{32}(\mathbf{x}) - \overline{\sigma} \leq 0 \\
& g_7(\mathbf{x}) = -\overline{\sigma} - \sigma_{12}(\mathbf{x}) \leq 0 \\
& g_8(\mathbf{x}) = -\overline{\sigma} - \sigma_{22}(\mathbf{x}) \leq 0 \\
& g_9(\mathbf{x}) = \underline{A} - A_1 \leq 0 \\
& g_{10}(\mathbf{x}) = \underline{A} - A_2 \leq 0 \\
& g_{11}(\mathbf{x}) = \underline{A} - A_3 \leq 0
\end{aligned} \tag{6}$$

To fix the topology, we set a lower bound $\underline{A} = 0.01A_0$ for the member areas. Applying a sequential quadratic programming method gives the following solution:

$$A_1 = 2.40910A_0 \qquad A_2 = 0.19613A_0 \qquad A_3 = 2.11116A_0 \qquad V^* = 7.82542LA_0 \tag{7}$$

At the optimum, the stresses σ_{11} and σ_{32} reach the tensile limit. Thus, the solution is not a fully stressed design. It can be seen that $V^* < V_0$, so the global optimum is attained by the statically indeterminate topology. Note that the solution of Eq. (7) is not necessarily a global optimum. However, the statically determinate topologies cannot give better solutions than the one above.

3.3 Compliance minimization

3.3.1 Problem formulation

The multicriterion formulation for the three-bar truss corresponding to P_C is

$$\begin{aligned} \min_{\mathbf{x}\in D} \quad & \{\, c_1(\mathbf{x}) \quad c_2(\mathbf{x}) \,\} \\ \text{s.t} \quad & \mathbf{K}(\mathbf{x})\mathbf{u}_j = \mathbf{p}_j,\ j=1,2 \\ & V(\mathbf{x}) \le V_{\max} \\ & 0 \le A_i,\ i=1,2,3 \end{aligned} \tag{8}$$

The material volume upper bound is expressed as $V_{\max} = kLA_0$, where $k > 0$. The domain D is as in Eq. (2).

In order to apply the KKT-conditions, we solve the nodal displacements from the stiffness equation and substitute them into the expressions of the compliances. Then the following analytical expressions are obtained:

$$\text{LC 1:} \qquad c_1(\mathbf{x}) = \frac{16\sqrt{2}A_1 + 144A_2 + 4(6+3\sqrt{3})A_3}{4\sqrt{2}A_1A_2 + (2\sqrt{2}+\sqrt{6})A_1A_3 + 6A_2A_3}\frac{F^2L}{E} \tag{9}$$

$$\text{LC 2:} \qquad c_2(\mathbf{x}) = \frac{18(2\sqrt{2}A_1 + 8A_2 + A_3)}{4\sqrt{2}A_1A_2 + (2\sqrt{2}+\sqrt{6})A_1A_3 + 6A_2A_3}\frac{F^2L}{E} \tag{10}$$

The problem formulation can be tidied up by scaling the variables and the criteria. Furthermore, since the material volume constraint is active at all Pareto optimal solutions, one variable can be solved from the equation $V(\mathbf{x}) = V_{\max}$.

The variables are scaled by the factor kA_0, which leads to dimensionless design variables $x_i = A_i/(kA_0)$. Then, the compliances are scaled by the factor $F^2L/(kEA_0)$. Variable x_1 is solved from the active material volume constraint:

$$x_1 = \frac{1}{\sqrt{2}}(1 - x_2 - 2x_3)$$

Substituting this and the other scalings to the expressions of the compliances, Eq. (9) and Eq. (10) yields

$$f_1(x_2,x_3) = \frac{4(-4+(2-3\sqrt{3})x_3 - 32x_2)}{4x_2^2 + (4+2\sqrt{3})x_3^2 + (4+\sqrt{3})x_2x_3 - (2+\sqrt{3})x_3 - 4x_2}$$

$$f_2(x_2,x_3) = \frac{18(-6x_2+3x_3-2)}{4x_2^2 + (4+2\sqrt{3})x_3^2 + (4+\sqrt{3})x_2x_3 - (2+\sqrt{3})x_3 - 4x_2}$$

The remaining constraints are the non-negativity conditions of the variables. Thus, the following standard form is obtained for the minimum compliance problem

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in D'} & \{\, f_1(\mathbf{x}) \quad f_2(\mathbf{x})\,\} \\ & g_1(\mathbf{x}) = \frac{\sqrt{2}}{2}(x_2+2x_3-1) \le 0 \\ & g_2(\mathbf{x}) = -x_2 \le 0 \\ & g_3(\mathbf{x}) = -x_3 \le 0 \end{array} \qquad (P_{st})$$

Here, the set D' is

$$D' = \{\, \mathbf{x} \ge \mathbf{0} \mid g_1 g_2 > 0 \vee g_1 g_3 > 0 \vee g_2 g_3 > 0 \,\}$$

which is equivalent to stating that only one member can be removed from the ground structure in order to keep the truss statically feasible.

3.3.2 Pareto optimal solutions

The KKT-conditions of Problem P_{st} are

$$\alpha_1 \nabla f_1(\mathbf{x}) + \alpha_2 \nabla f_2(\mathbf{x}) + \mu_1 \nabla g_1(\mathbf{x}) + \mu_2 \nabla g_2(\mathbf{x}) + \mu_3 \nabla g_3(\mathbf{x}) = 0 \qquad (11)$$

$$\mu_i g_i(\mathbf{x}) = 0,\ i = 1,2,3 \qquad (12)$$

$$g_i(\mathbf{x}) \le 0,\ i = 1,2,3 \qquad (13)$$

$$\alpha_j \ge 0,\ \mu_i \ge 0,\ (\alpha,\mu) \ne \mathbf{0} \qquad (14)$$

In solving the KKT-conditions, the matter of active constraints is important, since for inactive constraints ($g_i(\mathbf{x}) < 0$), the Lagrange multiplier is zero, that is $\mu_i = 0$, which simplifies the solution of the gradient equation, Eq. (11).

Finding solutions of the KKT-conditions involves the following steps. First, the active constraints are chosen and the Lagrange multipliers of the inactive constraints are set to zero. Then, the gradient equation Eq. (11) is considered. In this equation, some of the remaining Lagrange multipliers may be zero, so each combination of multiplier values must be checked separately. Assuming some Lagrange multipliers to be zero and the others positive, the gradient equation can be solved for the non-zero multipliers or their ratio. Finally, enforcing the non-negativity or positivity condition of the Lagrange multipliers, values of x_2 and x_3 can be found such that the KKT-conditions are satisfied. This procedure is repeated for all feasible combinations of active constraints.

For the three-bar truss problem, there are four cases to be considered: one of the constraints is active (3 cases) or all are inactive (1 case). Note that two constraints cannot be active simultaneously, since this would lead to a truss with only a single member, which cannot support either of the loading conditions.

$g_1 = 0$, $g_2 < 0$, $g_3 < 0$. Now $x_2 + 2x_3 - 1 = 0$. From the positivity of x_2 and x_3 it follows that

$$0 < x_3 < \frac{1}{2}, \quad x_2 = 1 - 2x_3 \quad \text{and} \quad 0 < x_2 < 1$$

Since $\mu_2 = \mu_3 = 0$, the KKT-conditions reduce to

$$\alpha_1 \nabla f_1(\mathbf{x}) + \alpha_2 \nabla f_2(\mathbf{x}) + \mu_1 \nabla g_1(\mathbf{x}) = 0$$

Since both α_1 and α_2 cannot be zero, all possible combinations of zero Lagrange multipliers need to be considered separately.

1. $\mu_1 = 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$.

   Now the KKT-conditions reduce to a pair of linear equations with respect to α_1 and α_2. This system of linear equations has a non-trivial solutions only if the determinant of the matrix, whose columns are the gradients of the criteria, is zero. This leads to the following equations

$$\begin{aligned} f_{1,2}\, f_{2,3} - f_{1,3}\, f_{2,2} &= 0 \\ x_2 &= 1 - 2x_3 \end{aligned}$$

   where the condition $g_1 = 0$ has been employed.

   This pair of equations has no real-valued solutions.

2. $\alpha_1 = 0$, $\mu_1 > 0$, $\alpha_2 > 0$.

   The KKT-conditions reduce to

$$\begin{bmatrix} f_{2,2} & \frac{1}{\sqrt{2}} \\ f_{2,3} & \sqrt{2} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \mu_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   The values of x_3 leading to non-trivial solutions are

$$x_3 = \frac{2}{3} \quad \text{or} \quad x_3 = \frac{2}{5}$$

   of which the latter is feasible. Then $x_2 = \frac{1}{5}$. The KKT-conditions are satisfied, if $\mu_1/\alpha_2 = -\sqrt{2} f_{2,2} > 0$. When $x_2 = 1/5$ and $x_3 = 2/5$ are substituted, we get

$$\frac{\mu_1}{\alpha_2} = -294.5144651 < 0$$

   Thus, the point $(\frac{1}{5}, \frac{2}{5})$ does not satisfy the KKT-conditions.

3. $\alpha_2 = 0$, $\mu_1 > 0$, $\alpha_1 > 0$.

   The gradient equation becomes

$$\begin{bmatrix} f_{1,2} & \frac{1}{\sqrt{2}} \\ f_{1,3} & \sqrt{2} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \mu_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Non-trivial solutions with respect to μ_1 and α_1 are obtained when

$$x_3 = \frac{\sqrt{3} - 9}{\sqrt{3} - 22} \approx 0.3586 \quad \text{or} \quad x_3 = -\frac{15 + \sqrt{3}}{-22 + \sqrt{3}} \approx 0.8255$$

The first solution is feasible. The ratio of the Lagrange multipliers is

$$\frac{\mu_1}{\alpha_1} = -366.4565 < 0$$

so this solution does not satisfy the KKT-conditions.

4. $\alpha_1 > 0$, $\alpha_2 > 0$, $\mu_1 > 0$.

   In this case there are three unknowns and two equations. Therefore, only the ratios $z_1 = \alpha_1/\mu_1$ and $z_2 = \alpha_2/\mu_1$ of the Lagrange multipliers can be solved from

$$\begin{bmatrix} f_{1,2} & f_{2,2} \\ f_{1,3} & f_{2,3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\sqrt{2} \end{bmatrix}$$

   With the condition $x_2 = 1 - 2x_3$, the solutions of this equation are

$$\frac{\alpha_1}{\mu_1} = p_1(x_3) = -\frac{3(-14\sqrt{2}+3\sqrt{6})x_3(30x_3^3 - 47x_3^2 + 24x_3 - 4)}{4(1014x_3^2 - (997+64\sqrt{3})x_3 + 264 + 40\sqrt{3})}$$
$$\frac{\alpha_2}{\mu_1} = p_2(x_3) = \frac{(-317\sqrt{2}+3\sqrt{6})(962x_3^4 - C_1x_3^3 + C_2x_3^2 - C_3x_3)}{962(1014x_3^2 - (997+64\sqrt{3})x_3 + 264 + 40\sqrt{3})}$$

   where

$$C_1 = 1537 + 48\sqrt{3} \qquad C_2 = 792 + 36\sqrt{3} \qquad C_3 = 132 + 6\sqrt{3}$$

   Since all Lagrange multipliers have to be positive, the conditions $p_1 > 0$, $p_2 > 0$ must hold. The simplest way of examining this is to plot p_1 and p_2 in the interval $0 < x_3 < 1/2$ as shown in Fig. 2. From the figure it can be seen that the ratios are never simultaneously positive, which means that all Lagrange multipliers cannot be positive. Thus, $g_1 = 0$ yields no Pareto optimal solutions.

$g_2 = 0$, $g_1 < 0$, $g_3 < 0$. Now $x_2 = 0$, and $0 < x_3 < \frac{1}{2}$. The gradient equation becomes

$$\alpha_1 \nabla f_1(\mathbf{x}) + \alpha_2 \nabla f_2(\mathbf{x}) + \mu_2 \nabla g_2(\mathbf{x}) = 0$$

Again, the combinations of the Lagrange multipliers need to be considered.

1. $\mu_2 = 0$, $\alpha_1 > 0$, $\alpha_2 > 0$.

   The KKT-conditions reduce to a pair of linear equations with respect to α_1 and α_2. Requiring a non-trivial solution and substituting $x_2 = 0$, it can be shown that the equation

$$f_{1,2}\, f_{2,3} - f_{1,3}\, f_{2,2} = 0$$

   has no real-valued solutions.

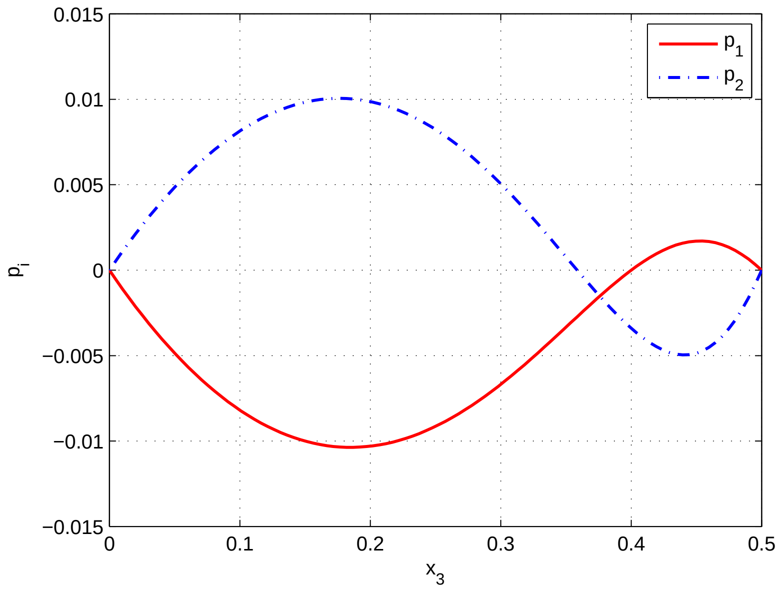

Figure 2: Ratios of the Lagrange multipliers, when $g_1 = 0$.

2. $\alpha_1 = 0$, $\mu_2 > 0$, $\alpha_2 > 0$.

   Now the gradient equation is

$$\begin{bmatrix} f_{2,2} & -1 \\ f_{2,3} & 0 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Requiring non-trivial solutions gives

$$x_3 = 1 \quad \text{or} \quad x_3 = x_3^{\mathrm{B}} = \frac{1}{3}$$

   of which the latter is feasible. Then $x_1 = \frac{\sqrt{2}}{6}$. The KKT-conditions are satisfied, if $\mu_2/\alpha_2 = f_{2,2} > 0$. When $x_2 = 0$ and $x_3 = 1/3$ are substituted, we get

$$\frac{\mu_2}{\alpha_2} = 41.73762175 > 0$$

   Thus, the solution $x_2 = 0$, $x_3 = \frac{1}{3}$ satisfies the KKT-conditions.

   The Pareto optimal solution found is

$$x_1 = \frac{\sqrt{2}}{6} \approx 0.2357 \qquad x_2 = 0 \qquad x_3 = \frac{1}{3} \approx 0.3333$$

   The values of the scaled compliances are

$$f_1 = \frac{12(10+3\sqrt{3})}{2+\sqrt{3}} \approx 48.86156123 \qquad f_2 = \frac{162}{2+\sqrt{3}} \approx 43.40776917$$

   The solution is the minimum of the compliance of LC2.

3. $\alpha_2 = 0$, $\mu_2 > 0$, $\alpha_1 > 0$.

   The gradient equation becomes

$$\begin{bmatrix} f_{1,2} & -1 \\ f_{1,3} & 0 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Non-trivial solutions for α_2 and μ_2 are obtained, when

$$x_3 = x_3^{\mathsf{A}} = \frac{\sqrt{3}-1}{4\sqrt{3}+5} \approx 0.2290412693 \quad \text{or} \quad x_3 = -\left(\frac{9\sqrt{3}+17}{4\sqrt{3}+5}\right)$$

   The first solution is feasible. The ratio of the Lagrange multipliers is

$$\frac{\mu_2}{\alpha_1} = 39.28938110 > 0$$

   so the point is Pareto optimal. The values of the design variables are

$$x_1 = \frac{(3+2\sqrt{3})\sqrt{2}}{2(4\sqrt{3}+5)} \approx 0.38319 \qquad x_2 = 0 \qquad x_3 = \frac{\sqrt{3}-1}{4\sqrt{3}+5} \approx 0.22904$$

   The scaled values of the compliances are

$$f_1 = \frac{4(5+4\sqrt{3})(27+17\sqrt{3})}{(3+2\sqrt{3})(\sqrt{3}+1)(2+\sqrt{3})} \approx 40.86156123$$
$$f_2 = \frac{18(5+4\sqrt{3})(7+5\sqrt{3})}{(3+2\sqrt{3})(\sqrt{3}+1)(2+\sqrt{3})} \approx 51.01546433$$

   This solution is the minimizer of the compliance of LC1.

4. $\alpha_1 > 0$, $\alpha_2 > 0$, $\mu_2 > 0$.

   Again, only the ratios of the Lagrange multipliers, $z_1 = \alpha_1/\mu_2$ and $z_2 = \alpha_2/\mu_2$ can be solved from

$$\begin{bmatrix} f_{1,2} & f_{2,2} \\ f_{1,3} & f_{2,3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   Substituting $x_2 = 0$, we get

$$\frac{\alpha_1}{\mu_2} = p_1(x_3) = \frac{(19+8\sqrt{3})(3x_3^2-4x_3+1)(1-2x_3)x_3}{4(1014x_3^2+(332\sqrt{3}-1155)x_3+408-184\sqrt{3})}$$
$$\frac{\alpha_2}{\mu_2} = p_2(x_3) = -\frac{(34+41\sqrt{3})(46x_3^3+D_1x_3^2-D_2x_3+D_3)x_3}{414(1014x_3^2+(332\sqrt{3}-1155)x_3+408-184\sqrt{3})}$$

where

$$D_1 = 9 + 48\sqrt{3} \qquad D_2 = 24 + 36\sqrt{3} \qquad D_3 = 4 + 6\sqrt{3}$$

p_1 and p_2 are plotted in Fig. 3. It can be seen, that both p_1 and p_2 are positive in the interval $x_3^A < x_3 < x_3^B$. Since the end points of this interval are also Pareto optimal, it can be deduced, that the following points are Pareto optimal, when $g_2 = 0$:

$$x_1 = \frac{1}{\sqrt{2}}(1 - 2x_3) \qquad x_2 = 0 \qquad x_3 \in \left[\frac{7-\sqrt{3}}{23}, \frac{1}{3}\right]$$

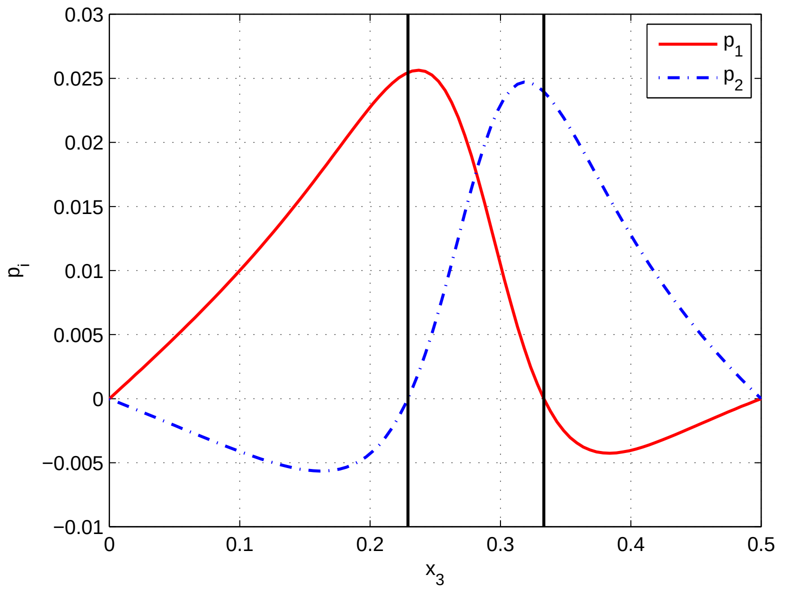

Figure 3: Ratios of the Lagrange multipliers, when $g_2 = 0$.

$g_3 = 0$, $g_1 < 0$, $g_2 < 0$. Now $x_3 = 0$, and $0 < x_2 < 1$. The gradient equation of the KKT-conditions becomes

$$\alpha_1 \nabla f_1(\mathbf{x}) + \alpha_2 \nabla f_2(\mathbf{x}) + \mu_3 \nabla g_3(\mathbf{x}) = 0$$

Again, the combinations of the Lagrange multipliers have to be considered.

1. $\mu_3 = 0$, $\alpha_1 > 0$, and $\alpha_2 > 0$.
   As in the previous cases, the equation

   $$f_{1,2}\, f_{2,3} - f_{1,3}\, f_{2,2} = 0$$

   has no real-valued solutions.

2. $\alpha_1 = 0$, $\mu_3 > 0$, $\alpha_2 > 0$.

   The gradient equation becomes

$$\begin{bmatrix} f_{2,2} & 0 \\ f_{2,3} & -1 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Non-trivial solutions are obtained, when

$$x_2 = -1 \quad \text{or} \quad x_2 = \frac{1}{3}$$

   of which the latter is feasible. The ratio of the Lagrange multipliers is

$$\frac{\mu_3}{\alpha_2} = -226.7220866 < 0$$

   so the KKT-conditions are not satisfied.

3. $\alpha_2 = 0$, $\mu_1 > 0$, $\alpha_1 > 0$.

   The gradient equation becomes

$$\begin{bmatrix} f_{1,2} & 0 \\ f_{1,3} & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Requiring non-trivial solutions for α_1 and μ_3 gives

$$x_2 = -\frac{1}{2} \quad \text{or} \quad x_2 = \frac{1}{4}$$

   of which the latter is feasible. The ratio of the Lagrange multipliers is

$$\frac{\mu_3}{\alpha_1} = -179.1384388 < 0$$

   so the KKT-conditions are not satisfied.

4. $\alpha_1 > 0$, $\alpha_2 > 0$, $\mu_3 > 0$.

   Solving the ratios of the Lagrange multipliers, $z_1 = \alpha_1/\mu_3$ and $z_2 = \alpha_2/\mu_3$ from the equation

$$\begin{bmatrix} f_{1,2} & f_{2,2} \\ f_{1,3} & f_{2,3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   with $x_3 = 0$, gives the following:

$$\frac{\alpha_1}{\mu_3} = p_1(x_2) = \frac{(3\sqrt{3}-14)(3x_2^2+2x_2-1)(x_2-1)x_2}{507x_2^2+(50\sqrt{3}-177)x_2+66+10\sqrt{3}}$$
$$\frac{\alpha_2}{\mu_3} = p_2(x_2) = \frac{4(14-3\sqrt{3})(8x_2^3-6x_2^2-3x_2+1)x_2}{9(507x_2^2+(50\sqrt{3}-177)x_2+66+10\sqrt{3})}$$

   The solutions p_1 and p_2 are plotted in Fig. 4. It can be seen, that both solutions are not positive simultaneously. Therefore, no positive Lagrange multipliers can be found, and there are no Pareto optimal solutions for this combination of multipliers.

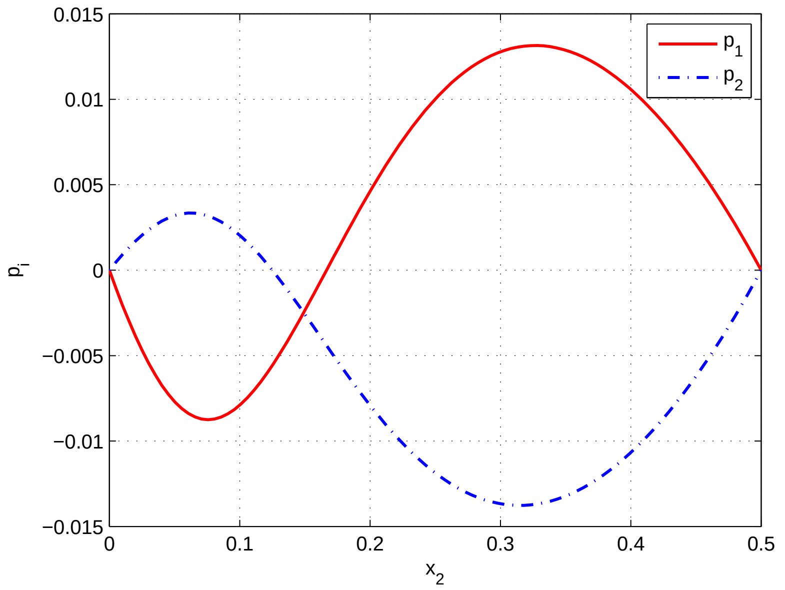

Figure 4: Ratios of the Lagrange multipliers, when $g_3 = 0$.

$g_1 < 0, g_2 < 0, g_3 < 0$. Now $\mu_i = 0$ for all $i = 1,2,3$. The gradient equation reduces to

$$\alpha_1 \nabla f_1(\mathbf{x}) + \alpha_2 \nabla f_2(\mathbf{x}) = 0$$

This equation has a non-trivial solution, if and only if

$$f_{1,2}\, f_{2,3} - f_{1,3}\, f_{2,2} = 0$$

This equation has four solutions, which are denoted by $\hat{x}_3^i = r_i(x_2)$, $i = 1,2,3,4$. Each r_i has to satisfy the following conditions:

i) $\hat{x}_3^i > 0$ (i.e. $x_3 > 0$)

ii) $g_1(x_2, r_i(x_2)) < 0$ (i.e. $x_1 > 0$)

iii) $-f_{2,2}(r_i(x_2), x_2)/f_{1,2}(r_i(x_2), x_2) > 0$

1. $r_1(x_2) = Ax_2 + \frac{1}{4} + \frac{1}{4}\sqrt{Bx_2^2 + Cx_2 + 1}$, where

$$A = \frac{2\sqrt{3}-5}{4} \approx -0.3840 \quad B = 12\sqrt{3} - 27 \approx -6.2154 \quad C = 54 - 23\sqrt{3} \approx 5.5026$$

   This solution is positive, so the condition i) is satisfied. However, $g_1 > 0$ for all $0 < x_2 < 1$, so r_1 gives no feasible solutions.

2. $r_2(x_2) = Ax_2 + \frac{1}{4} + \frac{1}{4}\sqrt{Bx_2^2 + Cx_2 + 1}$.
   This solution does not satisfy condition i).

3.

$$r_3(x_2) = Ax_2 + \frac{1155 - 332\sqrt{3}}{2028} + \frac{1}{2028}\sqrt{Dx_2^2 + Ex_2 + F}$$
$$r_4(x_2) = Ax_2 + \frac{1155 - 332\sqrt{3}}{2028} - \frac{1}{2028}\sqrt{Dx_2^2 + Ex_2 + F}$$

where

$$D = 3084588\sqrt{3} - 6940323 \qquad E = 301158 - 468468\sqrt{3} \qquad F = 9849 - 20616\sqrt{3}$$

These solutions are not real-valued for $0 < x_2 < 1$.

3.3.3 Summary

The Pareto optimal solutions are

$$x_1 = \frac{1}{\sqrt{2}}(1 - 2x_3) \qquad x_2 = 0 \qquad x_3 \in \left[\frac{7 - \sqrt{3}}{23}, \frac{1}{3}\right]$$

This means that only the statically determinate topology, where member 2 is removed, produces Pareto optimal designs.

4 Unequal allowable stresses in tension and in compression

In the previous section, there was only a single Pareto optimal topology. In order to study the more common case, where several Pareto optimal topologies appear, consider the truss depicted in Fig. 5. The ground structure is subject to two loading conditions. In loading condition LC1, a vertical load with magnitude F and a horizontal load acting to the right with magnitude $F/10$ are applied. In loading condition LC2, only the horizontal load acting to the left with magnitude F is applied.

Except for the loading conditions, the structure is identical to the one in the previous section. In the following, the minimum weight and minimum compliance designs are solved.

4.1 Minimum weight design

The stiffness and stress matrices are identical to the ones in the previous section in Eq. (3) and Eq. (5), respectively. In this case, however, the stress bounds for all members are $\overline{\sigma}_i = \frac{F}{A_0}$, and $\underline{\sigma}_i = -\overline{\sigma}_i/10$.

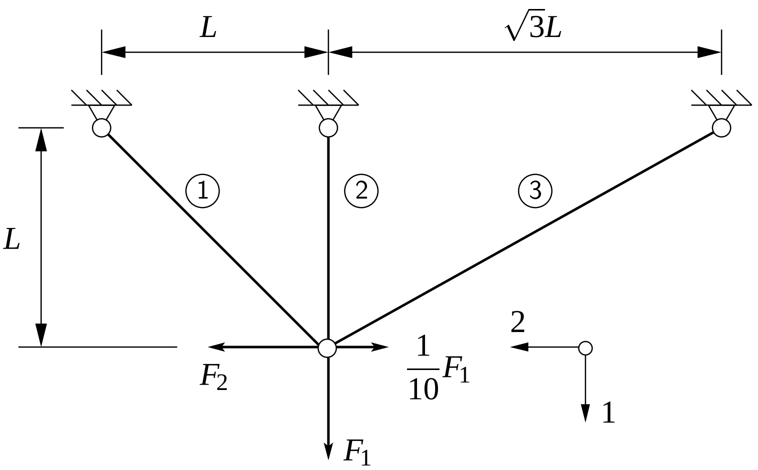

Figure 5: Three-bar truss. The loading conditions are: LC 1: $F_1 = F$, $F_2 = 0$; LC 2: $F_1 = 0$, $F_2 = F$.

The nodal displacements are

$$\text{LC1:} \quad u_{11}(\mathbf{x}) = \frac{18\sqrt{2}A_1 + (30+\sqrt{3})A_3}{5d(A_1,A_2,A_3)}\frac{FL}{E}$$

$$u_{21}(\mathbf{x}) = \frac{18\sqrt{2}A_1 - 8A_2 - (10\sqrt{3}+1)A_3}{5d(A_1,A_2,A_3)}\frac{FL}{E}$$

$$\text{LC2:} \quad u_{12}(\mathbf{x}) = \frac{4\sqrt{2}A_1 - 2\sqrt{3}A_3}{d(A_1,A_2,A_3)}\frac{FL}{E}$$

$$u_{22}(\mathbf{x}) = \frac{4\sqrt{2}A_1 + 16A_2 + 2A_3}{d(A_1,A_2,A_3)}\frac{FL}{E}$$

where

$$d(A_1,A_2,A_3) = 4\sqrt{2}A_1A_2 + (2\sqrt{2}+\sqrt{6})A_1A_3 + 6A_2A_3$$

The member stresses are

$$\text{LC1:} \quad \sigma_{11}(\mathbf{x}) = \frac{8A_2 + (31+11\sqrt{3})A_3}{10d(A_1,A_2,A_3)}F \qquad \text{(tension)}$$

$$\sigma_{21}(\mathbf{x}) = \frac{18\sqrt{2}A_1 + (30+\sqrt{3})A_3}{5d(A_1,A_2,A_3)}F \qquad \text{(tension)}$$

$$\sigma_{31}(\mathbf{x}) = \frac{(9\sqrt{2}+9\sqrt{6})A_1 - 4\sqrt{3}A_2}{10d(A_1,A_2,A_3)}F$$

$$\text{LC2:} \quad \sigma_{12}(\mathbf{x}) = -\frac{8A_2 + (1+\sqrt{3})A_3}{d(A_1,A_2,A_3)}F \qquad \text{(compression)}$$

$$\sigma_{22}(\mathbf{x}) = \frac{4\sqrt{2}A_1 - 2\sqrt{3}A_3}{d(A_1,A_2,A_3)}F$$

$$\sigma_{32}(\mathbf{x}) = \frac{\sqrt{2}(1+\sqrt{3})A_1 + 4\sqrt{3}A_2}{d(A_1,A_2,A_3)}F \qquad \text{(tension)}$$

The minimum material volume can be found by solving the linear programming problem P_{LP}, where member normal forces and areas are the variables and nodal equilibrium equations are the constraints. This formulation, where the compatibility conditions have been neglected, is

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{z} \\ \text{s.e.} \quad & \mathbf{C}\mathbf{z} = \mathbf{p} \\ & \mathbf{R}\mathbf{z} \leq \mathbf{0} \\ & 0 \leq A_i \end{aligned} \tag{P_L}$$

where

$$\mathbf{z} = \{\, A_1\ A_2\ A_3\ N_{11}\ N_{21}\ N_{31}\ N_{12}\ N_{22}\ N_{32} \,\} \in \mathbb{R}^9$$
$$\mathbf{c} = \{\, \sqrt{2}\ 1\ 2\ 0\ 0\ 0\ 0\ 0\ 0 \,\} L \in \mathbb{R}^9$$
$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B} \end{bmatrix} \in \mathbb{R}^{4\times 9} \quad \mathbf{B} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{2\times 3}$$
$$\mathbf{R} = \begin{bmatrix} -\overline{\sigma}\mathbf{I} & \mathbf{I} & \mathbf{0} \\ \underline{\sigma}\mathbf{I} & -\mathbf{I} & \mathbf{0} \\ -\overline{\sigma}\mathbf{I} & \mathbf{0} & \mathbf{I} \\ \underline{\sigma}\mathbf{I} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \in \mathbb{R}^{12\times 9} \quad \mathbf{p} = F \begin{bmatrix} -\frac{1}{10} \\ 1 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^4$$

The matrix **B** is obtained from the force equilibrium equations of the loaded node. The rows of matrix **R** are obtained from the stress constraints as follows:

$$\sigma_{ij} \leq \overline{\sigma} \;\Rightarrow\; \frac{N_{ij}}{A_i} \leq \overline{\sigma} \;\Rightarrow\; N_{ij} \leq \overline{\sigma} A_i \;\Rightarrow\; -\overline{\sigma} A_i + N_{ij} \leq 0$$
$$\sigma_{ij} \geq \underline{\sigma} \;\Rightarrow\; \frac{N_{ij}}{A_i} \geq \overline{\sigma} \;\Rightarrow\; N_{ij} \geq \overline{\sigma} A_i \;\Rightarrow\; \overline{\sigma} A_i - N_{ij} \leq 0$$

The solution of problem P_L is

$$A_1 = 0 \qquad A_2 = \frac{10}{\sqrt{3}} A_0 \approx 5.7735 A_0 \qquad A_3 = \frac{2}{\sqrt{3}} A_0 \approx 1.1547 A_0 \qquad V = \frac{14}{\sqrt{3}} L A_0 \approx 8.0829 L A_0$$

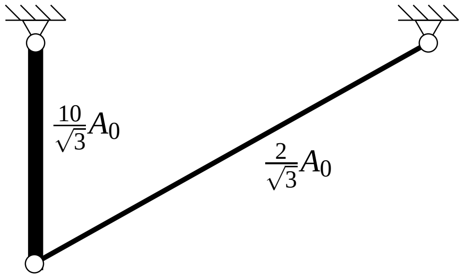

Figure 6: Minimum weight design.

At the optimum, the member stresses are

$$\begin{aligned}
\text{LC1:}\quad & \sigma_{11} = \frac{71\sqrt{3}+33}{600}\frac{F}{A_0} & \sigma_{21} &= \frac{10\sqrt{3}+1}{100}\frac{F}{A_0} & \sigma_{31} &= -\frac{1}{10}\frac{F}{A_0} \\
& \approx 0.2600\frac{F}{A_0} & &\approx 0.1832\frac{F}{A_0} & & \\
\text{LC2:}\quad & \sigma_{12} = -\frac{3+41\sqrt{3}}{60}\frac{F}{A_0} & \sigma_{22} &= -\frac{1}{10}\frac{F}{A_0} & \sigma_{32} &= \frac{F}{A_0} \\
& \approx -1.2334\frac{F}{A_0} & & & &
\end{aligned}$$

The stresses indicate, that the area of member 2 is determined by LC1 and the area of member 3 by LC2. There seems to be non-zero stress in the removed member 1. This false result is due to the discontinuity of the stress function at zero cross-sectional area [4].

Since the optimum solution is statically determinate, it satisfies the compatibility conditions and it is therefore also the global optimum of Problem P_V. However, it is still possible that the other topologies could give the same minimum material volume.

The uniqueness of the solution of an LP problem in standard form can be investigated by the following test [2]. Let

$$\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\
\text{such that} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned} \tag{P_0}$$

and let $\mathbf{x}^*$ be the solution, with $z^* = \mathbf{c}^{\mathrm{T}}\mathbf{x}^*$. Denote by T the indices of the non-basic variables at $\mathbf{x}^*$. We solve the following problem:

$$\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{d}^{\mathrm{T}}\mathbf{x} \\
\text{such that} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{c}^{\mathrm{T}}\mathbf{x} = z^* \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned} \tag{P_0'}$$

where $d_j = -1$, if $j \in T$ and $d_j = 0$ otherwise. The maximum of Problem P_0' is 0, and it is obtained at $\mathbf{x}^*$. All other basic feasible solutions have $x_j > 0$ for some $j \in T$, and in this case the objective function value decreases. Hence, if the minimum of Problem P_0' equals 0, then $\mathbf{x}^*$ is the unique optimum of Problem P_0. Otherwise there exist multiple global optima.

When the LP-formulation P_L is transferred into standard form and problem P_0' is solved, it can be seen that the solution is unique. Therefore, we can conclude that a unique globally optimal minimum weight design has been found.

4.2 Compliance minimization

The compliance minimization problem in standard form is as in P_{st}. Only the expressions of the compliances need to be refined to match the loadings of the present case. The expressions of the compliances are

$$\text{LC 1:} \qquad c_1(\mathbf{x}) = \frac{162\sqrt{2}A_1 + 8A_2 + (301+20\sqrt{3})A_3}{50(4\sqrt{2}A_1A_2 + (2\sqrt{2}+\sqrt{6})A_1A_3 + 6A_2A_3)}\frac{F^2L}{E}$$

$$\text{LC 2:} \qquad c_2(\mathbf{x}) = \frac{4\sqrt{2}A_1 + 16A_2 + 2A_3}{4\sqrt{2}A_1A_2 + (2\sqrt{2}+\sqrt{6})A_1A_3 + 6A_2A_3}\frac{F^2L}{E}$$

When the design variables are scaled by kA_0 and the compliances by $F^2L/(kEA_0)$, and the scaled variable x_1 is solved from the active material volume constraint, the following expressions are obtained:

$$f_1(x_2,x_3) = \frac{162 + (-23+20\sqrt{3})x_3 - 154x_2}{50(-4x_2^2 + (-4-2\sqrt{3})x_3^2 + (-4-\sqrt{3})x_2x_3 + (2+\sqrt{3})x_3 + 4x_2)}$$

$$f_2(x_2,x_3) = \frac{12x_2 - 6x_3 + 4}{-4x_2^2 + (-4-2\sqrt{3})x_3^2 + (-4-\sqrt{3})x_2x_3 + (2+\sqrt{3})x_3 + 4x_2}$$

Solving the KKT-conditions follows the same steps as in the previous section.

4.2.1 $g_1 = 0$, $g_2 < 0$, $g_3 < 0$

Now $x_2 + 2x_3 - 1 = 0$. From $x_2 > 0$ and $x_3 > 0$ it follows that

$$0 < x_3 < \frac{1}{2}, \quad x_2 = 1 - 2x_3$$

Furthermore, $\mu_2 = \mu_3 = 0$, so the gradient equation becomes

$$\alpha_1 \nabla f_1(\mathbf{x}) + \alpha_2 \nabla f_2(\mathbf{x}) + \mu_1 \nabla g_1(\mathbf{x}) = 0$$

We investigate all possible values of the Lagrange multipliers.

1. $\mu_1 = 0$, $\alpha_i > 0$.

   Now the equations to be solved are

$$\begin{aligned} f_{1,2}\, f_{2,3} - f_{1,3}\, f_{2,2} &= 0 \\ x_2 &= 1 - 2x_3 \end{aligned}$$

The solutions are

$$x_3 = 1.2159 \quad \text{or} \quad x_3 = -0.5293$$

Neither of these is feasible.

2. $\alpha_1 = 0$, $\mu_1 > 0$, $\alpha_2 > 0$.

   The gradient equation becomes

$$\begin{bmatrix} f_{2,2} & \frac{1}{\sqrt{2}} \\ f_{2,3} & \sqrt{2} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \mu_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Non-trivial solutions are obtained, when

$$x_3 = \frac{2}{3} \quad \text{or} \quad x_3 = \frac{2}{5}$$

   The latter of these is feasible. Then $x_2 = \frac{1}{5}$. The KKT-conditions require that $\mu_1/\alpha_2 = -\sqrt{2} f_{2,2} > 0$. Substituting $x_2 = 1/5$ and $x_3 = 2/5$, we get

$$\frac{\mu_1}{\alpha_2} = -32.72382945 < 0$$

   Therefore the KKT-conditions are not satisfied.

3. $\alpha_2 = 0$, $\mu_1 > 0$, $\alpha_1 > 0$.

   The gradient equation becomes

$$\begin{bmatrix} f_{1,2} & \frac{1}{\sqrt{2}} \\ f_{1,3} & \sqrt{2} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \mu_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Requiring non-trivial solutions with respect to μ_1 and α_1 gives

$$x_3 = \frac{4\sqrt{3} - 2}{55} \approx 0.0896 \quad \text{or} \quad x_3 = -\left(\frac{20\sqrt{3} + 6}{291}\right)$$

   The former is feasible. The ratio of the Lagrange multipliers is then

$$\frac{\mu_1}{\alpha_1} = -6.521271232 < 0$$

   Thus, the KKT-conditions are not satisfied.

4. $\alpha_1 > 0$, $\alpha_2 > 0$, $\mu_1 > 0$.

   In this case we can solve the ratios $z_1 = \alpha_1/\mu_1$ and $z_2 = \alpha_2/\mu_1$ from

$$\begin{bmatrix} f_{1,2} & f_{2,2} \\ f_{1,3} & f_{2,3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\sqrt{2} \end{bmatrix}$$

Employing the condition $x_2 = 1-2x_3$, we get

$$\frac{\alpha_1}{\mu_1} = p_1(x_3) = \frac{45\sqrt{2}(5+\sqrt{3})x_3(30x_3^3-47x_3^2+24x_3-4)}{8(16\sqrt{3}+33-99x_3^2+(42+15\sqrt{3}x_3)}$$
$$\frac{\alpha_2}{\mu_1} = p_2(x_3) = \frac{-3\sqrt{2}(27+7\sqrt{3})x_3(-32010x_3^3+A_1x_3^2+A_2x_3-228+16\sqrt{3})}{15520(16\sqrt{3}+33-99x_3^2+(42+15\sqrt{3}x_3)}$$

where

$$A_1 = 14181+128\sqrt{3}$$
$$A_2 = 1368-96\sqrt{3}$$

The ratios are plotted in Fig. 7, from which it can be seen that all Lagrange multipliers cannot be positive for feasible values of x_3. Therefore, no Pareto optimal solutions are obtained.

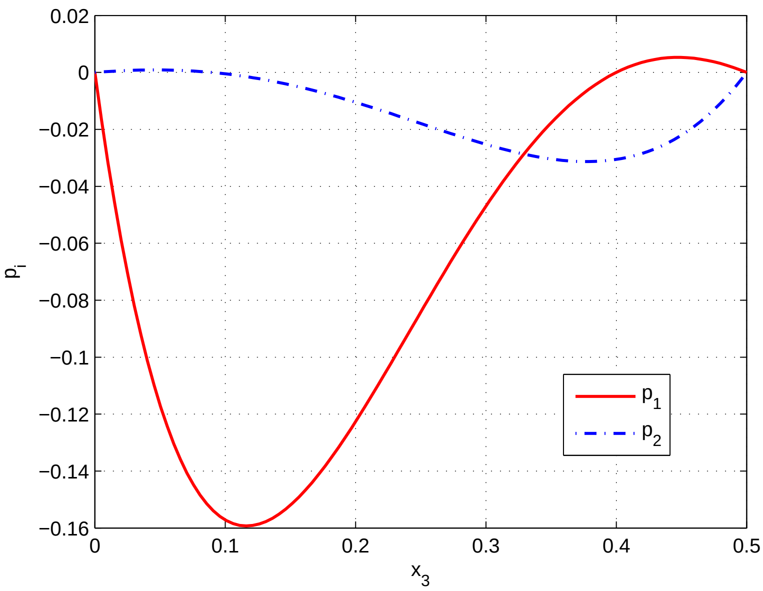

Figure 7: Ratios of the Lagrange multipliers, when $g_1 = 0$.

4.2.2 $g_2 = 0, g_1 < 0, g_3 < 0$

Now $x_2 = 0$, and $0 < x_3 < \frac{1}{2}$. The gradient equation reduces to

$$\alpha_1 \nabla f_1(\mathbf{x}) + \alpha_2 \nabla f_2(\mathbf{x}) + \mu_2 \nabla g_2(\mathbf{x}) = 0$$

Again, we investigate the combinations of Lagrange multipliers.

1. $\mu_2 = 0$, $\alpha_i > 0$.

The gradient equation has a non-trivial solution if and only if

$$f_{1,2}\, f_{2,3} - f_{1,3}\, f_{2,2} = 0$$

The solutions of this equation are

$$x_3 = B + C\sqrt{D} \approx 7.20 \quad \text{or} \quad x_3 = B - C\sqrt{D} \approx 0.3232$$

where

$$B = \frac{139}{44} + \frac{23\sqrt{3}}{66} \approx 3.7627 \qquad C = \frac{1}{132} \approx 0.007576$$
$$D = 123213 + 47868\sqrt{3} \approx 2.0612 \cdot 10^5$$

The latter is in the interval $(0, \frac{1}{2})$. The ratio of the Lagrange multipliers is 0.0822, so the KKT-conditions are satisfied.

We denote this Pareto optimum as point B:

$$x_1^{\mathsf{B}} = \frac{1}{\sqrt{2}} - \sqrt{2}(B - C\sqrt{D}) \approx 0.2500 \qquad x_2^{\mathsf{B}} = 0 \qquad x_3^{\mathsf{B}} = B - C\sqrt{D} \approx 0.3232$$

The compliances at this point are

$$f_1^{\mathsf{B}} = 7.773619182 \qquad f_2^{\mathsf{B}} = 4.831695915$$

2. $\alpha_1 = 0$, $\mu_2 > 0$, $\alpha_2 > 0$.

   The gradient equation becomes

$$\begin{bmatrix} f_{2,2} & -1 \\ f_{2,3} & 0 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Non-trivial solutions are obtained by

$$x_3 = 1 \quad \text{or} \quad x_3 = \frac{1}{3}$$

   The latter is feasible. Then $x_1 = \frac{\sqrt{2}}{6}$. Substituting these values to $\mu_2/\alpha_2 = f_{2,2}$, we get

$$\frac{\mu_2}{\alpha_2} = 4.637513528 > 0$$

   so another KKT-point has been found.

This Pareto optimum is denote point A:

$$x_1^{\mathsf{A}} = \frac{\sqrt{2}}{6} \approx 0.2357 \qquad x_2^{\mathsf{A}} = 0 \qquad x_3^{\mathsf{A}} = \frac{1}{3} \approx 0.3333$$

The values of the compliances at this point are

$$f_1^{\mathsf{A}} = 8.000550504 \qquad f_2^{\mathsf{A}} = 4.823085463$$

This is the minimum of the compliance of LC2.

3. $\alpha_2 = 0$, $\mu_2 > 0$, $\alpha_1 > 0$.

   The gradient equation becomes

$$\begin{bmatrix} f_{1,2} & -1 \\ f_{1,3} & 0 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Non-trivial solutions for α_2 and μ_2 are obtained, when

$$x_3 = \frac{171 - 90\sqrt{3}}{61} \approx 0.2478 \quad \text{or} \quad x_3 = -\frac{153 + 90\sqrt{3}}{11}$$

   The first solution is feasible. The ratio of the Lagrange multipliers is

$$\frac{\mu_2}{\alpha_1} = -45.69771162 < 0$$

   so the KKT-conditions are not satisfied.

4. $\alpha_1 > 0$, $\alpha_2 > 0$, $\mu_2 > 0$.

   Now we solve the ratios $z_1 = \alpha_1/\mu_2$ and $z_2 = \alpha_2/\mu_2$ from

$$\begin{bmatrix} f_{1,2} & f_{2,2} \\ f_{1,3} & f_{2,3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   Substituting $x_2 = 0$, we get

$$\frac{\alpha_1}{\mu_2} = p_1(x_3) = \frac{5(13 + 7\sqrt{3})x_3(6x_3^3 - 11x_3^2 + 6x_3 - 1)}{4(216 - 36\sqrt{3} + 66x_3^2 - (417 + 46\sqrt{3})x_3)}$$

$$\frac{\alpha_2}{\mu_2} = p_2(x_3) = \frac{(11 + 9\sqrt{3})x_3(2x_3 - 1)(671x_3^2 + B_1x_3 - B_2))}{4880(216 - 36\sqrt{3} + 66x_3^2 - (417 + 46\sqrt{3})x_3)}$$

   where

$$B_1 = 7452 + 6480\sqrt{3}$$
$$B_2 = 1863 + 1620\sqrt{3}$$

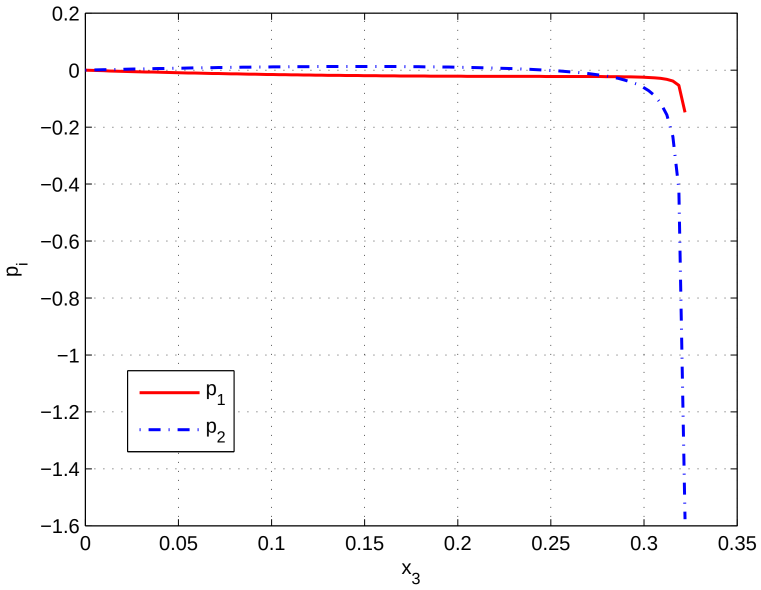

Figure 8: Ratios of the Lagrange multipliers, when $g_2 = 0$, in the interval $(0, B - C\sqrt{D})$.

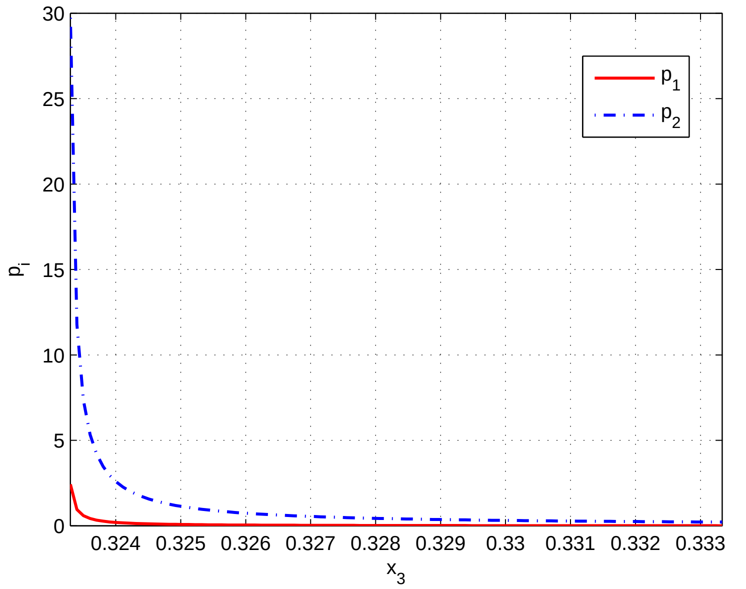

Figure 9: Ratios of the Lagrange multipliers, when $g_2 = 0$, in the interval $(B - C\sqrt{D}, 1/3)$.

These solutions are plotted in Fig. 8 and Fig. 9. It can be seen from these figures, that the ratios are positive at the same time in the open interval $(B - C\sqrt{D}, 1/3)$.

Thus, for the topology, where member 2 is removed, the following points are Pareto optimal:

$$x_1 = \frac{1}{\sqrt{2}}(1 - 2x_3) \qquad x_2 = 0 \qquad x_3 \in \left[B - C\sqrt{D}, \frac{1}{3}\right]$$

4.2.3 $g_3 = 0, g_1 < 0, g_2 < 0$

Now $x_3 = 0$, and $0 < x_2 < 1$. The gradient equation becomes

$$\alpha_1 \nabla f_1(\mathbf{x}) + \alpha_2 \nabla f_2(\mathbf{x}) + \mu_3 \nabla g_3(\mathbf{x}) = 0$$

In the following, we investigate the combinations of the Lagrange multipliers.

1. $\mu_3 = 0$, $\alpha_i > 0$.

   The gradient equation has a non-trivial solution if and only if

   $$f_{1,2}\, f_{2,3} - f_{1,3}\, f_{2,2} = 0$$

   with $x_3 = 0$. The solutions are

   $$x_2 = Q + R\sqrt{S} \approx 6.62 \quad \text{or} \quad x_2 = Q - R\sqrt{S} \approx 0.6564$$

   where

   $$Q = \frac{75 + 26\sqrt{3}}{33} \approx 3.6374 \qquad R = \frac{2}{33} \approx 0.06061$$
   $$S = 1245 + 678\sqrt{3} \approx 2419.3304$$

   The latter is in the interval $(0, 1)$. The ratio of the Lagrange multipliers is 20.47, so the KKT-conditions are satisfied.

   We denote this Pareto optimal solution as point C:

   $$x_1^C = \frac{1}{\sqrt{2}}(1 - (Q - R\sqrt{S})) \approx 0.2423 \qquad x_2^C = Q - R\sqrt{S} \approx 0.6564 \qquad x_3^C = 0$$

   The values of the scaled compliances are

   $$f_1^C = 1.350476870 \qquad f_2^C = 13.16370143$$

2. $\alpha_1 = 0$, $\mu_3 > 0$, $\alpha_2 > 0$.

   The gradient equation becomes

$$\begin{bmatrix} f_{2,2} & 0 \\ f_{2,3} & -1 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Non-trivial solutions with respect to α_2 and μ_3 are obtained, when

$$x_2 = -1 \quad \text{and} \quad x_2 = \frac{1}{3}$$

   The latter is feasible. Then

$$\frac{\mu_3}{\alpha_2} = -25.19134295 < 0$$

   so the KKT-conditions are not satisfied.

3. $\alpha_2 = 0$, $\mu_1 > 0$, $\alpha_1 > 0$.

   The gradient equation is

$$\begin{bmatrix} f_{1,2} & 0 \\ f_{1,3} & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Requiring non-trivial solutions yields

$$x_2 = \frac{9}{7} \quad \text{or} \quad x_2 = \frac{9}{11}$$

   The latter is feasible. The ratio of the Lagrange multipliers is

$$\frac{\mu_3}{\alpha_1} = 2.338945369 > 0$$

   so the KKT-conditions are satisfied.

   This Pareto optimal solution is denoted point D:

$$x_1^{\mathrm{D}} = \frac{\sqrt{2}}{11} \approx 0.1286 \qquad x_2^{\mathrm{D}} = \frac{9}{11} \approx 0.8182 \qquad x_3^{\mathrm{D}} = 0$$

   The scaled compliances at this point are

$$f_1^{\mathrm{D}} = \frac{121}{100} \approx 1.2100 \qquad f_2^{\mathrm{D}} = \frac{209}{9} \approx 23.2222$$

   This point gives the minimum of the compliance of LC1.

4. $\alpha_1 > 0$, $\alpha_2 > 0$, $\mu_3 > 0$.

   The ratios $z_1 = \alpha_1/\mu_3$ and $z_2 = \alpha_2/\mu_3$ of the Lagrange multipliers are solved from

$$\begin{bmatrix} f_{1,2} & f_{2,2} \\ f_{1,3} & f_{2,3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   Substituting $x_3 = 0$ gives

$$\frac{\alpha_1}{\mu_3} = p_1(x_2) = \frac{5(5+\sqrt{3})x_2(x_2-1)(3x_2^2+2x_2-1)}{33x_2^2-(150+52\sqrt{3})x_2+81+36\sqrt{3}}$$
$$\frac{\alpha_2}{\mu_3} = p_2(x_2) = \frac{(5+\sqrt{3})x_2(77x_2^3-239x_2^2+243x_2-81)}{20(33x_2^2-(150+52\sqrt{3})x_2+81+36\sqrt{3})}$$

   This time we investigate the positivity of p_1 and p_2 algebraically in the open interval $(0,1)$. First, note that the nominators have zeros in this interval as follows

$$p_1 = 0 \Rightarrow x_2 = \frac{1}{3} \quad p_2 = 0 \Rightarrow x_2 = \frac{9}{11}$$

   Furthermore, both have a singularity at $x_2 = Q - R\sqrt{S} \approx 0.6564$.

   Since the functions are continuous everywhere except at the singularity, it suffices to calculate their signs at three points, namely between their zeros and the singularity. We choose $x_2 = 0.2$, $x_2 = 0.5$ and $x_2 = 0.9$. In the table below, the first row indicates the interval and the second and third rows are the values of p_1 and p_2 at the chosen points in the corresponding intervals. The sign of p_1 and p_2 in each given interval is determined by the sign at the chosen point.

$(0,1/3)$	$(1/3, Q-R\sqrt{S})$	$(Q-R\sqrt{S}, 9/11)$	$(9/11,1)$
$p_1(0.2) = 0.02674$	$p_1(0.5) = -0.200$	$p_1(0.8) = 0.492$	$p_1(0.9) = 0.213$
$p_2(0.2) = -0.02879$	$p_2(0.5) = -0.051$	$p_2(0.8) = 0.0013$	$p_2(0.9) = -0.0016$

   The only interval, where both p_1 and p_2 are positive is $(Q-R\sqrt{S}, 9/11)$. In this interval, all Lagrange multipliers can take positive values. Therefore, the following points are Pareto optimal for this topology:

$$x_1 = \frac{1}{\sqrt{2}}(1-x_2) \qquad x_2 \in \left[Q-R\sqrt{S}, \frac{9}{11}\right] \qquad x_3 = 0$$

4.2.4 $g_1 < 0, g_2 < 0, g_3 < 0$

Now all Lagrange multipliers corresponding to the constraints are zero, and the gradient equation becomes

$$\alpha_1 \nabla f_1(\mathbf{x}) + \alpha_2 \nabla f_2(\mathbf{x}) = 0$$

Requiring a non-trivial solutions leads to the equation

$$f_{1,2}\, f_{2,3} - f_{1,3}\, f_{2,2} = 0$$

This equation has four solutions, that we denote $\hat{x}_3^i = r_i(x_2)$, $i = 1,2,3,4$. The following conditions must be met:

i) $\hat{x}_3^i > 0$

ii) $x_1(x_2, r_i(x_2)) > 0$

iii) $-f_{1,1}(r_i(x_2), x_2)/f_{1,2}(r_i(x_2), x_2) > 0$

The first and second condition ensure that x_3 and x_1 are positive, and the third condition enforces the Lagrange multipliers to be positive.

1. $r_1(x_2) = Ax_2 + Bk + C\sqrt{Dk^2 + Ekx_2 + Fx_2^2}$, where the constants B, C, and D are as earlier. The other constants are

   $$A = \frac{\sqrt{3}}{2} - \frac{5}{4} \approx -0.3840 \qquad E = 32670 - 8712\sqrt{3} \approx 17580.3734$$
   $$F = 13068\sqrt{3} - 29403 \approx -6768.5600$$

   In the interval $(0,1)$, the condition ii) ($g_1 < 0$) is not satisfied. Therefore, this solution yields no feasible points.

2. $r_2(x_2) = Ax_2 + Bk - C\sqrt{Dk^2 + Ekx_2 + Fx_2^2}$.

   This solution provides feasible points in the interval $(0, B - C\sqrt{D})$. In this interval, there are no zeros or singularities in the ratio of the Lagrange multipliers. Furthermore, the ratio in this interval is positive, so in this interval, the points are Pareto optimal.

3. $r_3(x_2) = Ax_2 + \frac{1}{4} + \frac{1}{4}\sqrt{1 + (54 - 28\sqrt{3})x_2 + (12\sqrt{3} - 27)x_2^2}$.

   Now $r_3 > 0$ is satisfied in $(0, 1/2)$, but $x_1(r_3(x_2), x_2) > 0$ for all $x_2 \in (0, 1/2)$, so this solution is not feasible.

4. $r_4(x_2) = Ax_2 + \frac{1}{4} - \frac{1}{4}\sqrt{1 + (54 - 28\sqrt{3})x_2 + (12\sqrt{3} - 27)x_2^2}$.

   Since $r_4 < 0$ at all points of the interval $(0,1)$, so this solution gives no Pareto optimal points.

The following points are Pareto optimal for this topology:

$$x_1 = \frac{1}{\sqrt{2}}(1 - x_2 - 2x_3)$$
$$x_2 \in \left(0, Q - R\sqrt{S}\right)$$
$$x_3 = Ax_2 + B - C\sqrt{D + Ex_2 + Fx_2^2}$$

4.2.5 Summary

The Pareto optimal solutions and the corresponding minimal points are depicted in Fig. 10 and Fig. 11, respectively. In both figures, scaled variables and compliances are shown.

It is worth noting that the set of Pareto optimal points is connected, which means that the change of optimal topology happens continuously. However, to show that this property holds in general would require a more thorough study.

Finally, the compliances of the stress-constrained minimum weight design, $\mathbf{x}^*$, are computed, and the resulting point is shown is Fig. 11 as point P. The scaled values of the compliances are

$$f_1(\mathbf{x}^*) = \frac{2387}{1500} + \frac{7\sqrt{3}}{75} \approx 1.7530 \qquad f_2(\mathbf{x}^*) = \frac{287}{15} \approx 19.1333$$

This point is clearly dominated.

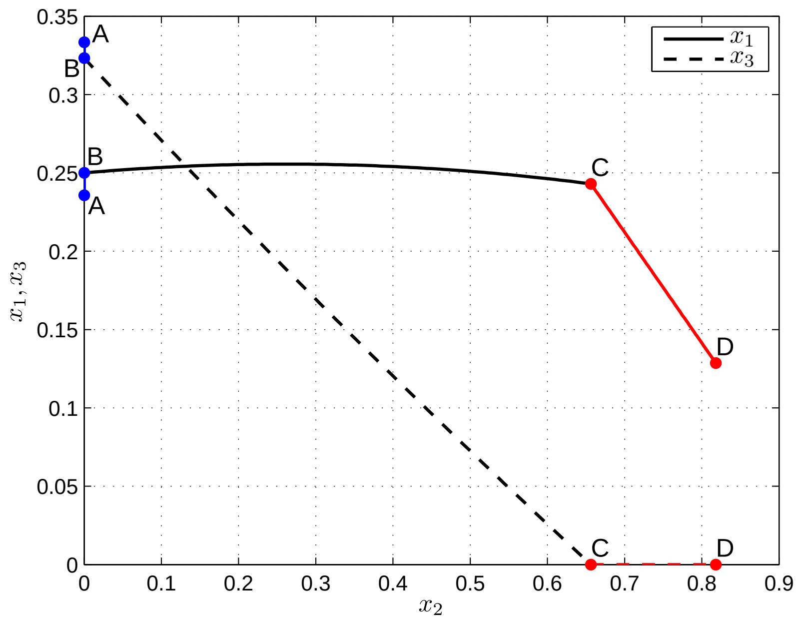

Figure 10: Pareto optimal points of the three-bar truss with unequal allowable stresses in tension and in compression. The labels of the points correspond with the labels in Fig. 11.

5 Conclusions

In this study, a three-bar truss test problem was solved for minimum weight and minimum compliance under multiple loading conditions. In the first case, the allowable stresses in tension and in compression were equal, and in the second case they were unequal. In both cases the stress-constrained minimum weight topology did not coincide with the Pareto optimal topologies of the compliance minimization problem.

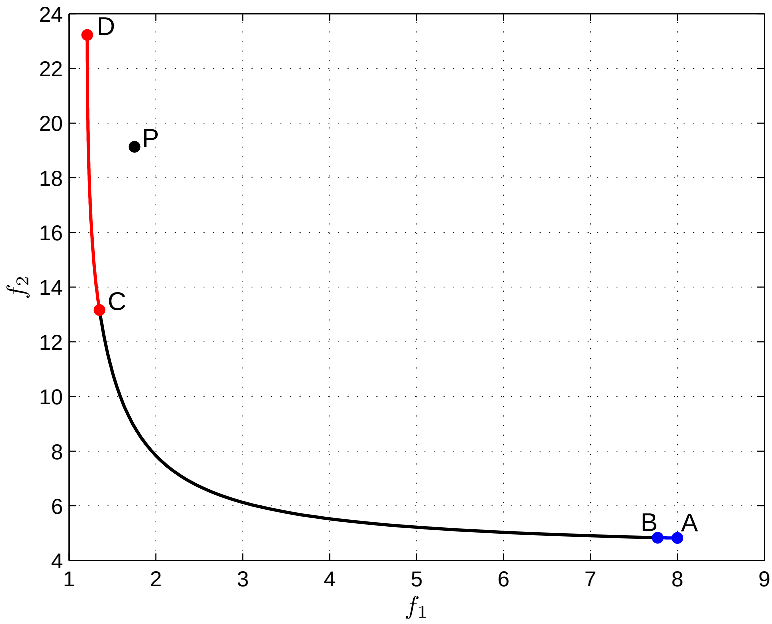

Figure 11: Minimal points of the three-bar truss with unequal allowable stresses in tension and in compression. The optimal topology changes in points B and C. The minimum weight solution (point P) is clearly dominated.

Since all the problems have been solved in detail using analytical expressions, they provide reliable test cases for future research, where different aspects of truss topology optimization are studied.

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