Author(s) Paunonen, Lassi; Zwart, Hans

Title A Lyapunov approach to strong stability of semigroups

Citation Paunonen, Lassi; Zwart, Hans 2013. A Lyapunov approach to strong stability of semigroups. Systems and Control Letters vol. 62, num. 8, 673-678.

Year 2013

DOI http://dx.doi.org/10.1016/j.sysconle.2013.05.001

Version Post-print

URN http://URN.fi/URN:NBN:fi:tty-201312161489

A Lyapunov Approach to Strong Stability of Semigroups

L. Paunonen[a], H. Zwart[b]

[a] *Department of Mathematics, Tampere University of Technology. PO. Box 553, 33101 Tampere, Finland.*
[b] *Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands.*

Abstract

In this paper we present Lyapunov based proofs for the well-known Arendt-Batty-Lyubich-Vũ Theorem for strongly continuous and discrete semigroups. We also study the spectral properties of the limit isometric groups used in the proofs.

Keywords: Strongly continuous semigroup, strong stability, Lyapunov equation.
2010 MSC: 47D06, 93D20, 39B42

1. Introduction

In this paper we study the asymptotic behavior of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X. Recall that $(T(t))_{t\geq 0}$ is exponentially stable if there exist $M>0$ and $\omega>0$ such that $\|T(t)\|\leq Me^{-\omega t}$ for $t\geq 0$. It is strongly stable if for all $x_0\in X$ we have $T(t)x_0\to 0$ for $t\to\infty$. There are several well-known characterizations for exponential stability of a semigroup on a Hilbert space. Characterizing strong stability of a semigroup is also possible [1, Thm 3.1], but the known characterizations are usually difficult to verify in practical applications. Because of this, the strong stability of a semigroup is usually shown using more easily applicable sufficient conditions. The most well-known of such sufficient conditions is the Arendt-Batty-Lyubich-Vũ Theorem, [2, 3].

Theorem 1 (Arendt-Batty-Lyubich-Vũ Theorem)**.** *Let $(T(t))_{t\geq 0}$ be a uniformly bounded semigroup on a Hilbert space. If $\sigma_p(A)\cap i\mathbb{R}=\varnothing$ and if the intersection $\sigma(A)\cap i\mathbb{R}$ is countable, then $(T(t))_{t\geq 0}$ is strongly stable.*

The above result is also valid in reflexive Banach spaces, see [4, Corollary V.2.22]. In this case there is an additional condition that A may have no residual spectrum on the imaginary axis.

Although there are several proofs available, none of them make use of Lyapunov techniques. Lyapunov functions are common in proving stability of nonlinear differential equations, but they can also be used to characterize stability properties of semigroups, see [5, Theorem 5.1.3] for exponential stability. In [6] the following necessary and sufficient condition for strong stability is given.

Theorem 2. *Let $(T(t))_{t\geq 0}$ be a uniformly bounded semigroup on the Hilbert space X and let A with domain $\mathcal{D}(A)$ be the infinitesimal generator of $(T(t))_{t\geq 0}$. Then $(T(t))_{t\geq 0}$ is strongly stable if and only if the (unique) solutions of the Lyapunov equations*

$$\langle x_1, Q_\sigma(A-\sigma I)x_2\rangle+\langle (A-\sigma I)x_1, Q_\sigma x_2\rangle=-\langle x_1,x_2\rangle, \quad (1)$$

with $\sigma>0$, and $x_1,x_2\in\mathcal{D}(A)$, satisfy

$$\lim_{\sigma\downarrow 0}\langle x,\sigma Q_\sigma x\rangle=0, \qquad x\in X. \quad (2)$$

The theorem shows that it is possible to characterize strong stability on a Hilbert space using Lyapunov functions. Unfortunately, solving the Lyapunov equations (1) is not an easy task. Because of this, applying Theorem 2 will be difficult in many practical situations. However, we will see that this characterization can be used to prove the ABLV Theorem.

In particular, the use of Lyapunov equations clarifies the construction of the *limit isometric group* used in studying the stability of semigroups, and in particular, in the proof of the ABLV Theorem [4, Sec. IV.2b], [3, 1]. We show that the abstract *Banach limit* used in the construction of the limit isometric group can be replaced with a positive operator determined by the Lyapunov equations (1). This modification allows us to show a concrete spectral relationship between the generators of the limit isometric group and of the original semigroup $(T(t))_{t\geq 0}$.

In this paper we also apply Lyapunov techniques to present a proof for the analogue of Theorem 1 for discrete semigroups $(A^n)_{n\in\mathbb{N}}$ with $A\in\mathcal{L}(X)$ on a Hilbert space X.

In Section 2 we give some more properties of the Lyapunov equation (1) and construct the limit isometric group. In Section 3 we study its spectral properties and present the new proof of Theorem 1. The Arendt-Batty-Lyubich-Vũ Theorem for discrete semigroups is proved in Section 4.

Email addresses: `lassi.paunonen@tut.fi` (L. Paunonen), `h.j.zwart@utwente.nl` (H. Zwart)

It is straightforward to show that any strongly stable semigroup must be uniformly bounded. Hence, throughout this paper we assume that the semigroup $(T(t))_{t\geq 0}$ on the separable Hilbert space X is uniformly bounded. The generator of $(T(t))_{t\geq 0}$ is denoted by $A : \mathcal{D}(A) \subset X \to X$.

2. The Lyapunov Equations and the Limit Isometric Group

From [6] we have the following results concerning the solutions of the Lyapunov equations in Theorem 2.

Lemma 3. *Let M be such that $\|T(t)\| \leq M$ for $t \geq 0$, and let Q_σ be the solution of the Lyapunov equation (1). Then Q_σ is a bounded, self-adjoint, non-negative operator, and is given by*

$$\langle x, Q_\sigma x\rangle = \int_0^\infty e^{-2\sigma t}\|T(t)x\|^2 dt. \tag{3}$$

Furthermore, the norm of Q_σ satisfies

$$\|Q_\sigma\| \leq \frac{M}{2\sigma}. \tag{4}$$

By (4) and [5, Thm. A.3.39] we have that for a (positive) sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ with $\sigma_n \to 0$ there exists a bounded linear operator Q such that for all $x_1, x_2 \in X$

$$\langle x_1, Qx_2\rangle = \lim_{n\to\infty} \sigma_n \langle x_1, Q_{\sigma_n} x_2\rangle. \tag{5}$$

Lemma 4. *The operator Q as defined in (5) is self-adjoint, non-negative and satisfies*

$$\langle x_1, QAx_2\rangle + \langle Ax_1, Qx_2\rangle = 0, \qquad x_1, x_2 \in \mathcal{D}(A). \tag{6}$$

Furthermore,

$$\|Q^{\frac{1}{2}}T(t)x\| = \|Q^{\frac{1}{2}}x\|, \qquad x \in X, t \geq 0; \tag{7}$$

and for the kernel of Q we have

$$\mathcal{N}(Q) = \{x \in X \mid \lim_{t\to\infty} T(t)x = 0\}. \tag{8}$$

Proof. The operator theoretic properties of Q follow directly from the corresponding properties of Q_σ and (5). Multiplying equation (1) (with $\sigma = \sigma_n$) by σ_n and using (5) gives (6). For all $x \in \mathcal{D}(A)$ and $t > 0$ we have by (6) that

$$\begin{aligned}
&\|Q^{\frac{1}{2}}T(t)x\|^2 - \|Q^{\frac{1}{2}}x\|^2 = \langle T(t)x, QT(t)x\rangle - \langle x, Qx\rangle \\
&= \int_0^t \Big(\frac{d}{ds}\langle T(s)x, QT(s)x\rangle\Big) ds \\
&= \int_0^t \Big(\langle T(s)x, QAT(s)x\rangle + \langle AT(s)x, QT(s)x\rangle\Big) ds = 0.
\end{aligned}$$

Since $\mathcal{D}(A)$ is dense in X, this implies (7).

The inclusion "$\supset$" in (8) follows from (7). So it remains to show that the converse inclusion holds. Let $x_0 \in \mathcal{N}(Q)$ and define $t_n = \sigma_n^{-1}$. Then

$$\begin{aligned}
t_n\langle y, e^{-\sigma_n t_n}T(t_n)x_0\rangle &= \int_0^{t_n} \langle y, e^{-\sigma_n t_n}T(t_n)x_0\rangle ds \\
&= \int_0^{t_n} \langle e^{-\sigma_n(t_n-s)}T(t_n-s)^* y, e^{-\sigma_n s}T(s)x_0\rangle ds \\
&\leq \sqrt{\int_0^{t_n} \|e^{-\sigma_n(t_n-s)}T(t_n-s)^*y\|^2 ds} \\
&\quad\times \sqrt{\int_0^{t_n} \|e^{-\sigma_n s}T(s)x_0\|^2 ds} \\
&\leq \frac{M\|y\|}{\sqrt{2\sigma_n}}\sqrt{\langle x_0, Q_{\sigma_n}x_0\rangle},
\end{aligned}$$

where we have used the uniform bound of the semigroup and relation (3). Since $\sigma_n t_n = 1$ by definition, we have

$$\begin{aligned}
\|T(t_n)x_0\| &= \sup_{y\neq 0} \frac{\langle y, T(t_n)x_0\rangle}{\|y\|} \\
&= \sup_{y\neq 0} \frac{1}{\|y\|}\sigma_n t_n e\langle y, e^{-\sigma_n t_n}T(t_n)x_0\rangle \\
&\leq \frac{Me}{\sqrt{2}}\sqrt{\sigma_n}\sqrt{\langle x_0, Q_{\sigma_n}x_0\rangle}.
\end{aligned} \tag{9}$$

Let $\varepsilon > 0$ be given. Since $x_0 \in \mathcal{N}(Q)$, by (5) there exists an N such that for $n > N$ we have $\sigma_n\langle x_0, Q_{\sigma_n}x_0\rangle \leq \varepsilon^2$. Since $\sigma_n \to 0$, we also have $t_n \to \infty$. By (9) we see that $\|T(t_n)x_0\| \leq \frac{M}{\sqrt{2}}\varepsilon$ for all $n > N$. Hence $T(t)x_0$ converges to zero along the unbounded sequence t_n. Since $(T(t))_{t\geq 0}$ is uniformly bounded, we conclude that $\lim_{t\to\infty} T(t)x_0 = 0$. □

The characterization for the kernel of the limit operator Q in Lemma 4 shows that the uniformly bounded semigroup $(T(t))_{t\geq 0}$ is strongly stable if and only if $Q = 0$.

We denote by $j : X \to X/\mathcal{N}(Q)$ the canonical quotient map. The space $X/\mathcal{N}(Q)$ is a Banach space with the norm $\|\cdot\|_{X/\mathcal{N}(Q)}$ defined by

$$\|j(x)\|_{X/\mathcal{N}(Q)} = \operatorname{dist}(x, \mathcal{N}(Q)), \qquad x \in X.$$

We define the inner product and induced norm on $X/\mathcal{N}(Q)$ by

$$\begin{aligned}
\langle j(x), j(y)\rangle_Q &= \langle x, Qy\rangle_X, \\
\|j(x)\|_Q^2 &= \langle x, Qx\rangle_X = \|Q^{1/2}x\|_X^2.
\end{aligned}$$

If we decompose $X = \mathcal{N}(Q) \oplus \mathcal{N}(Q)^\perp$ into orthogonal subspaces and write $x = x_0 + x_1$ according to this decomposition, then we can see that the norm $\|\cdot\|_Q$ satisfies

$$\begin{aligned}
\|j(x)\|_Q &= \|Q^{1/2}x\|_X = \|Q^{1/2}x_1\|_X \\
&\leq \|Q^{1/2}\| \cdot \|x_1\|_X = \|Q^{1/2}\| \cdot \|j(x)\|_{X/\mathcal{N}(Q)},
\end{aligned}$$

where $\|x_1\|_X = \operatorname{dist}(x, \mathcal{N}(Q)) = \|j(x)\|_{X/\mathcal{N}(Q)}$ follows from the orthogonality of the decomposition $x = x_0 + x_1$.

We define the space X_Q as the completion of $X/\mathcal{N}(Q)$ with respect to the norm $\|\cdot\|_Q$, i.e.

$$X_Q = \overline{X/\mathcal{N}(Q)}^{\|\cdot\|_Q}.$$

It is then immediate that $(X_Q, \|\cdot\|_Q)$ is a Hilbert space.

By (7) or (8) we see that the kernel of Q is $T(t)$-invariant for all $t \geq 0$ and thus we may consider the quotient semigroup $(T_0(t))_{t\geq 0}$ on $X/\mathcal{N}(Q)$ defined by [4, Par. I.5.13]

$$T_0(t)j(x) = j(T(t)x) \qquad \forall x \in X.$$

Lemma 4 now implies that for any $x \in X$ we have

$$\begin{aligned}\|T_0(t)j(x)\|_Q^2 &= \|j(T(t)x)\|_Q^2 = \|Q^{1/2}T(t)x\|_X^2 \\ &= \|Q^{1/2}x\|_X^2 = \|j(x)\|_Q^2.\end{aligned}$$

This shows that $(T_0(t))_{t\geq 0}$ is an isometric semigroup on $X/\mathcal{N}(Q)$. Finally, we complete this construction by extending the quotient semigroup to a semigroup $(T_Q(t))_{t\geq 0}$ on the whole space X_Q. The fact that the quotient semigroup is isometric on $X/\mathcal{N}(Q)$ implies that also its extension to X_Q isometric.

Corollary 5. *The semigroup $(T_Q(t))_{t\geq 0}$ is isometric.*

The construction of the semigroup $(T_Q(t))_{t\geq 0}$ is similar to the one carried out in [1, Sec. 2]. The main difference is that instead of using the abstract concept of Banach limit, we have used the operator Q to divide the space X into parts based on whether or not the orbits $t \mapsto T(t)x$ originating from the elements $x \in X$ decay to zero asymptotically.

We will now turn to the study of the properties of the semigroup $(T_Q(t))_{t\geq 0}$ and its generator.

2.1. Properties of the Semigroup $(T_Q(t))_{t\geq 0}$

Our main interests in this section are the spectral properties of the generator of the semigroup $(T_Q(t))_{t\geq 0}$. In particular we show that if $\rho(A) \cap i\mathbb{R} \neq \varnothing$, then $(T_Q(t))_{t\geq 0}$ extends to an isometric group and thus by Stone's theorem its generator A_Q is a skew-adjoint operator. The spectral theory of skew-adjoint operators allows us to obtain information on the stability properties of original uniformly bounded semigroup $(T(t))_{t\geq 0}$.

We will start with the following auxiliary lemma.

Lemma 6. *The generator A_Q of $(T_Q(t))_{t\geq 0}$ has the property that $j(\mathcal{D}(A)) \subset \mathcal{D}(A_Q)$ and $A_Q j(x) = j(Ax)$ for all $x \in \mathcal{D}(A)$. Furthermore, $j(\mathcal{D}(A))$ is a core of A_Q.*

Proof. The first claim follows directly from the fact that for any $x \in \mathcal{D}(A) \subset X$ we have

$$\begin{aligned}&\left\| \frac{T_Q(t)j(x) - j(x)}{t} - j(Ax) \right\|_Q \\ &= \left\| j\left(\frac{T(t)x - x}{t}\right) - j(Ax) \right\|_Q \\ &= \left\| Q^{1/2}\left(\frac{T(t)x - x}{t} - Ax\right) \right\|_X \\ &\leq \|Q^{1/2}\| \cdot \left\| \frac{T(t)x - x}{t} - Ax \right\|_X \to 0\end{aligned}$$

as $t \downarrow 0$.

It remains to show that $j(\mathcal{D}(A))$ is a core of A_Q. By [4, Prop. II.1.7] it suffices to show that $j(\mathcal{D}(A))$ is $T_Q(t)$-invariant for all $t \geq 0$ and $\|\cdot\|_Q$-dense in X_Q. Let $j(x) \in j(\mathcal{D}(A))$ and $t \geq 0$. Since on the space $X/\mathcal{N}(Q)$ the semigroup $(T_Q(t))_{t\geq 0}$ coincides with the quotient semigroup $(T_0(t))_{t\geq 0}$, we have

$$T_Q(t)j(x) = T_0(t)j(x) = j(T(t)x) \in j(\mathcal{D}(A)),$$

since $T(t)x \in \mathcal{D}(A)$. To show the density of $j(\mathcal{D}(A))$ in X_Q, let $x_Q \in X_Q$ and $\varepsilon > 0$. Since X_Q is the completion of $X/\mathcal{N}(Q)$, there exists $x \in X$ such that $\|x_Q - j(x)\|_Q < \frac{\varepsilon}{2}$. Furthermore, since $\mathcal{D}(A)$ is $\|\cdot\|_X$-dense in X, there now exists $y \in \mathcal{D}(A)$ such that

$$\|x - y\|_X < \frac{\varepsilon}{2\|Q^{1/2}\|}.$$

Combining these two estimates and using the definition of $\|\cdot\|_Q$ we obtain

$$\begin{aligned}\|x_Q - j(y)\|_Q &\leq \|x_Q - j(x)\|_Q + \|j(x) - j(y)\|_Q \\ &< \frac{\varepsilon}{2} + \|Q^{1/2}(x-y)\|_X < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This concludes that $j(\mathcal{D}(A))$ is a core of A_Q. ☐

Now we can formulate and prove the first result on the relation between the spectra of A and A_Q. In the case where $(T_Q(t))_{t\geq 0}$ extends to a group, the extension $(T_Q(t))_{t\in\mathbb{R}}$ is called the *limit isometric group.*

Theorem 7. *The spectrum of the operator A_Q satisfies $\sigma(A_Q) \subset \sigma(A)$. If $\rho(A) \cap i\mathbb{R} \neq \varnothing$, then $(T_Q(t))_{t\geq 0}$ extends to an isometric group $(T_Q(t))_{t\in\mathbb{R}}$ on X_Q and A_Q is a skew-adjoint operator whose spectrum satisfies $\sigma(A_Q) \subset \sigma(A) \cap i\mathbb{R}$.*

Proof. Let $\lambda \in \rho(A)$ and denote by $R(\lambda) : X/\mathcal{N}(Q) \to X/\mathcal{N}(Q)$ the linear operator defined by

$$R(\lambda)j(x) = j(R(\lambda, A)x), \qquad x \in X. \tag{10}$$

Now for any $x \in X$ we have by (3) and (5) that

$$\begin{aligned}\|R(\lambda)j(x)\|_Q^2 &= \|j(R(\lambda, A)x)\|_Q^2 = \|Q^{1/2}R(\lambda, A)x\|^2 \\ &= \lim_{\sigma_n \to 0} \sigma_n \int_0^\infty e^{-2\sigma_n t}\|T(t)R(\lambda, A)x\|^2 dt \\ &= \lim_{\sigma_n \to 0} \sigma_n \int_0^\infty e^{-2\sigma_n t}\|R(\lambda, A)T(t)x\|^2 dt \\ &\le \|R(\lambda, A)\|^2 \cdot \lim_{\sigma_n \to 0} \sigma_n \int_0^\infty e^{-2\sigma_n t}\|T(t)x\|^2 dt \\ &= \|R(\lambda, A)\|^2\|Q^{1/2}x\|^2 = \|R(\lambda, A)\|^2\|j(x)\|_Q^2.\end{aligned}$$

This shows that $R(\lambda)$ is bounded on $X/\mathcal{N}(Q) \subset X_Q$ and thus it extends to a bounded operator on X_Q. Using (10) and Lemma 6 we can see that for any $x \in X$ and $y \in \mathcal{D}(A)$

$$\begin{aligned}(\lambda I - A_Q)R(\lambda)j(x) &= (\lambda I - A_Q)j(R(\lambda, A)x) \\ &= j((\lambda I - A)R(\lambda, A)x) = j(x) \\ R(\lambda)(\lambda I - A_Q)j(y) &= R(\lambda)j((\lambda I - A)y) \\ &= j(R(\lambda, A)(\lambda I - A)y) = j(y).\end{aligned}$$

Since $X/\mathcal{N}(Q)$ is dense in X_Q and since $j(\mathcal{D}(A))$ is a core of A_Q, the above identities are satisfied on X_Q and $\mathcal{D}(A_Q)$, respectively. This implies $\lambda \in \rho(A_Q)$ and $R(\lambda) = R(\lambda, A_Q)$. This concludes that $\rho(A) \subset \rho(A_Q)$, or equivalently $\sigma(A_Q) \subset \sigma(A)$.

If $\rho(A) \cap i\mathbb{R} \neq \varnothing$, we also have $\rho(A_Q) \cap i\mathbb{R} \neq \varnothing$. This implies that the isometric semigroup $(T_Q(t))_{t\geq 0}$ extends to an isometric group on X_Q [4, Lem. IV.2.19]. By Stone's Theorem [4, Thm. II.3.24] its generator A_Q is a skew-adjoint operator. In particular, $\sigma(A_Q) \subset i\mathbb{R}$. □

Remark 8. It should be noted that if the condition $\rho(A) \cap i\mathbb{R} \neq \varnothing$ is not satisfied, then either $\sigma(A_Q) = i\mathbb{R}$ and $(T_Q(t))_{t\geq 0}$ extends to an isometric group on X_Q, or $\sigma(A_Q) = \overline{\mathbb{C}^-}$ [4, Lem. IV.2.19]. However, even in the latter case it is still possible to extend $(T_Q(t))_{t\geq 0}$ to an isometric group on a larger space $\tilde{X}_Q$, see [1, p. 67] and references therein.

Equation (10) in the proof of the previous theorem also establishes the following relationship between the resolvent operators of A and A_Q.

Lemma 9. *For all $\lambda \in \rho(A)$ we have*

$$R(\lambda, A_Q)j(x) = j(R(\lambda, A)x) \qquad x \in X.$$

The fact that A_Q is a skew-adjoint operator enables us to use the well-known spectral theory of skew-adjoint and self-adjoint operators [7, 8] in the study of the properties of the semigroup $(T(t))_{t\geq 0}$. First of all, Theorem 7 can be used to show the stability of the semigroup $(T(t))_{t\geq 0}$ in the case where the entire imaginary axis belongs to the resolvent set of A.

Theorem 10. *If $(T(t))_{t\geq 0}$ is uniformly bounded and $\sigma(A) \cap i\mathbb{R} = \varnothing$, then $(T(t))_{t\geq 0}$ is strongly stable.*

Proof. By Theorem 7 we have $i\mathbb{R} \subset \rho(A) \subset \rho(A_Q)$. The spectral theory of skew-adjoint operators implies that this is only possible if $X_Q = \{0\}$. This, on the other hand, only possible if $\mathcal{N}(Q) = X$. By Lemma 4 we conclude that $(T(t))_{t\geq 0}$ is strongly stable. □

3. The Arendt-Batty-Lyubich-Vũ Theorem

In this section we present a proof for the Arendt-Batty-Lyubich-Vũ Theorem. In addition, we also show a detailed spectral relationship between the operators A and A_Q. The following lemma is needed in the proofs. In a more general form this result is known as the Mean Ergodic Theorem [9, 4, 10].

Lemma 11. *Let $(S(t))_{t\geq 0}$ be a uniformly bounded semigroup on a Hilbert space Y with infinitesimal generator A_S. Then*

$$Y = \mathcal{N}(A_S) \oplus \overline{\mathcal{R}(A_S)}. \tag{11}$$

If $\mathcal{N}(A_S) = \{0\}$, then for all $y \in Y$ we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(s)y ds = 0. \tag{12}$$

Proof. Since Y is a Hilbert space, Theorem 2.25 in [10] shows that (11) holds.

For $y \in \mathcal{R}(A_S)$, we have that

$$\frac{1}{t}\int_0^t S(s)y ds = \frac{1}{t}\int_0^t S(s)A_S z ds = \frac{1}{t}(S(t)z - z).$$

Combining this with the uniform boundedness of $(S(t))_{t\geq 0}$ we see that (12) holds for $y \in \mathcal{R}(A_S)$. By assumption and (11) this set is dense in Y. By the uniform boundedness of $(S(t))_{t\geq 0}$ we have that the operators $y \mapsto \frac{1}{t}\int_0^t S(s)y ds$ are uniformly bounded. Combining these observations, we conclude that (12) holds for all $y \in Y$. □

For the purpose of proving the ABLV Theorem, the most essential of our spectral results is that the point spectrum of A_Q coincides with $\sigma_p(A) \cap i\mathbb{R}$.

Theorem 12. *The point spectrum of A_Q satisfies $\sigma_p(A_Q) = \sigma_p(A) \cap i\mathbb{R}$.*

Proof. We will first show that $\sigma_p(A) \cap i\mathbb{R} \subset \sigma_p(A_Q)$. Let $i\omega \in \sigma_p(A) \cap i\mathbb{R}$ and let $x \in \mathcal{D}(A)$ be such that $x \neq 0$ and $(A - i\omega I)x = 0$. We have from Lemma 6 that $j(x) \in \mathcal{D}(A_Q)$ and

$$A_Q j(x) = j(Ax) = i\omega j(x).$$

It only remains to show that $j(x) \neq 0$. We have that $T(t)x = e^{i\omega t}x$ for all $t \geq 0$. This immediately implies that

$$\|T(t)x\| = \|x\| \not\to 0 \quad \text{as } t \to \infty.$$

We therefore have from Lemma 4 that $x \notin \mathcal{N}(Q)$ and thus $j(x) \neq 0$.

To prove the converse inclusion, we will first show that $\sigma_p(A_Q)$ is a subset of the imaginary axis. To this end, let $\lambda \in \sigma_p(A_Q)$ and let $x \in \mathcal{D}(A_Q)$ be such that $x \neq 0$ and $(A_Q - \lambda I)x = 0$. In this case we also have $T_Q(t)x = e^{\lambda t}x$ and since the semigroup $(T_Q(t))_{t\geq 0}$ is isometric by Corollary 5, we have

$$\|x\|_Q = \|T_Q(t)x\|_Q = e^{\operatorname{Re}\lambda t}\|x\|_Q, \qquad \forall t \geq 0.$$

Since $x \neq 0$ this can only be satisfied if $\operatorname{Re}\lambda = 0$, and thus $\sigma_p(A_Q) \subset i\mathbb{R}$.

It remains to show that $\sigma_p(A_Q) \subset \sigma_p(A)$. To this end, let $\omega \in \mathbb{R}$ be such that $i\omega \notin \sigma_p(A)$. We will show that this implies $\mathcal{N}(A_Q - i\omega I) = \{0\}$, or equivalently $i\omega \notin \sigma_p(A_Q)$.

Since $A - i\omega I$ is the infinitesimal generator of the uniformly bounded semigroup $(e^{-i\omega t}T(t))_{t\geq 0}$ on the Hilbert space X, and since it is by assumption injective, Lemma 11 shows that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-i\omega s}T(s)x ds = 0, \qquad \forall x \in X. \tag{13}$$

For $x_Q \in \mathcal{N}(A_Q - i\omega I)$ and $\varepsilon > 0$ choose an $x \in X/\mathcal{N}(Q)$ such that $\|x_Q - j(x)\|_Q < \varepsilon/2$. This is possible since $X/\mathcal{N}(Q)$ is $\|\cdot\|_Q$-dense in X_Q. By (13) there exists a $t_\varepsilon > 0$ such that

$$\left\|\frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} e^{-i\omega s}T(s)x ds\right\|_X < \frac{\varepsilon}{2\|Q^{1/2}\|}.$$

The fact that $(A_Q - i\omega I)x_Q = 0$ implies $e^{-i\omega t}T_Q(t)x_Q = x_Q$, and thus

$$\frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} e^{-i\omega s}T_Q(s)x_Q ds = \frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} x_Q ds = x_Q.$$

Using this we get

$$\begin{aligned}
\|x_Q\|_Q &= \left\|\frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} e^{-i\omega s}T_Q(s)x_Q ds\right\|_Q \\
&\leq \left\|\frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} e^{-i\omega s}T_Q(s)(x_Q - j(x)) ds\right\|_Q \\
&\qquad + \left\|\frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} e^{-i\omega s}T_Q(s)j(x) ds\right\|_Q \\
&\leq \frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} \|e^{-i\omega s}T_Q(s)(x_Q - j(x))\|_Q ds \\
&\qquad + \left\|j\left(\frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} e^{-i\omega s}T(s)x ds\right)\right\|_Q \\
&= \frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} \|x_Q - j(x)\|_Q ds \\
&\qquad + \left\|Q^{1/2}\left(\frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} e^{-i\omega s}T(s)x ds\right)\right\|_X \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we must have $x_Q = 0$. Since x_Q was an arbitrary element of $\mathcal{N}(A_Q - i\omega I)$ we find that $i\omega \notin \sigma_p(A_Q)$. This concludes that $\sigma_p(A_Q) \subset \sigma_p(A)$. □

Using the spectral theory of skew-adjoint operators and the above relationship between the eigenvalues of A and A_Q we can easily prove the ABLV Theorem.

Theorem 13. *If $(T(t))_{t\geq 0}$ is a uniformly bounded semigroup such that $\sigma_p(A) \cap i\mathbb{R} = \varnothing$ and the intersection $\sigma(A) \cap i\mathbb{R}$ is countable, then $(T(t))_{t\geq 0}$ is strongly stable.*

Proof. The fact that $\sigma(A) \cap i\mathbb{R}$ is countable implies that $\rho(A) \cap i\mathbb{R} \neq \varnothing$, and since $\sigma(A_Q) \subset \sigma(A) \cap i\mathbb{R}$ by Theorem 7, we have that also $\sigma(A_Q)$ must be countable. All isolated spectral points of a skew-adjoint operator are eigenvalues [11, Sec. V.3.5]. Since A_Q is skew-adjoint and has a countable spectrum, $\sigma(A_Q)$ must consist of eigenvalues of A_Q and their accumulation points. However, if $\lambda \in \sigma_p(A_Q)$, then Theorem 12 would imply $\lambda \in \sigma_p(A) \cap i\mathbb{R} = \varnothing$, which is impossible. This concludes that $\sigma_p(A_Q) = \varnothing$, and

$$\sigma(A_Q) = \overline{\sigma_p(A_Q)} = \varnothing.$$

A skew-adjoint operator can have empty spectrum only if $X_Q = \{0\}$. By construction this is only possible if $Q = 0$, and $\mathcal{N}(Q) = X$. Finally, $\mathcal{N}(Q) = X$ together with Lemma 4 concludes that $(T(t))_{t\geq 0}$ is strongly stable. □

If $\rho(A) \cap i\mathbb{R} \neq \varnothing$, then combining the previous results also leads to the inclusion

$$\sigma_c(A_Q) \subset \sigma_c(A) \cap i\mathbb{R}.$$

Indeed, this is a direct consequence of the observation that due to Lemma 11 we must have $\sigma(A) \cap i\mathbb{R} \subset \sigma_p(A) \cup \sigma_c(A)$, and of the spectral relationships in Theorems 7 and 12. The following corollary collects the relationships between the spectra of A_Q and A. The description of the spectrum of A_Q is now complete, because for a skew-adjoint operator we have $\sigma(A_Q) = \sigma_p(A_Q) \cup \sigma_c(A_Q)$.

Corollary 14. *If $T(t)$ is uniformly bounded and $\rho(A) \cap i\mathbb{R} \neq \varnothing$, then*

$$\sigma(A_Q) \subset \sigma(A) \cap i\mathbb{R}$$
$$\sigma_p(A_Q) = \sigma_p(A) \cap i\mathbb{R}$$
$$\sigma_c(A_Q) \subset \sigma_c(A) \cap i\mathbb{R}.$$

4. The Arendt-Batty-Lyubich-Vũ Theorem for Discrete Semigroups

In this section we use the Lyapunov approach to present a proof for the Arendt-Batty-Lyubich-Vũ Theorem for discrete semigroups [10, Thm. 2.18]. By a discrete semigroup we mean a family $(A^n)_{n\in\mathbb{N}}$ of operators, where $A \in \mathcal{L}(X)$ and $\mathbb{N} = 0, 1, 2, \ldots$. We denote the unit circle of $\mathbb{C}$ by $\mathbb{T}$, and call the semigroup $(A^n)_{n\in\mathbb{N}}$ *power bounded* if $\sup_{n\in\mathbb{N}}\|A^n\| < \infty$.

Theorem 15 (Arendt-Batty-Lyubich-Vũ Theorem)**.** *Let $(A^n)_{n\in\mathbb{N}}$ be a power bounded discrete semigroup on a Hilbert space. If $\sigma_p(A) \cap \mathbb{T} = \varnothing$ and if the intersection $\sigma(A) \cap \mathbb{T}$ is countable, then $(A^n)_{n\in\mathbb{N}}$ is strongly stable.*

We prove Theorem 15 using similar steps as in the proof of Theorem 1. Most notably, we use solutions of Lyapunov equations to construct a discrete limit isometric group $(A_Q^n)_{n\in\mathbb{Z}}$ and use its properties in the proving the strong stability of $(A^n)_{n\in\mathbb{N}}$. Because of the same approach, most of the techniques used in this section are direct counterparts of the ones in Sections 2 and 3.

Lyapunov equations can be used in characterizing the strong stability of a discrete semigroup in the following way.

Theorem 16 ([6, Thm. 3.5])**.** *Let $(A^n)_{n\in\mathbb{N}}$ be a power bounded discrete semigroup on a Hilbert space X. Then $(A^n)_{n\in\mathbb{N}}$ is strongly stable if and only if the (unique) positive solutions Q_r of the Lyapunov equations*

$$r^2\langle Ax_1, Q_rAx_2\rangle - \langle x_1, Q_rx_2\rangle = -\langle x_1, x_2\rangle, \qquad (14)$$

with $r\in(0,1)$, and $x_1, x_2 \in X$, satisfy

$$\lim_{r\uparrow 1}(1-r)\langle x, Q_rx\rangle = 0, \qquad x\in X.$$

We begin the proof of Theorem 15 by constructing the limit isometric group. We have from [6] that if $(A^n)_{n\in\mathbb{N}}$ is power bounded such that $\|A^n\| \le M$ for some $M>0$ and for all $n\in\mathbb{N}$, then for every $r\in(0,1)$ the equation (14) has a bounded, self-adjoint, non-negative solution Q_r satisfying

$$\langle x, Q_rx\rangle = \sum_{k=0}^{\infty}\|r^kA^kx\|^2. \qquad (15)$$

Moreover, the norm of Q_r satisfies $\|Q_r\| \le M^2/(2(1-r))$.

Similarly as in Section 2, Theorem A.3.39 in [5] implies that there exists a sequence $\{r_n\}_{n\in\mathbb{N}} \subset (0,1)$ with $r_n \uparrow 1$ and a self-adjoint non-negative operator Q such that

$$\langle x_1, Qx_2\rangle = \lim_{n\to\infty}(1-r_n)\langle x_1, Q_{r_n}x_2\rangle \qquad (16)$$

for all $x_1, x_2\in X$.

Lemma 17. *The operator Q has the following properties.*

$$\langle Ax_1, QAx_2\rangle - \langle x_1, Qx_2\rangle = 0, \qquad x_1,x_2\in X, \qquad (17)$$

$$\|Q^{\frac{1}{2}}A^nx\| = \|Q^{\frac{1}{2}}x\|, \qquad x\in X, n\in\mathbb{N}, \qquad (18)$$

$$\mathcal{N}(Q) = \{x\in X \mid \lim_{n\to\infty}A^nx = 0\}. \qquad (19)$$

Proof. Equation (17) follows from multiplying equation (14) (with $r=r_n$) by $1-r_n$ and using (16). We now have

$$\|Q^{\frac{1}{2}}Ax\|^2 = \sqrt{\langle Ax, QAx\rangle} = \sqrt{\langle x, Qx\rangle} = \|Q^{\frac{1}{2}}x\|,$$

which further implies $\|Q^{\frac{1}{2}}A^nx\| = \|Q^{\frac{1}{2}}A^{n-1}x\| = \cdots = \|Q^{\frac{1}{2}}x\|$ for $n\in\mathbb{N}$.

The inclusion "$\supset$" in (19) follows from (18) and $\mathcal{N}(Q) = \mathcal{N}(Q^{\frac{1}{2}})$, and the converse inclusion can be shown as in the proof of [6, Thm. 3.4]. $\square$

The next step in constructing the limit isometric group is the construction of the space X_Q with the norm $\|\cdot\|_Q$. The definitions are identical to the ones in the continuous time case, and consequently the properties of $X_Q = \overline{X/\mathcal{N}(Q)}^{\|\cdot\|_Q}$ and $\|\cdot\|_Q$ are identical to those presented in Section 2.

We define a linear operator $A_0 : X/\mathcal{N}(Q) \to X/\mathcal{N}(Q)$ by

$$A_0j(x) = j(Ax) \qquad \forall x\in X.$$

Using (18) in Lemma 17 we have that

$$\begin{aligned}\|A_0j(x)\|_Q^2 &= \|j(Ax)\|_Q^2 = \|Q^{1/2}Ax\|_X^2\\ &= \|Q^{1/2}x\|_X^2 = \|j(x)\|_Q^2\end{aligned}$$

for all $x\in X$. Therefore A_0 is a unitary operator on $X/\mathcal{N}(Q)$, and it has a unique unitary extension $A_Q \in \mathcal{L}(X_Q)$. The discrete group $(A_Q^n)_{n\in\mathbb{Z}}$ is called the *limit isometric group.*

As in Section 2.1 we can show that the spectrum of the operator A_Q satisfies $\sigma(A_Q) \subset \sigma(A)\cap\mathbb{T}$ and $R(\lambda, A_Q)j(x) = j(R(\lambda,A)x)$ for all $\lambda\in\rho(A)$ and $x\in X$. Indeed, since A_Q is a unitary operator, its spectrum satisfies $\sigma(A_Q)\subset\mathbb{T}$ [7, Thm. 10.5-1]. The inclusion $\sigma(A_Q)\subset\sigma(A)$ can now be proved as in Lemma 7, since for all $x\in X$ we have

$$\begin{aligned}\|R(\lambda)j(x)\|_Q^2 &= \|j(R(\lambda,A)x)\|_Q^2 = \|Q^{1/2}R(\lambda,A)x\|^2\\ &= \lim_{n\to\infty}(1-r_n)\sum_{k=0}^{\infty}\|r_n^kA^kR(\lambda,A)x\|^2\\ &\le \|R(\lambda,A)\|^2\cdot\lim_{n\to\infty}(1-r_n)\sum_{k=0}^{\infty}\|r_n^kA^kx\|^2\\ &= \|R(\lambda,A)\|^2\|Q^{1/2}x\|^2 = \|R(\lambda,A)\|^2\|j(x)\|_Q^2.\end{aligned}$$

In order to prove Theorem 15 we need a version of Lemma 11 for discrete semigroups.

Lemma 18. *Let $(B^n)_{n\in\mathbb{N}}$ be a power bounded discrete semigroup on a Hilbert space Y. Then*

$$Y = \mathcal{N}(I-B)\oplus\overline{\mathcal{R}(I-B)}. \qquad (20)$$

If $\mathcal{N}(I-B) = \{0\}$, then for all $y\in Y$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}B^ny = 0. \qquad (21)$$

Proof. Since Y is a Hilbert space, Theorem 2.9 and Corollary 2.11 in [10] imply that (20) holds.

Now assume $\mathcal{N}(I-B) = \{0\}$. For $y\in\mathcal{R}(I-B)$ we have

$$\frac{1}{N}\sum_{k=0}^{N-1}B^ky = \frac{1}{N}\sum_{k=0}^{N-1}B^k(I-B)z = \frac{1}{N}(z-B^Nz)\to 0$$

as $N \to \infty$ since $(B^n)_{n\in\mathbb{N}}$ is power bounded. Thus (21) holds for all $y \in \mathcal{R}(I-B)$. By assumption and (20) this set is dense in Y. By the power boundedness of $(B^n)_{n\in\mathbb{N}}$ we have that the operators $y \mapsto \frac{1}{N}\sum_{k=0}^{N-1} B^k y$ are uniformly bounded with respect to N. Combining these observations we can conclude that (21) holds for all $y \in Y$. □

As in the continuous time case, we have that the point spectrum of A_Q coincides with $\sigma_p(A) \cap \mathbb{T}$.

Theorem 19. *The point spectrum of A_Q satisfies $\sigma_p(A_Q) = \sigma_p(A) \cap \mathbb{T}$.*

Proof. The inclusion $\sigma_p(A) \cap \mathbb{T} \subset \sigma_p(A_Q)$ can be proved similarly as in the proof of Theorem 12. To prove the converse inclusion, we again note that $\sigma_p(A_Q) \subset \mathbb{T}$ since A_Q is a unitary operator. It therefore remains to show that $\sigma_p(A_Q) \subset \sigma_p(A)$. To this end, let $\mu \in \mathbb{T}$ be such that $\mu \notin \sigma_p(A)$. We will show that this implies $\mathcal{N}(A_Q - \mu I) = \{0\}$, or equivalently $\mu \notin \sigma_p(A_Q)$.

The operator $\overline{\mu}A$ is a power bounded and by assumption we have $\mathcal{N}(I - \overline{\mu}A) = \mathcal{N}(\overline{\mu}(\mu I - A)) = \mathcal{N}(\mu I - A) = \{0\}$ since $|\mu|^2 = 1$. From Lemma 18 we have that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \overline{\mu}^k A^k x = 0, \qquad \forall x \in X. \tag{22}$$

Let $x_Q \in \mathcal{N}(A_Q - \mu I)$ and $\varepsilon > 0$. Choose $x \in X/\mathcal{N}(Q)$ such that $\|x_Q - j(x)\|_Q < \varepsilon/2$. This is possible since $X/\mathcal{N}(Q)$ is $\|\cdot\|_Q$-dense in X_Q. By (22) there exists a $N_\varepsilon \in \mathbb{N}$ such that

$$\left\| \frac{1}{N_\varepsilon}\sum_{k=0}^{N_\varepsilon-1} \overline{\mu}^k A^k x \right\|_X < \frac{\varepsilon}{2\|Q^{1/2}\|}.$$

The equation $(A_Q - \mu I)x_Q = 0$ implies

$$\overline{\mu}A_Q x_Q = \overline{\mu}\mu x_Q = |\mu|^2 x_Q = x_Q$$

and thus $\overline{\mu}^n A_Q^n x_Q = x_Q$ for all $n \in \mathbb{N}$. Using this we get

$$\begin{aligned}
\|x_Q\|_Q &= \left\| \frac{1}{N_\varepsilon}\sum_{k=0}^{N_\varepsilon-1} x_Q \right\|_Q = \left\| \frac{1}{N_\varepsilon}\sum_{k=0}^{N_\varepsilon-1} \overline{\mu}^k A_Q^k x_Q \right\|_Q \\
&\le \left\| \frac{1}{N_\varepsilon}\sum_{k=0}^{N_\varepsilon-1} \overline{\mu}^k A_Q^k (x_Q - j(x)) \right\|_Q + \left\| \frac{1}{N_\varepsilon}\sum_{k=0}^{N_\varepsilon-1} \overline{\mu}^k A_Q^k j(x) \right\|_Q \\
&\le \frac{1}{N_\varepsilon}\sum_{k=0}^{N_\varepsilon-1} \|\overline{\mu}^k A_Q^k (x_Q - j(x))\|_Q + \left\| j\left(\frac{1}{N_\varepsilon}\sum_{k=0}^{N_\varepsilon-1} \overline{\mu}^k A^k x \right) \right\|_Q \\
&= \frac{1}{N_\varepsilon}\sum_{k=0}^{N_\varepsilon-1} \|x_Q - j(x)\|_Q + \left\| Q^{\frac{1}{2}} \left(\frac{1}{N_\varepsilon}\sum_{k=0}^{N_\varepsilon-1} \overline{\mu}^k A^k x \right) \right\|_Q \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

where we have used the fact that $\overline{\mu}A_Q$ is unitary. Because $\varepsilon > 0$ was arbitrary we must have $x_Q = 0$. Since x_Q was an arbitrary element of $\mathcal{N}(A_Q - \mu I)$ we find that $\mu \notin \sigma_p(A_Q)$. This concludes $\sigma_p(A_Q) \subset \sigma_p(A)$. □

We can now prove the Arendt-Batty-Lyubich-Vũ Theorem for discrete semigroups.

Proof of Theorem 15. Since $\sigma(A) \cap \mathbb{T}$ is countable and $\sigma(A_Q) \subset \sigma(A) \cap \mathbb{T}$, we have that also $\sigma(A_Q)$ must be countable. The spectral theorem for unitary operators [7, Thm 10.5-4] implies that all isolated points in $\sigma(A_Q)$ are eigenvalues of A_Q. Since A_Q is unitary and has a countable spectrum, $\sigma(A_Q)$ must consist of eigenvalues of A_Q and their accumulation points. However, if we would have $\mu \in \sigma_p(A_Q)$, then Theorem 12 implies $\mu \in \sigma_p(A) \cap \mathbb{T} = \varnothing$. This concludes that $\sigma_p(A_Q) = \varnothing$, and

$$\sigma(A_Q) = \overline{\sigma_p(A_Q)} = \varnothing.$$

A unitary operator can have empty spectrum only if $X_Q = \{0\}$. By construction this is only possible if $Q = 0$, and $\mathcal{N}(Q) = X$. Finally, $\mathcal{N}(Q) = X$ together with (19) concludes that $(A^n)_{n\in\mathbb{N}}$ is strongly stable. □

We conclude this section by presenting properties of the spectrum of the limit isometric group. Let Y be a Hilbert space. If $(B^n)_{n\in\mathbb{N}}$ with $B \in \mathcal{L}(Y)$ is a power bounded discrete semigroup and $\mu \in \mathbb{T}$, then also $((\overline{\mu}B)^n)_{n\in\mathbb{N}}$ is power bounded. If $\mu \notin \sigma_p(B)$, or equivalently

$$\{0\} = \mathcal{N}(\mu I - B) = \mathcal{N}(\mu(I - \overline{\mu}B)) = \mathcal{N}(I - \overline{\mu}B)$$

(since $|\mu|^2 = 1$), Lemma 11 implies that

$$\mathcal{R}(I - \overline{\mu}B) = \mathcal{R}(\mu(I - \overline{\mu}B)) = \mathcal{R}(\mu I - B)$$

is dense in Y. This implies that the spectrum of a power bounded operator satisfies $\sigma(B) \cap \mathbb{T} \subset \sigma_p(B) \cup \sigma_c(B)$. Using this and the previous spectral results leads to the following spectral inclusions.

Corollary 20. *If $(A^n)_{n\in\mathbb{N}}$ is power bounded, then the spectra of A and A_Q satisfy*

$$\begin{aligned}
\sigma(A_Q) &\subset \sigma(A) \cap \mathbb{T} \\
\sigma_p(A_Q) &= \sigma_p(A) \cap \mathbb{T} \\
\sigma_c(A_Q) &\subset \sigma_c(A) \cap \mathbb{T}.
\end{aligned}$$

Acknowledgement

The authors would like to thank the reviewer for his or her valuable suggestions.

References

[1] Y. Tomilov, A resolvent approach to stability of operator semigroups, Journal of Operator Theory 46 (2001) 63–98.

[2] W. Arendt, C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, Trans. Amer. Math. Soc. 306 (1988) 837–841.

[3] Y. I. Lyubich, V. Q. Phông, Asymptotic stability of linear differential equations in Banach spaces, Studia Mathmatica 88 (1988) 37–42.

[4] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, New York, 2000.

[5] R. F. Curtain, H. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, Springer-Verlag, New York, 1995.

[6] B. Guo, H. Zwart, On the relation between stability of continuous- and discrete-time evolution equations via the Cayley transform, Integral Equations Operator Theory 54 (2006) 349–383.

[7] E. Kreyszig, Introductory functional analysis with applications, Wiley Classics Library, John Wiley & Sons Inc., New York, 1989.

[8] N. Dunford, J. T. Schwartz, Linear Operators, Part II: Spectral Theory, Interscience Publishers, Inc., New York, 1963.

[9] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, Vector-Valued Laplace Transforms and Cauchy Problems, Birkhäuser, Basel, 2001.

[10] T. Eisner, Stability of operators and operator semigroups, volume 209 of *Operator Theory: Advances and Applications*, Birkhäuser Verlag, Basel, 2010.

[11] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1966.