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On the Uniform One-Dimensional Fragment over Ordered Models

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Tiivistelmä

Uniformi, yksiulotteinen fragmentti U_1 on hiljan esitelty kahden muuttujan logiikan FO^2 ekstensio. Logiikka U_1 mahdollistaa mitä tahansa ariteettia olevien relaatiosymbolien käytön ja näin ollen laajentaa FO^2:n sovellusalaa. Tässä tutkielmassa me osoitamme, että logiikan U_1 toteutuvuus- ja äärellinen toteutuvuusongelma lineaarisesti järjestettyjen mallien suhteen ovat NExpTime-täydellisiä. Kahden muuttujan logiikan vastaavat toteutuvuusongelmat ovat niin ikään NExpTime-täydellisiä, joten siirtymä logiikasta FO^2 logiikkaan U_1 järjestettyjen mallien tapauksessa ei kasvata kompleksisuutta. Vastakohtana edellä mainituille ratkeavuustuloksille osoitamme myös, että U_1 kahdella epäuniformilla sisäänrakennetulla lineaarijärjestyksellä on ratkeamaton.

Abstract

The uniform one-dimensional fragment U_1 is a recently introduced extension of the two-variable fragment FO^2. The logic U_1 enables the use of relation symbols of all arities and thereby extends the scope of applications of FO^2. In this thesis we show that the satisfiability and finite satisfiability problems of U_1 over linearly ordered models are NExpTime-complete. The corresponding problems for FO^2 are likewise NExpTime-complete, so the transition from FO^2 to U_1 in the ordered realm causes no increase in complexity. To contrast our results, we also establish that U_1 with an unrestricted use of two built-in linear orders is undecidable.

Preface

As this thesis manifests a rather small part of the work done in my personal journey towards mathematical maturity, I feel somewhat obliged to say a few words of how I ended up doing what this thesis represents.

I think it all began when I read Alan Turing's paper with the title "On Computable Numbers, with an Application to the Entscheidungsproblem" in my senior year of high school. I cannot say that I understood much about the paper, but it somehow initiated an idea — I want to be able to "program" mathematics. I shall not explain what I mean by "being able to program mathematics," as I never fully explained it to myself. The idea was and still is more or less intuitive, and those, such as my thesis advisor, for whom it says something, do not need an explanation anyway.

After high school I did various jobs in the IT field before I started computer science studies in the University of Tampere. I must say that I have never been an "orthodox" student. My formal education is everything else than a textbook example; I have always studied what I want rather than what I am told to. Having obtained some experience in the IT field, I began to demand a more fundamental understanding of things I was dealing with. In the university I realized — after studying one year and working as a research assistant for four years — that standard contemporary computer science would not provide the required fundamental understanding I was looking for. This realization is partially due to the fact that every time I wanted to know something thoroughly, I found myself reading mathematics instead of reading standard computer science textbooks. (Personally, I regard theoretical computer science as part of mathematics.) Thus I finished my computer science studies as a B.Sc and pursued a master's degree in mathematics.

Since my formal education was what it was, especially in relation to mathematics, I basically started my master's degree studies in mathematics from the starting level comparable to first-year university students. I knew I could somehow catch up with other students, and the situation improved radically when Antti Kuusisto became my master's thesis advisor. Regarding the amount of time and effort Kuusisto has given me, I think it is fair to say that he pretty much took me from the ground level to the advanced level I (sometimes stubbornly) required. It must not have been an easy task. At the time I asked Kuusisto to be my thesis advisor, he was no longer a staff member in the University of Tampere. For this reason, most of the advising was done online. In addition to the online advising, I visited two universities where Kuusisto was working at the time: one week in the university of Stockholm and one week in the university of Bremen. Kuusisto also occasionally visited Tampere for short periods of time.

All in all, this thesis is the result of the process not necessarily so typical for master's theses. While I did not know what the word Entscheidungsproblem (decision problem) meant when I tried to pronounce it for the first time, this thesis now deals with several decision problems in relation to fragments of first-order logic. (As elementary concepts will mostly not be covered, the reader is assumed to possess at least an elementary knowledge of both mathematical logic and computational complexity theory.)

This thesis indeed addresses several decision problems. However, there is still one personal problem that will remain undecidable, namely, how do I thank my thesis advisor Antti Kuusisto. As currently I could not find any sufficient way to thank him, I decided (being logical :) not to thank him at all. He will surely appreciate this kind of a move. However, I hope I will find a concrete way to thank him in the future.

Professor Lauri Hella deserves thanks for being supportive of the research process and reading the thesis under a very tight schedule. In addition, as a special group, I want to thank the following people: Miikka Ojala, John Miller, and Brian Carroll.

Finally, I thank my family for all support.

Contents

Chapter 1

Introduction

Many questions regarding the foundations of mathematics were addressed in the early the 20th century. One of the questions, which will be of particular interest to us, is the *decision problem* or the Entscheidungsproblem as posed by David Hilbert in his famous program (Hilbert's program) [27]. Unfortunately, in this introduction, we shall not dive into the history of mathematics, as intriguing it is, but we shall merely give a rather informal definition of this particular problem — the decision problem.

The decision problem can be defined as follows. For a given first-order formula, decide whether the formula is satisfiable. Alternatively, decide whether the formula is valid. Here the first-order formula belongs to the language of *first-order* (FO) logic, the definition of which we shall not give here. The formula is satisfiable if it has a model and valid if every model where the formula is defined satisfies it. See any standard textbook on logic for the definition of first-order logic.

To try to decide the satisfiability of a first-order formula, we could use an algorithm designed and implemented to solve the satisfiability of FO-formulae. (Note that in this thesis, the notion of algorithm is assumed known by the reader. Furthermore, the reader is assumed to have at least an elementary knowledge of computational complexity theory.) That is to say, an algorithm solving the satisfiability of FO-formulae, would be a solution to the decision problem. Therefore, the question is now whether there exists such an algorithm. Before revealing the existence or non-existence of such an algorithm, let us suppose that we have, indeed, an algorithm called the *decision algorithm* that takes as input an FO-formula and determines whether it is satisfiable.

First, intuitively speaking, assume that we could express all mathematical problems as FO-formulae. Now what one would need to do, in order to solve any mathematical problem, is to formulate an FO-formula expressing the problem and input it to the decision algorithm which, in turn, would determine the satisfiability or validity of the formula. Regardless of the complexity of translating mathematical problems into FO-formulae, which is not necessarily a trivial task, we may say that the existence of the decision algorithm would take care of a great part of mathematical inquiry, at least the mechanical aspect of it. In this regard, there must have been a concern among some mathematicians in the early the 20th century.[1] One could also say, however, that this concern only emphasized the importance of the decision problem, and there were mathematicians such as Hilbert, Ackermann, Herbrand, and Ramsey, among others, who found the decision problem the *main problem* of mathematical logic [10].

Now to increase the level of formality, yet keeping things somewhat informal, let us rephrase what we just said above. Let T be a theory consisting of a finite number of FO-sentences. Recall that an FO-formula is called a sentence, if it does not contain free variables. Let φ be the conjunction of the sentences (also called axioms) in T and ψ some FO-sentence. Now, if we want to know whether ψ is implied by the theory T, i.e., if ψ is a theorem of T, we set χ to be the implication $\varphi \rightarrow \psi$ and input χ to the decision algorithm. If the decision algorithm determines χ to be valid, then ψ is a theorem of T, otherwise not.

Having given an idea of what we could do with the decision algorithm, it is now time to reveal what the reader may have already anticipated. The *negative answer* to the decision problem was independently established by Alonzo Church [12] and Alan Turing [55] in 1936. In other words, there is no decision algorithm for first-order logic. Despite the negative answer, research around first-order logic had already provided many results regarding certain sublogics of first-order logic, which we shall introduce next.

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[1] This is author's interpretation, and admittedly it is somewhat provocative. In the words of G. H. Hardy: "There is of course no such theorem, and this is very fortunate, since if there were we should have a mechanical set of rules for the solution of all mathematical problems, and our activities as mathematicians would come to an end."

1.1 Prefix-vocabulary classes

A class X of formulae of first-order logic denoted $X \subseteq$ FO is called a *fragment* of first-order logic. Prior to the negative answer to the decision problem, many fragments of first-order logic were shown to be either decidable, i.e., they have a decision algorithm, or as hard as the decision problem itself. In order for a fragment X of FO to be as hard as the problem itself, there must exist some algorithm A that maps every FO-formula φ to some formula in X such that φ is satisfiable if and only if $A(\varphi)$ is. In other words, the satisfiability of FO-formulae is reduced to satisfiability of formulae in X. Such fragments are called *reduction classes* for satisfiability. Note that if one had a decision algorithm for a reduction class, then this decision algorithm could be used to solve the decision problem. As the decision problem is undecidable, the existence of a decision algorithm of any reduction class would lead to a contradiction. Consequently, every reduction class is undecidable.

The first fragments, which were shown to be either decidable or reduction classes (before Church's and Turing's results) are called *prefix-vocabulary* classes. Informally, prefix-vocabulary classes can be defined as follows. Let X be a class of sentences such that every sentence in X is of the following form: a sentence starts with a quantifier block (prefix) generated by a regular expression such as $\forall\exists\exists$ or $\forall\exists\forall^*$, where * means that any number of $\forall$-symbols may follow the $\exists$-symbol. After the quantifier block, there is a quantifier-free FO-formula of a certain vocabulary. The vocabulary may contain function and relation symbols (but no constant symbols, i.e., nullary function symbols). Furthermore, sentences may contain identity symbols $(=)$. In other words, each prefix-vocabulary class is associated with a prefix, vocabulary, and information whether sentences may contain identity symbols. To exemplify the above informal definition, let us give an example. The prefix-vocabulary class X denoted by $[\forall^*\exists^*, (0,2), (1)]_=$ consists of sentences starting with any number of universal quantifiers followed by any number of existential quantifiers. In the notation $[\forall^*\exists^*, (0,2), (1)]_=$, $(0,2)$ means that exactly two fixed binary relation symbols may occur in the sentences in X. No other relation symbols may occur. The part (1) in the notation means that exactly one unary function symbol is allowed to occur in the sentences in X. Furthermore, the presence of the identity symbol in the notation indicates that identity symbols may occur. Let R and S be binary relation symbols and f an unary function symbol. The quantifier-free part of the sentences in the class X may contain identity, R, S and f symbols, but no other non-logical symbols. For example $\forall x \forall y \forall z(\,(Rxy \land Rxz) \to y = z\,)$ and $\forall x \exists y(\,Sxfx \land Rfxy\,)$ are in X, but $\exists x \forall y(\,Tx \land Uxyg(y)x\,)$ is not.

We shall now mention two prefix-vocabulary classes, one of which is decidable and another which is a reduction class. The Löwenheim class is the class $[all, (\omega), (0)]_=$, where *all* denotes that any kind of quantifier prefixes may be used and ω denotes that any number of unary relation symbols may be used. This class is also known as monadic first-order logic (or monadic predicate calculus). It was in 1915 that Löwenheim [45] provided a decision algorithm for monadic first-order logic. He also showed that allowing the use of binary relation symbols (without unary relation symbols), would result in a prefix-vocabulary class $[all, (0, \omega), (0)]$ that is a reduction class. These results were sharpened many times later. For example, by extending the equality-free Löwenheim class $[all, (\omega), (0)]$ to the Löb-Gurevich class $[all, (\omega), (\omega)]$ or to Rabin class $[all, (\omega), (1)]_=$, we get classes which preserve decidability. On the other hand, the Kalmár-Surányi class $[\forall^* \exists, (0, 1), (0)]$ or the Denton class $[\forall \exists \forall^*, (0, 1), (0)]$ are reduction classes which are contained in $[all, (0, \omega), (0)]$. In addition to the classes mentioned above, there are many other prefix-vocabulary classes shown to be either decidable or undecidable. These results concerning prefix-vocabulary classes are due to a great research effort made during several decades, and thus the amount of related material is immense. We can only scratch the surface of all material available, but for readers who are interested in these rather historical fragments of first-order logic, we recommend the book *The Classical Decision Problem* [10].

As final words concerning prefix-vocabulary classes, we ask, why prefix-vocabulary classes and why were they studied for so many decades. We justify these questions by noting that not only is there an uncountable number of fragments of first-order logic[2] from where to choose, but also a relative lack of applications (at least currently) in fields other than mathematical logic can be seen as unattractive. Obviously, one reason could simply be historical. At the advent of first-order logic, there were not that many “real life” applications motivating research in the field of mathematical logic. Another reason could be the simple syntactic form of the prefix-vocabulary classes that simplifies their classification, and also the hopes that a full classification is obtainable. In the next three sections, we will introduce more modern fragments of first-order logic that have various applications in, e.g., database

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[2] Most of these obviously have a non-recursive syntax, as the number of algorithms is countable.

theory and beyond.

1.2 The two-variable fragment

One way, and surely a simple way, to restrict first-order logic is to restrict the number of variables which may occur in formulae. Fragments with a fixed number of variables are called *variable-bounded* (finite-variable) fragments of first-order logic. Henkin is considered one of the first who did a systematic study on them [26]. Let us denote by FO^k the k-variable-bounded fragment of first-order logic meaning that the formulae of FO^k may only contain the variables $x_1, \ldots, x_k$. In addition to the variable restriction, formulae of variable bounded fragments are relational, that is, they may only contain relation symbols, but not function or constant symbols. In contrast to prefix-vocabulary classes, formulae of variable-bounded fragments do not need to be in prenex normal form, allowing the reuse ("recycling") of variables in nested subformulae in formulae. (An FO-formula is in prenex normal form, if all quantifiers occurring in it appear at the beginning of the formula (prefix part) followed by quantifier free part (matrix part).)

Variable-bounded fragments have many applications in various fields such as finite model theory, database theory, knowledge representation (AI), and model checking [10].

In the case of variable-bounded fragments, the undecidability of FO^k for $k \geq 3$ follows directly from the undecidability of the conservative reduction class $[\forall\exists\forall, (\omega, 1), (0)]$, as it is properly contained in FO^k for every $k \geq 3$. Note that this holds even without the identity symbol.

It was Mortimer who first showed that FO^2, i.e., the *two-variable fragment* of first-order logic, is decidable [47]. The result was established by showing that FO^2 has the finite model property, that is, every FO^2-formula has a model if and only if it has a finite model. Note that in addition to the variable restriction, FO^2 is a relational fragment of FO, meaning that no function symbols may occur in the formulae of FO^2.[3] Adding just one function symbol would result in a fragment that contains e.g. the Gurevich class $[\forall^2, (0, 1), (1)]$ that is a conservative reduction class.

Note indeed that while we have only two distinct variables, say x and y, that can be used in FO^2-formulae, we can reuse them e.g. as follows:

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[3] Adding constant symbols would not change the decidability of FO^2, see [10]

$\exists x \exists y(Exy \wedge \exists x(Eyx \wedge \exists y(Exy)))$. This FO^2-sentence, where E is a binary relation representing edges in a directed graph, says that there is a directed path of length 3. Another example of variable reuse is the standard translation, which is a method for translating formulae of modal logic into FO-formulae: formulae of modal logic containing only unary modal diamond-operators can be translated into FO^2-formulae, see e.g. Chapter 2 in [7]. Consequently, for example standard *propositional modal logic* can be seen as a fragment of FO^2 due to the standard translation,

As a historical side-note, Scott [54] showed, before Mortimer, that FO^2 without equality is decidable. Scott essentially showed that FO^2-formulae without identity can be transformed into formulae in the Gödel class

$$[\exists^* \forall^2 \exists^*, (all), (0)].$$

At the time (1962) when Scott's result was published, it was thought that the Gödel class even with the identity would be decidable. The reason why it was thought to be the case was due to Gödel's claim [19]. Gödel claimed (without proof) that his decidability proof could be extended to deal with identity symbols, and thus the Gödel class with identity symbols would be decidable. However, due to Goldfarb, the class $[\forall^2 \exists^*, (\omega, 1), (0)]_=$ was shown to be undecidable [13]. Since the Goldfarb class is contained in the Gödel class with identity, the latter cannot also be decidable.

Complexity analysis of Mortimer's FO^2 decidability proof results in a 2NExpTime upper bound for the satisfiability problem of FO^2. This result was later sharpened to NExpTime-completeness in [15], along with a simpler proof for the finite model property.

Research concerning FO^2 has been, and still is, active. There are many extensions of FO^2 proved to be decidable or undecidable. For instance, the two-variable logic with counting quantifiers, FOC^2, was proved decidable in [16, 49] and its satisfiability problem was shown NExpTime-complete in [50]. As FOC^2 extends FO^2 by introducing new quantifiers, counting quantifiers, and thus extends the syntax of FO^2, we call FOC^2 a *syntactic extension* of FO^2. There of course are also undecidable syntactic extensions of FO^2, for instance *two-variable transitive closure logic*, TC^2. This extension along with many others were shown undecidable in [17]. In contrast to syntactic extensions, there are many decidable and undecidable extensions of FO^2 [48, 29, 46, 53, 34] which deal with certain restricted classes of structures rather than extending the syntax of FO^2. It is also worth pointing out some recent studies on the two-variable logic FO^2 such as [5, 4, 8, 33, 9] among others. Recent research on two-variable logic includes even investigations in non-classical frameworks (e.g., [41, 35, 36]).

1.3 The guarded fragment

Another fragment (or type of fragment) of first-order logic worth mentioning is the *guarded fragment* (GF) of first-order logic. This fragment was introduced in [1], and like FO^2, GF is also a relational fragment. All relational, quantifier-free FO-formulae are GF-formulae, and GF-formulae with quantifiers are of the following form. Let $\bar{x} = x_1, \ldots, x_k$ and $\bar{y} = y_1, \ldots, y_l$ be sequences of variables, $\varphi(\bar{x}, \bar{y})$ a GF-formula and $\alpha(\bar{x}, \bar{y})$ an atomic formula. Now $\exists \bar{y}(\alpha(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{y}))$ and $\forall \bar{y}(\alpha(\bar{x}, \bar{y}) \to \varphi(\bar{x}, \bar{y}))$ are GF-formulae; here the atomic formula α is called a *guard*, and it contains all free variables of φ. In other words, quantifiers in GF-formulae must be relativized by atomic formulae. Note also that the inspiration for the "guarded" quantification seems to come from the standard translation of modal logic.

The satisfiability problem for GF was shown to be 2ExpTime-complete in [14]. In the same paper, the satisfiability problem for fragments of GF, which have a bounded number of variables or only relation symbols of bounded arity, was shown to be ExpTime-complete.

The guarded fragment has been extended many times since its introduction. There is, for example, the *guarded negation fragment* of first-order logic, GNFO, introduced and shown to be 2NExpTime-complete in [3], and many other variants, see [18, 28].

Andréka et al. proved in [1] that variable bounded fragments, including FO^2, do not have all the "nice" model-theoretic properties possessed by modal logics. What are these nice properties (model-theoretic or modal behavior) are intentionally left somewhat vague, as we shall not analyze these properties much here. For the readers interested in this, we suggested to begin with the article [1]. Here we only aim to give a minimal background on modern fragments of first-order logic in order to motivate the reader for the work below. Moreover, the reason why FO^2 and GF in particular were introduced here is that current research regarding first-order fragments seems to be very active on these two fragments in particular.

One of the reasons FO^2 and GF in particular are important is their direct relation to modal logic. Modal logic has well-known applications in several fields, including specification and verification, knowledge representation, and even distributed computing. For the most recent research direction in the intersection of modal logic and distributed computing, see [51, 21, 38, 39]. Modal logic has also important applications in more theoretical frameworks. For example provability logic and intuitionist logics are very closely related to modal logic. Also concerning theoretical work, both FO^2 and GF have often proved directly useful when developing the theory of modal-logic-based systems. Typical examples of this include for example the direct extraction of upper bounds (for satisfiability problems of modal logics). See, e.g., [11, 25] for examples of this.

1.4 The uniform one-dimensional fragment

The *equality-free uniform one-dimensional fragment*, denoted $\mathrm{U}_1(wo =)$, of first-order logic was introduced in [23]. This relational fragment allows the use of relation symbols of arbitrary arity with certain restrictions. These restrictions are the *uniformity* and *one-dimensionality* conditions which can be described as follows. The one-dimensionality condition restricts quantification to blocks of existential (universal) quantifiers such that at most one variable may remain free in the quantified formula. The uniformity condition restricts the use of atomic formulae such that if $k > 1$ and $l > 1$, then Boolean combinations of atoms $Rx_1 \dots x_k$ and $Sy_1 \dots y_l$ are allowed only if the sets $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_l\}$ of variables are equal. Moreover, $\mathrm{U}_1(wo =)$-formulae do not contain identity symbols ($w$ith$o$ut $=$). However, Boolean combinations of formulae with at most one free variable can be formed freely.

In [23], the authors proved decidability of $\mathrm{U}_1(wo =)$ by a direct reduction to monadic first-order logic. The argument was based on extending the approach developed in [22, 24] and Chapter 2 of [37]. In [23], it was also shown that relaxing either the one-dimensionality or uniformity condition would result in undecidable extensions of $\mathrm{U}_1(wo =)$. More precisely, the *general one-dimensional fragment* GF_1, where uniformity is relaxed, and *strongly uniform two-dimensional fragment* SUF_2, where the dimensionality condition now concerns two free variables instead of one, were shown to be undecidable in [23]. In addition to the above results, $\mathrm{U}_1(wo =)$ was shown to be incomparable in expressivity with both FOC^2 and GNFO in [23], meaning that there are properties expressible in $\mathrm{U}_1(wo =)$ but not it in FOC^2 (GNFO) and vice versa.

It is also worth noting that uniformity, along with one-dimensionality, seems to be quite a crucial condition in terms of the decidability of $\mathrm{U}_1(wo =)$. If just one binary relation is allowed to be used in a non-uniform way, the resulting extension is undecidable. This result follows directly due to the class $[\forall\exists \wedge \forall^3, (\omega, 1)]$, which is a conservative reduction class [10]. This class consists of conjunctions $\psi \wedge \varphi$ such that $\psi \in [\forall\exists, (\omega, 1)]$ and $\varphi \in [\forall^3, (\omega, 1)]$.

The *uniform one-dimensional fragment* of first-order logic, U_1, was studied in [30]. This fragment extends $\mathrm{U}_1(wo =)$ by allowing the non-uniform use of identity symbols. In [30] a finite model property for U_1 was established and the satisfiability problem for U_1 was shown to be NExpTime-complete. The attempt to extend U_1 even further by adding counting quantifiers results in a fragment called *uniform one-dimensional fragment with counting quantifiers*, UC_1, which was shown undecidable in [30]. In addition to the above results, it was shown in [30] that $\mathrm{FO}^2 < \mathrm{U}_1 < \mathrm{FOC}^2$ when signatures contain only unary and binary relation symbols. In other words, the expressivity of U_1 over structures containing only unary or binary relations lies strictly between FO^2 and FOC^2. Furthermore, in [43] it was shown that the *fully uniform one-dimensional fragment* of first-order logic, FU_1, and FO^2 are *equi-expressive*, when signatures contain only unary or binary relation symbols. The logic FU_1 is a fragment of U_1, and full uniformity means that equality is also subject to the uniformity condition, just like all binary relations are. Due to the properties listed above, we may say that U_1, and especially FU_1, is a canonical, decidable extension of FO^2.

The paper [43] also works as a survey of the research on U_1. In addition to the survey nature of the paper [43], it presents some new results, inter alia showing that GNFO and U_1 are incomparable in expressivity. Furthermore, it also introduces a novel *description logic* $\mathcal{DL}_{\mathrm{FU}_1}$ that is shown to be expressively equivalent to FU_1 and also argued to be a natural generalization of the description logic $\mathcal{ALBO}^{id}$ [52] to higher arity contexts. Description logics are a family of knowledge representation languages with various applications in database theory as well as the theory of knowledge bases. Most description logics can be seen as fragments of first-order logic [2], and in particular, fragments of decidable fragments of FO such as two-variable logic and guarded fragments. As U_1 is a decidable extension of FO^2, it can also be seen as a potential formalism for description logic studies [43]. Those readers interested in description logics see [2] for an introduction on the subject. In any case, one of the main motivations for studying U_1 is the fact that U_1 extends the scope of (the very active) research program on FO^2 to the context with higher arity relations, and this, in turn, can be seen as crucial especially from the point of view of database theory.

Extensions of U_1, in addition to the one with counting quantifiers, have also been studied. In [32], *uniform one-dimensional fragment with one equivalence relation*, $\mathrm{U}_1[\sim]$, was shown to be decidable, and its satisfiability problem was shown to be 2NExpTime-complete. The binary equivalence symbol $\sim$ in $\mathrm{U}_1[\sim]$ is a so-called *non-uniform built-in relation*, meaning that it can be used in $\mathrm{U}_1[\sim]$-formulae in the same way as the identity symbol can be used in U_1, that is in a non-uniform way. The extension $\mathrm{U}_1[\sim_1, \sim_2]$ with two built-in equivalence relations increases expressive power such that it no longer preserves decidability [32]. In addition to the above results, the authors of [32] also studied some natural fragments of U_1 and proved that a certain restriction of U_1 that still contains FO^2, is only NExpTime-complete in the presence of a single non-uniform built-in equivalence. Also, U_1 with one built-in transitive relation was shown undecidable.

In this thesis, we continue research on extensions of U_1 started in [32]. We show that U_1 over ordered structures, denoted $\mathrm{U}_1(<)$, is decidable and its satisfiability problem is NExpTime-complete. Here the built-in linear order relation $<$ is like any other binary symbol in the sense that it is used only uniformly. Despite this, many interesting properties concerning the interplay of $<$ with even ternary and higher arity relations, are expressible in $\mathrm{U}_1(<)$. The syntax of U_1 is not extended, but we in fact deal with a collection of classes of structures, namely, finite linearly ordered, well-ordered, and linearly ordered classes of structures. In addition to the order relation $<$, structures may of course contain an arbitrary number of other relation symbols of any arity.

In contrast to the case of $\mathrm{U}_1(<)$, we also show that uniform one-dimensional fragment with two non-uniform built-in order relations, $\mathrm{U}_1[<_1, <_2]$, is undecidable. Note indeed that here the binary relation symbols $<_1$ and $<_2$ may be freely (non-uniformly) used. We point out, as suggested future work, that decidability of $\mathrm{U}_1(<_1, <_2)$ over ordered domains, with two built-in linear order relations $<_1$ and $<_2$ that are used uniformly, as well as $\mathrm{U}_1[<]$, where $<$ is a non-uniform built-in linear order relation, remain unsolved.

We have now introduced the three modern, decidable fragments FO^2, GF, and U_1 of first-order logic, where U_1 is the most recent one. They all have their place in research and potential for applications, and there is no reason to put them in any clear order of preference. This is partially justified already by the fact that FO^2 and U_1 are incomparable with GF (and even with GNFO) in expressivity. Furthermore, while FO^2 is properly contained in U_1, its extension FOC^2 is incomparable with U_1.

1.5 The structure of the thesis

The structure of the thesis is the following. In Chapter 2, we properly define some of the notions mentioned above. Moreover, more definitions and notations are introduced, and thus very little background information is needed to understand this thesis. Chapters 3 and 4 together present the main results of the thesis, namely the fact that U_1 is decidable over different kinds of classes of linearly ordered structures. Following the decidability results, the complexity of the related satisfiability problems for U_1 over ordered domains is given in Chapter 5. As a final result, Chapter 6 presents the undecidability result of U_1 with two non-uniform built-in linear orders. Note that all results in this theses are novel and not published yet anywhere. The argument leading to the main result of this work uses new methods in addition to methods introduced in [48] and [30]. The research results presented in this work are joint work with Antti Kuusisto. Chapter 7 is the last chapter concluding the thesis.

Chapter 2

Preliminaries

We let $\mathbb{Z}_+$ denote the set of positive integers. If f is a function with a domain S, we define

$$img(f) := \{\, f(s) \mid s \in S \,\}.$$

An *ordered set* is a structure $(A, <)$ where A is a set and $<$ a linear order on A. We call a subset I of A an *interval* if for all $a, c \in I$ and all $b \in A$, it holds that if $a < b < c$, then $b \in I$. A *permutation of a tuple* $(u_1, \dots, u_k)$ is a tuple $(u_{f(1)}, \dots, u_{f(k)})$ for some bijection $f : \{1, \dots, k\} \to \{1, \dots, k\}$. A *trivial tuple* is a tuple $(u_1, \dots, u_k)$ such that $u_i = u_j$ for all $i, j \in \{1, \dots, k\}$.

We let VAR denote the set $\{v_1, v_2, \ldots\}$ of first-order variable symbols. We mostly use metavariables x, y, z, x_1, y_1, z_1, etc., to denote the variables in VAR. Note that for example the metavariables x and y may denote the same variable symbol v_i, while v_i and v_j for $i \neq j$ are always different symbols. Let R be a k-ary relation symbol. An atomic formula $Rx_1...x_k$ is called an X-atom if $X = \{x_1, \dots, x_k\}$. For example, if x, y, z are distinct variables, then Syx and $Rxyxxy$ are $\{x, y\}$-atoms while Px and $Txzy$ are not. $Txyz$ and $Syyxz$ are $\{x, y, z\}$-atoms. For technical reasons, atoms $x = y$ with an equality symbol are *not* $\{x, y\}$-atoms.

Let τ be a relational vocabulary. A k-ary τ-atom is an atomic τ-formula that mentions exactly k variables: for example, if x, y, z are distinct variables and $R, T \in \tau$ relation symbols with arities 5 and 3, respectively, then the atoms $Txxy$ and $x = y$ are binary τ-atoms and $Rxxyzx$ and $Txyz$ ternary τ-atoms. If $P, S \in \tau$ are relation symbols of arities 1 and 2, respectively, then Px and $x = x$ are unary τ-atoms and Sxy a binary τ-atom.

Let τ_m denote a countably infinite relational vocabulary in which every relation symbol is of the arity m. Let $\mathcal{V}$ be *a complete relational vocabulary*, that is $\mathcal{V} = \bigcup_{m \in \mathbb{Z}_+} \tau_m$. In this thesis we consider models and logics with relation symbols only; function and constant symbols will not be considered. (The identity symbol is considered a logical constant and is therefore not a relation symbol.) We denote models by $\mathfrak{A}$, $\mathfrak{B}$ etcetera. The domain of these models is then denoted by A and B, respectively. If τ is a vocabulary, then a τ-model interprets all the relation symbols in τ and no other relation symbols. A τ-formula is a formula whose relation symbols are contained in τ. If $\mathfrak{A}$ is a τ-model and $\mathfrak{B}$ a τ'-model such that $\tau \subseteq \tau'$ and $\mathfrak{A} = \mathfrak{B} \upharpoonright \tau$, then $\mathfrak{B}$ is an *expansion* of $\mathfrak{A}$ and $\mathfrak{A}$ is the *τ-reduct* of $\mathfrak{B}$. The notion of a *substructure* is defined in the usual way, and if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ (written: $\mathfrak{A} \subseteq \mathfrak{B}$), then $\mathfrak{B}$ is an *extension* of $\mathfrak{A}$.

Consider a vocabulary $\tau \subseteq \mathcal{V}$. The set of τ-formulae of the *equality-free uniform one-dimensional fragment* $\mathrm{U}_1(wo{=})$ is the smallest set $\mathcal{F}$ such that the following conditions hold.

1. Every unary τ-atom is in $\mathcal{F}$.

2. If $\varphi \in \mathcal{F}$, then $\neg\varphi \in \mathcal{F}$.

3. If $\varphi, \psi \in \mathcal{F}$, then $(\varphi \wedge \psi) \in \mathcal{F}$.

4. Let $X' := \{x_0, ..., x_k\} \subseteq \mathrm{VAR}$ and $X \subseteq X'$. Let φ be a Boolean combination of X-atoms and formulae in $\mathcal{F}$ whose free variables (if any) are in the set X'. Then the formulae $\exists x_1 ... \exists x_k \varphi$ and $\exists x_0 ... \exists x_k \varphi$ are in $\mathcal{F}$.

In addition to the logical symbols $\neg$ and $\wedge$, we use the following abbreviations: $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \rightarrow \psi := \neg\varphi \vee \psi$, $\varphi \leftrightarrow \psi := \neg(\varphi \vee \psi) \vee \neg(\neg\psi \vee \neg\varphi)$, and $\forall x \varphi := \neg\exists x \neg\varphi$. We usually omit the parentheses around $(\varphi \wedge \psi)$ and write $\varphi \wedge \psi$, if there is no risk that omitting parentheses would cause confusion. For example, $\exists x \exists y \exists z(\neg Rxyzxy \wedge \neg Tyxz \wedge Px \vee Qy)$ and $\exists x \forall y \forall z(\neg Sxy \rightarrow \exists u \exists v Tuvz)$ are formulae of $\mathrm{U}_1(wo{=})$. If $\psi(y)$ is a formula of $\mathrm{U}_1(wo{=})$, then $\exists y \exists z(Txyz \wedge Rzxyzz \wedge \psi(y))$ is as well. However, the formula $\exists x \exists y \exists z(Sxy \vee Sxz)$ is *not* a formula of $\mathrm{U}_1(wo{=})$ because $\{x, y\} \neq \{x, z\}$. The formula is said to violate the uniformity condition, i.e., the syntactic restriction that the relational atoms of higher arity bind the same set of variables. The formula $\forall y(Py \wedge \exists x Txyz)$ is not a formula of $\mathrm{U}_1(wo{=})$ because it violates one-dimensionality, as $\exists x Txyz$ has two free variables. Perhaps the simplest formula of $\mathrm{U}_1(wo{=})$ that can be expressed in neither two-variable logic with counting quantifiers FOC^2 nor in the guarded negation fragment GNFO is the formula $\exists x \exists y \exists z \neg Txyz$.

The set of formulae of the *fully uniform one-dimensional fragment* FU_1 is obtained from the set of formulae of $\mathrm{U}_1(wo{=})$ by allowing the substitution of any binary relation symbols in a formula of $\mathrm{U}_1(wo{=})$ by the equality symbol $=$. If restricted to vocabularies with at most binary symbols, FU_1 is exactly as expressive as FO^2 [43].

The set of τ-formulae of the *uniform one-dimensional fragment* U_1 is the smallest set $\mathcal{F}$ obtained by adding to the four above clauses that define $\mathrm{U}_1(wo{=})$ the following additional clause:

5. Every equality atom $x = y$ is in $\mathcal{F}$.

For example $\exists y \exists z (Txyz \wedge Qy \wedge x \neq y)$ as well as the formula $\exists x \exists y \exists z (x \neq y \wedge y \neq z \wedge z \neq x)$ are U_1-formulae. The latter formula is an example of a (counting) condition that is well known to be undefinable in FO^2. A more interesting example of a condition not expressible in FO^2 (cf. [43]) is defined by the U_1-formula $\exists x \forall y \forall z (Syz \rightarrow (x = y \vee x = z))$, which expresses that some element is part of every tuple of S. For more examples and background intuitions, see the survey [43].

Let $\bar{x}$ be a tuple of variables. Let $\exists \bar{x} \varphi$ be a U_1-formula which is formed by applying the rule 4 of the syntax above. Recall the set X used in the formulation. If φ does not contain any relational atom (other than equality) with at least two distinct variables, we define $L_\varphi := \emptyset$, and otherwise we define $L_\varphi := X$. We call the set L_φ the set of *live variables* of φ. For example, in $\exists y \exists z \exists u (Txyz \wedge Rxxyyz \wedge x = u \wedge Q(u))$ the set of live variables is $\{x, y, z\}$.

A quantifier-free subformula of a U_1-formula is called a U_1-matrix. Let $\psi(x_1, \dots, x_k)$ be a U_1-matrix with exactly the distinct variables $x_1, \dots, x_k$. Let $\mathfrak{A}$ be a model with domain A, and let $a_1, \dots, a_k \in A$ be (not necessarily distinct) elements. Let T be the smallest subset of $\{a_1, ..., a_k\}$ such that for every $x_i \in L_\psi$, we have $a_i \in T$, that is $T = \{a_i \mid x_i \in L_\psi\}$. We denote T by $\mathit{live}\big(\psi(x_1, ..., x_k)[a_1, ..., a_k]\big)$. Let us have an example of this notation.

Example 1.

$$\begin{aligned}&\text{If } \psi(v_1, v_2, v_3, v_4) := (Rv_2v_3v_2 \wedge Pv_4 \wedge v_1 = v_2),\\ &\text{then } \mathit{live}\big(\psi(v_1, v_2, v_3, v_4)[a, b, c, b]\big) = \{b, c\}.\end{aligned}$$

We shall shorten the notation $live\big(\psi(x_1,...,x_k)[a_1,\dots,a_k]\big)$ to $live\big(\psi[a_1,...,a_k]\big)$ when there is no possibility of confusion.

2.1 Generalized Scott normal form

A U_1-formula φ is in *generalized Scott normal form*, if

$$\varphi = \bigwedge_{1\le i\le m_\exists} \forall x\exists y_1\dots\exists y_{k_i}\varphi_i^\exists(x,y_1,\dots,y_{k_i}) \wedge \bigwedge_{1\le i\le m_\forall} \forall x_1\dots\forall x_{l_i}\varphi_i^\forall(x_1,\dots,x_{l_i}),$$

where the formulae $\varphi_i^\exists$ and $\varphi_i^\forall$ are *quantifier-free* U_1*-matrices.* Henceforth by a normal form we always mean generalized Scott normal form. The formulae $\forall x\exists y_1...\exists y_{k_i}\varphi_i^\exists(x,y_1,...,y_{k_i})$ are called *existential conjuncts* and the formulae $\forall x_1...\forall x_{l_i}\varphi_i^\forall(x_1,...,x_{l_i})$ *universal conjuncts* of φ. Let n be the maximum number of the set $\{k_i+1\}_{1\le i\le m_\exists}\cup\{l_i\}_{1\le i\le m_\forall}$. We call n the *width* of the sentence φ. The quantifier-free part of an existential (universal) conjunct is called an *existential (universal) matrix.* We often do not properly differentiate between existential conjuncts and existential matrices when there is no risk of confusion. The same holds for universal matrices and universal conjuncts.

Proposition 2 ([30]). *Every* U_1*-formula* φ *can be translated in polynomial time to a* U_1*-formula* φ' *in generalized Scott normal form that is equisatisfiable with* φ *in the following sense. If* $\mathfrak{A}\models\varphi$*, then* $\mathfrak{A}^*\models\varphi'$ *for some expansion* $\mathfrak{A}^*$ *of* $\mathfrak{A}$*, and vice versa, if* $\mathfrak{B}\models\varphi'$*, then* $\mathfrak{B}'\models\varphi$ *for some reduct* $\mathfrak{B}'$ *of* $\mathfrak{B}$*. The vocabulary of* φ' *expands the vocabulary of* φ *with fresh unary relation symbols only.*

Let $\mathfrak{A}$ be a model satisfying a normal form sentence φ of U_1. Let $a,a_1,\dots,a_{k_i}\in A$, and let $\forall x\exists y_1...\exists y_{k_i}\varphi_i^\exists(x,y_1,...,y_{k_i})$ be an existential conjunct of φ such that $\mathfrak{A}\models\varphi_i^\exists(a,a_1,...,a_{k_i})$. Then we define

$$\mathfrak{A}_{a,\varphi_i^\exists} := \mathfrak{A}\restriction\{a,a_1,\dots,a_{k_i}\}$$

and we call $\mathfrak{A}_{a,\varphi_i^\exists}$ a *witness structure* for the pair $(a,\varphi_i^\exists)$. The elements of the witness structure are called *witnesses.* In addition, we define

$$\bar{\mathfrak{A}}_{a,\varphi_i^\exists} := \mathfrak{A}_{a,\varphi_i^\exists}\restriction live(\varphi_i^\exists[a,a_1,\dots,a_{i_k}])$$

and we call it the *live part* of $\mathfrak{A}_{a,\varphi_i^\exists}$. If the live part $\bar{\mathfrak{A}}_{a,\varphi_i^\exists}$ does not contain a, then it is called *free*. The remaining part $\mathfrak{A}_{a,\varphi_i^\exists} \restriction (A_{a,\varphi_i^\exists} \setminus \bar{A}_{a,\varphi_i^\exists})$ of $\mathfrak{A}_{a,\varphi_i^\exists}$ is called the *dead part* of the witness structure. In other words, the witness structure consists of the two parts: the live part and the dead part.

2.2 Structure classes

Fix a binary relation $<$. Throughout the thesis, we let $\mathcal{O}$ denote the class of all structures $\mathfrak{A}$ such that $\mathfrak{A}$ is a τ-structure for some $\tau \subseteq \mathcal{V}$ with $< \in \tau$, and the symbol $<$ is interpreted as a *linear order* over A. (Note that the vocabulary is not required to be the same for all models in $\mathcal{O}$.) The class $\mathcal{WO}$ is defined similarly, but this time $<$ is interpreted as a well-ordering of A, i.e., a linear order over A such that each nonempty subset of A has a least element w.r.t. $<$. Similarly, $\mathcal{O}_{fin}$ is the subclass of $\mathcal{O}$ where every model is finite.

Consider a class $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$. The *satisfiability problem of* U_1 *over* $\mathcal{K}$ asks, given a formula of U_1, whether φ has a model in $\mathcal{K}$. The set of relation symbols in the input formula φ is not limited in any way.

If R_1 and R_2 are binary relation symbols, we let $\mathrm{U}_1[R_1, R_2]$ be the extension of U_1 such that φ is a formula of $\mathrm{U}_1[R_1, R_2]$ iff it can be obtained from some formula of U_1 by replacing any number of equality symbols with R_1 or R_2.

Example 3. The sentence $\forall x \forall y \forall z((R_1xy \wedge R_1yz) \to R_1xz)$ is obtained from the U_1-formula $\forall x \forall y \forall z((x = y \wedge y = z) \to x = z)$ in the way described above.

Such extensions of U_1 are said to allow non-uniform use of R_1 and R_2 in formulae. At the end of this thesis we investigate $\mathrm{U}_1[<_1, <_2]$ over structures where $<_1$ and $<_2$ both denote linear orders.

2.3 Types and tables

Let τ be a finite relational vocabulary. A 1-type (over τ) is a maximally consistent set of τ-atoms and negated τ-atoms in the single variable v_1. We denote 1-types by α and the set of all 1-types over τ by $\boldsymbol{\alpha}_\tau$. If there is no risk of confusion, we may write $\boldsymbol{\alpha}$ instead of $\boldsymbol{\alpha}_\tau$. The size of $\boldsymbol{\alpha}_\tau$ is clearly bounded by $2^{|\tau|}$. We often identify a 1-type α with the conjunction of its elements, thereby considering $\alpha(x)$ as simply a formula in the single variable x. (Note that here we used x instead of the official variable v_1 with which the 1-type α was defined.)

Let $\mathfrak{A}$ be a τ-model and α a 1-type over τ. The type α is said to be *realized* in $\mathfrak{A}$ if there is some $a \in A$ such that $\mathfrak{A} \models \alpha(a)$. We say that the point a realizes the 1-type α in $\mathfrak{A}$ and write $tp_{\mathfrak{A}}(a) = \alpha$. Note that every element of $\mathfrak{A}$ realizes exactly one 1-type over τ. We let $\boldsymbol{\alpha}_{\mathfrak{A}}$ denote the set of all 1-types over τ that are realized in $\mathfrak{A}$. It is worth noting that 1-types do not only involve unary relations: for example an atom $Rxxx$ can be part of a 1-type.

Let $k \geq 2$ be an integer. A *k-table* over τ is a maximally consistent set of $\{v_1, \dots, v_k\}$-atoms and negated $\{v_1, \dots, v_k\}$-atoms over τ. Moreover, 2-tables do not contain identity atoms or negated identity atoms.

Example 4. Using the meta-variables x, y instead of v_1, v_2, the set

$$\{Rxxy, Rxyx, \neg Ryxx, Ryyx, \neg Ryxy, Rxyy, x < y, \neg\, y < x\}$$

is a 2-table over $\{R, <, P\}$, where R is a ternary, $<$ binary and P a unary symbol.

We denote k-tables by β. Similarly to what we did with 1-types, a k-table β can be identified with the conjunction of its elements, denoted by $\beta(x_1, \dots, x_k)$. If $a_1, \dots, a_k \in A$ are *distinct* elements such that $\mathfrak{A} \models \beta(a_1, \ldots, a_k)$, we say that $(a_1, \dots, a_k)$ *realizes* the table β and write

$$tb_{\mathfrak{A}}(a_1, \dots, a_k) = \beta.$$

Every tuple of k distinct elements in the τ-structure $\mathfrak{A}$ realizes exactly one k-table β over τ.

Let α be a 1-type. We define the formulae

$$min_\alpha(x) := \alpha(x) \wedge \forall y\big((\alpha(y) \wedge x \neq y) \rightarrow x < y\big) \text{ and}$$
$$max_\alpha(x) := \alpha(x) \wedge \forall y\big((\alpha(y) \wedge x \neq y) \rightarrow y < x\big)$$

for later use. An element $a \in A$ is called a *minimal* (resp., *maximal*) *realization* of α in $\mathfrak{A}$ iff $\mathfrak{A} \models min_\alpha(a)$ (resp., $\mathfrak{A} \models max_\alpha(a)$). This definition holds even if $\mathfrak{A}$ interprets $<$ as a binary relation that is *not* a linear order; at a certain very clearly marked stage of the investigations below, the symbol $<$

is used over a model $\mathfrak{B}$ where it is not necessarily interpreted as an order but is instead simply a binary relation.

Let φ be a normal form sentence of U_1 over τ and let $\mathfrak{A}$ be a τ-model. Let n be the width of φ. A 1-type α over τ is called *royal* (in $\mathfrak{A}$ and w.r.t. φ) if there are at most $n-1$ elements in A realizing α. Elements in A that realize a royal 1-type are called *kings* (w.r.t. φ). Other elements in A are *pawns* (w.r.t. φ). If $K_{\mathfrak{A}}$ denotes the set of kings in $\mathfrak{A}$, then $K_{\mathfrak{A}}$ is bounded by

$$(n-1)|\boldsymbol{\alpha}| = (n-1)2^{|\tau|},$$

where $\boldsymbol{\alpha}$ is the set of all 1-types over τ.

Now recall the notion of a witness structure $\mathfrak{A}_{a,\varphi_i^{\exists}}$ in a model $\mathfrak{A}$ for a pair $(a, \varphi_i^{\exists})$, where $a \in A$ is an element and $\varphi_i^{\exists}$ an existential conjunct of a normal form formula. Let α be a 1-type. By a *witness structure of* $(\alpha, \varphi_i^{\exists})$ we mean a witness structure $\mathfrak{A}_{a',\varphi_i^{\exists}}$ for some pair $(a', \varphi_i^{\exists})$ such that $a' \in A$ realizes α.

Chapter 3

Analysing ordered structures

Let φ be a normal form sentence of U_1 and τ the set of relation symbols in φ. Assume that the symbol $<$ occurs in φ. Let r be the highest arity occurring in the symbols in τ, and let n be the width of φ. Denote $min\{r, n\}$ by m. Let $\mathfrak{A} \in \mathcal{O}$ be a τ-model that satisfies φ. Let $P \subseteq A$ be the set of all pawns (w.r.t. φ) of $\mathfrak{A}$. Thus, for every $p \in P$, there are at least n elements in A realizing the 1-type of p. Let $\mathbf{c} \geq 3$ be an integer. The $\mathbf{c}$-*cloning extension of* $\mathfrak{A}$ *with respect to* φ is a linearly ordered *extension* $\mathfrak{A}'$ of $\mathfrak{A}$ defined by the following process.

1. Defining an ordered domain for $\mathfrak{A}'$: For each $p \in P$, let $Cl(p)$ be a set $\{p_0\} \cup \{p_2, \dots, p_{\mathbf{c}-1}\}$ of fresh elements. The domain of $\mathfrak{A}'$ is the set

$$A' = A \cup \bigcup_{p \in P} Cl(p).$$

For each $p \in P$, the elements $\{p_2, \dots, p_{\mathbf{c}-1}\}$ are placed immediately after p while the element p_0 is inserted immediately before p, so

$$\{p_0\} \cup \{p\} \cup \{p_2, \dots, p_{\mathbf{c}-1}\}$$

becomes an interval with $\mathbf{c}$ elements such that

$$p_0 < p < p_2 < \dots < p_{\mathbf{c}-1}.$$

The reason why we place the element p_0 before p and the other elements after it will become clear later on.

2. Cloning stage: For every $p \in P$, every $p' \in Cl(p)$, and every subset $S \subseteq A \setminus \{p\}$ such that $1 \leq |S| \leq m - 1$, we define

$$tp_{\mathfrak{A}'}(p') := tp_{\mathfrak{A}}(p)$$

and

$$tb_{\mathfrak{A}'}(p', \bar{s}) := tb_{\mathfrak{A}}(p, \bar{s}),$$

where $\bar{s}$ is an $|S|$-tuple that enumerates the elements of S.

3. Completion stage: For each $p \in P$, let I_p denote the interval

$$\{p_0\} \cup \{p\} \cup \{p_2, \ldots, p_{\mathbf{c}-1}\}.$$

We call the intervals I_p *clone intervals* and define

$$\mathbf{I} := \bigcup_{p \in P} I_p.$$

Now define P_2 to be the set of all pairs (α_1, α_2) of 1-types such that we have

$$\mathfrak{A}' \models \alpha_1(u) \wedge \alpha_2(u') \wedge u < u'$$

for some elements $u, u' \in A'$. (Note that α_1 and α_2 are allowed to be the same type.) Then define a function $t_2 : P_2 \to A^2$ that maps every pair (α_1, α_2) in P_2 to some pair $(w, w') \in A^2$ such that

$$tp_{\mathfrak{A}}(w) = \alpha_1,\ tp_{\mathfrak{A}}(w') = \alpha_2, \text{ and } w <^{\mathfrak{A}} w'.$$

We then do the following.

Assume $u, u' \in \mathbf{I}$ such that $u <^{\mathfrak{A}'} u'$. Let α_1 and α_2 denote the 1-types of u and u', respectively, and assume no table has been defined over (u, u') or (u', u) in the *cloning stage*. Then we define

$$tb_{\mathfrak{A}'}(u, u') := tb_{\mathfrak{A}}(t_2(\alpha_1, \alpha_2)).$$

Now recall $m = min\{n, r\}$. Assume $k \in \{3, \ldots, m\}$, and let P_k be the set of tuples $(\alpha_1, \ldots, \alpha_k)$ of 1-types (repetitions of types allowed) such that

$$\mathfrak{A}' \models \alpha_1(u_1) \wedge \ldots \wedge \alpha_k(u_k)$$

for some elements $u_1, \ldots, u_k \in A'$ such that

$$u_1 <^{\mathfrak{A}'} u_2 <^{\mathfrak{A}'} \ldots <^{\mathfrak{A}'} u_k.$$

Define a function $t_k : P_k \to A^k$ that maps every tuple $(\alpha_1, \ldots, \alpha_k)$ in P_k to some tuple $(w_1, \ldots, w_k) \in A^k$ of *distinct* elements such that

$$tp_{\mathfrak{A}}(w_j) = \alpha_j$$

for each $j \in \{1, \ldots, k\}$. Note that the order of the elements $w_1, \ldots, w_k$ in $\mathfrak{A}$ does not matter, and note also that it is indeed always possible to find k suitable elements because each pawn in $\mathfrak{A}$ has at least $n \geq m \geq k$ occurrences in $\mathfrak{A}$. Now consider every tuple $(u_1, \ldots, u_k) \in A'^k$ of elements such that

$$u_1 <^{\mathfrak{A}'} u_2 <^{\mathfrak{A}'} \ldots <^{\mathfrak{A}'} u_k$$

and such that we have not defined any table in the *cloning stage* over $(u_1, \ldots, u_k)$ or over any permutation of $(u_1, \ldots, , u_k)$, and define

$$tb_{\mathfrak{A}'}(u_1, \ldots, , u_k) := tb_{\mathfrak{A}}(t_k(\alpha_1, \ldots, \alpha_k)),$$

where α_j denotes the type of u_j for each j. Do this procedure for each $k \in \{3, \ldots, m\}$. Finally, *over tuples with more than m distinct elements*, we define arbitrarily the interpretations (in $\mathfrak{A}'$) of relation symbols of arities greater than m. This completes the definition of $\mathfrak{A}'$.

Lemma 1. *Let $\mathfrak{A} \in \mathcal{O}$ be a model and $\mathfrak{A}'$ its* **c**-*cloning extension w.r.t. φ. Now, if $\mathfrak{A} \models \varphi$, then $\mathfrak{A}' \models \varphi$.*

Proof. It is easy to show that the existential conjuncts are dealt with in the *cloning stage* of the construction of $\mathfrak{A}'$, so we only need to argue that for all universal conjuncts χ of φ, if $\mathfrak{A} \models \chi$, then $\mathfrak{A}' \models \chi$. To see that $\mathfrak{A}'$ satisfies the universal conjuncts, consider such a conjunct $\forall x_1 \ldots \forall x_k \psi(x_1, \ldots, x_k)$, where $\psi(x_1, \ldots, x_k)$ is quantifier free, and let $(a_1, \ldots, a_k)$ be a tuple of elements from A', with possible repetitions. We must show that

$$\mathfrak{A}' \models \psi(a_1, \ldots, a_k).$$

Let

$$\{u_1, \ldots, u_{k'}\} := live(\psi(x_1, \ldots, x_k)[a_1, \ldots, a_k])$$

and call $V := \{a_1, \dots, a_k\}$. The table $tb_{\mathfrak{A}'}(u_1, \dots, u_{k'})$ has been defined either in the *cloning stage* or the *completion stage* to be $tb_{\mathfrak{A}}(b_1, \dots, b_{k'})$ for some distinct elements $b_1, \dots, b_{k'} \in A$. Furthermore, since $\mathfrak{A}'$ and $\mathfrak{A}$ have exactly the same number of realizations of each royal 1-typep and since both models have at least $n \geq k$ realizations of each pawn, it is easy to define an injection f from V into A that preserves 1-types and such that $f(u_i) = b_i$ for each $i \in \{1, \dots, k'\}$. Therefore

$$\mathfrak{A}' \models \psi(a_1, \ldots, a_k) \text{ iff } \mathfrak{A} \models \psi(f(a_1), \ldots, f(a_k)).$$

Since $\mathfrak{A} \models \varphi$, we have $\mathfrak{A} \models \psi(f(a_1), \dots, f(a_k))$ and therefore $\mathfrak{A}' \models \psi(a_1, \dots, a_k)$. $\square$

We now fix a sentence φ of U_1 with the set τ (with $< \in \tau$) of relation symbols occurring in it. We also fix a τ-model $\mathfrak{A} \in \mathcal{O}$. We assume $\mathfrak{A} \models \varphi$ and fix a 3-cloning extension $\mathfrak{A}'$ of $\mathfrak{A}$ w.r.t. φ. We let n be the width of φ and $m_\exists$ the number of existential conjuncts in φ. The models $\mathfrak{A}$ and $\mathfrak{A}'$ as well as the sentence φ will remain fixed in the next two sections (3.1 and 3.2). In the two sections we will study these two models and the sentence φ and isolate some constructions and concepts that will be used later on.

3.1 Identification of a court

Let K denote the set of kings of $\mathfrak{A}'$ (w.r.t. φ). Thus K is also the set of kings of $\mathfrak{A}$. We next identify a finite substructure $\mathfrak{C}$ of $\mathfrak{A}$ called a *court of* $\mathfrak{A}$ *with respect to* φ. We note that a court of $\mathfrak{A}$ w.r.t. φ can in general be chosen in several ways.

Before defining $\mathfrak{C}$, we construct a certain set $D \subseteq A$. Consider a pair $(\alpha, \varphi_i^\exists)$, where α is a 1-type (over τ) and $\varphi_i^\exists$ an existential conjunct of φ. If there exists a *free* witness structure in $\mathfrak{A}$ for $\varphi_i^\exists$ and some element $a \in A$ realizing 1-type α, then pick exactly one such free witness structure $\mathfrak{A}_{a,\varphi_i^\exists}$ and define

$$D(\alpha, \varphi_i^\exists) := \bar{A}_{a,\varphi_i^\exists},$$

i.e., the set $D(\alpha, \varphi_i^\exists)$ is the domain of the *live part* $\bar{\mathfrak{A}}_{a,\varphi_i^\exists}$ of $\mathfrak{A}_{a,\varphi_i^\exists}$. Otherwise define $D(\alpha, \varphi_i^\exists) = \emptyset$. Define D to be the union of the sets $D(\alpha, \varphi_i^\exists)$ for each 1-type α (over τ) and each existential conjunct $\varphi_i^\exists$ of φ. The size of D is bounded by $m_\exists|\boldsymbol{\alpha}|n$.

Now, for each $a \in (K \cup D) \subseteq A$ and each $\varphi_i^{\exists}$, let $\mathfrak{C}_{a,\varphi_i^{\exists}}$ be some witness structure for the pair $(a, \varphi_i^{\exists})$ in $\mathfrak{A}$. Define the domain C of $\mathfrak{C}$ as follows:

$$C := \bigcup_{a \in K \cup D,\ 1 \leq i \leq m_{\exists}} C_{a,\varphi_i^{\exists}}.$$

Note that K and D are both subsets of C. We define $\mathfrak{C}$ to be the substructure of $\mathfrak{A}$ induced by C, i.e., $\mathfrak{C} := \mathfrak{A} \upharpoonright C$. Thus $\mathfrak{C}$ is also a substructure of $\mathfrak{A}'$. An upper bound for the size of C is obtained as follows, where $\boldsymbol{\alpha}$ denotes $\boldsymbol{\alpha}_{\tau}$.

$$\begin{aligned}
|C| &\leq |D \cup K| n m_{\exists} \\
&\leq (n m_{\exists} |\boldsymbol{\alpha}| + n|\boldsymbol{\alpha}|) n m_{\exists} \\
&\leq (|\varphi|^2 |\boldsymbol{\alpha}| + |\varphi||\boldsymbol{\alpha}|)|\varphi|^2 \\
&\leq (|\varphi|^4 + |\varphi|^3)|\boldsymbol{\alpha}| \\
&\leq 2|\varphi|^4 |\boldsymbol{\alpha}|.
\end{aligned}$$

We call $\mathfrak{C}$ the court of $\mathfrak{A}$ (w.r.t. φ). Note that we could have chosen the court $\mathfrak{C}$ in many ways from $\mathfrak{A}$. Here we choose a single court $\mathfrak{C}$ for $\mathfrak{A}$ and fix it for Section 3.2.

3.2 Partitioning cloning extensions into intervals

In this section we partition the 3-cloning extension $\mathfrak{A}'$ of the ordered structure $\mathfrak{A}$ into a finite number of non-overlapping intervals. Roughly speaking, the elements of the court $\mathfrak{C}$ of $\mathfrak{A}$ will all create a singleton interval and the remaining interval bounds will indicate the least upper bounds and greatest lower bounds of occurrences of 1-types in $\mathfrak{A}'$. We next define the partition formally; we call the resulting family of intervals $I_s \subseteq A'$ the *canonical partition of $\mathfrak{A}'$ with respect to $\mathfrak{C}$.*

We begin with some auxiliary definitions. Recall that $\boldsymbol{\alpha}_{\mathfrak{A}}$ denotes the set of 1-types realized in $\mathfrak{A}$, and thus $\boldsymbol{\alpha}_{\mathfrak{A}} = \boldsymbol{\alpha}_{\mathfrak{A}'}$. For each non-royal 1-type α in

$\boldsymbol{\alpha}_{\mathfrak{A}}$, define the sets

$$\begin{aligned} A'_\alpha &= \{a \in A' \mid tp_{\mathfrak{A}'}(a) = \alpha\}, \\ \mathcal{D}^-_\alpha &= \bigcup_{a \in A'_\alpha} \{b \in A' \mid a \leq b\}, \text{ and} \\ \mathcal{D}^+_\alpha &= \bigcup_{a \in A'_\alpha} \{b \in A' \mid b \leq a\}. \end{aligned}$$

In an ordered set $(L, <)$, an *interval bound* is defined to be a nonempty set $S \subsetneq L$ that is downwards closed ($u' < u \in S \Rightarrow u' \in S$). A finite number of interval bounds define a partition of an ordered set into a finite number of intervals in a natural way. We define the following finite collection of interval bounds for $\mathfrak{A}'$.

- Every $c \in C$ defines two interval bounds, $\{u \in A' \mid u < c\}$ and $\{u \in A' \mid u \leq c\}$. Thereby each $c \in C$ forms a singleton interval $\{c\}$.
- Each non-royal 1-type α creates two interval bounds: the sets $A' \setminus \mathcal{D}^-_\alpha$ and $\mathcal{D}^+_\alpha$.

This creates a finite family of intervals $(I_s)_{1 \leq s \leq N}$ that partitions A'. Here N is the finite total number of intervals in the family. The intervals I_s in the family are enumerated in the natural way, so if $s < s'$ for some $s, s' \in \{1, \dots, N\}$, then $u < u'$ for all $u \in I_s$ and $u' \in I_{s'}$.

We obtain an upper bound for N as follows. Observe that the number of interval bounds is bounded from above by $2(|C| + |\boldsymbol{\alpha}|)$, where $\boldsymbol{\alpha}$ denotes the set $\boldsymbol{\alpha}_\tau$ of all 1-types over τ. Thus the number of intervals is definitely bounded from above by $2(|C| + |\boldsymbol{\alpha}|) + 1$. Since we know from the previous section that $|C| \leq 2|\varphi|^4|\boldsymbol{\alpha}|$, we obtain that

$$\begin{aligned} N &\leq 2(2|\varphi|^4|\boldsymbol{\alpha}| + |\boldsymbol{\alpha}|) + 1 \\ &= (4|\varphi|^4 + 2)|\boldsymbol{\alpha}| + 1 \\ &\leq 6|\varphi|^4|\boldsymbol{\alpha}|. \end{aligned}$$

3.3 Defining admissibility tuples

Let χ be a normal form sentence of U_1 with the set σ of relation symbols. Assume $< \in \sigma$. We now define the notion of an *admissibility tuple for* χ. At this stage we only give a formal definition of admissibility tuples. The point is to capture enough information of ordered models of χ to the admissibility tuples for χ so that satisfiability of U_1 over ordered structures can be reduced to satisfiability of U_1 over general structures in Section 4. In particular, our objective is to facilitate Lemma 5. Once we have given the formal definition of an admissibility tuple, we provide an example how a concrete linearly ordered model of a U_1-sentence can be canonically associated with an admissibility tuple for that sentence, thereby providing background intuition related to admissibility tuples. Indeed, the reader may find it helpful to refer to that part while internalising the formal definitions.

Consider a tuple $\Gamma_\chi := (\mathfrak{C}^*, (\boldsymbol{\alpha}_{\sigma,s})_{1\leq s\leq N^*}, \boldsymbol{\alpha}_\sigma^K, \boldsymbol{\alpha}_\sigma^\perp, \boldsymbol{\alpha}_\sigma^\top, \delta, F)$ such that the following conditions hold.

- $\mathfrak{C}^*$ is a linearly ordered σ-structure, and the size of the domain C^* of $\mathfrak{C}^*$ is bounded by $2|\chi|^4|\boldsymbol{\alpha}_\sigma|$. Compare this to the bound $2|\varphi|^4|\boldsymbol{\alpha}_\tau|$ for the size of $\mathfrak{C}$ from Section 3.1. We call $\mathfrak{C}^*$ the *court structure of* Γ_χ.
- $N^* \in \mathbb{Z}_+$ is an integer such that $|C^*| \leq N^* \leq 6|\chi|^4|\boldsymbol{\alpha}_\sigma|$, and $(\boldsymbol{\alpha}_{\sigma,s})_{1\leq s\leq N^*}$ is a family of sets $\boldsymbol{\alpha}_{\sigma,s} \subseteq \boldsymbol{\alpha}_\sigma$ of 1-types such that we have $\boldsymbol{\alpha}_{\sigma,s} \subseteq \{\alpha \in \boldsymbol{\alpha}_\sigma \mid \neg(v_1 < v_1) \in \alpha\}$ for each $s \in \{1, \ldots, N^*\}$; recall here that v_1 is the variable with which we formally speaking specify 1-types, and recall also that in addition to ordered models, we will ultimately also consider model classes where $<$ is simply a binary symbol not necessarily interpreted as an order. Compare the bound $6|\chi|^4|\boldsymbol{\alpha}_\sigma|$ to the bound $6|\varphi|^4|\boldsymbol{\alpha}_\tau|$ for N from Section 3.2. We call N^* the *index of* Γ_χ.
- $\boldsymbol{\alpha}_\sigma^K \subseteq \boldsymbol{\alpha}_\sigma$ and also $\boldsymbol{\alpha}_\sigma^\perp \subseteq \boldsymbol{\alpha}_\sigma$ and $\boldsymbol{\alpha}_\sigma^\top \subseteq \boldsymbol{\alpha}_\sigma$
- δ is an injective mapping from C^* to N^* (i.e. from C^* to $\{1, \ldots, N^*\}$).
- F is a subset of the set $\boldsymbol{\alpha}_\sigma \times \Phi^\exists$, where $\Phi^\exists$ is the set of all existential conjuncts of χ.

Note that we could have chosen the tuple Γ_χ above in multiple ways. We denote the set of all tuples Γ_χ that satisfy the above conditions by $\hat{\Gamma}_\chi$. The tuples in $\hat{\Gamma}_\chi$ are called *admissibility tuples* for χ.

Lemma 2. *The (binary) description of* Γ_χ *is bounded exponentially in* $|\chi|$.

Proof. Let χ be a sentence and σ the set of relation symbols in χ with $< \in \sigma$. Consider an arbitrary tuple

$$\Gamma_\chi = (\mathfrak{C}^*, (\boldsymbol{\alpha}_{\sigma,s})_{1\leq s\leq N^*}, \boldsymbol{\alpha}_\sigma^K, \boldsymbol{\alpha}_\sigma^\perp, \boldsymbol{\alpha}_\sigma^\top, \delta, F)$$

such that $\Gamma_\chi \in \hat{\Gamma}_\chi$. To prove Lemma 2, we show that each of the seven elements in Γ_χ has a binary description whose length is exponentially bounded in $|\chi|$. This clearly suffices to prove the lemma.

For describing the model $\mathfrak{C}^*$, we use the straightforward convention from Chapter 6 of [44] according to which the unique description of $\mathfrak{C}^*$ with some ordering of σ is of the length

$$|C^*| + 1 + \sum_{i=1}^{|\sigma|} |C^*|^{ar(R_i)},$$

where $ar(R_i)$ is the arity of $R_i \in \sigma$. Since we have

$$|C^*| \leq 2|\chi|^4|\boldsymbol{\alpha}_\sigma|$$

by definition of the tuples in $\hat{\Gamma}_\chi$, and since we clearly have

$$|\boldsymbol{\alpha}_\sigma| \leq 2^{|\chi|},$$

we observe that $|C^*|$ is exponentially bounded in $|\chi|$. Since $ar(R_i) \leq |\chi|$, each term $|C^*|^{ar(R_i)}$ is likewise exponentially bounded in $|\chi|$. Furthermore, as $|\sigma| \leq |\chi|$, we conclude that the description of $\mathfrak{C}^*$ is exponentially bounded by $|\chi|$.

As

$$|\boldsymbol{\alpha}_\sigma| \leq 2^{|\chi|},$$

and as each 1-type $\alpha \in \boldsymbol{\alpha}_\sigma$ can clearly be encoded by a string whose length is polynomial in $|\chi|$, we can describe $\boldsymbol{\alpha}_\sigma$ with a description that is exponentially bounded in $|\chi|$, and as $\boldsymbol{\alpha}_\sigma^K$, $\boldsymbol{\alpha}_\sigma^\perp$, and $\boldsymbol{\alpha}_\sigma^\top$ are subsets of $\boldsymbol{\alpha}_\sigma$, their descriptions are also exponentially bounded in $|\chi|$. Moreover, the same upper bound bounds each member $\boldsymbol{\alpha}_{\sigma,s}$ of the family $(\boldsymbol{\alpha}_{\sigma,s})_{1\leq s\leq N^*}$. Therefore, as we have

$$N^* \leq 6|\chi|^4|\boldsymbol{\alpha}_\sigma|$$

by the definition of tuples in $\hat{\Gamma}_\chi$, we observe that

$$N^* \leq 6|\chi|^4 \cdot 2^{|\chi|},$$

and therefore the length of the description of $(\boldsymbol{\alpha}_{\sigma,s})_{1\leq s\leq N^*}$ is also exponentially bounded in $|\chi|$.

Due to the bounds for $|C^*|$ and $|N^*|$ identified above, the function $\delta : C^* \to N^*$ can clearly be encoded by a description bounded exponentially in $|\chi|$.

Let $m_\exists$ denote the number of existential conjuncts in χ. Thus we have

$$|F| \leq m_\exists|\boldsymbol{\alpha}_\sigma| \leq |\chi| \cdot 2^{|\chi|},$$

so the description of F can clearly be bounded exponentially in $|\chi|$. $\square$

For each $s \in \{1, \ldots, N^*\}$, let $\boldsymbol{\alpha}^-_{\sigma,s}$ and $\boldsymbol{\alpha}^+_{\sigma,s}$ be the subsets of $\boldsymbol{\alpha}_{\sigma,s}$ defined as follows.

$$\boldsymbol{\alpha}^-_{\sigma,s} := \boldsymbol{\alpha}_{\sigma,s} \setminus \bigcup_{i<s} \boldsymbol{\alpha}_{\sigma,i} \text{ and } \boldsymbol{\alpha}^+_{\sigma,s} := \boldsymbol{\alpha}_{\sigma,s} \setminus \bigcup_{i>s} \boldsymbol{\alpha}_{\sigma,i}.$$

The following definition provides an important classification of admissibility tuples.

Definition 5. Consider the set $\hat{\Gamma}_\chi$ of admissibility tuples for χ. We define the following six conditions, called *admissibility conditions* for χ, in order to classify the set $\hat{\Gamma}_\chi$ into different sets of admissibility tuples.

i. The sets $\boldsymbol{\alpha}^K_\sigma$, $\boldsymbol{\alpha}^\top_\sigma$ and $\boldsymbol{\alpha}^\perp_\sigma$ are subsets of $\bigcup_{1\leq s\leq N^*} \boldsymbol{\alpha}_{\sigma,s}$.

ii. If $\boldsymbol{\alpha}_{\sigma,s} \cap \boldsymbol{\alpha}^K_\sigma \neq \emptyset$, then $s = \delta(c)$ for some $c \in C^*$. Also, for every $c \in C^*$, it holds that $\boldsymbol{\alpha}_{\sigma,\delta(c)} = \{tp_{\mathfrak{C}^*}(c)\}$, and furthermore, $tp_{\mathfrak{C}^*}(c) \in \boldsymbol{\alpha}^K_\sigma$ or $\boldsymbol{\alpha}^-_{\sigma,\delta(c)} = \emptyset = \boldsymbol{\alpha}^+_{\sigma,\delta(c)}$.

iii. $|\boldsymbol{\alpha}^-_{\sigma,s}| \leq 1$ for all $s \in \{1, \ldots, N^*\}$

iv. $\boldsymbol{\alpha}^\perp_\sigma = \bigcup_{1\leq s\leq N^*} \boldsymbol{\alpha}_{\sigma,s}$

v. $|\boldsymbol{\alpha}^+_{\sigma,s}| \leq 1$ for all $s \in \{1, \ldots, N^*\}$

vi. $\boldsymbol{\alpha}^\top_\sigma = \bigcup_{1\leq i\leq N^*} \boldsymbol{\alpha}_{\sigma,s}$

Definition 6. An admissibility tuple Γ_χ is admissible for $\mathcal{O}$ if the conditions i and ii in Definition 5 are satisfied. It is admissible for $\mathcal{WO}$ if the four conditions i-iv in Definition 5 are satisfied. Finally, it is admissible for $\mathcal{O}_{fin}$ if all the six conditions i-vi in Definition 5 are satisfied. We call admissibility for $\mathcal{O}$ the lowest degree of admissibility and admissibility for $\mathcal{O}_{fin}$ the highest.

Let φ be a U_1-sentence containing $<$ and $\mathfrak{A} \models \varphi$ a model. Let $\mathfrak{C}$ be a court of $\mathfrak{A}$ w.r.t. φ and $\mathfrak{A}'$ a 3-cloning extension of $\mathfrak{A}$ w.r.t. φ. Let $(I_s)_{1 \leq s \leq N}$ be the canonical partition of $\mathfrak{A}'$ w.r.t. $\mathfrak{C}$. We will next specify a tuple

$$\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'} := (\mathfrak{C}, (\boldsymbol{\alpha}_{\mathfrak{A}',s})_{1 \leq s \leq N}, \boldsymbol{\alpha}_{\mathfrak{A}'}^{K}, \boldsymbol{\alpha}_{\mathfrak{A}'}^{\perp}, \boldsymbol{\alpha}_{\mathfrak{A}'}^{\top}, \delta, F)$$

which we call a *canonical admissibility tuple of* $\mathfrak{A}'$ *w.r.t* $(\mathfrak{C}, \mathfrak{A}, \varphi)$ (cf. Lemma 3 below).

We now specify the elements of the tuple $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$ above; note that $\mathfrak{C}$ has already been specified to be a court of $\mathfrak{A}$. Recall that $(I_s)_{1 \leq s \leq N}$ is the canonical partition of $\mathfrak{A}'$ w.r.t. $\mathfrak{C}$ and define the family $(\boldsymbol{\alpha}_{\mathfrak{A}',s})_{1 \leq s \leq N}$ such that $\boldsymbol{\alpha}_{\mathfrak{A}',s} := \{tp_{\mathfrak{A}'}(a) \mid a \in I_s\}$ for all $s \in \{1, \ldots, N\}$. Let $\boldsymbol{\alpha}_{\mathfrak{A}'}^{K} \subseteq \boldsymbol{\alpha}_{\mathfrak{A}'}$ be the set of the royal 1-types realized in $\mathfrak{A}'$, and define $\boldsymbol{\alpha}_{\mathfrak{A}'}^{\perp} \subseteq \boldsymbol{\alpha}_{\mathfrak{A}'}$ (respectively, $\boldsymbol{\alpha}_{\mathfrak{A}'}^{\top} \subseteq \boldsymbol{\alpha}_{\mathfrak{A}'}$) to be the set of 1-types that have a minimal (resp., maximal) realization in $\mathfrak{A}'$. Note that if $\mathfrak{A}'$ is in $\mathcal{WO}$, we have $\boldsymbol{\alpha}_{\mathfrak{A}'}^{\perp} = \boldsymbol{\alpha}_{\mathfrak{A}'}$, and if $\mathfrak{A}'$ is also in $\mathcal{O}_{fin}$, then $\boldsymbol{\alpha}_{\mathfrak{A}'}^{\perp} = \boldsymbol{\alpha}_{\mathfrak{A}'}^{\top} = \boldsymbol{\alpha}_{\mathfrak{A}'}$. For every c in the domain C of $\mathfrak{C}$, we define $\delta(c) := j \in \{1, \ldots, N\}$ such that $I_j = \{c\}$. We let F be the set of those pairs $(\alpha, \varphi_i^{\exists})$ that have a witness structure in $\mathfrak{A}'$ whose live part is free.

Lemma 3. *Let* $\mathfrak{A} \in \mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$ *and suppose* $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$ *is a canonical admissibility tuple for* $\mathfrak{A}'$ *w.r.t* $(\mathfrak{C}, \mathfrak{A}, \varphi)$*. Then* $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'} \in \hat{\Gamma}_{\varphi}$ *and* $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$ *is admissible for* $\mathcal{K}$*.*

Proof. Note that by definition, since $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$ is canonical admissibility tuple for $\mathfrak{A}'$ w.r.t. $(\mathfrak{C}, \mathfrak{A}, \varphi)$, the structure $\mathfrak{C}$ is a court of $\mathfrak{A}$ w.r.t. φ and we have $\mathfrak{A} \models \varphi$, and furthermore, the set of relation symbols in φ (to be denoted by τ) contains $<$. We let N denote the index of $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$. We note that N is the number of intervals in the canonical partition of $\mathfrak{A}'$ w.r.t. $\mathfrak{C}$.

By the discussion in Section 3.1, $\mathfrak{C}$ is an ordered structure whose size is bounded by $2|\varphi|^4|\boldsymbol{\alpha}|$ where $\boldsymbol{\alpha}$ is the set of all 1-types over τ. By Section 3.2, we have

$$|C| \leq N \leq 6|\varphi|^4|\boldsymbol{\alpha}|.$$

Thus the admissibility condition ii from Definition 5 is the only non-trivial remaining condition to show in order to conclude that $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$ is an admissibility tuple in $\hat{\Gamma}_{\varphi}$ admissible for each $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$ such that $\mathfrak{A} \in \mathcal{K}$. We next argue that this condition indeed holds.

First assume that $\boldsymbol{\alpha}_{\mathfrak{A}',s} \cap \boldsymbol{\alpha}_{\mathfrak{A}'}^{K} \neq \emptyset$. Thus $\alpha \in \boldsymbol{\alpha}_{\mathfrak{A}',s}$ for some royal 1-type α realized in $\mathfrak{A}'$. Therefore the interval $I_s \subseteq A'$ contains a king c of $\mathfrak{A}'$ that realizes α. Since kings of $\mathfrak{A}'$ are in singleton intervals of the family $(I_t)_{1\leq t\leq N}$, we have $I_s = \{c\}$. Furthermore, since kings of $\mathfrak{A}'$ are all in $\mathfrak{C}$, we have c in the domain of δ, and thus, by the definition of δ, we have $I_{\delta(c)} = \{c\}$. Thus we have $I_s = \{c\} = I_{\delta(c)}$, whence $s = \delta(c)$. Thus the first part of admissibility condition ii is satisfied. To prove the second condition, assume $c \in C$. Therefore the set $\{c\}$ was appointed, as described in Section 3.2, to be a singleton interval $I_{\delta(c)}$ in the family $(I_s)_{1\leq s\leq N}$. Thus $\boldsymbol{\alpha}_{\mathfrak{A}',\delta(c)} = \{tp_{\mathfrak{C}}(c)\}$. To show that, furthermore, we have $tp_{\mathfrak{C}}(c) \in \boldsymbol{\alpha}^K_{\mathfrak{A}'}$ or $\boldsymbol{\alpha}^-_{\mathfrak{A}',\delta(c)} = \emptyset = \boldsymbol{\alpha}^+_{\mathfrak{A}',\delta(c)}$, we consider two cases, the case where c is a king and the case where it is a pawn. If c is a king, then $tp_{\mathfrak{C}}(c) \in \boldsymbol{\alpha}^K_{\mathfrak{A}'}$ by the definition of $\boldsymbol{\alpha}^K_{\mathfrak{A}'}$. On the other hand, if $c \in C$ is a pawn, we argue as follows. Now, as $C \subseteq A \subseteq A'$, we know that c has two elements $u, u' \in A'$ of the same 1-type (as c itself) immediately before and after c that were introduced when constructing the 3-cloning extension $\mathfrak{A}'$ of $\mathfrak{A}$ (see the beginning of Chapter 3). Therefore every 1-type has neither its first nor last realization in $\mathfrak{A}'$ in the interval $I_{\delta(c)} = \{c\}$, and hence $\boldsymbol{\alpha}^-_{\mathfrak{A}',\delta(c)} = \emptyset = \boldsymbol{\alpha}^+_{\mathfrak{A}',\delta(c)}$, as required. $\square$

3.4 Pseudo-ordering axioms

Let χ be a normal form sentence of U_1 with the set σ of relation symbols. We assume that the symbol $<$ occurs in χ. Let r be the highest arity occurring in the symbols in σ, and let n be the width of χ. Let $m_\exists$ be the number of existential conjuncts in χ. Assume

$$\Gamma_\chi = (\mathfrak{C}, (\boldsymbol{\alpha}_{\sigma,s})_{1\leq s\leq N}, \boldsymbol{\alpha}^K_\sigma, \boldsymbol{\alpha}^\perp_\sigma, \boldsymbol{\alpha}^\top_\sigma, \delta, F)$$

is some admissibility tuple in $\hat{\Gamma}_\chi$. In this section we construct a certain large sentence $Ax(\Gamma_\chi)$ that axiomatizes structures with properties given by Γ_χ. The ultimate use of the sentence $Ax(\Gamma_\chi)$ will be revealed by the statement of Lemma 5, which is one or our main technical results. Note that in that lemma, satisfiability of $Ax(\Gamma_\chi)$ is considered in relation to classes of models where the symbol $<$ is not necessarily interpreted as a linear order.

Let K, D, $P_\perp$, $P_\top$, and U_s for each $s \in \{1, \dots, N\}$ be fresh unary relation symbols, where N is the size of the family $(\boldsymbol{\alpha}_{\sigma,s})_{1\leq s\leq N}$ in Γ_χ. Intuitively, the relation symbols K and D correspond to a set of kings and a set of domains of free witness structures, respectively, as we shall see. The symbols U_s, for $1 \leq s \leq N$, correspond to intervals, but this intuition is not precise as we shall interpret the predicates U_s over models where $<$ is not assumed to be a linear order. The predicates $P_\perp$ and $P_\top$ will be axiomatized to contain the minimal and the maximal realization of each 1-type belonging to $\boldsymbol{\alpha}_\sigma^\perp$ and $\boldsymbol{\alpha}_\sigma^\top$, respectively.

Let σ' be the vocabulary

$$\sigma \cup \{K, D, P_\perp, P_\top\} \cup \{U_s \mid 1 \leq s \leq N\}.$$

We define the *pseudo-ordering axioms for* Γ_χ (over σ') as follows. For most axioms we also give an informal description of its meaning (when interpreted together with the other pseudo-ordering axioms). Each of the 16 axioms is a U_1-sentence in normal form.

1. χ

2. The predicates U_s $(1 \leq s \leq N)$ partition the universe:
$\bigwedge\limits_{s} \exists x\, U_s x \;\wedge\; \forall x\big(\bigvee\limits_{s}(U_s x \wedge \bigwedge\limits_{t\neq s} \neg U_t x)\big)$

3. For all $s \in \{1, \dots, N\}$, the elements in U_s realize exactly the 1-types (over σ) in $\boldsymbol{\alpha}_{\sigma,s}$:
$\bigwedge\limits_{1\leq s\leq N} \forall x(\, U_s x \;\leftrightarrow\; \bigvee_{\alpha\in\boldsymbol{\alpha}_{\sigma,s}} \alpha(x)\,)$
Note indeed that the 1-types α in $\boldsymbol{\alpha}_{\sigma,s}$ are with respect to the vocabulary σ, and thus are definitely not 1-types with respect to the extended vocabulary σ'.

4. Each predicate $U_{\delta(c)}$, where c is an element in the domain C of $\mathfrak{C}$, is a singleton set containing an element that realizes $\alpha = tp_{\mathfrak{C}}(c)$:
$\bigwedge\limits_{c\in C,\, \alpha=tp_{\mathfrak{C}}(c)} \big(\exists y(\, U_{\delta(c)} y \wedge \alpha(y)\,) \;\wedge\; \forall x \forall y(\,(\, U_{\delta(c)} x \wedge U_{\delta(c)} y\,) \rightarrow x = y\,)\big)$

5. Each $\alpha \in (\bigcup \boldsymbol{\alpha}_{\sigma,s} \setminus \boldsymbol{\alpha}_\sigma^K)$ is realized at least n times (recall that n is the width of χ):
$\bigwedge\limits_{\alpha\in(\bigcup \boldsymbol{\alpha}_{\sigma,s}\setminus\boldsymbol{\alpha}_\sigma^K)} \exists x_1 ... \exists x_n(\, \bigwedge\limits_{i\neq j}(x_i \neq x_j) \wedge \bigwedge\limits_{i} \alpha(x_i)\,)$

6. Each $\alpha \in \boldsymbol{\alpha}_\sigma^K$ is realized at least once but at most $n-1$ times:
$\bigwedge\limits_{\alpha\in\boldsymbol{\alpha}_\sigma^K} \exists y\, \alpha(y) \wedge \forall x_1 ... \forall x_n\big(\,(\,\bigwedge\limits_{i} \alpha(x_i)\,) \rightarrow \bigvee\limits_{j\neq k} x_j = x_k\big)$

7. K is the set of realizations of types in $\boldsymbol{\alpha}_\sigma^K$:
$\forall x\big(\,(\, \bigvee\limits_{\alpha\in\boldsymbol{\alpha}_\sigma^K} \alpha(x)\,) \leftrightarrow Kx\big)$

8. In order to define the next axiom, we begin with an auxiliary definition. For each existential matrix $\chi_i^{\exists}(x, y_1, \ldots, y_{k_i})$ in χ, let the set $\{z_1, \ldots, z_{l_i}\} \subseteq \{x, y_1, \ldots, y_{k_i}\}$ be the set of live variables of $\chi_i^{\exists}(x, y_1, \ldots, y_{k_i})$. We then define the following axiom which asserts that the set F is the set of all pairs $(\alpha, \chi_i^{\exists})$ that have a witness structure whose live part is free, and furthermore, the set D contains, for each pair $(\alpha, \chi_i^{\exists}) \in F$, the live part of at least one free witness structure for $(\alpha, \chi_i^{\exists})$.
$$\bigwedge_{(\alpha,\chi_i^{\exists})\in F} \exists x \exists y_1 \ldots \exists y_{k_i} \big(\alpha(x) \wedge \chi_i^{\exists}(x, y_1, \ldots, y_{k_i}) \wedge \bigwedge_{1\leq j\leq l_i} (z_j \neq x \wedge D z_j) \big)$$
$$\wedge \bigwedge_{(\alpha,\chi_i^{\exists})\notin F} \forall x \forall y_1 \ldots \forall y_{k_i} \big(\neg (\alpha(x) \wedge \chi_i^{\exists}(x, y_1, \ldots, y_{k_i}) \wedge \bigwedge_{1\leq j\leq l_i} z_j \neq x) \big)$$

9. Axioms 6 and 7 define the set K, and D is described by the previous axiom. The next axiom says that every element $c \in (K \cup D)$ is in $\bigcup_{c\in C} U_{\delta(c)}$:
$$\forall x((Kx \vee Dx) \rightarrow \bigvee_{c\in C} U_{\delta(c)} x)$$

10. There is a witness structure for every $c \in (K \cup D)$ such that each element of the witness structure is in $\bigcup_{c\in C} U_{\delta(c)}$:
$$\bigwedge_{1\leq i\leq m_{\exists}} \forall x \exists y_1 \ldots \exists y_{k_i} \big((Kx \vee Dx) \rightarrow$$
$$((\bigwedge_{1\leq j\leq k_i} \bigvee_{c\in C} U_{\delta(c)} y_j) \wedge \chi_i^{\exists}(x, y_1, \ldots, y_{k_i})) \big)$$

11. The next axiom ensures that there exists an isomorphic copy of $\mathfrak{C}$ in the model considered. Let $m = min\{n, r\}$, where r is the maximum arity of relation symbols that occur in χ. For each $k \in \{1, \ldots, m\}$, let $\mathcal{C}_k$ denote the set of all subsets of size k of the domain C of $\mathfrak{C}$. Let $\bar{\mathcal{C}_k}$ denote the set obtained from $\mathcal{C}_k$ by replacing each set $C_k \in \mathcal{C}_k$ by exactly one tuple $(c_1, \ldots, c_k)$ that enumerates the elements of C_k in some arbitrarily chosen order. (Thus $|\mathcal{C}_k| = |\bar{\mathcal{C}_k}|$.) For each tuple $(c_1, \ldots, c_k) \in \bar{\mathcal{C}_k}$, let $\beta_{[(c_1,\ldots,c_k)]}$ denote the table $tb_{\mathfrak{C}}(c_1, \ldots, c_k)$. We define the required axiom as follows:
$$\bigwedge_{1\leq k\leq m} \bigwedge_{(c_1,\ldots,c_k)\in\bar{\mathcal{C}_k}} \forall x_1 \ldots \forall x_k \big((\bigwedge_{j\in\{1,\ldots,k\}} U_{\delta(c_j)} x_j) \rightarrow \beta_{[(c_1,\ldots,c_k)]}(x_1, \ldots, x_k) \big)$$
Note that strictly speaking the axiom ignores sets of size greater than m.

12. The relation symbol $<$ is interpreted to be a tournament:
$$\forall x \forall y (x < y \vee y < x \vee x = y) \wedge \forall x \forall y \neg (x < y \wedge y < x)$$

13. Together with the previous axiom, the *first three* big conjunctions of the next axiom imply that for all $\alpha \in \boldsymbol{\alpha}_\sigma^\perp$ there exists a point in $P_\perp$ that realizes α, and furthermore, $P_\perp$ is true at a point u iff there exists a 1-type α such that $\alpha \in \boldsymbol{\alpha}_\sigma^\perp$ and u is the unique minimal realization of that 1-type. The *last* big conjunction of the axiom implies that if $\alpha \in \boldsymbol{\alpha}_{\sigma,s}^- \cap \boldsymbol{\alpha}_\sigma^\perp$ for some $s \in \{1, \dots, N\}$, then there exists a point u' which is the minimal realization of α and satisfies U_s:
$$\begin{aligned}
&\bigwedge_{\alpha \in \boldsymbol{\alpha}_\sigma \setminus \boldsymbol{\alpha}_\sigma^\perp} \forall x \neg\, (\, \alpha(x) \wedge P_\perp(x)\,) \\
&\wedge \bigwedge_{\alpha \in \boldsymbol{\alpha}_\sigma^\perp} \big(\, \exists x (\, \alpha(x) \wedge P_\perp x\,)\big) \\
&\wedge \bigwedge_{\alpha \in \boldsymbol{\alpha}_\sigma^\perp} \forall x \forall y \big(\, (\, P_\perp x \wedge \alpha(x) \wedge \alpha(y) \wedge y \neq x\,) \rightarrow x < y \big) \\
&\wedge \bigwedge_{\alpha \in \boldsymbol{\alpha}_{\sigma,s}^- \cap \boldsymbol{\alpha}_\sigma^\perp} \exists x (\, P_\perp x \wedge \alpha(x) \wedge U_s x\,)
\end{aligned}$$

14. The next axiom is analogous to the previous one:
$$\begin{aligned}
&\bigwedge_{\alpha \in \boldsymbol{\alpha}_\sigma \setminus \boldsymbol{\alpha}_\sigma^\top} \forall x \neg\, (\, \alpha(x) \wedge P_\top(x)\,) \\
&\wedge \bigwedge_{\alpha \in \boldsymbol{\alpha}_\sigma^\top} \big(\, \exists x (\, \alpha(x) \wedge P_\top x\,)\big) \\
&\wedge \bigwedge_{\alpha \in \boldsymbol{\alpha}_\sigma^\top} \forall x \forall y \big(\, (\, P_\top x \wedge \alpha(x) \wedge \alpha(y) \wedge y \neq x\,) \rightarrow x > y \big) \\
&\wedge \bigwedge_{\alpha \in \boldsymbol{\alpha}_{\sigma,s}^+ \cap \boldsymbol{\alpha}_\sigma^\top} \exists x (\, P_\top x \wedge \alpha(x) \wedge U_s x)
\end{aligned}$$

The last two axioms below are technical assertions about the predicates U_s, the relation $<$ and 1-types. The significance of these axioms becomes clarified in the related proofs.

15. $\bigwedge_{1 \leq s < t \leq N} \forall x \forall y \big(\, (\, U_s x \wedge U_t y\,) \;\rightarrow\; x < y \big)$

16. $\bigwedge_{s \in \{1,\dots,N\} \setminus img(\delta)} \; \bigwedge_{\alpha \in \boldsymbol{\alpha}_{\sigma,s}^+} \; \bigwedge_{\alpha' \in \boldsymbol{\alpha}_{\sigma,s}} \exists x \exists y \big(\, \alpha(x) \wedge \alpha'(y) \wedge U_s x \wedge U_s y \wedge y < x\, \big)$

We denote the conjunction of the above 16 pseudo-ordering axioms over σ' for the admissibility tuple Γ_χ by $Ax(\Gamma_\chi)$. We note that $Ax(\Gamma_\chi)$ is a normal form sentence of U_1 over the vocabulary σ' which expands the vocabulary σ of χ. The formulae $Ax(\Gamma_\chi)$ play a central role in the reduction of ordered satisfiability to standard satisfiability based on Lemma 5.

Lemma 4. *Let φ be a* U_1 *formula with the set τ of relation symbols, $< \in \tau$. Let $\mathfrak{A} \models \varphi$ be a τ-model. Let $\mathfrak{C}$ be a court of $\mathfrak{A}$ w.r.t. φ and $\mathfrak{A}'$ a* 3*-cloning extension of $\mathfrak{A}$ w.r.t. φ. Let $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$ be a canonical admissibility tuple for $\mathfrak{A}'$ w.r.t. $(\mathfrak{C}, \mathfrak{A}, \varphi)$ and N the index of $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$. Then the τ-model $\mathfrak{A}'$ has an expansion $\mathfrak{A}''$ to the vocabulary $\tau \cup \{K, D, P_{\perp}, P_{\perp}\} \cup \{U_s \,|\, 1 \leq s \leq N\}$ such that $\mathfrak{A}'' \models Ax(\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'})$.*

Proof. Recall the τ-formula φ and the τ-model $\mathfrak{A} \models \varphi$ fixed in the first section of Chapter 3. Recall also the 3-cloning extension $\mathfrak{A}'$ of $\mathfrak{A}$ w.r.t. φ and the court $\mathfrak{C}$ of $\mathfrak{A}$ w.r.t. φ fixed in the first section of Chapter 3. Let $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$ be the canonical admissibility tuple of $\mathfrak{A}'$ w.r.t. $(\mathfrak{C}, \mathfrak{A}, \varphi)$. Note that by Lemma 3, we have $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'} \in \hat{\Gamma}_{\varphi}$, and furthermore, $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$ is admissible for each $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$ such that $\mathfrak{A} \in \mathcal{K}$. We will show that $\mathfrak{A}'$ has an expansion $\mathfrak{A}''$ such that $\mathfrak{A}'' \models Ax(\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'})$. As φ, $\mathfrak{A}$, $\mathfrak{A}'$ and $\mathfrak{C}$ were fixed arbitrarily, this proves the current lemma (Lemma 4).

Let N be the index of $\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$, in other words, N is the size of the family $(I_s)_{1 \leq s \leq N}$ of intervals fixed in Section 3.2. Thus we now must prove that $\mathfrak{A}'$ has an expansion $\mathfrak{A}''$ to the vocabulary $\tau \cup \{K, D, P_{\perp}, P_{\perp}\} \cup \{U_s \,|\, 1 \leq s \leq N\}$ such that $\mathfrak{A}'' \models Ax(\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'})$. We let $\mathfrak{A}''$ be the expansion of $\mathfrak{A}'$ obtained by interpreting the extra predicates $\{K, D, P_{\perp}, P_{\top}\} \cup \{\, U_s \mid 1 \leq s \leq N\}$ as follows.

1. $K^{\mathfrak{A}''}$ and $D^{\mathfrak{A}''}$ are defined as K and D in the Section 3.1, respectively. Thus $K^{\mathfrak{A}''} \subseteq A$ is the set of kings in $\mathfrak{A}'$ (and $\mathfrak{A}$) and $D^{\mathfrak{A}''} \subseteq A$ is a set that contains, for every pair $(\alpha, \varphi_i^{\exists})$ that has a free witness structure in $\mathfrak{A}$, the free part of at least one such witness structure (cf. Section 3.1).

2. $P_{\perp}^{\mathfrak{A}''}$ is defined to satisfy the pseudo-ordering axiom 13; we let $P_{\perp}^{\mathfrak{A}''}$ be true at a point u iff there is some 1-type α such that u is the minimal realization of α. $P_{\top}^{\mathfrak{A}''}$ is defined analogously to satisfy axiom 14.

3. Each predicate $U_s^{\mathfrak{A}''}$ is defined to be the interval $I_s \subseteq A'$ identified in Section 3.2.

Next we show that $\mathfrak{A}'' \models Ax(\Gamma_{\varphi}^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'})$. As it is easy to see that $\mathfrak{A}''$ satisfies axioms 1-7 and 9-16, it suffices to show that $\mathfrak{A}''$ satisfies axiom 8. Recalling the definition of $D^{\mathfrak{A}''}$, this can clearly be done by proving the following claim. (Recall (cf. Section 3.3) that F is the set of those pairs $(\alpha, \varphi_i^{\exists})$ that have a free witness structure in $\mathfrak{A}'$.)

Claim: $\mathfrak{A}$ has a free witness structure for a pair $(\alpha, \varphi_i^{\exists})$ iff $(\alpha, \varphi_i^{\exists}) \in F$.

As $\mathfrak{A}'$ is a 3-cloning extension of $\mathfrak{A}$, it is clear that $\mathfrak{A}'$ has a free witness structure for a pair $(\alpha, \varphi_i^{\exists})$ if $\mathfrak{A}$ has. Suppose now that for some $a \in A'$, $\mathfrak{A}'$ has a free witness structure $\mathfrak{A}'_{a,\varphi_i^{\exists}}$ for some $(\alpha, \varphi_i^{\exists}) \in F$ and $\mathfrak{A}$ does not have a free witness structure for this pair. Let $\mathfrak{A}'_{a,\varphi_i^{\exists}} \models \varphi_i^{\exists}(a, a_1, \dots, a_{k_i})$ for some points $a_1, \dots, a_{k_i} \in A'$, which are not necessarily distinct. Let $u_1, \dots, u_l \in (A'_{a,\varphi_i^{\exists}} \setminus \{a\})$ be the distinct points forming the live part of $\mathfrak{A}'_{a,\varphi_i^{\exists}}$. Thus some points $a_1, \dots, a_{k'} \in (A'_{a,\varphi_i^{\exists}} \setminus \{u_1, \dots, u_l\})$ together with a form the dead part of $\mathfrak{A}'_{a,\varphi_i^{\exists}}$.

The table $tb_{\mathfrak{A}'}(u_1, \dots, u_l)$ has been defined either in the *cloning stage* or the *completion stage* to be $tb_{\mathfrak{A}}(b_1, \dots, b_l)$ for some distinct elements $b_1, \dots, b_l \in A$. Furthermore, since $\mathfrak{A}'$ and $\mathfrak{A}$ have exactly the same number of realizations of each royal 1-type and since both models have at least $n \geq k_i+1$ realizations of each pawn, it is easy to define an injection f from $A'_{a,\varphi_i^{\exists}}$ into A that preserves 1-types and such that $f(u_i) = b_i$ for each $i \in \{1, \dots, l\}$. Therefore

$$\mathfrak{A}' \models \varphi_i^{\exists}(a, a_1, \dots, a_{k_i}) \text{ iff } \mathfrak{A} \models \varphi_i^{\exists}(f(a), f(a_1), \dots, f(a_{k_i})),$$

whence we have $\mathfrak{A} \models \varphi_i^{\exists}(f(a), f(a_1), \dots, f(a_{k_i}))$. Therefore, as f is injective, we see that $\mathfrak{A}$ has a free witness structure for $(\alpha, \varphi_i^{\exists})$. This contradicts the assumption that $\mathfrak{A}$ does not have a free witness structure for the pair $(\alpha, \varphi_i^{\exists})$. □

Chapter 4

Reducing ordered satisfiability to standard satisfiability

In this section we establish decidability of the satisfiability problems of U_1 over $\mathcal{O}$, $\mathcal{WO}$ and $\mathcal{O}_{fin}$. The next lemma (Lemma 5) is the main technical result needed for the decision procedure. Note that satisfiability in the case (b) of the lemma is with respect to general rather than ordered models. In the lemma we assume w.l.o.g. that φ contains $<$.

Lemma 5. *Let φ be a U_1-sentence containing the symbol $<$. Let $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$. The following conditions are equivalent:*

(a) $\varphi \in sat_{\mathcal{K}}(\mathrm{U}_1)$.

(b) $Ax(\Gamma_\varphi) \in sat(\mathrm{U}_1)$ for some admissibility tuple $\Gamma_\varphi \in \hat{\Gamma}_\varphi$ that is admissible for $\mathcal{K}$.

Proof. In order to prove the implication from (a) to (b), suppose that $\varphi \in sat_{\mathcal{K}}(\mathrm{U}_1)$. Thus there is a structure $\mathfrak{A} \in \mathcal{K}$ such that $\mathfrak{A} \models \varphi$. As $\mathfrak{A} \models \varphi$, there exists a court $\mathfrak{C}$ of $\mathfrak{A}$ w.r.t. φ. Now let $\mathfrak{A}'$ be a 3-cloning extension of $\mathfrak{A}$ w.r.t. φ, and let $\Gamma_\varphi^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'}$ be the canonical admissibility tuple of $\mathfrak{A}'$ w.r.t. $(\mathfrak{C}, \mathfrak{A}, \varphi)$. By Lemma 3, the canonical tuple is in $\hat{\Gamma}_\varphi$ and admissible for $\mathcal{K}$. By Lemma 4, $\mathfrak{A}'$ has an expansion $\mathfrak{A}''$ such that $\mathfrak{A}'' \models Ax(\Gamma_\varphi^{\mathfrak{C},\mathfrak{A},\mathfrak{A}'})$. The proof for the direction from (b) to (a) is given next.

Please note that the proof for the direction from (b) to (a) of Lemma 5 below spans all of the current chapter, ending at the end of Chapter 4. We deal with the three cases $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$ in parallel. We let τ denote the set of relation symbols in φ.

To prove the implication from (b) to (a), assume that $\mathfrak{B} \models Ax(\Gamma_\varphi)$ for some τ'-model $\mathfrak{B}$ and some admissibility tuple

$$\Gamma_\varphi = (\mathfrak{C}, (\boldsymbol{\alpha}_{\tau,s})_{1\leq s\leq N}, \boldsymbol{\alpha}_\tau^K, \boldsymbol{\alpha}_\tau^\perp, \boldsymbol{\alpha}_\tau^\top, \delta, F) \in \hat{\Gamma}_\varphi$$

that is admissible for the class $\mathcal{K}$. Here

$$\tau' = \tau \cup \{K, D, P_\perp, P_\top\} \cup \{U_s \mid s \in \{1, \ldots, N\}\}.$$

Note that while $\mathfrak{B}$ interprets the symbol $<$, it is not assumed to be an ordered model. Based on $\mathfrak{B}$ and Γ_φ, we will construct an ordered τ-model $\mathfrak{A} \in \mathcal{K}$ such that $\mathfrak{A} \models \varphi$. The construction of $\mathfrak{A}$ consists of the following (informally described) four steps; each step is described in full detail in its own subsection below.

1) We first construct the domain A of $\mathfrak{A}$ and define a linear order $<$ over it. We also label the elements of A with 1-types in $\boldsymbol{\alpha}_\tau$. After this stage the relations of $\mathfrak{A}$ (other than $<$) contain no tuples other than trivial tuples, i.e., tuples $(u, \ldots, u)$ with u repeated.

2) We then copy a certain substructure $\mathfrak{C}$ of $\mathfrak{B}$ into $\mathfrak{A}$; the structure $\mathfrak{C}$ is the set of points in B that satisfy some predicate U_s with $s \in img(\delta)$. This step introduces fresh non-trivial tuples into the relations of $\mathfrak{A}$.

3) We then define a witness structure for each element $a \in A$ and each existential conjunct $\varphi_i^\exists$ of φ. As the above step, this step introduces non-trivial tuples into the relations of $\mathfrak{A}$.

4) Finally, we complete the construction of $\mathfrak{A}$ by making sure that $\mathfrak{A}$ also satisfies all universal conjuncts $\varphi_i^\forall$ of φ. Also this step involves introducing non-trivial tuples.

1) Constructing an ordered and labelled domain for $\mathfrak{A}$ Before defining an ordered domain $(A, <)$ for $\mathfrak{A}$, we construct an ordered set $(I_s, <)$ for each $s \in \{1, \ldots, N\}$ based on the set $\boldsymbol{\alpha}_{\tau,s} \in (\boldsymbol{\alpha}_{\tau,s})_{1\leq s\leq N}$ of Γ_φ. Once we have the ordered sets defined, the ordered domain $(A, <)$ is defined to be the ordered sum

$$(A, <) = \Sigma_{1\leq s\leq N}(I_s, <),$$

i.e., the ordered sets $(I_s, <)$ are simply concatenated so that the elements of I_t are before the elements of $I_{t'}$ iff $t < t'$. Thus the ordered sets $(I_s, <)$ become intervals in $(A, <)$.

However, we will not only construct an ordered domain $(A,<)$ in the current subsection 1) (Constructing an ordered and labelled domain for $\mathfrak{A}$), we will also label the elements of A by 1-types over τ. Thus, by the end of the current subsection, the structure $\mathfrak{A}$ will be a linearly ordered structure with the 1-types over τ defined. Each interval I_s will be labelled such that exactly all the 1-types in the set $\boldsymbol{\alpha}_{\tau,s}$ given in Γ_φ are satisfied by the elements of I_s.

Let $s \in \{1,\dots,N\}$. We now make use of the admissibility tuple Γ_φ as follows. If $\boldsymbol{\alpha}_{\tau,s} \cap \boldsymbol{\alpha}_\tau^K \neq \emptyset$, then by the admissibility condition ii from Definition 5, we have $s = \delta(c)$ for some $c \in C$ where C is the domain of the structure $\mathfrak{C}$ from Γ_φ. Furthermore, we infer, using the admissibility condition ii, that $\boldsymbol{\alpha}_{\tau,\delta(c)}$ must in fact be a singleton $\{\alpha_s\}$ such $\alpha_s = tp_{\mathfrak{C}}(c)$. We define I_s to be a singleton set, and we label the unique element u in I_s by the type α_s by defining $lab(u) = \alpha_s$ where lab denotes a labelling function $lab : A \to \boldsymbol{\alpha}_\tau$ whose definition will become fully fixed once we have dealt with all the intervals $(I_s,<)$.

Having discussed the case where $\boldsymbol{\alpha}_{\tau,s} \cap \boldsymbol{\alpha}_\tau^K \neq \emptyset$, we assume that $\boldsymbol{\alpha}_{\tau,s} \cap \boldsymbol{\alpha}_\tau^K = \emptyset$. We divide the analysis of this case into three subcases (see below) depending on the degree of admissibility of Γ_φ (cf. Definition 6). Before dealing with the cases, we define some auxiliary ordered sets that will function as building blocks when we construct the intervals $(I_s,<)$.

Fix n to be the width of φ and $m_\exists$ the number of existential conjuncts in φ. By a $3(m_\exists+n)$-*block* we mean a finite ordered set that consists of $3(m_\exists+n)$ elements. A $3(m_\exists+n)$-block divides into into three disjoint sets that we call the *E-part, F-part and G-part*. Each of the parts contains $m_\exists+n$ consecutive elements in the block such that the sets E, F and G appear in the given order. We will define the remaining intervals $(I_s,<)$ below using $3(m_\exists+n)$-blocks. For each $3(m_\exists+n)$-block $(U,<)$ we use, the elements in U will be labelled with a single 1-type, i.e., we will define $lab(u) = lab(u')$ for all $u,u' \in U$. Therefore we in fact (somewhat informally) talk about $3(m_\exists+n)$-blocks $(U,<)$ *of* 1-*type* α. This means that while $(U,<)$ is strictly speaking only an (unlabelled) ordered set with $3(m_\exists+n)$ elements, we will ultimately set $lab(u) = \alpha$ for all $u \in U$.

Let $(J,<)$ be a finite, ordered set consisting of several $3(m_\exists+n)$-blocks such that there is one $3(m_\exists+n)$-block for each 1-type $\alpha \in \boldsymbol{\alpha}_{\tau,s}$ and no other blocks; the order in which the blocks $(U,<)$ for different 1-types appear in $(J,<)$ is chosen arbitrarily. Similarly, let $(J^-,<)$ contain a $3(m_\exists+n)$-block for each $\alpha \in \boldsymbol{\alpha}_{\tau,s}^-$ in some order and no other blocks. Let $(J^+,<)$ contain a block for each $\alpha \in \boldsymbol{\alpha}^{+}_{\tau,s}$ in some order and no other blocks. Note that J^- and J^+ may be empty. We define the ordered interval $(I_s, <)$ as follows:

1. Assume Γ_φ is admissible for $\mathcal{O}$ but not for $\mathcal{WO}$. We define $(I_s, <)$ to be the ordered set consisting of three parts $(I_s, <)_1$, $(I_s, <)_2$ and $(I_s, <)_3$ in the given order and defined as follows.
   (a) $(I_s, <)_1$ consists of a countably infinite number of copies of $(J^-, <)$ such that the different copies are ordered as the negative integers, i.e., $(I_s, <)_1$ can be obtained by ordering $\mathbb{Z}_{neg} \times J^-$ lexicographically, where $\mathbb{Z}_{neg}$ denotes the negative integers; schematically, $(I_s, <)_1 := \;...\cdot (J^-, <) \cdot (J^-, <) \cdot (J^-, <)$ where " $\cdot$ " denotes concatenation.
   (b) $(I_s, <)_2 := (J, <)$.
   (c) $(I_s, <)_3$ consists of a countably infinite number of copies of $(J^+, <)$ such that the different copies are ordered as the positive integers.

   Schematically, $(I_s, <)$ is therefore the structure
   $$\ldots \cdot (J^-, <) \cdot (J^-, <) \cdot (J, <) \cdot (J^+, <) \cdot (J^+, <) \cdot \ldots$$

2. Assume Γ_φ is admissible for $\mathcal{WO}$ but not for $\mathcal{O}_{\mathit{fin}}$. Again the interval $(I_s, <)$ is the concatenation of three parts $(I_s, <)_1, (I_s, <)_2, (I_s, <)_3$ in that order, but while $(I_s, <)_2$ and $(I_s, <)_3$ are the same as above, now $(I_s, <)_1 := (J^-, <)$. Thus, $(I_s, <)$ is the structure
   $$(J^-, <) \cdot (J, <) \cdot (J^+, <) \cdot (J^+, <) \cdot \ldots$$

3. Assume Γ_φ is admissible for $\mathcal{O}_{fin}$. In this case we define $(I_s, <)$ to be the structure
   $$(J^-, <) \cdot (J, <) \cdot (J^+, <).$$

Note that since we already associated each $3(m_\exists + n)$-block in each of the structures $(J, <), (J^-, <), (J^+, <)$ with a labelling with 1-types, we have now also defined the 1-types over the interval $(I_s, <)$. Therefore we have now shown how to construct an ordered domain $(A, <)$ for $\mathfrak{A}$ and also defined a labelling of A with 1-types.

2) Copying $\mathfrak{C}$ into $\mathfrak{A}$ Due to axiom 11, the structure $\mathfrak{B}$ contains an isomorphic copy $\mathfrak{C}_{\mathfrak{B}}$ of the structure $\mathfrak{C}$ from Γ_φ, that is, $\mathfrak{B}$ has a substructure $\mathfrak{C}'_{\mathfrak{B}}$ such that $\mathfrak{C}'_{\mathfrak{B}} \upharpoonright \tau$ is isomorphic to $\mathfrak{C}$ and $\mathfrak{C}_{\mathfrak{B}} := \mathfrak{C}'_{\mathfrak{B}} \upharpoonright \tau$. The domain $C_{\mathfrak{B}}$ of $\mathfrak{C}_{\mathfrak{B}}$ is the union of the sets $U^{\mathfrak{B}}_{\delta(c)}$ for all $c \in C$; recall that by axiom 4, each $U^{\mathfrak{B}}_{\delta(c)}$, for $c \in C$, is a singleton.

Let g be the isomorphism from $\mathfrak{C}_{\mathfrak{B}}$ to $\mathfrak{C}$. (The isomorphism is unique since $\mathfrak{C}$ is an ordered set.) We shall create an isomorphic copy of $\mathfrak{C}$ into $\mathfrak{A}$ by introducing tuples to the relations of $\mathfrak{A}$; no new points will be added to A. We first define an injective mapping h from $C_{\mathfrak{B}}$ to A as follows. Let $b \in C_{\mathfrak{B}}$, and denote $\delta(g(b))$ by s. Now, if b realizes a 1-type $\alpha \in \boldsymbol{\alpha}_{\tau,s} \cap \boldsymbol{\alpha}^K$, then we recall from the subsection 1) that $I_s \subseteq A$ is a singleton interval that realizes the type α. We let h map b to the element in $I_s \subseteq A$. Otherwise b realizes a 1-type $\alpha \in \boldsymbol{\alpha}_{\tau,s}$ such that $\alpha \notin \boldsymbol{\alpha}^K$. Then, by admissibility condition ii, (see Definition 5), $\boldsymbol{\alpha}^-_{\tau,s}$ and $\boldsymbol{\alpha}^+_{\tau,s}$ are empty. Therefore, using the notation from the subsection 1), we have $(I_s, <) = (J, <)$ as J^- and J^+ are empty. Therefore, and since $\boldsymbol{\alpha}_{\tau,s}$ is a singleton (by admissibility condition ii), we observe that $(I_s, <)$ consists of a single $3(m_\exists + n)$-block of elements realizing α. We let h map b to the first element in $I_s \subseteq A$.

Denote the set $img(h)$ by $C_{\mathfrak{A}}$. Hence h is a bijection from $C_{\mathfrak{B}}$ onto $C_{\mathfrak{A}}$ that preserves 1-types over τ. Due to the construction of the order $<^{\mathfrak{A}}$ and axiom 15, it is easy to see that h also preserves order, i.e., we have

$$b < b' \text{ iff } h(b) < h(b')$$

for all $b, b' \in C_{\mathfrak{B}}$.

Now let r' denote the highest arity of the relation symbols in φ. Let $\{b_1, \ldots, b_j\} \subseteq C_{\mathfrak{B}}$ be a set with $j \in \{2, \ldots, r'\}$ elements. We define

$$tb_{\mathfrak{A}}(h(b_1), \ldots, h(b_j)) := tb_{\mathfrak{B}\upharpoonright\tau}(b_1, \ldots, b_j)$$

and repeat this for each subset of $C_{\mathfrak{A}}$ of size from 2 up to r'. By construction, h is an isomorphism from $\mathfrak{C}_{\mathfrak{B}} \upharpoonright \tau$ to $\mathfrak{C}_{\mathfrak{A}}$.

3) Finding witness structures Recalling the function h from the previous section, we define

$$K_{\mathfrak{A}} := \{h(k) \mid k \in K^{\mathfrak{B}}\} \text{ and } D_{\mathfrak{A}} := \{h(d) \mid d \in D^{\mathfrak{B}}\}.$$

By axiom 9 and due to the definition of the domain $C_{\mathfrak{B}}$ (cf. subsection above), we have

$$(K^{\mathfrak{B}} \cup D^{\mathfrak{B}}) \subseteq C_{\mathfrak{B}} \subseteq B.$$

Moreover, by axiom 10 and how $C_{\mathfrak{B}}$ was defined, there is a witness structure in $\mathfrak{C}_{\mathfrak{B}}$ for every $b \in (K^{\mathfrak{B}} \cup D^{\mathfrak{B}}) \subseteq C_{\mathfrak{B}}$ and every existential conjunct $\varphi_i^{\exists}$ of φ. As $\mathfrak{C}_{\mathfrak{A}}$ is isomorphic to $\mathfrak{C}_{\mathfrak{B}}$, there is a witness structure in $\mathfrak{C}_{\mathfrak{A}}$ for every $a \in (K_{\mathfrak{A}} \cup D_{\mathfrak{A}}) \subseteq C_{\mathfrak{A}}$ and every conjunct $\varphi_i^{\exists}$ of φ. In this section we show how to define, for each element $a \in A \setminus (K_{\mathfrak{A}} \cup D_{\mathfrak{A}})$ and each existential conjunct $\varphi_i^{\exists}$ of φ, a witness structure in $\mathfrak{A}$. This consists of the following steps, to be described in detail later on.

1. We first choose, for each $a \in A \setminus (K_{\mathfrak{A}} \cup D_{\mathfrak{A}})$, a *pattern element* b_a of the same 1-type (over τ) from $\mathfrak{B}$.
2. We then locate, for each pattern element b_a and each existential conjunct $\varphi_i^{\exists}$, a witness structure $\mathfrak{B}_{b_a,\varphi_i^{\exists}}$ in $\mathfrak{B}$.
3. We then find, for each element b' of the live part $\bar{\mathfrak{B}}_{b_a,\varphi_i^{\exists}}$ of $\mathfrak{B}_{b_a,\varphi_i^{\exists}}$, a corresponding $3(m_{\exists} + n)$-block of elements from $\mathfrak{A}$. The elements of the block satisfy the same 1-type as b'. We denote the block by $bl(b')$.
4. After this, we locate from each block $bl(b')$ an element corresponding to b'. We then construct from these elements a live part $\bar{\mathfrak{A}}_{a,\varphi_i^{\exists}}$ of a witness structure for a and $\varphi_i^{\exists}$.
5. These live parts are then, at the very end of our procedure, completed to full witness structures by locating suitable dead parts from $\mathfrak{A}$.

Let $a \in A \setminus (K_{\mathfrak{A}} \cup D_{\mathfrak{A}})$ and let $s_a \in \{1, \dots, N\}$ denote the index of the interval I_{s_a} such that $a \in I_{s_a}$. Let $\alpha \in \boldsymbol{\alpha}_{\tau,s_a} \setminus \boldsymbol{\alpha}^K$ be the 1-type of a over τ. We next show how to select a pattern element b_a for a. The pattern element b_a will be selected from the set $U_{s_a}^{\mathfrak{B}} \subseteq B$.

1. Firstly, if $a \in C_{\mathfrak{A}}$, then we let $b_a := h^{-1}(a) \in C_{\mathfrak{B}}$, where h is the bijection from $C_{\mathfrak{B}}$ to $C_{\mathfrak{A}}$. Otherwise we consider the following cases 2-4.
2. Assume Γ_{φ} is admissible for $\mathcal{O}$ but not for $\mathcal{WO}$ (and thus not for $\mathcal{O}_{fin}$ either). Then we let b_a be an arbitrary realization of α in $U_{s_a}^{\mathfrak{B}}$.
3. Assume that Γ_{φ} is admissible for $\mathcal{WO}$ but not for $\mathcal{O}_{fin}$. Then, if $\alpha \notin \boldsymbol{\alpha}_{\tau,s_a}^{-}$, we let b_a be an arbitrary realization of α in $U_{s_a}^{\mathfrak{B}}$. If $\alpha \in \boldsymbol{\alpha}_{\tau,s_a}^{-}$, we let b_a be the element in $U_{s_a}^{\mathfrak{B}}$ that satisfies $min_{\alpha}(x)$; this is possible due to admissibility condition iv and axiom 13.

4. Assume Γ_φ is admissible for $\mathcal{O}_{fin}$. Now, if we have $\alpha \notin \boldsymbol{\alpha}^-_{\tau,s_a} \cup \boldsymbol{\alpha}^+_{\tau,s_a}$, we let b_a be an arbitrary realization of α in $U^{\mathfrak{B}}_{s_a}$. If $\alpha \in \boldsymbol{\alpha}^-_{\tau,s_a} \setminus \boldsymbol{\alpha}^+_{\tau,s_a}$, then we let b_a be the element in $U^{\mathfrak{B}}_{s_a}$ that satisfies $min_\alpha(x)$, which is possible due to the admissibility condition iv and axiom 13. If $\alpha \in \boldsymbol{\alpha}^+_{\tau,s_a} \setminus \boldsymbol{\alpha}^-_{\tau,s_a}$, then we let b_a be the element in $U^{\mathfrak{B}}_{s_a}$ that satisfies $max_\alpha(x)$, which is possible due to the admissibility condition vi and axiom 14. Finally, if $\alpha \in \boldsymbol{\alpha}^-_{\tau,s_a} \cap \boldsymbol{\alpha}^+_{\tau,s_a}$, then there are the following two cases: If a is not in the last $3(m_\exists + n)$-block in I_{s_a}, then we choose b_a as in the case $\alpha \in \boldsymbol{\alpha}^-_{\tau,s_a} \setminus \boldsymbol{\alpha}^+_{\tau,s_a}$. If a is in the last $3(m_\exists + n)$-block in I_{s_a}, then we choose b_a as in the case $\alpha \in \boldsymbol{\alpha}^+_{\tau,s_a} \setminus \boldsymbol{\alpha}^-_{\tau,s_a}$.

We have now a pattern element b_a for each a in $A \setminus (K_{\mathfrak{A}} \cup D_{\mathfrak{A}})$. Let a denote an arbitrary element in $A \setminus (K_{\mathfrak{A}} \cup D_{\mathfrak{A}})$ and let $\varphi^\exists_i$ be an arbitrary existential conjunct of φ. By axiom 1, we have $\mathfrak{B} \models \varphi$, and thus we find a witness structure $\mathfrak{B}_{b_a,\varphi^\exists_i}$ in $\mathfrak{B}$ for the pair $(b_a, \varphi^\exists_i)$. Next we consider a number of cases based on what the live part $\bar{\mathfrak{B}}_{b_a,\varphi^\exists_i}$ of the witness structure $\mathfrak{B}_{b_a,\varphi^\exists_i}$ is like and how the live part is oriented in relation to $\mathfrak{B}_{b_a,\varphi^\exists_i}$. In each case, we ultimately define a live part $\bar{\mathfrak{A}}_{a,\varphi^\exists_i}$ for some witness structure $\mathfrak{A}_{a,\varphi^\exists_i}$. The dead part of the witness structure $\mathfrak{A}_{a,\varphi^\exists_i}$ will be found at a later stage of our construction. In many of the cases, the identification of the live part $\bar{\mathfrak{A}}_{a,\varphi^\exists_i}$ requires that we first identify suitable $3(m_\exists + n)$-blocks $bl(b')$ for the elements b' of $\bar{\mathfrak{B}}_{b_a,\varphi^\exists_i}$, and only after finding the blocks, we identify suitable elements from the blocks in order to construct $\bar{\mathfrak{A}}_{a,\varphi^\exists_i}$.

Case '*empty live part*': If the live part $\bar{\mathfrak{B}}_{b_a,\varphi^\exists_i}$ of the witness structure $\mathfrak{B}_{b_a,\varphi^\exists_i}$ is empty, we let the live part $\bar{\mathfrak{A}}_{a,\varphi^\exists_i}$ of a witness structure for $(a, \varphi^\exists_i)$, whose dead part will be constructed later, be empty.

Case '*free live part*': Assume that b_a does not belong to the (non-empty) live part $\bar{\mathfrak{B}}_{b_a,\varphi^\exists_i}$ of the witness structure $\mathfrak{B}_{b_a,\varphi^\exists_i}$. By axiom 8, there is a witness structure for $(\alpha, \varphi^\exists_i)$ in $\mathfrak{B}$ whose live part is in the set $D^{\mathfrak{B}} \subseteq C_{\mathfrak{B}} \subseteq B$. Let $d_1, \dots, d_k \in D^{\mathfrak{B}}$ be the elements of $\bar{\mathfrak{B}}_{b_a,\varphi^\exists_i}$ (so $\bar{\mathfrak{B}}_{b_a,\varphi^\exists_i}$ contains exactly $k \geq 1$ elements). According to axiom 8, as $C_{\mathfrak{B}}$ and $C_{\mathfrak{A}}$ are isomorphic (via the bijection h), it is clear that

$$tb_{\mathfrak{A}}(h(d_1), \dots, h(d_k)) = tb_{\mathfrak{B}\restriction\tau}(d_1, \dots, d_k).$$

Therefore we let $\{h(d_1), \dots, h(d_k)\}$ be the domain of the live part $\bar{\mathfrak{A}}_{a,\varphi_i^{\exists}}$ of a witness structure for $(a, \varphi_i^{\exists})$, whose dead part will be constructed later; we note that $a \notin D_{\mathfrak{A}}$ due to our assumption that $a \notin K_{\mathfrak{A}} \cup D_{\mathfrak{A}}$, so $\bar{\mathfrak{A}}_{a,\varphi_i^{\exists}}$ is free w.r.t. a, i.e., $a \notin \bar{A}_{a,\varphi_i^{\exists}}$.

Case '*local singleton live part*': Assume that b_a is alone in the live part $\bar{\mathfrak{B}}_{b_a,\varphi_i^{\exists}}$ of the witness structure $\mathfrak{B}_{b_a,\varphi_i^{\exists}}$, i.e., $|\bar{B}_{b_a,\varphi_i^{\exists}}| = 1$. We recall that $tb_{\mathfrak{A}}(a) = tb_{\mathfrak{B}\restriction\tau}(b_a)$, and we let $\{a\}$ be the domain of the live part $\bar{\mathfrak{A}}_{a,\varphi_i^{\exists}}$ of a witness structure for $(a, \varphi_i^{\exists})$, whose dead elements will be identified later.

Case '*local doubleton live part*': Assume that b_a and some other element $b' \neq b_a$ in B form the live part $\bar{\mathfrak{B}}_{b_a,\varphi_i^{\exists}}$ of the witness structure $\mathfrak{B}_{b_a,\varphi_i^{\exists}}$. Thus $|\bar{B}_{b_a,\varphi_i^{\exists}}| = 2$. Let $t_{b'} \in \{1, \dots, N\}$ be the index such that $b' \in U^{\mathfrak{B}}_{t_{b'}} \subseteq B$. Next we consider several subcases of the case *local doubleton live part.*

In the following subcases 1 and 2, we assume that $t_{b'} \neq s_a$; recall that $b_a \in U^{\mathfrak{B}}_{s_a}$ and $b' \in U^{\mathfrak{B}}_{t_{b'}}$. We first note that if $t_{b'} < s_a$ (respectively, if $s_a < t_{b'}$), then by axiom 15, we have $\mathfrak{B} \models b' < b_a$ (resp., $\mathfrak{B} \models b_a < b'$).

1. If $b' \in C_{\mathfrak{B}}$, then we define

   $$tb_{\mathfrak{A}}(a, h(b')) := tb_{\mathfrak{B}\restriction\tau}(b_a, b').$$

   We note that in the special case where $a \in C_{\mathfrak{A}}$, as we have $b' \in C_{\mathfrak{B}}$, both elements a and $h(b')$ are in $C_{\mathfrak{A}}$, and therefore we have actually already defined the table $tb_{\mathfrak{A}}(a, h(b'))$ when $\mathfrak{C}_{\mathfrak{B}}$ was copied into $\mathfrak{A}$.

2. If $b' \notin C_{\mathfrak{B}}$, then we select some $3(m_{\exists} + n)$-block $bl(b')$ of elements in $I_{t_{b'}} \subseteq A$ realizing the 1-type $tp_{\mathfrak{B}\restriction\tau}(b')$; this is possible as for all $s \in \{1, \dots, N\}$, the interval $I_s \subseteq A$ has been constructed so that it realizes exactly the same 1-types over τ as the set $U^{\mathfrak{B}}_s$, and furthermore, for the following reason: Since $b' \notin C_{\mathfrak{B}}$, we have $b' \notin K^{\mathfrak{B}}$, and thus (by axiom 7) we have $tp_{\mathfrak{B}\restriction\tau}(b') \notin \boldsymbol{\alpha}^K_{\tau}$, whence it follows from the construction of the domain A that the interval $I_{t_{b'}}$ contains at least one $3(m_{\exists}+n)$-block of each 1-type realized in the interval. With the block $bl(b')$ chosen, we will later on show how to choose an element $a' \in bl(b') \subseteq A$ in order to construct a full live part of a witness structure for $(a, \varphi_i^{\exists})$. After that we will identify related dead elements in order to ultimately complete the live part into a full witness structure. (Strictly speaking, rather than

seeking full definitions of witness structures, we will always define only a table for the live part of a witness structure in addition to making sure that suitable elements for the dead part can be found.)

In the following subcases 3 and 4 of the case *local doubleton free-part*, we assume that $t_{b'} = s_a$, i.e., $b_a, b' \in U_{s_a}^{\mathfrak{B}}$. It follows from axiom 12 that either $\mathfrak{B} \models b_a < b'$ or $\mathfrak{B} \models b' < b_a$ but not both. In both subcases 3 and 4, we locate only a $3(m_\exists + n)$-block $bl(b') \subseteq A$ of elements of 1-type $tp_{\mathfrak{B}\restriction\tau}(b')$. Once again we will only later find elements from the block $bl(b')$ in order to identify a live part of a witness structure for $(a, \varphi_i^\exists)$, and after that we ultimately complete the live part to a full witness structure by finding suitable dead elements.

Note that since b_a and $b' \neq b_a$ are both in $U_{s_a}^{\mathfrak{B}}$, the set $U_{s_a}^{\mathfrak{B}}$ is not a singleton and thus $U_{s_a}^{\mathfrak{B}} \cap C_{\mathfrak{B}} = \emptyset$. Therefore, $b' \notin K^{\mathfrak{B}}$ and by axiom 7, $tp_{\mathfrak{B}\restriction\tau}(b') \notin \boldsymbol{\alpha}_\tau^K$. Now it follows from the construction of the domain A that the interval I_{s_a} contains at least one $3(m_\exists + n)$-block of 1-type $tp_{\mathfrak{B}\restriction\tau}(b')$.

3. Assume that $\mathfrak{B} \models b' < b_a$. Let α' denote the 1-type $tp_{\mathfrak{B}\restriction\tau}(b')$ of b'. If $\alpha' \notin \boldsymbol{\alpha}_{\tau,s_a}^-$, then we must have $\alpha' \in \boldsymbol{\alpha}_{\tau,t}$ for some $t < s_a$. Thus, and as $tp_{\mathfrak{B}\restriction\tau}(b') \notin \boldsymbol{\alpha}_\tau^K$, $I_t \subseteq A$ contains at least one block $bl(b')$ of elements realizing the 1-type α'. We choose the block $bl(b')$ to be the desired block to be used later. If, on the other hand, we have $\alpha' \in \boldsymbol{\alpha}_{\tau,s_a}^-$, we proceed as follows.

   a) Assume Γ_φ is admissible only for $\mathcal{O}$ and not for $\mathcal{WO}$ (and thus not for $\mathcal{O}_{fin}$ either). Then, due to the way we have defined the interval $I_{s_a} \subseteq A$ and labelled its elements by 1-types, there exists a $3(m_\exists + n)$-block $bl(b') \subseteq I_{s_a}$ of elements of type α' such that $bl(b')$ precedes the block in I_{s_a} that contains a. We appoint $bl(b')$ to be the desired block to be used later.

   b) Assume Γ_φ is admissible for $\mathcal{WO}$ and not for $\mathcal{O}_{fin}$. Assume first that $\alpha \notin \boldsymbol{\alpha}_{\tau,s_a}^-$ (where we recall that α is the 1-type of a and b_a over τ). Since $\alpha' \in \boldsymbol{\alpha}_{\tau,s_a}^-$ and $\alpha \notin \boldsymbol{\alpha}_{\tau,s_a}^-$, we observe that the interval $I_{s_a} \subseteq A$ has been defined such that there exists a $3(m_\exists + n)$-block $bl(b') \subseteq I_{s_a}$ of elements of type α' such that $bl(b')$ precedes the block in I_{s_a} that contains a. We appoint $bl(b')$ to be the block to be used later.

      Assume then that $\alpha \in \boldsymbol{\alpha}_{\tau,s_a}^-$. In this case, we have chosen the pattern element b_a to be the minimal realization of α in $\mathfrak{B}$. Since

$\mathfrak{B} \models b' < b_a$, we must have $tp_{\mathfrak{B}\restriction\tau}(b') \neq tp_{\mathfrak{B}\restriction\tau}(b_a)$. Thus we must have $\alpha' = tp_{\mathfrak{B}\restriction\tau}(b') \notin \boldsymbol{\alpha}^{-}_{\tau,s_a}$ by the admissibility condition iii (which states that $|\boldsymbol{\alpha}^{-}_{\tau,s_a}| \leq 1$). This contradicts the assumption that $\alpha' \in \boldsymbol{\alpha}^{-}_{\tau,s_a}$, so this case is in fact impossible and can thus be ignored.

c.1) Assume Γ_φ is admissible for $\mathcal{O}_{fin}$. Furthermore, assume that one of the following conditions holds.

  c.1.1) $\alpha \notin \boldsymbol{\alpha}^{-}_{\tau,s_a}$ (but α may be in $\boldsymbol{\alpha}^{+}_{\tau,s_a}$).

  c.1.2) $\alpha \in \boldsymbol{\alpha}^{-}_{\tau,s_a} \cap \boldsymbol{\alpha}^{+}_{\tau,s_a}$ and a is in the last block in I_{s_a}.

  Now, since $\alpha' \in \boldsymbol{\alpha}^{-}_{\tau,s_a}$ we observe that the interval $I_{s_a} \subseteq A$ has been defined such that there is a $3(m_\exists + n)$-block $bl(b') \subseteq I_{s_a}$ of elements of type α' such that $bl(b')$ precedes the block in I_{s_a} that contains a. We appoint $bl(b')$ to be the block to be used later.

c.2) Now assume Γ_φ is admissible for $\mathcal{O}_{fin}$, and furthermore, assume that one of the following conditions holds.

  c.2.1) $\alpha \in \boldsymbol{\alpha}^{-}_{\tau,s_a} \setminus \boldsymbol{\alpha}^{+}_{\tau,s_a}$.

  c.2.2) $\alpha \in \boldsymbol{\alpha}^{-}_{\tau,s_a} \cap \boldsymbol{\alpha}^{+}_{\tau,s_a}$ and a is not in the last block in I_{s_a}.

  In these cases we have chosen the pattern element b_a to be the minimal realization of α in $\mathfrak{B}$. Since $\mathfrak{B} \models b' < b_a$, we must have $tp_{\mathfrak{B}\restriction\tau}(b') \neq tp_{\mathfrak{B}\restriction\tau}(b_a)$. Thus we must have $\alpha' = tp_{\mathfrak{B}\restriction\tau}(b') \notin \boldsymbol{\alpha}^{-}_{\tau,s_a}$ by the admissibility condition iii (which states that $|\boldsymbol{\alpha}^{-}_{\tau,s_a}| \leq 1$). This contradicts the assumption that $\alpha' \in \boldsymbol{\alpha}^{-}_{\tau,s_a}$, so this case is in fact impossible and can thus be ignored.

4. Assume that $\mathfrak{B} \models b_a < b'$. Again we let α' denote $tp_{\mathfrak{B}\restriction\tau}(b')$. If $\alpha' \notin \boldsymbol{\alpha}^{+}_{\tau,s_a}$, then we have $\alpha' \in \boldsymbol{\alpha}_{\tau,t}$ for some $t > s_a$. We choose $bl(b')$ to be some block of elements realizing the 1-type α' from the interval $I_t \subseteq A$. If $\alpha' \in \boldsymbol{\alpha}^{+}_{\tau,s_a}$, we proceed as follows.

   a) Assume that Γ_φ is *not* admissible for $\mathcal{O}_{fin}$ but *is* admissible for $\mathcal{O}$ or even for $\mathcal{WO}$. Then, due to the way we defined 1-types over the interval I_{s_a}, there exists a block $bl(b') \subseteq I_{s_a}$ of type α' following the block that contains a in I_{s_a}. We appoint the block $bl(b')$ to be used later.

   b.1) Assume that Γ_φ is admissible for $\mathcal{O}_{fin}$. Furthermore, recall that α is the 1-type of a and assume that one of the following conditions holds.

b.1.1) $\alpha \notin \boldsymbol{\alpha}^{-}_{\tau,s_a} \cup \boldsymbol{\alpha}^{+}_{\tau,s_a}$

b.1.2) $\alpha \in \boldsymbol{\alpha}^{-}_{\tau,s_a} \setminus \boldsymbol{\alpha}^{+}_{\tau,s_a}$

b.1.3) $\alpha \in \boldsymbol{\alpha}^{-}_{\tau,s_a} \cap \boldsymbol{\alpha}^{+}_{\tau,s_a}$ and a is not in the last block in I_{s_a}.

Now, since $\alpha' \in \boldsymbol{\alpha}^{+}_{\tau,s_a}$ and due to admissibility condition v and the way we defined 1-types over the interval I_{s_a}, the last block in I_{s_a} is of 1-type α'. Clearly this last block comes after the block that contains a in I_{s_a}. We call this last block $bl(b')$ and appoint it for later use.

b.2) Assume Γ_φ is admissible for $\mathcal{O}_{fin}$ and that one of the following cases holds.

b.2.1) $\alpha \in \boldsymbol{\alpha}^{+}_{\tau,s_a} \setminus \boldsymbol{\alpha}^{-}_{\tau,s_a}$

b.2.2) $\alpha \in \boldsymbol{\alpha}^{-}_{\tau,s_a} \cap \boldsymbol{\alpha}^{+}_{\tau,s_a}$ and a is in the last block in I_{s_a}.

Then we have chosen the pattern element b_a to be the maximal realization of α in B, i.e., it satisfies $max_\alpha(x)$. As admissibility for $\mathcal{O}_{fin}$ implies that $|\boldsymbol{\alpha}^{+}_{\tau,s_a}| \leq 1$, we have $\alpha = \alpha'$. As we have assumed that $\mathfrak{B} \models b_a < b'$, we observe that this case is in fact impossible and can thus be ignored.

Now recall that when constructing the domain A of $\mathfrak{A}$ using $3(m_\exists + n)$-blocks, we defined the *E-part* of a $3(m_\exists + n)$-block to be the set that contains the first $(m_\exists + n)$ elements of the block. Similarly, we defined the *F-part* to be the set with the subsequent $(m_\exists + n)$ elements immediately after the E-part, and the G-part was defined to be the set with the last $(m_\exists + n)$ elements. Below, we let $E \subseteq A$ denote the union of the E-parts of all the $3(m_\exists + n)$-blocks used in the construction of A. Similarly, we let F and G denote the unions of the F-parts and G-parts, respectively.

Now, in the subcases 2-4 of the case *doubleton live part*, we located a $3(m_\exists + n)$-block $bl(b') \subseteq A$ of elements of type $\alpha' = tp_{\mathfrak{B}\restriction\tau}(b')$. Let $t \in \{1, \dots, N\}$ be the index of the interval $I_t \subseteq A$ where the block $bl(b')$ is. Next we will select an element a' from $bl(b') \subseteq I_t$ in order to define the domain of a live part of a witness structure for $(a, \varphi^{\exists}_{i})$ in A; note that in the subcase 1, such an element was already chosen. Now, if $a \in E$, we let a' be the i-th element (where i is the index of $\varphi^{\exists}_{i}$) realizing α' in $F \cap bl(b')$. Similarly, if $a \in F$ (respectively, if $a \in G \cup (C_{\mathfrak{A}} \setminus (K_{\mathfrak{A}} \cup D_{\mathfrak{A}}))$), we choose a' to be the i-th element in $G \cap bl(b')$ (resp., in $E \cap bl(b')$). Then we define

$$tb_{\mathfrak{A}}(a, a') := tb_{\mathfrak{B}\restriction\tau}(b, b'),$$

thereby possibly creating new tuples into the relations of $\mathfrak{A}$. Now $\{a, a'\}$ is the domain of the live part of a witness structure for $(a, \varphi_i^{\exists})$. Assigning 2-tables in this cyclic way prevents conflicts, as each pair $(a, a') \in A^2$ is considered at most once.

We then proceed to considering the case where b_a and *at least two other elements* in B form the live part $\bar{\mathfrak{B}}_{b_a,\varphi_i^{\exists}}$ of the witness structure $\mathfrak{B}_{b_a,\varphi_i^{\exists}}$. The sets $E, F, G \subseteq A$ defined above will play a role here as well.

Case '*local large live part*': Assume indeed that the live part $\bar{\mathfrak{B}}_{b_a,\varphi_i^{\exists}}$ has at least three elements, i.e., $|\bar{B}_{b_a,\varphi_i^{\exists}}| \geq 3$. Let $r_1, \dots, r_k$ (possibly $k = 0$) be the elements in $\bar{B}_{b_a,\varphi_i^{\exists}}$ that belong also to $K^{\mathfrak{B}}$, and let $b_a, b_1, \dots, b_l$ (possibly $l = 0$) be the remaining elements of $\bar{B}_{b_a,\varphi_i^{\exists}}$. As $|\bar{B}_{b_a,\varphi_i^{\exists}}| \geq 3$, we have $k+l \geq 2$. Now let $j \in \{1, \dots, l\}$ and identify, in an arbitrary way, a $3(m_{\exists} + n)$-block $bl(b_j) \subseteq A$ of elements that realize the same 1-type as b_j does. We let α_j be the 1-type of b_j, i.e., $\alpha_j = tp_{\mathfrak{B}\restriction\tau}(b_j)$, and we also let $t_{b_j} \in \{1, \dots, N\}$ denote the index of the interval where $bl(b_j)$ is. Then, with the blocks $bl(b_j)$ chosen for each j, we move on to considering the following subcases of the case *local large live part* in order to define a live part of a witness structure for $(a, \varphi_i^{\exists})$ in A.

1. Assume $l = 0$ and $a \in C_{\mathfrak{A}}$ (whence $k \geq 2$). We let $\{a, h(r_1), \dots, h(r_k)\}$ (where h is the bijection from $C_{\mathfrak{B}}$ to $C_{\mathfrak{A}}$ we defined above) be the domain of the desired live part. We note that $tb_{\mathfrak{A}}(a, h(r_1), \dots, h(r_k))$ has already been defined when $\mathfrak{C}_{\mathfrak{B}}$ was copied into $\mathfrak{A}$.

2. Assume $l = 0$ and $a \notin C_{\mathfrak{A}}$ (whence $k \geq 2$). Let $\{a, h(r_1), \dots, h(r_k)\}$ be the domain of the desired live part and define

   $$tb_{\mathfrak{A}}(a, h(r_1), \dots, h(r_k)) := tb_{\mathfrak{B}\restriction\tau}(b_a, r_1, \dots, r_k).$$

   Note here that the mapping h is injective and $a \notin img(h) = C_{\mathfrak{A}}$.

3. Assume $l > 0$ and $a \in E$. We will next define elements $a_1, \dots, a_l \in A$ corresponding to $b_1, \dots, b_l$. We first let a_1 be the i-th (where $i \leq m_{\exists}$ is the index of $\varphi_i^{\exists}$) element in $bl(b_1) \cap F$. Then, if $l > 1$, we define the elements $a_2, \dots, a_l$ to be *distinct* elements such that a_j is, for an arbitrary $p \in \{m_{\exists}+1, \dots, m_{\exists}+n\}$, the p-th element in $bl(b_j) \cap F$. Note that $l < n$, so it is easy to ensure the elements $a_2, \dots, a_l$ are distinct even if chosen from a single block. We let $\{a, h(r_1), \dots, h(r_k), a_1, \dots, a_l\}$

be the domain of the desired live part of a witness structure, and we define

$$tb_{\mathfrak{A}}(a, h(r_1), \ldots, h(r_k), a_1, \ldots, a_l) := tb_{\mathfrak{B}\restriction\tau}(b_a, r_1, \ldots, r_k, b_1, \ldots, b_l),$$

thereby possibly creating new tuples to the relations of $\mathfrak{A}$.

4. Assume $l > 0$ and $a \in F$. Then we proceed as in the previous case, but we take the elements $a_1, \ldots, a_l$ from G. Similarly, if $l > 0$ and $a \in G \cup (C_{\mathfrak{A}} \setminus (K_{\mathfrak{A}} \cup D_{\mathfrak{A}}))$, we take the elements $a_1, \ldots, a_l$ from E. As before, we let $\{a, h(r_1), \ldots, h(r_k), a_1, \ldots, a_l\}$ be the domain of the desired live part of a witness structure, and we then define

$$tb_{\mathfrak{A}}(a, h(r_1), \ldots, h(r_k), a_1, \ldots, a_l) := tb_{\mathfrak{B}\restriction\tau}(b_a, r_1, \ldots, r_k, b_1, \ldots, b_l),$$

thus again possibly creating new tuples to relations.

We have now considered several cases and defined the live part $\bar{\mathfrak{A}}_{a,\varphi_i^\exists}$ of a witness structure $\mathfrak{A}_{a,\varphi_i^\exists}$ in each case (or rather a table over the elements of the live part). We next show how to complete the definition of $\mathfrak{A}_{a,\varphi_i^\exists}$ by finding a suitable dead part for it. We have defined $\bar{\mathfrak{A}}_{a,\varphi_i^\exists}$ in each case so that there is a bijection from $\bar{B}_{b_a,\varphi_i^\exists} \cup \{b_a\}$ onto $\bar{A}_{a,\varphi_i^\exists} \cup \{a\}$; note that b_a (respectively, a) may or may not be part of the live part $\bar{\mathfrak{B}}_{b_a,\varphi_i^\exists}$ (resp., $\bar{\mathfrak{A}}_{a,\varphi_i^\exists}$) depending on whether the live part is free, and it holds that $b_a \in \bar{B}_{b_a,\varphi_i^\exists} \Leftrightarrow a \in \bar{A}_{a,\varphi_i^\exists}$. The task is now to extend this bijection to a map that maps injectively from $B_{b_a,\varphi_i^\exists}$ into A and preserves 1-types over τ. This will complete the construction of $\mathfrak{A}_{a,\varphi_i^\exists}$. This is very easy to do: Note first that since n is the width of φ, we have $|B_{b_a,\varphi_i^\exists}| \leq n$. Now recall that in $\mathfrak{A}$, each pawn is part of some $3(m_\exists + n)$-block of elements of the same 1-type, so there are at least $3(m_\exists + n)$ elements of that type in $\mathfrak{A}$. Furthermore, the elements of $\mathfrak{B}$ with a 1-type (over τ) that is royal in $\mathfrak{A}$ are all in $K^{\mathfrak{B}} \subseteq C_{\mathfrak{B}}$, and $\mathfrak{A}$ contains the copy $\mathfrak{C}_{\mathfrak{A}}$ of $\mathfrak{C}_{\mathfrak{B}}$ as a substructure. Thus it is easy to extend the bijection in the required way.

4) Completion procedure Let r be the highest arity occurring in the symbols in τ and n the width of φ. Define $m := min\{r, n\}$ and $k \in \{2, \ldots, m\}$. Let $S \subseteq A$ be a set with k-elements. Assume that $tb_{\mathfrak{A}}(\overline{s})$ has not been defined for any k-tuple $\overline{s}$ enumerating the elements of S when copying $\mathfrak{C}$ into $\mathfrak{A}$ and when finding witness structures in $\mathfrak{A}$; thus we still need to define some k-table for some tuple $\overline{s}$ that enumerates the points in S. We do this next.

Assume first that $k = 2$. Assume $S = \{a_1, a_2\}$ such that $a_1 < a_2$ and such that $tp_{\mathfrak{A}}(a_1) = \alpha_1$ and $tp_{\mathfrak{A}}(a_2) = \alpha_2$. Let $s, t \in \{1, \dots, N\}$ be the indices such that $a_1 \in I_s$ and $a_2 \in I_t$. Due to the way we constructed the intervals of A in the subsection 1), we know that $\alpha_1 \in \boldsymbol{\alpha}_{\tau,s}$ and $\alpha_2 \in \boldsymbol{\alpha}_{\tau,t}$. Furthermore, as $a_1 < a_2$, we know that either I_s is an interval preceding the interval I_t and thus $s < t$, or I_s and I_t are the same interval and thus $s = t$.

If $s < t$, then by axioms 3 and 15, we find from $\mathfrak{B}$ a point $b_1 \in U_s^{\mathfrak{B}}$ realizing α_1 and a point $b_2 \in U_t^{\mathfrak{B}}$ realizing α_2 such that $b_1 <^{\mathfrak{B}} b_2$. We set

$$tb_{\mathfrak{A}}(a_1, a_2) := tb_{\mathfrak{B}\restriction\tau}(b_1, b_2).$$

Now assume that $s = t$. We consider the two cases where $\alpha_2 \notin \boldsymbol{\alpha}_{\tau,s}^{+}$ and $\alpha_2 \in \boldsymbol{\alpha}_{\tau,s}^{+}$. If $\alpha_2 \notin \boldsymbol{\alpha}_{\tau,s}^{+}$, then there is some $t' \in \{1, \dots, N\}$ such that $s < t'$ and $\alpha_2 \in \boldsymbol{\alpha}_{\tau,t'}$. Thus, again by axioms 3 and 15, we find from $\mathfrak{B}$ a point $b_1 \in U_s^{\mathfrak{B}}$ realizing α_1 and a point $b_2 \in U_{t'}^{\mathfrak{B}}$ realizing α_2 such that $b_1 <^{\mathfrak{B}} b_2$. We set

$$tb_{\mathfrak{A}}(a_1, a_2) := tb_{\mathfrak{B}\restriction\tau}(b_1, b_2).$$

Assume then that $\alpha_2 \in \boldsymbol{\alpha}_{\tau,s}^{+}$. We consider the two subcases where $s \notin img(\delta)$ and $s \in img(\delta)$; recall the definition of δ from Section 3.3. If $s \notin img(\delta)$, then by axioms 3 and 16, there is in $\mathfrak{B}$ a point $b_1 \in U_s^{\mathfrak{B}}$ realizing α_1 and a point $b_2 \in U_s^{\mathfrak{B}}$ realizing α_2 such that $b_1 <^{\mathfrak{B}} b_2$. Once again we set

$$tb_{\mathfrak{A}}(a_1, a_2) := tb_{\mathfrak{B}\restriction\tau}(b_1, b_2).$$

If $s \in img(\delta)$, then, by admissibility condition ii, either I_s is a singleton with an element with a royal type or $\boldsymbol{\alpha}_{\tau,s}^{-} = \emptyset = \boldsymbol{\alpha}_{\tau,s}^{+}$. If I_s is a singleton, then the assumption $a_1 < a_2$ fails, so we must have $\boldsymbol{\alpha}_{\tau,s}^{-} = \emptyset = \boldsymbol{\alpha}_{\tau,s}^{+}$. Thus the assumption $\alpha_2 \in \boldsymbol{\alpha}_{\tau,s}^{+}$ fails, and thus this case is in fact impossible and can thus be ignored.

Assume then that $k > 2$. We select distinct elements $b_1, \dots, b_k$ in B such that $tp_{\mathfrak{A}}(a_i) = tp_{\mathfrak{B}\restriction\tau}(b_i)$ for each $i \in \{1, \dots, k\}$; this is possible because every king of $\mathfrak{A}$ is in $C_{\mathfrak{A}}$ and thus there exists a corresponding point in $C_{\mathfrak{B}}$, and furthermore, by axiom 5, for each pawn u of $\mathfrak{A}$, there exist at least $n \geq k$ points of the 1-type (over τ) of u in $\mathfrak{B}$. Now we set

$$tb_{\mathfrak{A}}(a_1, \dots, a_k) := tb_{\mathfrak{B}\restriction\tau}(b_1, \dots, b_k).$$

Finally, if the maximum arity r of relations in τ is greater than n, then the tables of $\mathfrak{A}$ over sets with more than n elements are defined arbitrarily.

The model $\mathfrak{A}$ is now fully defined. To finish the proof of Lemma 5, we argue that $\mathfrak{A} \models \varphi$. The fact that $\mathfrak{A}$ satisfies all the existential conjuncts of φ was established in the subsection 3). To see that $\mathfrak{A}$ satisfies also the universal conjuncts, consider such a conjunct $\forall x_1 \ldots \forall x_k \psi(x_1, \ldots, x_k)$, and let $(a_1, \ldots, a_k)$ be a tuple of elements from A, with possible repetitions. We must show that $\mathfrak{A} \models \psi(a_1, \ldots, a_k)$. Let

$$\{u_1, \ldots, u_{k'}\} := \mathit{live}(\psi(x_1, \ldots, x_k)[a_1, \ldots, a_k]),$$

and let

$$V := \{a_1, \ldots, a_k\} \setminus \{u_1, \ldots, u_{k'}\}.$$

The table $tb_{\mathfrak{A}}(u_1, \ldots, u_{k'})$ has been defined either when finding witness structures or in the above completion construction based on some table $tb_{\mathfrak{B} \restriction \tau}(b_1, \ldots, b_{k'})$ of distinct elements. We now observe the following.

1. All the kings of $\mathfrak{A}$ are in $C_{\mathfrak{A}}$ and thereby have corresponding elements in $C_{\mathfrak{B}}$ that satisfy the same 1-type over τ.
2. For each pawn u of $\mathfrak{A}$, there exist at least n elements of the same 1-type over τ as u in $\mathfrak{B}$ (by axiom 5).
3. The set $V \cup \{u_1, \ldots, u_{k'}\} = \{a_1, \ldots, a_k\}$ has at most n elements.

Based on the above, it is easy to see that we can define an injection f from $\{u_1, \ldots, u_{k'}\} \cup V$ into B that preserves 1-types (over τ) and satisfies $f(u_i) = b_i$ for each $i \in \{1, \ldots, k'\}$. Therefore

$$\mathfrak{A} \models \psi(a_1, \ldots, a_k) \text{ iff } \mathfrak{B} \models \psi(f(a_1), \ldots, f(a_k)).$$

Since $\mathfrak{B} \models \varphi$, we have $\mathfrak{B} \models \psi(f(a_1), \ldots, f(a_k))$ and therefore $\mathfrak{A} \models \psi(a_1, \ldots, a_k)$. □

The following gives a brief description of the decision process which is also outlined in Algorithm 1. A complete and rigorous treatment of related details is given in Chapter 5 which is devoted to the proof of Theorem 7.

1. An input to the problem is a sentence ψ' of U_1, which is immediately converted into a normal form sentence ψ of U_1 (cf. Proposition 2).

2. Based on ψ, an admissibility tuple $\Gamma_\psi \in \hat{\Gamma}_\psi$ is guessed non-deterministically. The size of the tuple is exponential in $|\psi|$ (cf. Lemma 2). It is then checked whether the tuple is admissible for the class $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$ whose decision problem we are considering.

3. Based on Γ_ψ, the sentence $Ax(\Gamma_\psi)$ is produced. The length of $Ax(\Gamma_\psi)$ is exponential in $|\psi|$ (cf. Lemma 6).

4. Then a model $\mathfrak{B}$, whose description is exponential in $|\psi|$ (cf. Lemma 8), is guessed. It is then checked whether $\mathfrak{B} \models Ax(\Gamma_\psi)$, which can be done in exponential time in $|\psi|$.

Theorem 7. *Let $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$. The satisfiability problem for* U_1 *over $\mathcal{K}$ is* NExpTime*-complete.*

Proof. The lower bound (for each of the three decision problems) follows immediately from [48]. The remaining part of the proof is given in the next chapter. □

Algorithm 1 Solving satisfiability of U_1 over $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$.
The symbol $\triangleright$ indicates comment.

```
procedure SATISFIABILITY($\psi'$) over $\mathcal{K}$.
    $\triangleright$ The $\mathrm{U}_1$-sentence $\psi'$ is an input to the algorithm. Here $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$, so we are outlining three procedures in parallel.
    Construct a normal form sentence $\psi$ of $\mathrm{U}_1$ from $\psi'$. Let $\tau$ be the vocabulary consisting of all the relation symbols occurring in $\psi$.    $\triangleright$ By Proposition 2, it holds that $\psi$ is satisfiable iff $\psi'$ is satisfiable.
    Guess $\Gamma_\psi \in \hat{\Gamma}_\psi$ and check that $\Gamma_\psi$ is an admissibility tuple admissible for $\mathcal{K}$.
    Let $\tau' := \tau \cup \{U_s \mid s \in \{1, \dots, N\}\} \cup \{K, D, P_\perp, P_\top\}$. Formulate the pseudo-ordering axioms for $\Gamma_\psi$ over $\tau'$ and let $Ax(\Gamma_\psi)$ be the conjunction of these axioms.    $\triangleright$ Note that $Ax(\Gamma_\psi)$ is in normal form.
    Guess a potential model $\mathfrak{B}$ of $Ax(\Gamma_\psi)$ whose size is exponentially bounded in $|\psi|$.    $\triangleright$ In the next lines it is checked whether $\mathfrak{B} \models Ax(\Gamma_\psi)$. Note that by Lemma 5, if $\mathfrak{B} \models Ax(\Gamma_\psi)$, then $\psi \in sat_{\mathcal{K}}(\psi)$.

    for all $b \in B$ do
        for all existential conjuncts $\chi := \forall x \exists y_1 \dots \exists y_l\, \beta(x, y_1, \dots, y_l)$ of $Ax(\Gamma_\psi)$ do
            Guess elements $b'_1, \dots, b'_l$ in $B$ to form a witness structure $\mathfrak{B}_{b,\chi}$ and
            check whether $\mathfrak{B} \models \beta(b, b'_1, \dots, b'_l)$.
        end for
    end for
    for all universal conjuncts $\forall x_1 \dots \forall x_{l'} \beta'(x_1, \dots x_{l'})$ of $Ax(\Gamma_\psi)$ do
        for all tuples $(b_1, \dots, b_{l'})$ of elements of $B$, do
            Check whether $\mathfrak{B} \models \beta'(b_1, \dots, b_{l'})$.
        end for
    end for
end procedure
```

Chapter 5

Complexity

In this chapter we study the complexity of the algorithm outlined in Algorithm 1 and establish that it runs in NEXPTIME in all cases $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$. We now fix some $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$ and study only the algorithm for the class $\mathcal{K}$; below we call the algorithm *Algorithm 1.*

Let ψ' be a U_1-sentence given as an input to Algorithm 1. It follows from Proposition 2 that ψ' can be translated in polynomial time in $|\psi'|$ to a normal form sentence ψ such that ψ' is satisfiable in some model $\mathfrak{M} \in \mathcal{K}$ iff ψ is satisfiable in some expansion $\mathfrak{M}^* \in \mathcal{K}$ of $\mathfrak{M}$. The formula ψ is the normal form sentence of U_1 constructed at line 2 of Algorithm 1. Let τ be the vocabulary consisting of the relation symbols in ψ. We assume w.l.o.g. that $< \in \tau$.

At line 3 we guess some $\Gamma_\psi \in \hat{\Gamma}_\psi$ and check that Γ_ψ is indeed an admissibility tuple admissible for $\mathcal{K}$. The length of Γ_ψ is bounded exponentially in $|\psi|$ by Lemma 2, and checking admissibility of Γ_ψ for $\mathcal{K}$ can be done in polynomial time in $|\Gamma_\psi|$.

At line 4 we let τ' be the vocabulary

$$\tau \cup \{U_s \mid s \in \{1, \ldots, N\}\} \cup \{K, D, P_\perp, P_\top\}$$

and formulate the conjunction $Ax(\Gamma_\psi)$ of the pseudo-ordering axioms for Γ_ψ over τ'.

Lemma 6. *Consider a normal form sentence* χ *of* U_1 *and a related admissibility tuple. The size of the sentence* $Ax(\Gamma_\chi)$ *is exponentially bounded in* $|\chi|$.

Proof. Let N be the index of Γ_χ and C the domain of the court structure of Γ_χ. Let σ be the vocabulary of χ. Now let β be some axiom from the list of 16 axioms that make $Ax(\Gamma_\chi)$, see Section 3.4. The sentence β is a normal form sentence with some number $m_{\exists,\beta}$ of existential conjuncts and some number $m_{\forall,\beta}$ of universal conjuncts. Now, by inspection of the pseudo-ordering axioms, the sum $m_{\exists,\beta}+m_{\forall,\beta}$ is bounded above by the very generous[1] bound

$$\mathit{const}\cdot|\chi|\cdot N^2\cdot|\boldsymbol{\alpha}_\sigma|^2\cdot|C|^{|\chi|}+\mathit{const}$$

for some constant *const*. Recalling from Section 3.3 that

$$|C|\leq 2|\chi|^4|\boldsymbol{\alpha}_\sigma| \text{ and } N\leq 6|\chi|^4|\boldsymbol{\alpha}_\sigma|,$$

we get that $m_{\exists,\beta}+m_{\forall,\beta}$ is bounded by

$$\mathit{const}\cdot|\chi|\cdot(6|\chi|^4|\boldsymbol{\alpha}_\sigma|)^2\cdot|\boldsymbol{\alpha}_\sigma|^2\cdot(2|\chi|^4|\boldsymbol{\alpha}_\tau|)^{|\chi|}+\mathit{const}.$$

Since $|\boldsymbol{\alpha}_\sigma|\leq 2^{|\chi|}$, it is therefore easy to see that this bound is exponential in $|\chi|$. Therefore, to conclude our proof, it suffices to find some bound $\mathcal{B}$ exponential in $|\chi|$ such that the length of each existential conjunct as well as the length of each universal conjunct in $Ax(\Gamma_\chi)$ is bounded above by $\mathcal{B}$.

To find such a bound $\mathcal{B}$, we first investigate axiom 11. We note that each formula $\beta_{[c_1,\ldots,c_k]}(x_1,\ldots,x_k)$ in axiom 11 is a k-table and therefore consists of a conjunction over a set such as—to give a possible example—the one given in Example 4. The *number of conjuncts* in $\beta_{[c_1,\ldots,c_k]}(x_1,\ldots,x_k)$ is therefore definitely bounded above by the bound $|\chi|\cdot|\chi|^{|\chi|}$. Thus it is easy to see that there exists a term $\mathcal{B}_{(11)}$ exponential in $|\chi|$ such that the length of each universal conjunct of axiom 11 is bounded above by $\mathcal{B}_{(11)}$. To cover the existential and universal conjuncts in the other axioms, we investigate each axiom individually and easily conclude that there exists a term $\mathcal{B}_{(i)}$ for each axiom $i\in\{1,\ldots,16\}$ such that the length of each existential and universal conjunct in the axiom (i) is bounded above by $\mathcal{B}_i$, and furthermore, $\mathcal{B}_i$ is exponential in $|\chi|$. By taking the product of the terms $\mathcal{B}_{(i)}$, we find a uniform exponential bound for the length of all existential and universal conjuncts in $Ax(\Gamma_\chi)$. ☐

---

[1] We shall not seek minimal or in any sense canonical bounds. Instead we settle with "clearly sufficient" bounds. This applies here as well as later on.

At line 5 of Algorithm 1 we guess a τ'-model $\mathfrak{B}$ whose domain size is exponential in $|\psi|$ (rather than exponential in $|Ax(\Gamma_\psi)|$); a sufficient bound is established below (Lemma 8), and furthermore, it is shown that not only the domain size but even the full description of $\mathfrak{B}$ can be bounded exponential in $|\psi|$. (Recall that $\mathfrak{B}$ does not have to interpret the binary relation symbol $<$ as and order.) We now begin the process of finding an exponential upper bound (in $|\psi|$) for the size of $\mathfrak{B}$ and show that this bound is indeed sufficient. We also establish that, indeed, the full description of $\mathfrak{B}$ likewise has a bound exponential in $|\psi|$. To achieve these goals, we first analyze below the proof of Theorem 8; this theorem is Theorem 2 in the article [30] (and Theorem 3.4 in [31] due to different numbering). The original proof is given in detail in Section 3 of both [30] and [31]. We state the theorem exactly as in [30] and [31], and thus note that $\mathrm{UF}_1^=$ denotes U_1 in the theorem. (Note that obviously the theorem concerns general U_1 as opposed to U_1 over ordered structures.)

Theorem 8 ([30]). $\mathrm{UF}_1^=$ *has the finite model property. Moreover, every satisfiable* $\mathrm{UF}_1^=$*-formula* φ *has a model whose size is bounded exponentially in* $|\varphi|$.

It follows from Theorem 8 that $Ax(\Gamma_\psi)$ has a model $\mathfrak{M}$ whose size is exponential in $|Ax(\Gamma_\psi)|$, but since $|Ax(\Gamma_\psi)|$ is exponential in $|\psi|$, the size of the model $\mathfrak{M}$ is double exponential in $|\psi|$. This is not the desired result. To lower the bound to exponential, we now analyze the proof of Theorem 8 given in Section 3 of [30] and [31]. This will result in the following lemma which follows directly and very easily from [30, 31] but is implicit there, i.e., not stated as an explicit lemma. Recall here that $\boldsymbol{\alpha}_{\mathfrak{A}}$ denotes the 1-types realized in $\mathfrak{A}$.

Lemma 7. *Let* φ *be a normal form sentence of* U_1. *Let* $m_\exists > 0$ *be the number of existential conjuncts in* φ. *Let* $n \geq 2$ *be the width of* φ *and* σ *the vocabulary of* φ. *If* φ *is satisfiable, then it is satisfiable in some model* $\mathfrak{M}$ *such that* $|M| \leq 8m_\exists^2 n^2 \boldsymbol{\alpha}_{\mathfrak{M}}$ *where* $\boldsymbol{\alpha}_{\mathfrak{M}} \subseteq \boldsymbol{\alpha}_\sigma$.

Proof. Let φ, $n \geq 2$, σ and $m_\exists \neq 0$ be as specified above. Assume φ is satisfiable. The claim of the current lemma follows directly by inspection of the relatively short argument in Section 3 of [30, 31], but we shall anyway outline here why there exists a model $\mathfrak{M}$ with the given limit $8m_\exists^2 n^2 \boldsymbol{\alpha}_{\mathfrak{M}}$ on domain size.

Assume $\mathfrak{A}$ is a σ-model such that $\mathfrak{A} \models \varphi$. The original proof constructs from the σ-model[2] $\mathfrak{A}$ of φ a new σ-model $\mathfrak{A}'$ whose domain A' consists of the union of four sets C, E, F, G, where the set C is constructed with the help of two sets K and D. Now, while it is stated in [30] that

$$|K| \leq (n-1)|\boldsymbol{\alpha}_\sigma| \text{ and } |D| \leq (n-1)m_\exists|\boldsymbol{\alpha}_\sigma|,$$

it is straightforward to observe that in fact

$$|K| \leq (n-1)|\boldsymbol{\alpha}_\mathfrak{A}| \text{ and } |D| \leq (n-1)m_\exists|\boldsymbol{\alpha}_\mathfrak{A}|.$$

(Note that we use $\boldsymbol{\alpha}_\mathfrak{A}$ instead of $\boldsymbol{\alpha}_{\mathfrak{A}'}$ here.) It is also easily seen that $|C| \leq n|K \cup D|m_\exists$, and thus we can calculate, using the above bounds for K and D, that

$$\begin{aligned} C &\leq n|K \cup D|m_\exists \\ &\leq n((n-1)|\boldsymbol{\alpha}_\mathfrak{A}| + (n-1)m_\exists|\boldsymbol{\alpha}_\mathfrak{A}|)m_\exists \\ &\leq (n^2|\boldsymbol{\alpha}_\mathfrak{A}| + n^2 m_\exists|\boldsymbol{\alpha}_\mathfrak{A}|)m_\exists \\ &\leq 2n^2 m_\exists^2|\boldsymbol{\alpha}_\mathfrak{A}|. \end{aligned}$$

We then consider the sets E, F, G. The article [30] gives a bound $(n+m_\exists)|\boldsymbol{\alpha}_\sigma|$ for each of these sets, but it is immediate that in fact $(n+m_\exists)|\boldsymbol{\alpha}_\mathfrak{A}|$ suffices.

Putting all the above together, we calculate

$$\begin{aligned} |C \cup E \cup F \cup E| &\leq 2n^2 m_\exists^2|\boldsymbol{\alpha}_\mathfrak{A}| + 3(n+m_\exists)|\boldsymbol{\alpha}_\mathfrak{A}| \\ &\leq 3n^2 m_\exists^2|\boldsymbol{\alpha}_\mathfrak{A}| + 3(n^2 m_\exists^2)|\boldsymbol{\alpha}_\mathfrak{A}| \\ &\leq 8n^2 m_\exists^2|\boldsymbol{\alpha}_\mathfrak{A}|. \end{aligned}$$

It is also immediate that $\boldsymbol{\alpha}_{\mathfrak{A}'} \subseteq \boldsymbol{\alpha}_\mathfrak{A}$, so the domain of $\mathfrak{A}'$, i.e., the set $C \cup E \cup F \cup E$, is bounded above by $8n^2 m_\exists^2|\boldsymbol{\alpha}_{\mathfrak{A}'}|$. $\square$

Lemma 8. *Let $\Gamma_\varphi \in \hat{\Gamma}_\varphi$ be some tuple admissible for $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$ such that $Ax(\Gamma_\varphi)$ is satisfiable. Then $Ax(\Gamma_\varphi)$ has a model $\mathfrak{A}$ whose size is bounded exponentially in $|\varphi|$. Moreover, even the length of the description of $\mathfrak{A}$ is bounded exponentially in $|\varphi|$.*

---

[2] The vocabulary used in the original proof is denoted by τ instead of σ. We use σ here because τ is in 'global' use by Algorithm 1.

Proof. Let σ be the vocabulary of φ. Let n be the width of φ and $m_\exists$ the number of existential conjuncts in φ. Let N be the index of Γ_φ and

$$\sigma' := \sigma \cup \{U_s \mid 1 \leq s \leq N\} \cup \{K, D, P_\perp, P_\top\}$$

the vocabulary of $Ax(\Gamma_\varphi)$. Let C be the domain of the court structure in Γ_φ. Assume $\mathfrak{M} \models Ax(\Gamma_\varphi)$. Recalling from Section 3.3 that $N \leq 2|\varphi|^4|\boldsymbol{\alpha}_\sigma|$ and thus clearly $N \leq 2|\varphi|^4 \cdot 2^{|\varphi|}$, we have

$$\begin{aligned}
|\sigma'| &= |\sigma| + |\{U_s \mid 1 \leq s \leq N\}| + |\{K, D, P_\perp, P_\top\}| \\
&= |\sigma| + N + 4 \\
&\leq |\varphi| + 2|\varphi|^4 \cdot 2^{|\varphi|} + 4
\end{aligned}$$

Thus $|\boldsymbol{\alpha}_{\sigma'}|$ is bounded by $2^{|\varphi|+2|\varphi|^4 \cdot 2^{|\varphi|}+4}$. This is double exponential in $|\varphi|$. However, the upper bound for $|\boldsymbol{\alpha}_{\mathfrak{M}}|$ (i.e., the number of 1-types over σ' realized in $\mathfrak{M}$) is exponentially bounded in $|\varphi|$ for the following reason.

Since the predicates U_s, where $s \in \{1, \ldots, N\}$, partition the domain M, each element in M satisfies exactly one of the predicates U_s. Therefore, letting

$$\sigma'' := \sigma' \setminus \{U_s \,|\, 1 \leq s \leq N\},$$

we have

$$|\boldsymbol{\alpha}_{\mathfrak{M}}| \leq N|\boldsymbol{\alpha}_{\sigma''}|.$$

On the other hand,

$$|\boldsymbol{\alpha}_{\sigma''}| \leq 2^{|\sigma|+4} \leq 2^{|\varphi|+4}.$$

Combining these, we obtain that $|\boldsymbol{\alpha}_{\mathfrak{M}}| \leq N \cdot 2^{|\varphi|+4}$. Recalling (from a few lines above) that $N \leq 2|\varphi|^4 \cdot 2^{|\varphi|}$, we get

$$|\boldsymbol{\alpha}_{\mathfrak{M}}| \leq 2|\varphi|^4 \cdot 2^{|\varphi|} \cdot 2^{|\varphi|+4} = 2|\varphi|^4 \cdot 2^{2|\varphi|+4}.$$

This is exponential in $|\varphi|$.

As $Ax(\Gamma_\varphi)$ is satisfiable, it follows from Lemma 7 that $\mathfrak{A} \models Ax(\Gamma_\varphi)$ for some σ'-structure $\mathfrak{A}$ whose size is bounded by $8\hat{m}_\exists \hat{n}^2 |\boldsymbol{\alpha}_{\mathfrak{A}}|$, where $\hat{m}_\exists$ is the number of existential conjuncts in $Ax(\Gamma_\varphi)$ and $\hat{n}$ is the width of $Ax(\Gamma_\varphi)$. On the other hand, by the result from the previous paragraph, we have

$$|\boldsymbol{\alpha}_{\mathfrak{A}}| \leq 2|\varphi|^4 \cdot 2^{2|\varphi|+4}.$$

Therefore, to show that the domain of $\mathfrak{A}$ is bounded exponentially in $|\varphi|$, it suffices to show that $\hat{m}_\exists$ and $\hat{n}$ are exponentially bounded in $|\varphi|$. This follows immediately by Lemma 6.

We then show that even the length of the description of $\mathfrak{A}$ is, likewise, exponentially bounded in $|\varphi|$. For describing models, we use the straightforward convention from Chapter 6 of [44], according to which the unique description of $\mathfrak{A}$ with some ordering of σ' is of the length

$$|A| + 1 + \sum_{i=1}^{|\sigma'|} |A|^{ar(R_i)},$$

where $ar(R_i)$ is the arity of $R_i \in \sigma'$. Since $|A|$ is exponential in $|\varphi|$ and $ar(R_i) \leq |\varphi|$, each term $|A|^{ar(R_i)}$ is likewise exponentially bounded in $|\varphi|$. Furthermore, at the beginning of the current proof we calculated that

$$|\sigma'| \leq |\varphi| + 2|\varphi|^4 \cdot 2^{|\varphi|} + 4.$$

Thus we conclude that the description of $\mathfrak{A}$ exponentially bounded in $|\varphi|$. $\square$

Once we have guessed the exponentially bounded model $\mathfrak{B}$ at line 5 of Algorithm 1, the remaining part of the algorithm is devoted for checking that $\mathfrak{B} \models Ax(\Gamma_\psi)$. At lines 6-11 we scan each $b \in B$ and each existential conjunct of $Ax(\Gamma_\psi)$. Then at lines 12-16 we check the universal conjuncts by checking all tuples of length at most n' in B, where n' is the width of $Ax(\Gamma_\psi)$. Noting that $n' \leq n + 1$, where n is the width of ψ, the procedure at lines 5-16 can be carried out in exponential time in $|\psi|$.

We have now proved the following theorem, which is a restatement of Theorem 7. (Recall that the lower bound is obtained because FO^2 is NExpTime-complete for all the classes $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$ [48].)

Theorem 9. (Restatement of Theorem 7): *Let* $\mathcal{K} \in \{\mathcal{O}, \mathcal{WO}, \mathcal{O}_{fin}\}$. *The satisfiability problem for* U_1 *over* $\mathcal{K}$ *is* NExpTime*-complete.*

Chapter 6

Undecidable extensions

The satisfiability problem for $\mathrm{FO}^2(<_1, <_2, <_3)$ over structures with three linear orders is undecidable [29]. In addition, the finite satifiability problem for $\mathrm{FO}^2(<_1, +1_1, <_2, +1_2)$ over the structures with two linear orders and their induced successors is undecidable [46]. On the other hand, while the finite satisfiability problem for FO^2 over structures with two linear orders is decidable and in 2NExpTime [57], the general satisfiability problem for FO^2 with two linear orders (and otherwise unrestricted vocabulary) is open. These results raise the question whether the satisfiability problem for the extension $\mathrm{U}_1[<_1, <_2]$ of U_1 (see Section 2.2) over structures with two linear orders is decidable. We use tiling arguments to answer this question in the negative.

We note that $\mathrm{U}_1[\sim_1, \sim_2]$, where $\sim_1$ and $\sim_2$ denote non-uniform built-in equivalence relations, is undecidable [32].

6.1 Two linear orders

Theorem 10. *The satisfiability problem for* $\mathrm{U}_1[<_1, <_2]$ *is undecidable.*

Before giving the proof, we introduce some definitions and lemmas used in the proof.

A *domino system* $\mathcal{D}$ is a structure (D, H_{do}, V_{do}), where D is a finite set (of dominoes) and $H_{do}, V_{do} \subseteq D \times D$. We say that a mapping $\tau : \mathbb{N} \times \mathbb{N} \to D$ is a *$\mathcal{D}$-tiling* of $\mathbb{N} \times \mathbb{N}$, if for every $i, j \in \mathbb{N}$, it holds that

$$(\tau(i, j), \tau(i + 1, j)) \in H_{do}$$

and

$$(\tau(i,j), \tau(i,j+1)) \in V_{do}.$$

The *tiling problem* asks, given a domino system $\mathcal{D}$ as an input, whether there exists a $\mathcal{D}$-tiling of $\mathbb{N} \times \mathbb{N}$. Due to [6] the tiling problem is undecidable.

Let $\mathfrak{G}_{\mathbb{N}} = (\mathbb{N} \times \mathbb{N}, H, V)$ be the *standard grid*, where $H = \{\big((i,j),(i+1,j)\big) \mid i,j \in \mathbb{N}\}$ and $V = \{\big((i,j),(i,j+1)\big) \mid i,j \in \mathbb{N}\}$ are binary relations.

Let $\mathfrak{A} = (A, H, V)$ and $\mathfrak{B} = (B, H, V)$ be $\{H, V\}$-structures, where H and V are binary relations. The structure $\mathfrak{A}$ is *homomorphically embeddable* into $\mathfrak{B}$, if there is a homomorphism $h : A \to B$ defined in the usual way.

Definition 11. A structure $\mathfrak{G} = (G, H, V)$ is called *grid-like*, if there exists a homomorphism from $\mathfrak{G}_{\mathbb{N}}$ to $\mathfrak{G}$, i.e., $\mathfrak{G}_{\mathbb{N}}$ is homomorphically embeddable into $\mathfrak{G}$.

Let $\mathfrak{G}$ be a $\{H, V\}$-structure with two binary relations H and V. We say that H is *complete over* V, if $\mathfrak{G}$ satisfies the formula $\forall xyzt(\,(\,Hxy \wedge Vxt \wedge Vyz\,) \to Htz\,)$.

The following lemma is from [48]. Note that FO^2 is contained in U_1.

Lemma 9 ([48])**.** *Let $\mathfrak{G} = (G, H, V)$ be a structure satisfying the FO^2-axiom $\forall x(\,\exists y Hxy \wedge \exists y Vxy\,)$. If H is complete over V, then $\mathfrak{G}$ is grid-like.*

Let $\mathcal{D}$ be a domino system, and let $(P_d)_{d \in D}$ be a set of unary relation symbols. Assume that there is a $\mathcal{D}$-tiling of $\mathbb{N} \times \mathbb{N}$. The correctness of the $\mathcal{D}$-tiling can be expressed by the FO^2-sentence $\varphi_{\mathcal{D}} := \forall x(\bigvee_d P_d x \wedge \bigwedge_{d \neq d'} \neg(P_d x \wedge P_{d'} x)\,) \wedge \forall xy(\,Hxy \to \bigvee_{(d,d') \in H_{do}}(\,P_d x \wedge P_{d'} y\,)\,) \wedge \forall xy(\,Vxy \to \bigvee_{(d,d') \in V_{do}}(\,P_d x \wedge P_{d'} y\,)\,)$.

Lemma 10. *Let $\mathcal{D}$ be a domino system, and let $\mathcal{G}$ be a class of grid-like structures such that $\mathfrak{G}_{\mathbb{N}} \in \mathcal{G}$. Then there exists a $\mathcal{D}$-tiling of $\mathbb{N} \times \mathbb{N}$ iff there exists $\mathfrak{G} \in \mathcal{G}$ that can be expanded to $\mathfrak{G}' = (G, H, V, (P_d)_{d \in D})$ such that $\mathfrak{G}' \models \varphi_{\mathcal{D}}$.*

Proof. Assume first that there exists a $\mathcal{D}$*-tiling* of $\mathbb{N} \times \mathbb{N}$. Then, as $\mathfrak{G}_{\mathbb{N}} \in \mathcal{G}$, we expand $\mathfrak{G}_{\mathbb{N}}$ to $\mathfrak{G}'_{\mathbb{N}} = (\mathbb{N} \times \mathbb{N}, H, V, (P_d)_{d \in D})$ in the obvious way, whence $\mathfrak{G}'_{\mathbb{N}} \models \varphi_{\mathcal{D}}$.

Assume then that there exists $\mathfrak{G} \in \mathcal{G}$ that can be expanded to $\mathfrak{G}' = (G, H, V, (P_d)_{d \in D})$ such that $\mathfrak{G}' \models \varphi_{\mathcal{D}}$. As $\mathfrak{G}$ is grid-like, it follows from

Definition 11 that there is a homomorphism $h : \mathfrak{G}_{\mathbb{N}} \to \mathfrak{G}$. We define $\tau : \mathbb{N} \times \mathbb{N} \to D$ such that

$$\tau(i, j) = d, \text{ if } h(i, j) \in P_d$$

for some $d \in D$. Now the mapping τ is a $\mathcal{D}$-*tiling* of $\mathbb{N} \times \mathbb{N}$. $\square$

Proof of Theorem 10. Let $\tau = \{H, V\}$. Recall that the standard grid $\mathfrak{G}_{\mathbb{N}}$ is a τ-structure. Let $\tau' = \tau \cup \{<_1, <_2, N\}$, where $<_1$ and $<_2$ are binary symbols and N is a 4-ary symbol. Let us first informally outline the proof. First the standard grid $\mathfrak{G}_{\mathbb{N}}$ is expanded to τ'-structure $\mathfrak{G}'_{\mathbb{N}}$. Expanding $\mathfrak{G}_{\mathbb{N}}$ to $\mathfrak{G}'_{\mathbb{N}}$ amounts to describing how the new symbols $<_1$, $<_2$, and N are interpreted in $\mathfrak{G}'_{\mathbb{N}}$. A fragment of the intended structure can be seen in Figure 6.1. Then we axiomatize some important properties of $\mathfrak{G}'_{\mathbb{N}}$ such that the structures that interpret $<_1$ and $<_2$ as linear orders and satisfy the axioms, resemble $\mathfrak{G}'_{\mathbb{N}}$ closely enough. Now, let $\mathcal{G}$ be the class of τ-reducts of τ'-structures that interpret $<_1$ and $<_2$ as linear orders and satisfy the axioms. In particular, $\mathfrak{G}_{\mathbb{N}}$ is in $\mathcal{G}$. We show that every structure in $\mathcal{G}$ satisfies the local criterion that H is complete over V. It will then follow from Lemma 9 that every structure in $\mathcal{G}$ is grid-like. Then the undecidability of the general satisfiability problem for $\mathrm{U}_1[<_1, <_2]$ follows from Lemma 10.

We now go to the details of the proof. We define the τ'-expansion $\mathfrak{G}'_{\mathbb{N}}$ of $\mathfrak{G}_{\mathbb{N}}$ as follows. The linear order $<_1$ follows a lexicographical order such that for all $(i, j), (i', j') \in \mathbb{N}^2$, we have $(i, j) <_1 (i', j')$ if and only if $j < j'$ or ($j = j'$ and $i < i'$). In the linear order $<_2$, the roles of i and j are swapped, i.e., for all $(i, j), (i', j') \in \mathbb{N}^2$, we have $(i, j) <_2 (i', j')$ if and only if $i < i'$ or ($i = i'$ and $j < j'$).

The relation N is defined as follows. For all points a, b, c, d in $\mathbb{N}^2$, we have $Nabdc$ if and only if Hab, Hcd, Vac, and Vbd; see Figure 6.1.

Next we define a few auxiliary formulae. For $i \in \{1, 2\}$, let

$$x \leq_i y := x = y \vee x <_i y.$$

Define also

$$\sigma_i(x, y, z) := x <_i y \wedge (z \leq_i x \vee y \leq_i z).$$

We are now ready to give the desired axioms defining a class of τ'-structures. Let η be the conjunction of the following sentences.

$\eta_G \;= \forall x(\,\exists y Hxy \wedge \exists y Vxy\,)$.

$\eta_H \;= \forall xyz\big(\,Hxy \to \sigma_1(x,y,z)\,\big)$. Together with the previous axiom, this axiom forces H to be a kind of an "induced successor relation" of the linear order $<_1$. It is worth noting that H is subject to the uniformity condition of U_1, i.e., H cannot be used freely in quantifier-free $U_1[<_1,<_2]$-formulae, but the order symbols $<_1, <_2$ can.

$\eta_V \;= \forall xyz\big(\,Vxy \to \sigma_2(x,y,z)\,\big)$. This is analogous to η_H.

$\eta_{N\exists} \;= \forall x \exists yzt(\,Nxyzt\,)$. This axiom states that each point is a first coordinate in some 4-tuple in N. We call the 4-tuples in N *quasi-squares.*

$\eta_{N\forall} \;= \forall xyztu\big(\,Nxyzt \to (\sigma_1(x,y,u) \wedge \sigma_2(x,t,u) \wedge \sigma_2(y,z,u) \wedge \sigma_1(t,z,u))\,\big)$. The points of the quasi-squares are connected via the induced successors of $<_1$ and $<_2$; see the dash-dotted curves representing tuples in N, Figure 6.1.

Thus we have $\eta := \eta_G \wedge \eta_H \wedge \eta_V \wedge \eta_{N_\exists} \wedge \eta_{N_\forall}$. It is readily checked that the expansion $\mathfrak{G}'_{\mathbb{N}}$ of the standard grid $\mathfrak{G}_{\mathbb{N}}$ satisfies the sentence η. Let

$$\mathcal{G} = \{\mathfrak{G}' \upharpoonright \tau \mid \mathfrak{G}' \text{ is } \tau'\text{-model s.t. } <_1^{\mathfrak{G}'} \text{ and } <_2^{\mathfrak{G}'} \text{ are linear orders and } \mathfrak{G}' \models \eta\}.$$

Next we need to show that every structure $\mathfrak{G} \in \mathcal{G}$ is grid-like. This can be done by applying Lemma 9: as every structure $\mathfrak{G} \in \mathcal{G}$ satisfies η_G, it suffices to show that for every structure $\mathfrak{G} \in \mathcal{G}$, H is complete over V.

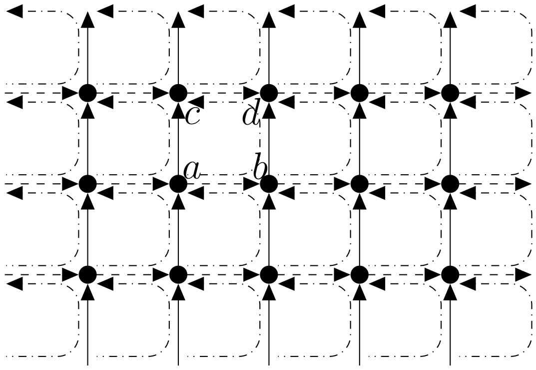

Figure 6.1: A finite fragment of the intended structure. The dashed arrows represent the H-relations and the solid arrows the V-relations. The dash-dotted curves represent the N-relations, e.g. Nabdc.

To show that H is complete over V in every structure in $\mathcal{G}$, let $\mathfrak{G}'$ be a τ'-structure interpreting $<_1$ and $<_2$ as linear orders and satisfying η. For convenience, for $i \in \{1, 2\}$, let

$$\beta_i(x, y) := \forall z\,(\sigma_i(x, y, z)).$$

Let $a \in G'$. From η_G, we get points $b, c, d \in G'$ such that

$$Hab \wedge Vac \wedge Vbd.$$

As $Hab \wedge Vac \wedge Vbd$, we conclude that

$$\beta_1(a, b) \wedge \beta_2(a, c) \wedge \beta_2(b, d)$$

from η_H and η_V. From $\eta_{N\exists}$, we get $Nab'd'c'$ for some $b', c', d' \in G'$. As $Nab'd'c'$, we conclude that

$$\beta_1(a, b') \wedge \beta_2(a, c') \wedge \beta_2(b', d') \wedge \beta_1(c', d')$$

from $\eta_{N\forall}$. The following claim is clear.

Claim. If $\beta_1(a, b) \wedge \beta_1(a, b')$, then $b = b'$.

As $\beta_1(a, b) \wedge \beta_1(a, b')$, it follows from the claim that $b = b'$. We then conclude similarly that $c = c'$ and $d = d'$ (recalling that $b = b'$). From η_G, we get a point $d'' \in G'$ such that Hcd'' and then conclude that $\beta_1(c, d'')$ from η_H. Furthermore, as $\beta_1(c', d')$, $c = c'$ and $d = d'$, we have $\beta_1(c, d) \wedge \beta_1(c, d'')$. Now, analogously to the claim, we have $d = d''$ (See Figure 6.2). Therefore, as Hcd'', we have Hcd.

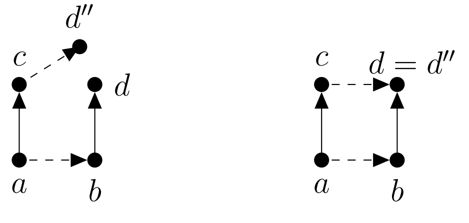

Figure 6.2: H is complete over V

Let $\mathfrak{G} := \mathfrak{G}' \upharpoonright \tau$. Thus for $\mathfrak{G} \in \mathcal{G}$, it holds that H is complete over V. Now it follows from Lemma 9 that $\mathfrak{G}$ is grid-like.

As $\mathfrak{G}'_{\mathbb{N}} \models \eta$, the standard grid $\mathfrak{G}_{\mathbb{N}}$ is also in $\mathcal{G}$. It now follows from Lemma 10 that the (general) satisfiability problem for $\mathrm{U}_1[<_1, <_2]$ over structures with linear orders $<_1$ and $<_2$ is undecidable. $\square$

Theorem 12. *The finite satisfiability problem for* $\mathrm{U}_1[<_1, <_2]$ *is undecidable.*

Proof. This follows from the fact that (finite case) $\mathrm{FO}^2(<_1, +1_1, <_2, +1_2)$ is undecidable and the fact that $\mathrm{U}_1[<_1, <_2]$ can express the successors of $<_1$ and $<_2$ as shown in the general case. $\square$

Chapter 7

Conclusion

We have shown that U_1 is NExpTime-complete over ordered, well-ordered and finite ordered structures. To contrast these results, we have established that $\mathrm{U}_1[<_1, <_2]$ is undecidable. The results here are the first results concerning U_1 with built-in linear orders. Several open problems remain, e.g., investigating U_1 with combinations of equivalence relations and linear orders. Such results would contribute in a natural way to the active research program concerning FO^2 with built-in relations and push the field towards investigating frameworks with relation symbols of arbitrary arity.

While various interesting research directions remain in the field of first-order fragments, it would also make sense—as suggested in [40, 42]—to expand the related studies into fragments of the Turing complete logic of [40].

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