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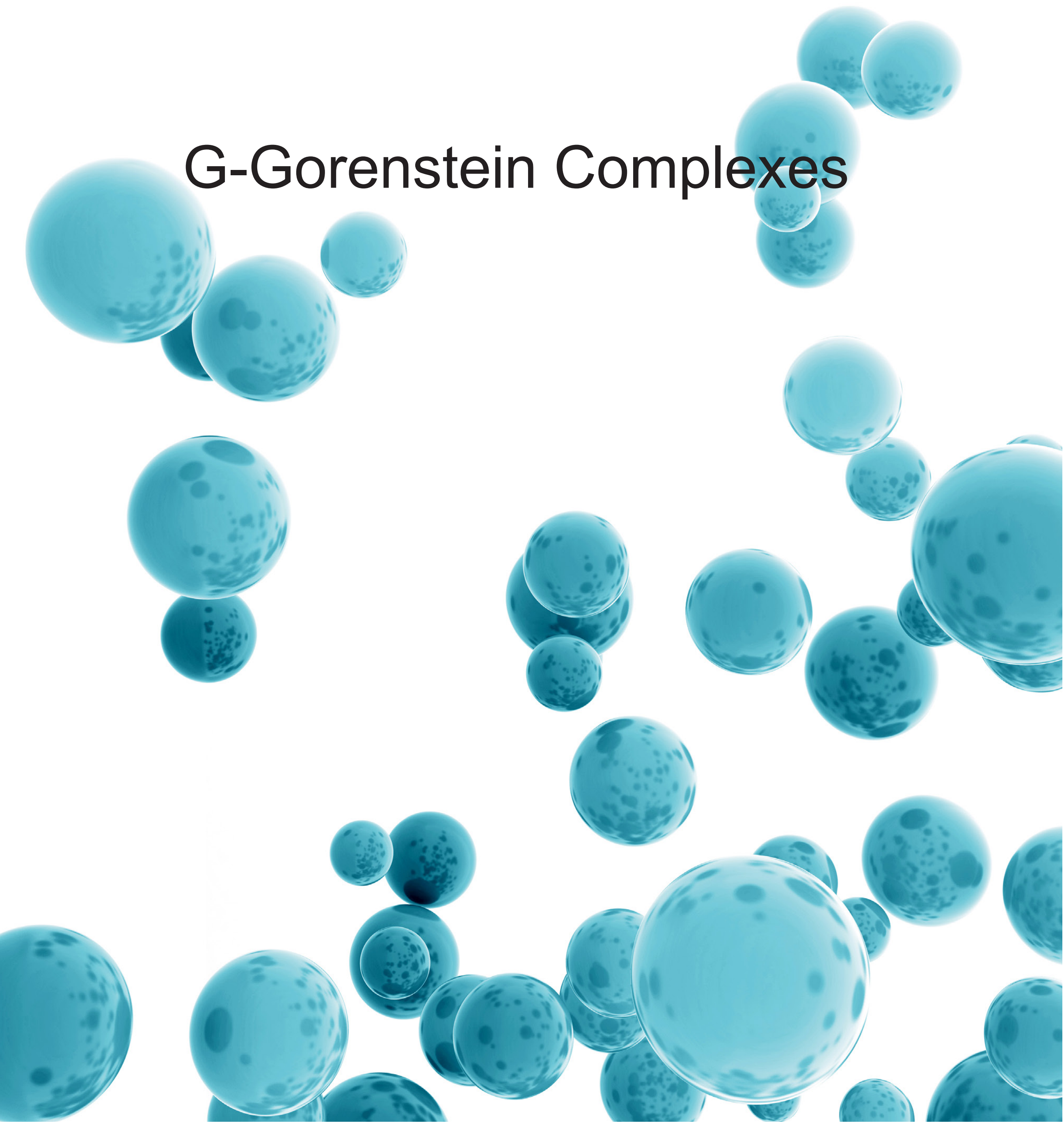
MARYAM AKHAVIN

G-Gorenstein Complexes



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ACADEMIC DISSERTATION
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For “Is” and “Is-not” thought with Rule and Line
And “Up” and “Down” by Logic I define,
Of all that one should care to fathom,
Was never deep in anything but−−Wine.

O. Khayyam (1048–1131)

Abstract

The aim of this thesis is to present in the context of Gorenstein homological algebra the notion of a "G-Gorenstein complex" as the counterpart of the classical notion of a Gorenstein complex. We investigate the structure of a G-Gorenstein complex. We will also find out in which extent classical results about Gorenstein complexes generalize to this case. We establish equivalences between the category of G-Gorenstein complexes of a fixed dimension and the G-class of modules. In particular, the first category turns out to be equivalent with a category of Cousin complexes whose terms are Gorenstein injective and homology bounded and finitely generated.

One of our main tools is the notion of the canonical module of a complex. We consider Serre's conditions for a complex and study their relationship to the local cohomology of the canonical module and its ring of endomorphisms. We characterize complexes satisfying Serre's conditions in terms of the homology of their Cousin complex.

Key words and phrases. G-Gorenstein complex, Serre's conditions, Gorenstein complex, Cohen-Macaulay complex, modules of deficiency, Cousin complex.

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Last but not least, I am deeply indebted to my motherland, the land of my roots, where I was aided by many people and granted many opportunities throughout my education. This dissertation, together with my sincerest respect, is dedicated to “all who are in love with our great and ancient Iran”.

Tampere, October 2016

Maryam Akhavin

Contents

Introduction

The language of homological algebra is eminently categorical. Gorenstein homological algebra is the relative version of homological algebra, where classical injective and projective modules are replaced by Gorenstein injective and Gorenstein projective modules, respectively. The study of this theory goes back to the work of Auslander and Bridger in [4]. They introduced the notion of a Gorenstein dimension of a finitely generated module over a commutative Noetherian ring. Gorenstein dimension characterizes Gorenstein rings like projective dimension does for regular rings.

The purpose of this thesis is to introduce an analogue of the notion of a Gorenstein complex in the context of Gorenstein homological algebra. We follow thereby the maxim "Every result in classical homological algebra has a counterpart in Gorenstein homological algebra" suggested by Holm in [38]. In particular, we can extend several properties of Gorenstein modules proved by Sharp to the case of G-Gorenstein complexes. Our work also generalizes the earlier work of Aghajani and Zakeri on G-Gorenstein modules (see [1], and also [42]).

Gorenstein complexes, defined in [36], play a crucial role in Grothendieck's theory of duality in the derived category of sheaves of modules over a locally Noetherian scheme. He described an equivalence between the category of Gorenstein complexes and the category of Cousin complexes whose terms are injective and cohomology is bounded (see e.g. [22, Theorem 3.1.3]). In fact, this equivalence is a restriction of an equivalence between the category of Cohen-Macaulay complexes and the category of Cousin complexes discovered by Suominen (see [59, Theorem 3.9]). Sharp initiated in [55] the study of Gorenstein modules from the point of view of commutative algebra. His way to characterize Cohen-Macaulay modules and Gorenstein modules in terms of their Cousin complexes (see [54] and [55]) reflects the above ideas. Gorenstein complexes have also been studied by Roberts in [49].

We will now describe our results in more detail. Let R be a commutative Noetherian ring. The derived category of bounded complexes of R-modules with finitely generated homology is denoted by $D_b^f(R)$. Generalizing the definition of a Gorenstein complex given in [36] we define a complex $M \in D_b^f(R)$ to be G-Gorenstein if it is Cohen-Macaulay and the local cohomology modules $\mathrm{H}^i_{pR_p}(M_p)$ are Gorenstein injective for all $i \in \mathbb{Z}$ and prime ideals $p \in \operatorname{Spec} R$.

From now on we assume that (R, m) is a local ring admitting a dualizing complex. We denote by D_R the dualizing complex normalized with $\sup D_R = \dim R$. It comes out in Proposition 3.1.7 that the G-Gorensteineness of M is equivalent to $\dim_R M = \operatorname{depth}_R M = \operatorname{Gid}_R M$. This is further equivalent to M being of finite Gorenstein injective dimension and having $\operatorname{depth}_R M = \operatorname{depth} R - \inf M$. Recall the open question concerning the analogue of Bass's theorem in Gorenstein homological algebra: Does the existence of an R-module of finite Gorenstein injective dimension imply that R is Cohen-Macaulay (see [19, Question 3.26])? Regarding this question we point out in Corollary 3.2.10 that if R satisfies Serre's condition (S_2), then the existence of a G-Gorenstein module always implies that R is Cohen-Macaulay.

If $M \in D_b^f(R)$ is a complex of finite Gorenstein injective dimension, then the biduality morphism $L \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(L, M), M)$ can not be an isomorphism for $L \in D_b^f(R)$ unless M is a dualizing complex. This was observed by Christensen in [15, Proposition 8.4]. Nevertheless, it turns out that if M is G-Gorenstein, then biduality preserves depth. In fact, we prove in our first main result Theorem 3.1.13 that among complexes of finite Gorenstein injective dimension G-Gorenstein complexes are characterized by the equality

$$\operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(L, M), M) = \operatorname{depth}_R L$$

for all complexes $L \in D_b^f(R)$ of finite projective or injective dimension.

Let $M \in D_b^f(R)$. Our Theorem 3.2.2, Theorem 3.2.4 and Theorem 3.2.15 show that the following conditions are equivalent:

1) M is a G-Gorenstein complex of dimension t;

2) $M \simeq \sum^{-t} \mathrm{Hom}_R(K, D_R)$ for some $K \in G(R)$;

3) $M \simeq \sum^{-t} D_R \otimes_R N$ for some $N \in G(R)$;

4) $\mathbf{R}\mathrm{Hom}_R(D_R, M) \simeq \sum^{-t} N$ for some $N \in G(R)$;

5) $M \simeq C$ for some $C \in \mathrm{GIcz}(\mathcal{D}^t, R)$.

Here $G(R)$ is the G-class modules, and $\mathrm{GIcz}(\mathcal{D}^t, R)$ denotes the category of Cousin complexes with respect to the filtration $\mathcal{D}^t = (D_i^t)_{i \in \mathbb{Z}}$, defined by

$$D_i^t = \{p \in \operatorname{Spec} R \mid i \leq t - \dim R/p\} \quad (i \in \mathbb{Z}),$$

for which all terms are Gorenstein injective, and the homology is bounded and finitely generated. As usual, the symbol "$\simeq$" indicates an isomorphism in $D(R)$.

Let $D_{t-GGor}(R)$ denote the full subcategory of $D_b^f(R)$ of G-Gorenstein complexes of dimension t. In more abstract terms, we can then say that there is a diagram

$$\begin{array}{ccc}
D_{t-GGor} & \underset{\sum^{-t}\mathbf{R}\mathrm{Hom}_R(-,D_R)}{\overset{\mathrm{H}_{-t}\,\mathbf{R}\mathrm{Hom}_R(-,D_R)}{\rightleftarrows}} & \mathrm{G}(R)^{opp} \\
\mathrm{Id}\ \Big\uparrow\!\Big\downarrow & & \Big\uparrow\!\Big\downarrow\ \mathrm{Hom}_R(-,R) \\
D_{t-GGor} & \underset{\sum^{-t} D_R\otimes_R^L -}{\overset{\mathrm{H}_{-t}\,\mathbf{R}\mathrm{Hom}_R(D_R,-)}{\rightleftarrows}} & \mathrm{G}(R)
\end{array}$$

of equivalences of categories, where the horizontal arrows are quasi-inverses of each other. The diagram is commutative up to canonical isomorphisms. The upper equivalence is the restriction of an equivalence between the full subcategory of $D_b^f(R)$ of Cohen-Macaulay complexes of dimension t and the category of finitely generated R-modules. The latter equivalence was first observed by Yekutieli and Zhang in [61] and later utilized by Lipman, Nayak and Sastry in [47]. The lower equivalence comes from Foxby equivalence

$$\mathrm{A}(R) \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes_R^L -}{\rightleftarrows}} \mathrm{B}(R),$$

between the Auslander and the Bass classes. Moreover, we see that there exists an equivalence of categories

$$\mathrm{GIcz}(\mathcal{D}^t, R) \underset{Q(-)}{\overset{\mathrm{E}_{\mathcal{D}^t}(-)}{\rightleftarrows}} D_{t-GGor}(R)\ .$$

Inspired by the theory of Gorenstein objects in triangulated categories developed by Asadollahi and Salarian in [3], we want to consider G-Gorenstein complexes as Gorenstein objects. Let $t \in \mathbb{Z}$. Set $D = \sum^{-t} D_R$. We look at towers

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & D^{\oplus n_{i+1}} & & \xrightarrow{d_{i+1}} & & D^{\oplus n_i} & & \xrightarrow{d_i} & & D^{\oplus n_{i-1}} & \longrightarrow \cdots \\
\nearrow & & \searrow^{g_{i+1}} & & \nearrow^{f_i} & & \searrow^{g_i} & & \nearrow^{f_{i-1}} & & \searrow \\
M_{i+1} & \dashleftarrow & & M_i & & \dashleftarrow & M_{i-1} & & \dashleftarrow & & M_{i-2}
\end{array}$$

of exact triangles in $D_b^f(R)$, where $d_i = f_{i-1}g_i$. It then comes out in Theorem 3.2.27 that a complex $M \in D_b^f(R)$ is a G-Gorenstein complex of dimension t if and only if $M \simeq M_i$ for some i in a tower of triangles, where the triangles are both $\mathrm{Hom}_{D(R)}(D,-)$-exact and $\mathrm{Hom}_{D(R)}(-,D)$-exact (see Definition 3.2.21). In Corollary 3.2.28 we look at the special case where R is Cohen-Macaulay with the canonical module K_R. Then a finitely generated R-module M is G-Gorenstein if and only if M appears as a kernel in an exact complex of R-modules

$$\cdots \to K_R^{\oplus n_{i+1}} \xrightarrow{d_{i+1}} K_R^{\oplus n_i} \xrightarrow{d_i} K_R^{\oplus n_{i-1}} \to \cdots$$

which is both $\mathrm{Hom}_R(K_R,-)$-exact and $\mathrm{Hom}_R(-,K_R)$-exact. This means that G-Gorenstein modules are exactly the K_R-Gorenstein projective modules in the sense of [25].

The new notion of a module of deficiency of a complex is an important tool in this thesis. Generalizing the work of Schenzel in [51], we define for any complex $M \in D_b^f(R)$ and any $i \in \mathbb{Z}$ the i-th module of deficiency K_M^i by setting $K_M^i = \mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(M, D_R))$. The canonical module of M is $K_M = K_M^{\dim_R M}$. The canonical module of a module always satisfies Serre's condition (S_2). This does not necessarily hold for $M \in D_b^f(R)$ even if M is a Cohen-Macaulay complex (see Example 2.1.3 and Example 2.2.16). We will see in Proposition 2.1.6 that

$$\begin{aligned}\mathrm{Ass}_R K_M &= \left\{p \in \mathrm{Supp}_R M \mid \dim R/p = \dim_{R_p} M_p + \inf M_p\right\} \\ &= \bigcup_{i\in\mathbb{Z}} (\mathrm{Ass}_R \mathrm{H}_i(M))_{i+\dim_R M}.\end{aligned}$$

This leads us also to study the concept of Serre's condition for complexes. Given $k \in \mathbb{Z}$, we say that a complex M satisfies Serre's condition (S_k) if

$$\mathrm{depth}_{R_p} M_p \geq \min\left\{k - \inf M_p, \dim_{R_p} M_p\right\}$$

for all prime ideals $p \in \mathrm{Supp}_R M$. It is convenient to consider equidimensional complexes i.e. complexes satisfying the condition

$$\dim_R M = \dim_{R_p} M_p + \dim R/p$$

for all $p \in \mathrm{Supp}_R M$ (see page 11 and Lemma 2.2.4). It then follows from Proposition 2.3.12 that (S_k) is equivalent to the natural homomorphism

$$\mathrm{Ext}_R^{-i}(M,M) \to K^{i+\dim_R M}_{M\otimes^L_R K_M} \tag{1}$$

being bijective for all $i > -k+2$, and injective for $i = -k+1$. Note that $K_{M\otimes^L_R K_M} \cong \operatorname{Hom}_R(K_M, K_M)$. It makes also sense, for any $l \in \mathbb{Z}$ to look at the condition $(S_{k,l})$ saying that

$$\operatorname{depth}_{R_p} M_p \geq \min\left\{k - l, \dim_{R_p} M_p\right\}$$

for all prime ideals $p \in \operatorname{Supp}_R M$. Observe that Serre's condition (S_k) implies $(S_{k,\sup M})$. We reprove a result of Lipman, Nayak and Sastry ([45, Proposition 9.3.5]) saying that the Cousin complex of M

$$E_{\mathcal{D}^{\dim_R M}}(M) \cong \sum\nolimits^{-\dim_R M} \operatorname{Hom}_R(K_M, D_R).$$

In particular, there is a natural morphism $h_M\colon M \to E_{\mathcal{D}^{\dim_R M}}(M)$. It then comes out in Corollary 2.4.2 that a complex M satisfying condition $(S_{k,l})$ is equivalent to the map $\mathrm{H}_i(h_M)$ being bijective for $i \geq l-k+2$ and injective for $i = l-k+1$. Recalling that $E_{\mathcal{D}^{\dim_R M}}(M)$ is always Cohen-Macaulay, this shows that in order to see how close M is to being Cohen-Macaulay, it is enough to know how close h_M is to an invertible morphism.

We then want to understand the relationship between the Cousin complex of M and that of $\mathrm{H}_{\sup M}(M)$. Assume that M is an equidimensional complex satisfying condition (S_1). Then $\operatorname{Ass}_R \mathrm{H}_{\sup M}(M) = \operatorname{Assh}_R \mathrm{H}_{\sup M}(M)$. Moreover, if M satisfies condition (S_2), then $\mathrm{H}_{\sup M}(M) \cong K_{K_M}$ and

$$\operatorname{Hom}_{D(R)}(M, M) \cong \operatorname{Hom}_R(K_M, K_M)$$

(see Corollary 2.3.16). Set $M^\dagger = \operatorname{Hom}_R(M, D_R)$. Suppose that $\sup M_p = \sup M$ for all $p \in \operatorname{Supp}_R M$. If $M^\dagger$ satisfies Serre's condition (S_2), then $K_M \cong K_{\mathrm{H}_s(M)}$, where $s = \sup M$. More precisely, it comes out in Corollary 2.4.4 that

$$E_{\mathcal{D}^{\dim_R M}}(M) \cong \sum\nolimits^{s} E_{\mathcal{D}^{\dim_R \mathrm{H}_s(M)}}(\mathrm{H}_s(M)).$$

Finally, in Proposition 3.2.17 we look at the special case where M may not be G-Gorenstein but its Cousin complex is G-Gorenstein. Suppose that R satisfies Serre's condition (S_2). Assume also that M is equidimensional, and that $\operatorname{Supp}_R \mathrm{H}_s(M) = \operatorname{Spec} R$ where $s = \sup M$. If either, both M and K_M satisfy (S_2) or both $M^\dagger$ and $\mathrm{H}_s(M)$ satisfy (S_2), we can show that $E_{\mathcal{D}^{\dim_R M}}(M)$ is a complex of Gorenstein injective modules if and only if $\mathrm{H}_s(M) \cong K_F$ for some $F \in G(R)$. This generalizes [23, Theorem 3.3] of Dibaei.

We will now describe the contents of this thesis. In Chapter 1 we recall some preliminaries of hyperhomological algebra and Gorenstein homological algebra. In Chapter 2 we investigate properties of modules of deficiency of a complex. We also define Serre's conditions for complexes. We then study the Cousin functor for complexes. Finally, in Chapter 3 we turn to consider G-Gorenstein complexes.

Chapter 1

Preliminaries

In this chapter we fix some notation and recall some basic facts and theorems which we shall often use in the sequel.

We will always use the letter "R" to denote a commutative Noetherian ring with non-zero identity. In the case R is local, m and k denote the unique maximal ideal and the residue field R/m, respectively. The set of all prime ideals of R is denoted by $\operatorname{Spec} R$.

1.1 Categories of Complexes

Throughout this thesis we work within the derived category $D(R)$ of R-modules. Acquaintance with derived categories is assumed. As usual, $C(R)$ denotes the category of complexes of R-modules. The localization functor is $Q\colon K(R) \to D(R)$ where $K(R)$ is the homotopy category. We use homological grading so that the objects of $D(R)$ are complexes of R-modules of the form

$$M\colon \qquad \ldots \to M_{i+1} \stackrel{d_{i+1}}{\to} M_i \stackrel{d_i}{\to} M_{i-1} \stackrel{d_{i-1}}{\to} \ldots \quad .$$

The derived category is triangulated, the suspension functor $\sum$ being defined by the formulas $(\sum M)_n = M_{n-1}$ and $d_n^{\sum M} = -d_{n-1}$. We use the symbol "$\simeq$" for isomorphisms in $D(R)$. For any $i \in \mathbb{Z}$, the i-th homology functor is denoted by $\mathrm{H}_i(-)$. The homological supremum and infimum of a complex M are defined by:

$$\sup M = \sup \{i \in Z \mid H_i(M) \neq 0\}, \quad \inf M = \inf \{i \in Z \mid H_i(M) \neq 0\}.$$

The *amplitude* is

$$\operatorname{amp} M = \sup M - \inf M.$$

We denote by D_+ and D_- the full subcategories of $D(R)$, for which $\inf M > -\infty$ and $\sup M < \infty$, respectively. We use the subscript "b" to denote the homological boundness and the superscript "f" to denote the homological finiteness. So the full subcategory of $D(R)$ consisting of complexes with finitely generated homology modules is denoted by $D^f(R)$. We use the standard notations $- \otimes_R^L -$ and $\mathbf{R}\mathrm{Hom}_R(-,-)$ for the derived tensor product functor and the derived homomorphism functor, respectively.

Let $M, N, K \in D(R)$. Then the following functorial isomorphisms exist in $D(R)$:

(Adjointness)

$$\mathbf{R}\mathrm{Hom}_R(M \otimes_R^L N, L) \simeq \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_R(N, L)); \tag{1.1}$$

(Swap)

$$\mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_R(N, L)) \simeq \mathbf{R}\mathrm{Hom}_R(N, \mathbf{R}\mathrm{Hom}_R(M, L)). \tag{1.2}$$

Moreover, there are the following functorial morphisms:

(Tensor evaluation)

$$\alpha_{M,N,L}\colon \mathbf{R}\mathrm{Hom}_R(M, N) \otimes_R^L L \longrightarrow \mathbf{R}\mathrm{Hom}_R(M, N \otimes_R^L L); \tag{1.3}$$

(Homomorphism evaluation)

$$\beta_{M,N,L}\colon M \otimes_R^L \mathbf{R}\mathrm{Hom}_R(N, L) \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, N), L). \tag{1.4}$$

The morphism $\alpha_{M,N,L}$ is an isomorphism, if $M \in D_b^f(R)$, $N \in D_b(R)$ and either M has finite projective dimension or L has finite flat dimension, whereas the morphism $\beta_{M,N,L}$ is an isomorphism, when $M \in D^f(R)$, $N \in D_b(R)$ and either M is of finite projective dimension or L is of finite injective dimension.

Krull Dimension and Support

Let $M \in D_b(R)$, and let $p \in \operatorname{Spec} R$. The localization of M at p is defined by $M_p = R_p \otimes_R M$. The following inequalities hold:

$$\sup M_p \leq \sup M \qquad \text{and} \qquad \inf M \leq \inf M_p.$$

Furthermore, the *support* of a complex $M \in D(R)$ is the set

$$\mathrm{Supp}_R\, M = \{p \in \operatorname{Spec} R \mid M_p \not\simeq 0\}.$$

The *Krull dimension* of M is

$$\dim_R M = \sup\{\dim R/p - \inf M_p \mid p \in \mathrm{Supp}_R\, M\}. \tag{1.5}$$

One also has for any $s \in \mathbb{Z}$,

$$\dim_R \sum^s M = -s + \dim_R M.$$

It has been shown in [18, Lemma 6.3.5]) that

$$\dim_R M = \sup\{\dim_R \mathrm{H}_i(M) - i \mid i \in \mathbb{Z}\}. \tag{1.6}$$

Obviously,

$$-\inf M \leq \dim_R M \leq \dim R - \inf M. \tag{1.7}$$

Depth and Width

Let (R, m) be a local ring, and let $M \in D_b(R)$. One defines the *depth* and the *width* of M by the formulas

$$\operatorname{depth}_R M = -\sup \mathbf{R}\mathrm{Hom}_R(k, M) \quad \text{and} \quad \operatorname{width}_R M = \inf k \otimes^L_R M,$$

respectively. More generally, for any ideal $I \subseteq R$

$$\operatorname{depth}_R(I, M) = \inf\left\{\operatorname{depth}_{R_p} M_p \mid p \in \mathbb{V}(I)\right\} \tag{1.8}$$

where $\mathbb{V}(p)$ denotes the set of all prime ideals containing I (see [40, Proposition 5.4]). The following inequalities hold:

$$\operatorname{depth}_R M \geq -\sup M \quad \text{and} \quad \operatorname{width}_R M \geq \inf M.$$

The first inequality turns to an equality if and only if $m \in \mathrm{Ass}_R \mathrm{H}_{\sup M}(M)$, while the second one is an equality if and only if $k \otimes \mathrm{H}_{\inf M}(M) \neq 0$ (see [18, Observation 5.2.2 and Observation 5.2.5]. Finally, note that for any $s \in \mathbb{Z}$,

$$\operatorname{depth}_R \sum^s M = -s + \operatorname{depth}_R M.$$

Equidimensionality

Let $M \in D_b(R)$. Recall from [16] that a prime ideal $p \in \mathrm{Supp}_R M$ is called an *associated prime* of M if $\operatorname{depth}_{R_p} M_p = -\sup M_p$. The set of all associated primes of M is denoted by $\mathrm{Ass}_R M$. Furthermore, if $M \not\simeq 0$, then by [14, A.6.1.2],

$$p \in \mathrm{Ass}_R \mathrm{H}_{\sup M}(M) \quad \text{if and only if} \quad \operatorname{depth}_{R_p} M_p = -\sup M. \tag{1.9}$$

A prime ideal $p \in \mathrm{Supp}_R M$ is called an *anchor prime* of M, if $\dim_{R_p} M_p = -\inf M_p$. The set of all anchor primes of M is denoted by $\mathrm{Anc}_R M$. The anchor primes play the role of minimal primes for complexes (see [17]). Set also

$$\mathrm{W}_0(M) = \{p \in \operatorname{Supp}_R M \mid \dim_R M - \dim R/p + \inf M_p = 0\}\,.$$

One has

$$\operatorname{Min}\operatorname{Supp}_R(M) \subseteq \operatorname{Anc}_R M \qquad \text{and} \qquad \mathrm{W}_0(M) \subseteq \operatorname{Anc}_R(M).$$

In the case (R, m) is a local ring we say that the complex $M \in D_b^f(R)$ is *equidimensional*, when $\mathrm{W}_0(M) = \operatorname{Anc}_R(M)$.

Local cohomology

For any ideal $J \subset R$, the derived section functor is denoted by $\mathbf{R}\Gamma_J(-)$. As usual, the i-th hypercohomology functor is $\mathrm{H}^i_J(-) = \mathrm{H}_{-i}(\mathbf{R}\Gamma_J(-))$. Furthermore, when (R, m) is a local ring, the following equalities hold (see e.g. [31, p. 8, 2.4]):

$$-\inf \mathbf{R}\Gamma_m(M) = \dim_R M \quad \text{and} \quad -\sup \mathbf{R}\Gamma_m(M) = \operatorname{depth}_R M. \tag{1.10}$$

1.2 Dualities

In this section we review some basic definitions and known results about dualizing complexes and recall their application in the local duality. Our main reference here is [36].

Dualizing Complexes

Let R be a ring. A complex $C \in D_b^f(R)$ is said to be a *semi-dualizing complex*, if the homothety morphism $C\colon R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism. A semi-dualizing complex $D \in D_b^f(R)$ is called a *dualizing complex* if D has finite injective dimension.

Let (R, m) be a local ring. Then the following statements hold:

- If a dualizing complex exists, then R is catenary;
- Every two dualizing complexes are isomorphic up to a suspension;
- Let $D \in D_b^f(R)$. Then D is a dualizing complex if and only if for some $n \in \mathbb{Z}$,
$$\mathbf{R}\mathrm{Hom}_R(k, D) \simeq \textstyle\sum^n k.$$

Moreover, a dualizing complex is said to be *normalized*, when $n = 0$. In this case

$$D_i = \bigoplus_{\dim R/p=i} E_R(R/p).$$

Here $E_R(R/p)$ denotes the injective envelope of R/p (see [36, V.3.1, V.3.4 and V.7.2]). From now on, we denote the normalized dualizing complex by D_R. If $M \in D_b^f(R)$, the *dagger dual* of M is defined by $M^\dagger = \mathbf{R}\mathrm{Hom}_R(M, D_R)$. We also observe that by [51, Lemma 1.3.3]

$$(M_p)^{\dagger p} \simeq \sum\nolimits^{-\dim R/p}(M^\dagger)_p. \tag{1.11}$$

Here the dagger dual on the left-hand side is taken with respect to the normalized dualizing complex of the localization R_p.

Local Duality

Let (R, m) be a local ring admitting a dualizing complex. For $M \in D_b^f(R)$, the local duality (see [36, V.6.2]) says that

$$\mathbf{R}\Gamma_m(M) \simeq \mathrm{Hom}_R(M^\dagger, E_R(k)). \tag{1.12}$$

Taking the homology gives

$$\mathrm{H}^i_m(M) = \mathrm{Hom}_R(H_i(M^\dagger), E_R(k)) \tag{1.13}$$

for all $i \in \mathbb{Z}$.

Dagger Duality

Let (R, m) be a local ring, and let $M \in D_b^f(R)$. It follows from formula (1.12) together with formula (1.10) that

$$\sup M^\dagger = \dim_R M \quad \text{and} \quad \inf M^\dagger = \operatorname{depth}_R M. \tag{1.14}$$

Therefore, $M^\dagger \in D_b^f(R)$. The canonical morphism $M \to M^{\dagger\dagger}$ induces now an isomorphism

$$M \simeq M^{\dagger\dagger}. \tag{1.15}$$

In other words, there is an equivalence of categories

$$D_b^f(R) \underset{(-)^\dagger}{\overset{(-)^\dagger}{\rightleftarrows}} D_b^f(R)\ ,$$

which is called the *dagger duality*.

1.3 Gorenstein Dimensions

In the current section we recall some basic notions of Gorenstein homological algebra. This material is essential for Chapter 3.

Gorenstein Injective Dimension

Recall that an R-module N is called *Gorenstein injective*, if there is an exact complex I of injective R-modules such that the complex $\operatorname{Hom}_R(J, I)$ is exact for every injective R-module J, and that N appears as a kernel in I. For $M \in D_b(R)$, the Gorenstein injective dimension of M, denoted by $\operatorname{Gid}_R M$, is defined as the infimum of all integers n such that there exists a complex I of Gorenstein injective R-modules for which $I \simeq M$ in $D(R)$, and $I_i = 0$ for $i > -n$. For details, see [20, 1.8]. Note that $\operatorname{Gid}_R M \leq \inf M$. We also have

$$\operatorname{Gid}_R \textstyle\sum^s M = -s + \operatorname{Gid}_R M$$

for any $s \in \mathbb{Z}$.

Gorenstein Projective Dimension

The definition of a *Gorenstein projective* module is dual to that of the Gorenstein injective one. For $M \in D_b(R)$, the Gorenstein projective dimension of M, denoted by $\operatorname{Gpd}_R M$, is defined as the infimum of all integers n such that there exists a complex P of Gorenstein projective R-modules for which $M \simeq P$, and $P_i = 0$ if $i > n$. We have $\operatorname{Gpd}_R M \geq \sup M$. Observe that

$$\operatorname{Gpd}_R \textstyle\sum^s M = s + \operatorname{Gpd}_R M$$

for any $s \in \mathbb{Z}$.

Gorenstein Flat Dimension

An R-module N is called *Gorenstein flat*, if there is an exact complex F of flat R-modules such that the complex $J \otimes_R F$ is exact for every injective R-module J, and that N appears as a kernel in F. The notion of Gorenstein flat dimension of a complex $M \in D_b(R)$ is defined analogously to the previous Gorenstein dimensions.

G-class of Modules

The *G-class* of modules, denoted by $G(R)$, consists of all finitely generated Gorenstein projective, or, equivalently, Gorenstein flat R-modules.

Auslander class and Bass class

Let R be a ring admitting a dualizing complex D. Consider the pair of adjoint functors $(D \otimes_R^L -, \mathbf{R}\mathrm{Hom}_R(D, -))$. Let

$$\epsilon_-^D \colon D \otimes_R^L \mathbf{R}\mathrm{Hom}_R(D, -) \to \mathrm{Id} \quad \text{and} \quad \gamma_-^D \colon \mathrm{Id} \to \mathbf{R}\mathrm{Hom}_R(D, D \otimes_R^L -)$$

denote the unit and the counit of adjunction, respectively. The *Auslander class* $\mathrm{A}(R)$ and the *Bass class* $\mathrm{B}(R)$ with respect to D are the full subcategories of $D_b(R)$ defined by:

- A complex $M \in D_b(R)$ is in $\mathrm{A}(R)$ if and only if $D \otimes_R^L M \in D_b(R)$ and γ_M^D is an isomorphism;
- A complex $N \in D_b(R)$ is in $\mathrm{B}(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(D, N) \in D_b(R)$ and ϵ_N^D is an isomorphism.

It is now easy to see that we obtain an equivalence of categories

$$\mathrm{A}(R) \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes_R^L -}{\rightleftarrows}} \mathrm{B}(R),$$

which is called the *Foxby Duality*. If $\mathrm{A}^f(R)$ and $\mathrm{B}^f(R)$ denote the restrictions of $D_b^f(R)$ to the categories $\mathrm{A}(R)$ and $\mathrm{B}(R)$, this induces further an equivalence $\mathrm{A}^f(R) \to \mathrm{B}^f(R)$.

Finally, it is an important fact that in the case of a local ring the objects of $\mathrm{A}(R)$ are the complexes $M \in D_b(R)$ of finite Gorenstein projective (or equivalently of finite Gorenstein flat) dimension. Moreover, the dual statement holds for $\mathrm{B}(R)$ (see [20, Theorem 4.4 and Theorem 4.1]).

Chapter 2

Serre's Conditions for Complexes

The aim of this chapter is to generalize the notion of Serre's condition to a complex. This generalization is expected to give a criterion for evaluating how far a complex is from being Cohen-Macaulay.

2.1 Modules of Deficiency of a Complex

In this section we introduce the notion of a module of deficiency of a complex as a technical tool which will be used throughout the rest of this thesis.

In the module case this was done by P. Schenzel in [51].

Definition 2.1.1. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$. For every $i \in \mathbb{Z}$, set $K_M^i = \mathrm{H}_i(M^\dagger)$. The modules K_M^i are called the modules of deficiency of the complex M. Moreover, we set $K_M = K_M^{\dim_R M}$, and say that K_M is the canonical module of M.*

Remark 2.1.2. *Obviously, the modules of deficiency are finitely generated. Using formula* (1.14)*, we get*

$$\operatorname{depth}_R M = \inf \left\{ i \in \mathbb{Z} \mid K_M^i \neq 0 \right\} \quad \text{and} \quad \dim_R M = \sup \left\{ i \in \mathbb{Z} \mid K_M^i \neq 0 \right\}.$$

Example 2.1.3. *Any finitely generated R-module is canonical module of a complex. Indeed, if K is a finitely generated module and $t \in \mathbb{Z}$, set $M = \sum^{-t} K^\dagger$. Since $\dim_R M = t$ by formula* (1.14)*, it now follows by biduality that*

$$K_M = \mathrm{H}_t(\textstyle\sum^t K) = K.$$

Lemma 2.1.4. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$. Then the following statements hold:*

a) $(K_M^i)_p \cong K_{M_p}^{i-\dim R/p}$ *for every* $p \in \operatorname{Supp}_R M$*;*

b) If $p \in \operatorname{Supp}_R M$ *with* $\dim_R M = \dim_{R_p} M_p + \dim R/p$*, then* $K_{M_p} \cong (K_M)_p$.

Proof. a) Using formula (1.11) we get

$$(K_M^i)_p \cong \mathrm{H}_i((M^\dagger)_p) \cong \mathrm{H}_{i-\dim R/p}({M_p}^{\dagger p}) = K_{M_p}^{i-\dim R/p}.$$

b) Part a) immediately implies that

$$K_{M_p} \cong (K_M^{\dim_{R_p} M_p + \dim R/p})_p = (K_M)_p.$$

□

Our next aim is to investigate the associated primes of modules of deficiency. From now on we set

$$(X)_i = \{p \in X \mid \dim R/p = i\}$$

for every $X \subseteq \operatorname{Spec} R$ and all $i \in \mathbb{Z}$.

Lemma 2.1.5. *Let* (R, m) *be a local ring admitting a dualizing complex, and let* $M \in D_b^f(R)$*. Then the following statements hold for all* $i \in \mathbb{Z}$*:*

a) $\dim_R K_M^i \leq i + \sup M$*;*

b) $(\operatorname{Ass}_R K_M^{i-s})_i = (\operatorname{Ass}_R \mathrm{H}_s(M))_i$ *where* $s = \sup M$*;*

c) $(\operatorname{Ass}_R K_M)_i = (\operatorname{Ass}_R \mathrm{H}_{i-\dim_R M}(M))_i$*;*

Proof. a) Using formula (1.6) we have

$$\dim_R M^\dagger = \sup\{\dim_R K_M^i - i \mid i \in \mathbb{Z}\}.$$

Therefore $\dim_R K_M^i \leq i + \dim_R M^\dagger$. This implies the claim, since $\dim_R M^\dagger = \sup M$ by formula (1.14).
b) By a) we have $\dim_R K_M^{i-s} \leq i$. Hence,

$$(\operatorname{Ass}_R K_M^{i-s})_i = (\operatorname{Supp}_R K_M^{i-s})_i.$$

It is then is enough to prove that

$$(\operatorname{Supp}_R K_M^{i-s})_i = (\operatorname{Ass}_R \mathrm{H}_s(M))_i.$$

Take first $p \in (\operatorname{Supp}_R K_M^{i-s})_i$. Then $K_{M_p}^{-s} \neq 0$ by Lemma 2.1.4 a). Therefore $\mathrm{H}_s(\mathbf{R}\Gamma_{pR_p}(M_p)) \neq 0$ implying that

$$s \leq \sup \mathbf{R}\Gamma_{pR_p}(M_p).$$

On the other hand, we have

$$\sup \mathbf{R}\Gamma_{pR_p}(M_p) = -\operatorname{depth}_{R_p} M_p$$

by formula (1.10). It now follows that

$$s \leq \sup \mathbf{R}\Gamma_{pR_p}(M_p) = -\operatorname{depth}_{R_p} M_p \leq \sup M_p \leq s.$$

Therefore, $-\operatorname{depth}_{R_p} M_p = s = \sup M_p$. By formula (1.9) this means that $p \in \operatorname{Ass}_R \mathrm{H}_s(M)$. So

$$(\operatorname{Supp}_R K_M^{i-s})_i \subseteq (\operatorname{Ass}_R \mathrm{H}_s(M))_i.$$

Conversely, let $p \in (\operatorname{Ass}_R \mathrm{H}_s(M))_i$. Then $\operatorname{depth}_{R_p} M_p = -s$ by formula (1.9). Hence $-\sup \mathbf{R}\Gamma_{pR_p}(M_p) = -s$ implying that $K_{M_p}^{-s} \neq 0$. By Lemma 2.1.4 a) this means that $(K_M^{i-s})_p \neq 0$. Thus $p \in \operatorname{Supp}_R K_M^{i-s}$. Therefore

$$(\operatorname{Ass}_R \mathrm{H}_s(M))_i \subseteq (\operatorname{Supp}_R K_M^{i-s})_i.$$

c) This follows by applying b) to $M^\dagger$, because $K_{M^\dagger} \cong \mathrm{H}_s(M)$ by formula (1.14). □

We can now identify the associated primes and the support of the canonical module of a complex.

Proposition 2.1.6. *Let (R,m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$. Then*

a) $\operatorname{Ass}_R K_M = \{p \in \operatorname{Supp}_R M \mid \dim R/p = \dim_R M + \inf M_p\}$*;*

b) $\operatorname{Supp}_R K_M = \{p \in \operatorname{Supp}_R M \mid \dim_R M = \dim_{R_p} M_p + \dim R/p\}$.

Proof. a) Let $p \in \operatorname{Supp}_R M$. We apply formula (1.9) to $M^\dagger$. Because $\sup M^\dagger = \dim_R M$ by formula (1.14), it thus follows that $p \in \operatorname{Ass}_R K_M$ if and only if $\operatorname{depth}_{R_p}(M^\dagger)_p = -\dim_R M$. Using formulas (1.11) and (1.14) we get

$$\begin{aligned}\operatorname{depth}_{R_p}(M^\dagger)_p &= \operatorname{depth}_{R_p}(M_p)^{\dagger p} - \dim R/p \\ &= \inf M_p - \dim R/p.\end{aligned}$$

Hence $p \in \operatorname{Ass}_R K_M$ if and only if

$$\inf M_p - \dim R/p = -\dim_R M.$$

This proves the claim.
b) Let $p \in \operatorname{Supp}_R K_M$. Note first that $\operatorname{Supp}_R M^\dagger = \operatorname{Supp}_R M$. Indeed, $\operatorname{Supp}_R M^\dagger \subseteq \operatorname{Supp}_R M$ which implies that $\operatorname{Supp}_R M \subseteq \operatorname{Supp}_R M^\dagger$ by biduality. Since $\operatorname{Supp}_R K_M \subseteq \operatorname{Supp}_R M^\dagger$, we then have $p \in \operatorname{Supp}_R M$. Take $q \in \operatorname{Ass}_R K_M$ such that $q \subseteq p$. Then

$$\dim R/q = \dim_R M + \inf M_q$$

by a). Hence

$$\begin{aligned} \dim_{R_p} M_p &\geq \operatorname{height} p/q - \inf M_q \\ &= \dim R/q - \dim R/p - \inf M_q \\ &= \dim_R M - \dim R/p, \end{aligned}$$

where the first inequality is clear by the definition of Krull dimension and the subsequent equality holds true, since R/q is a catenary integral domain. Taking into account the inequality $\dim_R M \geq \dim_{R_p} M_p + \dim R/p$ (see [18, Lemma 6.3.4]), we now get

$$\dim_R M = \dim R/p + \dim_{R_p} M_p.$$

Suppose then that $p \in \operatorname{Supp}_R M$ with $\dim_R M = \dim_{R_p} M_p + \dim R/p$. By Lemma 2.1.4 we have $(K_M)_p \cong K_{M_p} \neq 0$. Thus $p \in \operatorname{Supp}_R K_M$, and we are done. □

Remark 2.1.7. *We observe that by Proposition 2.1.6 a) and Lemma 2.1.5 c)*

$$\{p \in \operatorname{Supp}_R M \mid \dim R/p = \dim_R M + \inf M_p\} = \bigcup_{i \in \mathbb{Z}} (\operatorname{Ass}_R \operatorname{H}_i(M))_{i + \dim_R M}.$$

This can also be proved directly by using formula 1.5 together with formula 1.6.

Corollary 2.1.8. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$. Set $t = \dim_R M$ and $s = \sup M$. Then $\dim_R K_M = s + t$ if and only if $\dim_R \operatorname{H}_s(M) = s + t$.*

Proof. *a)* By Lemma 2.1.5 a) $\dim_R K_M \leq s + t$. Since $\operatorname{H}_s(M) \cong K_{M^\dagger}$, we also have $\dim_R \operatorname{H}_s(M) \leq s + t$. Because $(\operatorname{Ass}_R K_M)_{s+t} = (\operatorname{Ass}_R \operatorname{H}_s(M))_{s+t}$ by Lemma 2.1.5 b), the claim follows. □

2.2 Cohen-Macaulay Complexes

There are two possibilities to introduce the notion of a Cohen-Macaulay complex. In this section we will utilize the notion of the Cohen-Macaulay defect. The other possibility will be discussed in the next section.

We start by recalling the following definition (see [7, p. 6]):

Definition 2.2.1. *Let (R, m) be a local ring. The Cohen-Macaulay defect of a complex $M \in D_b(R)$ is defined by*

$$\operatorname{cmd}_R M = \dim_R M - \operatorname{depth}_R M.$$

It can be shown that if $M \in D_b^f(R)$ and $M \not\simeq 0$ then $0 \leq \operatorname{cmd}_R M$ (see [18, 6.3.8]).

Proposition 2.2.2. *Let (R, m) be a local ring, and let $M \in D_b^f(R)$. Then the following statements are equivalent:*

a) $\operatorname{cmd}_R M = 0$;

b) $\operatorname{cmd}_{R_p} M_p = 0$ *for every* $p \in \operatorname{Supp}_R M$.

Proof. Note first that by [18, Lemma 6.1.11]

$$\operatorname{depth}_R M \leq \operatorname{depth}_{R_p} M_p + \dim R/p,$$

and by [18, Lemma 6.3.4]

$$\dim_R M \geq \dim_{R_p} M_p + \dim R/p.$$

To see the equivalence of a) and b), it is now enough to observe that by the above two inequalities

$$0 \leq \operatorname{cmd}_{R_p} M_p \leq \operatorname{cmd}_R M.$$

□

Lemma 2.2.3. *Let (R, m) be a local ring, and let $M \in D_b^f(R)$. If $\operatorname{cmd}_R M = 0$, then the following statements hold:*

a) M is equidimensional;

b) $\dim_R M = \dim_{R_p} M_p + \dim R/p$ *for every* $p \in \operatorname{Supp}_R M$.

Proof. *a*) This is proved in [17, Theorem 2.3 (d)].

b) Since $\operatorname{depth}_{R_p} M_p \leq \dim_{R_p} M_p$, putting together [18, Lemma 6.1.11] and [18, Lemma 6.3.4], one obtains

$$\operatorname{depth}_R M \leq \operatorname{depth}_{R_p} M_p + \dim R/p \leq \dim_{R_p} M_p + \dim R/p \leq \dim_R M.$$

Now because $\dim_R M = \operatorname{depth}_R M$, it follows from the above inequalities that

$$\dim_R M = \dim_{R_p} M_p + \dim R/p.$$

□

In this context we one to mention the following general fact.

Lemma 2.2.4. *Let (R, m) be a catenary local ring, and let $M \in D^f_+(R)$. Then the following conditions are equivalent:*

a) M is equidimensional;

b) $\dim_R M = \dim_{R_p} M_p + \dim R/p$ for every $p \in \operatorname{Supp}_R M$.

Proof. *a*) ⇒ *b*) : Let $p \in \operatorname{Supp}_R M$. By [18, Lemma 6.3.4] we have

$$\dim_{R_p} M_p \geq \dim_{R_q} M_q + \dim R_p/qR_p$$

for every $qR_p \in \operatorname{Supp}_{R_p} M_p$. Take now $q \in \operatorname{Min}\operatorname{Supp}_R M$ such that $q \subseteq p$. Then

$$\begin{aligned}\dim_{R_p} M_p + \dim R/p &\geq \dim_{R_q} M_q + \dim R_p/qR_p + \dim R/p\\ &= \dim_{R_q} M_q + \dim R/q\\ &= -\inf M_q + \dim R/q\\ &= \dim_R M.\end{aligned}$$

Here the first equality holds true, since R/q is a catenary integral domain. The second equality comes from the fact that $\operatorname{Min}\operatorname{Supp}_R M \subseteq \operatorname{Anc}_R M$ (see [17, Theorem 2.3 (a)]). The last equality then follows from the equidimensionality of M. Since the converse inequality comes from inequality [18, Lemma 6.3.4], we are done.
b) ⇒ *a*) : This is clear, since now

$$-\inf M_p = \dim_{R_p} M_p = \dim_R M - \dim R/p$$

for every $q \in \operatorname{Anc}_R M$. □

Definition 2.2.5. *Let R be a ring, and let $M \in D_b^f(R)$. Then M is called Cohen-Macaulay if $M \not\simeq 0$ and*

$$\operatorname{cmd}_{R_m} M_m = 0$$

for all $m \in \operatorname{Max}(R) \cap \operatorname{Supp}_R M$.

The following two results are well known for specialists. We write them here for the convenience of the reader.

Proposition 2.2.6. *Let R be a ring, and let $M \in D_b^f(R)$. Then the following statements are equivalent:*

a) M is Cohen-Macaulay;

b) $\mathrm{H}^i_{pR_p}(M_p) = 0$ for all $p \in \operatorname{Supp}_R M$ and $i \neq \dim_{R_p} M_p$;

c) $\mathrm{H}^i_{mR_m}(M_m) = 0$ for all $m \in \operatorname{Max} R$ and $i \neq \dim_{R_m} M_m$.

Proof. $a) \Rightarrow b)$: There is nothing to prove unless $p \in \operatorname{Supp}_R M$. By Proposition 2.2.2 b) we have $\operatorname{depth}_{R_p} M_p = \dim_{R_p} M_p$ for every $p \in \operatorname{Supp}_R M$. It thus follows from formula (1.10) that

$$\sup \mathbf{R}\Gamma_{pR_p}(M_p) = \inf \mathbf{R}\Gamma_{pR_p}(M_p) = -\dim_R M_p$$

implying b).
$b) \Rightarrow c)$: This is trivial.
$c) \Rightarrow a)$: We now have $\sup \mathbf{R}\Gamma_{R_m}(M_m) = \inf \mathbf{R}\Gamma_{R_m}(M_m)$. Then using again formula (1.10) we get the claim. □

Notation 2.2.7. *Let R be a ring. Let $t \in \mathbb{Z}$. We denote by $D_{t-CM}(R)$ the full subcategory of $D_b^f(R)$ of Cohen-Macaulay complexes of dimension t.*

Corollary 2.2.8. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$. Then the following statements are equivalent:*

a) M is Cohen-Macaulay;

b) $M^\dagger \simeq \sum^{\dim_R M} K_M$;

c) $M^\dagger \simeq \sum^t N$ for some finitely generated R-module N and $t \in \mathbb{Z}$.

It follows that the functors

$$D_{t-CM}(R) \underset{\sum^{-t}(-)^\dagger}{\overset{K_-}{\rightleftarrows}} (\textit{finitely generated } R\textit{-modules})^{opp}$$

are quasi-inverses of each other, and thus provide an equivalence of categories.

Proof. *a)* $\Rightarrow$ *b)* : Because $\mathrm{H}^i_m(M) = 0$ for $i \neq \dim_R M$ by Proposition 2.2.6 c), it follows from formula (1.13) that $K^i_M = 0$ for $i \neq \dim_R M$. This gives $M^\dagger \simeq \sum^{\dim_R M} K_M$.

b) $\Rightarrow$ *c)* : This is trivial.

c) $\Rightarrow$ *a)* : Using formula (1.14) one now has $\dim_R M = \operatorname{depth}_R M$ as desired. □

The above equivalence of categories is due to Yekutieli and Zhang (see [61, Theorem 6.2]). In particular, we also recover the following (see [61, Remark 6.3]):

Corollary 2.2.9. *Let (R, m) be a local ring admitting a dualizing complex. Then $D_{t-CM}(R)$ is an abelian subcategory of $D^f_b(R)$.*

2.2.1 Canonical Modules of Cohen-Macaulay Complexes

In this section we discuss how the Cohen-Macaulayness of a complex can be seen in the canonical module. This is motivated by the equivalence of categories stated in Corollary 2.2.8.

Remark 2.2.10. *Let (R, m) be a local ring, and let $M \in D^f_b(R)$. Recall that the Bass series and the Poincaré series of M are the series*

$$I^M_R(x) = \sum_{i\in\mathbb{Z}} \mu^i_R(M)x^i \quad \text{and} \quad P^R_M(x) = \sum_{i\in\mathbb{Z}} \beta^R_i(M)x^i,$$

respectively. Here the Bass numbers, $\mu^i_R(M)$, and the Betti numbers, $\beta^R_i(M)$, are defined by the formulas

$$\mu^i_R(M) = \dim_k \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(k, M)) \quad \text{and} \quad \beta^R_i(M) = \dim_k \mathrm{H}_i(k \otimes^L_R M)$$

for every $i \in \mathbb{Z}$.

Proposition 2.2.11. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_{t-CM}(R)$. Then the following statements hold for every $i \in \mathbb{Z}$:*

a) $\mu^i_R(M) = \beta^R_{i-t}(K_M)$;

b) $\beta^R_i(M) = \mu^{i+t}_R(K_M)$.

Proof. *a*) We will first show that $I_R^M(x) = x^t P_{K_M}^R(x)$. Since $M \simeq \sum^{-t}(K_M)^\dagger$ by Corollary 2.2.8 b), we have

$$I_R^M(x) = I_R^{\sum^{-t}(K_M)^\dagger}(x) = x^t I_R^{(K_M)^\dagger}(x) = x^t P_{K_M}^R(x) I_R^{D_R}(x) = x^t P_{K_M}^R(x).$$

Here the third equality comes from [18, Lemma 6.2.10], whereas in the last equality we use the fact that $I_R^{D_R}(x) = 1$ (see [18, Proposition 7.1.11 a)]). A comparison of the coefficients now gives the claim.

b) The argument is similar to a), but we now use [18, Lemma 6.2.12] to see that $P_M^R(x) = x^{-t} I_R^{K_M}(x)$. □

Proposition 2.2.12. *Let (R, m) be a local ring, and let $M \in D_b^f(R)$ be a Cohen-Macaulay complex. Then*

$$\operatorname{Ass}_R \operatorname{H}_s(M) = \{p \in \operatorname{Supp}_R M \mid \dim R/p = \dim_R M + s\},$$

where $s = \sup M$. In particular, $\operatorname{Ass}_R \operatorname{H}_s(M) = \operatorname{Assh}_R \operatorname{H}_s(M)$.

Proof. Let $p \in \operatorname{Supp}_R M$. By formula (1.9), $p \in \operatorname{Ass}_R \operatorname{H}_s(M)$ if and only if $\operatorname{depth}_{R_p} M_p = -s$. Since M is Cohen-Macaulay, we have $\operatorname{depth}_{R_p} M_p = \dim_{R_p} M_p$. It then follows from Lemma 2.2.3 b) that $p \in \operatorname{Ass}_R \operatorname{H}_s(M)$ if and only if $\dim_R M = \dim R/p - s$.

□

Proposition 2.2.13. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$ be a Cohen-Macaulay complex. Then the following statements hold:*

a) $\operatorname{Supp}_R M = \operatorname{Supp}_R K_M$;

b) $(K_M)_p \cong K_{M_p}$ *for every* $p \in \operatorname{Spec} R$;

c) $\operatorname{Anc}_R M = \operatorname{Ass}_R K_M$.

Proof. *a*) By Corollary 2.2.8 b) $M^\dagger \simeq \sum^{\dim_R M} K_M$. Hence $\operatorname{Supp}_R K_M = \operatorname{Supp}_R M^\dagger$. On the other hand, $\operatorname{Supp}_R M^\dagger = \operatorname{Supp}_R M$ by formula (1.15). Then

$$\operatorname{Supp}_R M = \operatorname{Supp}_R M^\dagger = \operatorname{Supp}_R K_M.$$

b) If $p \notin \operatorname{Supp}_R K_M$, then $p \notin \operatorname{Supp}_R M$ so that $K_{M_p} = 0$. For $p \in \operatorname{Supp}_R K_M$, the claim follows from a) together with Lemma 2.2.3 b) and Lemma 2.1.4 b).

c) Because $\operatorname{Anc}_R M = \operatorname{W}_0(M)$ by [17, Theorem 2.3 d)], the claim immediately follows from Proposition 2.1.6 a). □

Corollary 2.2.14. *Let* (R, m) *be a local ring admitting a dualizing complex, and let* $M \in D_b^f(R)$ *be a Cohen-Macaulay complex. Then*

a) $\dim_R K_M = \dim_R M + \sup M$*;*

b) $\operatorname{depth}_R K_M = \dim_R M + \inf M$*.*

In particular, K_M *is Cohen-Macaulay if and only if* M *is a module up to a suspension.*

Proof. Since $K_M \simeq \sum^{-\dim_R M} M^\dagger$ by Corollary 2.2.8 b), the claim follows from formula (1.14). □

The following result is an immediate consequence of Proposition 2.2.13 a) and Corollary 2.2.14 a) together with Proposition 2.2.12.

Corollary 2.2.15. *Let* (R, m) *be a local ring admitting a dualizing complex, and let* $M \in D_b^f(R)$ *be a Cohen-Macaulay complex. Then* $\operatorname{Ass}_R \operatorname{H}_s(M) = \operatorname{Assh}_R K_M$*, where* $\sup M = s$*.*

If M is Cohen-Macaulay module, then so is K_M. We end up this subsection with the following example exhibiting that this does not necessarily hold for complexes.

Example 2.2.16. *Set* $R = K[X, Y, Z]_m/(XY, XZ)_m$ *where* K *is a field and* $m = (X, Y, Z)$*. Then* R *is a non-Cohen-Macaulay ring of dimension two, which admits a dualizing complex. Set* $M = \sum^{-2} D_R$*. Then* $K_M = R$*. Although* M *is a Cohen-Macaulay complex by Corollary 2.2.8,* K_M *is not even equidimensional.*

2.2.2 Cohen-Macaulayness with respect to a filtration

In this section we discuss the notion of a Cohen-Macaulay complex with respect to a filtration, as defined by Grothendieck (see [36]).

Let X be a Noetherian topological space (i.e. it satisfies the descending chain condition for closed subsets). Assume further that any irreducible closed subset of X has a unique generic point (a point in a topological space is called a generic point if its closure coincides with the whole space). Recall from [36] that a filtration of X is a descending sequence

$$\mathcal{F} = (\mathcal{F}_i)_{i \in \mathbb{Z}}: \quad \ldots F_{i-1} \supseteq F_i \supseteq F_{i+1} \supseteq \ldots$$

of subsets of X which satisfies the following conditions:

a) $F_i = X$ for some $i \in \mathbb{Z}$;

b) $\bigcap_{i\in\mathbb{Z}} F_i = \emptyset$;

c) For any $i \in \mathbb{Z}$, if $q, q' \in F_i \backslash F_{i+1}$ and q is a specialization of q' (i.e. q is in the closure of $\{q'\}$) , then $q = q'$;

d) Each F_i is stable under specialization (i.e. if $q' \in F_i$ and $q \in X$ is a specialization of q' , then $q \in F_i$).

From now on we set $\delta_i\mathcal{F} := F_i \backslash F_{i+1}$ for every $i \in \mathbb{Z}$.

Let R be a ring. We always assume that $\operatorname{Spec} R$ is equipped with the Zariski topology. Note that by [11, Chap II, 401-4.3] any irreducible closed subset of $\operatorname{Spec} R$ has a unique generic point. In this case a subset $F \subseteq \operatorname{Spec} R$ is stable under specialization, if $\mathbb{V}(p) \subseteq F$ for any $p \in F$. It is easy to see that a descending sequence $\mathcal{F}$ of subsets of $\operatorname{Spec} R$ is a filtration of a subspace $X \subset \operatorname{Spec} R$ if and only if $F_i = X$ for some $i \in \mathbb{Z}$, $\bigcap_{i\in\mathbb{Z}} F_i = \emptyset$, and each element of $\delta_i\mathcal{F}$ is a minimal element of F_i with respect to inclusion.

Lemma 2.2.17. *Let R be a ring, and let $M \in D_b^f(R)$. Set*

$$H_i = \left\{p \in \operatorname{Supp}_R M \mid i \leq \dim_{R_p} M_p\right\}.$$

Then $\mathcal{H}(M) = (H_i)_{i\in\mathbb{Z}}$ is a filtration of $\operatorname{Supp}_R M$.

Proof. For every $p \in \operatorname{Supp}_R M$, we have

$$-\sup M \leq -\sup M_p \leq -\inf M_p \leq \dim_{R_P} M_p$$

by formula (1.7). Thus $H_{-\sup M} = \operatorname{Supp}_R M$. Since $\dim_{R_p} M_p$ is finite by formula (1.7), then $\bigcap_{i\in\mathbb{Z}} H_i = \emptyset$. Suppose next that $p \in \delta_i\mathcal{H}(M)$. To see the minimality of p in H_i, assume for a contradiction that $q \subset p$ for some $q \in H_i$. Then

$$\dim_{R_q} M_q \leq \dim_{R_q} M_q + \dim R_p/qR_p \leq \dim_{R_p} M_p = i$$

where the last inequality is by [18, Lemma 6.3.4]. This contradicts $q \in H_i$, and we are done. □

Remark 2.2.18. *Let R be a ring of finite dimension and let $t \in \mathbb{Z}$. Set*

$$D_i^t = \{p \in \operatorname{Spec} R \mid i \leq t - \dim R/p\}$$

for all $i \in \mathbb{Z}$. Then $\mathcal{D}^t = (D_i^t)_{i\in\mathbb{Z}}$ is a filtration of $\operatorname{Spec} R$. In particular, if $M \in D_b^f(R)$ and $t = \dim_R M$, we call this filtration as the "M-dimension-filtration", and denote it by $\mathcal{D}(M)$.

Definition 2.2.19. *(Compare [36, p. 238]) Let R be a ring, and let $\mathcal{F}$ be a filtration of* $\operatorname{Spec} R$*. Then $M \in D_b^f(R)$ is called a Cohen-Macaulay complex with respect to $\mathcal{F}$ if $\mathrm{H}^n_{pR_p}(M_p) = 0$ for all $n \neq i$ and $p \in \delta_i\mathcal{F}$. The full subcategory of $D_b^f(R)$ of Cohen-Macaulay complexes with respect to $\mathcal{F}$ is denoted by $D_{\mathcal{F}-CM}$.*

Proposition 2.2.20. *Let R be a ring, and let $M \in D_b^f(R)$. Then the following conditions are equivalent:*

a) M is Cohen-Macaulay;

b) M is Cohen-Macaulay with respect to the filtration $\mathcal{H}(M)$.

Moreover, if (R, m) is a local ring, then the above conditions are equivalent to:

c) M is Cohen-Macaulay with respect to the filtration $\mathcal{D}(M)$.

In particular, $D_{t-CM}(R) = D_{\mathcal{D}^t-CM}(R)$ for any $t \in \mathbb{Z}$. Furthermore, any dualizing complex D is Cohen-Macaulay with respect to the filtration $\mathcal{D}(D)$.

Proof. The equivalence of a) and b) follows directly from the equivalence of a) and b) in Proposition 2.2.6.

$b \Rightarrow c$) : Since M is now Cohen-Macaulay by a), it follows from Lemma 2.2.3 that $\dim_R M - \dim R/p = \dim_{R_p} M_p$ for every $p \in \operatorname{Supp}_R M$. Hence the claim follows from the definition.

$c \Rightarrow a$) : Because $m \in \delta_{\dim_R M}\mathcal{D}(M)$, one now has $\mathrm{H}^i_m(M) = 0$ for $i \neq \dim_R M$. Thus M is Cohen-Macaulay by Proposition 2.2.6.

Furthermore, the uniqueness of the dualizing complex implies that $D \simeq \sum^{-t'} D_R$ for some $t' \in \mathbb{Z}$. Hence D is Cohen-Macaulay of dimension t'. It thus follows from c) that D is Cohen-Macaulay with respect to $\mathcal{D}(D)$. □

2.3 Serre's Conditions

In this section we will generalize Serre's conditions to complexes. It is convenient to begin with the following very general definition:

Definition 2.3.1. *Let R be a ring. Let $k \in \mathbb{N}$ and $N \in D_b^f(R)$. We say that a complex $M \in D_b^f(R)$ satisfies Serre's condition $(S_{k,N})$, if*

$$\operatorname{depth}_{R_p} \mathbf{R}\mathrm{Hom}_{R_p}(N_p, M_p) \geq \min\left\{k, \dim_{R_p} M_p + \inf N_p\right\}$$

for all $p \in \operatorname{Spec} R$.

Remark 2.3.2. *Since*

$$\operatorname{depth}_{R_p} \mathbf{R}\mathrm{Hom}_{R_p}(N_p, M_p) = \inf N_p + \operatorname{depth}_{R_p} M_p \tag{2.1}$$

for all $p \in \operatorname{Spec} R$ *by [30, Proposition 4.6], it is clear that* $(S_{k,N})$ *is equivalent to having*

$$\operatorname{depth}_{R_p} M_p \geq \min\left\{k - \inf N_p, \dim_{R_p} M_p\right\} \tag{2.2}$$

for all $p \in \operatorname{Spec} R$.

The following proposition is now immediate:

Proposition 2.3.3. *Let* R *be a ring. Let* $k \in \mathbb{Z}$ *and* $N \in D_b^f(R)$. *Then a complex* $M \in D_b^f(R)$ *satisfies the condition* $(S_{k,N})$, *if and only if* M_p *is Cohen-Macaulay for every* $p \in \operatorname{Spec} R$ *with* $\operatorname{depth}_{R_p} M_p < k - \inf N_p$.

Notation 2.3.4. *If* $N = M$ *or* $N = \sum^l R$ *for some* $l \in \mathbb{Z}$, *we will speak about the condition* (S_k) *or* $(S_{k,l})$, *respectively. In other words, the complex* M *satisfies* (S_k) *if and only if*

$$\operatorname{depth}_{R_p} M_p \geq \min\left\{k - \inf M_p, \dim_{R_p} M_p\right\}$$

for all $p \in \operatorname{Spec} R$. *Similarly,* M *is said to satisfy* $(S_{k,l})$ *if and only if*

$$\operatorname{depth}_{R_p} M_p \geq \min\left\{k - l, \dim_{R_p} M_p\right\}$$

for all $p \in \operatorname{Spec} R$.

Remark 2.3.5. *Let* R *be a ring, and let* $M \in D_b^f(R)$. *Since* $\inf M_p \leq \sup M_p \leq \sup M$ *for all* $p \in \operatorname{Supp}_R M$, *we see that* (S_k) *always implies* $(S_{k,\sup M})$.

Remark 2.3.6. *Note that our* (S_k) *differs from the condition given by Celikbas and Piepmeyer in [13, 2.4] according to which a complex* $M \in D_b^f(R)$ *satisfies* (S_k) *if*

$$\operatorname{depth}_{R_p} M_p + \inf M_p \geq \min\left\{k, \operatorname{height} p\right\}$$

for every $p \in \operatorname{Supp}_R M$. *However, because*

$$\dim_{R_p} M_p + \inf M_p \leq \operatorname{height} p - \inf M_p$$

by formula (1.7), *we observe that the condition of Celikbas and Piepmeyer implies our* (S_k).

Given an integer n, recall that the *soft truncation* of a complex M above at n is the complex

$$M_{\subseteq n}\colon\ \ldots \to 0 \to \operatorname{Coker} d^M_{n+1} \to M_{n-1} \to M_{n-2} \to \ldots \qquad .$$

In $D(R)$ we now have an exact triangle

$$\textstyle\sum^n \mathrm{H}_n(M) \to M_{\subseteq n} \to M_{\subseteq n-1} \to \sum^{n+1} \mathrm{H}_n(M). \tag{2.3}$$

Note that if $n \geq \sup M$, then the natural morphism $M \to M_{\subseteq n}$ becomes an isomorphism in $D(R)$.

The next definition was given for modules by P. Schenzel in [53, Definition 4.1].

Definition 2.3.7. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D^f_b(R)$. We call the complex $(M^\dagger)_{\subseteq \dim_R M-1}$ as the complex of deficiency of M, and denote it by C_M.*

Remark 2.3.8. *Clearly*

$$\mathrm{H}_i(C_M) \cong \begin{cases} K^i_M, & \text{if } i \leq \dim_R M; \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

In particular, when $M \not\simeq 0$, we have $C_M \simeq 0$ if and only if M is Cohen-Macaulay. If this is not the case, then $\inf C_M = \operatorname{depth}_R M$.

Remark 2.3.9. *Because* $\sup M^\dagger = \dim_R M$ *by formula* (1.14), *we obtain an exact triangle*

$$\textstyle\sum^{\dim_R M} K_M \to M^\dagger \to C_M \to \sum^{\dim_R M+1} K_M. \tag{2.5}$$

An application of the functor $(-)^\dagger$ to (2.5) *yields an exact triangle*

$$C^\dagger_M \to M \to \textstyle\sum^{-\dim_R M} K^\dagger_M \to \sum^1 C^\dagger_M. \tag{2.6}$$

From now on we denote the natural morphism $M \to \sum^{-\dim_R M} K^\dagger_M$ by h_M.

Lemma 2.3.10. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D^f_b(R)$. If M is equidimensional, then $(C^\dagger_M)_p \simeq C^{\dagger p}_{M_p}$ for every $p \in \operatorname{Spec} R$.*

Proof. Since $\operatorname{Supp}_R C^\dagger_M \subseteq \operatorname{Supp}_R M$, there is nothing to say if $p \notin \operatorname{Supp}_R M$. Assume then that $p \in \operatorname{Supp}_R M$. Set $\dim_R M = t$. We observe first that

$(C_M)_p \simeq \sum^{\dim R/p} C_{M_p}$. Indeed, by using formula (1.11) and Lemma 2.2.4, we get

$$\begin{aligned}((M^\dagger)_{\subseteq t-1})_p &\cong ((M^\dagger)_p)_{\subseteq t-1}\\ &\simeq (\textstyle\sum^{\dim R/p} {M_p}^{\dagger p})_{\subseteq t-1}\\ &= \textstyle\sum^{\dim R/p}(({M_p}^{\dagger p})_{\subseteq t-\dim R/p-1})\\ &= \textstyle\sum^{\dim R/p}(({M_p}^{\dagger p})_{\dim_{R_p} M_p-1}).\end{aligned}$$

Then

$$(C_M)_p^{\dagger p} \simeq \textstyle\sum^{-\dim R/p} C_{M_p}^{\dagger p},$$

so that by formula (1.11) $(C_M^\dagger)_p \simeq C_{M_p}^{\dagger p}$ as wanted. □

Lemma 2.3.11. *Let (R, m) be a local ring admitting a dualizing complex, and let M, $N \in D_b^f(R)$. Suppose that M is an equidimensional complex. Then the following conditions are equivalent:*

a) *M satisfies condition $(S_{k,N})$;*

b) $\inf C_{M_p} \geq k - \inf N_p$ *for all* $p \in \operatorname{Spec} R$;

c) $\sup \mathbf{R}\mathrm{Hom}_R(N, C_M^\dagger) \leq -k$;

d) $\dim_R \mathrm{H}_i(N) \otimes_R K_M^j \leq i + j - k$ *for all* $i, j \in \mathbb{Z}$, $j < \dim_R M$;

e) $\dim R/p \leq n - k + \inf N_p$ *for every* $p \in \operatorname{Supp}_R K_M^n$ *and all* $n < \dim_R M$;

f) $\operatorname{depth}_R(a_M^n, N^\dagger) \geq k - n$ *for all* $n < \dim_R M$ *where* $a_M^n = (0 :_R K_M^n)$;

g) $\sup \mathbf{R}\mathrm{Hom}_R(K_M^n, N^\dagger) \leq n - k$ *for all* $n < \dim_R M$.

Proof. *a)* ⇔ *b)* : Let $p \in \operatorname{Spec} R$. We want to show that the conditions

$$\operatorname{depth}_{R_p} M_p \geq \min\left\{k - \inf N_p, \dim_{R_p} M_p\right\}$$

and $\inf C_{M_p} \geq k - \inf N_p$ are equivalent. If M_p is Cohen-Macaulay, this is clear. Suppose thus that M_p is not Cohen-Macaulay. By Remark 2.3.8 the latter condition now means that $\operatorname{depth}_{R_p} M_p \geq k - \inf N_p$. But because $\operatorname{depth}_{R_p} M_p < \dim_{R_p} M_p$, this implies the desired equivalence.

b) ⇔ *c)* : It is enough to observe that by Lemma [18, Lemma 5.2.8], Lemma 2.3.10 and formula (1.14) we have

$$\begin{aligned}-\sup \mathbf{R}\mathrm{Hom}_R(N, C_M^\dagger) &= \inf\left\{\operatorname{depth}_{R_p}(C_M^\dagger)_p + \inf N_p \mid p \in \operatorname{Spec} R\right\}\\ &= \inf\left\{\operatorname{depth}_{R_p} {C_{M_p}}^{\dagger p} + \inf N_p \mid p \in \operatorname{Spec} R\right\}\\ &= \inf\left\{\inf C_{M_p} + \inf N_p \mid p \in \operatorname{Spec} R\right\}.\end{aligned}$$

$c) \Leftrightarrow d)$: Using adjointness and formula (1.14) we get

$$\begin{aligned}-\sup \mathbf{R}\mathrm{Hom}_R(N, C_M^\dagger) &= \sup(N \otimes_R^L C_M)^\dagger \\ &= \dim_R N \otimes_R^L C_M.\end{aligned}$$

The claim follows, since by [18, Lemma 6.3.9 b), (E.6.3.1)] and Remark 2.3.8 we have

$$\begin{aligned}\dim_R N \otimes_R^L C_M &= \sup\left\{\dim_R \mathrm{H}_i(N) \otimes_R^L C_M - i \mid i \in \mathbb{Z}\right\} \\ &= \sup\left\{\sup\left\{\dim_R \mathrm{H}_i(N) \otimes_R^L \mathrm{H}_j(C_M) - j \mid j \in \mathbb{Z}\right\} - i \mid i \in \mathbb{Z}\right\} \\ &= \sup\left\{\dim_R \mathrm{H}_i(N) \otimes_R^L \mathrm{H}_j(C_M) - i - j \mid i, j \in \mathbb{Z}\right\} \\ &= \sup\left\{\dim_R \mathrm{H}_i(N) \otimes_R \mathrm{H}_j(C_M) - i - j \mid i, j \in \mathbb{Z}\right\} \\ &= \sup\left\{\dim_R \mathrm{H}_i(M) \otimes_R K_M^j - i - j \mid i, j \in \mathbb{Z}, j < \dim_R M\right\}.\end{aligned}$$

$a) \Leftrightarrow e)$ Suppose first that M satisfies condition $(S_{k,N})$. Assume for the sake of contradiction that there would exist $p \in \mathrm{Supp}_R K_M^n$ with

$$n - \dim R/p < k - \inf N_p.$$

By Lemma 2.2.4

$$n - \dim R/p < \dim_R M - \dim R/p = \dim_{R_p} M_p.$$

On the other hand, because $K_{M_p}^{n-\dim R/p} \neq 0$ by 2.1.4 a), we have $\mathrm{depth}_{R_p} M_p \leq n - \dim R/p$. It now follows that

$$\mathrm{depth}_{R_p} M_p < \min\left\{k - \inf N_p, \dim_{R_p} M_p\right\}.$$

This contradicts with formula (2.2).

Conversely, suppose that e) holds. If M would not satisfy condition $(S_{k,N})$, then we could find $p \in \mathrm{Supp}_R M$ with

$$\mathrm{depth}_{R_p} M_p < \min\left\{k - \inf N_p, \dim_{R_p} M_p\right\}.$$

Put $n = \mathrm{depth}_{R_p} M_p + \dim R/p$. One has

$$n < k - \inf N_p + \dim R/p. \qquad (*)$$

On the other hand, by Lemma 2.1.4 a)

$$(K_M^n)_p \cong K_{M_p}^{n-\dim R/p} = K_{M_p}^{\mathrm{depth}_{R_p} M_p} \neq 0.$$

Hence $p \in \operatorname{Supp}_R K_M^n$. Because $n < \dim_R M$ by [18, Lemma 6.3.4], it now follows from the assumption that $\dim R/p - \inf N_p \leq n - k$. This contradicts with the inequality $(*)$ so that M satisfies condition $(S_{k,N})$.

e) $\Leftrightarrow$ *f)* Note that

$$\operatorname{depth}_{R_p}(N^\dagger)_p = -\dim R/p + \operatorname{depth}_{R_p}(N_p)^{\dagger p} = -\dim R/p + \inf N_p,$$

where the equalities are by formulas (1.11) and (1.14), respectively. Hence using formula (1.8)

$$\operatorname{depth}_R(a_M^n, N^\dagger) = \inf\left\{-\dim R/p + \inf N_p \mid p \in \mathbb{V}(a_M^n)\right\},$$

implying that f) and e) are equivalent.

f) $\Leftrightarrow$ *g)* : Since by formula (1.8) and [18, Lemma 5.2.8]

$$\operatorname{depth}_R(a_M^n, N^\dagger) = -\sup \mathbf{R}\mathrm{Hom}_R(K_M^n, N^\dagger)$$

the claim follows. □

Proposition 2.3.12. *Let (R, m) be a local ring admitting a dualizing complex. Let $k \in \mathbb{Z}$ and M, $N \in D_b^f(R)$. Set $\dim_R M = t$. If M is equidimensional, then the following conditions are equivalent:*

a) M satisfies condition $(S_{k,N})$;

b) The natural homomorphism $\operatorname{Ext}_R^{-i}(N, M) \to K^{i+t}_{N \otimes_R^L K_M}$ is bijective for all $i \geq -k+2$, and injective for $i = -k+1$.

Proof. By applying the functor $\mathbf{R}\mathrm{Hom}_R(N, -)$ on (2.6) we get the exact triangle

$$\mathbf{R}\mathrm{Hom}_R(N, C_M^\dagger) \to \mathbf{R}\mathrm{Hom}_R(N, M) \to \sum\nolimits^{-t} \mathbf{R}\mathrm{Hom}_R(N, K_M^\dagger) \to \sum\nolimits^{1} \mathbf{R}\mathrm{Hom}_R(N, C_M^\dagger).$$

Observe that $\mathbf{R}\mathrm{Hom}_R(N, K_M^\dagger) \simeq (N \otimes_R^L K_M)^\dagger$ by adjointness. Since $(S_{k,N})$ is by Lemma 2.3.11 equivalent to $\sup \mathbf{R}\mathrm{Hom}_R(N, C_M^\dagger) \leq -k$ a look at the corresponding long exact sequence of homology implies the claim. □

In particular, if we take $N = M$, this applies to Serre's condition (S_k). For this case, we observe the following

Proposition 2.3.13. *Let (R, m) be a local ring admitting a dualizing complex. Let $M \in D_b^f(R)$. Set $t = \dim_R M$. Then $\dim_R M \otimes_R^L K_M = t$. Moreover, the natural homomorphism*

$$\operatorname{Ext}^{-i}(K_M, K_M) \to K^{i+t}_{M \otimes_R^L K_M}$$

is an isomorphism for $i > \sup C_M - t$. In particular,

$$K_{M \otimes_R^L K_M} = \operatorname{Hom}_R(K_M, K_M).$$

Proof. By adjointness

$$(M \otimes_R^L K_M)^\dagger \simeq \mathbf{R}\operatorname{Hom}_R(K_M, M^\dagger).$$

Since $\operatorname{Hom}_R(K_M, K_M) \neq 0$, it follows from formula (1.14) and [14, Proposition A.4.6] that

$$\dim_R M \otimes_R^L K_M = \sup \mathbf{R}\operatorname{Hom}_R(K_M, M^\dagger) = \sup M^\dagger - \inf K_M = \dim_R M.$$

An application of the functor $\mathbf{R}\operatorname{Hom}_R(K_M, -)$ on (2.5), yields the exact triangle

$$\textstyle\sum^{-t-1} \mathbf{R}\operatorname{Hom}_R(K_M, C_M) \to \mathbf{R}\operatorname{Hom}_R(K_M, K_M) \to \sum^{-t} \mathbf{R}\operatorname{Hom}_R(K_M, M^\dagger) \to \sum^{-t} \mathbf{R}\operatorname{Hom}_R(K_M, C_M).$$

The desired isomorphism now follows from the corresponding long exact sequence of homology, because $\sup \mathbf{R}\operatorname{Hom}_R(K_M, C_M) \leq \sup C_M$ by [14, Proposition A.4.6]. □

Let us then consider the conditions $(S_{k,l})$.

Corollary 2.3.14. *Let (R, m) be a local ring admitting a dualizing complex. Let $k, l \in \mathbb{Z}$ and $M \in D_b^f(R)$. Set $t = \dim_R M$ and $s = \sup M$. If M is equidimensional, then the following conditions are equivalent:*

a) M satisfies condition $(S_{k,l})$;

b) The natural homomorphism $\mathrm{H}_i(h_M)\colon \mathrm{H}_i(M) \to K^{i+t}_{K_M}$ is bijective for $i \geq l - k + 2$, and injective for $i = l - k + 1$;

c) The natural homomorphism $\mathrm{H}^i_m(c_{K_M})\colon \mathrm{H}^{i+t}_m(K_M) \to \mathrm{H}_i(M)^\vee$ is bijective for $i \geq l - k + 2$, and surjective for $i = l - k + 1$.

Here $(-)^\vee = \operatorname{Hom}_R(-, E_R(k))$.

Proof. The equivalence of a) and b) follows from Proposition 2.3.12 by taking $N = \sum^l R$, whereas the homomorphism of c) is by local duality the Matlis-dual of that b).

□

If R is a ring and N is an R-module, we use the notation

$$\operatorname{Assh}_R N = \{p \in \operatorname{Supp}_R N \mid \dim(R/p) = \dim_R N\}.$$

Corollary 2.3.15. *Let (R, m) be a local ring admitting a dualizing complex. Let $M \in D_b^f(R)$ be an equidimensional complex. Set $t = \dim_R M$ and $s = \sup M$. If M satisfies Serre's condition (S_1), then*

a) $\dim_R \mathrm{H}_s(M) = \dim_R K_M = s + t$;

b) $\operatorname{Ass}_R \mathrm{H}_s(M) = \operatorname{Assh}_R \mathrm{H}_s(M)$.

Proof. *a*) Recall that (S_1) implies $(S_{1,s})$. By Corollary 2.3.14 the natural homomorphism $\mathrm{H}_s(M) \to K_{K_M}^{s+t}$ is injective. Then $K_{K_M}^{s+t} \neq 0$ so that by Lemma 2.1.5 a) we must have $\dim_R K_M = s + t$. It now follows from Corollary 2.1.8 a) that $\dim_R \mathrm{H}_s(M) = s + t$ too.
b) Because $\dim_R \mathrm{H}_s(M) = s + t$ by a), it is enough to show that

$$\operatorname{Ass}_R \mathrm{H}_s(M) = (\operatorname{Ass}_R \mathrm{H}_s(M))_{s+t}.$$

By a) we also have an injective homomorphism $\mathrm{H}_s(M) \to K_{K_M}$. So

$$\operatorname{Ass}_R \mathrm{H}_s(M) \subseteq \operatorname{Ass}_R K_{K_M} = (\operatorname{Ass}_R K_M)_{s+t},$$

where the last equality comes from [53, Proposition 2.3 b)]. Since

$$(\operatorname{Ass}_R K_M)_{s+t} = (\operatorname{Ass}_R \mathrm{H}_s(M))_{s+t}$$

by Lemma 2.1.5 b), we get

$$\operatorname{Ass}_R \mathrm{H}_s(M) = (\operatorname{Ass}_R \mathrm{H}_s(M))_{s+t}$$

as wanted. □

Corollary 2.3.16. *Let (R, m) be a local ring admitting a dualizing complex. Let $M \in D_b^f(R)$ be an equidimensional complex. Set $s = \sup M$. If M satisfies Serre's condition (S_2), then*

$$\operatorname{Hom}_{D(R)}(M, M) \cong \operatorname{Hom}_R(K_M, K_M)$$

and $\mathrm{H}_s(M) \cong K_{K_M}$. Moreover, if K_M is equidimensional and satisfies Serre's condition (S_2), then $K_M \cong K_{\mathrm{H}_s(M)}$.

Proof. Recall first that

$$\mathrm{Hom}_{D(R)}(M, M) \cong \mathrm{Ext}^0_R(M, M).$$

The desired isomorphism

$$\mathrm{Hom}_{D(R)}(M, M) \cong \mathrm{Hom}_R(K_M, K_M)$$

comes from Proposition 2.3.12 by taking $N = M$ and Proposition 2.3.13 whereas Corollary 2.3.14 and Corollary 2.3.15 provide the isomorphism $\mathrm{H}_s(M) \to K_{K_M}$. Combining the latter with [52, Theorem 1.14 (ii)], shows that $K_{\mathrm{H}_s(M)} \cong K_{K_{K_M}} \cong K_M$. □

We now turn to look at the dagger dual:

Proposition 2.3.17. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D^f_b(R)$. The complex $M^\dagger$ satisfies Serre's condition (S_k) if and only if* $\sup M_p = \inf M_p$ *for every* $p \in \mathrm{Supp}_R M$ *with* $\mathrm{depth}_{R_p} M_p + \inf M_p < k$.

Proof. Note that by formula (1.11) together with formula (1.14) we get

$$\dim_{R_p}(M^\dagger)_p = -\dim R/p + \sup M_p$$

and

$$\mathrm{depth}_{R_p}(M^\dagger)_p = -\dim R/p + \inf M_p.$$

We also have

$$\inf(M^\dagger)_p = \dim R/p + \inf M_p^{\dagger p} = \dim R/p + \mathrm{depth}_{R_p} M_p.$$

The claim then follows from Proposition 2.3.3. □

In order to apply Proposition 2.3.12 in this case, we need

Lemma 2.3.18. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D^f_b(R)$. Then* $\mathrm{Ext}^i_R(M^\dagger, M^\dagger) \cong \mathrm{Ext}^i(M, M)$ *for all* $i \in \mathbb{Z}$.

Proof. By "swap" and dagger duality

$$\mathbf{R}\mathrm{Hom}_R(M^\dagger, M^\dagger) \simeq \mathbf{R}\mathrm{Hom}_R(M, M^{\dagger\dagger}) \simeq \mathbf{R}\mathrm{Hom}_R(M, M).$$

The claim now follows by taking homology. □

Let us then observe the following:

Proposition 2.3.19. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D^f_b(R)$. Then the following statements are equivalent:*

a) $M^\dagger$ *is equidimensional;*

b) $\sup M = \sup M_p$ *for any* $p \in \operatorname{Supp}_R M$*.*

Proof. We know by Lemma 2.2.4 together with formula (1.14) that $M^\dagger$ being equidimensional is equivalent to

$$\dim_{R_p}(M^\dagger)_p = -\dim R/p + \sup M$$

for every $p \in \operatorname{Supp}_R M$. On the other hand by using formula (1.11) we get

$$\begin{aligned}\dim_{R_p}(M^\dagger)_p &= -\dim R/p + \dim_{R_p}(M_p)^{\dagger p}\\ &= -\dim R/p + \sup M_p\end{aligned}$$

where the last equality comes from formula (1.14). So a) and b) are equivalent. □

We can now state

Corollary 2.3.20. *Let* (R, m) *be a local ring admitting a dualizing complex. Let* $k \in \mathbb{Z}$ *and let* $M \in D_b^f(R)$*. Set* $s = \sup M$*. If* $M^\dagger$ *is equidimensional, then the following conditions are equivalent:*

a) $M^\dagger$ *satisfies condition* (S_k)*;*

b) The natural homomorphism $\operatorname{Ext}_R^{-i}(M, M) \to K^{i+s}_{\mathbf{R}\mathrm{Hom}_R(\mathrm{H}_s(M),M)}$ *is bijective for all* $i \geq -k+2$*, and injective for* $i = -k+1$*.*

Proof. Note that $\dim_R M^\dagger = s$ by formula (1.14). By dagger duality we have $K_{M^\dagger} = \mathrm{H}_s(M)$. By adjointness and biduality we then get

$$(M^\dagger \otimes^L_R K_{M^\dagger})^\dagger \simeq \mathbf{R}\mathrm{Hom}_R(K_{M^\dagger}, M^{\dagger\dagger}) \simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_s(M), M).$$

The claim is then a direct consequence of Lemma 2.3.18 and Proposition 2.3.12. □

In a similar way, Corollary 2.3.14 yields

Corollary 2.3.21. *Let* (R, m) *be a local ring admitting a dualizing complex,* $k, l \in \mathbb{Z}$*, and let* $M \in D_b^f(R)$*. Set* $s = \sup M$*. If* $M^\dagger$ *is equidimensional, then the following conditions are equivalent:*

a) $M^\dagger$ *satisfies Serre's condition* $(S_{k,l})$*;*

b) The natural homomorphism $K^i_M \to K^{i+s}_{\mathrm{H}_s(M)}$ *is bijective for* $i \geq l-k+2$*, and injective for* $i = l-k+1$*;*

c) The natural homomorphism $\mathrm{H}_m^{i+s}(\mathrm{H}_s(M)) \to \mathrm{H}_m^i(M)$ *is bijective for* $i \geq l-k+2$*, and surjective for* $i = l-k+1$*.*

Corollary 2.3.15 and Corollary 2.3.16 have now the following analogues:

Corollary 2.3.22. *Let* (R, m) *be a local ring admitting a dualizing complex, and let* $M \in D_b^f(R)$*. Suppose that* $M^\dagger$ *is equidimensional. Set* $s = \sup M$*. If* $M^\dagger$ *satisfies Serre's condition* (S_1)*, then*

a) $\dim_R \mathrm{H}_s(M) = \dim_R K_M = s + t$*;*

b) $\mathrm{Ass}_R K_M = \mathrm{Assh}_R K_M$*.*

Corollary 2.3.23. *Let* (R, m) *be a local ring admitting a dualizing complex, and let* $M \in D_b^f(R)$*. Suppose that* $M^\dagger$ *is equidimensional. Set* $s = \sup M$*. If* $M^\dagger$ *satisfies Serre's condition* (S_2)*, then*

$$\mathrm{Hom}_{D(R)}(M, M) \cong \mathrm{Hom}_R(\mathrm{H}_s(M), \mathrm{H}_s(M))$$

and $K_M \cong K_{\mathrm{H}_s(M)}$*. Moreover, if* $\mathrm{H}_s(M)$ *is equidimensional and satisfies Serre's condition* (S_2)*, then* $\mathrm{H}_s(M) \cong K_{K_M}$*.*

2.3.1 Cousin Functor

In this subsection we recall the notion of a Cousin complex, which was introduced by Grothendieck (see [36]). Some of the results are not new, merely restated in the language of commutative algebra from [36], [45] and [59].

Throughout this section $\operatorname{Spec} R$ is assumed to be equipped with the Zariski topology.

Definition 2.3.24. *Let R be a ring, and let $\mathcal{F}$ be a filtration of* $\operatorname{Spec} R$. *A complex $M \in C(R)$ is said to be a* Cousin complex *with respect to $\mathcal{F}$, if for all $i \in \mathbb{Z}$*

$$M_{-i} = \bigoplus_{p \in \delta_i \mathcal{F}} M_{(p)},$$

where every $M_{(p)}$ is an R_p-module such that all elements of $M_{(p)}$ are pR_p-torsion.

Notation 2.3.25. *We denote by* $\operatorname{Coz}(\mathcal{F}, R)$ *the full subcategory of $C(R)$ of Cousin complexes with respect to $\mathcal{F}$. Observe that every Cousin complex is bounded above.*

Remark 2.3.26. *Let $i \in \mathbb{Z}$, and $M, N \in \operatorname{Coz}(\mathcal{F}, R)$. It is easily checked that $\operatorname{Hom}_R(M_{(p)}, N_{(q)}) = 0$ for every $p \in \delta_i \mathcal{F}$ and $q \in \delta_{i-1}\mathcal{F}$. Therefore, if f and g are two homotopic homomorphisms from M to N, then $f = g$. Hence $\operatorname{Coz}(\mathcal{F}, R)$ can be considered as a full subcategory of $K(R)$.*

Example 2.3.27. *Let K is a finitely generated R-module and let $t \in \mathbb{Z}$. Set $N = \sum^{-t} \operatorname{Hom}_R(K, D_R)$. Then $N \in \operatorname{Coz}(\mathcal{D}^t, R)$. Indeed, for any $i \in \mathbb{Z}$,*

$$N_{-i} = \bigoplus_{\dim R/p = -i+t} \operatorname{Hom}_R(K, E_R(R/p)) = \bigoplus_{p \in \delta_i \mathcal{D}^t} \operatorname{Hom}_R(K, E_R(R/p)),$$

where each R_p-module $\operatorname{Hom}_R(K, E_R(R/p))$ is supported at pR_p.

Let R be a ring. Let $Z \subseteq \operatorname{Spec} R$. Recall that for any R-module N, *the section module with support in Z* is defined by the formula

$$\Gamma_Z(N) = \{n \in N \mid \operatorname{Supp}_R \langle n \rangle \subseteq Z\}.$$

More generally, if $Z' \subset Z$, we set

$$\Gamma_{Z/Z'}(N) = \Gamma_Z(N)/\Gamma_{Z'}(N).$$

In this way we obtain additive functors $\Gamma_Z(-)$ and $\Gamma_{Z/Z'}(-)$ on the category of R-modules. We set

$$\mathrm{H}^i_Z(-) = \mathrm{H}_{-i}(\mathbf{R}\Gamma_Z(-)) \quad \text{and} \quad \mathrm{H}^i_{Z/Z'}(-) = \mathrm{H}_{-i}(\mathbf{R}\Gamma_{Z/Z'}(-))$$

for all $i \in \mathbb{Z}$.

Lemma 2.3.28. *Let R be a ring, and let $Z \subseteq \operatorname{Spec} R$. Then*

$$\mathrm{H}^i_Z(M) = \varinjlim_{\mathbb{V}(a)\subseteq Z} \mathrm{H}^i_a(M),$$

where $a \subseteq R$ denotes an ideal, for all $i \in \mathbb{Z}$ and $M \in D_-(R)$.

Proof. It is enough to observe that if N is an R-module, then

$$\Gamma_Z(N) = \varinjlim_{\mathbb{V}(a)\subseteq Z} \Gamma_a(N).$$

Indeed, if $n \in \Gamma_Z(N)$, then $\mathbb{V}(\operatorname{Ann} n) = \operatorname{Supp}_R\langle n\rangle \subset Z$ so that

$$\Gamma_Z(N) = \bigcup_{\mathbb{V}(a)\subseteq Z} \mathbb{V}(a).$$

□

Remark 2.3.29. *Let R be a ring. Let $Z' \subseteq Z \subseteq \operatorname{Spec} R$. Assume that every element in $Z\backslash Z'$ is minimal in Z (with respect to inclusion). Let N be an R-module. Consider the natural map*

$$\Gamma_Z(N) \to \prod_{p\in Z\backslash Z'} N_p.$$

If $n \in \Gamma_Z(N)$ with $0 \neq n/1 \in N_p$ for some $p \in Z\backslash Z'$, then $p \in \operatorname{Min}\operatorname{Supp}_R\langle n\rangle$. Therefore, the image of the above homomorphism is contained in

$$\bigoplus_{p\in Z\backslash Z'} \Gamma_{pR_p}(N_p).$$

Furthermore, if $0 = n/1 \in N_p$ for all $p \in Z\backslash Z'$, then $\operatorname{Supp}_R\langle n\rangle \subseteq Z'$ so that $n \in \Gamma_{Z'}(N)$. It thus follows that the kernel of the above homomorphism is $\Gamma_{Z'}(N)$. There thus exists an injective natural homomorphism

$$\xi_N \colon \Gamma_{Z/Z'}(N) \to \bigoplus_{p\in Z\backslash Z'} N_p.$$

Lemma 2.3.30. *Let R be a ring. Let I be an injective R-module, and let $p \in \operatorname{Supp}_R I$. Then*

a) The natural map $I \to I_p$ is surjective;

b) The natural map $\Gamma_p(I) \to \Gamma_{pR_p}(I_p)$ is surjective.

Proof. For any $s \in R$, let I_s denote the localization of I with respect to the multiplicatively closed subset $S = \{s^i \mid i \in \mathbb{N}\}$ of R. Because I is injective, we know by [37, III Lemma 3.3] that the natural map $I \to I_s$ is surjective. This implies a). By [12, Proposition 2.1.4] the module $\Gamma_p(I)$ is also injective. Therefore we can apply a) to get b). $\square$

Lemma 2.3.31. *Let R be a ring, and let $Z' \subseteq Z \subseteq \operatorname{Spec} R$. Assume that every element in $Z\backslash Z'$ is minimal in Z (with respect to inclusion). If I is an injective module, then the natural homomorphism*

$$\xi_I \colon \Gamma_{Z/Z'}(I) \to \bigoplus_{p \in Z\backslash Z'} \Gamma_{pR_p}(I_p) \tag{2.7}$$

of Remark 2.3.29 is an isomorphism.

Proof. By Remark 2.3.29, it is enough to show that ξ_I is surjective. Let

$$y = (y_p)_{p \in Z\backslash Z'} \in \bigoplus_{p \in Z\backslash Z'} \Gamma_{pR_p}(I_p),$$

and let $p_0 \in Z\backslash Z'$. We look first at the special case where $y_p = 0$ for all $p \neq p_0$. It follows from Lemma 2.3.30 b) that $y_{p_0} = i/1$ for some $i \in \Gamma_{p_0}(I)$. We will show that $\xi_I(i + \Gamma_{Z'}(I)) = y$. To see this, we need to prove that if $0 \neq i/1 \in I_p$ for some $p \in Z\backslash Z'$, then $p = p_0$. Suppose thus that $(0 :_R i) \subseteq p$. Because $i \in \Gamma_{p_0}(I)$, we have $p_0 \subseteq p$. The ideal p being minimal implies that $p = p_0$. Thus $y = \xi_I(i + \Gamma_{Z'}(I))$ as claimed. It is now easy to see that ξ_I is surjective.

$\square$

Using Lemma 2.3.31, we see that there is natural isomorphism

$$\mathbf{R}\Gamma\xi_M \colon \mathbf{R}\Gamma_{Z/Z'}(M) \to \bigoplus_{p \in Z\backslash Z'} \mathbf{R}\Gamma_{pR_p}(M_p) \tag{2.8}$$

for every $M \in D_-(R)$. In particular,

$$\mathrm{H}^i_{Z/Z'}(M) \cong \bigoplus_{p \in Z\backslash Z'} \mathrm{H}^i_{pR_p}(M_p). \tag{2.9}$$

for all $i \in \mathbb{Z}$.

Let $\mathcal{F}$ be a filtration of $\operatorname{Spec} R$, and let $M \in D_-(R)$. Suppose that I is an injective resolution of M, which is bounded above. Then corresponding to $\mathcal{F}$, there exists the following filtration of the complex I:

$$(\Gamma_{F_i}(I))_{i \in \mathbb{Z}} : \ldots \supseteq \Gamma_{F_{i-1}}(I) \supseteq \Gamma_{F_i}(I) \supseteq \Gamma_{F_{i+1}}(I) \supseteq \ldots \quad . \tag{2.10}$$

This determines a canonical spectral sequence $(E^r_{i,j}, d^r_{i,j})$ with

$$E^0_{-i,-j} = \Gamma_{F_i}(I_{-i-j})/\Gamma_{F_{i+1}}(I_{-i-j}) = \Gamma_{F_i/\Gamma_{F_{i+1}}}(I_{-i-j})$$

and

$$E^1_{-i,-j} = \mathrm{H}_{-i-j}(E^0_{-i,\bullet}) = \mathrm{H}^{i+j}_{F_i/F_{i+1}}(M).$$

It is well known that every boundary map

$$d^1_{-i,-j} \colon \mathrm{H}^{i+j}_{F_i/F_{i+1}}(M) \to \mathrm{H}^{i+j+1}_{F_{i+1}/F_{i+2}}(M)$$

is a connecting homomorphism in the long exact sequence of homology associated to the short exact sequence

$$0 \to \Gamma_{F_{i+1}/F_{i+2}}(I) \to \Gamma_{F_i/F_{i+2}}(I) \to \Gamma_{F_i/F_{i+1}}(I) \to 0 \tag{2.11}$$

(see [60, Lemma 8.24], for example). Moreover, if $(\Gamma_{F_i}(I))_{i\in\mathbb{Z}}$ is bounded, then the above spectral sequence converges:

$$E^2_{i,j} \overset{i}{\to} \mathrm{H}_n(M) \quad \text{where} \quad n = i + j. \tag{2.12}$$

Remark 2.3.32. *Let (R, m) be a local ring, and let $\mathcal{F}$ be a filtration of* $\operatorname{Spec} R$*. Set $m \in \delta_t F$ for some $t \in \mathbb{Z}$. Since m is the minimal element of F_t, then $F_{t+1} = \emptyset$. Therefore $\mathcal{F}$ is bounded so that the above spectral sequence converges.*

The Cousin complex of M corresponding to the filtration $\mathcal{F}$ is now the complex

$$\dots \longrightarrow \mathrm{H}^{i-1}_{F_{i-1}/F_i}(M) \overset{d^1_{-i+1,0}}{\longrightarrow} \mathrm{H}^{i}_{F_i/F_{i+1}}(M) \overset{d^1_{-i,0}}{\longrightarrow} \mathrm{H}^{i+1}_{F_{i+1}/F_{i+2}}(M) \overset{d^1_{-i-1,0}}{\longrightarrow} \dots$$

consisting of the $E^1_{-i,0}$ terms of the above spectral sequence. It is denoted by $E_{\mathcal{F}}(M)$. Note that by formula (2.9) there is a natural isomorphism

$$E_{\mathcal{F}}(M)_{-i} = \bigoplus_{p\in\delta_i\mathcal{F}} \mathrm{H}^i_{pR_p}(M_p) \tag{2.13}$$

Also observe that

$$\mathrm{H}_n\left(E_{\mathcal{F}}(M)\right) = E^2_{n,0}.$$

Let $\eta\colon M \to N$ be a morphism in $D_-(R)$. Take bounded above injective resolutions I and J for M and N, respectively. Then η can be represented as morphism of complexes $I \to J$, which is unique up to homotopy. Moreover, this morphism is a morphism of filtered complexes inducing a morphism between the corresponding spectral sequences. Therefore we obtain a morphism of complexes

$$E_{\mathcal{F}}(\eta)\colon E_{\mathcal{F}}(M) \to E_{\mathcal{F}}(M).$$

In fact,

$$E_{\mathcal{F}}(\eta) = (\mathrm{H}_{-i}(\mathbf{R}\Gamma_{F_i/F_{i+1}}(\eta)))_{i\in\mathbb{Z}}.$$

In this way, we obtain a covariant functor $E_{\mathcal{F}}(-)\colon D_{-}(R) \to \mathrm{Coz}(\mathcal{F}, R)$.

Remark 2.3.33. *Let $\eta\colon M \to N$ be a morphism in $D_{-}(R)$. By using the natural isomorphism* (2.8) *we see that $E_{\mathcal{F}}(\eta)$ is isomorphism, if*

$$\mathrm{H}^i_{pR_p}(\eta_p)\colon \mathrm{H}^i_{pR_p}(M_p) \to \mathrm{H}^i_{pR_p}(N_p)$$

is isomorphism for all $i \in \mathbb{Z}$.

Lemma 2.3.34. *Let R be a ring and let $\mathcal{F}$ be a filtration of* $\operatorname{Spec} R$*. If $M \in \mathrm{Coz}(\mathcal{F}, R)$ and $i, j \in \mathbb{Z}$, then*

a) $\Gamma_{F_i}(M_{-j}) = M_{-j}$ for $i \le j$ and $\Gamma_{F_i}(M_{-j}) = 0$ for $i > j$;

b) $\mathrm{H}^n_{F_i/F_{i+1}}(M_{-j}) = 0$ for all $n > 0$;

c) $\mathbf{R}\Gamma_{F_i/F_{i+1}}(M) \cong \Gamma_{F_i/F_{i+1}}(M)$;

d) $E_{\mathcal{F}}(Q(M)) \cong M$.

Here $Q(-)$ denotes the localization functor from $K(R)$ to $D(R)$.

Proof. *a*) Let $i > j$. Now if $p \in \delta_j F$, then $p \notin F_i$. Take $\alpha \in \Gamma_{F_i}(M_{-j})$. Then by definition of a Cousin complex $\alpha = (\alpha_p)_{p\in\delta_j\mathcal{F}}$, where every α_p is pR_p-torsion. If we would have $\alpha_p \neq 0$ for some $p \in \delta_j\mathcal{F}$, then $p \in \mathrm{Supp}_R \langle\alpha_p\rangle \subseteq \mathrm{Supp}_R \langle\alpha\rangle$ would imply $p \in F_i$, which is a contradiction. Therefore $\Gamma_{F_i}(M_{-j}) = 0$. In the case $i \le j$, one has $\Gamma_{F_i}(M_{-j}) = M_{-j}$, because $\mathrm{Supp}_R(M_{-j}) \subseteq F_j$.

b) By formula (2.9) it is enough to show that $\mathrm{H}^n_{pR_p}((M_{-j})_p) = 0$ for all $p \in \delta_i\mathcal{F}$. Let $M_{-j} = \oplus_{q\in\delta_j F} M(q)$. Assume that $\mathrm{H}^n_{pR_p}(M(q)_p) \neq 0$ for some $n \ge 0$. We will show that $p = q$. Since now $M(q)_p \neq 0$, we have $q \subseteq p$. On the other hand, $\mathrm{H}^n_p(M(q))$ is an R_q-module so that $\mathrm{H}^n_p(M(q)) \cong \mathrm{H}^n_{pR_q}((M(q))_q)$. Therefore the module $\mathrm{H}^n_p(M(q))$ being non-zero implies that $p \subseteq q$. Hence $p = q$ as claimed. Now, in the case $p = q$, $\mathrm{H}^n_{qR_q}(M(q)_q) = 0$ for all $n > 0$, because the elements of $M(q)$ are qR_q-torsion, and we are done.

c) Because the Cousin complex is $\Gamma_{F_i/F_{i+1}}$-acyclic by b), we can by [44, Proposition 2.2.6] use it as a resolution of itself to compute $\mathbf{R}\Gamma_{F_i/F_{i+1}}$.

d) By c) we now have $E_{\mathcal{F}}(M)_{-i} = H_i(\Gamma_{F_i/F_{i+1}}(M))$ for all $i \in \mathbb{Z}$. But by a), we see that $\Gamma_{F_i/F_{i+1}}(M)) = \sum^{-i} M_{-i}$ so that $E_{\mathcal{F}}(M)_{-i} = M_{-i}$. To get the boundary maps, we look at the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_{F_{i+1}/F_{i+2}}(M) & \longrightarrow & \Gamma_{F_i/F_{i+2}}(M) & \longrightarrow & \Gamma_{F_i/F_{i+1}}(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \sum^{-i-1} M_{-i-1} & \longrightarrow & X & \longrightarrow & \sum^{-i} M_{-i} & \longrightarrow & 0 \end{array}$$

of exact sequences, where every vertical map is an identity morphism. Here X denotes the complex

$$\ldots \to 0 \to M_{-i} \stackrel{d^M_{-i}}{\to} M_{-i-1} \to 0 \to \ldots$$

concentrated in degrees $-i$ and $-i-1$. From the long exact sequences of homology associated to the above exact sequences we get the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i_{F_i/F_{i+1}}(M) & \xrightarrow{d^1_{-i,0} = d^{E_{\mathcal{F}}(M)}_{-i}} & \mathrm{H}^{i+1}_{F_{i+1}/F_{i+2}}(M) \\ \mathrm{Id}\downarrow & & \downarrow\mathrm{Id} \\ M_{-i} & \xrightarrow{\delta_{-i}} & M_{-i-1}. \end{array}$$

It is straightforward to directly determine the connecting homomorphisms of the bottom sequence to obtain $\delta_{-i} = d^M_{-i}$. This now implies the claim.

□

The following result has been proved by Lipman, Nayak and Sastry in [45, Proposition 9.3.5]. Similar results have been obtained in the module case by Dibaei and Tousi in [24, Theorem 1.4], and by Kawasaki in [41, Theorem 5.4].

Proposition 2.3.35. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D^f_b(R)$. Then*

$$E_{\mathcal{D}(M)}(M) \cong \sum^{-\dim_R M} \mathrm{Hom}_R(K_M, D_R).$$

Proof. Set $N = \sum^{-\dim_R M} \mathrm{Hom}_R(K_M, D_R)$. Then $N \in \mathrm{Coz}(\mathcal{D}(M), R)$ by Example 2.3.27. Lemma 2.3.34 implies that $E_{\mathcal{D}(M)}(N) \cong N$. To see that

$E_{\mathcal{D}(M)}(M) \cong N$, it is enough to show that the natural morphism $h_M \colon M \to N$ induces an isomorphism

$$E_{\mathcal{D}(M)}(M) \cong E_{\mathcal{D}(M)}(N).$$

We will utilize Remark 2.3.33. Let $i \in \mathbb{Z}$ and suppose that $p \in \delta_i \mathcal{D}(M)$. Localizing the triangle (2.6) at p and applying the functor $\mathbf{R}\Gamma_{pR_p}(-)$ yields the exact triangle

$$\mathbf{R}\Gamma_{pR_p}((C_M^\dagger)_p) \to \mathbf{R}\Gamma_{pR_p}(M_p) \overset{\mathbf{R}\Gamma_{pR_p}((h_M)_p)}{\to} \mathbf{R}\Gamma_{pR_p}(N_p) \to \textstyle\sum^1 \mathbf{R}\Gamma_{pR_p}(C_M^\dagger)_p).$$

Using formula (1.10), we get

$$\begin{aligned} -\inf \mathbf{R}\Gamma_{pR_p}((C_M^\dagger)_p) &= \dim_{R_p}(C_M^\dagger)_p \\ &\leq \dim_R C_M^\dagger - \dim R/p \\ &= \sup C_M - \dim R/p \\ &\leq \dim_R M - 1 - \dim R/p \\ &= i - 1. \end{aligned}$$

Here the first inequality comes from [18, Lemma 6.3.4]. The second equality is by formula (1.14), while the last one is because $p \in \delta_i \mathcal{D}(M)$. Since $\inf \mathbf{R}\Gamma_{pRp}((C_M^\dagger)_p) \leq -i+1$, it follows from the long exact sequence of homology corresponding to the above triangle that

$$\mathrm{H}^i_{pR_p}((h_M)_p) \colon \mathrm{H}^i_{pR_p}(M_p) \to \mathrm{H}^i_{pR_p}(N_p)$$

is an isomorphism, as desired. $\square$

2.3.2 Sharp's Cousin complex

Let R be a ring, and let $\mathcal{F} = (F_i)_{i \geq 0}$ be a filtration $\operatorname{Spec} R$. Suppose that M is an R-module with $\operatorname{Supp}_R M \subseteq F_0$. Sharp introduced a commutative algebra analogue of the Cousin complex. Sharp's Cousin complex is a complex

$$C(\mathcal{F}, M): \quad \cdots \to 0 \overset{d_2}{\to} M \overset{d_1}{\to} M_0 \overset{d_0}{\to} M_{-1} \overset{d_{-1}}{\to} \ldots \quad ,$$

where

$$M_{-i} = \bigoplus_{p \in \delta_i \mathcal{F}} (\operatorname{Coker} d_{-i+2})_p$$

for all $i \geq 0$. For details of the construction of this complex, we refer to [54] and [57]. Note, however, that to get a Cousin complex in the sense of Definition 2.3.24 one has to look at the complex

$$C(\mathcal{F}, M)': \quad \ldots \to 0 \to 0 \to M_0 \overset{d_0}{\to} M_{-1} \overset{d_{-1}}{\to} \ldots \quad .$$

More generally, a complex

$$X = (X_n)_{n\le 2}\colon \quad 0 \xrightarrow{d_2} M \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} \dots$$

is said to be of "Cousin type for M with respect to $\mathcal{F}$", if it satisfies the following conditions for all $n \in \mathbb{N}$:

a) $\operatorname{Supp}_R X_{-n} \subseteq F_n$;

b) $\operatorname{Supp}_R \mathrm{H}_{-n+1}(X) \subseteq F_{n+1}$;

c) $\operatorname{Supp}_R \operatorname{Coker} d_{-n+2} \subseteq F_n$;

d) The natural homomorphism $X_{-n} \to \bigoplus_{p\in\delta_n\mathcal{F}} (X_{-n})_p$ is an isomorphism

(see [48]). In particular, it was shown in [58] that $C(\mathcal{F}, M)$ is of Cousin type for M with respect to $\mathcal{F}$.

We want to compare Sharp's Cousin complex $C(\mathcal{F}, M)$ and the Cousin complex $E_{\mathcal{F}}(M)$. Let $d\colon M \to \mathrm{H}^0_{F_0/F_1}(M)$ be the natural homomorphism. We will first show that

$$E_{\mathcal{F}}(M)^*\colon 0 \to M \xrightarrow{d} E_{\mathcal{F}}(M)_0 \xrightarrow{d_0} E_{\mathcal{F}}(M)_{-1} \xrightarrow{d_{-1}} E_{\mathcal{F}}(M)_{-2} \dots \tag{2.14}$$

is a complex of Cousin type. The following result, proved by Riley, Sharp and Zakeri in [48, Theorem 3.3], will then yield $E_{\mathcal{F}}(M)^* \cong C(\mathcal{F}, M)$.

Theorem 2.3.36. *Let $\mathcal{F} = (F_i)_{i\ge 0}$ be a filtration of $\operatorname{Spec} R$, and let M be an R-module with $\operatorname{Supp}_R M \subseteq F_0$. Suppose that X and Y are two complexes of Cousin type for M with respect to $\mathcal{F}$. Then there exists a unique isomorphism of complexes $\phi\colon X \to Y$ with $\phi_1 = \mathrm{Id}_M$.*

To proceed, we first need two auxiliary results.

Lemma 2.3.37. *Let R be a ring, and let $\mathcal{F} = (F_i)_{i\ge 0}$ be a filtration of $\operatorname{Spec} R$. Suppose M is an R-module. Then the following statements hold:*

a) If $p \in \delta_i\mathcal{F}$, then $i \ge \operatorname{height} p$;

b) $\mathrm{H}^t_{pR_p}(M_p) = 0$ for any $p \in \delta_i\mathcal{F}$ and for any $t > i$.

Proof. a) We will show that if $q \subset p$ for $p \in \delta_i\mathcal{F}$ and $q \in \delta_j\mathcal{F}$, then $j < i$. Assume for the sake of contradiction that $i \le j$. Because now $F_j \subseteq F_i$, $q \in F_i$. This is impossible, since p is a minimal element in F_i, and we are done.

Suppose next that height $p = n$. Let $p_0 \subset p_1 \subset \ldots \subset p_{n-1} \subset p_n = p$ be a saturated chain of prime ideals with $p_j \in \delta_{t_j}\mathcal{F}$ for any j. Then,

$$t_0 < t_1 < \ldots < t_n = i$$

which is an increasing chain of length $n+1$. Since $t_0 \geq 0$, it thus follows that $i \geq n$, as wanted.

b) Because $p \in \delta_i\mathcal{F}$, we have $t \geq \dim R_p$ by a). Hence $t \geq \dim_{R_p} M_p$ so that $\mathrm{H}^t_{pR_p}(M_p) = 0$ by formula (1.10). □

Proposition 2.3.38. *Let R be a ring, and let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of* $\operatorname{Spec} R$*. If M is an R-module, then*

$$\operatorname{Supp}_R \mathrm{H}_{-n}(E_{\mathcal{F}}(M)) \subseteq F_{n+2}.$$

Proof. Let $p \in \operatorname{Spec} R$. Set $\mathcal{F}' = (F'_n)_{n\geq 0}$, where

$$F'_n = \{qR_p \in \operatorname{Spec} R_p \mid q \in F_n\}.$$

Using formula (2.8) we get that

$$(\mathbf{R}\Gamma_{F_n/F_{n+1}}(M))_p \cong \mathbf{R}\Gamma_{F'_n/F'_{n+1}}(M_p).$$

Therefore $E_{\mathcal{F}}(M)_p \cong E_{\mathcal{F}'}(M_p)$. Hence $p \in \operatorname{Supp}_R \mathrm{H}_{-n}(E_{\mathcal{F}}(M))$ if and only if $pR_p \in \operatorname{Supp}_{R_p} \mathrm{H}_{-n}(E_{\mathcal{F}'}(M_p))$. So it is enough to prove the claim for $E_{\mathcal{F}'}(M_p)$. We may thus assume without loss of generality that R is local.

We will use decreasing induction on r to show that $\operatorname{Supp}_R E^r_{-n,0} \subseteq F_{n+2}$. By Remark 2.3.32 the spectral sequence now converges to the homology of M. Thus $E^\infty_{-n+i,-i}$ can be considered as a factor module of a bounded filtration of $\mathrm{H}_{-n}(M)$. Since $\mathrm{H}_{-n}(M) = 0$, $E^\infty_{-n+i,-i} = 0$ for all $i \in \mathbb{Z}$. Hence $E^r_{-n+i,-i} = 0$ for large enough r (see [50, Theorem 10.14], for example). Therefore we may assume that $\operatorname{Supp}_R E^{r+1}_{-n,0} \subseteq F_{n+2}$ for some $r > 1$, which forms the basis of the induction. Note also that

$$E^1_{-n+r,-r+1} = \bigoplus_{p\in\delta_{n-r}\mathcal{F}} \mathrm{H}^{n-1}_{pR_p}(M_p) = 0$$

by formula (2.9) and Lemma 2.3.37 b). Hence $E^r_{-n+r,-r+1} = 0$. Consequently, the complex

$$\ldots \to E^r_{-n+r,-r+1} \overset{d^r_{-n+r,-r+1}}{\to} E^r_{-n,0} \overset{d^r_{-n,0}}{\to} E^r_{-n-r,r-1} \overset{d^r_{-n-r,r-1}}{\to} \ldots$$

gives the exact sequence

$$0 \to E^{r+1}_{-n,0} \to E^r_{-n,0} \overset{d^r_{-n,0}}{\to} E^r_{-n-r,r-1}.$$

Therefore using the induction assumption for $r+1$, we obtain

$$\operatorname{Supp}_R E^r_{-n,0} \subseteq F_{n+2} \cup \operatorname{Supp}_R E^r_{-n-r,r-1}.$$

It is then enough to check that $\operatorname{Supp}_R E^r_{-n-r,r-1} \subseteq F_{n+r}$. To see this, assume that I is an injective resolution for M. Then

$$\begin{aligned} \operatorname{Supp}_R E^1_{-n-r,r-1} &= \operatorname{Supp}_R \mathrm{H}^{n+1}_{F_{n+r}/F_{n+r+1}}(I) \\ &\subseteq \operatorname{Supp}_R \Gamma_{F_{n+r}}(I_{-n-1}) \\ &\subseteq F_{n+r}, \end{aligned}$$

which completes the proof. □

We are now ready to show that

$$E_{\mathcal{F}}(M)^*\colon 0 \to M \xrightarrow{d} E_{\mathcal{F}}(M)_0 \xrightarrow{d_0} E_{\mathcal{F}}(M)_{-1} \xrightarrow{d_{-1}} E_{\mathcal{F}}(M)_{-2} \dots \qquad (2.15)$$

is a complex with $\operatorname{Supp}_R \mathrm{H}_1(E_{\mathcal{F}}(M)^*) \subseteq F_1$ and $\operatorname{Supp}_R \mathrm{H}_0(E_{\mathcal{F}}(M)^*) \subseteq F_2$.

Suppose that I is an injective resolution of M, which is bounded above. Consider the commutative diagram of complexes

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & \Gamma_{F_2}(I) & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & \Gamma_{F_1}(I) & \longrightarrow & \Gamma_{F_0}(I) & \longrightarrow & \Gamma_{F_0/F_1}(I) & \longrightarrow & 0 \\ & & \downarrow j & & \downarrow & & \downarrow \mathrm{Id} & & \\ 0 & \longrightarrow & \Gamma_{F_1/F_2}(I) & \longrightarrow & \Gamma_{F_0/F_2}(I) & \longrightarrow & \Gamma_{F_0/F_1}(I) & \longrightarrow & 0. \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

Taking homology we get the commutative diagram of modules

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Gamma_{F_1}(M) & \longrightarrow & M & \xrightarrow{d} & E^1_{0,0} & \xrightarrow{\gamma} & \mathrm{H}^1_{F_1}(M) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow \mathrm{Id} & & \downarrow f & & \\ 0 & \longrightarrow & \mathrm{H}^0_{F_1/F_2}(M) & \longrightarrow & \mathrm{H}^0_{F_0/F_2}(M) & \longrightarrow & E^1_{0,0} & \xrightarrow{d_0} & E^1_{-1,0} & \longrightarrow & \dots \end{array} .$$

One clearly has $d_0 d = 0$. Moreover $\ker d \cong \Gamma_{F_1}(M)$ so that

$$\operatorname{Supp}_R \mathrm{H}_1(E_{\mathcal{F}}(M)^*) \subseteq F_1.$$

Let us then show that $\operatorname{Supp}_R \mathrm{H}_0(E_{\mathcal{F}}(M)^*) \subseteq F_2$. Note that the natural map

$$\mathrm{H}_0\left(E_{\mathcal{F}}(M)^*\right) \to \ker f$$

which sends $e + \operatorname{Im} d$ to $\gamma(e)$ for any $e \in \ker d_0$, makes sense. This is injective so that $\operatorname{Supp}_R \mathrm{H}_0(\mathrm{E}_{\mathcal{F}}(M)^*) \subseteq \operatorname{Supp}_R \ker f$. On the other hand, because $f = \mathrm{H}_{-1}(j)$, we get $\operatorname{Supp}_R \ker f \subseteq F_2$. This shows that

$$\operatorname{Supp}_R \mathrm{H}_0(E_{\mathcal{F}}(M)^*) \subseteq F_2,$$

as desired.

Hence, by Proposition 2.3.38 we now have $\operatorname{Supp}_R \mathrm{H}_{-n+1}(E_{\mathcal{F}}(M)^*) \subseteq F_{n+1}$ for all $n \in \mathbb{N}$. We then look at the short exact sequences

$$0 \to \mathrm{H}_{-n+1}(E_{\mathcal{F}}(M)^*) \to \operatorname{Coker} d_{-n+2} \to \operatorname{Im} d_{-n+1} \to 0.$$

Since $\operatorname{Supp}_R \operatorname{Im} d_{-n+1} \subseteq F_n$, then $\operatorname{Supp}_R \operatorname{Coker} d_{-n+2} \subseteq F_n$ for all $n \in \mathbb{N}$.

In order to complete the proof that $E_{\mathcal{F}}(M)^*$ is a complex of Cousin type, we still need to show that the natural homomorphism

$$E_{\mathcal{F}}(M)_{-n} \to \bigoplus_{p \in \delta_n \mathcal{F}} (E_{\mathcal{F}}(M)_{-n})_p$$

is an isomorphism for all $n \in \mathbb{N}$. But by formula (2.13)

$$E_{\mathcal{F}}(M)_{-n} = \bigoplus_{p \in \delta_n \mathcal{F}} \mathrm{H}^n_{pR_p}(M_p)$$

so that

$$\begin{aligned}(E_{\mathcal{F}}(M)_{-n})_p &= \bigoplus_{q \in \delta_n \mathcal{F}} (\mathrm{H}^n_{qR_q}(M_q))_p \\ &= (\mathrm{H}^n_{pR_p}(M_p))_p \\ &= \mathrm{H}^n_{pR_p}(M_p)\end{aligned}$$

for all $p \in \delta_n \mathcal{F}$.

2.3.3 Cousin Functor for Complexes of Sheaves of Modules

Finally, we want briefly sketch the relationship between the Cousin functor constructed in this work and the one defined in [36]. Let R be a ring, and let $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ be the associated affine scheme. One denotes by $D(\mathcal{O}_{\operatorname{Spec} R})$ the derived category of the category of $\mathcal{O}_{\operatorname{Spec} R}$-modules.

Let $Z \subseteq \operatorname{Spec} R$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_{\operatorname{Spec} R}$-modules. We denote by $\underline{\Gamma}_Z(\mathcal{G})$ the subsheaf of $\mathcal{G}$ whose sections over any open subset $U \subseteq \operatorname{Spec} R$ consist of those sections of $\mathcal{G}(U)$ whose support lies in Z. This gives a functor $\underline{\Gamma}_Z(-)$. Moreover, given $Z' \subseteq Z$, by setting $\underline{\Gamma}_{Z/Z'}(U) = \underline{\Gamma}_Z(U)/\underline{\Gamma}_Z(U)$ for any open subset $U \subseteq \operatorname{Spec} R$, we obtain a functor $\underline{\Gamma}_{Z/Z'}(-)$. Finally, set

$$\underline{\mathrm{H}}^i{}_Z(-) = \mathrm{H}_{-i}(\mathbf{R}\underline{\Gamma}_Z(-)) \quad \text{and} \quad \underline{\mathrm{H}}^i_{Z/Z'}(-) = \mathrm{H}_{-i}(\mathbf{R}\underline{\Gamma}_{Z/Z'}(-))$$

for all $i \in \mathbb{Z}$.

Note that for any sheaf $\mathcal{G}$ of $\mathcal{O}_{\operatorname{Spec} R}$-modules,

$$\underline{\Gamma}_Z(\mathcal{G}) = \varinjlim_{F \subseteq Z} \underline{\Gamma}_F(\mathcal{G}),$$

where $F \subseteq Z$ is closed. In particular, if M is an R-module, we obtain

$$\begin{aligned}\underline{\Gamma}_Z(\widetilde{M}) &\cong \varinjlim_{\mathbb{V}(a) \subseteq Z} \underline{\Gamma}_{\mathbb{V}(a)}(\widetilde{M}) \\ &\cong \varinjlim_{\mathbb{V}(a) \subseteq Z} (\Gamma_a(M))^{\sim} \\ &\cong (\varinjlim_{\mathbb{V}(a) \subseteq Z} \Gamma_a(M))^{\sim} \\ &\cong (\Gamma_Z(M))^{\sim}.\end{aligned}$$

Here the second isomorphism follows from [34, Exposé II. Corollaire 4]. The second last one holds true, since the functor $(-)^{\sim}$ is exact, and the last one follows from Lemma 2.3.28. More generally, if $M \in D_{-}(R)$ and I is an injective resolution of M, then by [36, II. Corollary 7.14] $\widetilde{I}$ is an injective resolution of $\widetilde{M}$. It follows that

$$\underline{\mathrm{H}}^i_Z(\widetilde{M}) = (\mathrm{H}^i_Z(M))^{\sim} \tag{2.16}$$

for all $i \in \mathbb{Z}$. Moreover, when $Z' \subset Z$, then also

$$\underline{\mathrm{H}}^i_{Z/Z'}(\widetilde{M}) = (\mathrm{H}^i_{Z/Z'}(M))^{\sim}. \tag{2.17}$$

Suppose then that $\mathcal{F}$ is a filtration of $\operatorname{Spec} R$. Recall that a quasi-coherent complex $\mathcal{M} \in D_b(\mathcal{O}_{\operatorname{Spec} R})$ is called a Cohen-Macaulay complex with respect to $\mathcal{F}$, if

$$\underline{\mathrm{H}}^i_{F_n/F_{n+1}}(\mathcal{M}) = 0$$

for every $i \neq n$ (see [45, p. 41]). However, taking into account our earlier Definition 2.2.19 , we prefer to look at bounded Cohen-Macaulay complexes with finitely generated cohomology, and denote by $D_{\mathcal{F}-CM}(\mathcal{O}_{\operatorname{Spec} R})$ the corresponding full subcategory of $D_b^f(\mathcal{O}_{\operatorname{Spec} R})$.

Recall from [36, p. 241] that a complex $\mathcal{N}$ of $\mathcal{O}_{\operatorname{Spec} R}$-modules is called a Cousin complex with respect to $\mathcal{F}$, if for every $n \in \mathbb{Z}$ there is a family $(M(p))_{p \in \delta_n F}$, of R_p-modules such that the elements of $M(p)$ are pR_p-torsion, and that

$$\mathcal{M}_{-n} \cong \bigoplus_{p \in \delta_n F} i_{p,*}(M(p))$$

where $i_p \colon \operatorname{Spec} R_p \to \operatorname{Spec} R$ is the natural morphism. It is well known that $i_{*,p}(M_p)$ are quasi-coherent (see [33, Proposition 7.24], for example). Hence any Cousin complex is a complex of quasi-coherent sheaves. The category of Cousin complexes with respect to $\mathcal{F}$ is denoted by $\operatorname{Coz}(\mathcal{F}, \mathcal{O}_{\operatorname{Spec} R})$.

The notion of the Cousin functor in $D_-(\mathcal{O}_{\operatorname{Spec} R})$ is defined in a similar way as it was done in $D_-(R)$, but now using the sheaf valued functors $\underline{\Gamma}_Z(-)$. For more details on this construction, one may consult [36, Chapter IV].

Remark 2.3.39. *Let $\mathcal{F}$ be a filtration of* $\operatorname{Spec} R$. *Suominen observed in [59, Theorem 3.9] that the Cousin functor $E_{\mathcal{F}}(-)$ together with the localization functor $Q(-)$ makes an equivalence of categories between the category of Cousin complexes and that of Cohen-Macaulay complexes. This now restricts to the equivalence*

$$\operatorname{Coz}_b^f(\mathcal{F}, \mathcal{O}_{\operatorname{Spec} R}) \underset{E_{\mathcal{F}}(-)}{\overset{Q(-)}{\rightleftarrows}} D_{\mathcal{F}-CM}(\mathcal{O}_{\operatorname{Spec} R}) .$$

Lemma 2.3.40. *Let R be a ring, and let* $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ *be the associated affine scheme. Suppose that $\mathcal{F}$ is a filtration on* $\operatorname{Spec} R$. *Then the following statements hold:*

a) If $M \in D_-(R)$, then $E_{\mathcal{F}}(\widetilde{M}) \cong (E_{\mathcal{F}}(M))^{\sim}$. In particular, $E_{\mathcal{F}}(M) \cong \Gamma_{\operatorname{Spec} R}(E_{\mathcal{F}}(\widetilde{M}))$;

b) A complex $M \in D_{\mathcal{F}-CM}(R))$ if and only if $\widetilde{M} \in D_{\mathcal{F}-CM}(\mathcal{O}_{\operatorname{Spec} R})$;

b') A quasi-coherent complex $\mathcal{M} \in D_{\mathcal{F}-CM}(\mathcal{O}_{\operatorname{Spec} R})$ if and only if $\Gamma_{\operatorname{Spec} R}(\mathcal{M}) \in D_{\mathcal{F}-CM}(R)$;

c) A complex $N \in \mathrm{Coz}(\mathcal{F}, R)$ *if and only if* $\widetilde{N} \in \mathrm{Coz}(\mathcal{F}, \mathcal{O}_{\mathrm{Spec}\, R})$*;*

c') A complex $\mathcal{N} \in \mathrm{Coz}(\mathcal{F}, \mathcal{O}_{\mathrm{Spec}\, R})$ *if and only if* $\Gamma_{\mathrm{Spec}\, R}(\mathcal{N}) \in \mathrm{Coz}(\mathcal{F}, R)$.

Proof. *a*) Suppose that I is an injective resolution for M, which is bounded above. Let $(E^r_{i,j}, d^r_{i,j})$ denote the spectral sequence associated to the filtration

$$(\Gamma_{F_i}(I))_{i\in\mathbb{Z}} : \ldots \supseteq \Gamma_{F_{i-1}}(I) \supseteq \Gamma_{F_i}(I) \supseteq \Gamma_{F_{i+1}}(I) \supseteq \ldots \qquad .$$

of I. By [36, II. Corollary 7.14] $\widetilde{I}$ is an injective resolution of $\widetilde{M}$. Since $\underline{\Gamma}_{F_i}(\widetilde{I}) = (\Gamma_{F_i})^{\sim}$ for all $i \in \mathbb{Z}$ and the functor $(-)^{\sim}$ is exact, it follows that $(\widetilde{E^r_{i,j}}, \widetilde{d^r_{i,j}})$ is now the spectral sequence associated to the filtration

$$(\underline{\Gamma}_{F_i}(\widetilde{I}))_{i\in\mathbb{Z}} : \ldots \supseteq \underline{\Gamma}_{F_{i-1}}(\widetilde{I}) \supseteq \underline{\Gamma}_{F_i}(\widetilde{I}) \supseteq \underline{\Gamma}_{F_{i+1}}(\widetilde{I}) \supseteq \ldots \qquad .$$

of $\widetilde{I}$. Then also $E_{\mathcal{F}}(\widetilde{M}) \cong (E_{\mathcal{F}}(M))^{\sim}$, and by [37, II. Proposition 5.1 (d)] $E_{\mathcal{F}}(M) \cong \Gamma_{\mathrm{Spec}\, R}(E_{\mathcal{F}}(\widetilde{M}))$.

b) By formula (2.17) $\mathrm{H}^n_{F_i/F_{i+1}}(M) = 0$ if and only if $\underline{\mathrm{H}}^n_{F_i/F_{i+1}}(\widetilde{M}) = 0$. The claim now follows by formula (2.9) from the definition of a Cohen-Macaulay complex.

b′) Note that by [37, II Corollary 5.5] the functor $(-)^{\sim}$ gives an equivalence between the derived category of R-modules and the derived category of quasi-coherent $\mathcal{O}_{\mathrm{Spec}\, R}$-modules. Hence $\mathcal{M} \simeq \widetilde{M}$ for some $M \in D_b(R)$. Moreover, $\Gamma_{\mathrm{Spec}\, R}(\mathcal{M}) \simeq M$. The claim thus follows from b).

c) Let $n \in \mathbb{Z}$. Suppose that $(M(p))_{p\in\delta_n F}$ is a family of R_p-modules such that the elements of $M(p)$ are pR_p-torsion. Now $((N(p))^{\sim} \cong i_{p,*}(N(p))$ (see [33, Proposition 7.24], for example). On the other hand, we have $\Gamma(\mathrm{Spec}\, R, i_{p,*}(N(p))) = N(p)$. So

$$N_{-n} \cong \bigoplus_{p\in\delta_n F} N(p)$$

if and only if

$$(N_{-n})^{\sim} \cong \bigoplus_{p\in\delta_n F} i_{p,*}(N(p)).$$

c′) Noting that $\mathcal{N}$ is a quasi-coherent complex, the claim follows as in b').

□

Remark 2.3.41. *Let $\mathcal{F}$ be a filtration of* $\operatorname{Spec} R$*. We look at the diagram*

$$\begin{array}{ccc}
\operatorname{Coz}_f^b(\mathcal{F}, \mathcal{O}_{\operatorname{Spec} R}) & \underset{E_{\mathcal{F}}(-)}{\overset{Q(-)}{\rightleftarrows}} & D_{\mathcal{F}-CM}(\mathcal{O}_{\operatorname{Spec} R}) \\
\widetilde{(-)} \uparrow \downarrow \Gamma_{\operatorname{Spec} R}(-) & & \widetilde{(-)} \uparrow \downarrow \Gamma_{\operatorname{Spec} R}(-) \\
\operatorname{Coz}_f^b(\mathcal{F}, R) & \underset{E_{\mathcal{F}}(-)}{\overset{Q(-)}{\rightleftarrows}} & D_{\mathcal{F}-CM}(R)
\end{array}$$

of categories and functors, where the upper arrows make the equivalence mentioned in Remark 2.3.39. Because $E_{\mathcal{F}}(\widetilde{N}) \cong (E_{\mathcal{F}}(N))^{\sim}$ by Lemma 2.3.40 a), the diagram is commutative. The vertical arrows are also equivalences of categories by Lemma 2.3.40 b) and c). It therefore follows that the lower arrows provide an equivalence of categories.

2.4 Conclusion

If M is a non-zero finitely generated module, it is known that M satisfying Serre's condition (S_n) is equivalent to Sharp's Cousin complex $C(\mathcal{H}(M), M)$ being exact at the spots $1, 0, -1, \ldots, n-2$ (see [56, Theorem 2.2] and [58, p. 516]). Inspired by this result, we state our next Theorem 2.4.1. It follows immediately from Proposition 2.3.14 and Proposition 2.3.35.

Theorem 2.4.1. *Let (R, m) be a local ring admitting a dualizing complex. Let $k \in \mathbb{Z}$ and $M, N \in D_b^f(R)$. If M is equidimensional, then the following conditions are equivalent:*

a) M satisfies Serre's condition $(S_{k,N})$;

b) The natural homomorphism $\operatorname{Ext}_R^{-i}(N, M) \to \operatorname{Ext}_R^{-i}(N, E_{\mathcal{H}(M)}(M))$ is bijective for all $i > -k+2$, and injective for $i = -k+1$.

In particular, this gives

Corollary 2.4.2. *Let (R, m) be a local ring admitting a dualizing complex. Let $k, l \in \mathbb{Z}$. If $M \in D_b^f(R)$ is equidimensional, then the following conditions are equivalent:*

a) M satisfies Serre's condition $(S_{k,l})$;

b) The morphism $\mathrm{H}_i(M) \to \mathrm{H}_i(E_{\mathcal{H}(M)}(M))$ is bijective for for all $i > l-k+2$, and injective for $i = l-k+1$.

We record here also the next two consequences of Proposition 2.3.35.

Corollary 2.4.3. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$. Set $s = \sup M$. If M and K_M are both equidimensional and satisfy Serre's condition (S_2), then*

$$E_{\mathcal{H}(M)}(M) \cong \textstyle\sum^s E_{\mathcal{H}(\mathrm{H}_s(M))}(\mathrm{H}_s(M)).$$

Proof. Set $t = \dim_R M$. We know by Corollary 2.3.15 that $\dim_R \mathrm{H}_s(M) = s + t$. Moreover, since $\mathrm{H}_s(M) \cong K_{K_M}$ by Corollary 2.3.16, then $\mathrm{H}_s(M)$ is equidimensional. Therefore by Proposition 2.3.35 we get

$$E_{\mathcal{H}(\mathrm{H}_s(M))}(\mathrm{H}_s(M)) \cong \textstyle\sum^{-s-t}(K_{\mathrm{H}_s(M)})^\dagger \cong \sum^{-s-t}(K_M)^\dagger \cong \sum^{-s} E_{\mathcal{H}(M)}(M)$$

where the second isomorphism comes from Corollary 2.3.16. □

Corollary 2.4.4. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$. Set $s = \sup M$. Suppose that $\sup_R M_p = \sup M$ for any $p \in \mathrm{Supp}_R M$. If $M^\dagger$ satisfies Serre's condition (S_2), then*

$$E_{\mathcal{D}(M)}(M) \cong \textstyle\sum^s E_{\mathcal{D}(\mathrm{H}_s(M))}(\mathrm{H}_s(M)).$$

Proof. Set $t = \dim_R M$. Note that $M^\dagger$ is equidimensional by Proposition 2.2.4, and $\dim_R \mathrm{H}_s(M) = s + t$ by Corollary 2.3.22. Since $K_{\mathrm{H}_s(M)} \cong K_M$ by Corollary 2.3.23, Proposition 2.3.35 gives

$$E_{\mathcal{D}(\mathrm{H}_s(M))}(\mathrm{H}_s(M)) \cong \textstyle\sum^{-s-t}(K_{\mathrm{H}_s(M)})^\dagger \cong \sum^{-s-t}(K_M)^\dagger \cong \sum^{-s} E_{\mathcal{D}(M)}(M).$$

□

Chapter 3

G-Gorenstein Complexes

3.1 Structure

In this chapter we will introduce the category of G-Gorenstein complexes in $D_b^f(R)$ which strictly includes the category of Gorenstein complexes. Recall from [36, p. 248] that a complex $M \in D_b^f(R)$ is called a Gorenstein complex if it is Cohen-Macaulay and the local cohomology modules $\mathrm{H}^i_{pR_p}(M_p)$ are injective R_p-modules for all $i \in \mathbb{Z}$ and $p \in \operatorname{Spec} R$. Motivated by this, we now give

Definition 3.1.1. *Let R be a ring. A complex $M \in D_b^f(R)$ is called a G-Gorenstein complex if it is a Cohen-Macaulay and the local cohomology modules $\mathrm{H}^i_{pR_p}(M_p)$ are Gorenstein injective R_p-modules for all $i \in \mathbb{Z}$ and $p \in \operatorname{Spec} R$.*

Taking into account that every Gorenstein complex is a G-Gorenstein complex, the following question immediately comes out:

Question 3.1.2. *Is there a G-Gorenstein complex that is not a Gorenstein complex?*

The above question will be answered by Proposition 3.2.3.

Remark 3.1.3. *Suppose that R admits a dualizing complex. Then $\mathrm{H}^i_{pR_p}(M_p)$ is Gorenstein injective as an R_p-module if and only if it is Gorenstein injective as an R-module (use [1, Lemma 3.2] and [20, Proposition 5.5]).*

We could reformulate Definition 3.1.1 in the presence of a dualizing complex as follows by using only maximal ideals:

Proposition 3.1.4. *Let R be a ring admitting a dualizing complex, and let $M \in D_b^f(R)$. Then M is a G-Gorenstein complex if and only if M is Cohen-Macaulay and the local cohomology modules $\mathrm{H}_m^i(M)$ are Gorenstein injective R_m-modules for all $m \in \operatorname{Max}(R)$ and $i \in \mathbb{Z}$.*

Proof. Let $m \in \operatorname{Max}(R)$ and $i \in \mathbb{Z}$. It is enough to show that if $\mathrm{H}_m^i(M)$ is Gorenstein injective, then $\mathrm{H}_{pR_p}^i(M_p)$ is Gorenstein injective for all $p \in \operatorname{Spec} R$ with $p \subseteq m$. Since R admits a dualizing complex, it follows from [20, Proposition 5.5] that $\mathrm{H}_{mR_m}^i(M_m) \cong (\mathrm{H}_m^i(M))_m$ is Gorenstein injective. We may thus assume that R is local. We have $\mathrm{H}_m^i(M) \cong \operatorname{Hom}_R(K_M^i, E_R(k))$. The module $\mathrm{H}_m^i(M)$ now being Gorenstein injective, this implies by [14, Theorem 6.4.2] that K_M^i is Gorenstein flat. By Lemma 2.1.4 a)

$$K_{M_p}^i \cong (K_M^{i+\dim R/p})_p.$$

So $K_{M_p}^i$ is Gorenstein flat. Using [14, Theorem 6.4.2] again shows that $\mathrm{H}_{pR_p}^i(M_p) \cong \operatorname{Hom}_{R_p}(K_{M_p}^i, E_{R_p}(R_p/pR_p))$ is Gorenstein injective as wanted. □

In analogy with Sharp's result [55, Theorem 3.11 (vi)] on Gorenstein modules, we want to characterize G-Gorenstein complexes in terms of Gorenstein injective dimension. First we need two lemmas.

Lemma 3.1.5. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$. Then $\operatorname{Gid}_R M = \operatorname{Gid}_R \mathbf{R}\Gamma_m(M)$.*

Proof. Since R admits a dualizing complex, we know by [20, Theorem 5.9 b)] that $\operatorname{Gid}_R \mathbf{R}\Gamma_m(M)$ and $\operatorname{Gid}_R M$ are simultaneously finite. So we can suppose that both of them are finite. We will use [20, Theorem 6.8)] according to which

$$\operatorname{Gid}_R N = \sup\left\{\operatorname{depth} R_p - \operatorname{width}_{R_p} N_p \mid p \in \operatorname{Spec} R\right\}$$

for any $N \in D_b(R)$. Here $\operatorname{width}_{R_p} N_p = \infty$ if $p \notin \operatorname{Supp}_R N$. Noting that $\operatorname{Supp}_R \mathbf{R}\Gamma_m(M) = \{m\}$, it then follows that

$$\operatorname{Gid}_R \mathbf{R}\Gamma_m(M) = \operatorname{depth} R - \operatorname{width}_R \mathbf{R}\Gamma_m(M).$$

Recall that $\mathbf{R}\Gamma_m(M) \simeq \mathrm{C}_m(R) \otimes_R^L M$, where $\mathrm{C}_m(R)$ denotes the Čech complex on m (see [43, Proposition 3.1.2], for example). Note that $\operatorname{width}_R \mathrm{C}_m(R) = 0$, because $\mathrm{C}_m(R) \otimes_R k \simeq k$ (this small sentence is added instead of the reference [A.6.5]). Therefore, $\operatorname{width}_R \mathbf{R}\Gamma_m(M) = \operatorname{width}_R M$. Furthermore, we have $\operatorname{width}_R M = \inf M$, since $M \in D_b^f(R)$. On the other hand, by [20, Theorem 6.3)] $\operatorname{Gid}_R M = \operatorname{depth} R - \inf M$. We can thus conclude that $\operatorname{Gid}_R \mathbf{R}\Gamma_m(M) = \operatorname{Gid}_R M$, as wanted. □

Lemma 3.1.6. *Let (R, m) be a local ring admitting a dualizing complex. If $M \in D_b^f(R)$ has finite Gorenstein injective dimension, then $\operatorname{Gid}_R M \geq \dim_R M$.*

Proof. One has $\operatorname{Gid}_R M = \operatorname{Gpd}_R M^\dagger$ by [20, Corollary 6.4]. Obviously we have $\operatorname{Gpd}_R M^\dagger \geq \sup M^\dagger$. So the claim results from formula (1.14). □

We are now ready to prove

Proposition 3.1.7. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$. Then the following statements are equivalent:*

a) M is a G-Gorenstein complex;

b) $\dim_R M = \operatorname{depth}_R M = \operatorname{Gid}_R M$;

c) The Gorenstein injective dimension of M is finite and

$$\operatorname{depth}_R M = \operatorname{depth} R - \inf M.$$

Proof. *a)* $\Leftrightarrow$ *b)* : Set $\dim_R M = t$. In any case, M is Cohen-Macaulay. So $\mathbf{R}\Gamma_m(M) \simeq \sum^{-t} \mathrm{H}_m^t(M)$ by Proposition 2.2.6. By Lemma 3.1.5 we then have

$$\operatorname{Gid}_R M = \operatorname{Gid}_R \textstyle\sum^{-t} \mathrm{H}_m^t(M) = t + \operatorname{Gid}_R \mathrm{H}_m^t(M).$$

This shows that $\mathrm{H}_m^t(M)$ is Gorenstein injective if and only if $\operatorname{Gid}_R M = t$, as needed.
b) $\Leftrightarrow$ *c)* : Because $\operatorname{Gid}_R M$ is finite, we know from [20, Theorem 6.3] that $\operatorname{Gid}_R M = \operatorname{depth} R - \inf M$. Since $\dim_R M \geq \operatorname{depth}_R M$, it follows from Lemma 3.1.6 that $\operatorname{depth}_R M = \operatorname{depth} R - \inf M$ if and only if $\dim_R M = \operatorname{depth}_R M = \operatorname{Gid}_R M$.

□

Proposition 3.1.8. *Let (R, m) be a local ring admitting a dualizing complex, and let M be a G-Gorenstein complex. Then*

$$\mathrm{W}_0(M) = \operatorname{Ass} R \cap \operatorname{Supp}_R M.$$

Proof. Let $p \in \operatorname{Supp}_R M$. Since M_p is G-Gorenstein, we now have

$$\dim_{R_p} M_p = \operatorname{depth} R_p - \inf M_p$$

by Proposition 3.1.7. Thus $p \in \operatorname{Ass} R$ if and only if $\dim_{R_p} M_p = -\inf M_p$. But M being Cohen-Macaulay, we know by [17, Theorem 2.3 (d)] that this is further equivalent to $\dim R/p - \inf M_p = \dim_R M$. □

3.1.1 Flat Base Change and the Dualizing Property

We first investigate the stability of G-Gorenstein complexes under flat base change.

Proposition 3.1.9. *Let $f\colon (R,m) \to (S,n)$ be a flat local homomorphism of local rings. Suppose that R admits a dualizing complex, and that the fiber S/mS is Gorenstein. Let $M \in D_b^f(R)$. Then M is G-Gorenstein as a complex of R-modules if and only if $M \otimes_R^L S$ is G-Gorenstein as a complex of S-modules.*

Proof. By [7, p. 60], we have

$$\dim_S M \otimes_R^L S = \dim_R M + \dim S/mS, \quad (\dagger)$$

and

$$\operatorname{depth}_S M \otimes_R^L S = \operatorname{depth}_R M + \operatorname{depth} S/mS.$$

The above equalities together with the Cohen-Macaulayness of S/mS now imply that $M \otimes_R^L S$ is Cohen-Macaulay if and only if M is Cohen-Macaulay. Note also that the existence of a dualizing complex was not yet needed.

Suppose that D is a dualizing complex for R. Since S/mS is Gorenstein, f is a Gorenstein homomorphism by [6, Proposition 4.2]. It therefore follows from [6, Theorem 5.1] that $D \otimes_R^L S$ is a dualizing complex for S.

We will use Proposition 3.1.7 b). We know by [20, Theorem 5.3] that $\operatorname{Gid}_R M$ and $\operatorname{Gid}_S M \otimes_R^L S$ are simultaneously finite. Moreover, $\inf M \otimes_R^L S = \inf M$, because S is a faithfully flat R-module. Thus [20, Theorem 6.3] implies that

$$\begin{aligned} \operatorname{Gid}_S M \otimes_R^L S &= \operatorname{depth} S - \inf M \otimes_R^L S \\ &= \operatorname{depth} R + \operatorname{depth} S/mS - \inf M \\ &= \operatorname{Gid}_R M + \dim S/mS. \end{aligned}$$

We now get by ($\dagger$) together with the above equality that $\operatorname{Gid}_S M \otimes_R^L S = \dim_S M \otimes_R^L S$ if and only if $\operatorname{Gid}_R M = \dim_R M$. This completes the proof. □

It is natural to ask when a G-Gorenstein complex can be considered as a semi-dualizing complex.

Proposition 3.1.10. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$ be a G-Gorenstein complex. Then the following statements are equivalent:*

a) M is a semi-dualizing complex;

b) M is a dualizing complex;

c) $K_M \cong R$;

d) K_M is a semi-dualizing module.

Proof. $a) \Rightarrow b)$: Because M has finite Gorenstein injective dimension by Proposition 3.1.7, we know by [15, Proposition 8.4] that M must be a dualizing complex.
$b) \Rightarrow c)$: By the uniqueness of the dualizing complex, we have $M \simeq \sum^{-t} D_R$ for some integer t. Then $M^\dagger \simeq \sum^t R$. Hence $\dim_R M = t$ by formula (1.14) so that

$$K_M \cong \mathrm{H}_t(\textstyle\sum^t R) \cong R.$$

$c) \Rightarrow d)$: This is clear.
$d) \Rightarrow a)$: By Corollary 2.2.8 $K_M \simeq \sum^{-\dim_R M} M^\dagger$. Using "swap" we then obtain

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(K_M, K_M) &\simeq \mathbf{R}\mathrm{Hom}_R(M^\dagger, M^\dagger) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(M, M^{\dagger\dagger}) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(M, M),
\end{aligned}$$

which implies the claim. □

3.1.2 A Formula for the Depth

Let (R, m) be a local ring admitting a dualizing complex and let $M \in D_b^f(R)$ be a G-Gorenstein complex. We know by Proposition 3.1.11 that the biduality morphism $L \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(L, M), M)$ cannot be an isomorphism for $L \in D_b^f(R)$ unless M is dualizing. As the main result of the current subsection we will show in Theorem 3.1.13 below that the G-Gorenstein complexes can be characterized as those complexes $M \in D_b^f(R)$ of finite Gorenstein injective dimension for which the equality

$$\operatorname{depth}_R L = \operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(L, M), M)$$

holds for all complexes L of finite injective or finite projective dimension.

We will use the following lemma in the proof of Theorem 3.1.13.

Lemma 3.1.12. *Let (R, m) be a local ring admitting a dualizing complex. If a complex $M \in D_b^f(R)$ has finite Gorenstein injective dimension, then*

$$\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(L, M) = \operatorname{depth}_R L - \operatorname{Gid}_R M$$

for all complexes $L \in D_b(R)$ of finite projective or injective dimension.

Proof. If L has finite injective dimension, then [21, Theorem 6.3 (iii)] and [20, Theorem 6.3] immediately yield

$$\begin{aligned} \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(L, M) &= \operatorname{depth}_R L + \operatorname{width}_R M - \operatorname{depth} R \\ &= \operatorname{depth}_R L - \operatorname{Gid}_R M. \end{aligned}$$

In the case L has finite projective dimension, we know by [21, Theorem 4.7(ii)] that $\operatorname{Gid}_R \mathbf{R}\mathrm{Hom}_R(L, M)$ has finite Gorenstein injective dimension. So another application of [21, Theorem 6.3 (iii)] gives

$$\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(D_R, \mathbf{R}\mathrm{Hom}_R(L, M)) = \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(L, M) - \operatorname{depth} R,$$

since $\operatorname{depth}_R D_R = 0$. On the other hand, by [21, Theorem 6.2 (ii)]

$$\begin{aligned} &\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(L, \mathbf{R}\mathrm{Hom}_R(D_R, M)) \\ &\qquad = \operatorname{depth}_R L + \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(D_R, M) - \operatorname{depth} R \\ &\qquad = \operatorname{depth}_R L - \operatorname{Gid}_R M - \operatorname{depth} R, \end{aligned}$$

where the second inequality is by the already established case (take $L = D_R$). Since

$$\mathbf{R}\mathrm{Hom}_R(D_R, \mathbf{R}\mathrm{Hom}_R(L, M)) \simeq \mathbf{R}\mathrm{Hom}_R(L, \mathbf{R}\mathrm{Hom}_R(D_R, M))$$

by "swap", we get $\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(L, M) = \operatorname{depth}_R L - \operatorname{Gid}_R M$, as wanted. □

Theorem 3.1.13. *Let (R, m) be a local ring admitting a dualizing complex and let $M \in D_b^f(R)$ be a complex of finite Gorenstein injective dimension. Then the following statements are equivalent:*

a) M is G-Gorenstein;

b) If $L \in D_b(R)$ has finite projective or injective dimension, then

$$\operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(L, M), M) = \operatorname{depth}_R L;$$

c) There exists a complex $L \in D_b(R)$ of finite projective or injective dimension such that $L \not\simeq 0$ and

$$\operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(L, M), M) \geq \operatorname{depth} L.$$

Proof. In order to see the equivalence of a) and b) note that

$$\begin{aligned} \operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(L, M), M) &= \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(L, M) + \operatorname{depth}_R M \\ &= \operatorname{depth}_R L - \mathrm{Gid}_R M + \operatorname{depth}_R M. \end{aligned}$$

The first equality comes from [30, Proposition 4.6] while the second one follows from Lemma 3.1.12. Hence the equation

$$\operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(L, M), M) = \operatorname{depth}_R L$$

is equivalent to $\operatorname{depth}_R M = \mathrm{Gid}_R M$. Noting that $\mathrm{Gid}_R M = \operatorname{depth} R - \inf M$ by [20, Theorem 6.3], the equivalence of a) and b) is then clear by Proposition 3.1.7.

As discussed above, the claim stated in c) is equivalent to $\mathrm{Gid}_R M \leq \operatorname{depth}_R M$. Hence using Proposition 3.1.6 we now have

$$\dim_R M \leq \mathrm{Gid}_R \leq \operatorname{depth}_R M.$$

This implies that $\dim_R M = \operatorname{depth}_R M = \mathrm{Gid}_R M$, and the claim results from Proposition 3.1.7. □

For the proof of coming Corollary 3.1.15, we need the following well-known lemma, which we prove here for the convenience of the reader.

Lemma 3.1.14. *Let (R, m) be a local ring admitting a dualizing complex. If $M \in D_b^f(R)$ then*

$$\mathbf{R}\mathrm{Hom}_R(E_R(k), M) \simeq \mathbf{R}\mathrm{Hom}_R(D_R, M) \otimes_R \hat{R}.$$

Proof. By local duality, adjointness, and tensor evaluation, we get

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(E_R(k), M) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_m(D_R), M)\\ &\simeq \mathbf{R}\mathrm{Hom}_R(D_R, \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_m(R), M))\\ &\simeq \mathbf{R}\mathrm{Hom}_R(D_R, M \otimes_R \hat{R})\\ &\simeq \mathbf{R}\mathrm{Hom}_R(D_R, M) \otimes_R \hat{R}.\end{aligned}$$

□

Corollary 3.1.15. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$ be a complex of finite Gorenstein injective dimension. Then the following statements are equivalent:*

a) M is G-Gorenstein;

b) $\mathrm{depth}_R\, \mathbf{R}\mathrm{Hom}_R(M, M) = \mathrm{depth}\, R$*;*

c) $\mathrm{depth}_R\, \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(D_R, M), M) = 0$*;*

d) $\mathrm{depth}_R\, \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(E_R(k), M), M) \geq 0$*;*

Proof. Using Theorem 3.1.13 we can take $L = R$ or $L = D_R$ to see that a), b) and c) are equivalent. In order to prove that a) and *d*) are equivalent set $X = \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(E_R(k), M), M)$. Then

$$X \simeq \mathbf{R}\mathrm{Hom}_R(\hat{R}, \mathbf{R}\mathrm{Hom}_R \mathbf{R}\mathrm{Hom}_R(D_R, M), M)$$

by Lemma 3.1.14 and adjointness. Using the above isomorphism it now follows from [30, Proposition 4.6] that

$$\begin{aligned}\mathrm{depth}_R\, X &= \mathrm{width}_R\, \hat{R} + \mathrm{depth}_R\, \mathbf{R}\mathrm{Hom}_R \mathbf{R}\mathrm{Hom}_R(D_R, M), M)\\ &= \mathrm{depth}_R\, \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(D_R, M), M).\end{aligned}$$

Hence the claim results from c) together with Theorem 3.1.13 c). □

3.1.2.1 Some Comments

The following result is known for Gorenstein modules and Gorenstein complexes (see [55, Theorem 3.11] and [49, Theorem 3.3]):

Theorem 3.1.16. *Let (R, m) be a local ring, and let $M \in D_b^f(R)$. Then the following statements are equivalent:*

a) M is a Gorenstein complex;

b) $\mathrm{Ext}^i_R(k, M) = 0$ *for* $i \neq t$ *and some* $t \in \mathbb{Z}$*;*

c) If $L \neq 0$ *is a finitely generated* R*-module, then* $\mathrm{Ext}^i_R(\mathrm{Ext}^j_R(L, M), M) = 0$ *for* $i < j$*.*

One can show that c) is further equivalent to

d) $\mathrm{Ext}^i_R(\mathrm{Ext}^j(k, M), M) = 0$ if $i < j$

(see the proof of [55, Theorem 3.11]).

In this subsection we will discuss the counterpart of the above result for G-Gorenstein complexes. We also present an example showing that the counterpart of b) does not hold for G-Gorenstein complexes.

Remark 3.1.17. *We know by [18, Lemma 5.2.10] that*

$$\sup \mathbf{R}\mathrm{Hom}_R(N, N') = \sup \{\sup \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_n(N), N') - n \mid n \in \mathbb{Z}\}$$

for any complexes $N \in D^f_+(R)$ *and* $N' \in D_-(R)$*. Let* $M \in D^f_b(R)$ *be a complex of finite injective dimension. Set*

$$X = \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(k, M), M).$$

It follows from the above formula that

$$\sup X = \sup \left\{\sup \mathbf{R}\mathrm{Hom}_R(\mathrm{Ext}^j_R(k, M), M) + j \mid j \in Z\right\}.$$

It is then straightforward to see that condition d) is equivalent to $\sup X \leq 0$*. On the other hand,* $\sup X = -\operatorname{depth}_R X$ *by [16, 13, 1.4.1]. Indeed, condition d) means that*

$$\operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(k, M), M) \geq 0.$$

We thus see that Corollary 3.1.15 d) is the counterpart of condition d) in the case that M *is of finite injective dimension.*

Recall that the *grade* of a complex $M \in D(R)$ is defined by the formula

$$\operatorname{grade}_R M = -\sup \mathbf{R}\mathrm{Hom}_R(M, R).$$

Definition 3.1.18. *(see e.g. [32, p. 5]) A complex* $M \in D^f_b(R)$ *is said to be G-perfect if* $\operatorname{grade}_R M = \operatorname{Gpd}_R M$*.*

Proposition 3.1.19. *Let* (R, m) *be a local ring admitting a dualizing complex. Let* $t \in \mathbb{Z}$*. If* $M \in D^f_b(R)$ *has finite Gorenstein injective dimension, then the following statements are equivalent:*

a) $\operatorname{Ext}^i_R(E_R(k), M) = 0$ *for every integer* $i \neq t$*;*

b) $M^\dagger$ *is G-perfect.*

Proof. It follows from Lemma 3.1.14 that

$$\operatorname{amp}_R \mathbf{R}\operatorname{Hom}_R(E_R(k), M) = \operatorname{amp}_R \mathbf{R}\operatorname{Hom}_R(D_R, M).$$

On the other hand,

$$\begin{aligned} \operatorname{Gid}_R M &= -\inf \mathbf{R}\operatorname{Hom}_R(D_R, M) \\ &\geq -\sup \mathbf{R}\operatorname{Hom}_R(D_R, M) \\ &= -\sup \mathbf{R}\operatorname{Hom}_R(M^\dagger, R) \\ &= \operatorname{grade}_R M^\dagger. \end{aligned}$$

Here the first equality comes from [20, Corollary 6.7], and the second one is by biduality and "swap". Thus $\operatorname{amp}_R \mathbf{R}\operatorname{Hom}_R(E_R(k), M) = 0$ if and only if $\operatorname{Gid}_R M = \operatorname{grad}_R M^\dagger$. Noting that $\operatorname{Gid}_R M = \operatorname{Gpd}_R M^\dagger$ by [20, Corollary 6.4], it now follows that the two statements a) and b) are equivalent.

□

In order to see that the counterpart of Theorem 3.1.16 b) does not hold for G-Gorenstein complexes, it is enough to provide an example of a complex $M \in D^f_b(R)$ which is not G-Gorenstein, but the complex $M^\dagger$ is G-perfect.

Example 3.1.20. *Let* (R, m) *be a local ring admitting a dualizing complex. Let* $x \in R$ *be a regular element, and set* $M = (R/xR)^\dagger$*. One may use the complex* $0 \to R \xrightarrow{x} R \to 0$ *as a projective resolution of* R/xR *to get*

$$\mathbf{R}\operatorname{Hom}_R(M^\dagger, R) \simeq \mathbf{R}\operatorname{Hom}_R(R/xR, R) \simeq \textstyle\sum^{-1} R/xR.$$

By [14, Corollary 2.3.8] we then obtain that

$$\operatorname{Gpd}_R M^\dagger = -\inf \mathbf{R}\operatorname{Hom}_R(M^\dagger, R) = 1.$$

Also

$$\operatorname{grade}_R M^\dagger = -\sup \mathbf{R}\operatorname{Hom}_R(M^\dagger, R) = 1$$

It thus follows that $M^\dagger$ *is G-perfect. Therefore* $\operatorname{amp}_R \mathbf{R}\operatorname{Hom}_R(E_R(k), M) = 0$ *by Lemma 3.1.19. In order to prove that* M *is not G-Gorenstein, observe that* $\operatorname{depth}_R M = \inf M^\dagger = 0$ *by formula* (1.14) *and* $\operatorname{Gid}_R M = \operatorname{Gpd}_R M^\dagger = 1$ *by [20, Corollary 6.4]. Then Proposition 3.1.7 implies the claim.*

3.2 Equivalences of Categories

In this section we will show that that the category of G-Gorenstein complexes of fixed dimension is equivalent to the G-class of modules. This equivalence follows by restriction either from the equivalence of Yekutieli and Zhang considered in Corollary 2.2.8 or the Foxby equivalence. In particular, we can characterize the rings whose class of G-Gorenstein complexes strictly includes the class of Gorenstein complexes.

Notation 3.2.1. *Let R be a ring. Let $t \in \mathbb{Z}$. We denote by $D_{t-GGor}(R)$ the full subcategory of $D_b^f(R)$ of G-Gorenstein complexes of dimension t.*

We are now ready to prove our main results. What follows also holds true after replacing G-Gorenstein complexes and the G-class by Gorenstein complexes and the class of finitely generated free modules, respectively.

Theorem 3.2.2. *Let (R, m) be a local ring admitting a dualizing complex. For any $t \in \mathbb{Z}$, the equivalence of Corollary 2.2.8 induces an equivalence of categories*

$$D_{t-GGor}(R) \underset{\sum^{-t}(-)^{\dagger}}{\overset{K_{-}}{\rightleftarrows}} \mathrm{G}(R)^{opp}.$$

Furthermore, the following statements are equivalent for a complex $M \in D_b^f(R)$:

a) $M \in D_{t-GGor}(R)$;

b) $M \simeq (\sum^t K_M)^{\dagger}$ and $K_M \in \mathrm{G}(R)$;

c) $M \simeq (\sum^t K)^{\dagger}$ for some $K \in \mathrm{G}(R)$.

Proof. The equivalence of a), b) and c) is clear as soon as we have established the claimed equivalence of categories. To do the latter, we need to show that the restriction of the equivalence of Corollary 2.2.8 makes sense.

Suppose therefore that $M \in D_{t-GGor}(R)$. Of course $M \in D_{t-CM}(R)$. Now $\mathrm{H}_m^t(M) = \mathrm{Hom}_R(K_M, E_R(k))$. Since $\mathrm{H}_m^t(M)$ is Gorenstein injective, an application of [14, Theorem 6.4.2] shows that K_M is Gorenstein flat. So $K_M \in G(R)$.

Conversely, take $K \in G(R)$ and set $M = \sum^{-t} K^{\dagger}$. Then $M \in D_{t-CM}(R)$. By local duality $\mathrm{H}_m^t(M) = \mathrm{Hom}_R(K, E_R(k))$, so that $\mathrm{H}_m^t(M)$ is Gorenstein injective by [14, Theorem 6.4.2]. Hence $M \in D_{t-GGor}(R)$ by Corollary 3.1.4 as wanted. □

We can now answer Question 3.1.2 in the following Proposition 3.2.3. It is an immediate consequence of Theorem 3.2.2.

Proposition 3.2.3. *Let (R, m) be a local ring admitting a dualizing complex. Then the class of all G-Gorenstein complexes strictly includes the class of all Gorenstein complexes if and only if the G-class of modules strictly includes the class of finitely generated free modules.*

Let us then consider the Foxby equivalence.

Theorem 3.2.4. *Let (R, m) be a local ring admitting a dualizing complex. For any $t \in \mathbb{Z}$, Foxby equivalence induces an equivalence of categories*

$$D_{t-GGor} \underset{\sum^{-t} D_R \otimes^L_R -}{\overset{\mathrm{H}_{-t}(\mathbf{R}\mathrm{Hom}_R(D_R,-))}{\rightleftarrows}} \mathrm{G}(R).$$

Furthermore, the following statements are equivalent for a complex $M \in D_b^f(R)$:

a) $M \in D_{t-GGor}(R)$;

b) $M \simeq \sum^{-t} D_R \otimes^L_R N$ for some $N \in \mathrm{G}(R)$;

c) $\mathbf{R}\mathrm{Hom}_R(D_R, M) \simeq \sum^{-t} N$ for some $N \in G(R)$.

Proof. Let us first check that the restiction of the Foxby equivalence makes sense. Take $M \in D_{t-GGor(R)}$. Since M by Proposition 3.1.7 is of finite Gorenstein injective dimension, we know that $M \in B^f(R)$. By Theorem 3.2.2 b) we have $M \simeq \sum^{-t} K_M^\dagger$, where $K_M \in G(R)$. By [29, Lemma 2.7] and [14, Proposition 2.2.2] we get

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_R(\textstyle\sum^{-t} D_R, M) &\simeq \mathbf{R}\mathrm{Hom}_R(D_R, K_M^\dagger) \qquad (\dagger) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(K_M, R) \\ &\simeq \mathrm{Hom}_R(K_M, R). \end{aligned}$$

This shows that $\mathrm{H}_{-t}(\mathbf{R}\mathrm{Hom}_R(D_R, M)) \in G(R)$, as desired.

Conversely, let $N \in G(R)$. Set $M = \sum^{-t} D_R \otimes^L_R N$. By [29, Lemma 2.7] and [14, Proposition 2.2.2]

$$M^\dagger \simeq \textstyle\sum^t (D_R \otimes^L_R N)^\dagger \simeq \sum^t \mathbf{R}\mathrm{Hom}_R(N, R) \simeq \sum^t \mathrm{Hom}_R(N, R) \qquad (\ddagger).$$

By formula (1.14) $\dim_R M = t$. Since $\mathrm{Hom}_R(N, R) \in G(R)$, we have $M \in D_{t-GGor}(R)$ by Theorem 3.2.2 b).

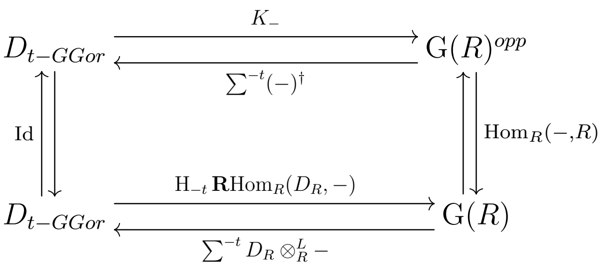
The equivalence of a) and b) is now immediate. It is also clear that b) implies c). To see the converse, recall from [14, Theorem 3.3.2 (b)] that $\mathbf{R}\mathrm{Hom}_R(\sum^{-t} D_R, M) \in \mathrm{A}(R)$ implies $M \in \mathrm{B}(R)$. By Foxby equivalence one then has

$$M \simeq \textstyle\sum^{-t} D_R \otimes^L_R \mathbf{R}\mathrm{Hom}_R(\sum^{-t} D_R, M).$$

□

The equivalences of Theorem 3.2.2 and Theorem 3.2.4 are compatible in the sense that the following diagram is commutative up to a canonical isomorphisms. In fact, this compatibility can also be seen as a special case of [29, Lemma 2.7]. We reprove it here for the convenience of the reader.

Proposition 3.2.5. *Let (R, m) be a local ring admitting a dualizing complex. Then the equivalences of Theorem 3.2.2 and Theorem 3.2.4 give raise to the following diagram, which is commutative up to canonical isomorphisms:*



Proof. The proof is straightforward. Taking the homology of (†) in the proof of Theorem 3.2.4 yields

$$\mathrm{H}_{-t}\,\mathbf{R}\mathrm{Hom}_R(D_R, M) \simeq \mathrm{Hom}_R(K_M, R)$$

for $M \in D_{t-GGor}(R)$, implying that the inner diagram commutes. Commutativity of the outer diagram can be deduced from (‡) by applying the dagger duality. Indeed, this gives

$$\textstyle\sum^t (D_R \otimes^L_R N)^\dagger \simeq \sum^t \mathrm{Hom}_R(N, R)$$

for any $N \in G(R)$, as needed. □

Inspired by [49, Theorem 3.3 (2)] we prove the following result.

Proposition 3.2.6. *Let (R, m) be a local ring admitting a dualizing complex. If $M \in D^f_b(R)$ has finite Gorenstein injective dimension, then the following statements are equivalent:*

a) M is G-Gorenstein of dimension t;

b) *There exists a Gorenstein injective module I and natural isomorphisms*

$$\mathbf{R}\mathrm{Hom}_R(L, M) \simeq \textstyle\sum^{-t} \mathrm{Hom}_R(L, I)$$

for all bounded complexes L with $\mathrm{Supp}_R\, L = \{m\}$ consisting of either injective modules or projective modules.

Proof. a) $\Rightarrow$ b) : Set $I = \mathrm{H}^t_m(M)$. We then know that I is Gorenstein injective and $\mathbf{R}\Gamma_m(M) \simeq \sum^{-t} I$. Now [43, Proposition 3.2.2] and [20, Corollary 2.12] yield

$$\mathbf{R}\mathrm{Hom}_R(L, M) \simeq \mathbf{R}\mathrm{Hom}_R(L, \mathbf{R}\Gamma_m(M)) \simeq \textstyle\sum^{-t} \mathrm{Hom}_R(L, I).$$

b) $\Rightarrow$ a) : We want to use Theorem 3.2.4 c). Therefore we need to show that $\mathbf{R}\mathrm{Hom}_R(D_R, M) \simeq \sum^{-t} N$ for some $N \in \mathrm{G}(R)$. We now have

$$\mathbf{R}\mathrm{Hom}_R(D_R, M) \otimes_R \hat{R} \simeq \mathbf{R}\mathrm{Hom}_R(E_R(k), M)$$

by Lemma 3.1.14. By assumption

$$\mathbf{R}\mathrm{Hom}_R(E_R(k), M) \simeq \textstyle\sum^{-t} \mathrm{Hom}_R(E_R(k), I).$$

It follows that $\mathbf{R}\mathrm{Hom}_R(D_R, M) \simeq \sum^{-t} N$ for some finitely generated R-module N. Now $\mathrm{Hom}_R(E_R(k), I)$ is Gorenstein flat by [21, Corollary 3.7 (c)]. So $N \otimes_R \hat{R}$ is Gorenstein flat as an R-module. By [21, Lemma 2.6 (a)] it is then Gorenstein flat also as an $\hat{R}$-module. Therefore $N \in \mathrm{G}(R)$ by [5, Theorem 8.7 (5)]. □

3.2.1 Application to Modules

By Proposition 3.1.7 we immediately recover [1, Theorem 3.8].

Corollary 3.2.7. *Let (R, m) be a local ring admitting a dualizing complex. If R is Cohen-Macaulay, then a finitely generated R-module is G-Gorenstein if and only if it is a maximal Cohen-Macaulay module of finite Gorenstein injective dimension.*

We also observe the following:

Corollary 3.2.8. *Let (R, m) be a local ring admitting a dualizing complex. If R admits a G-Gorenstein module with $\dim_R M = \dim R$, then R is Cohen-Macaulay.*

Proposition 3.2.9. *Let (R, m) be a local ring admitting a dualizing complex. If R satisfies Serre's condition (S_2) and $M \in D_b^f(R)$ is a G-Gorenstein complex, then* $\dim_R M = \dim R - \sup M$. *It follows that* $\operatorname{amp} M = \operatorname{cmd} R$. *In particular, if R is Cohen-Macaulay, then any G-Gorenstein complex is isomorphic to a module up to a suspension.*

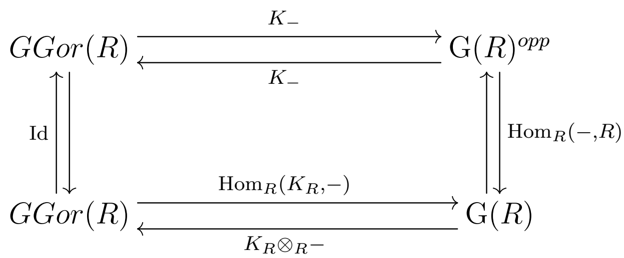
Proof. Recall first that Serre's condition (S_2) for R implies that $\operatorname{Ass} R = \operatorname{Assh} R$ (see e.g. [2, Lemma 1.1]). This together with Proposition 3.1.8 shows that $\dim R/p = \dim R$ for any $p \in \operatorname{Supp}_R M$ with $\dim R/p - \inf M_p = \dim_R M$. Because $\operatorname{Supp}_R M = \operatorname{Supp}_R K_M$ by Proposition 2.2.13 a), we get $\dim_R K_M = \dim R$. Thereby the desired formula $\dim_R M = \dim R - \sup M$ follows from Proposition 2.2.14 a). Since $\dim_R M = \operatorname{depth} R - \inf M$ by Proposition 3.1.7, this shows that $\operatorname{amp} M = \operatorname{cmd} R$. The last statement is now obvious. □

This gives immediately the following

Corollary 3.2.10. *Let (R, m) be a local ring admitting a dualizing complex and satisfying Serre's condition (S_2). If R admits a G-Gorenstein module, then R is Cohen-Macaulay.*

Notation 3.2.11. *If R is a ring, we denote by $GGor(R)$ the category of all G-Gorenstein modules.*

Corollary 3.2.12. *Let (R, m) be a Cohen-Macaulay local ring admitting a canonical module K_R. Then there exists a diagram*



of equivalences of categories, where the horizontal arrows are quasi-inverses of each other. The diagram is commutative up to canonical isomorphisms. Furthermore, if M is a finitely generated R-module, then the following statements are equivalent:

1) *M is a G-Gorenstein module;*

2) *M is an equidimensional module satisfying Serre's condition (S_2) and $K_M \in G(R)$;*

3) $M \cong K_R \otimes_R N$ *for some* $N \in G(R)$*;*

4) $\mathrm{Hom}_R(K_R, M) \in G(R)$*.*

Proof. Set $d = \dim R$. This is the diagram mentioned in Proposition 3.2.5 in the case $t = d$. Indeed, $D_R \simeq \sum^d K_R$ by the Cohen-Macaulayness of R. If $N \in G(R)$, then by the Auslander-Bridger formula (see [14, Theorem 1.4.8]) N is a Cohen-Macaulay module of dimension d. So $\sum^{-d} N^\dagger \simeq K_N$. Moreover, using [20, Corollary 2.12], we now observe that

$$\mathbf{R}\mathrm{Hom}_R(D_R, M) \simeq \mathrm{Hom}_R(D_R, M) \simeq \sum\nolimits^{-d} \mathrm{Hom}_R(K_R, M)$$

whereas by [20, Corollary 2.16]

$$\sum\nolimits^{-d} D_R \otimes^L_R N \simeq \sum\nolimits^{-d} D_R \otimes_R N \simeq K_R \otimes_R N$$

for all $N \in G(R)$.

To see the equivalence of a) and b), we can use the diagram. Indeed, if M is G-Gorenstein, then $M \cong K_{K_M}$, where $K_M \in G(R)$. Note that the module K_{K_M} is equidimensional and satisfies (S_2) by [52, Lemma 1.9, c) and e)]. Conversely, if M is an equidimensional module satisfying (S_2), then $M \cong K_{K_M}$ by [52, Proposition 1.1.4]. The equivalence of a), c) and d) follows directly from Theorem 3.2.4. □

Proposition 3.2.13. *Let* (R, m) *be a local ring admitting a dualizing complex, and let* $M \in D^f_b(R)$*. If* M *is a G-Gorenstein complex, then* $\mathrm{H}_s(M)_p$ *is G-Gorenstein as an* R_p*-module for all* $p \in \mathrm{Ass}_R \mathrm{H}_s(M)$*, where* $s = \sup M$*.*

Proof. Suppose $p \in \mathrm{Ass}_R \mathrm{H}_s(M)$. Then $\mathrm{depth}_{R_p} M_p = -\sup M_p$, because $\mathrm{Ass}_R \mathrm{H}_s(M) \subseteq \mathrm{Ass}_R M$ by [16, 2.4.1]. Also $\dim_{R_p} M_p = -\inf M_p$, because $\mathrm{Ass}_R M \subseteq \mathrm{Anc}_R M$ by [17, Theorem 2.3 (d)]. Noting that $\sup M_p = s$, it follows from Proposition 2.2.2 b) that $\inf M_p = \sup M_p = s$. This implies that

$$M_p \simeq \sum\nolimits^s \mathrm{H}_s(M)_p.$$

Therefore, using the fact that G-Gorenstein complexes are preserved by both suspensions and localization, we see that $\mathrm{H}_s(M)_p$ is G-Gorenstein as an R_p-module. □

3.2.2 Connection to Cousin Complexes

Notation 3.2.14. *Let R be a ring, and let $t \in \mathbb{Z}$. Consider the filtration $\mathcal{D}^t$. We look at the associated Cousin complexes, whose terms are Gorenstein injective, and whose homology is bounded and finitely generated. We denote by* $\mathrm{GIcz}(\mathcal{D}^t, R)$ *the corresponding full subcategory of the category* $\mathrm{Coz}_b^f(\mathcal{D}^t, R)$.

In this section we will show that the equivalence of Suominen, considered in Remark 2.3.41, restricts to an equivalence between the category $\mathrm{GIcz}(\mathcal{D}^t, R)$ and the category $D_{t-Ggor}(R)$.

Theorem 3.2.15. *Let (R, m) be a local ring admitting a dualizing complex. For any $t \in \mathbb{Z}$, the equivalence of Remark 2.3.41 induces an equivalence of categories*

$$\mathrm{GIcz}(\mathcal{D}^t, R) \underset{\mathrm{E}_{\mathcal{D}^t}(-)}{\overset{Q(-)}{\rightleftarrows}} D_{t-GGor}(R).$$

In particular, the following statements are equivalent for a complex $M \in D_b^f(R)$:

a) $M \in D_{\dim M-Ggor}(R)$;

b) $M \simeq \mathrm{E}_{\mathcal{D}(M)}(M)$, *and* $\mathrm{E}_{\mathcal{D}(M)}(M) \in \mathrm{GIcz}(\mathcal{D}(M), R)$.

Proof. We recall first that by [20, Theorem 6.9] and [39, Theorem 2.6] the class of Gorenstein injective R-modules is closed under direct sums and summands.

Let us then begin by checking that the restriction of the equivalence of Remark 2.3.41 makes sense. Take $N \in \mathrm{GIcz}(\mathcal{D}^t, R)$. Then $Q(N) \in D_{\mathcal{D}^t-CM}(R)$, and $Q(N)$ is Cohen-Macaulay of dimension t by Proposition 2.2.20. Therefore, it remains to show that the R_p-modules $\mathrm{H}^i_{pR_p}(Q(N)_p)$ are Gorenstein injective for all $i \in \mathbb{Z}$ and $p \in \mathrm{Spec}\, R$. By the Cohen-Macaulayness, we may assume that $i = \dim_{R_p} N_p$. Note that $i = t - \dim R/p$ by Lemma 2.2.3. Because $N \cong \mathrm{E}_{\mathcal{D}^t}(Q(N))$ by Lemma 2.3.34, formula (2.13) implies that

$$N_{-i} \cong \bigoplus_{i=t-\dim R/p} \mathrm{H}^i_{pR_p}(Q(N)_p).$$

Since N_{-i} is Gorenstein injective, it follows that each $\mathrm{H}^i_{pR_p}(Q(N)_p)$ is Gorenstein injective as an R-module. Since R admits a dualizing complex, it follows from Remark 3.1.3 that they are Gorenstein injective also as R_p-modules.

Conversely, let $M \in D_{t-GGor}(R)$. The modules $\mathrm{H}^i_{pR_p}(M_p)$ are then Gorenstein injective for every $p \in \mathrm{Spec}\, R$ and all $i \in \mathbb{Z}$. So is then also

$$(E_{\mathcal{D}^t}(M))_{-i} \cong \bigoplus_{i=t-\dim R/p} \mathrm{H}^i_{pR_p}(M_p)$$

Thus $M \in \mathrm{GIcz}(\mathcal{D}^t, R)$. The equivalence of a), b) immediately follows from the established equivalence of categories. $\square$

Remark 3.2.16. *Recall that a complex $M \in C(R)$ is called a* minimal complex, *in the sense of [8, Proposition 1.7], if every homomorphism $f \colon M \to M$ homotopic to Id_M is an isomorphism. Also note that by [8, Example 1.8] this agrees for 'injective resolutions' with the classical notion. It is clear by Remark 2.3.26 that any Cousin complex is minimal. Let (R, m) be a local ring admitting a dualizing complex. It then follows from Theorem 3.2.15 that if $M \in D_b^f(R)$ is a G-Gorenstein complex, then $\mathrm{E}_{\mathcal{D}(M)}(M)$ is a minimal complex with Gorenstein injective terms.*

The following Proposition generalizes a result of Dibaei (see [23, Theorem 3.3]) to the case of complexes.

Proposition 3.2.17. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D_b^f(R)$. Set $t = \dim_R M$ and $s = \sup M$. Suppose that M is equidimensional and that $\mathrm{Supp}_R \mathrm{H}_s(M) = \mathrm{Spec}(R)$. Assume that R satisfies Serre's condition (S_2), and that one of the following conditions holds:*

i) M and K_M satisfy Serre's condition (S_2);

ii) $M^\dagger$ and $\mathrm{H}_s(M)$ satisfy Serre's condition (S_2).

Then

a) Both K_M and $\mathrm{H}_s(M)$ are equidimensional;

b) $\mathrm{H}_s(M) \cong K_{K_M}$ and $K_M \cong K_{\mathrm{H}_s(M)}$.

Moreover, if one of the conditions i) or ii) holds, then the following statements are equivalent:

c) $\mathrm{E}_{\mathcal{D}(M)}(M) \in \mathrm{GIcz}(\mathcal{D}^{\dim M}, R)$;

d) $\mathrm{H}_s(M) \cong \mathrm{Hom}_R(F, K_R)$ for some $F \in G(R)$.

Proof. *a*) Note first that R is equidimensional by [35, Remark 2.4.1]. This implies that $\mathrm{H}_s(M)$ is equidimensional. We then have, in each situation *i*) or *ii*), that $\mathrm{H}_s(M) \cong K_{K_M}$ by Corollary 2.3.16 and Corollary 2.3.23, respectively. Therefore $\mathrm{Supp}_R K_M = \mathrm{Supp}_R \mathrm{H}_s(M)$, so that K_M is equidimensional.

b) This follows from part a), Corollary 2.3.16, and Corollary 2.3.23.

c) $\Rightarrow$ *d*) : Proposition 2.3.35 combined with [14, Theorem 6.4.2] implies that $\mathrm{E}_{\mathcal{D}(M)}(M) \in \mathrm{GIcz}(\mathcal{D}^{\dim M}, R)$ if and only if $K_M \in G(R)$. Because $\mathrm{H}_s(M) \cong$

K_{K_M} by b)), we have $\dim_R K_M = \dim R$. Hence $\mathrm{H}_s(M) \cong \mathrm{Hom}_R(K_M, K_R)$, which proves the claim.

d) $\Rightarrow$ *c)* : Note that $\mathrm{H}_s(M) \cong K_F$, since $\dim_R F = \dim R$. Using [14, Theorem 6.4.2], it is enough to show that $K_M \in G(R)$. Now

$$K_M \cong K_{\mathrm{H}_s(M)} \cong K_{K_F}$$

by b).

We will first show that F is equidimensional and satisfies Serre's condition (S_2). Because $\mathrm{Supp}_R F = \mathrm{Supp}_R \mathrm{H}_s(M)$, $\mathrm{Supp}_R F = \mathrm{Spec}\, R$ so that F is equidimensional. Take $p \in \mathrm{Spec}\, R$. By the Auslander-Bridger formula (see [14, Theorem 1.4.8]), we have

$$\mathrm{depth}_{R_p} F_p = \mathrm{depth}_{R_p} R_p \geq \min\left\{2, \dim_{R_p} R_p\right\} = \min\left\{2, \dim_{R_p} F_p\right\}.$$

So we have proved the above claim.

Since now $K_{K_F} = F$ by [52, Theorem 1.14], $K_M = F \in G(R)$, and we are done. □

Corollary 3.2.18. *Let (R, m) be a local ring admitting a dualizing complex, and let M be a finitely generated module. Suppose that R and M satisfy Serre's condition (S_2). If $(0 :_R M) = 0$, then the following conditions are equivalent:*

a) $\mathrm{E}_{\mathcal{D}(M)}(M) \in \mathrm{GIcz}(\mathcal{D}^{\dim M}, R)$*;*

b) $M \cong \mathrm{Hom}_R(F, K_R)$ *for some* $F \in G(R)$*.*

Proof. Since R is equidimensional by [35, Remark 2.4.1], the assumption $(0 :_R M) = 0$ implies that M is equidimensional. The claim is then an immediate consequence of Proposition 3.2.17. □

3.2.3 G-Gorenstein Complexes as Gorenstein objects

Let $\mathcal{C}$ be a class of objects in an abelian category $\mathcal{A}$. Consider an exact complex

$$X\colon \quad \cdots \to X_{i+1} \overset{d_{i+1}}{\to} X_i \overset{d_i}{\to} X_{i-1} \overset{d_{i-1}}{\to} \cdots$$

in $\mathcal{A}$, where $X_i \in \mathcal{C}$ for all $i \in \mathbb{Z}$. Recall that X is called $\mathcal{C}$-totally acyclic if it is both $\mathrm{Hom}_{\mathcal{A}}(\mathcal{C}, -)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{C})$-exact, i.e., the complexes $\mathrm{Hom}_{\mathcal{A}}(C, X)$ and $\mathrm{Hom}_{\mathcal{A}}(X, C)$ are exact in the category of abelian groups for any objects C in $\mathcal{C}$. A *$\mathcal{C}$-Gorenstein object* is an object in $\mathcal{A}$ appearing as a kernel in a $\mathcal{C}$-totally acyclic complex. In this section we want to show that in a certain sense G-Gorenstein complexes can be considered as Gorenstein objects in the nonabelian category $D(R)$.

We first need a suitable notion of exactness in a triangulated category. Our definition is a special case of the one Beligiannis gives in [10, Definition 4.7] (see also [3]). In the definition Δ refers to the class of all exact triangles in a triangulated category $\mathcal{D}$ (see [10, Example 2.3]). We will always denote the suspension functor by $\sum$.

Definition 3.2.19. *Let $\mathcal{D}$ be a triangulated category. A Δ-exact complex in $\mathcal{D}$ is a diagram*

$$X\colon \quad \cdots \to X_{i+1} \overset{d_{i+1}}{\to} X_i \overset{d_i}{\to} X_{i-1} \overset{d_{i-1}}{\to} \cdots$$

of objects and morphisms in $\mathcal{D}$ such that there exists for all $i \in \mathbb{Z}$ an exact triangle

$$M_i \overset{f_i}{\to} X_i \overset{g_i}{\to} M_{i-1} \to \sum M_i$$

where $d_i = f_{i-1}g_i$.

Remark 3.2.20. *By [3, Proposition 2.4 (a)], one has $d_{i-1}d_i = 0$ for all $i \in \mathbb{Z}$. Thus a diagram X as above is indeed a complex.*

The next two definitions are inspired by [3, Definition 3.2 and Definition 3.3].

Definition 3.2.21. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{C}$ be a class of objects in $\mathcal{D}$. We say that an exact triangle $N \to M \to L \to \sum N$ in $\mathcal{D}$ is* $\mathrm{Hom}_{\mathcal{D}}(\mathcal{C}, -)$*-exact if the induced sequence of abelian groups*

$$0 \to \mathrm{Hom}_{\mathcal{D}}(C, N) \to \mathrm{Hom}_{\mathcal{D}}(C, M) \to \mathrm{Hom}_{\mathcal{D}}(C, L) \to 0$$

is exact for all C in $\mathcal{C}$. The notion of a $\mathrm{Hom}_{\mathcal{D}}(-, C)$*-exact triangle is defined analogously.*

Definition 3.2.22. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{C}$ be a class of objects in $\mathcal{D}$.Consider a Δ-exact complex*

$$X\colon \quad \cdots \to X_{i+1} \overset{d_{i+1}}{\to} X_i \overset{d_i}{\to} X_{i-1} \overset{d_{i-1}}{\to} \cdots$$

in $\mathcal{D}$, where $X_i \in \mathcal{C}$ for all $i \in \mathbb{Z}$. We say that X is totally $\mathcal{C}$-acyclic if all the associated exact triangles

$$M_i \overset{f_i}{\to} X_i \overset{g_i}{\to} M_{i-1} \to \sum M_i$$

are both $\mathrm{Hom}_{\mathcal{D}}(\mathcal{C}, -)$*-exact and* $\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{C})$*-exact.*

Remark 3.2.23. *If X is a Δ-exact complex in $\mathcal{D}$ whose associated triangles are $\mathrm{Hom}_{\mathcal{D}}(\mathcal{C},-)$-exact (resp. $\mathrm{Hom}_{\mathcal{D}}(-,\mathcal{C})$-exact), then by pasting together the corresponding exact sequences of abelian groups, we see that the complex $\mathrm{Hom}_{\mathcal{D}}(C,X)$ (resp. $\mathrm{Hom}_{\mathcal{D}}(X,C)$) is exact for all $C \in \mathcal{C}$.*

Let R be a ring. Let $t \in \mathbb{Z}$. We aim next to investigate the relationship between the notion of Δ-exactness in $D_b^f(R)$ and the usual exactness in the abelian category $D_{t-CM}(R)$ of Cohen-Macaulay complexes of dimension t. For this we need some basic facts about t-structures.

Recall therefore from [9, Définition 1.3.1] that if $\mathcal{D}$ is a triangulated category, then a t-structure on $\mathcal{D}$ is a pair $(C_{\geq 0}, C_{\leq 0})$ of full subcategories of $\mathcal{D}$ satisfying the conditions:

1) $\sum C_{\geq 0} \subset C_{\geq 0}$ and $\sum^{-1} C_{\leq 0} \subset C_{\leq 0}$;

2) If $M \in C_{\geq 0}$ and $N \in \sum^{-1} C_{\leq 0}$ then $\mathrm{Hom}_{\mathcal{D}}(M,N) = 0$;

3) If $M \in \mathcal{D}$, then there exists an exact triangle $N \to M \to L \to \sum N$ with $N \in C_{\geq 0}$ and $L \in \sum^{-1} C_{\leq 0}$.

Set $C_{\geq n} = \sum^n C_{\geq 0}$ and $C_{\leq n} = \sum^n C_{\leq 0}$ for all $n \in \mathbb{Z}$. Then the *heart* of the above t-structure is $\mathcal{H} =: C_{\geq 0} \cap C_{\leq 0}$. The heart is an abelian category. For the proof of this and the following fact, we refer to [9, Théorème 1.3.6].

Fact 3.2.24. *A sequence*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $\mathcal{H}$ is exact if and only if there exists a morphism h such that

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \textstyle\sum X$$

is an exact triangle in $\mathcal{D}$.

Lemma 3.2.25. *Let (R,m) be a local ring admitting a dualizing complex. For any $t \in \mathbb{Z}$, there is a t-structure on $D_b^f(R)$ whose heart is $D_{t-CM}(R)$.*

Proof. Let $(D_{\geq t}, D_{\leq t})$ be the so called standard t-structure on $D_b^f(R)$, where

$$D_{\geq t} = \left\{ X \in D_b^f(R) \mid \mathrm{H}_i(X) = 0 \quad \text{for} \quad i < t \right\}$$

and

$$D_{\leq t} = \left\{ X \in D_b^f(R) \mid \mathrm{H}_i(X) = 0 \quad \text{for} \quad i > t \right\}.$$

By the dagger duality this gives raise to a t-structure $(D'_{\geq 0}, D'_{\leq 0})$, where

$$D'_{\geq t} = \left\{ X \in D_b^f(R) \mid X^\dagger \in D_{\geq t} \right\}$$

and

$$D'_{\leq t} = \left\{ X \in D_b^f(R) \mid X^\dagger \in D_{\leq t} \right\}.$$

Corollary 2.2.8 c) now implies that the heart of this t-structure is $D'_{\geq t} \cap D'_{\leq t} = D_{t-CM}(R)$. □

Proposition 3.2.26. *Let (R, m) be a local ring admitting a dualizing complex. Let $t \in \mathbb{Z}$, and set $D = \sum^{-t} D_R$. Consider a diagram*

$$X\colon \qquad \cdots \to X_{i+1} \overset{d_{i+1}}{\to} X_i \overset{d_i}{\to} X_{i-1} \overset{d_{i-1}}{\to} \cdots$$

of objects and morphisms in $D_{t-CM}(R)$. Then X is an exact complex in the abelian category $D_{t-CM}(R)$ if and only if it is a Δ-exact complex in $D_b^f(R)$ with $\mathrm{Hom}_{D(R)}(-, D)$-exact associated triangles. Moreover, the associated triangles are

$$M_i \overset{f_i}{\to} X_i \overset{g_i}{\to} M_{i-1} \to \sum M_i$$

where $d_i = f_{i-1} g_i$ and M_i denotes the kernel of d_i in $D_{t-CM}(R)$ for every $i \in \mathbb{Z}$.

Proof. Suppose first that X is exact in $D_{t-CM}(R)$. Let M_i denote the kernel of d_i in $D_{t-CM}(R)$ for every $i \in \mathbb{Z}$. By Fact 3.2.24 we get exact triangles

$$M_i \overset{f_i}{\to} X_i \overset{g_i}{\to} M_{i-1} \to \sum M_i,$$

where $d_i = f_{i-1} g_i$. So X is Δ-exact. Let us look at the long exact sequence of homology associated to the functor $\mathrm{Hom}_{D(R)}(-, D) = (\mathrm{H}_t(-)^\dagger)$. Since $M_i \in D_{t-CM}(R)$, we have $K^n_{M_i} = 0$ for all $n \neq t$. We thus obtain the exact sequences

$$0 \to K_{M_{i-1}} \to K_{X_i} \to K_{M_i} \to 0$$

showing that the triangles are indeed $\mathrm{Hom}_{D(R)}(-, D)$-exact.

Conversely, let X be Δ-exact complex in $D_b^f(R)$ with $\mathrm{Hom}_{D(R)}(-, D)$-exact associated exact triangles

$$M_i \overset{f_i}{\to} X_i \overset{g_i}{\to} M_{i-1} \to \sum M_i.$$

We will first show that every $M_i \in D_{t-CM}(R)$. Since $K^n_{X_i} = 0$ for $n \neq t$, the long exact sequence of homology associated to the functor $\mathrm{Hom}_{D(R)}(-, D)$ gives for any $n \neq t$ an isomorphism $K^n_{M_i} \cong K^{n-1}_{M_{i-1}}$ and an exact sequence

$$0 \to K^{t+1}_{M_i} \to K^{t}_{M_{i-1}} \to K_{X_i} \to K^{t}_{M_i} \to K^{t-1}_{M_{i-1}} \to 0.$$

Our triangle now being $\mathrm{Hom}_{D(R)}(-, D)$-exact, we must have $K^{t+1}_{M_i} = K^{t-1}_{M_{i-1}} = 0$. But then an easy induction shows that $K^{n}_{M_i} = 0$ for all $n \neq t$. Thus $M_i \in D_{t-CM}(R)$. Fact 3.2.24 then shows that the sequences

$$0 \to M_i \overset{f_i}{\to} X_i \overset{g_i}{\to} M_{i-1} \to 0$$

are exact in $D_{t-CM}(R)$. Finally, we observe that now $\mathrm{Ker}\, d_i = \mathrm{Ker}\, g_i$ and $\mathrm{Im}\, d_{i+1} = \mathrm{Im}\, f_i$ implying that X is an exact complex in $D_{t-CM}(R)$. □

We can now prove the promised main result of this section.

Theorem 3.2.27. *Let (R, m) be a local ring admitting a dualizing complex, and let $M \in D^f_b(R)$. Let $t \in \mathbb{Z}$, and set $D = \sum^{-t} D_R$. Then M is a G-Gorenstein complex of dimension t if and only if there exists a D-totally acyclic complex*

$$\cdots \to D^{\oplus n_{i+1}} \overset{d_{i+1}}{\to} D^{\oplus n_i} \overset{d_i}{\to} D^{\oplus n_{i-1}} \overset{d_{i-1}}{\to} \cdots$$

in $D^f_b(R)$ such that $M \simeq M_i$ where M_i belongs to some associated exact triangle

$$M_i \overset{f_i}{\to} D^{\oplus n_i} \overset{g_i}{\to} M_{i-1} \to \textstyle\sum^1 M_i.$$

Proof. By Theorem 3.2.2 we know that $M \in D_{t-GGor}(R)$ if and only if $M \in D_{t-CM}(R)$ and $K_M \in G(R)$. The latter means that K_M appears as a cokernel in a totally acyclic complex of finitely generated free R-modules. In the equivalence of categories of Corollary 2.2.8

$$D_{t-CM}(R) \underset{\sum^{-t}(-)^{\dagger}}{\overset{K_-}{\rightleftarrows}} (R - mod)^{opp}$$

this complex corresponds to a D-totally acyclic complex

$$\cdots \to D^{\oplus n_{i+1}} \overset{d_{i+1}}{\to} D^{\oplus n_i} \overset{d_i}{\to} D^{\oplus n_{i-1}} \overset{d_{i-1}}{\to} \cdots \qquad (*)$$

in $D_{t-CM}(R)$. It follows that $M \in D_{t-GGor}(R)$ if and only M is isomorphic to a kernel in this complex.

In light of Proposition 3.2.26 and Remark 3.2.23 it remains to show that if $(*)$ is D-totally acyclic complex in $D_{t-CM}(R)$, then the corresponding Δ-exact complex in $D_b^f(R)$ has $\mathrm{Hom}_{D(R)}(D,-)$-exact associated triangles. Consider thus the triangles

$$M_i \xrightarrow{f_i} D^{\oplus n_i} \xrightarrow{g_i} M_{i-1} \to \sum M_i,$$

where $d_i = f_{i-1}g_i$ and M_i is the kernel of d_i in $D_{t-CM(R)}$ for all $i \in \mathbb{Z}$. Because $K_{M_i} \in G(R)$, the complex M_i is G-Gorenstein. By Theorem 3.2.4 c) we then have $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(D, M_i)) = 0$ for all $i \neq 0$. The long exact sequence of homology associated to the functor $\mathrm{Hom}_{D(R)}(D,-) = \mathrm{H}_0(\mathbf{R}\mathrm{Hom}_R(D,-))$ therefore yields the exact sequences

$$0 \to \mathrm{Hom}_{D(R)}(D, M_i) \to \mathrm{Hom}_{D(R)}(D, D^{\oplus n_i}) \to \mathrm{Hom}_{D(R)}(D, M_{i-1}) \to 0$$

as needed. □

The following result is an immediate consequence of Theorem 3.2.27.

Corollary 3.2.28. *Let (R,m) be a Cohen-Macaulay local ring admitting a canonical module K_R. Let M be an R-module. Then M is a G-Gorenstein module if and only if M is a kernel in a totally K_R-acyclic complex*

$$\cdots \to K_R^{\oplus n_{i+1}} \xrightarrow{d_{i+1}} K_R^{\oplus n_i} \xrightarrow{d_i} K_R^{\oplus n_{i-1}} \to \cdots$$

of R-modules.

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