Derivation of arithmetical functions under the Dirichlet convolution

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Abstract

We present the group-theoretic structure of the classes of multiplicative and firmly multiplicative arithmetical functions of several variables under the Dirichlet convolution, and we give characterizations of these two classes in terms of a derivation of arithmetical functions.

Keywords: arithmetical function of several variables, multiplicative function, firmly multiplicative function, derivation, Dirichlet convolution

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1 Introduction

Let $R$ be an integral domain. Let $A_r(R) := \{f: R^r \to \mathbb{C}\}$ denote the set of all arithmetical functions of $r$ variables. An arithmetical function $f \in A_r(R)$ is said to be multiplicative if $f(1, \ldots, 1)$ is invertible and

$$f(m_1n_1, \ldots, m_rn_r) = f(m_1, \ldots, m_r)f(n_1, \ldots, n_r)$$
for all positive integers \( m_1, \ldots, m_r \) and \( n_1, \ldots, n_r \) with \((m_1 \cdots m_r, n_1 \cdots n_r) = 1\). Clearly, \( f(1, \ldots, 1) = 1 \) if \( f \) is multiplicative. Further, a multiplicative function is completely determined by its values at \((p^{s_1}, \ldots, p^{s_r})\), where \( p \) runs through all primes and \( s_1, \ldots, s_r \geq 0 \). This concept of a multiplicative function coincides with that presented in [9, 12] but differs from that used in [1].

We say that an arithmetical function \( f \in A_r(R) \) is firmly multiplicative if \( f(1, \ldots, 1) \) is invertible and

\[
f(m_1 n_1, \ldots, m_r n_r) = f(m_1, \ldots, m_r) f(n_1, \ldots, n_r)
\]

for all positive integers \( m_1, \ldots, m_r \) and \( n_1, \ldots, n_r \) with \((m_1, n_1) = \cdots = (m_r, n_r) = 1\). Clearly, \( f(1, \ldots, 1) = 1 \) if \( f \) is firmly multiplicative. Further, a firmly multiplicative function is completely determined by its values at \((1, \ldots, 1, p^s, 1, \ldots, 1)\), where \( p \) runs through all primes and \( s \geq 1 \), i.e., it is completely determined by its values at \((n_1, \ldots, n_r)\), where one of \( n_1, \ldots, n_r \) is a prime power (\( > 1 \)) and the others are \( = 1 \). The concept of a firmly multiplicative function coincides with the concept of a multiplicative function presented in [1].

For example, if \( f_1, \ldots, f_r \) are multiplicative functions of one variable, then the arithmetical function \( f \) of \( r \) variables defined as \( f(n_1, \ldots, n_r) = f(n_1) \cdots f(n_r) \) is firmly multiplicative. On the other hand, the function \( \gcd(n_1, \ldots, n_r) \) is multiplicative but not firmly multiplicative for \( r \geq 2 \). Further examples can be found, e.g., in [5, 10, 11]. A survey on multiplicative arithmetical functions of several variables is presented in [11].

An arithmetical function \( f \in A_r(R) \) is said to be additive if

\[
f(m_1 n_1, \ldots, m_r n_r) = f(m_1, \ldots, m_r) + f(n_1, \ldots, n_r)
\]

for all positive integers \( m_1, \ldots, m_r \) and \( n_1, \ldots, n_r \) with \((m_1 \cdots m_r, n_1 \cdots n_r) = 1\), and an arithmetical function \( f \in A_r(R) \) is said to be completely additive if

\[
f(m_1 n_1, \ldots, m_r n_r) = f(m_1, \ldots, m_r) + f(n_1, \ldots, n_r)
\]
for all positive integers $m_1, \ldots, m_r$ and $n_1, \ldots, n_r$. We say that an arithmetical function $f \in A_r(R)$ is \textit{firmly additive} if

$$f(m_1n_1, \ldots, m_rn_r) = f(m_1, \ldots, m_r) + f(n_1, \ldots, n_r)$$

for all positive integers $m_1, \ldots, m_r$ and $n_1, \ldots, n_r$ with $(m_1, n_1) = \cdots = (m_r, n_r) = 1$. Each completely additive function is firmly additive, and each firmly additive function is additive. Clearly, $f(1, \ldots, 1) = 0$ if $f$ is additive.

The Dirichlet convolution of $f, g \in A_r(R)$ is defined as

$$(f * g)(n_1, \ldots, n_r) = \sum_{d_1|n_1} \cdots \sum_{d_r|n_r} f(d_1, \ldots, d_r)g(n_1/d_1, \ldots, n_r/d_r).$$

Let $\delta \in A_1(R)$ be defined as $\delta(1) = 1$ and $\delta(n) = 0$ otherwise, and let $\delta \in A_r(R)$ be defined as $\delta(n_1, \ldots, n_r) = \delta(n_1) \cdots \delta(n_r)$. Then $\delta$ is the identity under the Dirichlet convolution, and it is firmly multiplicative. The Dirichlet inverse $f^{-1}$ of $f \in A_r(R)$ exists if and only if $f(1, \ldots, 1)$ is invertible.

Derivations for arithmetical functions have been presented, e.g., in [1, 2, 4, 3, 6]. A certain property of multiplicative type functions in terms of derivations is well known [1, 2, 4], see also [7, 8]. In this paper we adopt the derivation given in [1] and utilize the method of Rearick [7] to obtain the above mentioned derivation-related property for multiplicative and firmly multiplicative functions, see Theorems 3 and 4. We obtain a short proof for this property of firmly multiplicative functions given in [1, Theorem 5]. We also show that this property of firmly multiplicative functions actually holds only for firmly multiplicative functions and therefore is, in fact, a characterization of firmly multiplicative functions. An analogous characterization of multiplicative functions is also presented.

We begin by showing the group-theoretic structure of multiplicative and firmly multiplicative functions under the Dirichlet convolution.
2 Group-theoretic structure

Theorem 1. (a) The set of all arithmetical functions \( f \in A_r(R) \) with \( f(1,\ldots,1) \) invertible is an abelian group under the Dirichlet convolution.

(b) The set of all multiplicative functions forms a subgroup of the abelian group in (a).

(c) The set of all firmly multiplicative functions forms a subgroup of the abelian group in (b).

Proof. We prove only that if \( f \) is multiplicative, then \( f^{-1} \) is multiplicative. Let \( f \) be multiplicative. We proceed by induction on \( m_1\cdots m_r n_1\cdots n_r \) to prove that

\[
f^{-1}(m_1 n_1, \ldots, m_r n_r) = f^{-1}(m_1, \ldots, m_r) f^{-1}(n_1, \ldots, n_r) \tag{1}
\]

whenever \((m_1 \cdots m_r, n_1 \cdots n_r) = 1\).

Assume that \( m_1 \cdots m_r n_1 \cdots n_r = 1 \). Then \( m_1 = \cdots = m_r = n_1 = \cdots = n_r = 1 \), and since \( f \) is multiplicative, \( f(1,\ldots,1) = 1 \). This implies that \( f^{-1}(1,\ldots,1) = 1 \), and therefore both sides of (1) equal 1 and thus (1) holds.

Assume that \( m_1 \cdots m_r n_1 \cdots n_r \neq 1 \), \((m_1 \cdots m_r, n_1 \cdots n_r) = 1 \) and

\[
f^{-1}(m'_1 n'_1, \ldots, m'_r n'_r) = f^{-1}(m'_1, \ldots, m'_r) f^{-1}(n'_1, \ldots, n'_r) \tag{2}
\]

whenever \((m'_1 \cdots m'_r, n'_1 \cdots n'_r) = 1 \) and \( m'_1 \cdots m'_r n'_1 \cdots n'_r < m_1 \cdots m_r n_1 \cdots n_r \).

We show that (1) holds. If \( m_1 \cdots m_r = 1 \) or \( n_1 \cdots n_r = 1 \), then (1) holds. Assume that \( m_1 \cdots m_r \neq 1 \) and \( n_1 \cdots n_r \neq 1 \). Now, \((f \ast f^{-1})(m_1 n_1, \ldots, m_r n_r) = 0\), that is,

\[
f^{-1}(m_1 n_1, \ldots, m_r n_r) = - \sum_{d_1 | m_1 n_1, \ldots, d_r | m_r n_r, \frac{d_1 \cdots d_r}{d_1 \cdots d_r > 1}} f(d_1, \ldots, d_r) f^{-1}(m_1 n_1/d_1, \ldots, m_r n_r/d_r).
\]
Since \((m_1 \cdots m_r, n_1 \cdots n_r) = 1\), the above summation can be written as
\[
- \sum_{a_1|m_1, \ldots, a_r|m_r, b_1|n_1, \ldots, b_r|n_r \atop a_1 \cdots a_r b_1 \cdots b_r > 1} f(a_1, \ldots, a_r, b_1, \ldots, b_r) f^{-1}(m_1/a_1, \ldots, m_r/a_r)(n_1/b_1, \ldots, (n_r/b_r)).
\]

On the basis of multiplicativity of \(f\) and equation (2) this becomes
\[
- \sum_{a_1|m_1, \ldots, a_r|m_r, b_1|n_1, \ldots, b_r|n_r \atop a_1 \cdots a_r b_1 \cdots b_r > 1} f(a_1, \ldots, a_r) f(b_1, \ldots, b_r) f^{-1}(m_1/a_1, \ldots, m_r/a_r) f^{-1}(n_1/b_1, \ldots, n_r/b_r).
\]

Arranging the terms we obtain
\[
f^{-1}(m_1 n_1, \ldots, m_r n_r)
= -f^{-1}(m_1, \ldots, m_r) \sum_{b_1|n_1, \ldots, b_r|n_r \atop b_1 \cdots b_r > 1} f(b_1, \ldots, b_r) f^{-1}(n_1/b_1, \ldots, n_r/b_r)
- f^{-1}(n_1, \ldots, n_r) \sum_{a_1|m_1, \ldots, a_r|m_r \atop a_1 \cdots a_r > 1} f(a_1, \ldots, a_r) f^{-1}(m_1/a_1, \ldots, m_r/a_r)
- \sum_{a_1|m_1, \ldots, a_r|m_r \atop a_1 \cdots a_r > 1} f(a_1, \ldots, a_r) f^{-1}(m_1/a_1, \ldots, m_r/a_r)
\times \sum_{b_1|n_1, \ldots, b_r|n_r \atop b_1 \cdots b_r > 1} f(b_1, \ldots, b_r) f^{-1}(n_1/b_1, \ldots, n_r/b_r)
= f^{-1}(m_1, \ldots, m_r) f^{-1}(n_1, \ldots, n_r) + f^{-1}(m_1, \ldots, m_r) f^{-1}(n_1, \ldots, n_r)
- f^{-1}(m_1, \ldots, m_r) f^{-1}(n_1, \ldots, n_r)
= f^{-1}(m_1, \ldots, m_r) f^{-1}(n_1, \ldots, n_r).
\]

Thus (1) holds. This shows that \(f^{-1}\) is multiplicative.

### 3 Derivation

Let \(\psi \in A_r(R)\) be a completely additive function. We define the derivation \(D_\psi : A_r(R) \to A_r(R)\) by
\[
D_\psi(f)(n_1, \ldots, n_r) = f(n_1, \ldots, n_r) \psi(n_1, \ldots, n_r).
\]
In what follows we write \( n = (n_1, \ldots, n_r) \) for the sake of brevity.

We first present the basic properties of the derivation.

**Theorem 2.** For any \( f, g \in A_r(R) \) and \( c \in R \),

(a) \( D_\psi(f + g) = D_\psi(f) + D_\psi(g) \),

(b) \( D_\psi(f \ast g) = f \ast D_\psi(g) + g \ast D_\psi(f) \),

(c) \( D_\psi(cf) = cD_\psi(f) \).

**Proof.** See [1].

We next present the promised characterizations of multiplicative and firmly multiplicative functions in terms of the derivation.

**Theorem 3.** Let \( f \in A_r(R) \) be an arithmetical function with \( f(1, \ldots, 1) = 1 \). If \( f \) is multiplicative, then \( (D_\psi(f) \ast f^{-1})(n) = 0 \) whenever \( n \) is not of the form \((p^{s_1}, \ldots, p^{s_r})\), where \( p \) is a prime number and \( s_1, \ldots, s_r \geq 0 \). The converse holds provided that \( \psi(n) \neq 0 \) for all \( n \neq (1, \ldots, 1) \).

**Proof.** Assume that \( f \) is multiplicative. Assume also that \( n \) is not of the form \((p^{s_1}, \ldots, p^{s_r})\), where \( p \) is a prime number and \( s_1, \ldots, s_r \geq 0 \). Then \( n \) can be written as \( n = (k_1m_1, \ldots, k_r m_r) \), where \((k_1 \cdots k_r, m_1 \cdots m_r) = 1\) with \( k_1 \cdots k_r > 1 \) and \( m_1 \cdots m_r > 1 \). We show that \( (D_\psi(f) \ast f^{-1})(n) = 0 \). Since \( f \) is multiplicative, we have

\[
(D_\psi(f) \ast f^{-1})(n) = \sum_{d_1|k_1 m_1} \cdots \sum_{d_r|k_r m_r} f(d_1, \ldots, d_r) f^{-1}(k_1 m_1/d_1, \ldots, k_r m_r/d_r) \psi(d_1, \ldots, d_r)
\]

\[
= \sum_{a_1|k_1} \cdots \sum_{a_r|k_r} f(a_1, \ldots, a_r) f(b_1, \ldots, b_r) f^{-1}(k_1/a_1, \ldots, k_r/a_r)
\]

\[
\times f^{-1}(m_1/b_1, \ldots, m_r/b_r) \left( \psi(a_1, \ldots, a_r) + \psi(b_1, \ldots, b_r) \right).
\]
Rearranging the terms we obtain

\[
(D_\psi(f) * f^{-1})(n) = \sum_{a_1 | k_1} \cdots \sum_{a_r | k_r} f(a_1, \ldots, a_r) f^{-1}(k_1/a_1, \ldots, k_r/a_r) \psi(a_1, \ldots, a_r)
\]

\[
\times \sum_{b_1 | m_1} \cdots \sum_{b_r | m_r} f(b_1, \ldots, b_r) f^{-1}(m_1/b_1, \ldots, m_r/b_r)
\]

\[
+ \sum_{b_1 | m_1} \cdots \sum_{b_r | m_r} f(b_1, \ldots, b_r) f^{-1}(m_1/b_1, \ldots, m_r/b_r) \psi(b_1, \ldots, b_r)
\]

\[
\times \sum_{a_1 | k_1} \cdots \sum_{a_r | k_r} f(a_1, \ldots, a_r) f^{-1}(k_1/a_1, \ldots, k_r/a_r)
\]

\[
= (D_\psi(f) * f^{-1})(k) \delta(m) + (D_\psi(f) * f^{-1})(m) \delta(k).
\]

Since \( k \neq (1, \ldots, 1) \) and \( m \neq (1, \ldots, 1) \), we have \( \delta(k) = \delta(m) = 0 \), and therefore \( (D_\psi(f) * f^{-1})(n) = 0 \).

Conversely, assume that \((D_\psi(f) * f^{-1})(n) = 0 \) whenever \( n \) is not of the form \((p^{s_1}, \ldots, p^{s_r})\), where \( p \) is a prime number and \( s_1, \ldots, s_r \geq 0 \). We show that \( f \) is multiplicative. Let \( g \in A_r(R) \) be the arithmetical function defined as \( g(1, \ldots, 1) = 1 \) and

\[
g(n) = \prod_p f(p^{n_1(p)}, \ldots, p^{n_r(p)}),
\]

where \( n_i = \prod_p p^{n_i(p)} \) is the canonical factorization of \( n_i \) for \( i = 1, \ldots, r \). Then \( g \) is multiplicative. We show that \( f = g \).

We first show that \( D_\psi(f) * f^{-1} = D_\psi(g) * g^{-1} \). Clearly, \( f(n) = g(n) \) if \( n \) is of the form \((p^{s_1}, \ldots, p^{s_r})\), where \( p \) is a prime number and \( s_1, \ldots, s_r \geq 0 \). This implies that \( D_\psi(f)(n) = D_\psi(g)(n) \) and \( f^{-1}(n) = g^{-1}(n) \) if \( n \) is of the form \((p^{s_1}, \ldots, p^{s_r})\), where \( p \) is a prime number and \( s_1, \ldots, s_r \geq 0 \). Thus \( (D_\psi(f) * f^{-1})(n) = (D_\psi(g) * g^{-1})(n) \) if \( n \) is of the form \((p^{s_1}, \ldots, p^{s_r})\), where \( p \) is a prime number and \( s_1, \ldots, s_r \geq 0 \). Since \( g \) is multiplicative, on the basis of the first part of this theorem \( (D_\psi(g) * g^{-1})(n) = 0 \) if \( n \) is not of the form \((p^{s_1}, \ldots, p^{s_r})\), where \( p \) is a prime number and \( s_1, \ldots, s_r \geq 0 \). Thus, by the assumption on \( f \), \( (D_\psi(f) * f^{-1})(n) = (D_\psi(g) * g^{-1})(n) \) for all \( n \).
We now show that \( f(n) = g(n) \) for all \( n = (n_1, \ldots, n_r) \) by applying induction on \( \Omega(n_1) + \cdots + \Omega(n_r) \), where \( \Omega(n_i) \) is the total number of prime divisors of \( n_i \) each counted according to its multiplicity with \( \Omega(1) = 0 \). If \( \Omega(n_1) + \cdots + \Omega(n_r) = 0 \), then \( n = (1, \ldots, 1) \) and \( f(n) = g(n) = 1 \). Assume that \( f(n) = g(n) \) for \( \Omega(n_1) + \cdots + \Omega(n_r) < k \). Then \( f^{-1}(n) = g^{-1}(n) \) for \( \Omega(n_1) + \cdots + \Omega(n_r) < k \). Let \( m \) be such that \( \Omega(m_1) + \cdots + \Omega(m_r) = k \). We show that \( f(m) = g(m) \). We have shown that \( (D \psi(f) * f^{-1})(m) = (D \psi(g) * g^{-1})(m) \), which means that

\[
\sum_{d_1|m_1, \ldots, d_r|m_r} f(d_1, \ldots, d_r)f^{-1}(m_1/d_1, \ldots, m_r/d_r)\psi(d_1, \ldots, d_r)
\]

\[
= \sum_{d_1|m_1, \ldots, d_r|m_r} g(d_1, \ldots, d_r)g^{-1}(m_1/d_1, \ldots, m_r/d_r)\psi(d_1, \ldots, d_r).
\]

Since \( \psi(1, \ldots, 1) = 0 \), we have

\[
f(m)\psi(m) + \sum_{d_1|m_1, \ldots, d_r|m_r} f(d_1, \ldots, d_r)f^{-1}(m_1/d_1, \ldots, m_r/d_r)\psi(d_1, \ldots, d_r)
\]

\[
= g(m)\psi(m) + \sum_{d_1|m_1, \ldots, d_r|m_r} g(d_1, \ldots, d_r)g^{-1}(m_1/d_1, \ldots, m_r/d_r)\psi(d_1, \ldots, d_r).
\]

Since \( f(n) = g(n) \) and \( f^{-1}(n) = g^{-1}(n) \) for \( \Omega(n_1) + \cdots + \Omega(n_r) < k \), the summations above are equal. Thus \( f(m)\psi(m) = g(m)\psi(m) \), and since \( \psi(m) \neq 0 \), we have \( f(m) = g(m) \). We have thus shown by induction that \( f = g \). Since \( g \) is multiplicative, \( f \) is multiplicative. This completes the proof.

**Theorem 4.** Let \( f \in A_p(R) \) be an arithmetical function with \( f(1, \ldots, 1) = 1 \). If \( f \) is firmly multiplicative, then \( (D \psi(f) * f^{-1})(n) = 0 \) whenever \( n = (n_1, \ldots, n_r) \) is not of the form \( n = (1, \ldots, 1, p^s, 1, \ldots, 1) \), where \( p \) is a prime number and \( s \geq 0 \) (i.e., \( (D \psi(f) * f^{-1})(n) = 0 \) unless one of \( n_1, \ldots, n_r \) is a prime power \( \geq 1 \) and the others are \( = 1 \)). The converse holds provided that \( \psi(n) \neq 0 \) for all \( n \neq (1, \ldots, 1) \).
Proof. Assume that \( f \) is firmly multiplicative. Assume also that \( n \) is not of the form \((1, \ldots, 1, p^s, 1, \ldots, 1)\), where \( p \) is a prime number and \( s \geq 0 \). Then \( n \) can be written as \( n = (k_1m_1, \ldots, k_rm_r) \), where \((k_1, m_1) = \cdots = (k_r, m_r) = 1\) with \( k_1 \cdots k_r > 1 \) and \( m_1 \cdots m_r > 1 \). We show that \( (D_\psi(f) * f^{-1})(n) = 0 \).

Since \( f \) is firmly multiplicative, we have

\[
(D_\psi(f) * f^{-1})(n) = \sum_{d_1 | k_1} \cdots \sum_{d_r | k_r} f(d_1, \ldots, d_r) f^{-1}(k_1m_1/d_1, \ldots, k_rm_r/d_r) \psi(d_1, \ldots, d_r)
\]

where

\[
f(n) = \sum_{a_1 | k_1} \cdots \sum_{b_r | m_r} f(a_1, \ldots, a_r) f(b_1, \ldots, b_r) f^{-1}(k_1/a_1, \ldots, k_r/a_r)
\]

\[
\times f^{-1}(m_1/b_1, \ldots, m_r/b_r) \left( \psi(a_1, \ldots, a_r) + \psi(b_1, \ldots, b_r) \right)
\]

\[
= (D_\psi(f) * f^{-1})(k) \delta(m) + (D_\psi(f) * f^{-1})(m) \delta(k).
\]

Since \( k \neq (1, \ldots, 1) \) and \( m \neq (1, \ldots, 1) \), we have \( \delta(k) = \delta(m) = 0 \), and therefore \( (D_\psi(f) * f^{-1})(n) = 0 \).

Conversely, assume that \( (D_\psi(f) * f^{-1})(n) = 0 \) whenever \( n \) is not of the form \((1, \ldots, 1, p^s, 1, \ldots, 1)\), where \( p \) is a prime number and \( s \geq 0 \). We show that \( f \) is firmly multiplicative. Let \( g \in A_\psi(R) \) be the arithmetical function defined as \( g(1, \ldots, 1) = 1 \) and

\[
g(n) = \prod_p \left( f(p^{n_1(p)}, 1, \ldots, 1) f(1, p^{n_2(p)}, 1, \ldots, 1) \cdots f(1, \ldots, 1, p^{n_r(p)}) \right),
\]

where \( n_i = \prod_p p^{n_i(p)} \) is the canonical factorization of \( n_i \) for \( i = 1, 2, \ldots, r \). Then \( g \) is firmly multiplicative. We show that \( f = g \).

We first show that \( D_\psi(f) * f^{-1} = D_\psi(g) * g^{-1} \). Clearly, \( f(n) = g(n) \) if \( n \) is of the form \((1, \ldots, 1, p^s, 1, \ldots, 1)\), where \( p \) is a prime number and \( s \geq 0 \). This implies that \( D_\psi(f)(n) = D_\psi(g)(n) \) and \( f^{-1}(n) = g^{-1}(n) \) if \( n \) is of the form \((1, \ldots, 1, p^s, 1, \ldots, 1)\), where \( p \) is a prime number and \( s \geq 0 \). Thus \( (D_\psi(f) * f^{-1})(n) = (D_\psi(g) * g^{-1})(n) \) if \( n \) is of the form \((1, \ldots, 1, p^s, 1, \ldots, 1)\), where \( p \) is a prime number and \( s \geq 0 \). Since \( g \) is firmly multiplicative, on the basis of the first part of this theorem \( (D_\psi(g) * g^{-1})(n) = 0 \) if \( n \) is not of the
form \((1, \ldots, 1, p^s, 1, \ldots, 1)\), where \(p\) is a prime number and \(s \geq 0\). Thus, by the assumption on \(f\), 
\[ (D_{\psi}(f) * f^{-1})(n) = (D_{\psi}(g) * g^{-1})(n) \]
for all \(n\).

We can show that \(f(n) = g(n)\) for all \(n = (n_1, \ldots, n_r)\) applying induction on \(\Omega(n_1) + \cdots + \Omega(n_r)\) exactly in the same way as in the proof of Theorem 3. Thus \(f = g\), and since \(g\) is firmly multiplicative, \(f\) is firmly multiplicative. This completes the proof.

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References


