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Nonlinear Instabilities and Extreme Wave Localization in Fiber Optics

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ABSTRACT

Modulation instability and the existence of localized soliton structures are central phenomena of nonlinear systems governed by the focusing nonlinear Schrödinger equation such as Bose-Einstein condensates, plasmas and fluids. Yet their systematic study is often difficult because of the hostility of the associated environments towards direct controllability. Optical fibers, on the other hand, provide a mature technology where the controlled exploration of many fundamental effects is made possible. In this work we report on experimental, numerical and analytical studies of novel modulation instability and wave localization dynamics in optical fibers.

Modulation instability can be described in terms of exact analytical solutions of the nonlinear Schrödinger equation known as Akhmediev breathers. Such an analytical approach has, however, remained largely unnoticed in the context of optical fibers where approximative or purely numerical methods have been preferred. In this work we use the breather formalism to derive a general recipe for the generation of ultra-high repetition rate pulse trains from an initially weakly modulated field. We also describe a simple method to excite a nonlinear superposition of elementary instabilities. Such higher-order dynamics are verified experimentally and shown to be analytically describable with the mathematical method of Darboux transformation.

Soliton dynamics play a tremendous role in the spectacular phenomenon of fiber supercontinuum generation where narrowband laser light injected into a highly nonlinear fiber is transformed into a broad spectral continuum. In this work we present two novel soliton-related mechanisms that can contribute to the extent of the supercontinuum spectrum. Specifically, we show experimentally and numerically that the spontaneous nonlinear interaction of solitons with each other or with the residues of the input pump pulse can lead to the generation new frequency components.

The initial stages of long-pulse supercontinuum generation can be described in terms of noise-driven modulation instability, leading to the break-up of the temporally broad input pulse into a large number of interacting solitons. Significantly, this intrinsically noise-sensitive process has been shown to lead to the emergence of rare, abnormally red-shifted solitons whose statistical characteristics resemble those of the infamous oceanic rogue waves. Here we show that such optical rogue wave fluctuations can be observed also in the regime of femtosecond supercontinua but that, in fact, the quantified fluctuations do not necessarily imply the existence of solitons with abnormally large peak power. We find, however, that collisions of solitons do lead to the emergence of extremely localized waves whose amplitudes fulfill criteria often used to discriminate rogue waves in a hydrodynamic environment. It is expected that the studies performed in the framework of this thesis will lay the foundations and stimulate further research in nonlinear science.
PREFACE

Research contained in this thesis was performed in the Optics Laboratory of Tampere University of Technology during a two-year period from 2009 to 2011. I gratefully acknowledge the Graduate School of Tampere University of Technology, Emil Aaltosen säätiö and Tekniikan edistämissäätiö for financial support.

I would like to thank, most of all, my supervisor Docent Goëry Genty for providing me the opportunity to work with exciting topics in the field of nonlinear fiber optics. I also wish to express my sincere gratitude for his contributions to the actual research presented in this thesis, for his guidance in writing the publications and preparing the thesis as well as in inventing a remarkably motivating scientific reward system (from which he still owes me a couple of prizes). I would also like to express my deepest gratitude to Professor John M. Dudley for his considerable scientific contributions to all the publications contained in this thesis.

In addition to Docent Goëry Genty and Professor John M. Dudley, I am indebted to all my other co-authors: Benjamin Wetzel, Kamal Hammani, Bertrand Kibler, Christophe Finot and Nail Akhmediev. I also wish to thank the head of the Optics Laboratory Professor Martti Kauranen for providing an exquisitse environment for research on nonlinear optics and for motivating me to study nonlinear optics in the first place. Docent Juha Toivonen is also acknowledged for many useful discussions and for providing us a photonic crystal fiber used in surprisingly many experiments.

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Symbols and Abbreviations

Scalars are denoted by italic letters (e.g., \( r \)), whereas vectors and matrices are denoted by bold letters (e.g., \( \mathbf{r} \)). The real and imaginary parts of a complex variable \( r \) are denoted as \( \Re[r] \) and \( \Im[r] \), respectively.

**Symbols**

\[
\begin{align*}
V & \quad \text{Normalized frequency parameter} \\
\lambda & \quad \text{Wavelength of light} \\
a & \quad \text{Core radius of a fiber} \\
n_1 & \quad \text{Core refractive index} \\
n_2 & \quad \text{Cladding refractive index} \\
\Delta & \quad \text{Fractional refractive index difference of a fiber} \\
c & \quad \text{Speed of light} \\
\omega & \quad (\text{Angular}) \text{ frequency} \\
\bar{n} & \quad \text{Effective refractive index} \\
\omega_0 & \quad \text{Reference frequency} \\
v_g & \quad \text{Group velocity} \\
\beta & \quad \text{Propagation constant} \\
\beta_2 & \quad \text{Group-velocity dispersion parameter} \\
D & \quad \text{Dispersion parameter} \\
p & \quad \text{Pitch of a photonic crystal fiber} \\
d & \quad \text{Hole diameter of a photonic crystal fiber} \\
\mathbf{E}(\mathbf{r}, t) & \quad \text{Electric field vector} \\
\mathbf{P}(\mathbf{r}, t) & \quad \text{Material polarization vector} \\
\mu_0 & \quad \text{Vacuum permeability} \\
\epsilon_0 & \quad \text{Vacuum permittivity} \\
\chi^{(k)} & \quad \text{\( k \)th order material susceptibility tensor} \\
x, y & \quad \text{Transverse spatial coordinates} \\
F(x, y) & \quad \text{Spatial mode profile} \\
z & \quad \text{Physical longitudinal coordinate} \\
t & \quad \text{Physical temporal coordinate} \\
A(z, t) & \quad \text{Complex temporal envelope of the electric field} \\
\hat{A}(z, \omega - \omega_0) & \quad \text{Complex spectral envelope of the electric field} \\
T & \quad \text{Retarded time coordinate} \\
\alpha & \quad \text{Linear loss coefficient} \\
\hat{D} & \quad \text{Dispersion operator} \\
\gamma & \quad \text{Nonlinear coefficient} \\
\tau_{\text{shock}} & \quad \text{Temporal shock timescale} \\
R(T) & \quad \text{Nonlinear response function} \\
n_2 & \quad \text{Nonlinear refractive index} \\
A_{\text{eff}} & \quad \text{Effective mode area}
\end{align*}
\]
$f_r$  Fractional contribution of Raman response
$h_R(T)$  Raman response function
$\phi_{NL}$  Nonlinear phase shift
$\Delta k$  Four-wave mixing phase-mismatch
$\omega_p$  Pump frequency
$\omega_s$  Stokes frequency
$\Omega$  Frequency shift
$P_0$  Pump power
$T_0$  Pump pulse duration
$q$  Nonlinear phase shift of a soliton
$\bar{A}$  Steady state
$a_{1, 2}$  Complex sideband perturbation amplitude
$g$  Parametric gain
$\Omega_c$  Limit frequency of modulation instability gain
$\Omega_M$  Maximum modulation instability gain frequency
$\theta$  Pump-sideband phase difference
$\Omega_{mod}$  Modulation frequency
$T_{mod}$  Temporal period
$a$  Akhmediev breather modulation parameter
$b$  Akhmediev breather gain parameter
$L_{NL}$  Nonlinear length scale
$z'$  Akhmediev breather normalized distance
$z_0$  Akhmediev breather spatial center
$\psi_{-}$  Phase of an Akhmediev breather sideband at $-\infty$
$\psi_{+}$  Phase of an Akhmediev breather sideband at $\infty$
$\xi$  Normalized longitudinal coordinate
$\tau$  Normalized temporal coordinate
$\nu$  Spectral parameter
$\sigma$  Set of constants
$k$  Wave number
$\phi$  Phase
$N$  Soliton order
$M$  Sideband order
List of Publications

**Paper I**  

**Paper II**  

**Paper III**  

**Paper IV**  

**Paper V**  

**Paper VI**  

**Paper VII**  

**Related publications**

**RP1**  
Author’s Contribution

The subject of the articles included in this thesis and their key results are listed below.

**Paper I**
In this paper we use the Akhmediev breather theory to derive an estimate for the fiber length required to generate a pulse train from a weakly modulated continuous wave field. We also show numerically that the breather theory is a useful tool in describing modulation instability in many realistic experimental excitation conditions that do not possess the exact functional form of the breather solution.

**Paper II**
Here we show that the temporal splitting of a pulse train generated via induced modulation instability in an optical fiber can be analytically described as a process of higher-order modulation instability. The Darboux transformation method is used in the design and interpretation of experiments exciting a nonlinear superposition of Akhmediev breathers.

**Paper III**
In this paper we report on the experimental and numerical observation of generation of new spectral components due to nonlinear mixing between Raman-shifting solitons and residual pump radiation in supercontinuum generation.

**Paper IV**
We show experimental signatures of dispersive waves generated through soliton collisions. Two colliding solitons are generated from a single Ti:Sapphire oscillator followed by a Michelson interferometer.

**Paper V**
In this paper we use stochastic numerical simulations to show that spontaneous soliton collisions may lead to the generation of abnormally large dispersive waves in long-pulse supercontinuum generation.
**Paper VI**  Here we show experimentally that rogue-wave-like fluctuations can be observed in femtosecond supercontinuum generation. We also show that fluctuations in the long- and short-wavelength edges of the supercontinuum spectrum are coupled and lead to similar statistics. Finally, the statistics are shown to transform from $L$-shaped to quasi-Gaussian as the filter cutoff-wavelength is shifted towards the pump wavelength.

**Paper VII**  In this paper we use extensive numerical simulations to analyze optical rogue wave statistics. We find that the $L$-shaped peak power distribution obtained using a spectral edge-pass filter does not reflect the actual peak power distribution of e.g. solitons contained in the supercontinuum pulse. Moreover, we find that the abnormally large redshift experienced by a small number of solitons can be explained in terms of Raman-induced energy transfer during collisions of solitons.

Every publication contained in this thesis is a result of a collaborative international team effort. For Paper I the author provided the dominant contribution in developing the theory, performed the numerical simulations and prepared the manuscript. In Paper II the author formulated the higher-order modulation instability theory, performed part of the numerical simulations and contributed to the preparation of the manuscript. For Papers III and IV the author conducted the experiments, performed the numerical simulations and prepared the manuscripts. For Papers V and VII the author conducted the stochastic numerical simulations and prepared the manuscripts. In Paper VI the author performed the experiments.
1. Introduction

The development of low-loss silica optical fibers has played a crucial role in driving the advent of the modern information age. Yet the versatility of optical fibers extends far beyond mere communication purposes. For instance, they provide a convenient platform to study a wide range of fundamental nonlinear phenomena. Such phenomena include modulation instability (MI) [1–5], soliton dynamics [6–14], self-similarity [15–17] and even black hole event-horizons [18,19].

Modulation instability refers to the exponential amplification of a periodic modulation on a plane wave background. Originally observed in the context of hydrodynamics [20], MI has subsequently been studied in various distinct disciplines ranging from nonlinear optics [21–26] to plasma physics [27–29]. In optical fibers MI manifests in the temporal domain through the break-up of quasi-continuous wave (CW) radiation into a train of short pulses and has been explored as a possible means to generate pulse trains with ultra-high repetition rate [1,3,30–33]. In the spectral domain MI corresponds to degenerate four-wave mixing (FWM) phase-matched through self-phase modulation (SPM) and plays a key role in the realization of fiber-optic parametric amplifiers [34–38]. Whilst MI has been extensively studied in fiber-optics the vast majority of these studies have relied on analytical approximations [38–40] or purely numerical techniques [1,30,32]. An alternative approach which has remained largely unnoticed is one where the field evolution is described analytically, without approximations, in terms of spatially localized exact solutions of the underlying nonlinear Schrödinger equation (NLSE) known as Akhmediev breathers (ABs) [2,4,41].

Modulation instability is often considered as a precursor to the formation of solitons [38,42,43], localized waves that propagate undistorted due to an exact counteracting balance of linear and nonlinear effects [41,44,45]. Similarly to modulation instability, solitons are universal and, while first discovered in the context of water waves [46], they have subsequently been observed in numerous nonlinear systems extending from Bose-Einstein condensates [47–49] to biology [50,51]. The propagation of an optical soliton in a single-mode fiber was theoretically proposed in 1973 [7], and experimentally demonstrated in 1980 [8]. However, it was soon realized that, although theoretically the optical soliton may propagate undistorted over large distances, realistic fiber implementations contain disturbances that can cause the soliton to lose energy, emit radiation or experience spectral shifts [10,12,52,53]. While such effects may be detrimental for telecommunication applications, they can be used advantageously to perform wavelength conversion [54–57]. When pushed to the extreme these soliton dynamics can lead to the spectacular phenomenon of fiber supercontinuum (SC) generation [58].
1. Introduction

Supercontinuum generation refers to the process whereby a narrowband laser pulse injected into a nonlinear fiber is transformed into a broad spectral continuum and involves a remarkable number of distinct linear and nonlinear effects [59,60]. Although the principal mechanisms underlying the process were generally known already in the 1980s [61–65], the potential of SC generation was not fully realized until the advent of a novel type of optical fiber more suitable for SC generation in late 1990s known as the photonic crystal fiber (PCF) [66–70]. Indeed, following the first demonstration of SC generation in PCFs by Ranka et al. in 2000 [71], research in SC generation has attracted significant attention [59,60,72], giving rise to numerous applications in frequency metrology [73–76], spectroscopy [77–79] and medical imaging [80–83].

Supercontinuum generation in optical fibers has been studied for more than three decades [84], yet the immense richness of the dynamics behind the process continue to reveal unexpected temporal and spectral dynamics. Of particular recent interest has been the regime where long pulses or even continuous wave lasers have been used as the pump source. In this regime modulation instability truly acts as a precursor to soliton formation by providing exponential gain for broadband noise leading to subsequent break-up of the input field envelope into a distributed spectrum of interacting solitons [59,60,65,85–87]. While this intrinsic noise-sensitivity had been known to be a mechanism of SC decoherence [59,88], it was not until recently that these fluctuations were experimentally quantified on a shot-to-shot basis [89]. Rather surprisingly, it was discovered that a very small number of input pulses led to the emergence of localized soliton pulses with abnormal characteristics. Significantly, due to similarities in the statistical occurrence, these events were proposed as the optical analogues of the infamous oceanic rogue waves; devastating walls of water known to roam the oceans and considered responsible for several maritime disasters [90–92]. Because for the first time such extreme wave localization could be studied in a controlled laboratory environment this optical observation rapidly attracted significant interest and stimulated research on extreme events not only in optics [93–111], but also in a wide variety of other nonlinear systems [112–121].

1.1 Outline of the thesis

This thesis is a compilation of the research done by the author during a two-year period from 2009 to 2011. The main purpose of the work has been to study nonlinear instability and wave localization dynamics in an optical fiber context. The relevant topics investigated in this thesis are closely related, yet span a broad overall area of nonlinear fiber optics. These topics include modulation instability, soliton dynamics, supercontinuum generation and optical rogue waves.

This thesis is divided into six chapters, which are followed by seven original publications. The six introductory chapters aim to provide, in a self-consistent manner, the theoretical framework of nonlinear fiber optics necessary to understand the research presented in the original publications. Of course, the main results of the publications are also highlighted amidst the particular theory they aim to advance.
1. Introduction

In Chapter 2 we introduce the essential notions of nonlinear fiber optics. Specifically, we define concepts used throughout the thesis and discuss both linear and nonlinear optical effects that take place in single-mode silica fibers. In Chapter 3 we discuss the fundamental nonlinear phenomenon of modulation instability, starting from a simplified perturbation analysis before proceeding to discuss the process without approximations in terms of exact Akhmediev breather solutions of the nonlinear Schrödinger equation. In the end of Chapter 3 we introduce the mathematical concept of Darboux transformation and show how it can be used to describe complex temporal and spectral modulation instability related dynamics.

In Chapter 4 we summarize the physics behind several soliton-related propagation effects such as the Raman-induced self-frequency shift (SSFS) and the emission of Cherenkov-like dispersive wave radiation. We also provide a detailed description of effects arising from the collision of two solitons and from the nonlinear mixing of solitons and weak linear waves. Chapter 5 utilizes the theory presented in the previous chapters, describing long-pulse supercontinuum generation in terms an initial stage of modulation instability followed by the formation and propagation of solitons. Particular emphasis is given to interpreting and deconstructing the characteristics and formation dynamics of rogue-wave-like fluctuations in SC generation. Finally, a summary of the thesis is presented in Chapter 6.
Light propagation in single-mode optical fibers

2.1 Linear propagation

An optical fiber typically consists of a dielectric core surrounded by a dielectric cladding whose refractive index is lower than that of the core [122]. Due to this refractive index difference light incident on the core remains trapped through the process of total internal reflection (TIR), allowing for a long interaction length between light and the core medium.

Optical fibers can be divided into two categories: those that can sustain only a single and those that can sustain several electromagnetic eigenmodes [122]. In this context, an important parameter characterizing an optical fiber is the normalized frequency parameter $V$ defined as [38, 122]:

$$V = \frac{2\pi}{\lambda} a \sqrt{n_1^2 - n_2^2}.$$

(2.1)

Here $\lambda$ is the wavelength of light, $a$ the radius of the core and $n_1$ and $n_2$ are the indices of refraction of the core and the cladding, respectively. If the $V$-parameter fulfills the condition $V < 2.405$ the fiber only guides a single electromagnetic mode and is thus called a single-mode fiber (SMF). In contrast, fibers for which $V > 2.405$ are called multi-mode fibers and can guide light in two or more distinct modes. Typically both the cladding and the core are made out of fused silica glass (SiO$_2$) of high-chemical purity and the required refractive index difference is achieved by adding low-concentrations of dopants (such as titanium, germanium and boron) [122]. Consequently, the fractional refractive index difference is usually small, of the order of $\Delta = (n_1 - n_2)/n_1 \approx 0.003$. It is clear that the only physical difference of single- and multi-mode fibers is the radius of the core. For instance, for $\Delta = 0.003$ single-mode operation at $\lambda = 1.55 \mu m$ can be achieved when the core radius $a < 6 \mu m$ whereas for typical multi-mode fibers the core radius is $50 \mu m$.

The speed at which a monochromatic wave with frequency $\omega$ propagates in a medium is given by the phase velocity $c/n(\omega)$, where $c$ is the speed of light in vacuum and $n(\omega)$ the refractive index of the medium at the frequency $\omega$. In an optical fiber, however, the effective refractive index $\bar{n}(\omega)$ experienced by a guided mode is different from that of either the core or the cladding and varies for each mode sustained in the fiber. Aside from different effective refractive indices, different modes may also differ through their transverse spatial profiles. For instance, Fig. 2.1 displays six different spatial profiles all of which can be guided in a fiber whose core radius $a = 50 \mu m$ and fractional refractive index difference $\Delta = 0.002$. In a single-mode fiber only the mode with the
largest effective index is sustained. This mode is known as the fundamental mode and has a near-gaussian transverse profile (top left mode in Fig. 2.1) [38].

The fact that each mode experiences a different refractive index suggests that they also travel at different velocities resulting in the temporal broadening of short pulses [38]. Such broadening does not, however, cease to occur even in single-mode fibers due to intramodal dispersion which arises from the frequency-dependence of the effective modal index \( \bar{n}(\omega) \). Indeed, the spectrum of a short pulse is composed of a continuous band of spectral components all of which travel at different phase velocities \( c/\bar{n}(\omega) \). The pulse envelope itself, centered at frequency \( \omega_0 \), propagates at the group velocity

\[
v_g = \left( \frac{d\beta}{d\omega} \right)^{-1} \bigg|_{\omega=\omega_0}, \tag{2.2}
\]

where \( \beta(\omega) = \bar{n}(\omega)\omega/c \) is the propagation constant (wave number) of the mode. Usually the exact functional form of the propagation constant is not known and effects of intramodal dispersion are accounted for through a Taylor-series expansion centered about the center frequency \( \omega_0 \) of the electric field [38]

\[
\beta(\omega) = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2 + \cdots, \tag{2.3}
\]

where the \( \beta_k \) coefficients can be calculated from

\[
\beta_k = \left( \frac{d^k \beta}{d\omega^k} \right)_{\omega=\omega_0}. \tag{2.4}
\]

By comparison with Eq. (2.2) it is clear that \( \beta_1 = v_g^{-1} \). The coefficient \( \beta_2 \) is an important fiber-design parameter known as the group-velocity dispersion (GVD) parameter.
and it is precisely the GVD-parameter which plays the dominant role in the temporal broadening of short pulses. The dispersion is said to be normal if $\beta_2 > 0$, anomalous if $\beta_2 < 0$ and the wavelength at which $\beta_2 = 0$ is known as the zero-dispersion wavelength (ZDW). Traditionally in literature an alternative parameter called the dispersion parameter is also used and is defined by

$$D = -\frac{2\pi c}{\lambda^2} \beta_2.$$  \hfill (2.5)

Dispersion coefficients $\beta_k$ for which $k > 2$ are collectively known as higher-order dispersion (HOD) terms and their contribution is particularly important if the bandwidth of the pulse is large or if the pulse is centered close to the ZDW.

The frequency-dependence of the effective modal index arises from two contributions: (i) a material contribution reflecting the fact that the refractive indices of the cladding and the core are frequency-dependent and (ii) a waveguide contribution reflecting the frequency-dependence of the mode size. Due to the waveguide contribution the overall dispersion can be influenced by suitable choice of the fiber-design parameters. Figure 2.2(a) illustrates the wavelength-dependence of the refractive index of bulk fused silica calculated from a Sellmeier equation [38]. Correspondingly, Fig. 2.2(b) compares the dispersion parameter $D$ associated with bulk fused silica and the dispersion parameter of a typical single-mode fiber with $a = 4 \mu m$ and $\Delta = 0.002$. Whilst the dispersion parameters show good agreement over the entire wavelength range, the ZDW of the fiber can be seen to be slightly displaced towards longer wavelengths from that of bulk silica at $1.27 \mu m$. By suitably choosing the relative index difference $\Delta$ and the core radius $a$, the ZDW can be further pushed into the vicinity of $1.55 \mu m$ where fiber losses reach a minimum [123]. However, the dispersion profile can be tailored to a much more drastic extent if a microstructured refractive index profile is introduced into the cladding.

![Figure 2.2: Calculations showing (a) variation of the refractive index of fused silica with wavelength and (b) dispersion parameters of bulk fused silica (dashed line) and a single-mode fiber with $a = 4 \mu m$ and $\Delta = 0.002$ (solid line).](image-url)
2. Light propagation in single-mode optical fibers

2.2 Photonic crystal fibers

The use of a microstructured refractive index profile so as to influence the guidance characteristics of an optical fiber dates back to 1970s [124, 125]. However, it was not until the 1990s that such photonic crystal fibers became commercially accessible [66–70, 126–128].

Essentially, PCFs are composed of a periodic array of air holes embedded in glass matrix (typically fused silica) and can be classified in two categories: fibers whose guiding mechanism is total internal reflection [67, 68, 126, 129], and fibers where light is guided through the two-dimensional photonic band-gap (PBG) effect [128, 130–138]. The main structural difference between the two is that TIR-PCFs possess a solid core whereas the core of a PBG-PCF is hollow or is made of a material whose refractive index is smaller than that of the surrounding glass [137]. Figure 2.3 schematically illustrates the cross-sections of both types of PCFs with the defining structural parameters $p$ (pitch) and $d$ (hole diameter). Although PBG-PCFs have been successfully applied in fields as diverse as laser-induced guidance [139], nonlinear optics [140–142] and quantum optics [143–145], they are beyond the scope of the present thesis and only solid-core PCFs whose guiding mechanism is total internal reflection will be discussed in the following paragraphs. Moreover, from now on we assume the term photonic crystal fiber to refer exclusively to the afore-mentioned TIR-PCFs.

![Figure 2.3: Schematic illustration of the cross-section of a solid-core PCF (a) and a hollow-core PCF (b).](image)

Whereas the modal properties of standard optical fibers can be investigated analytically [38, 122], the guidance characteristics of PCFs are more difficult to analyze and numerical techniques are usually used. These include effective modal methods [68, 129, 146–148], multipole methods [149–151], the fully-vectorial plane-wave method [152–154], finite-difference methods [155–158] and the finite-element method [159, 160]. All of the methods possess distinct advantages as well as disadvantages that are well summarized in Ref. [161].

Photonic crystal fibers exhibit unique features compared to ordinary optical fibers. For instance, by a proper choice of the structural parameters $p$ and $d$ (see Fig. 2.3) PCFs can be designed to guide only the fundamental mode for an extremely broad...
wavelength range extending from near-ultraviolet to near-infrared and for various core sizes [67,68,129,151–153]. Hence, it is possible to design large area single-mode fibers in order to minimize nonlinear optical effects for high-power beam delivery applications [154,162,163]. Correspondingly, PCFs with very small mode areas can be fabricated as well, resulting in significantly increased optical nonlinearities [164–166]. Moreover, by implementing asymmetric air-hole patterns it is possible to obtain extremely strong birefringence for polarization-maintaining fibers [167–171].

One of the most important feature of PCFs is, however, that the microstructured cladding exhibits a particularly strong wavelength-dependence for the effective refractive index allowing for the realization of dispersion profiles inaccessible in standard fibers [172–174]. For instance, the ZDW of a PCF can be pushed down to the visible wavelengths well below 700 nm [172]. Furthermore, the dispersion can also be tailored to possess two zero-dispersion wavelengths [172,175–177], or to exhibit very low and flat dispersion over broad wavelength ranges [173,174,178,179]. Recently it has been suggested that it may even be possible to fabricate PCFs with three ZDWs [180]. Figure 2.4 displays the dispersion profiles of three PCFs with distinct design parameters calculated using the finite-element method.

![Figure 2.4: Calculated dispersion profiles of three PCFs with design parameters](image)

Figure 2.4: Calculated dispersion profiles of three PCFs with design parameters $d = 1.80 \, \mu m$ and $p = 2.00 \, \mu m$ (red, A); $d = 0.69 \, \mu m$ and $p = 1.24 \, \mu m$ (blue, B); $d = 1.70 \, \mu m$ and $p = 3.25 \, \mu m$ (black, C).

### 2.3 Nonlinear propagation

When an electromagnetic wave propagates in a medium it forces the bound electrons of the medium to oscillate giving rise to an induced material polarization $\mathbf{P}(\mathbf{r}, t)$. If the field is weak, the induced polarization is essentially proportional to the field amplitude and the light-matter interaction is linear, giving rise to effects such as refraction, reflection and dispersion. However, when the electric field amplitude is large the induced polarization is a nonlinear function of the excitation resulting in a wide variety of nonlinear optical effects [181].

Assuming no free currents nor charges, the evolution of the electric field $\mathbf{E}(\mathbf{r}, t)$ in a
2. Light propagation in single-mode optical fibers

medium is governed by the wave equation [38]:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}(\mathbf{r}, t)}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}(\mathbf{r}, t)}{\partial t^2},$$  \hspace{1cm} (2.6)

where $\mu_0$ is the vacuum permeability. The material polarization can be expanded as a power series of the electric field

$$\mathbf{P}(\mathbf{r}, t) = \varepsilon_0 \sum_{k=1}^{\infty} \chi^{(k)}(\mathbf{r}, t)^k,$$  \hspace{1cm} (2.7)

where $\varepsilon_0$ is the vacuum permittivity and $\chi^{(k)}$ are tensors of rank $(k+1)$ known as material susceptibilities [181]. For materials that possess inversion symmetry, such as silica, all even-order susceptibilities vanish identically and the magnitudes of the susceptibilities rapidly decrease with the order $k$ [181]. Consequently, nonlinear optical effects in silica-based optical fibers correspond predominantly to third-order effects described by the susceptibility tensor $\chi^{(3)}$. Whilst in general $\chi^{(3)}$ may possess 81 nonzero elements only 4 of them are independent in isotropic media [181]. Moreover, assuming the optical field to be linearly polarized and to maintain its polarization along propagation only a single tensorial element which we denote as $\chi_{xxxx}^{(3)}$ is required to describe the nonlinear dynamics. Of course, this scalar approach is strictly valid only for polarization maintaining fibers where the polarization of the field is aligned along one of the principal axes of the fiber.

Here, we define the forward-propagating electric field linearly polarized along the direction $\hat{x}$ as [38]:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2} \hat{x} \{ F(x, y) A(z, t) e^{i\beta_0 z - i\omega_0 t} + \text{c.c.} \},$$  \hspace{1cm} (2.8)

where $\omega_0$ is the carrier frequency, $F(x, y)$ is the transverse modal distribution associated with the propagating field, $\beta_0$ is the propagation constant of the mode at the carrier frequency (see Eq. (2.4)), and c.c. denotes the complex conjugate. $A(z, t)$ represents the temporal envelope normalized such that $|A(z, t)|^2$ yields the instantaneous power in Watts and its Fourier transform $\hat{A}(z, \omega - \omega_0)$ corresponds to the complex spectral envelope of the field [38, 59]. The temporal envelope $A(z, t)$ may be associated with a large number of separate spectral components and therefore the actual choice of the reference frequency $\omega_0$ in the field decomposition (2.8) is arbitrary, although it is computationally useful to choose it in the vicinity of the dominant spectral features.

By treating the nonlinear polarization as a perturbation in Eq. (2.6) a propagation equation for the temporal envelope $A(z, t)$ can be derived [38, 182–184]. With the change of variable $T = t - z/v_g$ the evolution equation can be written in a reference
2. Light propagation in single-mode optical fibers

frame moving with the group velocity $v_g$ at the reference frequency $\omega_0$ as [59]:

$$\frac{\partial A}{\partial z} + \frac{\alpha}{2} A - i \hat{D} \left( i \frac{\partial}{\partial T} \right) A = i \gamma \left( 1 + i \tau_{\text{shock}} \frac{\partial}{\partial T} \right) \left( A(z, T) \int_{-\infty}^{\infty} R(T') \, |A(z, T - T')|^2 dT' \right).$$  \hspace{1cm} (2.9)

On the one hand, the left-hand side of Eq. (2.9) models linear propagation effects where $\alpha$ is the linear loss coefficient and $\hat{D}$ the dispersion operator defined as

$$\hat{D} \left( i \frac{\partial}{\partial T} \right) = \sum_{k \geq 2} \frac{\beta_k}{k!} \left( i \frac{\partial}{\partial T} \right)^k.$$  \hspace{1cm} (2.10)

By comparison with Eq. (2.3) it is clear that $\hat{D} (\omega - \omega_0) = \beta(\omega) - \beta(\omega_0) - \beta_1 (\omega - \omega_0)$.

On the other hand, the right-hand side of Eq. (2.9) represents the nonlinear effects. The strength of the nonlinearity is described by the nonlinear coefficient $\gamma$:

$$\gamma = \frac{\omega_0 n_2}{c A_{\text{eff}}(\omega_0)},$$  \hspace{1cm} (2.11)

where $n_2 = 3/(8n) \text{Re} \chi^{(3)}_{xx}$ is the nonlinear refractive index and $A_{\text{eff}}$ is the effective area of the fiber mode evaluated at the carrier frequency $\omega_0$ [38,181]. For silica $n_2 \approx 2.6 \times 10^{-20} \text{ m}^2/\text{W}$ and the effective area can vary from 1 $\mu\text{m}^2$ for highly nonlinear PCFs [185], to several hundred $\mu\text{m}^2$ for large-mode area PCFs [186]. The nonlinear response function $R(T) = (1 - f_R)\delta(T) + f_R h_R(T)$ includes both the instantaneous electronic and delayed Raman contributions where $f_R = 0.18$ describes the fractional Raman contribution and $h_R(T)$ is the time-domain Raman response function [187]. The derivative term on the right-hand side models the dispersion of the nonlinearity characterized by a time scale $\tau_{\text{shock}} = 1/\omega_0$ [38,59]. Equation (2.9) can be numerically solved for arbitrary inputs using the well-known split-step Fourier method [38,182,188].

2.3.1 Kerr nonlinearity

The dominant nonlinear term in Eq. (2.9) is the Kerr nonlinearity which arises from quasi-instantaneous interaction of material electrons with the driving field [38]. The Kerr term can be isolated in Eq. (2.9) by neglecting linear propagation effects, delayed Raman effects and the dispersion of the nonlinearity. Indeed, by retaining only the first term in the left-hand side of Eq. (2.9) and setting $\tau_{\text{shock}} = f_R = 0$ the propagation equation reduces to

$$\frac{\partial A}{\partial z} = i \gamma |A|^2 A.$$  \hspace{1cm} (2.12)

Kerr nonlinearity gives rise to three important effects: self-phase modulation, cross-phase modulation (XPM) and four-wave mixing.
2. Light propagation in single-mode optical fibers

Self-phase modulation

Self-phase modulation results from the intensity-dependence of the refractive index. Specifically, the presence of a strong optical field modifies the refractive index of the medium such that the field acquires a nonlinear phase along propagation. This is particularly evident from the solution of Eq. (2.12): \( A(z, T) = A(0, T) \exp(i\phi_{NL}z) \), where the nonlinear phase \( \phi_{NL} \) is defined as

\[
\phi_{NL} = \gamma |A(0, T)|^2. \tag{2.13}
\]

It should be clear that SPM does not affect the temporal shape of the field (\(|A(z, T)|^2 = |A(0, T)|^2\)). However, for an optical pulse the nonlinear phase shift is time-dependent resulting in changes in the pulse spectrum. Although the exact effect of SPM depends drastically on the chirp and shape of the input pulse, it systematically leads to spectral broadening when the input is transform limited [38]. When combined with dispersion, the spectral chirp induced by SPM can exactly cancel the temporal broadening of pulses in the anomalous dispersion regime. This balance leads to the existence of pulses that can propagate for extended distances with no changes in either the time or the frequency domain [189]. Such pulses are known as solitons and will be discussed in detail in Chapter 4.

Cross-phase modulation

Cross-phase modulation describes the interaction of two fields at different frequencies \( \omega_1 \) and \( \omega_2 \) whereby the field at frequency \( \omega_1 \) modifies the refractive index experienced by the field at frequency \( \omega_2 \) and vice versa. Indeed, injecting a field of the form \( A = A_1 \exp(-j\omega_1 t) + A_2 \exp(-j\omega_2 t) \) into Eq. (2.12) yields terms oscillating at four different frequencies \( \omega_1, \omega_2, 2\omega_1 - \omega_2 \) and \( 2\omega_2 - \omega_1 \). Assuming that the spectra of the temporal envelopes \( A_1 \) and \( A_2 \) do not significantly overlap, the corresponding frequency components can be separated resulting in two coupled equations describing the evolution of \( A_1 \) and \( A_2 \) as:

\[
\frac{\partial A_1}{\partial z} = i\gamma(|A_1|^2 + 2|A_2|^2)A_1 \tag{2.14}
\]

\[
\frac{\partial A_2}{\partial z} = i\gamma(|A_2|^2 + 2|A_1|^2)A_2. \tag{2.15}
\]

It is easy to see that the coupled equations have a solution similar to that given above for SPM. However, now the nonlinear phases associated with fields \( A_1 \) and \( A_2 \) are given by \( \phi_{NL}^{(1)} = \gamma(|A_1(0, T)|^2 + 2|A_2(0, T)|^2) \) and \( \phi_{NL}^{(2)} = \gamma(|A_2(0, T)|^2 + 2|A_1(0, T)|^2) \), respectively. Clearly, the first term in the nonlinear phases corresponds to SPM and the second term to XPM which can be seen to be twice as efficient as SPM.
Four-wave mixing

The Kerr term also gives rise to parametric four-wave mixing [38, 181]. Namely, the interaction of waves at three frequencies $\omega_1$, $\omega_2$ and $\omega_3$ will drive the generation of a new spectral component at frequency $\omega_4$ such that energy is conserved: $\omega_1 + \omega_2 = \omega_3 + \omega_4$. In order for the process to be efficient the frequencies must satisfy a phase-matching condition $\Delta k = 0$, where $\Delta k = \beta(\omega_1) + \beta(\omega_2) - \beta(\omega_3) - \beta(\omega_4)$. In fact, FWM does not necessarily require three distinct waves to be present at the fiber input. Indeed, as seen above, the interaction of two frequencies $\omega_1$ and $\omega_2$ will drive the generation of new frequencies symmetrically distributed about the pumps: $\omega_3 = 2\omega_1 - \omega_2$ and $\omega_4 = 2\omega_2 - \omega_1$. Subsequently, the generated frequencies may interact with each other and the pump waves to produce a full cascade of equally separated spectral components [190–193]. Finally, even if a single frequency is injected into a fiber FWM can occur as a result of spontaneous growth of noise (see Chapter 5).

2.3.2 Stimulated Raman scattering

Raman scattering is an inelastic photon-phonon interaction whereby the frequency of a photon is downshifted (Stokes waves) [194]. Emission of a photon with higher frequency than that of the incident photon (anti-Stokes wave) is also possible but significantly less probable. For intense pump fields the Stokes wave can grow exponentially. This phenomenon is known as stimulated Raman scattering (SRS) [195, 196].

The Raman gain curve of fused silica is shown in Fig. 2.5. The maximum of the gain curve is located approximately at 13.2 THz and its bandwidth is roughly 30 THz. The physical meaning of the gain curve is that, if a probe wave at $\omega_s$ co-propagates with a strong pump wave at $\omega_p$, the probe wave will experience exponential amplification provided that the frequency detuning $\Omega/2\pi = \omega_p/2\pi - \omega_s/2\pi$ does not exceed the Raman gain bandwidth. If the pump wave alone is injected into the fiber, spontaneous Raman scattering or input noise can act as the probe and be amplified with propagation [38]. Although in this case photons with frequencies spanning the entire gain bandwidth are generated, the frequency for which the Raman gain is maximum is amplified most.

It should be noted that SRS can be strongly affected by parametric FWM [197, 198]. For instance when the linear phase-mismatch of the parametric Stokes/anti-Stokes interaction vanishes the growth of both waves can be suppressed [199]. Moreover, FWM coupling can act so as to introduce an enhancement on the SRS gain as well as to allow for the exponential growth of the anti-Stokes wave [200, 201].

In the induced nonlinear polarization model, SRS arises from the non-instantaneous part of the third-order susceptibility $\chi^{(3)}$. In the pulse propagation equation (2.9) effects arising from SRS are typically accounted for through a time-domain response function $h_R(t)$ which can be obtained from the experimentally measured SRS gain curve through the use of the Kramers-Kronig relations and inverse Fourier transformation [187].
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![Normalized Raman gain of silica](image)

Figure 2.5: Normalized Raman gain of silica

2.3.3 Noise sources

As mentioned above, in the absence of a distinct seed the SRS and FWM signals can grow from noise. It is therefore important to account for the presence of noise in the pulse propagation equation (2.9). The majority of studies assume negligible input classical noise and implement quantum shot-noise semiclassically through the addition of a noise seed of one photon with random phase for every spectral discretization bin [59]. Such an approach yields generally good results compared to experiments particularly for mode-locked lasers operating at high repetition rates. For continuous wave lasers, however, realistic noise implementation is more difficult and several approaches have been proposed [87, 106, 202–204]. Aside from input noise, fluctuations arising from spontaneous Raman scattering occurring along propagation can also be included in the propagation equation through the addition of a stochastic distance-dependent variable into the model [59, 205].
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3.1 Nonlinear Schrödinger Equation

When only the two dominant terms, group-velocity dispersion and Kerr-nonlinearity, are included, the pulse propagation equation (2.9) can be approximated as

\[ \frac{\partial A}{\partial z} + \beta_2 \frac{\partial^2 A}{\partial T^2} - i\gamma |A|^2 A = 0. \]  

(3.1)

Equation (3.1) has the form of the Schrödinger equation with a nonlinear potential and is therefore known as the nonlinear Schrödinger equation. The NLSE or variations thereof can be encountered in the description of various different systems. These include Bose-Einstein condensates [206–209], plasmas [27–29,210], and fluids [44,120,211]. The NLSE is an exactly integrable equation and can be solved using the inverse scattering transform (IST) [6,41,44,212,213]. However, in general the use of the IST for arbitrary initial conditions is highly non-trivial and solutions that can be expressed in closed analytical form are scarce [41].

Although strictly speaking the NLSE yields a good approximation for nonlinear pulse propagation in optical fibers only in a very restricted parameter regime [38], its analysis provides for considerable insight into other regimes as well. Indeed, although the inclusion of additional terms (compare Eqs. (3.1) and (2.9)) may destroy the integrability of the system, complicated spectral and temporal dynamics can be intuitively understood in the framework of exact analytical solutions of the NLSE. For instance, in this chapter we discuss how the fundamental nonlinear dynamics of modulation instability can be described in terms of exact analytical solutions of the NLSE known as Akhmediev breathers. Before studying the AB solutions, however, we begin by introducing the concept of modulation instability in terms of an approximative perturbation analysis.

3.2 Perturbation analysis

In certain parameter regimes the steady state solution of nonlinear systems governed by the NLSE is unstable against small fluctuations leading to the development of a temporally periodic modulation. This instability is known as modulation instability and has been studied in famous contexts including nonlinear optics [21–26], plasma physics [27–29] and fluid dynamics [20,214,215]. In optical fibers MI manifests through the break-up of an initially continuous or quasi-continuous wave into a train of ultrashort pulses [1,3,30–33,38,216–218], and it has been investigated using several tech-
3. Modulation instability

Techniques such as direct numerical simulations [1,30,32], linear stability analysis [38] or truncated three-wave models [39,40]. It should be stressed that four-wave mixing and modulation instability essentially describe the same effect in different Fourier domains. Indeed, whereas FWM is often envisaged as a spectral process, MI dynamics are typically described in the temporal domain [59].

The fundamental characteristics of MI can be understood by analyzing the steady state solution of the NLSE given by \[ \bar{A}(z) = \sqrt{P_0} \exp(i \phi_{NL} z) \]. Physically, \( \bar{A} \) corresponds to a monochromatic CW which acquires a nonlinear phase \( \phi_{NL} = \gamma P_0 \) along propagation due to SPM. The stability of the steady state can be analyzed by adding to the solution a weak periodic modulation and studying the evolution of the associated harmonic sidebands. Injecting the ansatz \[ A(z) = \bar{A}(z) + a_1(z) \exp(i \omega T) + a_2(z) \exp(-i \omega T) \] into the NLSE and linearizing with respect to \( a_1 \) and \( a_2 \) yields the following coupled equations for the complex sideband amplitudes:

\[ \frac{da_1}{dz} = i \kappa a_1 + i \gamma P_0 a^*_2 e^{2i \phi_{NL} z} \]  
\[ \frac{da_2^*}{dz} = -i \kappa a_2^* - i \gamma P_0 a_1 e^{-2i \phi_{NL} z}, \]

where \( \kappa = \Omega^2 \beta^2 / 2 + 2 \gamma P_0 \) and \( \Omega \) denotes the frequency detuning of the perturbations with respect to the pump frequency. It is important to note that in deriving Eqs. (3.2-3.3) the pump wave is assumed to remain undepleted (undepleted pump approximation). Writing \( a_k(z) = b_k(z) \exp(i \kappa z) \), with \( k = 1, 2 \), the solutions of Eqs. (3.2) and (3.3) are given by:

\[ b_1(z) = (b_{11} e^{gz} + b_{12} e^{-gz}) e^{i(\gamma P_0 - \kappa) z} \]  
\[ b_2(z) = (b_{21} e^{gz} + b_{22} e^{-gz}) e^{i(\gamma P_0 - \kappa) z}, \]

where \( g \) represents the parametric gain defined as:

\[ g = \frac{1}{2} \sqrt{-\beta_2^2 \Omega^4 - 4 \beta_2 \Omega^2 \gamma P_0} \]  

and the constants \( b_{11}, b_{12}, b_{21} \) and \( b_{22} \) can be determined from the initial conditions \( a_1(0) \) and \( a_2(0) \). It is clear from Eq. (3.6) that the perturbation can undergo monotonous growth only in the anomalous dispersion regime of the fiber (\( g \) is real only if \( \beta_2 < 0 \)) and for a limited range of frequency detunings \( \Omega \) which satisfy the condition

\[ |\Omega| < \Omega_c = \sqrt{\frac{4 \gamma P_0}{|\beta_2|}}. \]

The frequency which experiences the largest gain (and therefore the "fastest" growth) is given by \( \Omega_M = \pm \sqrt{2 \gamma P_0 / |\beta_2|} \) where the corresponding gain is \( g = \gamma P_0 \). An example of a MI gain curve corresponding to typical pump-fiber parameters is illustrated in Fig. 3.1. Unless specified, all computations in this chapter use the same set of parameters.
yet it should be stressed that all the results, when subject to proper scaling, are valid for other parameters as well.

Figure 3.1: Example of modulation instability gain. The pump-fiber parameters are: $\gamma = 1.2 \text{ km}^{-1}\text{W}^{-1}$, $\beta_2 = -21.4 \text{ ps}^2\text{km}^{-1}$ and $P_0 = 1 \text{ W}$.

The physical significance of the gain curve is that, if a single strong pump co-propagates with a weak probe in the anomalous dispersion regime, the probe wave can be exponentially amplified provided that its frequency detuning $\Omega$ from the pump wave satisfies Eq. (3.7). In other words, the steady state solution in the anomalous dispersion regime is unstable against perturbations with frequencies $|\Omega| < \Omega_c$. In the spectral domain this instability corresponds to the degenerate FWM process $2\omega_p = \omega_1 + \omega_2$, where $\omega_1 = \omega_p - \Omega$ and $\omega_2 = \omega_p + \Omega$ denote the absolute frequencies of the perturbation sidebands. Note that in the absence of SPM ($\gamma = 0$) the parametric gain $g$ is always purely imaginary [38].

Close inspection of Eqs (3.4) and (3.5) also suggests that, depending on the initial conditions, the perturbations $a_1$ and $a_2$ can undergo exponential decay as well (for instance if $b_{11} = 0$ in Eq. (3.4)). This can be seen more clearly by injecting $a_k = \sqrt{P_k(z)} \exp(i\phi_k(z))$, with $k = 1, 2$ into Eqs. (3.2) and (3.3). For simplicity it is assumed that $P_1 = P_2$, which physically corresponds to the case where a sinusoidally modulated continuous wave is injected into the fiber (e.g. intensity modulation). The evolution of the sidebands is then governed by two coupled differential equations:

$$\frac{dP_1}{dz} = \frac{dP_2}{dz} = -2\gamma P_0 P_1 \sin \theta \quad (3.8)$$

$$\frac{d\theta}{dz} = |\beta_2| \Omega^2 - 2\gamma P_0 - 2\gamma P_0 \cos \theta, \quad (3.9)$$

where $\theta(z) = 2\phi_{NL} z - \phi_1(z) - \phi_2(z)$ corresponds to the phase-difference of the pump and the sidebands and anomalous dispersion is assumed. It is clear from Eq. (3.8) that the power in the sidebands will undergo exponential growth if $\sin \theta < 0$ but if $\sin \theta > 0$ the power will exponentially decay. Careful inspection of Eq. (3.9) reveals that for
3. Modulation instability

Each frequency detuning $\Omega$ there are in fact two fixed points in the interval $[-\pi, \pi]$ for the phase-difference $\theta(z)$:

$$\theta_{\pm} = \pm \cos^{-1}\left(\frac{|\beta_2|\Omega^2 - 2\gamma P_0}{2\gamma P_0}\right).$$

(3.10)

The significance of the fixed points is that if, at the fiber input, the phase-difference $\theta = \theta_{\pm}$ it will remain constant and the sidebands will experience amplification for $\theta_-$ or decay for $\theta_+$ at a rate given by the parametric gain $g$. For any other initial value of the phase $\theta$ the evolution is more complicated and governed by Eqs. (3.8) and (3.9). However, simple linear stability analysis of Eq. (3.9) reveals that while the fixed point corresponding to exponential gain is stable, that associated with exponential decay is unstable. In fact, any initial condition such that $\theta \neq \theta_\pm$ will asymptotically tend towards the fixed point $\theta_-$, and even if at the input $\theta = \theta_+$ any small disturbance will cause the phase-difference to tend towards $\theta_-$. From an experimental point of view this means that the sidebands should always undergo exponential amplification when propagated long enough irrespective of the input phase-difference. Nonetheless it is worth stressing that, due to the phase-sensitivity of the process, the effective gain experienced by the sidebands coincides with the parametric gain given by Eq. (3.6) only if the initial pump-sideband phase difference $\theta = \theta_-$. 

3.3 Modulation instability beyond the undepleted pump approximation

The above perturbation analysis assumes the pump wave to remain undepleted. It should therefore be clear that the results are applicable only as long as the power in the sidebands is significantly smaller than the pump power. When the powers become comparable pump depletion must be accounted for and it is necessary to solve three coupled equations describing the general evolution of the pump and the sidebands [40]. While these coupled equations can be solved analytically to provide considerable insight [219], they are still approximative as only the interaction of three waves is considered [39,40]. Indeed, in general the interaction involves an infinite number of spectral sidebands generated through the cascaded FWM-process mentioned in section 2.3. Hence, only the direct integration of the NLSE can provide an exact solution to the problem.

To this end, Fig. 3.2 displays the spectral and temporal MI-dynamics when the NLSE is numerically integrated with an initial condition corresponding to a modulated CW signal: $A(0,T) = \sqrt{P_0}[1 + \delta \cos(\Omega_{\text{mod}} T)]$. The modulation frequency $\Omega_{\text{mod}} = 2\pi/T_{\text{mod}}$ was chosen to coincide with the maximum of the MI gain curve.

Several observations can be made from Fig. 3.2. In the early stages of the propagation all the harmonic sidebands can be seen to experience rapid growth leading, in the temporal domain, to the growth of the input modulation on top of the background CW. These early propagation dynamics are in good agreement with the above perturbation analysis as is highlighted in Fig. 3.3(a), where the power in the first sidebands
3. Modulation instability

Figure 3.2: Simulated spectral (a) and temporal (b) dynamics of modulation instability. The pump-fiber parameters correspond to those used in Fig. 3.1 and the initial modulation strength $\delta = 0.02$. The modulation frequency $\Omega_{\text{mod}} = 0.34$ THz corresponds to the peak of the MI gain and is associated with a temporal modulation period $T_{\text{mod}} = 18$ ps.

extracted from simulations is compared with that obtained directly from Eq. (3.4). Yet, the effect of pump depletion can also be clearly seen and, in particular, when the power in the first sideband becomes large its growth rate departs from exponential and the power reaches a maximum. At this point a well-defined pulse train with a repetition rate $\Omega_{\text{mod}}/2\pi$ and 100% contrast can be observed in the temporal domain (see Fig. 3.3(b)). Beyond this point the power in the sidebands starts flowing back to the pump, eventually leading to near-ideal restoration of the initial state.

Figure 3.3: (a) Evolution of the power in the pump and in the first spectral sidebands. Black solid line: NLSE simulation, red dashed line: undepleted pump approximation. (b) Simulated temporal profile at the distance of 4.1 km where the developing pulse train contrast is maximum.
3. Modulation instability

3.4 Akhmediev breathers

It is clear that the undepleted pump approximation fails to describe the full evolution cycle of MI-dynamics. Numerical simulations on the other hand are approximation free (at least in principle) and can provide useful insight into the general characteristics of MI. Significantly, MI-related dynamics can also be described using exact analytical solutions of the NLSE known as Akhmediev breathers. Although these solutions have been found more than 25 years ago, it is only recently that they have become of practical interest as they have been shown to provide for analytical insight into the initial stages of supercontinuum generation in the long-pulse regime as well as to allow for the experimental excitation of a central structure of nonlinear wave theory known as the Peregrine soliton [5,102,220–222].

The Akhmediev breather solution of the NLSE is given by

\[ A(z', T) = \sqrt{P_0} \left[ \frac{(1 - 4a) \cosh(bz') + ib \sinh(bz') + \sqrt{2a} \cos(\Omega T)}{\sqrt{2a} \cos(\Omega T) - \cosh(bz')} \right] e^{iz'}, \quad (3.11) \]

where the coefficients \( a \) and \( b \) are related to the modulation frequency \( \Omega \) and fiber parameters as \( 2a = 1 - (\Omega/\Omega_c)^2 \) and \( b = \sqrt{8a(1-2a)} \) with \( \Omega_c \) given by Eq. (3.7). Note that Eq. (3.11) introduces a normalized distance \( z' = (z - z_0)/L_{NL} \), where \( L_{NL} = (\gamma P_0)^{-1} \) defines a nonlinear length scale and \( z_0 \) specifies the spatial center of the solution. Equation (3.11) is valid for modulation frequencies experiencing nonzero MI gain such that \( 0 < a < 1/2 \). In fact, the equation remains valid also when \( a > 1/2 \) but in this case it corresponds to another solution known as the Ma-soliton which describes different physics that are beyond the scope of this thesis. The parameter \( b > 0 \) in Eq. (3.11) governs the growth rate of MI and is related to the parametric gain through \( g = b/L_{NL} \) [5,41].

Figure 3.4 shows the temporal evolution of Akhmediev breathers with \( z_0 = 0 \) for increasing values of the modulation parameter \( a \). In all the cases the longitudinal evolution consists of a single growth-return cycle suggesting that the ABs are perfectly localized along the propagation direction \( z \). The longitudinal evolution is non-periodic so that after a single growth the solution asymptotically decays back towards a plane wave. As the parameter \( a \) increases the temporal period of the emerging pulse train becomes longer, and the peak power at the stage of maximum temporal compression becomes larger. At the limit \( a \to 1/2 \) the AB solution approaches the Peregrine soliton which is localized both in time and space [102,120,220,221].

In order to show that equation (3.11) provides a good description of MI dynamics, Fig. 3.5 compares the simulation shown in Fig. 3.2 with the AB solution corresponding to the simulation parameters. In plotting the AB solution, the spatial center \( z_0 \) was fixed as \( z_0 = 4.1 \) km in order to match the point of maximum compression to that observed in the simulation. The fact that MI-induced growth and decay of a train of pulses can be described using the analytical Akhmediev breather solution renders the approach very powerful as it allows to directly link the evolution of a weakly
3. Modulation instability

Figure 3.4: Akhmediev breather solutions for varying modulation parameters: (a) \( a = 0.20 \), (b) \( a = 0.30 \), (c) \( a = 0.40 \) and (d) \( a = 0.46 \). Maximum temporal compression occurs at the spatial center \( z_0 = 0 \).

modulated CW signal with an AB whose characteristics can be analytically evaluated. For instance, from the AB solution it is a simple matter to compute the characteristics (temporal duration, peak power) of the high-repetition rate pulse train generated via modulation instability for given pump-fiber parameters or, vice versa, to deduce the required parameters to obtain a desired pulse train. Moreover, in Paper I we show that the AB approach can also be used to obtain an accurate expression for the distance of maximum compression \( z_0 \) associated with a given weakly modulated CW field.

In order to derive the distance of maximum compression and to gain additional insight it is useful to analyze the AB solution far from the point of maximum amplitude using a simple Taylor-series expansion. In particular, retaining only the first-order terms and dropping a constant phase term that affects all terms in the expansion equally, this yields:

\[
A\left(z' \to \pm \infty, T\right) \approx \sqrt{P_0} \left[ 1 + 2be^{-b|z'|}e^{-i\psi_{\pm}} \cos(\Omega T) \right],
\]

(3.12)

where \( \psi_{\pm} = \pm \tan^{-1}(b/(2 - 4a)) = \pm \tan^{-1} \sqrt{2a/(1 - 2a)} \). There are several points to be made. Firstly, the expansion clearly has the form of a weakly modulated plane wave where the sidebands experience uniform exponential gain or decay for increasing \( z' \) at a rate given by the parametric gain when \( z' < 0 \) or \( z' > 0 \), respectively. Secondly, the pump-sideband phase-difference \( \theta = 2\psi_{\pm} \) is invariant with respect to the longitudinal coordinate. In fact, comparison with Eq. (3.10) reveals that these phase-differences precisely coincide with the two fixed points found using the perturbation analysis: (i) for \( z' < 0 \) : \( 2\psi_- = \theta_- \) implying exponential gain and (ii) for \( z' > 0 \) : \( 2\psi_+ = \theta_+ \) implying exponential decay.
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![Figure 3.5](Image)

Figure 3.5: (a) Temporal dynamics of simulated modulation instability. (b) Akhmediev breather solution constructed using the simulation parameters with \( z_0 = 4.1 \) km. Colormap is the same as in Fig. 3.2(b). (c-d) Comparison of temporal (c) and spectral (d) profiles at \( z = z_0 \), respectively.

Whilst the pump-sideband phase difference in the expansion (3.12) is invariant with respect to the \( z \)-coordinate the modulation amplitude is not, but rather evolves according to \( \delta(z) = 2be^{-|z|}/L_{NL} \). Since the AB solution is centered at \( z_0 > 0 \), it should be clear that a weakly modulated field of the form

\[
A(z = 0, T) \approx \sqrt{P_0} \left[ 1 + \delta(0)e^{-i\psi} \cos(\Omega T) \right],
\]

(3.13)

injected into an optical fiber will therefore reach the stage of maximum compression at a distance of

\[
z_0 = -\frac{L_{NL}}{b} \ln \left( \frac{\delta(0)}{2b} \right).
\]

(3.14)

Although approximative forms for this characteristic compression distance have been derived earlier [223, 224], to our knowledge Eq. (3.14) as derived from the AB theory and presented in Paper I is the first to take into account the effect of pump depletion and therefore provides a considerably improved estimate for the compression distance.

From an experimental point of view, it is important to understand how deviations
3. Modulation instability

from the ideal AB expansion given by Eq. (3.12) influence the applicability of the breather theory in describing the field evolution. To this end we have, in Paper I, analyzed three distinct deviations commonly encountered in experimental situations: (i) a real input field where the pump-sideband phase-difference is zero corresponding e.g. to intensity modulation, (ii) a single-sideband input corresponding to the injection of a strong CW pump and a single weak CW probe and (iii) a weakly modulated pulsed input. The main effect arising from non-ideal initial pump-sideband phase difference is to induce changes in the distance of maximal compression. This is because, as discussed in section 3.3, the growth rate of the modulation is sensitive to the relative pump-sideband phase difference. In contrast, when only a single sideband is present at the fiber input the phase difference is automatically set such that it coincides with that of an AB whose modulation parameter $a = 0.25$ [37]. Therefore, for modulation frequencies close to the maximum of MI gain the distance of maximal compression for a single sideband input is nearly identical to that given by Eq. (3.14). In both cases, however, the temporal field characteristics closely match those predicted by the Akhmediev Breather solution. In contrast, it should be clear that modulation instability of short pulses cannot be described directly by the temporally periodic AB solutions. In Paper I we show, however, that the AB formalism remains a useful tool even in the pulsed regime, where the evolution of individual modulation cycles can in fact be described in terms of "localized" breather states. Specifically, such breather states correspond to isolated sub-pulses of ABs whose continuous wave power is given by the corresponding point along the input pulse envelope (see Paper I for details).

3.5 Recurrence phenomena

It should be clear from the Taylor-series expansion (see Eq. (3.12)) that the analytical AB solution is only approximately equal to a sinusoidally modulated CW field far from the maximal compression point. A direct consequence of this discrepancy is that when a modulated field is injected into an optical fiber it will not eventually stabilize towards a plane wave (as would an AB) but rather experiences multiple growth-return cycles [5]. In fact, extensive numerical simulations show that such periodicity is a general feature resulting from the change in sign of the pump-sideband phase-difference along propagation: $\theta(2z_0) = -\theta(0)$. Hence, if the phase-difference at fiber input does not coincide with $\theta_-$ such that $\theta(2z_0) = -\theta_- = \theta_+$ the evolution is expected to be approximately periodic also in the longitudinal coordinate. Yet it should be stressed that the fixed point $\theta_+$ is unstable and therefore that, from an experimental point of view, even if the initial phase-difference is $\theta_-$ multiple growth-cycles should be expected. For a real input field the initial phase-difference is zero and it follows that $\theta(2z_0) = \theta(0) = 0$ implying the longitudinal evolution to be $2z_0$ periodic. Such behavior can be observed in Fig. 3.6(a), where the field evolution of Fig. 3.2 is simulated in a fiber twice as long as in Fig. 3.2. Note that these dynamics are closely related to the celebrated Fermi-Pasta-Ulam (FPU) recurrence [225–229].

Whilst the AB solution does not reproduce such recurrence dynamics there are
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Figure 3.6: (a) Recurrence dynamics obtained from numerical integration of the NLSE. (b) Exact analytical solution of the NLSE constructed with the simulation parameters.

solutions of the NLSE that are periodic along both the spatial and temporal coordinates [41]. Such solutions can be constructed using the Jacobi elliptic functions and belong to the same class of a more general family of solutions as the ABs characterized by two constants $a_1$ and $a_2$ (for ABs $a_1 = a_2 = a$). Although a detailed discussion is beyond the scope of this thesis (see Ref. [41] for details), Fig. 3.6(b) illustrates a particular example of such a solution constructed using the simulation parameters of Fig. 3.6(a). The constants $a_1 = 0.257$ and $a_2 = 0.243$ were chosen so as to match the solution with the simulations and the longitudinal center of the solution was fixed as $z_0 = 4.1$ km.

3.6 Higher-order modulation instability

Figure 3.7 displays the temporal evolution of a weakly modulated continuous wave obtained by numerically integrating the NLSE with an initial condition of the form $A(0,T) = \sqrt{P_0} [1 + \delta \cos (\Omega_{\text{mod}}T)]$ for varying initial modulation frequency $\Omega_{\text{mod}}$: (a) $\Omega_{\text{mod}} = 0.367$ THz ($a = 0.200$), (b) $\Omega_{\text{mod}} = 0.259$ THz ($a = 0.350$), (c) $\Omega_{\text{mod}} = 0.120$ THz ($a = 0.468$) and (d) $\Omega_{\text{mod}} = 0.079$ THz ($a = 0.486$). Whilst for large modulation frequencies (see Fig. 3.7(a,b)) the periodic evolution resulting from the non-ideal initial conditions can be observed clear discrepancies arise for small modulation frequencies (see Fig. 3.7(c,d)). Specifically, the time-domain field can be seen to undergo a cascaded splitting process eventually leading to the growth of a pulse train whose
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repetition frequency is three (see Fig. 3.7(c)) and five (see Fig. 3.7(d)) times the original modulation frequency.

Figure 3.7: Simulated evolution of a weakly modulated continuous wave with (a) $\Omega_{\text{mod}} = 0.367$ THz ($a = 0.200$), (b) $\Omega_{\text{mod}} = 0.259$ THz ($a = 0.350$), (c) $\Omega_{\text{mod}} = 0.120$ THz ($a = 0.468$) and (d) $\Omega_{\text{mod}} = 0.079$ THz ($a = 0.486$). In all the cases the initial modulation strength $\delta = 0.02$.

To understand these frequency multiplication dynamics it is useful to inspect the spectra of the fields at a selected fiber length (see Fig. 3.8). The dashed vertical lines in Fig. 3.8 illustrate the limit-frequencies $\pm \Omega_c$ of MI (i.e. the frequencies between which the parametric gain $g$ is real). For large initial modulation frequency (Fig. 3.8(a,b)) only the first Fourier mode experiences MI gain whereas for small values of $\Omega$ also harmonics of the first Fourier mode up to third (Fig. 3.8(c)) and fifth (Fig. 3.8(d)) order (generated through cascaded FWM) are smaller than $\Omega_c$ and experience nonzero gain. Bifurcation into the frequency multiplication regime occurs precisely when the initial modulation frequency is $\Omega = \Omega_c/2$ such that the second harmonic of the initial modulation frequency generated through cascaded FWM falls within the MI gain curve [115,230]. In terms of the $a$-parameter the condition for observing frequency multiplication is thus given by $a > 0.375$.

Although the frequency multiplication dynamics may seem complex, we show in Paper II that they too can be described in terms of exact analytical solutions of the NLSE. To this end, it is useful to introduce a dimensionless canonical form of the equation valid in the anomalous dispersion regime:

$$\frac{i}{\xi} \frac{\partial \psi}{\partial \xi} + \frac{1}{2} \frac{\partial^2 \psi}{\partial r^2} + |\psi|^2 \psi = 0,$$

(3.15)
3. Modulation instability

where \( \psi = A/\sqrt{P_0} \) and the dimensionless temporal and spatial coordinates \( \tau = \Omega_c/2T \) and \( \xi = \gamma P_0 z \), respectively. Thus, given a solution \( \psi(\xi, \tau) \) of Eq. (3.15) a corresponding solution of the dimensional variant (3.1) can be constructed using:

\[
A(z, T) = \sqrt{P_0} \psi \left( \gamma P_0 z, \frac{\Omega_c}{2} T \right). \tag{3.16}
\]

Complex solutions of integrable nonlinear equations such as the NLSE can be constructed from trivial solutions of the same equation using dressing techniques [41]. One such method is known as the Darboux transformation, originally introduced in the context of the Sturm-Liouville equation by Darboux and extended to the NLSE by Sall’ [231, 232]. The Darboux transformation method is based upon the fact that the NLSE can be represented as the compatibility condition between the following linear set of equations [41]:

\[
\begin{align*}
R_{\tau} & = -JR\Lambda + UR, \\
R_{\xi} & = -JR\Lambda^2 + URA - \frac{1}{2}(JU^2 - JU_{\tau})R,
\end{align*} \tag{3.17}
\]

where \( R, J, U \) and \( \Lambda \) are 2 \times 2 matrices defined by

\[
\Lambda = \begin{bmatrix} \nu & 0 \\ 0 & \nu^* \end{bmatrix}, \quad J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad U = \begin{bmatrix} 0 & \psi \\ -\psi^* & 0 \end{bmatrix}. \tag{3.18}
\]
3. Modulation instability

and

\[ R(\sigma) = \begin{bmatrix} r(\sigma) & s^*(\sigma) \\ s(\sigma) & -r^*(\sigma) \end{bmatrix}. \] (3.19)

Here \( r(\sigma) = r(\xi, \tau; \sigma) \) and \( s(\sigma) = s(\xi, \tau; \sigma) \) are complex functions which depend on a set \( \sigma \) of three arbitrary constants: the spectral parameter \( \nu \) and two integration constants that define the spatial and temporal centers of the solution. The compatibility condition \( R_{\tau\xi} = R_{\xi\tau} \) reproduces Eq. (3.15) which implies that equations (3.17) are simultaneously satisfied for all \( \Lambda \) if and only if \( \psi(\xi, \tau) \) solves the NLSE. The linear set (3.17) is invariant with the Darboux transformation:

\[ R(\sigma) \rightarrow R_1(\sigma_1, \sigma) = R(\sigma)\Lambda - \kappa(\sigma_1)R(\sigma) \] (3.20)

\[ U \rightarrow U_1 = U - J\kappa(\sigma_1) + \kappa(\sigma_1)J, \] (3.21)

where \( \kappa(\sigma_1) = R(\sigma_1)\Lambda_1R^{-1}(\sigma_1) \). A consequence of the invariance is that if a solution \( \psi_0 \) of the NLSE and its corresponding functions \( r_0(\sigma) \) and \( s_0(\sigma) \) are known, a new analytical solution can be constructed from Eq. (3.21) as

\[ \psi_1 = \psi_0 + \frac{2i(\nu_1^* - \nu_1)r_0(\sigma_1)s_0^*(\sigma_1)}{|r_0(\sigma_1)|^2 + |s_0(\sigma_1)|^2}. \] (3.22)

Of course, since the transformation provides not only a new solution for the NLSE but also the corresponding functions satisfying the linear set (3.17) through Eq. (3.20), the transformation can be utilized iteratively to obtain solutions of arbitrary orders. The power of the Darboux transformation method lies in the fact that the linear set (3.17) needs to be solved only once and all subsequent steps consist solely of algebraic transformations. For example, starting from the trivial plane wave solution \( \psi = \exp(i\xi) \) of the NLSE, directly integrating (3.17) and using Eq. (3.22) the canonical AB solution is obtained. Subsequent iterations of the transformation then yield nonlinear superpositions of elementary ABs whose characteristics (modulation parameter \( a_i \) and temporal and spatial centers) are embedded in the sets \( \sigma_i \). Physically such nonlinear superpositions can be excited by imposing distinct modulation frequencies \( \Omega_i \) on top of a continuous wave such that all the frequencies are modulationally unstable (\(|\Omega_i| < \Omega_c\)) [41].

In Paper II we show that the frequency multiplication dynamics described above can be interpreted as a process of higher-order modulation instability consisting of a nonlinear superposition of elementary ABs whose modulation frequencies are harmonics of each other. As an example, Fig. 3.9 compares the temporal evolutions of the simulations shown in Fig. 3.7(c,d) with solutions obtained using the Darboux transformation method. The temporal center of each elementary AB was set to zero and the spatial centers were adjusted to yield the best fit with the simulations.

From an experimental viewpoint, the analysis presented in Paper II suggests that it should be possible to excite higher-order modulation instability simply by injecting an intensity modulated CW into an optical fiber, provided that the modulation frequency is sufficiently small allowing multiple harmonic sidebands to experience nonzero MI.
3. Modulation instability

Figure 3.9: (a,c) Simulated evolution of a weakly modulated continuous wave with (a) $a = 0.468$ and (c) $a = 0.486$. (b,d) Third (b) and fifth (d) order analytical solutions constructed using the Darboux method. Colormaps are the same as in Fig. 3.7.

Gain. This was demonstrated experimentally by coupling a modulated plane wave optical field at 1550 nm into a standard single-mode fiber. The field was intensity modulated at a frequency of 16 GHz which was sufficiently small to allow for the first, second and third Fourier modes to experience instability gain under the elementary MI gain curve. The longitudinal field evolution was extracted from the experiments using cutback measurements and shown to exhibit clear higher-order MI dynamics with splitting of the initial modulated field into two, then three subpulses (see Paper II for details). While similar dynamics have been experimentally identified earlier (see Ref. [221]) it should be stressed that describing these observations analytically in terms of nonlinear superpositions of ABs using the Darboux transformation has not been attempted before. In fact, to our knowledge the study presented in Paper II represents the first example in any NLSE system where the Darboux transformation is used in the interpretation and design of physical experiments.
4. Soliton dynamics in nonlinear fibers

Modulation instability discussed in the previous chapter is often considered as a precursor to solitons [38, 42, 43], waves that propagate undistorted at constant velocity as a result of exact balance of dispersive and nonlinear effects [41, 44, 45]. For instance, supercontinuum generation in the long-pulse regime can be described in terms of an initial stage of modulation instability followed by the emergence and subsequent propagation and interaction of solitons. While these dynamics will be discussed in detail in the next chapter, the present chapter concerns with isolating and describing various soliton-related effects observed in realistic nonlinear fibers.

Solitons are ubiquitous in nature and can be found in a wide variety of physical systems including fluids [46, 233–236], optical fibers [7, 8, 237], plasmas [238–240], Bose-Einstein condensates [47–49] et cetera. From a mathematical viewpoint, they represent a class of solutions of integrable nonlinear equations such as the Korteweg-de Vries equation [212, 235], the nonlinear Schrödinger equation [6], and the Sine-Gordon equation [241].

A fundamental optical fiber soliton centered at frequency $\omega_S$ is an exact analytical solution of the NLSE and its general mathematical form can be written as

$$A(z, T) = \sqrt{P_0} \text{sech} \left( \frac{T - \beta_2 \Omega z}{T_0} \right) e^{\frac{i}{2} \beta_2 \Omega z - \frac{i}{2} \Omega T + i q z}, \quad (4.1)$$

where $\Omega = \omega_S - \omega_0$ is the frequency shift of the soliton from the NLSE reference frequency $\omega_0$ and the parameter $q = \gamma P_0/2$ represents the nonlinear phase shift of the soliton. The peak power $P_0$ and duration $T_0$ of the soliton must satisfy the soliton condition:

$$\frac{\gamma P_0 T_0^2}{|\beta_2|} = 1. \quad (4.2)$$

It should be noted that the fact that Eq. (4.1) solves the NLSE for arbitrary frequency detuning $\Omega$ reflects the Galilean invariance of the equation [41]. In particular, when $\Omega \neq 0$ the soliton propagates with a nonzero group velocity $(\beta_2 \Omega)^{-1}$ in the reference frame of the NLSE, yet its shape and spectrum remain unchanged. Of course, in situations where the soliton spectral bandwidth is large it may be necessary to account for higher-order terms and to model the propagation using the generalized NLSE (see Eq. (2.9)). In this case, the soliton of the form of Eq. (4.1) does not solve the propagation model. The functional shape of the soliton, however, is approximately preserved and the main impact of perturbations such as higher-order dispersion and the Raman effect is to induce temporal and spectral deformations along propagation [242]. The effects of such perturbations on solitons have been widely investigated and are
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now well understood [58].

4.1 Soliton self-frequency shift

The spectrum of an ultrashort (< 100 fs) soliton is relatively broad (> 3 THz) and hence its high-frequency components can act as a Raman pump for the low-frequency components. This energy transfer results in a net shift of the spectral center of mass of the soliton towards lower frequencies (longer wavelengths), effect which is known as the soliton self-frequency shift [10,12]. The rate of SSFS depends inversely on the duration of the soliton and can be evaluated using an overlap integral of the soliton spectrum with the Raman gain [10]:

\[
\frac{d\omega_S}{dz} = -\frac{10^5 \lambda^2 D}{8cT_0^3} \int_0^\infty \Omega^3 \Im \left[ \tilde{h}_R \left( \frac{\Omega}{2\pi T_0} \right) \right] \frac{d\Omega}{\sinh^2(\pi\Omega/2)} \text{[THz/km]},
\]

where \(D\) is the dispersion parameter defined in Eq. (2.5); \(\lambda\), \(D\), \(c\) and \(T_0\) are in units of centimeters and picoseconds and \(\tilde{h}_R(\omega)\) is the Fourier transform of the time-domain Raman response function \(h_R(t)\).

Because of higher-order dispersion, the rate of SSFS is a dynamic quantity that does not remain constant during the propagation of the soliton. Indeed, as the soliton redshifts it experiences a varying GVD coefficient \(\beta_2(\lambda)\) and adjusts its parameters (duration, power) in order to fulfill the soliton condition (Eq. (4.2)). In a fiber with a single ZDW \(|\beta_2(\lambda)|\) increases with wavelength which means that the soliton must increase its duration and decrease its power to retain its functional shape. These changes in the soliton characteristics subsequently reduce the rate of SSFS as can be seen from Eq. (4.3), leading to an eventual stabilization of the soliton wavelength. Other effects such as linear attenuation or self-steepening both act similarly to reduce the rate of SSFS [243].

Figure 4.1 illustrates typical spectral and temporal SSFS dynamics as obtained by numerically integrating Eq. (2.9). The input is a 40 fs (FWHM) soliton centered at \(\lambda = 1200\) nm. The fiber dispersion corresponds to PCF C of Fig. 2.4 and the nonlinear coefficient \(\gamma = 0.01\) W\(^{-1}\)m\(^{-1}\). Unless otherwise mentioned, the same fiber parameters are assumed throughout this chapter. That the soliton does not stay stationary in the time domain results from the fact that as the soliton redshifts it experiences a reduced group velocity compared to the reference frequency at which the pulse propagation equation is derived. Thus, as long as the redshift persists the soliton experiences a corresponding deceleration leading to a quasi-parabolaic trajectory in the time domain [244,245].

4.2 Soliton interaction with linear waves

One of the reasons why solitons are stable in a pure NLSE system is that their wave numbers lie in a range which is forbidden for linear waves. This can be easily seen by considering a soliton whose center frequency \(\omega_S\) coincides with that of the NLSE.
reference frequency $\omega_0$. In this case the wave number of the soliton corresponds to the nonlinear contribution $k_S = q = \gamma P_0 / 2$ (see Eq. (4.1) with $\Omega = 0$). On the other hand, the wave number of a linear wave with center frequency $\omega_L$ is simply $k_L = \hat{D}(\omega_L - \omega_S)$, which in the case of the pure NLSE reduces to $k_L = \beta_2 (\omega_L - \omega_S)^2 / 2$. For solitons to exist $\beta_2 < 0$ and thus it is clear that $k_S > k_L$ for all $\omega_L$. As a consequence, linear waves can never be in resonance with solitons and therefore energy cannot be transferred from solitons to linear waves [41,53].

However, when higher-order dispersion is accounted for it is possible for the soliton to be in resonance with a linear wave [52]. The frequency $\omega_L$ of such resonant dispersive radiation can be found by searching for the frequency at which the soliton-linear wave phase-mismatch $\Delta \phi = \phi_S - \phi_L$ vanishes. Here, $\phi_S$ and $\phi_L$ represent the phases of the soliton and the linear dispersive wave, respectively, and they can be readily evaluated (in an arbitrary reference frame) from:

$$\phi_S = \beta(\omega_S) z - \omega_S t + q z$$  \hspace{1cm} (4.4) \\
$$\phi_L = \beta(\omega_L) z - \omega_L t.$$  \hspace{1cm} (4.5)

Since the soliton is highly localized in time the phases must be calculated at the time that coincides with the temporal center of the soliton: $t = \beta_1(\omega_S) z$. The resonance condition $\Delta \phi = 0$ then yields:

$$\begin{align*}
\beta(\omega_L) &= \beta(\omega_S) - \beta_1(\omega_S) \cdot (\omega_S - \omega_L) + q \\
\Leftrightarrow \hat{D}(\omega_L - \omega_0) &= \hat{D}(\omega_S - \omega_0) - \hat{D}_1(\omega_S) \cdot (\omega_S - \omega_L) + q,
\end{align*}$$  \hspace{1cm} (4.6)

(4.7)

where $\hat{D}_1(\omega_S) = d\hat{D}(\omega - \omega_0) / d\omega|_{\omega=\omega_S}$. If the reference frequency $\omega_0 = \omega_S$ it follows from Eq. (4.7) that the resonance condition reduces to $\hat{D}(\Omega) = q$, with $\Omega = \omega_L - \omega_S$. At this point it should be noted that, although not explicitly written, all the following
phase-matching conditions presented in this chapter depend solely on the dispersion operator $\hat{D}(\Omega)$ through an equivalence relation similar to Eq. (4.7). In other words, the resonance conditions do not depend on the absolute wave number $\beta_0$ or the group velocity $\beta^{-1}_1$ at the reference frequency $\omega_0$.

An important characteristic of the resonant dispersive wave radiation is that it does not grow from noise but requires that the spectrum of the soliton overlaps with the phase-matched frequency [53,188]. It is easy to show that this frequency must lie in the normal dispersion regime which means that in practice significant energy transfer to the linear wave is expected only when the soliton is spectrally close to the ZDW. As the soliton sheds energy into the normal dispersion regime it experiences spectral recoil hence shifting away from the ZDW, deeper into the anomalous dispersion regime [53]. This in turn reduces the amount of energy shed by the soliton and eventually a quasistationary situation is reached where the radiation emission is so weak that spectral recoil is negligible [53]. Spectral dynamics of a dispersive wave emission process are illustrated in Fig. 4.2(a). Here, the input 40 fs soliton at $\lambda = 1050$ nm is spectrally close to the zero-dispersion wavelength at 1030 nm and both self-steepening and SRS are neglected by setting $f_R = \tau_{\text{shock}} = 0$ in Eq. (2.9). The reference frequency of the simulation is chosen to coincide with that of the soliton. The resonant wavelength calculated by numerically solving the equation $\hat{D}(\Omega) = q$ is indicated in Fig. 4.2(a) as the dashed vertical line.

![Figure 4.2: Simulation results showing spectral dynamics of a soliton emitting a dispersive wave (DW) into the short- (a) and long-wavelength normal dispersion regime (b). The fiber in (a) possesses a single ZDW at 1030 nm whereas the fiber in (b) has two ZDWs at 760 nm and 1260 nm. The dashed white lines indicate the calculated resonant wavelengths.](image)

In fibers with two zero-dispersion wavelengths a soliton propagating in the anomalous dispersion region can coherently seed the generation of dispersive waves beyond the long-wavelength ZDW through the same mechanism described above. In this case, the emission of the dispersive wave acts back on the soliton with the stabilization of
4. Soliton dynamics in nonlinear fibers

the Raman-induced self-frequency shift [175, 246]. This is because when the soliton is sufficiently close to the ZDW the spectral recoil mechanism resulting from the dispersive wave emission exactly balances the SSFS [175, 246]. Figure 4.2(b) illustrates these dynamics, showing a numerical simulation of a 40 fs soliton launched between the zero-dispersion wavelengths (790 nm and 1260 nm) of PCF B shown in Fig. 2.4. It can be seen how the spectrum of the soliton redshifts from the input wavelength of 1100 nm to approximately 1200 nm where it stabilizes with the simultaneous emission of a dispersive wave into the long-wavelength normal dispersion regime. The dashed white curve indicates the resonance wavelength calculated from Eq. (4.6). That the resonance wavelength does not remain constant along propagation reflects the fact that as long as the soliton redshifts, the resonant wavelength given by Eq. (4.6) changes. It should be noted that similar SSFS stabilization can occur in a photonic band gap fiber [247–249]. In this case, however, the stabilization is not associated with the emission of a dispersive wave but rather arises due to the unique linear and nonlinear properties of PBG fibers in the vicinity of the band gap.

4.3 Resonant radiation from soliton-dispersive wave interaction

The dispersive wave emission process described above is directly driven by a soliton. When a weak linear wave co-propagates with a soliton, the Kerr term \(|A(z, T)|^2 A(z, T)|^2\) induces additional sources in the nonlinear polarization that can each generate similar phase-matched radiation at different frequencies [250–256].

Consider a field \(A(z, T) = S(z, T) + w(z, T)\) consisting of a soliton \(S(z, T)\) and a weak wave \(w(z, T)\) with \(|S|^2 \gg |w|^2\). Injecting \(A(z, T)\) into the Kerr-term and linearizing with respect to \(w(z, T)\) yields three source terms: (i) \(|S|^2 S\), (ii) \(2|S|^2 w\) and (iii) \(S^2 w^*\). All three terms can in principle drive the generation of phase-matched radiation. In fact, the first term (i) as described in the previous section is precisely that responsible for the generation of a linear wave by a soliton and governed by the phase-matching condition (4.6). The other two terms, on the other hand, result from the nonlinear mixing of the soliton and the weak wave and can lead to additional phase-matched radiation processes [250, 252]. The phases of the terms involving both the soliton \(S(z, T)\) and the weak wave \(w(z, T)\) are:

\[
\phi_{(ii)}(\omega_L) = \beta(\omega_W)z - \omega_W t, \tag{4.8}
\]

\[
\phi_{(iii)}(\omega_L) = 2\beta(\omega_S)z - 2\omega_S t + 2qz - \beta(\omega_W)z + \omega_W t, \tag{4.9}
\]

where the frequency of the weak wave is assumed to be \(\omega_W\). Similarly to the case of a dispersive wave generated by a single soliton, the nonlinear mixing of a soliton and a weak wave will generate a dispersive wave (at frequency \(\omega_L\)) provided that the phase \(\phi_{(ii)}\) or \(\phi_{(iii)}\) is equal to the phase \(\phi_L\) of the dispersive wave. This leads to the following
conditions when evaluated at the temporal center of the soliton:

\[
\beta(\omega_L) = \beta(\omega_W) - \beta_1(\omega_S) \cdot (\omega_W - \omega_L) \tag{4.10}
\]

\[
\beta(\omega_L) = 2\beta(\omega_S) - \beta(\omega_W) + 2q - \beta_1(\omega_S) \cdot (2\omega_S - \omega_W - \omega_L) \tag{4.11}
\]

It should be noted that the resonant process (4.11) is qualitatively distinct from the standard degenerate four-wave mixing process also driven by the term \(S^2w^*\). Indeed, in the standard FWM picture where the interacting waves are CWs the third wave is efficiently generated as a result of both energy \(2\omega_S = \omega_W + \omega_L\) and momentum \(2\beta(\omega_S) = \beta(\omega_W) + \beta(\omega_L)\) conservation. In contrast, here energy and momentum are shared between all the interacting waves (including all the Fourier components of the soliton) \([250, 252]\). As a consequence, although energy and momentum are conserved in the overall process, Eq. (4.11) cannot be directly associated with momentum conservation nor does it hold that \(2\omega_S = \omega_W + \omega_L\).

Figure 4.3 revisits the simulation shown in Fig. 4.2(b) over 1.2 m when a 50 W, 4 ps hyperbolic secant pulse centered at 960 nm is launched together with the soliton at 1100 nm (see Fig. 4.3(a) for a time-frequency representation of the input condition). As shown in Fig. 4.3(b), two distinct spectral components can be observed in the long-wavelength normal dispersion regime \((\lambda > 1260 \text{ nm})\) at the output of the fiber. The component around 1400 nm corresponds to the dispersive wave emitted by the soliton alone whereas the component around 1600 nm arises from the nonlinear mixing of the soliton and the weak pulse. To show this explicitly, the dashed red horizontal line indicates the resonant wavelength associated with the single-soliton emission process as calculated from Eq. (4.7), whilst the dashed white horizontal line indicates the resonant wavelength of the soliton-weak wave interaction as calculated from the FWM-like interaction (4.11).

Figure 4.3: Simulated time-frequency representations at fiber input (a) and output (b) when a 40 fs soliton at 1100 nm is launched together with a weak 4 ps pulse at 960 nm into a 1.2 m-long PCF with two ZDWs. Spectral components in the long-wavelength normal dispersion regime are generated by the soliton alone (dashed red line) and by the interaction of the soliton and the weak pulse (dashed white line) as described in the text. Colorbar on the right and the y-axis on the left apply to both plots.

Generation of phase-matched radiation arising from the interaction of a soliton and
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A weak wave can occur spontaneously when only a single wave is injected into a highly nonlinear fiber. Indeed, in Paper III we show experimentally and numerically that when a single temporally broad pulse is launched into a PCF with two ZDWs a situation equivalent with that shown in Fig. 4.3 can arise although no soliton is present at the fiber input. This is because solitons are spontaneously generated inside the fiber from the broad input pulse due to noise-driven modulation instability (see Chapter 5), and can subsequently interact with the pump pulse residue. In other words, the interaction can arise autonomously where, on the one hand, the soliton is generated from the input pulse and, on the other hand, the input pulse itself acts as the weak wave of the interaction (see Paper III). It is worth noting that previous studies have also identified the spontaneous mixing of a soliton and a dispersive wave emitted by another soliton as a spectral broadening mechanism in supercontinuum generation [251, 256]. This process is governed by the cross-phase modulation-like interaction $2|S|^2w$ (Eq. 4.10) and does not lead to a well-separated spectral peak but rather in the broadening of the already existing dispersive wave spectrum.

4.4 Dispersive wave trapping

The dispersive spreading of a linear wave in the normal dispersion regime can be suppressed when the wave co-propagates with a Raman redshifting soliton. Moreover, during the interaction the dispersive wave experiences a continuous blueshift such that its group-velocity remains equal to that of the solitons. It is then said that the dispersive wave is trapped by the soliton [58, 257–262].

This pulse trapping effect can be understood to arise from cross-phase modulation mediated interaction of the decelerating soliton and the dispersive wave. On the one hand, the soliton introduces an increase in the local refractive index profile which acts as a repulsive barrier to the dispersive wave on the front edge of the wave. On the other hand, the deceleration of the soliton due to SSFS introduces a gravity-like inertial force which opposes the spreading of the dispersive wave on the trailing edge [58, 260, 261]. The continuous blueshift of the dispersive wave has been described both in terms of cross-phase modulation [186, 257, 258, 263–265], and the resonant process governed by Eq. (4.10) [58, 254, 261]. However, it should be noted that both pictures arise from the same term $2|S|^2w$ (see the previous subsection) suggesting that the two interpretations are essentially similar.

An illustration of dispersive wave trapping dynamics is shown in Fig. 4.4 where a 40 fs soliton centered at 1200 nm co-propagates with a weak group-velocity matched 40 fs hyperbolic secant pulse at 885 nm in a 5-m-long PCF. The time-frequency representations of the field at selected propagation distances (Fig. 4.4(a-c)) show how the dispersive wave blueshifts as the soliton redshifts such that the group-velocity matching condition is fulfilled at all propagation distances. Moreover, the pulse in the normal dispersion regime is seen to remain as a confined wave packet with limited dispersive broadening in contrast to the situation where the weak pulse alone propagates inside the fiber (see Fig. 4.4(d)).
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Figure 4.4: (a-c) Simulated time-frequency representations of the field at selected propagation distances when a 4.3 kW peak power soliton co-propagates with a weak 0.5 kW peak power group-velocity matched pulse in the normal dispersion regime. In (d) the soliton is absent and the propagation of the weak pulse alone is simulated. Colorbar shown applies to all plots, all plots in the top panel have the same y-axis and all plots are normalized to the same value. Note, however, the different time axes. S: soliton, DW: dispersive wave.

4.5 Nonlinear superpositions of solitons

The Darboux transformation discussed in Chapter 3 can be used to construct nonlinear superpositions of solitons. Indeed, starting from the trivial solution $\psi = 0$, directly integrating (3.17) and using Eq. (3.22) the canonical single-soliton solution is obtained. Solutions given by subsequent iterations describe interactions of two or more fundamental solitons. All the elementary solitons are then uniquely specified by their amplitudes, frequencies, temporal centers and phases [41,266].

4.5.1 Higher-order solitons

Higher-order solitons are an important class of nonlinear superpositions of fundamental solitons. This is because at a particular point along their evolutionary cycle they attain the simple form of a hyperbolic secant: $A(T) = \sqrt{P_0 sech(T/T_0)}$, where the peak power and duration are related by

$$N^2 = \frac{\gamma P_0 T_0^2}{|\beta_2|},$$

and $N \in \mathbb{N}$ is known as the soliton order. Clearly the fundamental soliton is regained by setting $N = 1$. Higher-order solitons correspond to bound states of $N$ temporally
overlapping fundamental solitons with identical velocities (but alternating phases) [53, 266]. The peak powers and durations of the elementary solitons can be obtained from the relations [14,53,59]:

\[ P_n = \frac{P_0(2N - 2n + 1)^2}{N^2} \]  
\[ T_n = \frac{T_0}{2N - 2n + 1}, \]

where \( n = 1, ..., N \). In contrast with the fundamental soliton, higher-order solitons do not retain their shape during propagation but propagate in a complex albeit periodic manner with a period given by \( z_{sol} = \pi N^2/(2\gamma P_0) \). Figure 4.5 displays the spectral and temporal dynamics of a fourth-order soliton where the evolution is plotted for a single soliton period.

Figure 4.5: Evolution of the temporal (a) and spectral (b) profile of a fourth-order soliton solution over a single soliton period. The solution was constructed using the Darboux method and the relevant parameters are \( P_0 = 1600 \text{ W}, T_0 = 50 \text{ fs}, \gamma = 0.01 \text{ W}^{-1}\text{m}^{-1} \) and \( \beta_2 = -2.5 \text{ ps}^2/\text{km} \).

Higher-order solitons are unstable against perturbations [14,63,267]. This instability is known as soliton fission and manifests as the break-up of the higher-order soliton into its constituent fundamental solitons whose peak power and temporal duration are given by Eqs. (4.13) and (4.14), respectively [14,59]. It should be clear from Eq. (4.14) that by harnessing the process of soliton fission it is therefore possible to generate fundamental solitons whose duration is significantly smaller than the duration
of the input pulse. Moreover, by controlling the peak power of the input pulse the peak powers and durations of the emerging solitons can be systematically controlled. This experimental technique was exploited in Paper IV and will be discussed below.

### 4.5.2 Collisions of solitons

A particular interaction described by a nonlinear superposition of solitons is the collision of two solitons centered at different frequencies. In the case of the NLSE, solitons exhibit particle-like behavior and remain unaffected by the collision (aside from a phase shift), as illustrated in Fig. 4.6 which shows the exact analytical solution of the NLSE describing the collision of two solitons. The elasticity of soliton-soliton collisions is often considered as a defining property of solitons and lies at the root of the particle-like term "soliton" [45, 238]. However, in the presence of perturbations such as stimulated Raman scattering and higher-order dispersion collisions are no longer elastic and can lead to energy transfer between the interacting solitons [65, 87, 268–276] or to the emission of radiation [277–282].

![Figure 4.6: Analytical solution of the NLSE describing the collision of two 900 W peak power solitons with opposite velocities. The fiber parameters are $\beta_2 = -22$ ps$^2$/km and $\gamma = 0.01$ W$^{-1}$m$^{-1}$. The solution was constructed using the Darboux method.](image)

#### Effect of stimulated Raman scattering: energy transfer

Consider the collision of two solitons with different frequencies in an optical fiber. When the frequency separation of the colliding solitons is less than the Raman gain bandwidth, the high-frequency soliton can act as a Raman pump for the low-frequency soliton. This energy transfer leads to a corresponding change in the amplitudes and durations of the solitons but also affects their center frequencies. Such Raman-induced cross-frequency shift arises because of the spectral dependence of the Raman gain [65, 87, 268, 270–276]. For instance, if the frequency separation of the solitons is less than 13.2 THz the low-frequency components of the low-frequency soliton experience greater gain than the high-frequency components leading to a net downshift of the mean frequency [271]. Of course, the fact that the more redshifted soliton gains energy during the collision leads to a corresponding increase in its rate of self-frequency shift and it is therefore nontrivial to isolate the relative contribution of the Raman cross-frequency shift [276].

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Intuitively one would expect the largest energy transfer to occur when the frequency separation of the solitons coincides with the peak of Raman gain at 13.2 THz. This, however, is not true since larger frequency difference implies larger relative velocity which in turn decreases the interaction distance between the solitons. In fact, the energy transfer increases monotonously as the frequency separation of the solitons decreases [268]. When the frequency separation is small, however, the collision dynamics become sensitive to the relative phase of the solitons [274]. In paper VII we show that Raman-induced energy transfer during soliton collisions plays a key role in the emergence of abnormally redshifted "rogue" solitons in noise-driven supercontinuum generation. These dynamics will be discussed in detail in the next chapter.

Effect of higher-order dispersion: radiation through cascaded four-wave mixing

Higher-order dispersion can lead to the emission of linear waves during soliton collisions. Such emission can be mathematically treated by means of perturbed inverse scattering theories but this type of an approach does not provide an intuitive picture of the underlying physics [277–282]. On the other hand, physical insight into the emission process can be obtained using a four-wave mixing picture.

During the collision the Kerr-interaction of the solitons drives the generation of spectral sidebands due to four-wave mixing. In a pure NLSE system these sidebands exactly vanish after the collision (see Fig. 4.7(a-c) for an illustration) [189], but in the presence of higher-order dispersion the situation is different and one or more of the sidebands may be preserved after the collision.

Consider the collision of two solitons at \( \omega_{p1} \) and \( \omega_{p2} \) with \( \omega_{p1} = \omega_{p2} - \Omega \) and \( \Omega > 0 \). The (cascaded) FWM interaction will drive the growth of sidebands at \( \omega_{+M} = \omega_{p2} + M \Omega \) and \( \omega_{-M} = \omega_{p1} - M \Omega \), with \( M \in \mathbb{N} \). If the solitons are replaced with continuous waves it can be shown that a sideband at \( \omega_{+M} \) undergoes monotonous growth if the following cascaded phase-matching condition is satisfied [283]:

\[
\beta(\omega_{+M}) = (M+1)\beta(\omega_{p2}) - M \beta(\omega_{p1}) - (M+1)\phi_{p2} + M\phi_{p1}, \tag{4.15}
\]

where \( \phi_{pi} = \gamma P_{pi}, i = 1, 2 \) is the nonlinear phase of the \( i^{th} \) input wave. Similar condition is obtained for the low-frequency sideband by interchanging \( 1 \leftrightarrow 2 \). It can be envisaged that if the phase-matching condition given by Eq. (4.15) (with \( \phi_i = q_i = \gamma P_i/2 \)) holds when the interacting waves are solitons instead of CWs the phase-matched sideband may display unexpected behavior. The emission of radiation by soliton collision in the presence of third-order dispersion is illustrated in Fig. 4.7(e-f). The third-order dispersion parameter \( \beta_3 \) was chosen such that Eq. (4.15) is fulfilled for \( M = 2 \). It is clear that after the collision the \( M = 2 \) high-frequency sideband does not vanish, suggesting that the emission of radiation during a soliton-soliton collision in the presence of higher-order dispersion can be physically interpreted as a phase-matched cascaded FWM process. It is worth noting that the FWM components associated with the collision can persist even in the absence of higher-order dispersion provided that the system
contains other forms of perturbations. For example, in a soliton transmission system using lumped amplifiers the periodicity of the amplifiers can be in resonance with the FWM phase-mismatch leading to uncontrollable growth of the FWM sidebands [284].

Figure 4.7: Simulated spectra before (a, d), during (b, e) and after (c, f) the collision of two 200 fs solitons. In (a - c) the governing model is the pure NLSE with $\beta_2 = -24$ ps$^2$/km and $\gamma = 0.1$ W$^{-1}$m$^{-1}$ whereas in (e - f) third-order dispersion is included with $\beta_3 = 6.9 \times 10^{-2}$ ps$^3$/km.

In Paper IV we report experimental evidence of dispersive wave emission during a soliton-soliton collision in a fiber with two zero-dispersion wavelengths. The colliding solitons at different frequencies are generated from a single mode-locked Ti:Sapphire laser using a Michelson interferometer in conjunction with the nonlinear processes of soliton fission and soliton self-frequency shift. Specifically, 200 fs pulses from the oscillator are divided between two arms of a Michelson interferometer in which a variable attenuator is placed in the moveable arm. The interferometer output then consists of a pulse pair of variable relative peak power and temporal delay. The pulses are launched into a highly nonlinear PCF where they correspond to higher-order solitons and therefore fission into redshifting fundamental solitons whose peak power and temporal duration are given by Eqs. (4.13) and (4.14). Of course, since the pulses at the interferometer output have different peak powers, they will fission into solitons with different temporal durations. As a consequence, the generated solitons will undergo the SSFS at a different rate (see Eq. (4.3)) resulting in a frequency difference between the solitons. By controlling the relative temporal delay of the interferometer the solitons can then be made to collide inside the fiber resulting in a deterministic and controllable
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excitation of dispersive waves (for details, see Paper IV). It should be stressed that to our knowledge the experimental signatures presented in Paper IV represent the first experimental observation of radiation emission through soliton collisions in any NLSE system. Paper IV also shows indirect experimental signatures of the effect of soliton collision in enhancing the redshift of the solitons. Significantly, when the collision is adjusted to occur early on along the fiber, the enhanced redshift allows for the more redshifted soliton to reach the ZDW and emit dispersive radiation of its own. Similar signatures of collision-enhanced soliton redshift dynamics were observed also by Luan et al. [274].

In Paper V we use numerical simulations to show that soliton collisions and associated dispersive wave generation can also occur spontaneously in the context of supercontinuum generation when a single broad pulse is launched into a nonlinear fiber. In particular, we show that dispersive waves generated through the collision process possess abnormal characteristic such as very high peak power and considerably higher degree of temporal localization as compared to dispersive waves emitted by single solitons. These characteristics will be elaborated in the next chapter.
5. Optical rogue waves

Rogue waves refer to rare and enormous waves that appear without a warning on top of a sea surface and subsequently disappear without a trace [90,91]. While it has been suggested that such freak waves have been responsible for many catastrophic maritime disasters [91], their existence was not proven before one such wave was recorded hitting the Draupner platform in the North Sea on January 1, 1995 (happy new year!). Today the mechanism of rogue wave formation in oceans remains unclear and both linear and nonlinear processes have been suggested [92]. Significantly, similarly to optical fiber systems oceanic wave dynamics can, under certain conditions, be approximated by the nonlinear Schrödinger equation. In particular, the process of modulation instability that is often attributed to the emergence of freak ocean waves is common to both optics and hydrodynamics [20,92]. These phenomenological similarities suggest that it may be possible to observe analogous rogue wave formation also in an optical fiber context.

The first pioneering observation of optical rogue waves was reported in late 2007 by Solli et al., who showed that the shot-to-shot statistics of broadband supercontinuum spectra are associated with a small number of statistically rare rogue soliton events that have experienced enhanced Raman redshift [89]. Following this seminal experiment there has been considerable subsequent interest in the study of extreme events and wave localization in optics [93–111], and in a wide variety of other nonlinear systems [112–121]. It should be stressed that in an optical context the term "rogue wave" has been used to describe various distinct phenomena in a wide range of subdisciplines ranging from filamentation [96] to telecommunications [108] and fiber amplifiers [95]. Therefore it should be clear that a unified theory to describe rogue phenomena in optics is improbable and here we restrict our attention to rare events that emerge in the context of supercontinuum generation. We begin this chapter by a brief introduction to the process of fiber supercontinuum generation and link the dynamical field evolution with the regime where rogue-wave-like fluctuations are observed. We then proceed to analyze in detail the statistical characteristics and formation dynamics of optical rogue waves as originally measured experimentally by Solli et al. Specifically, we show that the interpretation of the existence of extreme waves on the basis of measurements carried out in the presence of spectrally selective optical elements is in fact inappropriate. Yet, we will also show that even in the absence of spectral selection, a supercontinuum field can contain highly localized and intense events. These events are not, however, connected in a transparent manner to the spectral fluctuations measured in the original optical rogue wave experiments.
5. Optical rogue waves

5.1 Supercontinuum generation

Supercontinuum generation refers to the dramatic broadening of the spectrum of narrowband laser light propagating in a nonlinear medium as a result of a myriad of nonlinear effects. Whilst originally observed in bulk glass [285, 286], the majority of subsequent SC-related studies have been conducted in an optical fiber context [59, 60, 62, 64, 65, 71, 72]. Indeed, despite the low nonlinearity of silica, the interaction length of light and matter can be made orders of magnitude larger in single-mode optical fibers as compared to bulk media, allowing for the efficient accumulation of nonlinear optical effects [59].

SC generation in optical fibers is a complex dynamical process involving a rich variety of phenomena [58–60]. In fact, all of the soliton effects described in the previous chapter as well modulation instability discussed in Chapter 3 can be observed in the context of SC generation. The specific dynamics as well as the characteristics of the resulting SC field crucially depend on the parameters of the fiber as well as those of the pumping laser source. Although all applications utilizing SC generation evidently benefit from the broad spectral bandwidth, different applications may require additional specific characteristics. For instance, telecommunication applications benefit from spectral flatness [287–292], while pulse-to-pulse stability is paramount for frequency metrology applications [73–76, 293].

The stability characteristics of SC sources are typically quantified through the complex degree of first-order coherence [88]:

\[
|g_{12}^{(1)}(\lambda)| = \left| \frac{\langle \tilde{E}_1(\lambda)\tilde{E}_2(\lambda) \rangle}{\sqrt{\langle |\tilde{E}_1(\lambda)|^2 \rangle \langle |\tilde{E}_2(\lambda)|^2 \rangle}} \right| ,
\]

where the angular brackets denote ensemble averaging over independently generated pairs of SC spectra \( \tilde{E}_i(\lambda) \). If all pulses launched into the fiber generate the exact same output it follows that \( \tilde{E}_1(\lambda)\tilde{E}_2(\lambda) = |\tilde{E}_1(\lambda)|^2 \). In this case \( |g_{12}^{(1)}| = 1 \) and the SC is fully coherent. In contrast if each input pulse generates a distinct output with a freely fluctuating pulse-to-pulse spectral phase it follows that \( |g_{12}^{(1)}| = 0 \) and the field is incoherent. The spectral degree of coherence \( |g_{12}^{(1)}| \) measures primarily phase stability [59]. Intensity fluctuations, on the other hand, must be quantified through the relative intensity noise that can be extracted from the radio-frequency spectrum of a given SC pulse train [294–297]. However, generally intensity and phase stability go hand-in-hand and typically only the spectral degree of coherence is analyzed [59].

When the pump wavelength lies deep in the normal dispersion regime the dominant spectral broadening mechanism (for short pump pulse) is self-phase modulation yielding supercontinua with quasi-perfect stability. The downfall of pumping in the normal dispersion regime is, however, that dispersive temporal broadening rapidly reduces the peak power of the pulse leading to considerably reduced bandwidths as compared to anomalous dispersion pumping [59]. Although a detailed discussion on normal dispersion supercontinua is out of the scope of this thesis it should be pointed out that recent
advances in the fabrication of all-normal dispersion photonic crystal fibers have allowed for the generation of fully coherent octave-spanning spectra with relatively low pulse energies [298–302].

Because of the significantly larger achievable bandwidth, the majority of fiber supercontinuum sources rely on anomalous dispersion pumping [59,72]. In this case, the coherence characteristics of the SC can be estimated from the soliton order $N$ of the input pulse (see Eq. (4.12)): for small $N$ significant coherence is expected whereas when $N \gg 10$ the SC exhibits large shot-to-shot fluctuations [59]. As a consequence, coherent supercontinua generally require the use of ultrashort femtosecond pump pulses whereas the use of picosecond (or longer) pulses leads to an incoherent supercontinuum [88]. These qualitative differences are highlighted in Figure 5.1, where the stability characteristics of simulated ensembles of supercontinua generated by femtosecond ($N = 4$) and picosecond ($N = 100$) pump pulses are compared. Both ensembles consist of 1000 realizations where the initial conditions of individual simulations are identical except for a random noise term representing quantum shot noise. While the femtosecond supercontinuum is clearly insensitive to input noise yielding the same output spectrum for each input, considerable spectral fluctuations are observed for the picosecond case (compare Fig. 5.1(a,b)). Moreover, the fact that each output spectrum is quasi-identical in the femtosecond case is clearly reflected in the complex degree of coherence which is close to unity for all wavelengths within the SC spectrum (see Fig. 5.1(c)). This is in contrast with the picosecond case where the complex degree of coherence is close to zero for all wavelengths as expected for an output containing considerable shot-to-shot fluctuations (see Fig. 5.1(d)).

5.2 Dynamical evolution of long-pulse supercontinuum

In the femtosecond regime supercontinuum generation can be described in terms of the soliton fission process which was briefly described in section 4.5. In this case the input pulse corresponding to a higher-order soliton breaks into ultrashort fundamental solitons that expand the SC spectrum towards longer wavelengths through the Raman-induced self-frequency shift. On the other hand, extension to shorter wavelengths occurs due to the interplay of the resonant radiation emission and dispersive wave trapping mechanisms discussed in sections 4.2 and 4.4, respectively [59,62,188,303]. As can be seen in Fig. 5.1(a,c), the soliton fission process is deterministic and cannot lead to the emergence of rogue events for fixed pump-fiber parameters as every output is virtually identical. This is in marked contrast with the long-pulse regime which gives rise to extreme fluctuations and optical turbulences that are favorable for the spontaneous and random emergence of large amplitude waves [89,93,94,98–100,104].

In fact, the soliton self-frequency and the emission and subsequent trapping of dispersive waves are also governing the spectral broadening in long-pulse supercontinuum [59,60]. It is rather the process responsible for the formation of solitons that is fundamentally different in the long- and short-pulse excitation regimes yielding dissimilar sensitivities towards input noise. In particular, in the long pulse regime spectral
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Figure 5.1: (a,b) Output spectra of 1000 individual simulations (gray) and the calculated mean spectrum (red) when the input is a shot-noise limited pulse centered at 1050 nm with (a) 50 fs duration and 1.6 kW peak power and (b) 5 ps duration and 315 W peak power. (c,d) Degree of coherence calculated from the simulated ensembles for (c) the 50 fs input pulse and (d) the 5 ps input pulse. Both ensembles use the fiber parameters of PCF C of Fig. 2.4.

broadening is triggered by the spontaneous growth of noise outside the input pulse spectrum due to modulation instability [5,59,304]: as discussed in Chapter 3, a weak signal wave co-propagating with a strong pump wave in the anomalous dispersion regime can undergo exponential growth due to four-wave mixing phase-matched through self-phase modulation. Although here only the strong pump wave is present at fiber input, it is the broadband noise that acts as the probe and is exponentially amplified. In the time domain the long-duration input field breaks into a train of short pulses which subsequently evolve into fundamental solitons. In contrast to the evolution of Akhmediev breathers discussed in section 3.4, here the emerging short-duration pulse train is noisy with significant amplitude and phase fluctuations from pulse to pulse. This is because the entire band of modulationally unstable frequencies with random phase relationships are simultaneously amplified, each frequency corresponding to a breather with specific characteristics [5].

Of course, since the modulation instability seed (input noise in this case) varies from shot-to-shot, the precise fashion how the initial pulse break-up takes place will correspondingly vary, leading to differences in the dynamics of the redshifting solitons for each pulse injected into the fiber. This is in contrast with the process of soliton fission governing the initial stages of femtosecond supercontinua where the soliton for-
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Information occurs in a deterministic and systematic fashion such that every pulse break-up is identical. Indeed, it is precisely this differentiating fact that explains the stability or instability of femtosecond and picosecond supercontinua, respectively.

An additional consequence of the different pulse break-up dynamics is that long-pulse supercontinuum typically exhibits a richer range of soliton effects than its femtosecond counterpart. For instance, in the femtosecond case solitons are ejected from the input pulse envelope in an ordered fashion such that the solitons with the largest peak power (shortest duration) are ejected first. This effectively prohibits the occurrence of solitonic collisions because the larger solitons also experience the largest Raman-induced deceleration. In contrast, in a supercontinuum driven by spontaneous modulation instability the solitons are formed quasi-simultaneously across the pump pulse. However, the solitons formed in the vicinity of the pulse envelope have the largest peak power and therefore undergo the most rapid Raman-induced redshift. These solitons also experience the largest deceleration and are delayed across the wing of the broad input pulse leading to multiple collisions with solitons formed near the trailing edge of the pulse. In effect, these collisions lead to a net transfer of energy as discussed in section 4.5 such that the more redshifted solitons gain energy thereby further enhancing their rate of redshift. In fact, it has been argued the the spectral bandwidth of long-pulse supercontinuums cannot be explained in the absence of collisions [65,273,305].

Figure 5.2 shows the spectral and temporal evolution of a particular realization of the simulated ensemble shown in in Fig. 5.1(b), illustrating the dynamics of long-pulse supercontinuum generation discussed above. The evolution of the spectral (5.2(a)) and temporal (5.2(b)) envelopes are visualized as a continuous density plot and the top curves in the plots display the corresponding profiles at the fiber output. The white arrows in Fig. 5.2(a) highlight that the spectral broadening indeed initiates outside the pulse spectrum from the noise-seeded growth of modulation instability sidebands. In the time domain (see Fig. 5.2(b)), the build-up of a fast modulation on top of the input pulse envelope and the subsequent formation of solitons can be clearly seen to occur after approximately 4 m. Following the formation of propagating solitons the extension of the SC spectral bandwidth towards longer and shorter wavelengths results from the soliton self-frequency shift and the dispersive wave trapping effect, respectively. Of course, if the fiber possesses two zero-dispersion wavelengths the soliton self-frequency shift is cancelled at the long-wavelength zero dispersion point, accompanied with the associated emission of dispersive waves into the long-wavelength normal dispersion regime (see section 4.2). It should be stressed that Fig. 5.2 shows the evolution of a single realization and that any change in the initial condition would induce considerable changes in the exact field evolution and output SC characteristics.

5.3 Rogue waves in supercontinuum generation

The relative intensity noise and the complex degree of coherence are average quantities and do not provide information on the shot-to-shot level. Therefore from these measures it is impossible to infer whether or not the output supercontinuum contains rare
5. Optical rogue waves

Figure 5.2: Simulation results showing the spectral (a) and temporal (b) dynamics of noise-driven supercontinuum generation in the anomalous dispersion regime. Top curves show the output spectral and temporal profiles and the white arrows in (a) highlight the noise-seeded growth of modulation instability sidebands.

Events with rogue-wave-like characteristics. Of course, were it possible to accurately measure the output field generated by every input pulse it would be straightforward to identify the presence of statistically rare events. Unfortunately, the large bandwidth associated with supercontinua typically preclude such single shot measurements.

Information can, nonetheless, be obtained at the shot-to-shot level in a straightforward manner if only a limited spectral band of the SC spectrum is analyzed. This can be done using e.g. a fast photodetector in conjunction with a wavelength selective element. Significantly, when fluctuations in the long-wavelength spectral edge of an incoherent SC spectrum are quantified in this manner, the shot-to-shot intensity distribution is found to be strongly non-gaussian [5, 89, 93, 98–100, 306]. The particular L-shape of the statistic resembles the wave height distribution of oceans where the rare outliers correspond to giant rogue wave events. Because of the statistical similarity of the two phenomena (and the mathematical resemblance of the two systems) these long-wavelength fluctuations were proposed by Solli et al. to be the optical counterparts of the oceanic freak waves [89].

As an example, Fig. 5.3(a) displays the shot-to-shot energy distribution at the extreme red edge of the SC spectrum shown in Fig. 5.1(b). The long-wavelength edge was isolated by numerically filtering away all spectral components smaller than 1300 nm corresponding to the experimental operation of long-pass filtering. The distribution clearly has a distinctive L-shape with a long tail indicating that extreme events are rare, yet considerably more frequent than predicted by e.g. Gaussian statistics. In Paper VI we show experimentally and numerically that, because of the particular nonlinear
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dynamics involved in the formation of broadband supercontinua, fluctuations in the long-wavelength spectral edge are reflected also in short-wavelength side giving rise to similar statistical distributions. Indeed, Fig. 5.3(b) shows the shot-to-shot energy distribution at the extreme blue edge of the SC spectrum and a clear resemblance can be identified with respect to the distribution characterizing the fluctuations in the long-wavelength side (see Fig. 5.3(a)).

![Histograms illustrating energy fluctuations in the red (a) and blue (b) edges of the SC spectrum. Insets show the results on a log-log scale. The red (blue) edge was isolated using a numerical long-pass (short-pass) filter with cutoff at 1300 nm (830 nm).](image)

It is interesting to note that the oscillator used in Paper VI delivered pulses whose duration was 200 fs, yet rogue-wave-like fluctuations were observed. This seemingly contradicts the above analysis on the absolute noise insensitivity of femtosecond supercontinua. However, it should be stressed that the transition from pure noise-insensitive soliton fission dynamics to those dominated by noise-sensitive modulation instability is not abrupt and intermediate levels exist where the two processes compete [59,307]. Hence, one of the key observations of Paper VI is that rogue-wave-like statistics can be observed also under conditions other than those corresponding to genuine long-pulse excitations (see Paper VI for details).

5.3.1 Interpretation of optical rogue wave statistics

The physical interpretation of the $L$-shaped distribution in the long-wavelength edge of the SC spectrum is that a very small portion of individual realizations contain a soliton that has undergone "larger than average" redshift. Therefore when a long-pass filter is utilized to isolate the spectral edge, the filtered waveform impinging on a photodetector will give rise to a nonzero photocurrent only in the case where the particular realization contains a soliton with significant spectral content beyond the cutoff wavelength of the filter. In most realizations, however, the solitons have undergone a lesser redshift and therefore lie in the stop-band of the filter leading to near-zero transmitted energy. Of course, enhanced redshift of a soliton also manifests in the high-frequency edge.
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through the dispersive wave trapping mechanism, explaining why similar statistics can be observed in both edges of the SC spectrum. This interpretation is highlighted in Fig. 5.4 where the time-frequency representation of a rogue event at the extreme tail of the distributions shown in Fig. 5.3 is compared with that of a typical median event. The white and red dashed lines indicate the cutoff wavelengths of the short- and long-pass filters used to construct the statistics in Fig. 5.3, respectively.

![Figure 5.4: Simulated time frequency-representations of a rogue event (a) and a median event (b) as described in the text. The dashed red and white lines indicate the locations of the long- and short-pass filters, respectively.](image)

In Paper VI we experimentally show that if the cutoff-wavelengths of the edge pass filters are moved towards the pump wavelength the statistical distributions transform into quasi-Gaussian. This is because in this case the filters capture more fully a wider range of redshifted solitons and blueshifted dispersive waves. For instance, if the cutoff wavelength of the long-pass filter used above was moved to 1200 nm a soliton would be captured also in the median event resulting in a considerably increased measured energy. However, this suggests then that the \(L\)-shaped statistical distributions shown in Fig. 5.3 do not necessarily imply the presence of waves with abnormal height or energy at all but rather quantify bandwidth fluctuations exclusively. In fact, in Paper VII we numerically show that, in the absence of any spectral selection, the statistical distributions of the peak power, energy and the central frequency of the most redshifted soliton
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across distinct realizations of a stochastically simulated SC ensemble are significantly less skewed. This indicates that the \( L \)-shaped experimentally quantified statistics arise as a result of the quasi-binary effect of the edge-pass filters which allow only a small number of events to be measured while a significant portion is completely neglected. Such an operation is qualitatively similar to assigning a zero value to all events smaller than a certain threshold in a Gaussian distribution which, of course, gives rise to an \( L \)-shaped distribution as illustrated in Fig. 5.5. Therefore a question arises whether the analogue to hydrodynamic rogue waves is in fact appropriate at all.

![Figure 5.5: Transformation of a normally distributed histogram (a) into an \( L \)-shaped distribution (b) when a selection which sets each negative element to zero is invoked. Before selection, the random-variable \( v \) is normally distributed with zero mean and unity standard deviation.](image)

5.3.2 Soliton collisions as rogue waves

Although hydrodynamic rogue waves are often perceived as enormous walls of water, they need not be the largest waves of the sea. Rather, rogue waves are statistically speaking defined as waves with large amplitude for a given sea state. A quantitative criterion commonly used for the classification of rogue events is that the height of a wave must exceed by more than twice the significant wave height (defined as the average wave height of the one-third largest waves) [92]. Importantly, in an oceanic context the definition of a rogue wave concerns strictly wave height and does not involve any spectral specificity. Therefore when pursuing analogous phenomena in optics (or other fields) the approach embraced should also be based solely on the wave height and not to be influenced by any spectral selection operation.

To this end, in Paper VII we performed a "full field anaysis" on a numerically simulated long-pulse SC ensemble and found that, when a wave height statistic is constructed out of all the solitons in the simulated ensemble in the complete absence of any wavelength specificity (e.g. spectral filtering), the corresponding distribution does posses a long tail. Significantly, the extreme outliers of the distribution also
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satisfy the hydrodynamic definition expressed above. These events, however, do not correspond to singular solitons but rather to the ongoing collision of two solitons. Of course, it should be stressed that a supercontinuum field contains not only solitons but also small-amplitude dispersive waves. If the statistic is generalized from containing only solitons to containing all the waves in the field the significant wave height will correspondingly decrease. In this case, it is possible that also individual solitons fulfill the hydrodynamic criterion described above. It should nonetheless be clear that the most intense events always correspond to collisions of solitons.

Figure 5.6(a) shows the statistical distribution of soliton wave heights (peak powers) when a "full field analysis" is performed on the simulated ensemble shown in Fig. 5.1(b). The solitons were isolated by neglecting spectral components in the normal dispersion regime and also by neglecting waves whose peak power is smaller than 315 W corresponding to the peak power of the broad input pulse (see Paper VII for a detailed description of the method). The statistic consist of more than 12000 individual pulses yielding a significant wave height of 1.9 kW and a corresponding criterion of 3.8 kW for an event to be categorized as rogue. This criterion is exceeded by 16 individual events, all of which have been verified to be associated with a solitonic collision. The time-frequency representation of a particular realization containing a pulse in the rogue regime is shown in 5.6(b) where the collision of two solitons yielding a total peak power of 4.8 kW can clearly be identified.

Aside from their intrinsically large amplitude, soliton collisions display other characteristics often associated with rogue waves as well. For instance, they are highly localized both in time and space and seem to appear from nowhere and disappear without a trace (see for instance Fig. 4.6). Due to their transient nature they are also exceedingly rare at the fiber output. Specifically, although collisions of solitons occur frequently during the development phase of the supercontinuum it is highly unlikely that one occurs precisely at the output of the fiber. Moreover, it is not yet sufficient
that two solitons merely collide in order to observe a giant amplitude wave. Indeed, if the solitons are out of phase at the moment of collision destructive interference prohibits the formation of a high-amplitude event. It should be noted that soliton collisions as optical rogue waves was first proposed, to our knowledge, by Mussot et al. in the context of continuous wave supercontinuum [99]. Yet we wish to stress that the collisions themselves do not directly populate the extreme tails of the experimentally quantified bandwidth fluctuations which rather correspond to abnormally redshifted solitons as discussed above and illustrated in Fig. 5.4. In fact, in a fully developed pulsed SC the collision events are generally not captured at all when a spectral filter is used to select only the extreme edge of the spectrum for measurement. This is because in this case the most redshifted solitons are already temporally well-separated from the rest, meaning that collisions are possible only between solitons far from the extreme SC edge. Of course, in a SC seeded by CW sources the most redshifted solitons can undergo collisions even when the SC is fully developed. However, the colliding solitons invariably have distinct center wavelengths implying that the spectral selection would only select the more redshifted one to be measured. Indeed, it is very unlikely that both colliding solitons would simultaneously have significant spectral contents beyond the cutoff wavelength of the filter.

Soliton collisions as procreators of abnormal redshift

In Paper VII we show that while the collision events themselves do not populate the extreme tails of the distributions describing SC bandwidth fluctuations (obtained through spectral filtering) they do provide a more indirect contribution. Specifically, the collisions are closely tied with the formation of abnormally redshifted solitons that do populate these tails.

Figure 5.7 shows the temporal and spectral dynamics of the SC realization whose output time-frequency representation is displayed in Fig. 5.4(a). In this case, it can be seen how after 8.7 m of propagation a localized structure is suddenly ejected from both the long and short wavelength edges of the SC spectrum. In the time domain a collision of large-amplitude solitons can be seen to occur exactly at the same distance. As discussed in section 4.5, during the collision the more redshifted soliton gains energy, undergoes the Raman cross-frequency shift and experiences an enhanced rate of self-frequency shift. These effects allow for the soliton to redshift beyond the cut-off wavelength of the long-pass filter as seen in the density plot in Fig. 5.7(a). Of course, this means then that, while the events in the tails of the distributions obtained via spectral selection may not be associated with abnormally large wave height (when compared e.g. to other solitons in the absence of spectral selection), as by-products of soliton collisions they can be regarded as an indirect signature of extremely localized, large-amplitude waves that have occurred in the past (collisions).
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Figure 5.7: Simulated spectral (a) and temporal (b) evolution of a realization associated with an abnormally broad supercontinuum spectrum. The dashed white vertical line in (a) indicates the location of the cut-off wavelength of the long-pass filter. The black horizontal line highlights that abrupt spectral broadening occurs at the same distance as two solitons collide in the time domain.

Soliton collisions as generators of rogue dispersive waves

In Paper V we show that because soliton collisions occur spontaneously during the development phase of long-pulse supercontinuum it is possible that new frequency components are generated via the phase-matched collision mechanism described in section 4.5. Although strictly speaking such collisions can radiate even in a fiber with a single ZDW (see e.g. Fig. 4.7), the fact that solitons spontaneously redshift renders the generation of observable radiation more probable in a fiber with a long-wavelength ZDW.

Figure 5.8(a-c) displays a particular simulation showing the generation of a dispersive wave during a soliton-soliton collision in noise-driven SC generation. The fiber of the simulation corresponds to fiber B of Fig. 2.4 and the 4 ps, 200 W pump pulse is centered at 900 nm. Figure 5.8(d) plots the linear phase-mismatch of the cascaded four-wave mixing process: 

$$\Delta \beta = \beta(\omega_{-M}) - (M + 1)\beta(\omega_{p1}) + M\beta(\omega_{p2})$$

where \(\omega_{p1}\) and \(\omega_{p2} > \omega_{p1}\) are the frequencies of the colliding solitons and \(\omega_{-M} = (M + 1)\omega_{p1} - M\omega_{p2}\) is the frequency of the \(M^{th}\) low-frequency cascaded sideband. The wavelengths of the solitons were extracted prior to the collision and are approximately \(\lambda_{p1} = 1122\) nm and \(\lambda_{p2} = 1107\) nm. The linear phase-mismatch can be seen to reach a minimum at the \(M = 20\) low-frequency sideband which corresponds to a wavelength of 1539 nm. This is in excellent agreement with the observed spectral location of the generated dispersive wave.

In long-pulse SC generation dispersive waves generated through the collision process can exhibit rogue-wave-like characteristics as compared to other dispersive waves (i.e. those emitted by single solitons). In particular, they can be associated with a
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Figure 5.8: (a-c) Simulated time frequency representations at selected propagation distances illustrating the generation of a dispersive wave due to a soliton collision in noise-driven SC generation. Insets show the temporal profile of the field in the region indicated by the dashed white rectangles and the colorbar applies to all density plots. (d) Calculated linear phase-mismatch of the cascaded FWM process as described in the text.

considerably larger wavelength detuning into the normal dispersion regime and also possess an order of magnitude greater peak power. Due to the transient nature of the collisions they are also highly localized and seem to appear from nowhere (see Fig. 5.8). Moreover, because these dispersive waves are emitted into the normal dispersion regime they undergo rapid dispersive broadening leading to a correspondingly rapid decrease in peak power. In other words, the dispersive waves generated by soliton collisions also seem to disappear without a trace (for details, see Paper V).

5.3.3 Effect of linear Raman gain approximation

Whilst the significant majority of numerical supercontinuum studies model the Raman effect using experimental gain measurements or accurate lorentzian approximations, a simplified first-order linear approximation has also been often employed. In this case, the frequency-domain Raman response function is approximated as \( \tilde{h}_R(\omega) = 1 + i\omega T_R/f_R \), where \( T_R \) is referred to as the characteristic Raman time scale and can be determined from the slope of the Raman gain curve. In Fig. 5.9(a) we compare the imaginary parts of the experimental response adopted from Ref. [187] and two distinct linear approximations with \( T_R = 3 \) fs and \( T_R = 5 \) fs.
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In Related Publication I we show that, when modeling broadband SC generation, the use of the linear Raman gain approximation can yield severe inaccuracies and result in incorrect physical interpretations. Significantly, this approach has also been used in studies related to optical rogue waves [89,107], and we next discuss how this simplified model can in fact artificially lead to the emergence of giant solitons with rogue-wave-like characteristics.

![Figure 5.9: (a) Imaginary parts of the Raman response function (note that \( \text{Im} \tilde{h}_R(\omega) \) is directly related to the Raman gain [187]). Black curve depicts the experimental response while the red and blue lines correspond to linear approximations with \( T_R = 3 \) fs and \( T_R = 5 \) fs, respectively. (b,c,d) Simulated time frequency representations at selected propagation distances when using the linear Raman gain approximation with \( T_R = 5 \) fs.](image)

To study the effect of the linear Raman gain approximation in long-pulse SC generation we use the simplified model and simulate the propagation of a 3 ps, 220 W pulse centered at 1550 nm in a fiber with \( \beta_2 = 1.13 \cdot 10^{-4} \) ps²/m, \( \beta_3 = 6.48 \cdot 10^{-5} \) ps³/m and \( \gamma = 10.66 \) W⁻¹/km. Note that although the value \( T_R = 5 \) fs is clearly not as good an approximation as \( T_R = 3 \) fs (see Fig. 5.9(a)), it is is this value that has been used in the context of optical rogue wave studies [89,107] and thus the one adopted here. Yet, it should be noted that the qualitative conclusions are similar for all \( T_R \). In Fig. 5.9(b,c,d) we plot the time-frequency representation of the evolving field at three distinct propagation lengths. The emergence of a single, extremely intense (>6 kW) and rapidly redshifting soliton can be observed. As the soliton is delayed across the trailing edge of the pump pulse the energy in the pump trailing edge can be
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seen to be quasi-depleted. This depletion occurs because, in the linear approximation, the Raman gain is unbounded and increases monotonously with frequency separation. Therefore the soliton experiences significant gain with respect to the pump trailing edge which then acts as a Raman pump for the soliton. In this way, the soliton captures a significant portion of the energy in the remains of the pulse envelope and grows extremely intense. Such pump depletion dynamics are of course unrealistic because the actual Raman gain possesses a finite bandwidth of approximately 30 THz. For example, in Fig. 5.9(c) the frequency separation of the soliton and the pump residue is approximately 50 THz which would realistically result in negligible Raman coupling between the pump residue and the soliton. Yet when modeling the field propagation using the linear Raman gain approximation, the gain experienced by the soliton is not only nonzero but very large. Finally, upon subsequent propagation these giant solitons can, in the linear Raman model, also disappear abruptly through a dramatic pulse collapse effect. Such collapse, to our knowledge, has not been observed in experiments nor in simulations that do not rely on the linear Raman gain approximation. In fact, in Related Publication I we show that this collapse requires not only the use of the approximated Raman model but also insufficient numerical gridding parameters. These findings clearly show that it is paramount to use the full Raman gain model when modeling broadband pulse propagation.
6. Summary

Optical fibers allow for the study and control of many interdisciplinary nonlinear phenomena. In particular, fundamental instability and wave localization phenomena that are common to many nonlinear systems can be conveniently studied both in situations where they manifest spontaneously or when they are directly stimulated. In this thesis we have studied modulation instability and wave localization dynamics in different contexts, ranging from those where they are externally excited to those wherein the phenomena arise through spontaneous nonlinear interactions. The main results of the thesis can be broadly divided into three intertwined regimes.

In this work we have adopted the analytical breather formalism originally introduced by Akhmediev and Korneev for the description of modulation instability and investigated its applicability under realistic excitation conditions. In particular, we have derived an improved estimate for the fiber length required to generate an ultrahigh-repetition-rate pulse train from an initially weakly modulated continuous wave and also shown that the breather formalism can be intuitively extended to describe modulation instability also in the pulsed regime. Whilst the breather theory provides an accurate representation of the initial phase of modulation instability, we have further shown that there exists parameter regimes where deviations from the ideal breather evolution are observed. Specifically, when the modulation frequency is sufficiently small allowing for multiple instability harmonics to experience nonzero gain the breather experiences complex dynamics resulting in a cascaded temporal splitting process. In this thesis we have shown experimentally, numerically and analytically that such temporal splitting dynamics can be described in terms of higher-order modulation instability corresponding to a nonlinear superposition of elementary breather solutions. To our knowledge this constitutes the first experimental identification of such a higher-order instability process in any physical system as well as the first utilization of the Darboux transformation method in the design and analysis of physical experiments.

Whilst the purely analytical breather formalism does provide an interesting perspective to modulation instability, it is rather the temporally localized soliton that is often heralded as the central structure of nonlinear science. Soliton propagation effects also lie at the focus of the present thesis because of their crucial role in the generation of broadband supercontinua in highly nonlinear fibers. In particular, here we have demonstrated experimentally and numerically novel soliton dynamics in long-pulse supercontinuum which lead to the generation of spectral components inaccessible through other spectral broadening processes. The first effect occurs due to the fact that ultrashort solitons ejected from the broad input pulse envelope temporally overlap with the trailing edge of the pump residue. Such a nonlinear superposition gives rise to a
phase-matched four-wave-mixing-like interaction of the soliton and the pump residue leading to the generation of narrowband spectral components clearly isolated from the continuum spectrum. In the second effect, the collision of two ultrashort solitons gives rise to a dispersive wave due to a phase-matched cascaded four-wave mixing process. Significantly, we have shown experimental signatures of such collision-generated dispersive waves by systematically exciting solitonic collisions using an interferometric setup. These signatures represent, to our knowledge, the first experimental observation of radiation emission by soliton collisions.

The initial stages of long-pulse supercontinuum generation are governed by noise-driven modulation instability which leads to the break-up of the input envelope into a train of short solitons. This intrinsically noise-sensitive process has been shown to lead to the emergence of rare, abnormally redshifted solitons that have been suggested as the optical analogues of the infamous oceanic rogue waves. In this thesis we have shown experimentally and numerically that similar optical-rogue-wave-like fluctuations can be observed in supercontinuum generation even under conditions other than those corresponding to genuine long-pulse excitation. However, we have also shown that the spectral selection technique originally used to capture optical rogue waves biases the measured statistical distributions and, in fact, does not necessarily reveal the presence of high-amplitude waves but instead quantifies SC bandwidth fluctuations exclusively. By performing a more detailed analysis in the absence of spectral selection we have, nonetheless, shown that the supercontinuum field does contain extreme events whose amplitudes fulfill criteria often used to discriminate rogue waves in a hydrodynamic environment. Such events are associated with the collision of solitons, displaying extreme wave localization both in time and space and exhibiting many of the features often attributed to oceanic rogue waves.

To summarize, this thesis has presented studies on nonlinear instability and wave localization dynamics in fiber optics, providing significant results both applied and fundamental that are expected to be of immediate use in the analysis of fiber-optic parametric amplifiers, in the design and implementation of ultrafast soliton-based all-optical switches and logic devices and in the design of broadband SC sources for specific applications. On a more fundamental note, this thesis contains several novel results concerning the existence and interactions of universal nonlinear structures. It is expected that these results, in particular, will evolve beyond fiber-optics and stimulate research of similar nature in the more general field of nonlinear science.
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