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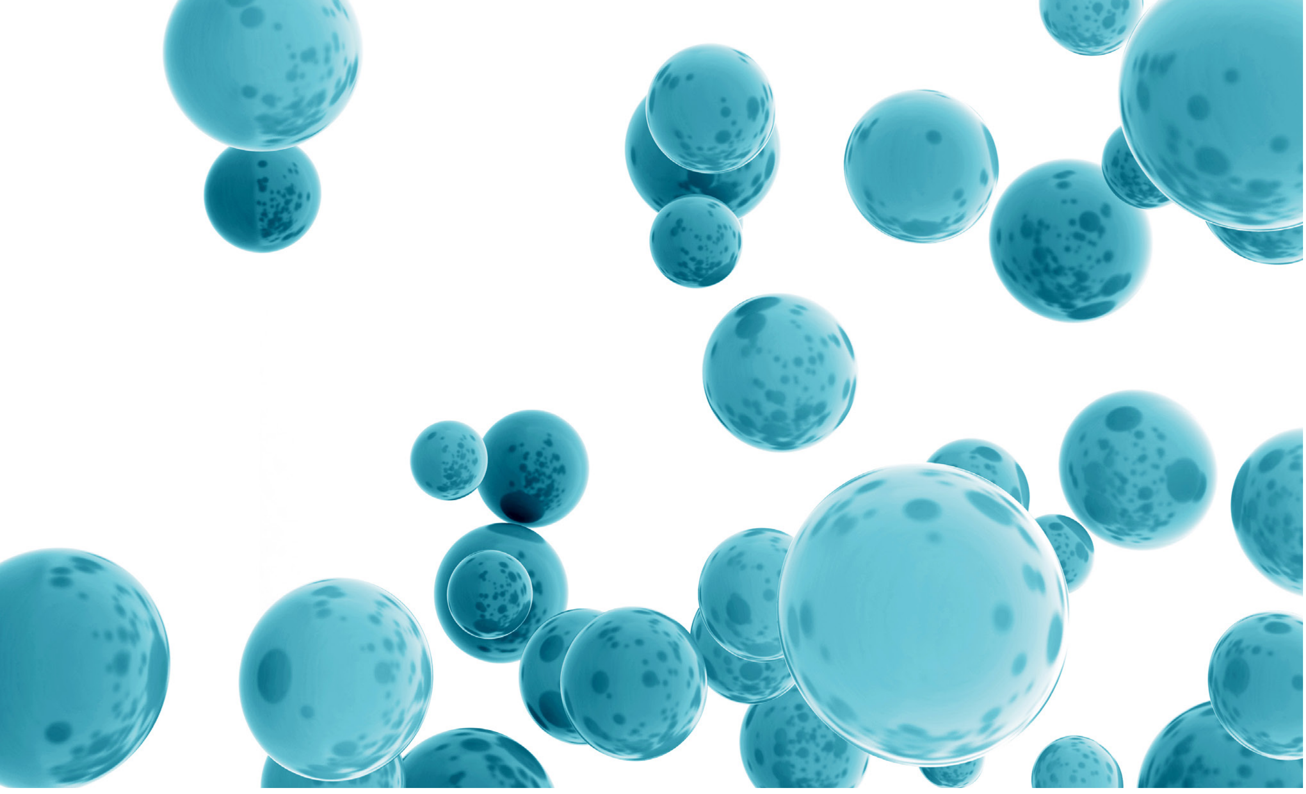
On the Invertibility and Eigenvalue Properties of Some Lattice-theoretic Matrices

Meet and join matrices studied via Möbius inversion



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ACADEMIC DISSERTATION
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UNIVERSITY OF TAMPERE

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Abstract

The main purpose of this thesis is to investigate the various properties of meet- and join-type matrices by using the Möbius inversion. We consider three distinct ways to generalize the concept of meet and join matrices and show that in each case the Möbius inversion can be used to study the determinant, inverse and sometimes even the eigenvalues of these matrices. We also turn our attention to the usual meet and join matrices and use this same method to study their positive definiteness. In the same context we give bounds for the eigenvalues of the usual meet and join matrices by making use of a particular method which does not require the use of the Möbius inversion.

Finally we conduct a more thorough investigation on the invertibility of join and LCM matrices via the Möbius inversion. We are going to see how our lattice-theoretic methods show their usefulness even when considering entirely number-theoretic LCM and power LCM matrices. We give a new lattice-theoretic proof for the well-known fact that the so-called Bourque-Ligh conjecture holds for GCD closed sets with less than 8 elements and does not hold in general for larger GCD closed sets. For the last we develop our method even further in order to study singular LCM and power LCM matrices. We show that, contrary to general expectations, so-called odd primitive singular numbers do exist. In addition, we are able to characterize all possible finite semilattice structures which can be used to generate GCD closed sets such that the power LCM matrix of this set is singular for some positive real exponent. At the same time we end up disproving several open conjectures presented by Hong.

KEYWORDS: meet matrix; meet semilattice; join matrix; join semilattice; lattice; incidence function; semimultiplicative function; determinant; inverse matrix; eigenvalue; positive definiteness; GCD matrix; LCM matrix; Smith's determinant; Bourque-Ligh conjecture

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Doctoral studies in mathematics could well be described as a random walk process that lasts over four years. Beforehand it is very hard to predict how the process will end and where the journey is eventually going to take you. Especially in the beginning it is difficult to tell what kind of results would be available in future work, since many "good" research ideas often turn out to be not working. With that being said, I believe I should consider myself lucky since my own random walk process with this thesis turned out to be quite pleasant and fruitful (instead of boring and frustrating). For that, I have many people to thank for.

The person that I definitely owe the most of my gratitude is my thesis supervisor Pentti Haukkanen. Pentti's contacts, vast knowledge in number theory and experience in scientific writing have benefited me tremendously throughout my doctoral studies. And, even more importantly, Pentti has continuously given me new ideas to work with. Pentti has indeed been an excellent supervisor and it has been a pleasure to work with him during these years.

When studying matrices without too much background in linear algebra it is useful to know people who are more familiar with this subject. Again I should feel myself lucky since I happen to know two such people: Jorma Merikoski and Simo Puntanen. I would like to thank them both for providing assistance with matrix analysis as well as bringing me into contact with other linear algebra researchers. Similarly, I wish to thank Eero Hyry for whom I have consulted on problems involving some serious algebra.

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I strongly believe that a research article cannot be good unless good language is maintained throughout the paper. I wish to thank Virginia Mattila who (along with Pentti) has helped me to improve the language and the grammar of my articles.

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Tampere, July 2015
Mika Mattila

Table of Contents

List of Abbreviations and Notations

Abbreviation	**Description**
$\mathbb{Z}^+$ or $\mathbb{Z}_+$	the set of positive integers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\|\|\|A\|\|\|_S$	the spectral norm of matrix A
$\|\|\|A\|\|\|_F$	the Frobenius norm of matrix A
$A \circ B$	the Hadamard product of matrices A and B
$\gcd(a,b)$ or (a,b)	the greatest common divisor of a and b $(a,b \in \mathbb{Z}^+)$
$\operatorname{lcm}(a,b)$ or $[a,b]$	the least common multiple of a and b $(a,b \in \mathbb{Z}^+)$
P	nonempty set
$S = \{x_1,\ldots,x_n\}$	finite subset of P
$\preceq$	partial order relation
$(P,\preceq)$	partially ordered set (or poset) P
$x \preceq y$	x precedes y
$x \prec y$	$x \preceq y$ and $x \neq y$
$x \npreceq y$	x does not precede y
$x \wedge y$	the meet (infimum) of x and y
$x \vee y$	the join (supremum) of x and y
$\bigwedge S$	the meet (infimum) of all the elements of S
$\bigvee S$	the join (supremum) of all the elements of S
$[\![x,y]\!]$	the closed interval between x and y (in P)
$]\!]x,y[\![$	the open interval between x and y (in P)
$\downarrow\! x$	the principal order ideal generated by x
$\uparrow\! x$	the principal order filter generated by x
$\operatorname{meetcl}(S)$	the meet closure generated by the set S
$\operatorname{joincl}(S)$	the join closure generated by the set S
$(P,\preceq) \cong (P',\preceq')$	posets P and P' are order isomorphic
f	real or complex-valued function on P
N	the function $\mathbb{Z}^+ \to \mathbb{R}$ with $N(m) = m$ for all $m \in \mathbb{Z}^+$
N^α	the function $\mathbb{Z}^+ \to \mathbb{R}$ with $N^\alpha(m) = m^\alpha$ for all $m \in \mathbb{Z}^+$

Abbreviation	**Description**
$\mu(\cdot)$	the number-theoretic Möbius function
$\mu_P(\cdot,\cdot)$	the Möbius function of the poset P
$(S)_f$ or $f(S)$	the meet matrix of the set S (associated with the function f)
$[S]_f$ or $f[S]$	the join matrix of the set S (associated with the function f)
$(S)_{N^\alpha}$ or $[(x_i,x_j)^\alpha]$	the power GCD matrix of the set $S \subset \mathbb{Z}^+$
$[S]_{N^\alpha}$ or $[[x_i,x_j]^\alpha]$	the power LCM matrix of the set $S \subset \mathbb{Z}^+$
(S) or $[(x_i,x_j)]$	the GCD matrix of the set $S \subset \mathbb{Z}^+$
$[S]$ or $[[x_i,x_j]]$	the LCM matrix of the set $S \subset \mathbb{Z}^+$

Publications

I. M. Mattila and P. Haukkanen, *Determinant and inverse of join matrices on two sets*, Linear Algebra and Its Applications 438 (2013) 3891–3904.

II. M. Mattila and P. Haukkanen, *Some properties of row-adjusted meet and join matrices*, Linear and Multilinear Algebra 60 (2012) 1211–1221.

III. M. Mattila, *On the eigenvalues of combined meet and join matrices*, Linear Algebra and Its Applications 466 (2015) 1–20.

IV. M. Mattila and P. Haukkanen, *On the positive definiteness and eigenvalues of meet and join matrices*, Discrete Mathematics 326 (2014) 9–19.

V. I. Korkee, M. Mattila and P. Haukkanen, *A lattice-theoretic approach to the Bourque-Ligh conjecture*, submitted to Discrete Mathematics.

VI. P. Haukkanen, M. Mattila and J. Mäntysalo, *Studying the singularity of LCM-type matrices via semilattice structures and their Möbius functions*, Journal of Combinatorial Theory, Series A 135 (2015) 181–200.

Chapter 1

Introduction

1.1 Lattice-theoretic background

Before introducing meet and join matrices we need some basic concepts and important terminology from lattice theory. Most of the terms are repeatedly used throughout this thesis. For more information about concepts in lattice theory, see e.g. Birkhoff [7] and Grätzer [9].

Let P be a nonempty set. A relation $\preceq$ on P is said to be a *partial ordering of* P if the following three conditions hold:

$$\begin{aligned} &\text{(Reflexivity)} && x \preceq x \text{ for all } x \in P, \\ &\text{(Antisymmetry)} && \text{if } x \preceq y \text{ and } y \preceq x \text{ for some } x, y \in P, \text{ then } x = y, \\ &\text{(Transitivity)} && \text{if } x \preceq y \text{ and } y \preceq z \text{ for some } x, y, z \in P, \text{ then } x \preceq z. \end{aligned}$$

The structure $(P, \preceq)$ is said to be *locally finite* if the interval

$$[\![x, y]\!] = \{z \in P \mid x \preceq z \preceq y\}$$

is a finite set for all $x, y \in P$. In this thesis we are only interested in locally finite partially ordered sets.

A partially ordered set P is a *meet semilattice* if for all $x, y \in P$ there exists $x \wedge y = \inf\{x, y\}$, the greatest common lower bound of x and y. Similarly, P is a *join semilattice* if for all $x, y \in P$ there exists $x \vee y = \sup\{x, y\}$, the smallest common upper bound of x and y. If $(P, \preceq)$ is both meet semilattice and join semilattice, then $(P, \preceq)$ is a *lattice*. It can be shown that if P is a finite meet semilattice with maximum element, then P is also a join semilattice.

If S is a subset of P and for all $x, y \in S$ we have either $x \preceq y$ or $y \preceq x$, then S is said to be a *chain*. If $x \wedge y \in S$ for all $x, y \in S$, then the set S is *meet closed*. And dually, if for all $x, y \in S$ we have $x \vee y \in S$, then the set S is *join closed.* If $y \preceq x \Rightarrow y \in S$ for all $x \in S$, then the set S is *lower closed.* And finally, the set S is *upper closed* if $x \preceq y \Rightarrow y \in S$ for all $x \in S$.

1.2 Meet and join matrices

Meet and join matrices are symmetric, real or complex square matrices. The exact definitions are as follows:

Definition 1. *Let $(P, \preceq)$ be a locally finite meet semilattice, f be a real or complex-valued function on P and $S = \{x_1, x_2, \ldots, x_n\}$ a finite subset of P with distinct elements such that $x_i \preceq x_j \Rightarrow i \leq j$. The* meet matrix *of the set S (associated with the function f) is the $n \times n$ matrix with $f(x_i \wedge x_j)$ as its ij entry.*

Definition 2. *Let $(P, \preceq)$ be a locally finite join semilattice, f be a real or complex-valued function on P and $S = \{x_1, x_2, \ldots, x_n\}$ a finite subset of P with distinct elements such that $x_i \preceq x_j \Rightarrow i \leq j$. The* join matrix *of the set S (associated with the function f) is the $n \times n$ matrix with $f(x_i \vee x_j)$ as its ij entry.*

A proper way to describe these matrices could be to say that in meet and join matrices the entries are determined partly by the function f, partly by the subset S and the underlying semilattice structure $(P, \preceq)$. For further information about meet and join matrices we refer to [17]. We are often interested in the cases when the set S is meet, join, lower or upper closed. Sometimes it is also useful to assume that $(P, \preceq)$ is a lattice and the function f is *semimultiplicative*, which means that

$$f(x)f(y) = f(x \wedge y)f(x \vee y)$$

for all $x, y \in P$. The reason is that in many cases this assumption about semimultiplicativity enables us to study join matrices via meet matrices and vice versa.

The research on meet and join matrices can be seen to originate in 1876 by the work of H.J.S. Smith. After letting $S = \{x_1, x_2, \ldots, x_n\}$ to be a finite ordered set of distinct positive integers Smith [23] investigated the $n \times n$ matrix with $\gcd(x_i, x_j)$ as it ij entry and developed an interesting product formula for its determinant. In fact, he was able to show that if the set S is *factor closed* (that is, $x \in S$ whenever $x \,|\, x_i$ for some $x_i \in S$), then the determinant of the matrix $(\gcd(x_i, x_j))$ is equal to

$$\phi(x_1)\phi(x_2)\cdots\phi(x_n),$$

where ϕ is the Euler totient function. Smith also considered more general GCD and LCM matrices associated with arithmetical functions. Also for the purposes of this thesis it is convenient to give a broader definition for GCD and LCM matrices:

Definition 3. *Let* $S = \{x_1, x_2, \ldots, x_n\}$ *be a finite subset* $\mathbb{Z}^+$ *with* $x_1 < x_2 < \cdots < x_n$ *and let* $f : \mathbb{Z}^+ \to \mathbb{C}$ *be an arithmetical function. The* GCD matrix of the set S (associated with f) *is the* $n \times n$ *matrix with* $f(\gcd(x_i, x_j))$ *as its* ij *entry. The* LCM matrix of the set S (associated with f) *is the* $n \times n$ *matrix with* $f(\text{lcm}(x_i, x_j))$ *as its* ij *entry.*

By letting $f(m) = m$ for all $m \in \mathbb{Z}^+$ (i.e. $f = N$) we obtain the usual GCD and LCM matrices in which the ij entries are $\gcd(x_i, x_j)$and $\text{lcm}(x_i, x_j)$, respectively. Thus the original Smith's matrix is in fact the GCD matrix of the set $S = \{1, 2, \ldots, n\}$. Despite of the rather simple appearance the usual GCD and LCM matrices have provided a fertile and nontrivial research field for many researchers over the years, see e.g. the references in [10].

Yet another important special cases of meet and join matrices are the so-called MIN and MAX matrices, which are obtained when the set S is a chain.

Definition 4. *Let* $S = \{x_1, x_2, \ldots, x_n\}$ *be a finite subset of* P *such that* $x_1 \prec x_2 \prec \cdots \prec x_n$ *and let* $f : P \to \mathbb{C}$ *be a function. The* MIN matrix of the set S (associated with f) *is the* $n \times n$ *matrix with* $f(\min(x_i, x_j))$ *as its* ij *entry. The* MAX matrix of the set S (associated with f) *is the* $n \times n$ *matrix with* $f(\max(x_i, x_j))$ *as its* ij *entry.*

In the special case when $(P, \preceq) = (\mathbb{Z}^+, \leq)$, $S = \{1, 2, \ldots, n\}$ and $f = N$ the MIN matrix of the set S is the matrix

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix},$$

which has been studied for example by Bhatia [6]. This particular matrix seems to have some relevance in statistics as well. As Bhatia states, this matrix is, up to a positive scalar, the covariance matrix of a stochastic process with increments which possess the same variance and are uncorrelated. This same matrix and its inverse also appear in a recent book about matrices in statistics, see [21, p. 252].

Chapter 2

The Möbius inversion

2.1 Arithmetical functions

The roots of Möbius inversion lie in the theory of arithmetical functions (for general accounts on arithmetical functions, see [3, 20, 22]). One way to approach this theme would be to consider the so-called sum functions. For a given arithmetical function $f : \mathbb{Z}^+ \to \mathbb{C}$ it is always possible to define a new arithmetical function F such that

$$F(n) = \sum_{d\,|\,n} f(d)$$

for all $n \in \mathbb{Z}^+$ (the function F is called the *sum function of* f). It is easy to see that in fact

$$F = f * \zeta,$$

where $*$ is the Dirichlet convolution and ζ is the arithmetical function with all values equal to 1 (the Dirichlet convolution $f * g$ of arithmetical functions f and g is defined as

$$(f * g)(n) = \sum_{d\,|\,n} f(d) g\left(\frac{n}{d}\right)$$

for all $n \in \mathbb{Z}^+$). But if F is the sum function of f and only the values of the function F are known, can they be used to calculate the values of the function f? The answer is positive. By using the number-theoretic Möbius function and the Dirichlet convolution we may write

$$f(n) = \sum_{d\,|\,n} F(d) \mu\left(\frac{n}{d}\right) = (F * \mu)(n), \tag{2.1}$$

since μ is the inverse of ζ with respect to the Dirichlet convolution and

$$f = F * \mu \Leftrightarrow F = f * \zeta.$$

The equation (2.1) is called the *classical Möbius inversion formula.* Although the underlying result is not very deep, the sigma-notation gives a quite impressive look for it. Trivial or not, the classical Möbius inversion formula is standard content in any textbook addressing arithmetical functions, see e.g. [3, Theorem 2.9] and [20, Theorem 1.3].

The values of the number-theoretic Möbius function are easy to determine by using its multiplicativity and the well-known fact that

$$\mu(p^n) = \begin{cases} 1 & \text{if } n = 0, \\ -1 & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases}$$

for all prime numbers p. Thus if $m = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where $p_1, \ldots, p_k$ are distinct primes, then by the multiplicativity of the function μ we have

$$\mu(m) = \mu(p_1^{n_1})\mu(p_2^{n_2})\cdots\mu(p_k^{n_k}).$$

2.2 Incidence functions

Let $(P, \leq)$ be a partially ordered set. An *incidence function on P* is any function $f : P \times P \to \mathbb{C}$ such that $f(x, y) = 0$ for all $x \nleq y$. The incidence functions have their own convolution operation, which is defined as

$$(f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y)$$

for all $x, y \in P$ (an empty sum is considered to be equal to zero). General material on incidence functions can be found in [1, 20, 24].

Incidence functions are sometimes referred to as generalized arithmetical functions (see [20]). The explanation is that the set of all arithmetical functions may be embedded into the set of incidence functions of $\mathbb{Z}^+ \times \mathbb{Z}^+$ with the mapping $f_A \mapsto f_I$, where

$$f_I(n, m) = \begin{cases} f_A\left(\frac{n}{m}\right) & \text{if } n \mid m, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, it is easy to see that the Dirichlet convolution $f_A * g_A$ of arithmetical functions f_A and g_A maps to the convolution $f_I * g_I$ of the incidence functions f_I and g_I.

Besides the convolution, there are also other quite naturally defined operations among the set of incidence functions. The usual sum and product of two incidence functions f and g are defined by

$$(f+g)(x,y) = f(x,y) + g(x,y) \quad \text{and} \quad (fg)(x,y) = f(x,y)g(x,y)$$

for all $x, y \in P$. The identity with respect to the usual product of incidence functions is clearly the function ζ for which

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

It is quite common to define the Möbius function μ_P of P as being the inverse of ζ with respect to the convolution. An alternative way would be to define this function recursively by using the formula

$$\mu_P(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum\limits_{x<z\leq y} \mu_P(z,y) = -\sum\limits_{x\leq z<y} \mu_P(x,z) & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

Computing the values of the Möbius function μ_P is often somewhat different as it was in the case of the number-theoretic Möbius function μ. If we know the prime factor decomposition of positive integer m, then it is completely trivial to calculate the value $\mu(m)$ of the number-theoretic Möbius function. But if we are studying the more general case when P is a poset and we wish to calculate the Möbius function value $\mu_P(x,y)$, then the most natural approach is to analyze the structure of the interval $[\![x,y]\!]$ and to use recursion and the above formula.

Also in the case of incidence functions it may be possible to define a sum function for a given function $f : P \to \mathbb{C}$. Of course, this requires that all of the necessary sums are finite (because of this we need to assume that our poset $(P, \leq)$ is locally finite). If the principal order ideal $\downarrow x$ is finite for all $x \in P$, we may define the *lower sum function* F_L *of* f by

$$F_L(x) = \sum_{y \leq x} f(y)$$

for all $x \in P$. Now if only the values of the lower sum function F_L are known, we may execute inversion from below and calculate the values of f by using the *Möbius inversion formula*

$$f(x) = \sum_{y \leq x} F_L(y)\mu_P(y,x)$$

for all $x \in P$ (see e.g. [1, Theorem 4.18 (i)] and [24, Proposition 3.7.1]). However, there may also be a possibility to execute the inversion from above. If the principal order filter $\uparrow x$ is finite for all $x \in P$, we may define the *upper sum function F_U of f* by

$$F_U(x) = \sum_{y \geq x} f(y)$$

for all $x \in P$. In this case the values of f may be calculated by using the *dual form of the Möbius inversion formula*

$$f(x) = \sum_{y \geq x} F_U(y)\mu_P(x, y)$$

for all $x \in P$ (see [1, Theorem 4.18 (ii)] and [24, Proposition 3.7.2]).

As it was in the case of arithmetical functions, both of these inversion formulas are direct consequences of the simple fact that the Möbius function μ_P is the inverse of the function ζ with respect to the convolution $*$. Although one may see them merely as a pair of two trivial observations, the true usefulness of these inversions lie in their numerous applications. As we are going to see, these two formulas play also a crucial role in our study of meet and join matrices.

Chapter 3

Summaries of the original articles

3.1 Generalizations of meet and join matrices

Although meet and join matrices itself are generalizations of number-theoretic GCD and LCM matrices, there are several ways to generalize these concepts even further. Quite obviously the generalized meet and join matrices are harder to study than the usual ones, since some of the methods that can be used in the study of traditional meet and join matrices do not work with the generalized matrices. However, the Möbius inversion remains effective at least to some extent.

In this section we summarize the first three articles of this thesis and take a closer look into three different generalizations of meet and join matrices one at a time. What is in common with these three articles is that the Möbius inversion plays a crucial role in every one of them.

3.1.1 Article I - Meet and join matrices on two sets

Meet and join matrices on two sets were introduced by Altinisik, Tuglu and Haukkanen [2]. In this case we have only one function $f : P \to \mathbb{C}$ but instead of having one single set S we now have two sets $X = \{x_1, \ldots, x_n\} \subseteq P$ and $Y = \{y_1, \ldots, y_m\} \subseteq P$ such that $x_i \leq x_j \Rightarrow i \leq j$ and $y_i \leq y_j \Rightarrow i \leq j$. The ij element of the meet matrix of the sets X and Y with respect to f is $f(x_i \wedge y_j)$, and the join matrix of the sets X and Y with respect to f has $f(x_i \vee y_j)$ as its ij element.

Altinisik et al. execute the Möbius inversion from below in order to obtain a factorization theorem for meet matrices on two sets. They use this formula and the Cauchy-Binet formula to calculate the determinant and inverse of a meet matrix on two sets. They also study the join matrix on two sets in the case when the function f is *semimultiplicative*.

In the present article the determinant and invertibility of join matrix on two sets are studied similarly, but in this case the Möbius inversion needs to be executed from above. This presents a problem, since usually we cannot assume that every principal order filter of $(P, \preceq)$ is finite (for example, if $(P, \preceq) = (\mathbb{Z}^+, |)$, then every principal order filter is infinite). The article presents a method which enables the use of Möbius inversion without making the assumption about the finiteness of the principal order filters.

3.1.2 Article II - Row-adjusted meet and join matrices

The motivation in defining row-adjusted meet matrices lies in Lindström's and Bege's generalizations of GCD matrices. Bege [4] proposed a question about the determinant of a generalized GCD matrix without knowing that this problem was in fact solved decades earlier by Lindström [18]. The present article gives a new proof for Lindström's determinant formula, and also other properties of these matrices are studied.

The main idea in row-adjusted meet matrices is that instead of having only one function $f : P \to \mathbb{C}$ we now have n different functions $f_1, \ldots, f_n : P \to \mathbb{C}$. In other words, everyone of the n rows has its own function f_i. The row-adjusted meet matrix of the set S with respect to the functions $f_1, \ldots, f_n$ has $f_i(x_i \wedge x_j)$ as its ij element and it is denoted by $(S)_{f_1,\ldots,f_n}$. One could also study the so-called *column-adjusted* meet matrix, but it would be only the transpose of the corresponding row-adjusted matrix.

Again, by executing the Möbius inversion from below for each function f_i it is possible to obtain a factorization for the matrix $(S)_{f_1,\ldots,f_n}$. In the case when the set S is meet closed it is then possible to estimate the rank of the matrix $(S)_{f_1,\ldots,f_n}$ as well as to study its determinant and inverse (unfortunately not much can be said about these properties if the set S is not meet closed, which makes this assumption inevitable). The *row-adjusted join matrix* $[S]_{f_1,\ldots,f_n}$, which has $f_i(x_i \vee x_j)$ as its ij element, is studied similarly by using Möbius inversion from above.

3.1.3 Article III - Combined meet and join matrices

In [16] Korkee studies the $n \times n$ matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ having

$$\frac{f(x_i \wedge x_j)^\alpha f(x_i \vee x_j)^\beta}{f(x_i)^\gamma f(x_j)^\delta}$$

as its ij element. In this case we have only one set S and one function f, but all the four terms $f(x_i \wedge x_j)$, $f(x_i \vee x_j)$, $f(x_i)$ and $f(x_j)$ are involved in the

ij element of the matrix instead of only one. Korkee derives a formula for the determinant and inverse of this matrix as well as presents two ways to factorize this matrix. One of the factorizations utilizes the usual meet matrix, the other the usual join matrix.

In the present article we make use of the two factorization theorems by Korkee. We continue by using the same methods as Hong and Loewy [14] and Ilmonen, Haukkanen and Merikoski [15] as we derive bounds for the eigenvalues of the matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ by using the Möbius inversion. However, some assumptions need to be made about the function f as well as about the parameters α, β, γ and δ. Under certain circumstances when the matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ is positive definite we are able to find a lower bound for the smallest eigenvalue of this matrix. In two other special cases we are able to define a region on the complex plane such that it contains all the eigenvalues of the matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta}$.

Our theorems concerning the bounds of the eigenvalues of the matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta}$ turn out to have interesting consequences as well. As one might expect, the eigenvalue bounds for usual meet and join matrices presented in [15] follow quite directly from the more general result. However, there exists also another class of matrices for which our results may quite naturally be applied. In [19] Mattila and Haukkanen derive bounds for the eigenvalues of the matrix $\mathbf{A}_n^{\alpha,\beta}$ with

$$(i,j)^\alpha [i,j]^\beta$$

as its ij element. In the present article it is shown that these results too are special cases of the respective theorems for the matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta}$.

In the end of the article a certain constant c_n is studied. The constant was originally defined by Hong and Loewy [14] and it was also considered by Ilmonen et al., but not much is known about it. In the present paper we are able to find a nontrivial lower bound for this constant. Such lower bound is needed when applying some of the eigenvalue theorems in practice. The article also shows how this lower bound can be improved if a certain conjecture by Ilmonen et al. holds.

3.1.4 Some comparisons between different matrix classes

As we have seen, there are three natural ways to generalize the concept of meet and join matrices. However, there are also numerous matrix classes which are closely related to meet and join matrices. Figure 3.1 shows how these different classes are related to each other. It also demonstrates how these three generalizations are entirely independent from each other. More detailed explanations for the different matrix classes can be found in Table 3.1.

Generalizations have also an effect on the various properties of the present matrix class. In Table 3.2 the basic properties of some of the most important matrix classes are compared with each other. As we can see, each matrix class differs from everyone else for more than one way.

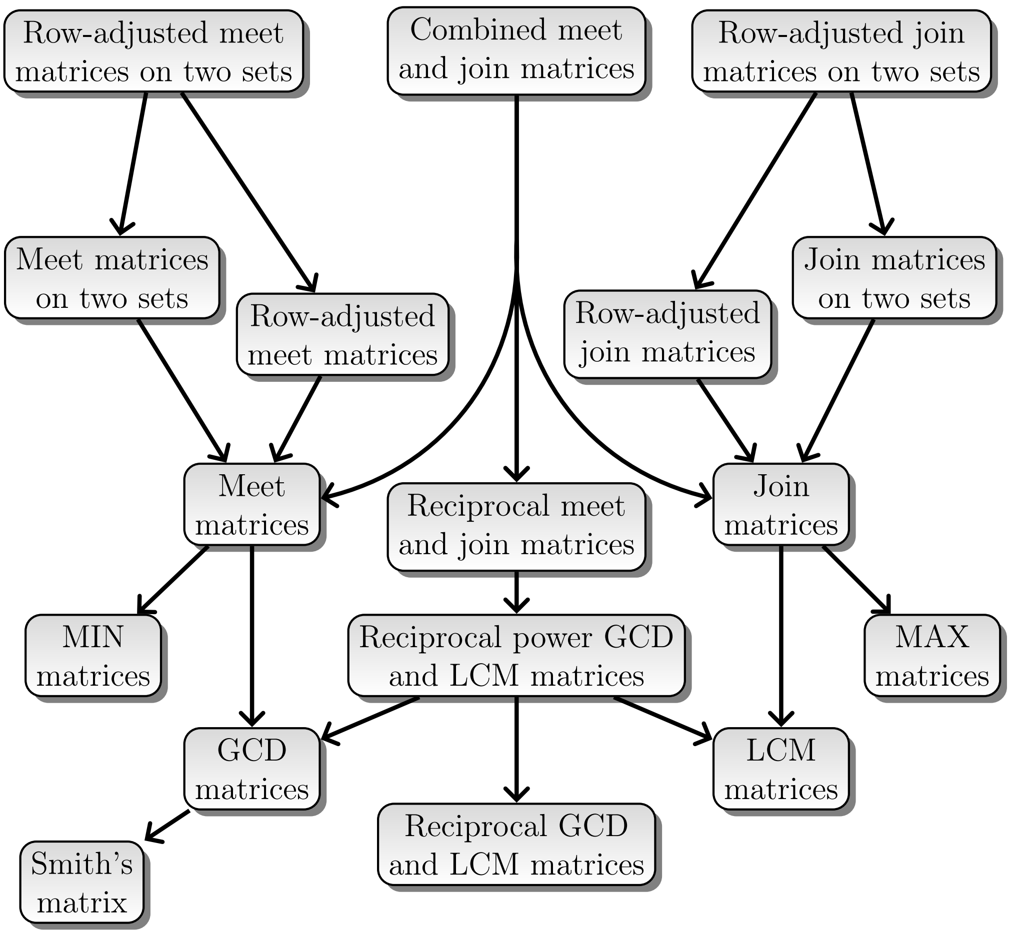

Figure 3.1: The connections between the most important meet and join related matrices.

3.2 Special cases of meet and join matrices

When studying more advanced matrix properties we may need to restrict ourselves to a more concise matrix class. For example, positive definiteness is a property which is defined only for Hermitean matrices and a complex symmetric meet matrix is Hermitean only if the elements of the matrix are entirely real. Also the question about the invertibility of meet and join

Name	Notation	ij entry
Row-adjusted meet matrix on two sets	$(X, Y)_{f_1,\ldots,f_n}$	$f_i(x_i \wedge y_j)$
Combined meet and join matrix	$M_{S,f}^{\alpha,\beta,\gamma,\delta}$	$\frac{f(x_i \wedge x_j)^\alpha f(x_i \vee x_j)^\beta}{f(x_i)^\gamma f(x_j)^\delta}$
Row-adjusted join matrix on two sets	$[X, Y]_{f_1,\ldots,f_n}$	$f_i(x_i \vee y_j)$
Meet matrix on two sets	$(X, Y)_f$	$f(x_i \wedge y_j)$
Row-adjusted meet matrix	$(S)_{f_1,\ldots,f_n}$	$f_i(x_i \wedge x_j)$
Row-adjusted join matrix	$[S]_{f_1,\ldots,f_n}$	$f_i(x_i \vee x_j)$
Join matrix on two sets	$[X, Y]_f$	$f(x_i \vee y_j)$
Meet matrix	$(S)_f$	$f(x_i \wedge x_j)$
Reciprocal meet and join matrix	–	$\frac{f(x_i \vee y_j)}{f(x_i \wedge x_j)}$ or $\frac{f(x_i \wedge x_j)}{f(x_i \vee y_j)}$
Join matrix	$[S]_f$	$f(x_i \vee x_j)$
MIN matrix	–	$f(\min(x_i, x_j))$
Reciprocal power GCD and LCM matrix	–	$\frac{\gcd(x_i,x_j)^\alpha}{\mathrm{lcm}(x_i,x_j)^\beta}$
MAX matrix	–	$f(\max(x_i, x_j))$
GCD matrix	(S)	$\gcd(x_i, x_j)$
Reciprocal GCD and LCM matrix	–	$\frac{\gcd(x_i,x_j)}{\mathrm{lcm}(x_i,x_j)}$ or $\frac{\mathrm{lcm}(x_i,x_j)}{\gcd(x_i,x_j)}$
LCM matrix	$[S]$	$\mathrm{lcm}(x_i, x_j)$
Smith's matrix	–	$\gcd(i, j)$

Table 3.1: Explanations of the matrix class names in Figure 3.1.

matrices makes more sense when considering more specific matrix classes. In this section we summarize the last three articles of this thesis and study positive definiteness, eigenvalues and invertibility of meet, join, reciprocal GCD and LCM matrices.

3.2.1 Article IV - Meet and join matrices associated with real valued functions

It is a well-known fact that GCD matrices are always positive definite whereas almost every LCM matrix is indefinite (the only exceptions are 1×1 matrices),

Property	Usual meet matrix	Row-adj. meet matrix	Meet matrix on two sets	Combined meet and join matrix
First appear in the literature	1991	2012 (1969)	2007	2005
Minimal requirement on $(P, \leq)$	Meet semilattice	Meet semilattice	Meet semilattice	Lattice
Number of subsets S of P	One	One	Two	One
Number of functions on P	One	n	One	One
Notation	$(S)_f$	$(S)_{f_1,\ldots,f_n}$	$(X,Y)_f$	$M_{S,f}^{\alpha,\beta,\gamma,\delta}$
ij element	$f(x_i \wedge x_j)$	$f_i(x_i \wedge x_j)$	$f(x_i \wedge y_j)$	$\frac{f(x_i \wedge x_j)^{\alpha} f(x_i \vee x_j)^{\beta}}{f(x_i)^{\gamma} f(x_j)^{\delta}}$
Size	$n \times n$	$n \times n$	$n \times m$	$n \times n$
Symmetricity	Yes	No	No	No
How it yields the usual meet matrix	–	Set $f_i = f$ for all $i = 1, \ldots n$	Set $X = Y = S$	Set $\alpha = 1$ and $\beta = \gamma = \delta = 0$
Dual concept	Join matrix	Row-adj. join matrix	Join matrix on two sets	Does not exist

Table 3.2: Some comparisons between meet matrices and generalized meet matrices.

see [5, 8]. In this fourth article we consider the positive definiteness of meet and join matrices with respect to a real-valued function. It turns out that in the case when the set S is meet closed, the positiveness of certain numbers obtained via executing the Möbius inversion determines the positive definiteness of the meet matrix $(S)_f$. A similar statement is made about the join matrix $[S]_f$. We also show that if the set S possesses a certain treelike structure, the positive definiteness of the matrix $(S)_f$ is equivalent to a certain monotonicity property of the function f. Again, a similar theorem is presented for join matrices.

In this article we also demonstrate how to find a lower bound for every eigenvalue of meet and join matrices without using the Möbius inversion. However, in these theorems the function f needs to be either *order-preserving* in the meet closure of the set S or *order-reversing* in the join closure of the set S.

3.2.2 Article V - The Bourque-Ligh conjecture

In 1992 Bourque and Ligh [8] conjectured that the LCM matrix of a GCD closed set is always invertible. In 1997 Haukkanen, Wang and Sillanpää [10] proved this wrong by presenting a GCD closed set S whose LCM matrix $[S]$ is singular. This counterexample contained nine elements. Two years later Hong [11] gave his own counterexample with only eight elements. By going through a vast amount of different cases he also proved number-theoretically that the Bourque-Ligh conjecture holds for GCD closed sets with at most seven elements.

In this fifth article we study the invertibilty of join matrices associated with semimultiplicative functions. Since the function N, where $N(m) = m$ for all $m \in \mathbb{Z}^+$, is trivially semimultiplicative, we are able to extend our results for the usual LCM matrices rather easily. At the same time we end up presenting a lattice-theoretic proof for the known fact that the Bourque-Ligh conjecture holds for all GCD closed sets with at most seven elements. The case when there are at least eight elements in the set S is also briefly addressed.

The key idea in this article is that by using the semimultiplicativity of the function f the matrix $[S]_f$ can be written as a product of three square matrices. By doing so we are able to revert the invertibility of the matrix $[S]_f$ to the invertibility of the reciprocal meet matrix $(S)_{1/f}$. This matrix is simplier to study, since the structure $(S, \preceq)$ is assumed to be a meet semilattice and thus the matrix $(S)_{1/f}$ can be factorized even further by using the Möbius function μ_S of the structure $(S, \preceq)$. The conditions for the invertibility of the matrices $[S]_f$ and $(S)_{1/f}$ (associated with any semimultiplicative function f with nonzero values) can then be found by examining all relevant semilattice structures with at most seven elements. By showing that the function N satisfies all these conditions it is then possible to get a new proof for the Bourque-Ligh conjecture in the case when there are less than eight elements in the set S.

3.2.3 Article VI - Singularity of LCM and power LCM matrices

In this sixth publication we develop further the Möbius function method presented in the previous article, but in this case we focus on the number-theoretic LCM and power LCM matrices. We begin by showing that if S and S' are GCD closed sets with 8 elements such that the LCM matrices $[S]$ and $[S']$ are singular, then the semilattices $(S, |)$ and $(S', |)$ are isomorphic and have the same cubelike structure. We also give an example from a GCD closed set S which consists of odd numbers and whose LCM matrix $[S]$ is singular. At the same time we are able to show that so-called odd primitive singular numbers do exist (contrary to a conjecture by Hong [13]).

In the second half of the article we turn our attention to power LCM matrices in which the ij entry is $[x_i, x_j]^\alpha$, where α is allowed to be any positive real number. It appears that some semilattice structures can be used to generate GCD closed sets S such that the power LCM matrix of the set S is singular for some $\alpha > 0$. However, there are also semilattice structures for which this is impossible. In the main result of this article we give a simple characterization which can be used to check whether a given semilattice structure can be used to generate a singular power LCM matrix or not. Finally we take a look at a couple of conjectures which concern singular power LCM matrices and were presented by Hong [12]. We are going to see that in the light of our new results it is easy to find counterexamples for them.

Chapter 4

Personal Contributions

In the following the contribution of the present author (and also the contribution of each of the other authors) is explained. The other authors Pentti Haukkanen, Ismo Korkee and Jori Mäntysalo are hereafter referred to as PH, IK and JM, respectively.

I. The ideas for the results in this article were given by PH. The present author worked all the details and wrote the paper. PH commented the manuscript.

II. PH gave the idea for the factorization theorem and for the determinant formula. The present author wrote the paper and developed the remaining results. PH commented the manuscript.

III. The author wrote this article on his own. PH gave some comments regarding the manuscript.

IV. PH gave the idea to use Hong's method in order to estimate the eigenvalues of meet and join matrices. The present author constructed these proofs. He also developed all the theorems concerning positive definiteness of meet and join matrices and took care of the writing of the paper. PH commented the manuscript.

V. This article is based on an old manuscript by IK and PH. The manuscript contained a couple of problematic issues that had prevented it from being published earlier. The present author solved these problems and revised the manuscript. PH and IK commented the manuscript. JM assisted with the parts that required the use of the program Sage.

VI. The present author developed the key results and wrote the paper. PH assisted in the writing of the proofs of the main results. JM took care of all the mathematical programming in this paper and conducted some testing with the program Sage (the existence of odd singular numbers was verified in these tests). PH and JM commented the manuscript. It should be noted that in this article the author names are listed in alphabetical order due to journal policy (i.e. in this article the ordering of the names does not reflect the contribution of each author as it is the case with the other articles).

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Publication I

Determinant and inverse of join matrices on two sets

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Determinant and inverse of join matrices on two sets

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ABSTRACT

Let $(P, \preceq)$ be a lattice and f a complex-valued function on P. We define meet and join matrices on two arbitrary subsets X and Y of P by $(X, Y)_f = (f(x_i \wedge y_j))$ and $[X, Y]_f = (f(x_i \vee y_j))$ respectively. Here we present expressions for the determinant and the inverse of $[X, Y]_f$. Our main goal is to cover the case when f is not semimultiplicative since the formulas presented earlier for $[X, Y]_f$ cannot be applied in this situation. In cases when f is semimultiplicative we obtain several new and known formulas for the determinant and inverse of $(X, Y)_f$ and the usual meet and join matrices $(S)_f$ and $[S]_f$. We also apply these formulas to LCM, MAX, GCD and MIN matrices, which are special cases of join and meet matrices.

1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function. Let $(S)_f$ denote the $n \times n$ matrix having $f((x_i, x_j))$, the image of the greatest common divisor of x_i and x_j, as its ij entry. Analogously, let $[S]_f$ denote the $n \times n$ matrix having $f([x_i, x_j])$, the image of the least common multiple of x_i and x_j, as its ij entry. That is, $(S)_f = (f((x_i, x_j)))$ and $[S]_f = (f([x_i, x_j]))$. The matrices $(S)_f$ and $[S]_f$ are referred to as the GCD and LCM matrices on S associated with f, respectively. The set S is said to be GCD-closed if $(x_i, x_j) \in S$ whenever $x_i, x_j \in S$, and the set S is said to be

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factor-closed if it contains every divisor of x for any $x \in S$. Clearly every factor-closed set is GCD-closed but the converse does not hold.

In 1875 Smith [31] calculated $\det(S)_f$ when S is factor-closed and $\det[S]_f$ in a more special case. Since then a large number of results on GCD and LCM matrices have been presented in the literature. See, for example [2,5–7,9–14,22]. Wilf [33] and Lindström [23] were the first to study the lattice-theoretic generalizations of GCD matrices already in the end of 1960s. A more extensive research on this topic was initiated three decades later by Haukkanen [8] when he generalized the concept of a GCD matrix into a meet matrix and later Korkee and Haukkanen [18] did the same with the concepts of LCM and join matrices. These generalizations happen as follows.

Let $(P, \preceq)$ be a locally finite lattice, let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P and let f be a complex-valued function on P. The $n \times n$ matrix $(S)_f = (f(x_i \wedge x_j))$ is called the meet matrix on S associated with f and the $n \times n$ matrix $[S]_f = (f(x_i \vee x_j))$ is called the join matrix on S associated with f. If $(P, \preceq) = (\mathbb{Z}^+, |)$, then meet and join matrices become respectively ordinary GCD and LCM matrices. However, some additional assumptions regarding the lattice $(P, \preceq)$ are still needed and we analyse these in Section 2.

The properties of meet and join matrices have been studied by many authors (see, e.g., [3,8,10,15,16,18,20,21,24,26,28,29]). Haukkanen [8] calculated the determinant of $(S)_f$ on an arbitrary set S and obtained the inverse of $(S)_f$ on a lower-closed set S and Korkee and Haukkanen [17] obtained the inverse of $(S)_f$ on a meet-closed set S. Korkee and Haukkanen [18] presented, among others, formulas for the determinant and inverse of $[S]_f$ on meet-closed, join-closed, lower-closed and upper-closed sets S.

Most recently, Altinisik et al. [4] generalized the concepts of meet and join matrices and defined meet and join matrices on two sets. Later these matrices were also treated in [19]. Next we present the same definitions.

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two subsets of P. We define the meet matrix on X and Y with respect to f as $(X, Y)_f = (f(x_i \wedge y_j))$. In particular, when $S = X = Y = \{x_1, x_2, \ldots, x_n\}$, we have $(S, S)_f = (S)_f$. Analogously, we define the join matrix on X and Y with respect to f as $[X, Y]_f = (f(x_i \vee y_j))$. In particular, $[S, S]_f = [S]_f$.

In [4] the authors presented formulas for the determinant and the inverse of the matrix $(X, Y)_f$. Applying these formulas they derived similar formulas for the matrix $[X, Y]_{1/f}$ with respect to semimultiplicative functions f with $f(x) \neq 0$ for all $x \in P$. The cases when f is not semimultiplicative or $f(x) = 0$ for some $x \in P$, however, were excluded from the examination. In this paper we give formulas that can also be used in these circumstances. We go through the same examinations presented in [4] but this time dually from the point of view of the matrix $[X, Y]_f$. That is, we present expressions for the determinant and the inverse of $[X, Y]_f$ on arbitrary sets X and Y. In the case when $X = Y = S$ we obtain a determinant formula for $[S]_f$ and a formula for the inverse of $[S]_f$ on arbitrary set S. We also derive formulas for the special cases when S is join-closed and upper-closed up to $\vee S$. Similar kind of determinant formulas for $(S)_f$ and $[S]_f$ have already been presented in [18], although they were obtained and presented by a different approach. By setting $(P, \preceq) = (\mathbb{Z}^+, |)$ we obtain corollaries for LCM matrices, and as another special case we also consider MAX and MIN matrices. In case when $(P, \preceq) = (\mathbb{Z}, \leqslant)$, where $\leqslant$ is the natural ordering of the integers, the MAX and MIN matrices of the set S are the matrices $[S]_f$ and $(S)_f$ respectively. MAX and MIN matrices have not been addressed before in this context.

2. Preliminaries

In the preceding section we defined the concept of GCD-closed set. Similarly, the set S is said to be LCM-closed if $[x_i, x_j] \in S$ whenever $x_i, x_j \in S$. Since the lattice $(\mathbb{Z}^+, |)$ does not have a greatest element, we need to define the dual concept for factor-closed set in a more special manner.

Definition 2.1. Let $\operatorname{lcm} S = [x_1, x_2, \ldots, x_n]$, and let

$$M_S = \{y \in \mathbb{Z}^+ \mid y \mid \operatorname{lcm} S \text{ and } x_i \mid y \text{ for some } x_i \in S\} = \bigcup_{i=1}^{n} [\![x_i, \operatorname{lcm} S]\!],$$

where $[\![x_i, \operatorname{lcm} S]\!]$ is the interval

$$\{y \in \mathbb{Z}^+ \mid x_i \mid y \text{ and } y \mid \operatorname{lcm} S\}.$$

We say that S is multiple-closed up to $\operatorname{lcm} S$ if

$$M_S = S,$$

that is, if $y \in S$ whenever $x_i \mid y \mid \operatorname{lcm} S$ for some $x_i \in S$.

Again, if S is multiple-closed up to $\operatorname{lcm} S$, then it is also LCM-closed, but an LCM-closed set is not necessarily multiple-closed up to $\operatorname{lcm} S$. Obviously the set M_S is multiple-closed up to $\operatorname{lcm} M_S = \operatorname{lcm} S$, and the semilattice $(M_S, |)$ also has the advantage of having the greatest element over the lattice $(\mathbb{Z}^+, |)$. Next we need corresponding definitions for a more general case.

Let $(P, \preceq)$ be a lattice. The set $S \subseteq P$ is said to be lower-closed (resp. upper-closed) if for every $x, y \in P$ with $x \in S$ and $y \preceq x$ (resp. $x \preceq y$), we have $y \in S$. The set S is said to be meet-closed (resp. join-closed) if for every $x, y \in S$, we have $x \wedge y \in S$ (resp. $x \vee y \in S$).

If every principal order filter of the lattice $(P, \preceq)$ is finite, the methods presented in the following sections can be applied to the lattice $(P, \preceq)$ directly. If the lattice $(P, \preceq)$ does not satisfy this property (which is the case when, for example, $P = \mathbb{Z}^+$ and $\preceq = |$), it is always possible to carry out the following procedures in an appropriate subsemilattice of $(P, \preceq)$. The most straightforward method is to generalize Definition 2.1 by simply translating it into terms of lattices so that the relation $|$ is replaced with the relation $\preceq$.

Definition 2.2. Let $\vee S = x_1 \vee x_2 \vee \cdots \vee x_n$ and let

$$P_S = \{y \in P \mid y \preceq \vee S \text{ and } x_i \preceq y \text{ for some } x_i \in S\} = \bigcup_{i=1}^{n} [\![x_i, \vee S]\!],$$

where $[\![x_i, \vee S]\!]$ is the interval

$$\{y \in P \mid x_i \preceq y \text{ and } y \preceq \vee S\}.$$

We say that S is upper-closed up to $\vee S$ if

$$P_S = S,$$

that is, if $y \in S$ whenever $x_i \preceq y \preceq \vee S$ for some $x_i \in S$.

Note that $S \subseteq P_S$ and $\vee P_S = \vee S$ for all finite sets S. Thus, $(P_S, \preceq)$ has the greatest element. If S is upper-closed up to $\vee S$, then S is join-closed, but the converse does not hold. A further rather trivial but important observation is that if $x, y \in P_S$, then

$$\mu_{P_S}(x, y) = \mu_P(x, y).$$

An alternative approach would be to restrict our consideration to $(\langle S \rangle, \preceq)$, the join-subsemilattice of $(P, \preceq)$ generated by the set S. Usually this would also reduce the number of computations needed. For example, the values of the Möbius function of $(\langle S \rangle, \preceq)$ are often much easier to calculate than the values of the Möbius function of $(P_S, \preceq)$ (see [1, Section IV.1]). And if we consider S as a subset of the meet-subsemilattice generated by itself, the set S is meet-closed iff it is lower-closed. Similarly, the terms join-closed and upper-closed coincide in the join-subsemilattice $(\langle S \rangle, \preceq)$. This is another benefit of restricting to $(\langle S \rangle, \preceq)$. This method is not, however, very convenient when considering the lattice $(\mathbb{Z}^+, |)$. The Möbius function of $(\langle S \rangle, \preceq)$, where $S \subset \mathbb{Z}^+$, has often very little in common with the number-theoretic Möbius function, which would likely cause confusion. For this reason we give our formulas in a form that fits both for the types of lattices defined in Definition 2.2 and for the lattice $(\langle S \rangle, \preceq)$.

Let $(P, \preceq)$ be a locally finite lattice, and let f be a complex-valued function on P. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two subsets of P. Let the elements of X and Y

be arranged so that $x_i \preceq x_j \Rightarrow i \leqslant j$. Let $D = \{d_1, d_2 \ldots, d_m\}$ be any subset of P containing the elements $x_i \vee y_j$, $i, j = 1, 2, \ldots, n$. Let the elements of D be arranged so that $d_i \preceq d_j \Rightarrow i \leqslant j$. Then we define the function $\Psi_{D,f}$ on D inductively as

$$\Psi_{D,f}(d_k) = f(d_k) - \sum_{d_k \prec d_v} \Psi_{D,f}(d_v) \tag{2.1}$$

or equivalently

$$f(d_k) = \sum_{d_k \preceq d_v} \Psi_{D,f}(d_v). \tag{2.2}$$

Then

$$\Psi_{D,f}(d_k) = \sum_{d_k \preceq d_v} f(d_v)\mu_D(d_k, d_v), \tag{2.3}$$

where μ_D is the Möbius function of the poset $(D, \preceq)$ (see [32, 3.7.2 Proposition]).

If D is join-closed, then

$$\Psi_{D,f}(d_k) = \sum_{\substack{d_k \preceq z \preceq \vee D \\ d_t \npreceq z \\ k<t}} \sum_{z \preceq w \preceq \vee D} f(w)\mu_{P_D}(z, w), \tag{2.4}$$

where μ_{P_D} is the Möbius function of $(P_D, \preceq)$, and if D is upper-closed up to $\vee D$, then

$$\Psi_{D,f}(d_k) = \sum_{d_k \preceq d_v} f(d_v)\mu_{P_D}(d_k, d_v), \tag{2.5}$$

where μ_{P_D} is the Möbius function of $(P_D, \preceq)$. Formula (2.5) follows trivially from Eq. (2.3) and the fact that in case when D is upper-closed we have $P_D = D$ and $\mu_{P_D}(d_k, d_v) = \mu_D(d_k, d_v)$ for all $d_k, d_v \in D$. We prove formula (2.4) by using methods similar to those in [8, Example 1].

Proof of (2.4)**.** Let $D = \{d_1, d_2, \ldots, d_m\}$ be join-closed. In order to prove that

$$\Psi_{D,f}(d_k) = \sum_{\substack{d_k \preceq z \\ d_t \npreceq z \\ k<t}} \sum_{z \preceq w \preceq \vee D} f(w)\mu_{P_D}(z, w)$$

we are going to show that $\Psi_{D,f}$ given in (2.4) satisfies (2.2), that is,

$$f(d_k) = \sum_{d_k \preceq d_v} \sum_{\substack{d_v \preceq z \\ d_t \npreceq z \\ v<t}} \sum_{z \preceq w \preceq \vee D} f(w)\mu_{P_D}(z, w). \tag{2.6}$$

We write $f(x) = \sum_{x \preceq z \preceq \vee D} g(z)$ or $g(x) = \sum_{x \preceq z \preceq \vee D} f(z)\mu_{P_D}(x, z)$ for all $x \in P_D$. We now have to prove that

$$\sum_{d_k \preceq z \preceq \vee D} g(z) = \sum_{d_k \preceq d_v} \sum_{\substack{d_v \preceq z \preceq \vee D \\ d_t \npreceq z \\ v<t}} g(z). \tag{2.7}$$

It is easy to see that the sums in (2.7) are non-repetitive, that is, each z is counted only once. Now, consider the sum on the right side of (2.7). Let $d_k \preceq d_v$ and $d_v \preceq z \preceq \vee D$ with $z \in P_D$. Then $d_k \preceq z \preceq \vee D$. Thus every z occurring on the right side of (2.7) occurs on the left side of (2.7). Conversely, consider the sum on the left side of (2.7). Suppose that $d_k \preceq z \preceq \vee D$. Let i be the greatest number such that $d_i \preceq z$. Then $d_t \npreceq z$ for $i < t$. Since S is join-closed, $d_k \vee d_i = d_r$ for some r with $i \leqslant r$. Since $d_k \preceq z$ and $d_i \preceq z$, we have $d_r \preceq z$. By maximality of i, we have $r = i$ and $d_r = d_i$. Therefore $d_k \preceq d_r$ means that $d_k \preceq d_i$. Thus every z occurring on the left side of (2.7) occurs on the right side of (2.7). This completes the proof of (2.7), that is, the proof of (2.4). □

Remark 2.1. If D is join-closed, then $D = \langle D \rangle$ and D is trivially upper-closed subset of $\langle D \rangle$ in $(\langle D \rangle, \preceq)$. Thus in this case we could also replace the μ_D in (2.3) and the μ_{P_D} in (2.5) by $\mu_{\langle D \rangle}$.

If $(P, \preceq) = (\mathbb{Z}^+, |)$ and D is multiple-closed up to lcm D, then $\mu_D(d_k, d_v) = \mu(d_v/d_k)$ (see [25, Chapter 7]), where μ is the number-theoretic Möbius function. In addition, for every $a \in \mathbb{Z}^+$ and arithmetical function f we may define another arithmetical function f_a, where

$$f_a(n) = f(an)$$

for every $n \in \mathbb{Z}^+$. Now from (2.3) we get

$$\begin{aligned} \Psi_{D,f}(d_k) &= \sum_{d_k \mid d_v} f(d_v)\mu\left(\frac{d_v}{d_k}\right) = \sum_{a \mid \frac{\operatorname{lcm} D}{d_k}} f(d_k a)\mu(a) \\ &= \sum_{a \mid \frac{\operatorname{lcm} D}{d_k}} (f_{d_k}\mu)(a) = [\zeta * (f_{d_k}\mu)]\left(\frac{\operatorname{lcm} D}{d_k}\right), \end{aligned} \tag{2.8}$$

where $*$ is the Dirichlet convolution of arithmetical functions.

Let $E(X) = (e_{ij}(X))$ and $E(Y) = (e_{ij}(Y))$ denote the $n \times m$ matrices defined by

$$e_{ij}(X) = \begin{cases} 1 & \text{if } x_i \preceq d_j, \\ 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

and

$$e_{ij}(Y) = \begin{cases} 1 & \text{if } y_i \preceq d_j, \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

respectively. Clearly $E(X)$ and $E(Y)$ also depend on D but for the sake of brevity D is omitted from the notation. We also denote

$$\Lambda_{D,f} = \operatorname{diag}(\Psi_{D,f}(d_1), \Psi_{D,f}(d_2), \ldots, \Psi_{D,f}(d_m)). \tag{2.11}$$

3. A structure theorem

In this section we give a factorization of the matrix $[X, Y]_f = \big(f(x_i \vee y_j)\big)$. A large number of similar factorizations is presented in the literature, for example in [16] the matrix $[S]_f$ is factorized in case when S is join-closed. The idea of this kind of factorization may be considered to originate from Pólya and Szegö [27].

Theorem 3.1

$$[X, Y]_f = E(X)\Lambda_{D,f}E(Y)^T. \tag{3.1}$$

Proof. By (2.2) the ij entry of $[X, Y]_f$ is

$$f(x_i \vee y_j) = \sum_{x_i \vee y_j \preceq d_v} \Psi_{D,f}(d_v). \tag{3.2}$$

Now, since $x_i, y_j \preceq d_v \Leftrightarrow x_i \vee y_j \preceq d_v$, by applying (2.9), (2.10) and (2.11) to (3.2) we obtain

$$\sum_{x_i \vee y_j \preceq d_v} \Psi_{D,f}(d_v) = \sum_{v=1}^{m} e_{iv}(X)\Psi_{D,f}(d_v)e_{jv}(Y) \tag{3.3}$$

and thus we have proven Theorem 3.1. □

Remark 3.1. The sets X and Y could be allowed to have distinct cardinalities in Theorems 3.1 and 6.1. However, in other results we must assume that these cardinalities coincide.

4. Determinant formulas

In this section we derive formulas for determinants of join matrices. In Theorem 4.1 we present an expression for $\det[X, Y]_f$ on arbitrary sets X and Y. Taking $X = Y = S = \{x_1, x_2, \ldots, x_n\}$ in Theorem 4.1 we obtain a formula for the determinant of usual join matrices $[S]_f$ on arbitrary set S, and further taking $(P, \preceq) = (\mathbb{Z}^+, |)$ we obtain a formula for the determinant of LCM matrices on arbitrary set S. In Theorems 4.2 and 4.3 respectively, we calculate $\det[S]_f$ when S is join-closed and upper-closed up to $\vee S$. Formulas similar to Theorems 4.2 and 4.3 but by different approach and notations are given in [18].

Theorem 4.1. *Let* $\operatorname{card}(X) = \operatorname{card}(Y) = n$ *and* $\operatorname{card}(D) = m$.

(i) If $n > m$, *then* $\det[X, Y]_f = 0$.
(ii) If $n \leqslant m$, *then*

$$\det[X, Y]_f = \sum_{1\leqslant k_1<k_2<\cdots<k_n\leqslant m} \det E(X)_{(k_1,k_2,\ldots,k_n)} \det E(Y)_{(k_1,k_2,\ldots,k_n)} \times \Psi_{D,f}(d_{k_1})\Psi_{D,f}(d_{k_2})\cdots\Psi_{D,f}(d_{k_n}). \tag{4.1}$$

Proof. By Theorem 3.1

$$\det[X, Y]_f = \det\left(E(X)\Lambda_{D,f}E(Y)^T\right). \tag{4.2}$$

Thus by the Cauchy–Binet formula we obtain Theorem 4.1. □

Theorem 4.2. *If S is join-closed, then*

$$\det[S]_f = \prod_{v=1}^{n} \Psi_{S,f}(x_v) = \prod_{v=1}^{n} \sum_{x_v\preceq x_t} f(x_t)\mu_S(x_v, x_t) = \prod_{v=1}^{n} \sum_{\substack{x_v\preceq z\preceq \vee S\\ x_t \npreceq z\\ v<t}} \sum_{z\preceq w\preceq \vee S} f(w)\mu_{P_S}(z, w). \tag{4.3}$$

Proof. We take $X = Y = S$ in Theorem 4.1. Since S is join-closed, we may further take $\langle D\rangle = D = S$. Then $m = n$ and $\det E(S)_{(k_1,k_2,\ldots,k_n)} = \det E(S)_{(1,2,\ldots,n)} = 1$ and so we obtain the first equality in (4.3). The second equality follows from Remark 2.1 and from Eq. (2.3), the third from (2.4). □

Remark 4.1. Theorem 4.2 can also be proved by taking $X = Y = S$ and $D = S$ in Theorem 3.1.

Example 4.1. Let $(P, \preceq) = (\mathbb{Z}, \leqslant)$, where $\leqslant$ is the natural ordering of the set of integers, and let $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{Z}$, where $x_1 < x_2 < \cdots < x_n$. Thus in this case $x_i \vee x_j = \max\{x_i, x_j\}$ for all $x_i, x_j \in S$. Let $t \in \mathbb{C}$ and $f : \mathbb{Z} \to \mathbb{C}$ be such function that $f(k) = k + t$ for all $k \in \mathbb{Z}$. Since the lattice $(\mathbb{Z}, \leqslant)$ is a chain, the set S is trivially both meet and join-closed. Now it follows from Theorem 4.2 that the determinant of the MAX matrix $[S]_f$ is

$$\begin{aligned}\det[S]_f &= \prod_{v=1}^{n} \sum_{x_v\preceq x_t} f(x_t)\mu_S(x_v, x_t)\\ &= (f(x_1) - f(x_2))(f(x_2) - f(x_3))\cdots(f(x_{n-1}) - f(x_n))f(x_n)\\ &= (x_1 - x_2)(x_2 - x_3)\cdots(x_{n-1} - x_n)(x_n + t).\end{aligned} \tag{4.4}$$

This result can easily be verified by using elementary methods. Since in this case the matrix $[S]_f$ is of the form

$$\begin{bmatrix} x_1+t & x_2+t & x_3+t & \cdots & x_n+t \\ x_2+t & x_2+t & x_3+t & \cdots & x_n+t \\ x_3+t & x_3+t & x_3+t & \cdots & x_n+t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n+t & x_n+t & x_n+t & \cdots & x_n+t \end{bmatrix}, \tag{4.5}$$

it is possible to obtain the same result for $\det[S]_f$ by using Gauss elimination process (first by subtracting the second row from the first, the third from the second etc.).

Theorem 4.3. *If S is upper-closed up to $\vee S$, then*

$$\det[S]_f = \prod_{v=1}^{n} \Psi_{S,f}(x_v) = \prod_{v=1}^{n} \sum_{x_v \preceq x_u} f(x_u)\mu(x_v, x_u). \tag{4.6}$$

Proof. The first equality in (4.6) follows from (4.3). The second equality follows from (2.5). □

Example 4.2. Let $(P, \preceq)$, S and f be as in Example 4.1 and let $x_i = x_1 + (i-1)$ for every $x_i \in S$. Now the set S is clearly upper-closed up to $x_n = x_1 + (n-1)$ and from Theorem 4.3 we get the determinant of the MAX matrix $[S]_f$ as

$$\begin{aligned} \det[S]_f &= \prod_{v=1}^{n} \sum_{x_v \preceq x_t} f(x_t)\mu(x_v, x_t) = (f(x_1) - f(x_2))(f(x_2) - f(x_3)) \cdots (f(x_{n-1}) - f(x_n))f(x_n) \\ &= (-1)^{n-1}(x_n + t). \end{aligned} \tag{4.7}$$

Note that this result can also be recovered easily by using the result in Example 4.1, since $x_{i-1} - x_i = -1$ for all $i = 2, \ldots, n$.

Corollary 4.1. *Let $(P, \preceq) = (\mathbb{Z}^+, |)$, let S be an LCM-closed set of distinct positive integers, and let f be an arithmetical function. Then the determinant of the LCM matrix $[S]_f$ is*

$$\det[S]_f = \prod_{v=1}^{n} \sum_{\substack{x_v \,|\, z \,|\, \mathrm{lcm}\, S \\ x_t \nmid z \\ v<t}} [\zeta * (f_z \mu)] \left(\frac{\mathrm{lcm}\, S}{z} \right). \tag{4.8}$$

Corollary 4.2. *Let $(P, \preceq) = (\mathbb{Z}^+, |)$, let S be a set of distinct positive integers which is multiple-closed up to* lcm S*, and let f be an arithmetical function. Then*

$$\det[S]_f = \prod_{v=1}^{n} [\zeta * (f_{x_v} \mu)] \left(\frac{\mathrm{lcm}\, S}{x_v} \right). \tag{4.9}$$

5. Inverse formulas

In this section we derive formulas for inverses of join matrices. In Theorem 5.1 we present an expression for the inverse of $[X, Y]_f$ on arbitrary sets X and Y, and in Theorem 5.2 we present an expression for the inverse of $[S]_f$ on arbitrary set S. Taking $(P, \preceq) = (\mathbb{Z}^+, |)$ we obtain a formula for the inverse of LCM matrices on arbitrary set S. Such formulas for the inverse of join or LCM matrices on an arbitrary set have not previously been presented in the literature. In Theorem 5.3 we calculate the inverse of $[S]_f$ on join-closed set S and in Theorem 5.4 we cover the case in which S is upper-closed up to $\vee S$.

Theorem 5.1. *Let $X_i = X \setminus \{x_i\}$ and $Y_i = Y \setminus \{y_i\}$ for $i = 1, 2, \ldots, n$. If $[X, Y]_f$ is invertible, then the inverse of $[X, Y]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \frac{(-1)^{i+j}}{\det[X, Y]_f} \sum_{1 \leqslant k_1 < k_2 < \cdots < k_{n-1} \leqslant m} \det E(X_j)_{(k_1, k_2, \ldots, k_{n-1})} \det E(Y_i)_{(k_1, k_2, \ldots, k_{n-1})} \times \Psi_{D,f}(d_{k_1}) \Psi_{D,f}(d_{k_2}) \cdots \Psi_{D,f}(d_{k_{n-1}}). \quad (5.1)$$

Proof. It is well known that

$$b_{ij} = \frac{\alpha_{ji}}{\det[X, Y]_f}, \quad (5.2)$$

where α_{ji} is the cofactor of the ji-entry of $[X, Y]_f$. It is easy to see that $\alpha_{ji} = (-1)^{i+j} \det[X_j, Y_i]_f$. By Theorem 4.1 we see that

$$\det[X_j, Y_i]_f = \sum_{1 \leqslant k_1 < k_2 < \cdots < k_{n-1} \leqslant m} \det E(X_j)_{(k_1, k_2, \ldots, k_{n-1})} \det E(Y_i)_{(k_1, k_2, \ldots, k_{n-1})} \times \Psi_{D,f}(d_{k_1}) \Psi_{D,f}(d_{k_2}) \cdots \Psi_{D,f}(d_{k_{n-1}}). \quad (5.3)$$

Combining the above equations we obtain Theorem 5.1. □

Theorem 5.2. *Let $S_i = S \setminus \{x_i\}$ for $i = 1, 2, \ldots, n$. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \frac{(-1)^{i+j}}{\det[S]_f} \sum_{1 \leqslant k_1 < k_2 < \cdots < k_{n-1} \leqslant m} \det E(S_j)_{(k_1, k_2, \ldots, k_{n-1})} \det E(S_i)_{(k_1, k_2, \ldots, k_{n-1})} \times \Psi_{D,f}(d_{k_1}) \Psi_{D,f}(d_{k_2}) \cdots \Psi_{D,f}(d_{k_{n-1}}). \quad (5.4)$$

Proof. Taking $X = Y = S$ in Theorem 5.1 we obtain Theorem 5.2. □

Theorem 5.3. *Suppose that S is join-closed. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \sum_{k=1}^{n} \frac{(-1)^{i+j}}{\Psi_{S,f}(x_k)} \det E(S_i^k) \det E(S_j^k), \quad (5.5)$$

where $E(S_i^k)$ is the $(n-1) \times (n-1)$ submatrix of $E(S)$ obtained by deleting the ith row and the kth column of $E(S)$, or

$$b_{ij} = \sum_{x_k \preceq x_i \wedge x_j} \frac{\mu_S(x_k, x_i) \mu_S(x_k, x_j)}{\Psi_{S,f}(x_k)}, \quad (5.6)$$

where μ_S is the Möbius function of the poset $(S, \preceq)$.

Proof. Since S is join-closed, we may take $D = S$. Then $E(S)$ is a square matrix with $\det E(S) = 1$. Further, $E(S)$ is the matrix associated with the zeta function of the finite poset $(S, \preceq)$. Thus the inverse of $E(S)$ is the matrix associated with the Möbius function of $(S, \preceq)$, that is, if $U = (u_{ij})$ is the inverse of $E(S)$, then $u_{ij} = \mu_S(x_i, x_j)$, see [1, p. 139]. On the other hand, $u_{ij} = \beta_{ij} / \det E(S) = \beta_{ij}$, where β_{ij} is the cofactor of the ij-entry of $E(S)$. Here $\beta_{ij} = (-1)^{i+j} \det E(S_i^j)$. Thus

$$(-1)^{i+j} \det E(S_i^j) = \mu_S(x_i, x_j). \quad (5.7)$$

Now we apply Theorem 5.2 with $D = S$. Then $m = n$, and using formulas (4.3) and (5.7) we obtain Theorem 5.3. □

Remark 5.1. Eq. (5.6) can also be proved by taking $X = Y = S$ and $D = S$ in Theorem 3.1 and then applying the formula

$$[S]_f^{-1} = (E(S)^T)^{-1}\Lambda_{S,f}^{-1}E(S)^{-1}.$$

Example 5.1. Let $(P, \preceq), f$ and S be as in Example 4.1. Let us denote $x_{n+1} = -t$. If $t = -x_{n+1} \neq -x_n$, then the matrix $[S]_f$ is invertible and the inverse of $[S]_f$ is the $n \times n$ tridiagonal matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \frac{1}{x_1 - x_2} & \text{if } i = j = 1, \\ \frac{1}{x_{i-1}-x_i} + \frac{1}{x_i - x_{i+1}} & \text{if } 1 < i = j \leqslant n, \\ \frac{1}{|x_i - x_j|} & \text{if } |i-j| = 1. \end{cases}$$

Theorem 5.4. *Suppose that S is upper-closed up to $\vee S$. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the $n \times n$ matrix $B = (b_{ij})$ with*

$$b_{ij} = \sum_{x_k \preceq x_i \wedge x_j} \frac{\mu(x_k, x_i)\mu(x_k, x_j)}{\Psi_{S,f}(x_k)}, \tag{5.8}$$

where μ is the Möbius function of $(P, \preceq)$.

Proof. Since S is upper-closed up to $\vee S$, we have $\mu_S = \mu$ on S, (apply [1, Proposition 4.6]). Thus Theorem 5.4 follows from Theorem 5.3. □

Example 5.2. Let $(P, \preceq), f$ and S be as in Example 4.2. If $t \neq -x_n$, then the matrix $[S]_f$ is invertible and the inverse of $[S]_f$ is the $n \times n$ tridiagonal matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 0 & \text{if } |i-j| > 1, \\ -1 & \text{if } i = j = 1, \\ -2 & \text{if } 1 < i = j < n, \\ -1 + \frac{1}{x_n + t} & \text{if } i = j = n, \\ 1 & \text{if } |i-j| = 1. \end{cases}$$

Corollary 5.1. *Let S be a set of distinct positive integers which is multiple-closed up to* lcm S*, and let f be an arithmetical function. If the LCM matrix $[S]_f$ is invertible, then its inverse is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \sum_{x_k | (x_i, x_j)} \frac{\mu(x_i/x_k)\mu(x_j/x_k)}{[\zeta * (f_{x_k}\mu)]\left(\frac{\operatorname{lcm} S}{x_k}\right)}. \tag{5.9}$$

Here μ is the number-theoretic Möbius function.

6. Formulas for meet matrices

Let f be a complex-valued function on P. We say that f is a semimultiplicative function if

$$f(x)f(y) = f(x \wedge y)f(x \vee y) \tag{6.1}$$

for all $x, y \in P$ (see [18]).

The notion of a semimultiplicative function arises from the theory of arithmetical functions. Namely, an arithmetical function f is said to be semimultiplicative if $f(r)f(s) = f((r, s))f([r, s])$ for all $r, s \in \mathbb{Z}^+$. For semimultiplicative arithmetical functions reference is made to the book by Sivaramakrishnan

[30], see also [9]. Note that a semimultiplicative arithmetical function f with $f(1) \neq 0$ is referred to as a quasimultiplicative arithmetical function. Quasimultiplicative arithmetical functions with $f(1) = 1$ are the usual multiplicative arithmetical functions.

In this section we show that meet matrices $(X, Y)_f$ with respect to semimultiplicative functions f possess properties similar to those given for join matrices $[X, Y]_f$ with respect to arbitrary functions f in Sections 3, 4 and 5. Since there already are several formulas for the determinant and the inverse of the matrix $(X, Y)_f$ (see [4,19]), the motivation in deriving new formulas probably needs clarification. The formulas of this section are especially useful when considering the matrix $(S)_f$, where the set S is either join-closed or upper-closed up to $\vee S$. That is, because in this case the formulas of this section result in shorter and simplier calculations. *Throughout this section f is a semimultiplicative function on P such that $f(x) \neq 0$ for all $x \in P$.*

Theorem 6.1

$$(X, Y)_f = \Delta_{X,f}[X, Y]_{1/f}\Delta_{Y,f} \tag{6.2}$$

or

$$(X, Y)_f = \Delta_{X,f}E(X)\Lambda_{D,1/f}E(Y)^T\Delta_{Y,f}, \tag{6.3}$$

where

$$\Delta_{X,f} = \operatorname{diag}(f(x_1), f(x_2), \ldots, f(x_n)) \tag{6.4}$$

and

$$\Delta_{Y,f} = \operatorname{diag}(f(y_1), f(y_2), \ldots, f(y_n)). \tag{6.5}$$

Proof. By (6.1) the *ij*-entry of $(X, Y)_f$ is

$$f(x_i \wedge y_j) = f(x_i)\frac{1}{f(x_i \vee y_j)}f(y_j). \tag{6.6}$$

We thus obtain (6.2), and applying Theorem 3.1 we obtain (6.3). □

From (6.2) we obtain

$$\det(X, Y)_f = \left(\prod_{\nu=1}^{n} f(x_\nu)f(y_\nu)\right)\det[X, Y]_{1/f} \tag{6.7}$$

and

$$(X, Y)_f^{-1} = \Delta_{Y,f}^{-1}[X, Y]_{1/f}^{-1}\Delta_{X,f}^{-1}. \tag{6.8}$$

Now, using (6.7), (6.8) and the formulas of Sections 4 and 5 we obtain formulas for meet matrices.

We first present formulas for the determinant of meet matrices. In Theorem 6.2 we give a formula for $\det(X, Y)_f$ on arbitrary sets X and Y. This is an alternative expression that given in [4]. In Theorems 6.3 and 6.4 respectively, we calculate $\det(S)_f$ when S is join-closed and upper-closed up to $\vee S$.

Theorem 6.2

(i) *If $n > m$, then* $\det(X, Y)_f = 0$.
(ii) *If $n \leqslant m$, then*

$$\det(X, Y)_f = \left(\prod_{\nu=1}^{n} f(x_\nu)f(y_\nu)\right)\left(\sum_{1\leqslant k_1<k_2<\cdots<k_n\leqslant m} \det E(X)_{(k_1,k_2,\ldots,k_n)} \det E(Y)_{(k_1,k_2,\ldots,k_n)} \times \Psi_{D,1/f}(d_{k_1})\Psi_{D,1/f}(d_{k_2})\cdots\Psi_{D,1/f}(d_{k_n})\right). \tag{6.9}$$

Theorem 6.3. *If S is join-closed, then*

$$\det(S)_f = \prod_{v=1}^{n} f(x_v)^2 \Psi_{S,1/f}(x_v) = \prod_{v=1}^{n} f(x_v)^2 \sum_{x_v \preceq x_u} \frac{\mu_S(x_v, x_u)}{f(x_u)}$$
$$= \prod_{v=1}^{n} f(x_v)^2 \sum_{\substack{x_v \preceq z \preceq \vee S \\ x_t \not\preceq z \\ v<t}} \sum_{z \preceq w \preceq \vee S} \frac{\mu(z, w)}{f(w)}. \tag{6.10}$$

Example 6.1. Let $(P, \preceq) = (\mathbb{Z}, \leqslant)$, $t \in \mathbb{C}$ a complex number such that $t \neq -x_i$ for all $x_i \in S$ and $f(x_i) = x_i + t$ for all $x_i \in S$. Since $(\mathbb{Z}, \leqslant)$ is a chain, the function f is trivially semimultiplicative. Now from Theorem 6.3 we get

$$\det(S)_f = \prod_{v=1}^{n} f(x_v)^2 \sum_{x_v \preceq x_u} \frac{\mu_S(x_v, x_u)}{f(x_u)} = \prod_{v=1}^{n-1} f(x_v)^2 \left(\frac{1}{f(x_v)} - \frac{1}{f(x_{v+1})} \right)$$
$$= \prod_{v=1}^{n-1} f(x_v)^2 \frac{f(x_{v+1}) - f(x_v)}{f(x_v) f(x_{v+1})}$$
$$= f(x_1)(f(x_2) - f(x_1))(f(x_3) - f(x_2)) \cdots (f(x_n) - f(x_{n-1}))$$
$$= (x_1 + t)(x_2 - x_1)(x_3 - x_2) \cdots (x_n - x_{n-1}).$$

It should be noted that under these assumptions the matrix $(S)_f$ is of the form

$$\begin{bmatrix} x_1+t & x_1+t & x_1+t & \cdots & x_1+t \\ x_1+t & x_2+t & x_2+t & \cdots & x_2+t \\ x_1+t & x_2+t & x_3+t & \cdots & x_3+t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1+t & x_2+t & x_3+t & \cdots & x_n+t \end{bmatrix}, \tag{6.11}$$

and as in Example 4.1, also in this case the determinant can easily be calculated by using Gauss elimination process.

Theorem 6.4. *If S is upper-closed up to $\vee S$, then*

$$\det(S)_f = \prod_{v=1}^{n} f(x_v)^2 \Psi_{S,1/f}(x_v) = \prod_{v=1}^{n} f(x_v)^2 \sum_{x_v \preceq x_u} \frac{\mu(x_v, x_u)}{f(x_u)}. \tag{6.12}$$

Example 6.2. Let $(P, \preceq) = (\mathbb{Z}, \leqslant)$, $S = \{x_1, x_1+1, x_1+2, \ldots, x_1+n-1\}$, $t \in \mathbb{C}$ a complex number such that $t \neq -x_i$ for all $x_i \in S$ and $f(x_i) = x_i + t$ for all $x_i \in S$. Now it follows from Theorem 6.4 that

$$\det(S)_f = \prod_{v=1}^{n} f(x_v)^2 \sum_{x_v \preceq x_u} \frac{\mu(x_v, x_u)}{f(x_u)} = \left(\prod_{v=1}^{n-1} f(x_v)^2 \left(\frac{1}{f(x_v)} - \frac{1}{f(x_{v+1})} \right) \right) f(x_n)$$
$$= \left(\prod_{v=1}^{n-1} f(x_v)^2 \frac{f(x_{v+1}) - f(x_v)}{f(x_v) f(x_{v+1})} \right) f(x_n)$$
$$= f(x_1)(f(x_2) - f(x_1))(f(x_3) - f(x_2)) \cdots (f(x_n) - f(x_{n-1}))$$
$$= (x_1 + t)(-1)^{n-1}.$$

Corollary 6.1. *Let S be an LCM-closed set of distinct positive integers, and let f be a quasimultiplicative arithmetical function such that $f(r) \neq 0$ for all $r \in \mathbb{Z}^+$. Then*

$$\det(S)_f = \prod_{v=1}^{n} f(x_v)^2 \sum_{\substack{x_v \mid z \mid \operatorname{lcm} S \\ x_t \nmid z \\ v<t}} \left[\zeta * \left(\frac{\mu}{f_z}\right)\right]\left(\frac{\operatorname{lcm} S}{z}\right). \tag{6.13}$$

Corollary 6.2. *Let S be a set of distinct positive integers which is multiple-closed up to* lcm *S, and let f be a quasimultiplicative arithmetical function such that $f(r) \neq 0$ for all $r \in \mathbb{Z}^+$. Then*

$$\det(S)_f = \prod_{v=1}^{n} f(x_v)^2 \left[\zeta * \left(\frac{\mu}{f_{x_v}}\right)\right]\left(\frac{\operatorname{lcm} S}{x_v}\right). \tag{6.14}$$

We next derive formulas for inverses of meet matrices. In Theorem 6.5 we give an expression for the inverse of $(X, Y)_f$ on arbitrary sets X and Y, and in Theorem 6.6 we give an expression for the inverse of $(S)_f$ on arbitrary set S. Taking $(P, \preceq) = (\mathbb{Z}^+, |)$ we could obtain a formula for the inverse of GCD matrices on arbitrary set S. In Theorems 6.7 and 6.8, respectively, we calculate the inverse of $(S)_f$ in cases when S is join-closed and upper-closed up to $\vee S$. Formulas similar to Theorems 6.7 and 6.8, although with stronger assumptions, have been presented earlier in [18].

Theorem 6.5. *Let $X_i = X \setminus \{x_i\}$ and $Y_i = Y \setminus \{y_i\}$ for $i = 1, 2, \ldots, n$. If $[X, Y]_f$ is invertible, then the inverse of $(X, Y)_f$ is the $n \times n$ matrix $B = (b_{ij})$ with*

$$\begin{aligned} b_{ij} = {} & \frac{(-1)^{i+j}}{f(x_j)f(y_i)\det(X, Y)_f}\left(\prod_{v=1}^{n} f(x_v)f(y_v)\right) \\ & \times\Bigg(\sum_{1 \leqslant k_1 < k_2 < \cdots < k_{n-1} \leqslant m} \det E(X_j)_{(k_1,k_2,\ldots,k_{n-1})} \det E(Y_i)_{(k_1,k_2,\ldots,k_{n-1})} \\ & \times \Psi_{D,1/f}(d_{k_1})\Psi_{D,1/f}(d_{k_2})\cdots\Psi_{D,1/f}(d_{k_{n-1}})\Bigg). \end{aligned} \tag{6.15}$$

Theorem 6.6. *Let $S_i = S \setminus \{x_i\}$ for $i = 1, 2, \ldots, n$. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the $n \times n$ matrix $B = (b_{ij})$ with*

$$\begin{aligned} b_{ij} = {} & \frac{(-1)^{i+j}}{f(x_i)f(x_j)\det(S)_f}\left(\prod_{v=1}^{n} f(x_v)^2\right) \\ & \times\Bigg(\sum_{1 \leqslant k_1 < k_2 < \cdots < k_{n-1} \leqslant m} \det E(S_i)_{(k_1,k_2,\ldots,k_{n-1})} \det E(S_j)_{(k_1,k_2,\ldots,k_{n-1})} \\ & \times \Psi_{D,1/f}(d_{k_1})\Psi_{D,1/f}(d_{k_2})\cdots\Psi_{D,1/f}(d_{k_{n-1}})\Bigg). \end{aligned} \tag{6.16}$$

Theorem 6.7. *Suppose that S is join-closed. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the $n \times n$ matrix $B = (b_{ij})$ with*

$$b_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{x_k \preceq x_i \wedge x_j} \frac{\mu_S(x_k, x_i)\mu_S(x_k, x_j)}{\Psi_{S,1/f}(x_k)}. \tag{6.17}$$

Here μ_S is the Möbius function of the poset $(S, \preceq)$.

Example 6.3. By Theorem 6.7, the inverse of the MIN matrix $(S)_f$ in Example 6.1 is the $n \times n$ tridiagonal matrix $B = (b_{ij})$ with

$$b_{ij} = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \frac{1}{x_2-x_1}\frac{x_2+t}{x_1+t} & \text{if } i = j = 1, \\ \frac{1}{x_i+t}\left(\frac{x_{i-1}+t}{x_i-x_{i-1}} + \frac{x_{i+1}+t}{x_{i+1}-x_i}\right) & \text{if } 1 < i = j < n, \\ \frac{1}{x_n+t}\left(\frac{x_{n-1}+t}{x_n-x_{n-1}} + 1\right) & \text{if } i = j = n, \\ \frac{-1}{|x_i-x_j|} & \text{if } |i-j| = 1. \end{cases}$$

Theorem 6.8. *Suppose that S is upper-closed up to $\vee S$. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{x_k \preceq x_i \wedge x_j} \frac{\mu(x_k, x_i)\mu(x_k, x_j)}{\Psi_{S,1/f}(x_k)}. \tag{6.18}$$

Here μ is the Möbius function of $(P, \preceq)$.

Example 6.4. By Theorem 6.8, the inverse of the MIN matrix $(S)_f$ in Example 6.2 is the $n \times n$ tridiagonal matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \frac{x_1+1+t}{x_1+t} & \text{if } i = j = 1, \\ 2 & \text{if } 1 < i = j < n, \\ 1 & \text{if } i = j = n, \\ -1 & \text{if } |i-j| = 1. \end{cases}$$

Corollary 6.3. *Let S be a set of distinct positive integers which is multiple-closed up to* lcm *S, and let f be a quasimultiplicative arithmetical function such that $f(r) \neq 0$ for all $r \in \mathbb{Z}^+$. If the GCD matrix $(S)_f$ is invertible, then its inverse is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{x_k \mid (x_i, x_j)} \frac{\mu(x_i/x_k)\mu(x_j/x_k)}{\left[\zeta * \left(\frac{\mu}{f_{x_k}}\right)\right]\left(\frac{\operatorname{lcm} S}{x_k}\right)}. \tag{6.19}$$

Here μ is the number-theoretic Möbius function.

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Publication II

Some properties of row-adjusted meet and join matrices

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Some properties of row-adjusted meet and join matrices

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Let $(P, \preceq)$ be a lattice, S a finite subset of P and $f_1, f_2, \ldots, f_n$ complex-valued functions on P. We define row-adjusted meet and join matrices on S by $(S)_{f_1,\ldots,f_n} = (f_i(x_i \wedge x_j))$ and $[S]_{f_1,\ldots,f_n} = (f_i(x_i \vee x_j))$. In this article, we determine the structure of the matrix $(S)_{f_1,\ldots,f_n}$ in general case and in the case when the set S is meet closed we give bounds for $\mathrm{rank}(S)_{f_1,\ldots,f_n}$ and present expressions for $\det(S)_{f_1,\ldots,f_n}$ and $(S)^{-1}_{f_1,\ldots,f_n}$. The same is carried out dually for row-adjusted join matrix of a join-closed set S.

Keywords: meet matrix; join matrix; GCD matrix; LCM matrix; Smith determinant

AMS Subject Classifications: 11C20; 15B36; 06B99

1. Introduction

In 1876 Smith [18] presented a formula for the determinant of the $n \times n$ matrix having (i, j), the greatest common divisor of i and j as its ij element. During the twentieth century many other results concerning matrices with similar structure were published, see for example [8,13,21]. In 1989 Beslin and Ligh [5] introduced the concept of a GCD matrix on a set S, where $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{Z}^+$ with $x_1 < x_2 < \cdots < x_n$ and the GCD matrix (S) has (x_i, x_j), the greatest common divisor of x_i and x_j as its ij entry. Since then numerous publications have appeared in order to universalize the concept of GCD matrix. For example, Haukkanen [6] and Luque [14] consider the determinants of multidimensional generalizations of GCD matrices and Hong et al. [9] study power GCD matrices for a unique factorization domain.

Poset theoretic generalizations of GCD matrices were first introduced by Lindström [12] and Wilf [20]. In these generalizations $(P, \preceq)$ is a meet semilattice, f is a function $P \to \mathbb{C}$, $S = \{x_1, x_2, \ldots, x_n\} \subset P$, $x_i \preceq x_j \Rightarrow i \leq j$ and $(S)_f$ is an $n \times n$ matrix with $f(x_i \wedge x_j)$ as its ij element (here $x_i \wedge x_j$ denotes the meet of x_i and x_j). These matrices are referred to as meet matrices. The papers by Lindström [12] and Wilf [20] arose from needs for combinatorics and became possible since Rota [17] had previously developed his famous theory on Möbius functions. Rajarama Bhat [16] and Haukkanen [7] were the first to investigate meet matrices systematically, presenting many important properties of ordinary GCD matrices in terms of meet matrices. In [11], Korkee and Haukkanen define and study the join matrix $[S]_f$ of the

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set S with respect to f, where $f(x_i \vee x_j)$ is the ij element of the matrix $[S]_f$. Here $x_i \vee x_j$ is the join of x_i and x_j.

During the past 10 years the concept of meet matrix has been generalized even further in many different ways. Korkee [10] studies the properties of a matrix $M_{S,f}^{\alpha,\beta,\gamma,\delta}$, which yields both the matrix $(S)_f$ and $[S]_f$ as its special case. A totally different approach is taken by Altinisik et al. in [2] when they define meet and join matrices on two subsets X and Y of P. In an upcoming paper Mattila and Haukkanen [15] give a more detailed treatment for join matrices on two sets. The methods we use in this article are similar to those occurring in papers [2,15].

A further idea of generalization is presented by Bege [3] as he studies yet another GCD related matrix $(F(i,(i,j)))$, where $F(m,n)$ is an arithmetical function of two variables. More recently Bege has also posted a paper [4] about similarly generalized LCM matrices. However, for present purposes it is convenient to use a slightly different notation. For every $i \in \mathbb{Z}^+$ we define an arithmetical function f_i of one variable by

$$f_i(m) = F(i,m) \quad \text{for all } m \in \mathbb{Z}^+. \tag{1.1}$$

With this notation Bege's matrix takes the form

$$\begin{bmatrix} f_1((1,1)) & f_1((1,2)) & \cdots & f_1((1,n)) \\ f_2((2,1)) & f_2((2,2)) & \cdots & f_2((2,n)) \\ \vdots & \vdots & \ddots & \vdots \\ f_n((n,1)) & f_n((n,2)) & \cdots & f_n((n,n)) \end{bmatrix}. \tag{1.2}$$

In order to distinguish between this and the numerous other generalizations of GCD matrices, this matrix is referred to as the *row-adjusted GCD matrix of the set* $\{1,2,\ldots,n\}$. This notation also enables us to define row-adjusted meet and join matrices.

Definition 1.1 Let $(P,\preceq)$ be a lattice, $S=\{x_1,x_2,\ldots,x_n\}$ be a finite subset of P with $x_i \preceq x_j \Rightarrow i \leq j$ and $f_1,f_2,\ldots,f_n$ be complex-valued functions on P. The row-adjusted meet matrix of the set S is the $n \times n$ matrix $(S)_{f_1,\ldots,f_n}$, which has $(f_i(x_i \wedge x_j))$ as its ij element. Similarly, the row-adjusted join matrix $[S]_{f_1,\ldots,f_n}$ has $(f_i(x_i \vee x_j))$ as its ij element.

More explicitly,

$$(S)_{f_1,\ldots,f_n} = \begin{bmatrix} f_1(x_1 \wedge x_1) & f_1(x_1 \wedge x_2) & \cdots & f_1(x_1 \wedge x_n) \\ f_2(x_2 \wedge x_1) & f_2(x_2 \wedge x_2) & \cdots & f_2(x_2 \wedge x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x_n \wedge x_1) & f_n(x_n \wedge x_2) & \cdots & f_n(x_n \wedge x_n) \end{bmatrix} \tag{1.3}$$

and

$$[S]_{f_1,\ldots,f_n} = \begin{bmatrix} f_1(x_1 \vee x_1) & f_1(x_1 \vee x_2) & \cdots & f_1(x_1 \vee x_n) \\ f_2(x_2 \vee x_1) & f_2(x_2 \vee x_2) & \cdots & f_2(x_2 \vee x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x_n \vee x_1) & f_n(x_n \vee x_2) & \cdots & f_n(x_n \vee x_n) \end{bmatrix}. \tag{1.4}$$

It turns out that there are some results concerning the matrix $(S)_{f_1,\ldots,f_n}$ to be found in the literature by Lindström [12] and Luque [14]. When the notation is the same as defined in (1.1), these results can be easily applied to Bege's matrix.

Unlike the ordinary meet and join matrices, the matrices $(S)_{f_1,\ldots,f_n}$ and $[S]_{f_1,\ldots,f_n}$ are usually not symmetric. There are also many other key properties of meet and join matrices that do not hold for row-adjusted meet and join matrices. Hence, neither the traditional methods of meet and join matrices works in the study of these row-adjusted matrices.

Remark 1.1 In the case when $f_1 = f_2 = \cdots = f_n = f$, we have $(S)_{f_1,\ldots,f_n} = (S)_f$ and $[S]_{f_1,\ldots,f_n} = [S]_f$.

Remark 1.2 Taking the transpose of a row-adjusted meet or join matrix results in a *column-adjusted* meet or join matrix. Therefore the results concerning row-adjusted meet and join matrices can be easily translated for column-adjusted meet and join matrices using this connection.

At the end of his paper Bege [3] presents an open problem regarding the structure and the determinant of the matrix $(F(i,(i,j)))$. It appears that the question about the determinant could be solved using Lindström's result in [12]. In this article, we present a more systematic investigation of the structure of $(S)_{f_1,\ldots,f_n}$ and $[S]_{f_1,\ldots,f_n}$ in general case. Then by using this knowledge we are able to find a different proof for Lindström's determinant formula and also prove some other results concerning the rank and inverse of these matrices.

2. Preliminaries

Let $(P, \preceq)$ be a lattice, $S = \{x_1, x_2, \ldots, x_n\}$ a finite subset of P and

$$f_1, f_2, \ldots, f_n \colon P \to \mathbb{C}$$

complex-valued functions on P (or functions from P to any field F). We also assume that the elements of S are distinct and arranged so that

$$x_i \preceq x_j \Rightarrow i \leq j.$$

The set S is said to be *meet closed* if $x \wedge y \in S$ for all $x, y \in S$. In other words, the structure $(S, \preceq)$ is a meet semilattice. The concept of *join-closed set* is defined dually.

Let $D = \{d_1, d_2, \ldots, d_m\}$ be another subset of P containing all the elements $x_i \wedge x_j$, $i, j = 1, 2, \ldots, n$, and having its elements arranged so that

$$d_i \preceq d_j \Rightarrow i \leq j.$$

Now for every $i = 1, 2, \ldots, n$ we define the function Ψ_{D,f_i} on D inductively as

$$\Psi_{D,f_i}(d_k) = f_i(d_k) - \sum_{d_v \prec d_k} \Psi_{D,f_i}(d_v), \tag{2.1}$$

or equivalently

$$f_i(d_k) = \sum_{d_v \preceq d_k} \Psi_{D,f_i}(d_v). \tag{2.2}$$

Thus we have

$$\Psi_{D,f_i}(d_k) = \sum_{d_v \preceq d_k} f_i(d_v)\mu_D(d_v, d_k), \tag{2.3}$$

where μ_D is the Möbius function of the poset $(D, \preceq)$, see [1, Section IV.1] and [19, 3.7.1 Proposition].

Let E_D be the $n \times m$ matrix defined as

$$(e_D)_{ij} = \begin{cases} 1 & \text{if } d_j \preceq x_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

The matrix E_D may be referred to as the incidence matrix of the set D with respect to the set S and the partial ordering $\preceq$.

Finally, we need another $n \times m$ matrix $\Upsilon = (\upsilon_{ij})$, where

$$\upsilon_{ij} = (e_D)_{ij}\Psi_{D,f_i}(d_j). \tag{2.5}$$

In other words, if Ξ is the $n \times m$ matrix having $\Psi_{D,f_i}(d_j)$ as its ij element, then $\Upsilon = E_D \circ \Xi$, the Hadamard product of the matrices E_D and Ξ.

3. A structure theorem

In this section we give a factorization of the matrix $(S)_{f_1,\ldots,f_n}$, which then enables us to derive formulae for the rank, the determinant and the inverse of the matrix $(S)_{f_1,\ldots,f_n}$. Unlike the theorems in Sections 4–6, Theorem 3.1 can be applied also in case when the set S is not meet closed.

THEOREM 3.1 *Let S and D be as in Section* 2. *We have*

$$(S)_{f_1,\ldots,f_n} = \Upsilon E_D^T = (E_D \circ \Xi)E_D^T, \tag{3.1}$$

where E_D^T means the transpose of the matrix E_D.

Proof By (2.2), (2.4) and (2.5) the ij element of $(S)_{f_1,\ldots,f_n}$ is

$$f_i(x_i \wedge x_j) = \sum_{d_v \preceq x_i \wedge x_j} \Psi_{D,f_i}(d_v) = \sum_{k=1}^{m} (e_D)_{ik}\Psi_{D,f_i}(e_D)_{jk}, \tag{3.2}$$

which is the ij element of the matrix ΥE_D^T. ■

Remark 3.1 Theorem 3.1 is the core of this article since theorems in Sections 4–6 essentially follow from this factorization.

Remark 3.2 It is possible to define row-adjusted meet and join matrices $(X, Y)_{f_1,\ldots,f_n}$ and $[X, Y]_{f_1,\ldots,f_n}$ on two sets X and Y by $((X, Y)_{f_1,\ldots,f_n})_{ij} = f_i(x_i \wedge y_j)$ and $([X, Y]_{f_1,\ldots,f_n})_{ij} = f_i(x_i \vee y_j)$. It would be possible to generalize Theorem 3.1 for these matrices, but the methods used in the proofs of the other theorems do not work in this general case.

Remark 3.3 In the case when the set S is meet closed Theorem 3.1 also provides an effective way to calculate all the necessary values $\Psi_{S,f_i}(x_j)$ as follows. In this case we may take $D=S$ in which case both E_S and Υ are square matrices of size $n\times n$. Since E_S is also invertible, from Equation (3.1) we obtain

$$\Upsilon=(S)_{f_1,\ldots,f_n}(E_S^T)^{-1}, \tag{3.3}$$

which gives the values of $\Psi_{S,f_i}(x_j)$. Here the matrix E_S^T is the matrix associated with the zeta function ζ_S of the set S [1, p. 139], and thus the matrix $(E_S^T)^{-1}$ is the matrix of the Möbius function of the set S and has $\mu_S(x_i,x_j)$ as its ij element.

The following example gives a solution for the first part of Bege's problem.

Example 3.1 The row-adjusted GCD matrix of the set $S=\{1,2,\ldots,n\}$ is the product of the matrices $\Upsilon=(\upsilon_{ij})$ and E_S^T, where

$$(e_S)_{ij}=\begin{cases}1 & \text{if } j\,|\,i,\\ 0 & \text{otherwise}\end{cases} \tag{3.4}$$

and

$$\upsilon_{ij}=(e_S)_{ij}\Psi_{S,f_i}(j)=(e_S)_{ij}\sum_{k\,|\,j}f_i(k)\mu\left(\frac{j}{k}\right)=(e_S)_{ij}(f_i*\mu)(j), \tag{3.5}$$

where $*$ is the Dirichlet convolution and μ is the number-theoretic Möbius function. It should be noted that here the notation $F(i,k)=f_i(k)$ is not only convenient but also enables the use of the Dirichlet convolution.

4. Rank estimations

In this section we derive bounds for $\operatorname{rank}(S)_{f_1,\ldots,f_n}$ in the case when the set S is meet closed. The rank of meet and join matrices or even GCD and LCM matrices has not been studied earlier in the literature.

THEOREM 4.1 *Let S be a meet-closed set and let k be the number of indices i with $\Psi_{S,f_i}(x_i)=0$. Then the following properties hold.*

(1) $\operatorname{rank}(S)_{f_1,\ldots,f_n}=0$ *iff* $f_i(x_i\wedge x_j)=0$ *for all* $i,j=1,\ldots,n$.
(2) *If* $k=0$, *then* $\operatorname{rank}(S)_{f_1,\ldots,f_n}=n$.
(3) *If* $k>0$, *then*

$$n-k\le\operatorname{rank}(S)_{f_1,\ldots,f_n}\le n-1. \tag{4.1}$$

Proof

(1) Follows trivially.
(2) By Theorem 3.1 we have

$$\operatorname{rank}(S)_{f_1,\ldots,f_n}=\operatorname{rank}\left(\Upsilon E_S^T\right). \tag{4.2}$$

Since in this case the matrices Υ and E_S are both triangular square matrices with full rank, the claim follows immediately.

(3) Since multiplying with the invertible matrix E_S^T does not change the rank, we have

$$\mathrm{rank}(S)_{f_1,\ldots,f_n} = \mathrm{rank}\,\Upsilon. \tag{4.3}$$

To obtain the latter inequality we only need to note that since at least one of the diagonal elements of Υ equals zero, the rows of Υ cannot be linearly independent and thereby Υ cannot have a full rank. On the other hand, the $n-k$ rows with nonzero diagonal elements constitute a linearly independent set, from which we obtain the first inequality. ■

In the case when the set S is meet closed and $f_1 = \cdots = f_n = f$ (i.e. in the case of ordinary meet matrix) the question of the rank becomes trivial. Namely, the matrix $(S)_f$ can be written as

$$(S)_f = E_S \Lambda E_S^T, \tag{4.4}$$

where $\Lambda = \mathrm{diag}(\Psi_{S,f}(x_1), \Psi_{S,f}(x_2), \ldots, \Psi_{S,f}(x_n))$, see [2, Theorem 3.1]. Now by the same argument as in the proof of Theorem 4.1 we have

$$\mathrm{rank}(S)_f = \mathrm{rank}\,\Lambda = n-k. \tag{4.5}$$

The following two examples show that the bounds in Theorem 4.1 are the best possible under these assumptions. They also show that a large value of k may indicate a large decline of the rank of the row-adjusted meet matrix, but not necessarily.

Example 4.1 Let $x_1 = x_i \wedge x_j$ for all $i,j = 1,\ldots,n$, which implies that x_1 is the smallest element of S and the set $S\backslash\{x_1\}$ is an antichain. Now the set S is clearly meet closed, and for every $i = 2,\ldots,n$ we have

$$\Psi_{S,f_i}(x_i) = f_i(x_i) - f_i(x_1). \tag{4.6}$$

If $i > 1$ and we set $f_i(x_i) = f_i(x_1)$, then the ith column of Υ becomes the zero vector and thus for every $i > 1$ we may reduce the rank of the matrix $(S)_{f_1,\ldots,f_n}$ by one. Therefore if the first diagonal element of Υ is not zero, then $\mathrm{rank}(S)_{f_1,\ldots,f_n} = n-k$.

Example 4.2 Let $(P, \preceq) = \mathcal{N}_5$ and $S = P$ as shown in Figure 1. Let

$$f_2(x_2) = f_3(x_1) = f_3(x_3) = f_4(x_3) = f_4(x_4) = f_5(x_4) = f_5(x_5) = 1 \tag{4.7}$$

and $f_i(x_j) = 0$ otherwise. Simple calculations show that $\Psi_{S,f_2}(x_2) = 1 \neq 0$,

$$\Psi_{S,f_1}(x_1) = \Psi_{S,f_3}(x_3) = \Psi_{S,f_4}(x_4) = \Psi_{S,f_5}(x_5) = 0, \tag{4.8}$$

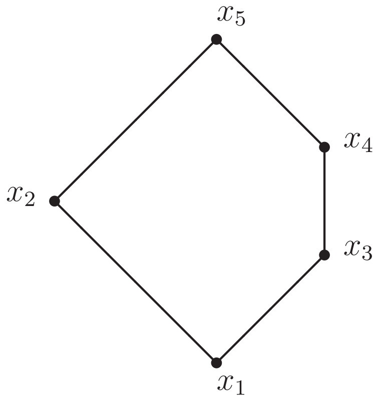

Figure 1. The lattice $\mathcal{N}_5$ and the choices of the elements of the set S.

and thereby $k = 4$. But, on the other hand, we have

$$(S)_{f_1,\ldots,f_n} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \tag{4.9}$$

and clearly $\operatorname{rank}(S)_{f_1,\ldots,f_n} = 4$.

5. Determinant formula

In this section we present a determinant formula for the matrix $(S)_{f_1,\ldots,f_n}$ when the set S is meet closed. This theorem is almost the same as that presented by Lindström [12]. It is possible to use the Cauchy–Binet equality to obtain a determinant formula for $(S)_{f_1,\ldots,f_n}$ in general case. Since it is similar to the case of usual meet matrix, we do not present it here.

THEOREM 5.1 [12, Theorem] *If the set S is meet closed, then*

$$\det(S)_{f_1,\ldots,f_n} = \prod_{i=1}^{n} \Psi_{S,f_i}(x_i) = \prod_{i=1}^{n} \sum_{x_j \preceq x_i} f_i(x_j)\mu_S(x_j, x_i). \tag{5.1}$$

Proof Since the set S is meet closed, we have $D = S$. Then the matrix E_S is a lower triangular square matrix having every main diagonal element equal to 1. The matrix Υ is a lower triangular square matrix with $\Psi_{S,f_1}(x_1), \Psi_{S,f_2}(x_2), \ldots, \Psi_{S,f_n}(x_n)$ as diagonal elements. Thus $\det E_S = 1$ and by Theorem 3.1 we have

$$\det(S)_{f_1,\ldots,f_n} = \det \Upsilon = \prod_{i=1}^{n} \Psi_{S,f_i}(x_i). \tag{5.2}$$

The second equality follows from (2.3). ■

Remark 5.1 The original theorem by Lindström [12] is slightly more general since it does not require the assumption $x_i \preceq x_j \Rightarrow i \leq j$. As he states, the rows and columns of $(S)_{f_1,\ldots,f_n}$ can always be permuted in a way that does not change the determinant but makes the matrix $(S)_{f_1,\ldots,f_n}$ to fulfil this condition.

The following example gives a solution to the second part of Bege's problem.

Example 5.1 For the row-adjusted GCD matrix on the set $S=\{1,2,\ldots,n\}$ we have

$$\det(\{1,2,\ldots,n\})_{f_1,\ldots,f_n} = \prod_{i=1}^{n} \Psi_{S,f_i}(i) = \prod_{i=1}^{n} \sum_{j\,|\,i} f_i(j)\mu\left(\frac{i}{j}\right) = \prod_{i=1}^{n} (f_i * \mu)(i). \tag{5.3}$$

6. Inverse formula

In this section we study the inverse of the matrix $(S)_{f_1,\ldots,f_n}$ when the set S is meet closed. A formula for $(S)^{-1}_{f_1,\ldots,f_n}$ in general case could be obtained with the aid of meet matrices on two sets and the Cauchy–Binet equation. We do not, however, present the details here.

THEOREM 6.1 *If the set S is meet closed, then the matrix $(S)_{f_1,\ldots,f_n}$ is invertible iff $\Psi_{S,f_i}(x_i) \neq 0$ for all $i=1,\ldots,n$. Furthermore, in this case the inverse of $(S)_{f_1,\ldots,f_n}$ is the $n \times n$ matrix $B=(b_{ij})$ with*

$$b_{ij} = \sum_{k=j}^{n} \mu_S(x_i, x_k)\theta_{kj}, \tag{6.1}$$

where the numbers $\theta_{jj}, \theta_{j+1,j}, \ldots, \theta_{nj}$ are defined recursively as

$$\theta_{kj} = \begin{cases} \dfrac{1}{\Psi_{S,f_j}(x_j)} & \text{if } k=j, \\ -\dfrac{1}{\Psi_{S,f_k}(x_k)} \displaystyle\sum_{u=j}^{k-1} e_{ku}\Psi_{S,f_k}(x_u)\theta_{uj} & \text{if } k>j. \end{cases} \tag{6.2}$$

Proof The first part follows directly from Theorem 5.1. To prove the second part we use Theorem 3.1 and we obtain

$$(S)^{-1}_{f_1,\ldots,f_n} = (E_S^T)^{-1}\Upsilon^{-1}. \tag{6.3}$$

In order to obtain the *ij* element of the matrix $(S)^{-1}_{f_1,\ldots,f_n}$ we only have to ascertain the *i*th row of $(E_S^T)^{-1}$ and the *j*th column of Υ^{-1}. As stated in Remark 3.3, the matrix $(E_S^T)^{-1}$ is the matrix associated with the Möbius function of the set S. Therefore its *i*th row is

$$\begin{bmatrix} 0 & \ldots & 0 & \underbrace{\mu_S(x_i,x_i)}_{=1} & \mu_S(x_i,x_{i+1}) & \ldots & \mu_S(x_i,x_n) \end{bmatrix}. \tag{6.4}$$

Now let $\Theta = (\theta_{ij})$ denote the inverse of Υ. By multiplying the jth row of Υ with the jth column of Θ, we obtain

$$\Psi_{S,f_j}(x_j)\theta_{jj} = 1. \tag{6.5}$$

Further, the multiplication of the kth row of Υ and the jth column of Θ results in

$$\sum_{u=j}^{k} e_{ku}\Psi_{S,f_k}(x_u)\theta_{uj} = 0. \tag{6.6}$$

Thus we obtain (6.2), and (6.1) follows when we multiply the matrices Θ and $(E_S^T)^{-1}$. ■

7. Formulae for row-adjusted join matrices

In this section, the results presented in previous sections are translated for row-adjusted join matrices. The proofs of these dual theorems are omitted for the sake of brevity. Similar methods as in [15] could be used to derive formulae for the matrix $[S]_{f_1,\ldots,f_n}$ in case when the set S is not join-closed. Row-adjusted join matrices (or even row-adjusted LCM matrices) have not previously been studied in the literature. As stated in Remark 1.2, the study of column-adjusted join matrices can be easily reverted to the study of row-adjusted join matrices via taking the transpose.

Let $S = \{x_1, x_2, \ldots, x_n\}$ and $D' = \{d'_1, d'_2, \ldots, d'_{m'}\}$ be a subset of P containing all the elements $x_i \vee x_j$, $i, j = 1, 2, \ldots, n$, and having its elements arranged so that

$$d'_i \preceq d'_j \Rightarrow i \leq j.$$

For every $i = 1, 2, \ldots, n$ we define the function Ψ'_{D',f_i} on D' inductively as

$$\Psi'_{D',f_i}(d'_k) = f_i(d'_k) - \sum_{d'_k \prec d'_v} \Psi'_{D',f_i}(d'_v), \tag{7.1}$$

or equivalently

$$f_i(d'_k) = \sum_{d'_k \preceq d'_v} \Psi'_{D',f_i}(d'_v). \tag{7.2}$$

Thus we have

$$\Psi'_{D',f_i}(d'_k) = \sum_{d'_k \preceq d'_v} f_i(d'_v)\mu_{D'}(d'_k, d'_v), \tag{7.3}$$

where $\mu_{D'}$ is the Möbius function of the poset $(D', \preceq)$, see [19, 3.7.2 Proposition].

Let $E'_{D'}$ be the $n \times m'$ matrix defined as

$$(e'_{D'})_{ij} = \begin{cases} 1 & \text{if } x_i \preceq d'_j, \\ 0 & \text{otherwise.} \end{cases} \tag{7.4}$$

Finally, let $\Upsilon' = (\upsilon'_{ij})$ be the $n \times m'$ matrix, where

$$\upsilon'_{ij} = (e'_{D'})_{ij}\Psi'_{D',f_i}(d'_j). \tag{7.5}$$

THEOREM 7.1 *Let S and D′ be as above. Then*

$$[S]_{f_1,\ldots,f_n} = \Upsilon'(E'_{D'})^T. \tag{7.6}$$

Remark 7.1 Theorem 7.1 has special importance like Theorem 3.1, since all of the proofs of Theorems 7.2, 7.3 and 7.4 make use of it.

THEOREM 7.2 *Let S be a join-closed set and let k be the number of indices i with* $\Psi'_{S,f_i}(x_i) = 0$. *Then the following properties hold.*

(1) $\mathrm{rank}[S]_{f_1,\ldots,f_n} = 0$ *iff* $f_i(x_i \vee x_j) = 0$ *for all* $i,j = 1,\ldots,n$.
(2) *If* $k = 0$, *then* $\mathrm{rank}[S]_{f_1,\ldots,f_n} = n$.
(3) *If* $k > 0$, *then*

$$n - k \le \mathrm{rank}[S]_{f_1,\ldots,f_n} \le n - 1. \tag{7.7}$$

THEOREM 7.3 *If the set S is join closed, then*

$$\det[S]_{f_1,\ldots,f_n} = \prod_{i=1}^{n} \Psi'_{S,f_i}(x_i) = \prod_{i=1}^{n} \sum_{x_i \preceq x_j} f_i(x_j)\mu_S(x_i, x_j). \tag{7.8}$$

THEOREM 7.4 *If the set S is join closed, then the* matrix$[S]_{f_1,\ldots,f_n}$ *is invertible iff* $\Psi'_{S,f_i}(x_i) \neq 0$ *for all* $i = 1,\ldots,n$. *Furthermore, in this case the inverse of* $[S]_{f_1,\ldots,f_n}$ *is the* $n \times n$ matrix $B' = (b'_{ij})$ *with*

$$b'_{ij} = \sum_{k=1}^{j} \mu_S(x_k, x_i)\theta'_{kj}, \tag{7.9}$$

where the numbers $\theta'_{jj}, \theta'_{j-1,j}, \ldots, \theta'_{1j}$ *are defined recursively as*

$$\theta'_{kj} = \begin{cases} \dfrac{1}{\Psi'_{S,f_j}(x_j)} & \text{if } k = j, \\[2ex] -\dfrac{1}{\Psi'_{S,f_k}(x_k)} \displaystyle\sum_{u=k+1}^{j} e'_{ku}\Psi'_{S,f_k}(x_u)\theta'_{uj} & \text{if } j > k. \end{cases} \tag{7.10}$$

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On the eigenvalues of combined meet and join matrices

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ABSTRACT

In this article we give bounds for the eigenvalues of a matrix, which can be seen as a common generalization of meet and join matrices and therefore also as a generalization of both GCD and LCM matrices. Although there are some results concerning the factorizations, the determinant and the inverse of this so-called combined meet and join matrix, the eigenvalues of this matrix have not been studied earlier. Finally we also give a nontrivial lower bound for a certain constant c_n, which is needed in calculating the above-mentioned eigenvalue bounds in practice. So far there are no such lower bounds to be found in the literature.

1. Introduction

The concept of a meet matrix was first defined by Indian mathematician Bhat in 1991 [3], whereas join matrices first appeared in a paper by Korkee and Haukkanen in 2003 [13]. There are also many other papers about these matrices by Haukkanen and Korkee, see e.g. the references in [17]. Meet and join matrices were also studied

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by Hong and Sun in 2004 [9]. Both concepts are natural generalizations of GCD and LCM matrices presented by Smith as early as in 1875 [20]. The definitions are as follows: Assume that $(P, \preceq)$ is a locally finite lattice, f is a real or complex-valued function on P and $S = \{x_1, x_2, \dots, x_n\}$ is a finite set of distinct elements of P such that

$$x_i \preceq x_j \quad \Rightarrow \quad i \leq j. \tag{1.1}$$

The $n \times n$ matrix having $f(x_i \wedge x_j)$ as its ij element is the *meet matrix* of the set S with respect to f and is denoted by $(S)_f$. Similarly, the $n \times n$ matrix having $f(x_i \vee x_j)$ as its ij element is the *join matrix* of the set S with respect to f and is denoted by $[S]_f$. When $(P, \preceq) = (\mathbb{Z}_+, |)$, where $|$ stands for the usual divisor relation of positive integers, the matrices $(S)_f$ and $[S]_f$ are referred to as the GCD and LCM matrices of the set S with respect to f. Another simple but important special case of meet and join matrices are MIN and MAX matrices, which are obtained when $(P, \preceq)$ is a chain. The MIN matrix of size $n \times n$ with $\min(i, j)$ as its ij element has been studied by Bhatia [4], for example, and this matrix can easily be seen as a meet matrix by setting $(P, \preceq) = (\mathbb{Z}_+, \leq)$, $S = \{1, 2, \dots, n\}$ and $f(m) = m$ for all $m \in \mathbb{Z}_+$.

There are several possible ways to further generalize the concept of meet and/or join matrices. One way to do this is to consider two sets instead of one set S (see [2,17]); another is to replace the function f with n functions $f_1, \dots, f_n$ (see [15]). Korkee [14] defines yet another distinct generalization: a combined meet and join matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$. What is special in this generalization is that it yields both meet and join matrices as its special cases, whereas the other generalizations yield only one of the two.

Although the structure, the determinant and the inverse of the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$ were studied by Korkee [14], there are currently no results concerning the eigenvalues of the general form of this matrix. Our main goal with this paper is to improve this situation. The task, however, is not very easy. Already in the case of more specific GCD and LCM matrices accessing the asymptotic behavior of the eigenvalues of these matrices requires some rather complicated methods, see e.g. [5,6,8]. In order to study the eigenvalues of a much more general matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$ we need to use at least as complicated methods at a more abstract level.

When studying a generalization of a matrix class, it is sometimes possible to extend some methods and results to consider the larger class (at least by making suitable assumptions). When Hong and Loewy obtained a lower bound for the smallest eigenvalue of certain GCD matrices (see [7, Theorem 4.2]), soon afterwards Ilmonen et al. [11] generalized this result to meet and join matrices. In this article, we show that, under certain circumstances, this method can be extended for the much more general matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$. The same goes for another method developed by Ilmonen et al., see [11, Theorem 4.1 and Theorem 6.1]. This is done in Sections 3 and 4.

In Section 5 we turn our attention to the special constants c_n originally defined by Hong and Loewy. Currently, no lower bounds are known for this constant for general n, which means that some of the results in [7] and in [11] cannot be applied in practice at all. It turns out that we were able to contribute something to this topic as well, in this article.

2. Preliminaries

Throughout this paper, $(P, \preceq)$ is a locally finite lattice, f is either a real or a complex-valued function on P and $S = \{x_1, x_2, \ldots, x_n\}$ is a finite set of distinct elements of P such that

$$x_i \preceq x_j \quad \Rightarrow \quad i \leq j. \tag{2.1}$$

In Proposition 2.3 and in Theorem 3.1 we also assume that P has $\hat{0}$ as its smallest element, and in Proposition 2.4 and in Theorem 3.2 P is supposed to have the largest element $\hat{1}$. These assumptions may, however, sound more restricting than they in fact are. If P does not have the smallest or the largest element, we may always restrict ourselves to the finite interval

$$\left[\!\left[\bigwedge S, \bigvee S \right]\!\right] = \left\{ z \in P \;\middle|\; \bigwedge S \preceq z \preceq \bigvee S \right\},$$

see e.g. [17, Section 2]. Furthermore, the set S is said to be *meet closed* if $x_i \wedge x_j \in S$ for all $x_i, x_j \in S$, or in other words, if the structure $(S, \preceq)$ is a meet semilattice. Similarly the set S is *join closed* if $x_i \vee x_j \in S$ for all $x_i, x_j \in S$ (i.e. $(S, \preceq)$ is a join semilattice).

Next let us recall the definition of a combined meet and join matrix by Korkee [14]:

Definition 2.1. (See [14], p. 76.) Let $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta} = [m_{ij}] \in \mathbb{C}^{n\times n}$ with

$$m_{ij} = \frac{f(x_i \wedge x_j)^\alpha f(x_i \vee x_j)^\beta}{f(x_i)^\gamma f(x_j)^\delta},$$

where α, β, γ, δ are real numbers such that the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$ exists.

In order for the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$ to exist whenever possible, we need to make the agreement that $0^0 = 1$, but even this does not entirely solve the problem. The following remark provides detailed criteria for the existence of the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$.

Remark 2.1. The matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$ exists if and only if the following conditions are satisfied:

1. If $f(x) = 0$ for some $x \in S$, then $\gamma = \delta = 0$,
2. If $f(x_i \wedge x_j) = 0$ for some $x_i, x_j \in S$, then $\alpha \geq 0$,
3. If $f(x_i \vee x_j) = 0$ for some $x_i, x_j \in S$, then $\beta \geq 0$.

By setting $\alpha = 1$ and $\beta = \gamma = \delta = 0$ we obtain $\boldsymbol{M}_{S,f}^{1,0,0,0} = (S)_f$. On the other hand, if $\beta = 1$ and $\alpha = \gamma = \delta = 0$, then $\boldsymbol{M}_{S,f}^{0,1,0,0} = [S]_f$. Thus the name *combined meet and join matrix* is well justified.

Next we present the two factorization theorems for the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$ given by Korkee [14]. The former makes use of the meet matrix $(S)_f$, whereas the latter uses the join matrix $[S]_f$. Here $\boldsymbol{A} \circ \boldsymbol{B}$ denotes the Hadamard product of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and f^α is simply the usual power of the function f with $f^\alpha(x) = [f(x)]^\alpha$ for all $x \in P$.

Proposition 2.1. *(See [14], Theorem 3.1 (meet-oriented structure theorem).) Let α, β, γ, δ be real numbers such that the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$ exists. Then*

$$\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta} = \boldsymbol{F}^{\beta-\gamma}\big((S)_{f^{\alpha-\beta}} \circ \boldsymbol{G}\big)\boldsymbol{F}^{\beta-\delta},$$

where $\boldsymbol{F} = \operatorname{diag}(f(x_1), f(x_2), \ldots, f(x_n))$ *and*

$$(\boldsymbol{G})_{ij} = \begin{cases} 1 & \text{if } x_i \preceq x_j \text{ or } x_j \preceq x_i, \\ \frac{f^\beta(x_i \wedge x_j) f^\beta(x_i \vee x_j)}{f^\beta(x_i) f^\beta(x_j)} & \text{otherwise.} \end{cases}$$

Proposition 2.2. *(See [14], Theorem 3.2 (join-oriented structure theorem).) Let α, β, γ, δ be such real numbers that the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$ exists. Then*

$$\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta} = \boldsymbol{F}^{\alpha-\gamma}\big([S]_{f^{\beta-\alpha}} \circ \boldsymbol{G}\big)\boldsymbol{F}^{\alpha-\delta},$$

where $\boldsymbol{F} = \operatorname{diag}(f(x_1), f(x_2), \ldots, f(x_n))$ *and*

$$(\boldsymbol{G})_{ij} = \begin{cases} 1 & \text{if } x_i \preceq x_j \text{ or } x_j \preceq x_i, \\ \frac{f^\alpha(x_i \wedge x_j) f^\alpha(x_i \vee x_j)}{f^\alpha(x_i) f^\alpha(x_j)} & \text{otherwise.} \end{cases}$$

After applying the previous two propositions, we also need to be able to factorize the usual meet and join matrices. The following four propositions help us with this. In order to shorten our notations, we introduce two so called *restricted incidence functions* as well as a convolution operation for incidence functions. The function f_d is defined on $\{\hat{0} \times P\}$, f_u on $P \times \{\hat{1}\}$ and

$$f_d(\hat{0}, z) = f(z) = f_u(z, \hat{1})$$

for all $z \in P$. The convolution of incidence functions f and g is the incidence function $f * g$ for which

$$(f * g)(x, y) = \sum_{x \preceq z \preceq y} f(x, z) g(z, y)$$

for all $x, y \in P$. Another thing that we need is the Möbius function μ_P of the poset P. The function μ_P is usually defined as being the inverse of certain incidence function ζ

with respect to the convolution (see [19, p. 296] and [1, p. 141]), but it may be more convenient to calculate its values recursively by using the formula

$$\mu_P(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \prec z \preceq y} \mu_P(z,y) = -\sum_{x \preceq z \prec y} \mu_P(x,z) & \text{if } x \prec y, \\ 0 & \text{otherwise,} \end{cases}$$

see e.g. [1, Proposition 4.6]. This enables us to write briefly by using the convolution $*$ as

$$\sum_{\hat{0} \preceq z \preceq w} f(z)\mu_P(z,w) = (f_d * \mu_P)(w) \quad \text{and} \quad \sum_{w \preceq z \preceq \hat{1}} f(z)\mu_P(w,z) = (\mu_P * f_u)(w).$$

Before going into the factorization theorems we need to deploy two concepts from lattice theory. First, let us assume that $\hat{0}$ is the smallest element of the lattice $(P, \preceq)$. The *order ideal generated by the set* S is the set

$$\{w \in P \mid \hat{0} \preceq w \preceq x_i \text{ for some } x_i \in S\} = \bigcup_{i=1}^{n} [\![\hat{0}, x_i]\!]$$

and it is denoted by ${\downarrow}S$. Similarly, if we assume that $\hat{1}$ is the largest element of the lattice $(P, \preceq)$, we may define the *order filter generated by the set* S as being the set

$$\{w \in P \mid x_i \preceq w \preceq \hat{1} \text{ for some } x_i \in S\} = \bigcup_{i=1}^{n} [\![x_i, \hat{1}]\!],$$

for which we use the notation ${\uparrow}S$.

Proposition 2.3. *(See [12], Lemma 3.2) Let* ${\downarrow}S = \{w_1, w_2, \ldots, w_m\}$ *and* $\boldsymbol{A} = (a_{ij})$ *be the* $n \times m$ *matrix with*

$$a_{ij} = \begin{cases} \sqrt{(f_d * \mu_P)(\hat{0}, w_j)} & \text{if } w_j \preceq x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(S)_f = \boldsymbol{A}\boldsymbol{A}^T$.

Proposition 2.4. *(See [13], Lemma 4.2.) Let* ${\uparrow}S = \{w_1, w_2, \ldots, w_m\}$ *and* $\boldsymbol{A} = (a_{ij})$ *be the* $n \times m$ *matrix with*

$$a_{ij} = \begin{cases} \sqrt{(\mu_P * f_u)(w_j, \hat{1})} & \text{if } x_i \preceq w_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $[S]_f = \boldsymbol{A}\boldsymbol{A}^T$.

Proposition 2.5. *(See [3], Theorem 12.) Let S be a meet closed set and let $\boldsymbol{E}$ and $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the $n \times n$ matrices with*

$$e_{ij} = \begin{cases} 1 & \text{if } x_j \preceq x_i, \\ 0 & \text{otherwise} \end{cases}$$

and

$$d_i = \sum_{\substack{z \preceq x_i \\ z \npreceq x_j \text{ for } j<i}} (f_d * \mu_P)(\hat{0}, z).$$

Then $(S)_f = \boldsymbol{E}\boldsymbol{D}\boldsymbol{E}^T$.

Proposition 2.6. *(See [11], Proposition 2.5.) Let S be a join closed set and let $\boldsymbol{E}$ and $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the $n \times n$ matrices with*

$$e_{ij} = \begin{cases} 1 & \text{if } x_j \preceq x_i, \\ 0 & \text{otherwise} \end{cases}$$

and

$$d_i = \sum_{\substack{x_i \preceq z \\ x_j \npreceq z \text{ for } i<j}} (\mu_P * f_u)(z, \hat{1}).$$

Then $[S]_f = \boldsymbol{E}^T\boldsymbol{D}\boldsymbol{E}$.

Before we can use these factorizations to estimate the eigenvalues of the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$, we also need the following lemma.

Lemma 2.1. *Let $\boldsymbol{A} = [a_{ij}]$, $\boldsymbol{B} = [b_{ij}]$, $\boldsymbol{C} = [c_{ij}]$, $\boldsymbol{D} = [d_{ij}] \in \mathbb{C}^{n\times n}$, where $\boldsymbol{C}$ and $\boldsymbol{D}$ are diagonal matrices. Then*

$$\boldsymbol{C}(\boldsymbol{A} \circ \boldsymbol{B})\boldsymbol{D} = \boldsymbol{B} \circ (\boldsymbol{C}\boldsymbol{A}\boldsymbol{D}).$$

Proof. Since

$$\begin{aligned} \big(\boldsymbol{C}(\boldsymbol{A} \circ \boldsymbol{B})\boldsymbol{D}\big)_{ij} &= \sum_{k=1}^{n} c_{ik}\big((\boldsymbol{A} \circ \boldsymbol{B})\boldsymbol{D}\big)_{kj} = \sum_{k=1}^{n} c_{ik}\left(\sum_{l=1}^{n} (\boldsymbol{A} \circ \boldsymbol{B})_{kl} d_{lj}\right) \\ &= \sum_{k=1}^{n} c_{ik}\left(\sum_{l=1}^{n} a_{kl} b_{kl} d_{lj}\right) = \sum_{k=1}^{n}\sum_{l=1}^{n} c_{ik} a_{kl} b_{kl} \cdot \underbrace{d_{lj}}_{\substack{=0 \\ \text{when } l \neq j}} \end{aligned}$$

$$= \sum_{k=1}^{n} \underbrace{c_{ik}}_{\substack{=0 \\ \text{when } i \neq k}} \cdot a_{kj} b_{kj} d_{jj} = c_{ii} a_{ij} b_{ij} d_{jj}$$

$$= b_{ij} \big((c_{ii} a_{ij}) d_{jj} \big) = b_{ij} \left(\left(\sum_{k=1}^{n} c_{ik} a_{kj} \right) d_{jj} \right) = b_{ij} \big((\boldsymbol{CA})_{ij} d_{jj} \big)$$

$$= b_{ij} \left(\sum_{k=1}^{n} (\boldsymbol{CA})_{ik} d_{kj} \right) = b_{ij} \big((\boldsymbol{CA}) \boldsymbol{D} \big)_{ij} = \big(\boldsymbol{B} \circ (\boldsymbol{CAD}) \big)_{ij},$$

the claim follows. □

In the following two sections we need to assume that our function f is *semimultiplicative*, which means that

$$f(x)f(y) = f(x \wedge y) f(x \vee y)$$

for all $x, y \in P$. We also adopt one constant c_n from Hong and Loewy [7] and another C_n from Ilmonen et al. [11]. Let $K(n)$ denote the set of all $n \times n$ lower triangular 0, 1 matrices with each main diagonal element equal to 1. Now for every positive integer n we define

$$c_n = \min\big\{ \lambda \bigm| \boldsymbol{X} \in K(n) \text{ and } \lambda \text{ is the smallest eigenvalue of } \boldsymbol{X}\boldsymbol{X}^T \big\}$$

and

$$C_n = \max\big\{ \lambda \bigm| \boldsymbol{X} \in K(n) \text{ and } \lambda \text{ is the largest eigenvalue of } \boldsymbol{X}\boldsymbol{X}^T \big\}.$$

Finally, we introduce some old and new notations concerning matrix analysis. We denote that $\boldsymbol{J}$ is the $n \times n$ matrix with all its elements equal to 1 (i.e. $\boldsymbol{J}$ is the identity element under the Hadamard product of complex $n \times n$ matrices). If $\boldsymbol{A}$ and $\boldsymbol{B}$ are real matrices, the notation $\boldsymbol{A} \leqslant \boldsymbol{B}$ is used for the componentwise inequality (that is, $a_{ij} \leq b_{ij}$ for all $i, j = 1, \ldots, n$). In this paper, $|\boldsymbol{A}|$ does not stand for the determinant of $\boldsymbol{A}$, but for the $n \times n$ matrix, with $|a_{ij}|$ as its ij element. The Frobenius and spectral norms of a given matrix $\boldsymbol{A}$ are denoted by $|||\boldsymbol{A}|||_F$ and $|||\boldsymbol{A}|||_S$ respectively. As usual, the spectral radius $\rho(\boldsymbol{A})$ of a matrix $\boldsymbol{A}$ is defined to be the maximum of the absolute values of the eigenvalues of $\boldsymbol{A}$. For the purposes of this paper, it is convenient to deploy similar notation for the smallest absolute value of the eigenvalues of the matrix $\boldsymbol{A}$. We denote

$$\kappa(\boldsymbol{A}) = \min\big\{ |\lambda| \bigm| \lambda \text{ is an eigenvalue of } \boldsymbol{A} \big\}.$$

For example, if $\boldsymbol{A}$ is invertible and Hermitean, then

$$\rho\big(\boldsymbol{A}^{-1}\big) = \big|\big|\big|\boldsymbol{A}^{-1}\big|\big|\big|_S = \frac{1}{\kappa(\boldsymbol{A})}.$$

3. Lower bound for the smallest eigenvalue of a positive definite combined meet and join matrix

Under suitable circumstances the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$ becomes positive definite and it is thus possible to find a real lower bound for its smallest eigenvalue by making use of the structure theorems presented earlier.

Theorem 3.1. *Let α, β, γ, δ be real numbers such that $\gamma = \delta$ and the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}$ exists. Let $f : P \to \mathbb{R} \setminus \{0\}$ be a semimultiplicative function and $\downarrow S = \{w_1, w_2, \ldots, w_m\}$. If $(f_d^{\alpha-\beta} * \mu_P)(\hat{0}, w_i) > 0$ for all $w_i \in \downarrow S$, then*

$$\kappa\big(\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}\big) \geq c_n \cdot \min_{1\leq i\leq n} \big(f_d^{\alpha-\beta} * \mu_P\big)(\hat{0}, x_i) \cdot \min_{1\leq i\leq n} \big[f^2(x_i)\big]^{\beta-\gamma}.$$

Proof. Let $\boldsymbol{A} = (a_{ij})$ be the $n \times m$ matrix with

$$a_{ij} = \begin{cases} \sqrt{(f_d^{\alpha-\beta} * \mu_P)(\hat{0}, w_j)} & \text{if } w_j \preceq x_i, \\ 0 & \text{otherwise,} \end{cases}$$

and $\boldsymbol{F} = \operatorname{diag}(f(x_1), \ldots, f(x_n))$. By Proposition 2.3 we have $(S)_{f^{\alpha-\beta}} = \boldsymbol{A}\boldsymbol{A}^T$. We may assume that $w_i = x_i$ for all $i \in \{1, 2, \ldots, n\}$, since rearranging the order of the elements of the set $\downarrow S$ corresponds to permuting some of the rows and respective columns of $(S)_f$, which does not affect the eigenvalues.

The matrix $\boldsymbol{A}$ can now be divided into blocks

$$\boldsymbol{A} = [\boldsymbol{B} \,|\, \boldsymbol{C}],$$

where $\boldsymbol{B}$ is an $n \times n$ matrix and $\boldsymbol{C}$ is of size $n \times (m-n)$. Since f is a semimultiplicative function, every element of the matrix $\boldsymbol{G}$ defined in Proposition 2.1 is equal to 1. By applying this proposition we obtain

$$\begin{aligned} \boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma} &= \boldsymbol{F}^{\beta-\gamma}\big((S)_{f^{\alpha-\beta}} \circ \boldsymbol{G}\big)\boldsymbol{F}^{\beta-\gamma} = \boldsymbol{F}^{\beta-\gamma}\big((S)_{f^{\alpha-\beta}} \circ \boldsymbol{J}\big)\boldsymbol{F}^{\beta-\gamma} \\ &= \boldsymbol{F}^{\beta-\gamma}(S)_{f^{\alpha-\beta}}\boldsymbol{F}^{\beta-\gamma} = \boldsymbol{F}^{\beta-\gamma}\big(\boldsymbol{A}\boldsymbol{A}^T\big)\boldsymbol{F}^{\beta-\gamma} \\ &= \boldsymbol{F}^{\beta-\gamma}\big([\boldsymbol{B} \,|\, \boldsymbol{C}][\boldsymbol{B} \,|\, \boldsymbol{C}]^T\big)\boldsymbol{F}^{\beta-\gamma} = \boldsymbol{F}^{\beta-\gamma}\left([\boldsymbol{B} \,|\, \boldsymbol{C}]\left[\frac{\boldsymbol{B}^T}{\boldsymbol{C}^T}\right]\right)\boldsymbol{F}^{\beta-\gamma} \\ &= \boldsymbol{F}^{\beta-\gamma}\big(\boldsymbol{B}\boldsymbol{B}^T + \boldsymbol{C}\boldsymbol{C}^T\big)\boldsymbol{F}^{\beta-\gamma} = \boldsymbol{F}^{\beta-\gamma}\boldsymbol{B}\boldsymbol{B}^T\boldsymbol{F}^{\beta-\gamma} + \boldsymbol{F}^{\beta-\gamma}\boldsymbol{C}\boldsymbol{C}^T\boldsymbol{F}^{\beta-\gamma} \\ &= \big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B}\big)\big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B}\big)^T + \big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{C}\big)\big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{C}\big)^T. \end{aligned} \quad (3.1)$$

Here the matrix $(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{C})(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{C})^T$ is clearly positive semidefinite, and thus [10, Corollary 4.3.12] implies that

$$\kappa\big(\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}\big) \geq \kappa\big(\big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B}\big)\big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B}\big)^T\big).$$

Let us then consider the $n \times n$ matrix $\boldsymbol{B} = (b_{ij})$ with

$$b_{ij} = \begin{cases} \sqrt{(f_d^{\alpha-\beta} * \mu_P)(\hat{0}, x_j)} & \text{if } x_j \preceq x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\boldsymbol{E}$ be the matrix defined in Proposition 2.5 and $\boldsymbol{D} = \operatorname{diag}(d_1, \ldots, d_n)$, where

$$d_i = \sqrt{(f_d^{\alpha-\beta} * \mu_P)(\hat{0}, x_i)}.$$

The matrix $\boldsymbol{B}$ can now be written as

$$\boldsymbol{B} = \boldsymbol{E}\boldsymbol{D}.$$

In addition,

$$\begin{aligned} \det(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B}) &= \det(\boldsymbol{F}^{\beta-\gamma}) \det(\boldsymbol{E}) \det(\boldsymbol{D}) \\ &= \prod_{i=1}^{n} [f(x_i)]^{\beta-\gamma} \cdot 1 \cdot \prod_{i=1}^{n} \sqrt{(f_d^{\alpha-\beta} * \mu_P)(\hat{0}, x_i)} \neq 0, \end{aligned}$$

which means that the matrix $\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B}$ is invertible. Therefore the greatest eigenvalue of the matrix

$$[(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})^T]^{-1} = ((\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})^{-1})^T (\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})^{-1}$$

is equal to

$$\rho([(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})^T]^{-1}) = |||[(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})^T]^{-1}|||_S.$$

Thus

$$\begin{aligned} \kappa((\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})^T) &= \frac{1}{\rho([(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})^T]^{-1})} \\ &= \frac{1}{|||[(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})^T]^{-1}|||_S}. \end{aligned}$$

The assumption about the positiveness implies that

$$\begin{aligned} |||(\boldsymbol{D}^2)^{-1}|||_S &= \left|\left|\left| \operatorname{diag}\left(\frac{1}{(f_d^{\alpha-\beta} * \mu_P)(\hat{0}, x_1)}, \ldots, \frac{1}{(f_d * \mu_P)(\hat{0}, x_n)} \right) \right|\right|\right|_S \\ &= \max_{1 \le i \le n} \frac{1}{(f_d^{\alpha-\beta} * \mu_P)(\hat{0}, x_i)} = \frac{1}{\min_{1 \le i \le n} (f_d^{\alpha-\beta} * \mu_P)(\hat{0}, x_i)}. \end{aligned}$$

Similarly,

$$\begin{aligned}\big|\!\big|\!\big|\big(\boldsymbol{F}^{2(\beta-\gamma)}\big)^{-1}\big|\!\big|\!\big|_S &= \left|\!\left|\!\left|\operatorname{diag}\left(\frac{1}{[f(x_1)]^{2(\beta-\gamma)}},\dots,\frac{1}{[f(x_n)]^{2(\beta-\gamma)}}\right)\right|\!\right|\!\right|_S \\ &= \max_{1\le i\le n}\frac{1}{[f(x_i)]^{2(\beta-\gamma)}} = \frac{1}{\min_{1\le i\le n}[f(x_i)]^{2(\beta-\gamma)}}.\end{aligned}$$

Applying the submultiplicativity of the spectral norm yields

$$\begin{aligned}&\big|\!\big|\!\big|\big[\big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B}\big)\big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B}\big)^T\big]^{-1}\big|\!\big|\!\big|_S \\ &\quad= \big|\!\big|\!\big|\big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{E}\boldsymbol{D}\boldsymbol{D}^T\boldsymbol{E}^T\big(\boldsymbol{F}^{\beta-\gamma}\big)^T\big)^{-1}\big|\!\big|\!\big|_S \\ &\quad= \big|\!\big|\!\big|\big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{E}\boldsymbol{D}^2\boldsymbol{E}^T\boldsymbol{F}^{\beta-\gamma}\big)^{-1}\big|\!\big|\!\big|_S \\ &\quad= \big|\!\big|\!\big|\big(\boldsymbol{F}^{\beta-\gamma}\big)^{-1}\big(\boldsymbol{E}^T\big)^{-1}\big(\boldsymbol{D}^2\big)^{-1}\boldsymbol{E}^{-1}\big(\boldsymbol{F}^{\beta-\gamma}\big)^{-1}\big|\!\big|\!\big|_S \\ &\quad\le \big|\!\big|\!\big|\big(\boldsymbol{F}^{\beta-\gamma}\big)^{-1}\big|\!\big|\!\big|_S\cdot\big|\!\big|\!\big|\big(\boldsymbol{E}^T\big)^{-1}\big|\!\big|\!\big|_S\cdot\big|\!\big|\!\big|\big(\boldsymbol{D}^2\big)^{-1}\big|\!\big|\!\big|_S\cdot\big|\!\big|\!\big|\boldsymbol{E}^{-1}\big|\!\big|\!\big|_S\cdot\big|\!\big|\!\big|\big(\boldsymbol{F}^{\beta-\gamma}\big)^{-1}\big|\!\big|\!\big|_S \\ &\quad= \big|\!\big|\!\big|\big(\boldsymbol{D}^2\big)^{-1}\big|\!\big|\!\big|_S\cdot\big(\big|\!\big|\!\big|\big(\boldsymbol{E}^{-1}\big)^T\big|\!\big|\!\big|_S\cdot\big|\!\big|\!\big|\boldsymbol{E}^{-1}\big|\!\big|\!\big|_S\big)\cdot\big|\!\big|\!\big|\big(\boldsymbol{F}^{\beta-\gamma}\big)^{-1}\big|\!\big|\!\big|_S^2 \\ &\quad= \big|\!\big|\!\big|\big(\boldsymbol{D}^2\big)^{-1}\big|\!\big|\!\big|_S\cdot\big|\!\big|\!\big|\big(\boldsymbol{E}^T\big)^{-1}\boldsymbol{E}^{-1}\big|\!\big|\!\big|_S\cdot\big|\!\big|\!\big|\big(\boldsymbol{F}^{2(\beta-\gamma)}\big)^{-1}\big|\!\big|\!\big|_S \\ &\quad= \big|\!\big|\!\big|\big(\boldsymbol{D}^2\big)^{-1}\big|\!\big|\!\big|_S\cdot\big|\!\big|\!\big|\big(\boldsymbol{E}\boldsymbol{E}^T\big)^{-1}\big|\!\big|\!\big|_S\cdot\big|\!\big|\!\big|\big(\boldsymbol{F}^{2(\beta-\gamma)}\big)^{-1}\big|\!\big|\!\big|_S.\end{aligned}$$

Since clearly $\boldsymbol{E}\in K(n)$, we must have $\kappa(\boldsymbol{E}\boldsymbol{E}^T)\ge c_n$. Thus

$$\big|\!\big|\!\big|\big(\boldsymbol{E}\boldsymbol{E}^T\big)^{-1}\big|\!\big|\!\big|_S = \rho\big(\big(\boldsymbol{E}\boldsymbol{E}^T\big)^{-1}\big) = \frac{1}{\kappa(\boldsymbol{E}\boldsymbol{E}^T)} \le \frac{1}{c_n},$$

and further

$$\frac{1}{|\!|\!|(\boldsymbol{E}\boldsymbol{E}^T)^{-1}|\!|\!|_S} \ge c_n.$$

Now combining all these results yields

$$\begin{aligned}\kappa\big(\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}\big) &\ge \kappa\big(\big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B}\big)\big(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B}\big)^T\big) = \frac{1}{|\!|\!|[(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{B})^T]^{-1}|\!|\!|_S} \\ &\ge \frac{1}{|\!|\!|(\boldsymbol{D}^2)^{-1}|\!|\!|_S\cdot|\!|\!|(\boldsymbol{E}\boldsymbol{E}^T)^{-1}|\!|\!|_S\cdot|\!|\!|(\boldsymbol{F}^{2(\beta-\gamma)})^{-1}|\!|\!|_S} \\ &= \frac{1}{|\!|\!|(\boldsymbol{E}\boldsymbol{E}^T)^{-1}|\!|\!|_S}\cdot\frac{1}{|\!|\!|(\boldsymbol{D}^2)^{-1}|\!|\!|_S}\cdot\frac{1}{|\!|\!|(\boldsymbol{F}^{2(\beta-\gamma)})^{-1}|\!|\!|_S} \\ &\ge c_n\cdot\min_{1\le i\le n}\big(f_d^{\alpha-\beta}*\mu_P\big)(\hat{0},x_i)\cdot\min_{1\le i\le n}\big[f(x_i)\big]^{2(\beta-\gamma)}. \qquad\square\end{aligned}$$

Example 3.1. If $\beta = 0$, we do not need to assume the semimultiplicativity of f in Theorem 3.1. Also, in this situation, $(P, \preceq)$ does not necessarily have to be a join semilattice, and neither is the assumption about the largest element $\hat{1}$ necessary. If $\beta = 0$, we have $\boldsymbol{G} = \boldsymbol{J}$ trivially. And further, if $\gamma = 0$, we can also allow f to have zero values and we simply have $\boldsymbol{F}^{\beta-\gamma} = \boldsymbol{F}^0 = \boldsymbol{I}$. Thus Theorem 4.1 in [11] is a corollary of Theorem 3.1 concerning the matrix $\boldsymbol{M}_{S,f}^{1,0,0,0} = (S)_f$.

Example 3.2. (See [16], Theorem 3.1.) Let $(P, \preceq, \hat{0}) = (\mathbb{Z}_+, |, 1)$. Consider the $n \times n$ matrix $\boldsymbol{A}_n^{\alpha,\beta}$ with

$$(i, j)^\alpha [i, j]^\beta$$

as its ij element. Suppose that $\alpha > \beta$. Clearly $\gamma = \delta = 0$, $S = \{1, \ldots, n\} = {\downarrow}S$ and $f = N$, where $N(m) = m$ for all $m \in \mathbb{Z}_+$. The function N is obviously semimultiplicative with nonzero values. In addition, since the set $\{1, \ldots, n\}$ is factor closed, we have

$$\mu_P(\hat{0}, w_i) = \mu(w_i/1) \quad \text{for all } 1 \le w_i \le n,$$

where μ denotes the number-theoretic Möbius function (see [19, Chapter 7]). Thus

$$\left(f_d^{\alpha-\beta} * \mu_P\right)(\hat{0}, w_i) = \left(N^{\alpha-\beta} * \mu\right)(w_i) = J_{\alpha-\beta}(w_i) = w_i^{\alpha-\beta} \prod_{p|w_i} \left(1 - \frac{1}{p^{\alpha-\beta}}\right) > 0,$$

where $J_{\alpha-\beta}$ denotes the generalized Jordan totient function and $*$ is the Dirichlet convolution. Furthermore, $\min_{1 \le i \le n} [f^2(x_i)]^{\beta-\gamma}$ is equal to either 1 or $n^{2\beta}$. Thus by Theorem 3.1 we have

$$\kappa\left(\boldsymbol{A}_n^{\alpha,\beta}\right) \ge c_n \cdot \min_{1 \le i \le n} J_{\alpha-\beta}(i) \cdot \min\left\{1, n^{2\beta}\right\} > 0.$$

The difference between this result and Theorem 3.1 of [16] is that in [16] the constant c_n is replaced with a larger constant t_n, which is obtained by calculating the smallest eigenvalue of the matrix $\boldsymbol{E}\boldsymbol{E}^T$, where $\boldsymbol{E}$ is the incidence matrix of the set $\{1, \ldots, n\}$ with respect to the divisor relation (which is not the matrix that yields the constant c_n).

Since we assume that $(P, \preceq)$ is not only a semilattice but a lattice, it is also possible to approach the eigenvalues of the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}$ via the join matrix $[S]_f$. In this case we just make use of Propositions 2.2 and 2.4 and then proceed as in the proof of Theorem 3.1.

Theorem 3.2. *Let α, β, γ, δ be real numbers such that $\gamma = \delta$ and the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}$ exists. Let $f : P \to \mathbb{R} \setminus \{0\}$ be a semimultiplicative function and ${\uparrow}S = \{w_1, w_2, \ldots, w_m\}$. If $(\mu_P * f_u^{\beta-\alpha})(w_i, \hat{1}) > 0$ for all $w_i \in {\uparrow}S$, then*

$$\kappa\left(\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}\right) \ge c_n \cdot \min_{1 \le i \le n} \left(\mu_P * f_u^{\beta-\alpha}\right)(x_i, \hat{1}) \cdot \min_{1 \le i \le n} \left[f^2(x_i)\right]^{\alpha-\gamma}.$$

Proof. The proof is similar to the proof of Theorem 3.1. □

Example 3.3. Theorem 5.1 in [11] follows directly from Theorem 3.2. In this case $\alpha = 0$, and therefore f does not need to be semimultiplicative, nor does $(P, \preceq)$ need to be a meet semilattice with $\hat{0}$ as the smallest element. If also $\gamma = 0$, then trivially $\boldsymbol{F}^{\alpha-\gamma} = \boldsymbol{I}$ and the image of f does not have to be restricted to nonzero values.

Theorems 3.1 and 3.2 provide two different approaches to the smallest eigenvalue of $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}$. It should be noted that the bounds obtained by using these theorems may differ greatly (provided that both theorems are applicable). For example, if the set $\downarrow S$ is much larger than the set $\uparrow S$, then the elements in the difference matrix $(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{C})(\boldsymbol{F}^{\beta-\gamma}\boldsymbol{C})^T$ in the proof of Theorem 3.1 are likely to be large, which also indicates much poorer lower bound. If the set $\uparrow S$ is large compared to $\downarrow S$, then the bound in Theorem 3.1 is likely to be much better.

4. Eigenvalue bound for the combined meet and join matrix of a meet or join closed set

So far we have been studying the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$ only under the circumstances that it is positive definite. Even if this is not the case, it may still be possible to define regions in the complex plain that contain the eigenvalues. It is then easy to apply these results, for example to a reciprocal matrix with

$$\frac{f(x_i \wedge x_j)}{f(x_i \vee x_j)} \quad \text{or} \quad \frac{f(x_i \vee x_j)}{f(x_i \wedge x_j)}$$

as its ij element. Next we consider the cases when the set S is closed under either operation $\wedge$ or $\vee$. The next theorem is in fact a generalization of Theorem 4.1 in [11].

Theorem 4.1. *Let S be a meet closed set, f be a function $P \to \mathbb{C}$ and α, β, γ, δ be real numbers such that $\gamma = \delta$ and the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}$ exists. If*

$$\left| \frac{f(x_i \wedge x_j) f(x_i \vee x_j)}{f(x_i) f(x_j)} \right|^{\beta} \leq 1 \tag{4.1}$$

for all $i, j \in \{1, 2, \ldots, n\}$, then all the eigenvalues of the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}$ lie in the region

$$\bigcup_{k=1}^{n} \left\{ z \in \mathbb{C} \;\middle|\; \left|z - f(x_k)^{\alpha+\beta-2\gamma}\right| \leq C_n \cdot \max_{1 \leq i \leq n} \left|f(x_i)\right|^{2(\beta-\gamma)} \cdot \max_{1 \leq i \leq n} |d_i| - \left|f(x_k)\right|^{\alpha+\beta-2\gamma} \right\},$$

where

$$d_i = \sum_{\substack{z \preceq x_i \\ z \not\preceq x_j \text{ for } j<i}} \left(f_d^{\alpha-\beta} * \mu_P\right)(\hat{0}, z).$$

Proof. It follows from condition (4.1) that the matrix $\boldsymbol{G} = [g_{ij}]$ defined in Proposition 2.1 satisfies

$$|g_{ij}| = \left|\frac{f(x_i \wedge x_j)f(x_i \vee x_j)}{f(x_i)f(x_j)}\right|^{\beta} \leq 1,$$

which implies that $|\boldsymbol{G}| \leqslant \boldsymbol{J}$. Let $\boldsymbol{E}$ now be the matrix defined in Proposition 2.5, $\boldsymbol{D} = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ and

$$\boldsymbol{\Lambda} = |\boldsymbol{D}|^{\frac{1}{2}} = \operatorname{diag}\bigl(\sqrt{|d_1|}, \sqrt{|d_2|}, \ldots, \sqrt{|d_n|}\bigr),$$

where

$$d_i = \sum_{\substack{z \preceq x_i \\ z \npreceq x_j \text{ for } j<i}} \bigl(f_d^{\alpha-\beta} * \mu_P\bigr)(\hat{0}, z).$$

According to Proposition 2.5, we have $(S)_{f^{\alpha-\beta}} = \boldsymbol{E}\boldsymbol{D}\boldsymbol{E}^T$. By using the above notations, Proposition 2.1 and Lemma 2.1 we obtain

$$\begin{aligned}
\bigl|\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}\bigr| &= \bigl|\boldsymbol{F}^{\beta-\gamma}\bigl((S)_{f^{\alpha-\beta}} \circ \boldsymbol{G}\bigr)\boldsymbol{F}^{\beta-\gamma}\bigr| = \bigl|\bigl(\boldsymbol{F}^{\beta-\gamma}(S)_{f^{\alpha-\beta}}\boldsymbol{F}^{\beta-\gamma}\bigr) \circ \boldsymbol{G}\bigr| \\
&= \bigl|\boldsymbol{F}^{\beta-\gamma}(S)_{f^{\alpha-\beta}}\boldsymbol{F}^{\beta-\gamma}\bigr| \circ |\boldsymbol{G}| \leqslant \bigl|\bigl(\boldsymbol{F}^{\beta-\gamma}(S)_{f^{\alpha-\beta}}\boldsymbol{F}^{\beta-\gamma}\bigr)\bigr| \circ \boldsymbol{J} \\
&= \bigl|\boldsymbol{F}^{\beta-\gamma}(S)_{f^{\alpha-\beta}}\boldsymbol{F}^{\beta-\gamma}\bigr| = \bigl|\boldsymbol{F}^{\beta-\gamma}\bigr|\bigl|(S)_{f^{\alpha-\beta}}\bigr|\bigl|\boldsymbol{F}^{\beta-\gamma}\bigr| \\
&= |\boldsymbol{F}|^{\beta-\gamma}\bigl|\boldsymbol{E}\boldsymbol{D}\boldsymbol{E}^T\bigr||\boldsymbol{F}|^{\beta-\gamma} \leqslant |\boldsymbol{F}|^{\beta-\gamma}\boldsymbol{E}|\boldsymbol{D}|\boldsymbol{E}^T|\boldsymbol{F}|^{\beta-\gamma} \\
&= |\boldsymbol{F}|^{\beta-\gamma}\boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{\Lambda}^T\boldsymbol{E}^T|\boldsymbol{F}|^{\beta-\gamma} = \bigl(|\boldsymbol{F}|^{\beta-\gamma}\boldsymbol{E}\boldsymbol{\Lambda}\bigr)\bigl(|\boldsymbol{F}|^{\beta-\gamma}\boldsymbol{E}\boldsymbol{\Lambda}\bigr)^T.
\end{aligned}$$

By Theorem 8.1.18 in [10] we now have

$$\rho\bigl(|\boldsymbol{F}|^{\beta-\gamma}\bigl|(S)_{f^{\alpha-\beta}}\bigr||\boldsymbol{F}|^{\beta-\gamma}\bigr) \leq \rho\bigl(|\boldsymbol{F}|^{\beta-\gamma}\boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{\Lambda}^T\boldsymbol{E}^T|\boldsymbol{F}|^{\beta-\gamma}\bigr).$$

In addition,

$$\begin{aligned}
\rho\bigl(|\boldsymbol{F}|^{\beta-\gamma}\boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{\Lambda}^T\boldsymbol{E}^T|\boldsymbol{F}|^{\beta-\gamma}\bigr) &= \bigl|\bigl|\bigl||\boldsymbol{F}|^{\beta-\gamma}\boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{\Lambda}^T\boldsymbol{E}^T|\boldsymbol{F}|^{\beta-\gamma}\bigr|\bigr|\bigr|_S \\
&\leq \bigl|\bigl|\bigl||\boldsymbol{F}|^{\beta-\gamma}\bigr|\bigr|\bigr|_S |||\boldsymbol{E}|||_S \bigl|\bigl|\bigl|\boldsymbol{\Lambda}\boldsymbol{\Lambda}^T\bigr|\bigr|\bigr|_S \bigl|\bigl|\bigl|\boldsymbol{E}^T\bigr|\bigr|\bigr|_S \bigl|\bigl|\bigl||\boldsymbol{F}|^{\beta-\gamma}\bigr|\bigr|\bigr|_S \\
&= \bigl|\bigl|\bigl||\boldsymbol{F}|^{2(\beta-\gamma)}\bigr|\bigr|\bigr|_S \bigl|\bigl|\bigl|\boldsymbol{E}\boldsymbol{E}^T\bigr|\bigr|\bigr|_S \bigl|\bigl|\bigl||\boldsymbol{D}|\bigr|\bigr|\bigr|_S \\
&\leq \max_{1\leq i\leq n}\bigl|f(x_i)\bigr|^{2(\beta-\gamma)} \cdot C_n \cdot \max_{1\leq i\leq n}|d_i|. \qquad (4.2)
\end{aligned}$$

Since $(\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma})_{ii} = f(x_i)^{\alpha+\beta-2\gamma}$ and

$$\bigl(|\boldsymbol{F}|^{\beta-\gamma}\bigl|(S)_{f^{\alpha-\beta}}\bigr||\boldsymbol{F}|^{\beta-\gamma}\bigr)_{ii} = \bigl|f(x_i)\bigr|^{\alpha+\beta-2\gamma},$$

by using (4.2) and by setting $\boldsymbol{A} = \boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\delta}$ and $\boldsymbol{B} = |\boldsymbol{F}|^{\beta-\gamma}|(S)_{f^{\alpha-\beta}}||\boldsymbol{F}|^{\beta-\gamma}$ in [10, Theorem 8.2.9] it now follows that all the eigenvalues of the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}$ belong to the above-mentioned region. □

Example 4.1. Theorem 4.1 in [11] is a consequence of Theorem 4.1. We only need to choose $\alpha = 1$ and $\beta = \gamma = \delta = 0$. Condition (4.1) is now trivially satisfied.

Example 4.2. Let S be meet closed. Let us consider the reciprocal matrix with $\frac{f(x_i \vee x_j)}{f(x_i \wedge x_j)}$ as its ij element. Thus in this case $\alpha = -1$, $\beta = 1$ and $\gamma = \delta = 0$. Now if

$$\left|\frac{f(x_i \wedge x_j)f(x_i \vee x_j)}{f(x_i)f(x_j)}\right| \leq 1$$

for all $i, j \in \{1, 2, \ldots, n\}$, then according to Theorem 4.1 all the eigenvalues of the matrix $\boldsymbol{M}_{S,f}^{-1,1,0,0}$ belong to the region

$$\bigcup_{k=1}^{n} \left\{ z \in \mathbb{C} \;\middle|\; |z - 1| \leq C_n \cdot \max_{1 \leq i \leq n} \left|f(x_i)\right|^2 \cdot \max_{1 \leq i \leq n} |d_i| - 1 \right\},$$

where

$$d_i = \sum_{\substack{z \preceq x_i \\ z \npreceq x_j \text{ for } j<i}} \left(f_d^{-2} * \mu_P\right)(\hat{0}, z).$$

Since every set in this union is a disc around 1, the one with the largest radius also contains all the eigenvalues of the matrix $\boldsymbol{M}_{S,f}^{-1,1,0,0}$.

Example 4.3. (See [16], Theorem 3.5.) Let $\boldsymbol{A}_n^{\alpha,\beta}$ be the matrix defined in Example 3.2. By applying Theorem 4.1 to this matrix, it is easy to see that all the eigenvalues of the matrix $\boldsymbol{A}_n^{\alpha,\beta}$ belong to the region

$$\bigcup_{k=1}^{n} \left\{ z \in \mathbb{C} \;\middle|\; \left|z - k^{\alpha+\beta}\right| \leq C_n \cdot \max\left\{1, n^{2\beta}\right\} \cdot \max_{1 \leq i \leq n} \left|J_{\alpha-\beta}(i)\right| - k^{\alpha+\beta} \right\}.$$

Proceeding now as in the proof of Theorem 3.5 in [16] it is possible to show that this union is in fact the real interval $[2\min\{1, n^{\alpha+\beta}\} - H_n, H_n]$, where $H_n = C_n \cdot \max\{1, n^{2\beta}\} \cdot \max_{1 \leq i \leq n} |J_{\alpha-\beta}(i)|$. Also in this case it would be possible to replace the constant C_n with a bit better (i.e. smaller) constant, which can be obtained by using the exact incidence matrix of the set $\{1, 2, \ldots, n\}$.

The next theorem is a result similar to Theorem 4.1, but it is for a join closed set S and is based on Propositions 2.2 and 2.6. The proof is omitted, as it is very similar to the proof of Theorem 4.1.

Theorem 4.2. *Let S be a join closed set, f be a function $P \to \mathbb{C}$ and α, β, γ, δ be real numbers such that $\gamma = \delta$ and the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}$ exists. If*

$$\left|\frac{f(x_i \wedge x_j)f(x_i \vee x_j)}{f(x_i)f(x_j)}\right|^{\alpha} \leq 1 \tag{4.3}$$

for all $i, j \in \{1, 2, \ldots, n\}$, then all the eigenvalues of the matrix $\boldsymbol{M}_{S,f}^{\alpha,\beta,\gamma,\gamma}$ belong to the region

$$\bigcup_{k=1}^{n}\Big\{z \in \mathbb{C} \;\Big|\; \big|z - f(x_k)^{\alpha+\beta-2\gamma}\big| \leq C_n \cdot \max_{1\leq i\leq n}\big|f(x_i)\big|^{2(\alpha-\gamma)} \cdot \max_{1\leq i\leq n}|d_i| - \big|f(x_k)\big|^{\alpha+\beta-2\gamma}\Big\},$$

where

$$d_i = \sum_{\substack{x_i \preceq z \\ x_j \npreceq z \text{ for } i<j}} \big(\mu_P * f_u^{\beta-\alpha}\big)(z, \hat{1}).$$

Example 4.4. Theorem 6.1 in [11] is a consequence of Theorem 4.2 and is obtained by setting $\beta = 1$ and $\alpha = \gamma = \delta = 0$. The condition (4.3) holds trivially.

Example 4.5. Let S be join closed. Consider the reciprocal matrix with $\frac{f(x_i \wedge x_j)}{f(x_i \vee x_j)}$ as its ij element. Now $\alpha = 1$, $\beta = -1$ and $\gamma = \delta = 0$. If also

$$\left|\frac{f(x_i \wedge x_j)f(x_i \vee x_j)}{f(x_i)f(x_j)}\right| \leq 1$$

for all $i, j \in \{1, 2, \ldots, n\}$, then all the eigenvalues of the matrix $\boldsymbol{M}_{S,f}^{1,-1,0,0}$ belong to the region

$$\bigcup_{k=1}^{n}\Big\{z \in \mathbb{C} \;\Big|\; |z - 1| \leq C_n \cdot \max_{1\leq i\leq n}\big|f(x_i)\big|^{2} \cdot \max_{1\leq i\leq n}|d_i| - 1\Big\},$$

where

$$d_i = \sum_{\substack{x_i \preceq z \\ x_j \npreceq z \text{ for } i<j}} \big(\mu_P * f_u^{-2}\big)(z, \hat{1}).$$

Just like in Example 4.2, also in this case we are able to define a disc around 1 that contains all the eigenvalues of $\boldsymbol{M}_{S,f}^{1,-1,0,0}$.

Remark 4.1. If the function f is semimultiplicative, then

$$\left|\frac{f(x_i \wedge x_j)f(x_i \vee x_j)}{f(x_i)f(x_j)}\right|^{\alpha} = \left|\frac{f(x_i \wedge x_j)f(x_i \vee x_j)}{f(x_i)f(x_j)}\right|^{\beta} = 1$$

for all $i, j \in \{1, 2, \dots, n\}$. Thus a semimultiplicative function automatically satisfies conditions (4.1) and (4.3).

We conclude this section by considering some classical examples.

Example 4.6. Wintner [21] and subsequently also Lindqvist and Seip [18] studied the $n \times n$ matrix with

$$\left(\frac{\gcd(i,j)}{\operatorname{lcm}(i,j)}\right)^{\alpha}$$

as its ij element ($\alpha \in \mathbb{R}$). Here we have $S = \{1, 2, \dots, n\}$ and $(P, \preceq)$ may be taken to be $(\mathbb{Z}_+, |)$. The set S is clearly meet closed. Further we have $\beta = -\alpha$, $\gamma = \delta = 0$ and $f = N$, which is trivially semimultiplicative. Thus condition (4.1) is satisfied and, by Theorem 4.1, all the eigenvalues of the matrix $\boldsymbol{M}_{S,f}^{\alpha,-\alpha,0,0}$ belong to the region

$$\bigcup_{k=1}^{n}\left\{z \in \mathbb{C} \;\middle|\; |z-1| \le C_n \cdot \max_{1\le i\le n} i^{-2\alpha} \cdot \max_{1\le i\le n} |d_i| - 1\right\},$$

where

$$d_i = \sum_{\substack{z \mid i \\ z \nmid j \text{ for } j<i}} \left(N^{2\alpha} * \mu\right)(z),$$

μ is the number-theoretic Möbius function and $*$ is the Dirichlet convolution. Since the only number z that satisfies $z \mid i$ and $z \nmid j$ when $j < i$ is the number i itself, d_i simplifies into

$$d_i = \left(N^{2\alpha} * \mu\right)(i) = J_{2\alpha}(i),$$

where $J_{2\alpha}$ is the generalized Jordan totient function. If $\alpha > 0$, we even have $J_{2\alpha}(i) > 0$ for all $i = 1, \dots, n$. As it was with the reciprocal matrices, also in this case this region is in fact a 1-centered disc. But since $\boldsymbol{M}_{S,f}^{\alpha,-\alpha,0,0}$ is real and symmetric, all the eigenvalues are real. Therefore the disc may be constricted into a real interval with 1 as its midpoint. Thus the eigenvalues of $\boldsymbol{M}_{S,f}^{\alpha,-\alpha,0,0}$ all belong to the interval

$$\left\{z \in \mathbb{R} \;\middle|\; |z-1| \le C_n \cdot \max_{1\le i\le n} i^{-2\alpha} \cdot \max_{1\le i\le n} J_{2\alpha}(i) - 1\right\}.$$

In the special case when $\alpha = \frac{1}{2}$ we have

$$N^{2\alpha} * \mu = N * \mu = \phi,$$

where ϕ is the Euler totient function. In this case the elements of $\boldsymbol{D}$ become

$$d_i = \phi(i) > 0.$$

Since for all $i \geq 2$ we have $\phi(i) \leq i-1$, it follows that $\max_{1\leq i\leq n} \phi(i) \leq n-1$. In addition, $\max_{1\leq i\leq n} i^{-1} = 1$, and this maximum is obtained when $i = 1$. Thus the eigenvalues of the matrix $\boldsymbol{M}_{S,f}^{\frac{1}{2},-\frac{1}{2},0,0}$ belong to the interval

$$\{z \in \mathbb{R} \mid |z-1| \leq C_n \cdot (n-1) - 1\} = [2 - C_n \cdot (n-1), C_n \cdot (n-1)].$$

5. Estimating the constant c_n

The constant c_n was originally defined by Hong and Loewy [7], but they did not give any approximations for it. Ilmonen et al. [11, Section 7] easily found a relatively good upper bound

$$T_n = \sqrt{(2n-1) + (2n-3) \cdot 4 + (2n-5) \cdot 9 + \cdots + 3 \cdot (n-1)^2 + n^2} \qquad (5.1)$$

for their other constant C_n, but they did not manage to prove anything about the constant c_n. Instead they ended up presenting the following conjecture.

Conjecture 5.1. *Let $\boldsymbol{Y}_0 = [(\boldsymbol{Y}_0)_{ij}]$, where*

$$(\boldsymbol{Y}_0)_{ij} = \begin{cases} 0 & \text{if } j > i, \\ 1 & \text{if } j = i, \\ 0 & \text{if } i > j \text{ and } i+j \text{ is even,} \\ 1 & \text{if } i > j \text{ and } i+j \text{ is odd.} \end{cases}$$

Then $c_n = \kappa(\boldsymbol{Y}_0 \boldsymbol{Y}_0^T)$.

Calculations have shown that this conjecture is true for $n = 2, 3, \ldots, 7$, but generally this problem is still open and appears to be quite hard to solve. However, the next theorem shows that it is possible to obtain a lower bound for c_n. Unfortunately this lower bound is far from accurate and thus for the most part is only of some theoretical interest.

Theorem 5.1. *The constant c_n is bounded below by $(\frac{6}{n^4+2n^3+2n^2+n})^{\frac{n-1}{2}}$.*

Proof. Let $\boldsymbol{X}_0 \in K(n)$ be the triangular 0, 1 matrix with $c_n = \kappa(\boldsymbol{X}_0 \boldsymbol{X}_0^T)$ and $\boldsymbol{M}_0 = \boldsymbol{X}_0 \boldsymbol{X}_0^T$. Let

$$g(\lambda) = \det(\boldsymbol{M}_0 - \lambda \boldsymbol{I}_n) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 \in \mathbb{Z}[\lambda]$$

be the characteristic polynomial of the matrix $\boldsymbol{M}_0$. Now

$$\begin{aligned} g(0) = a_0 = \det(\boldsymbol{M}_0) &= \det\big(\boldsymbol{X}_0\boldsymbol{X}_0^T\big) \\ &= \det(\boldsymbol{X}_0)\det\big(\boldsymbol{X}_0^T\big) = 1^n \cdot 1^n = 1, \end{aligned}$$

since all the diagonal elements of $\boldsymbol{X}_0$ are equal to 1. Since $\boldsymbol{M}_0$ is clearly positive definite, let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}_+$ be the eigenvalues of $\boldsymbol{M}_0$, where

$$0 < c_n = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le C_n.$$

Thus $g(\lambda)$ may be written as

$$g(\lambda) = (-1)^n \cdot (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n),$$

from which we obtain

$$1 = a_0 = \underbrace{\lambda_1}_{=c_n}\underbrace{\lambda_2}_{\le C_n}\cdots\underbrace{\lambda_n}_{\le C_n} \le c_n(C_n)^{n-1} \le c_n T_n^{n-1},$$

where T_n is the upper bound for C_n found in [11] and presented in (5.1). By dividing this last inequality by $(T_n)^{n-1} > 0$ we obtain $(\frac{1}{T_n})^{n-1} \le c_n$. The claim now follows by observing that

$$T_n = \sqrt{\frac{1}{6}n(n+1)\big(n^2+n+1\big)} = \sqrt{\frac{1}{6}\big(n^4+2n^3+2n^2+n\big)}$$

(this can easily be proven by induction, but we omit this for the sake of brevity). □

If Conjecture 5.1 holds, then we are able to slightly improve the lower bound presented in Theorem 5.1. We only need to calculate

$$\boldsymbol{Y}_0\boldsymbol{Y}_0^T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & & \vdots & \vdots \\ 1 & 0 & 1 & & & \\ \vdots & \vdots & \vdots & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & \cdots & \\ 0 & 1 & 1 & 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & 1 & 0 & 1 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ddots \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & \ddots \\ 0 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & \ddots \\ 1 & 1 & 1 & 3 & 1 & 2 & 1 & 2 & \ddots \\ 0 & 1 & 1 & 1 & 3 & 1 & 2 & 1 & \ddots \\ 1 & 1 & 1 & 2 & 1 & 4 & 1 & 3 & \ddots \\ 0 & 1 & 1 & 1 & 2 & 1 & 4 & 1 & \ddots \\ 1 & 1 & 1 & 2 & 1 & 3 & 1 & 5 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} = \boldsymbol{N}_0,$$

where the last row and column vectors are equal to

$$\begin{bmatrix} 1 & \underline{1} & 1 & \underline{2} & 1 & \underline{3} & \cdots & \underline{\frac{n}{2} - 2} & 1 & \underline{\frac{n}{2} - 1} & 1 & \frac{n}{2} + 1 \end{bmatrix}$$

when n is even, and equal to

$$\begin{bmatrix} \underline{0} & 1 & \underline{1} & 1 & \underline{2} & 1 & \underline{3} & \cdots & \underline{\frac{n-1}{2} - 2} & 1 & \underline{\frac{n-1}{2} - 1} & 1 & \frac{n+1}{2} \end{bmatrix}$$

when n is odd. Clearly

$$\rho\big(\boldsymbol{Y}_0 \boldsymbol{Y}_0^T\big) = \rho(\boldsymbol{N}_0) \le |||\boldsymbol{N}_0|||_F,$$

where $|||\boldsymbol{N}_0|||_F$ is the Frobenius norm of the matrix $\boldsymbol{N}_0$. It is now a cumbersome although an elementary task to show that

$$|||\boldsymbol{N}_0|||_F = \begin{cases} \sqrt{\frac{1}{48}(n^4 + 56n^2 + 48n)} & \text{if } n \text{ is even,} \\ \sqrt{\frac{1}{48}(n^4 + 50n^2 + 48n - 51)} & \text{if } n \text{ is odd.} \end{cases}$$

Then by replacing C_n with $\rho(\boldsymbol{N}_0)$ and T_n with $|||\boldsymbol{M}_0|||_F$ in the proof of Theorem 5.1 we are able to prove the following result:

Theorem 5.2. *If Conjecture 5.1 holds, then* $(\frac{48}{n^4+56n^2+48n})^{\frac{n-1}{2}}$ *is a lower bound for* c_n *when* n *is even, and* $(\frac{48}{n^4+50n^2+48n-51})^{\frac{n-1}{2}}$ *is a lower bound for* c_n *when* n *is odd.*

The following Table 1 shows the behavior of c_n and its lower bounds for $1 \le n \le 7$.

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Table 1
Some values of the constant c_n and its lower bounds.

n	Lower bound by Theorem 5.1	Lower bound by Theorem 5.2	Approximate value for c_n
1	1	1	1
2	0.377964	0.377964	0.381966
3	0.0384615	0.0769231	0.198062
4	0.00170747	0.00674936	0.0870031
5	$4.16233 \cdot 10^{-5}$	$5.40833 \cdot 10^{-4}$	0.0370683
6	$6.36185 \cdot 10^{-7}$	$2.05280 \cdot 10^{-5}$	0.0148276
7	$6.64148 \cdot 10^{-9}$	$8.16298 \cdot 10^{-7}$	0.00581700

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On positive definiteness and eigenvalues of meet and join matrices

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On the positive definiteness and eigenvalues of meet and join matrices

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ABSTRACT

In this paper we study the positive definiteness of meet and join matrices using a novel approach. When the set S_n is meet closed, we give a necessary and sufficient condition for the positive definiteness of the matrix $(f(S_n))$. From this condition we obtain some sufficient conditions for positive definiteness as corollaries. We also use graph theory and show that by making some graph theoretic assumptions on the set S_n we are able to reduce the assumptions on the function f while still preserving the positive definiteness of the matrix $(f(S_n))$. Dual theorems of these results for join matrices are also presented. As examples we consider the so-called power GCD and reciprocal power LCM matrices as well as MIN and MAX matrices. Finally we give bounds for the eigenvalues of meet and join matrices in cases when the function f possesses certain monotonic behaviour.

1. Introduction

The research of GCD and LCM matrices was initiated by H.J.S. Smith [41] in 1875 when he studied the determinant of the $n \times n$ matrix in which the ij element is the greatest common divisor (i, j) of i and j. He also considered the $n \times n$ matrix with the least common multiple $[i, j]$ of i and j as its ij element. During the next century the determinants of GCD-type matrices were a topic of interest for many number theorists and linear algebraists (see the references in [20]; the two articles [35,43] by Lindström and Wilf are especially relevant). In 1989 Beslin and Ligh [14] initiated a new wave of more intense research of GCD matrices, which soon led to poset-theoretic generalizations of GCD matrices. Rajarama Bhat [40] gave the definition of meet matrix, and Haukkanen [16] was the first to study these matrices systematically. Join matrices were defined later by Korkee and Haukkanen [33]. Since that, meet and join type matrices on posets have been studied in many papers, see e.g. [28,36,37].

Over the years many authors have considered the positive definiteness of GCD, LCM, meet and join matrices. In 1989 Beslin and Ligh [9] showed that the GCD matrix (S) of the set $S = \{x_1, \ldots, x_n\}$, in which the ij element is (x_i, x_j), is positive definite. Four years later Bourque and Ligh [13] proved that if f is an arithmetical function such that

$$d \mid x_i \quad \text{for some } x_i \in S \Rightarrow (f * \mu)(d) > 0,$$

then the matrix $(f(S))$ with $f((x_i, x_j))$ as its ij element is positive definite. In [14] Bourque and Ligh reported results concerning the positive definiteness of matrices associated with generalized Ramanujan's sums and in [15] they gave conditions under which the matrix $(f[S]) := (f[x_i, x_j])$ is positive definite. The LCM matrix $[S] := ([x_i, x_j])$ was also studied and it turned out that it is never positive definite (unless $n = 1$), see [12, p. 68] (in some cases the matrix $[S]$ is even singular,

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see [23] by Hong and [20] by Haukkanen et al.). In 2001 Korkee and Haukkanen [32] extended results by Hong [22] and gave a sufficient condition for positive definiteness of meet matrices, and in [33] they presented a similar condition for join matrices. A couple of years later Altinisik et al. [4] obtained a necessary and sufficient condition for positive definiteness of a matrix closely related to meet matrices. At the same time Ovall [39] went back to GCD and LCM matrices and showed that GCD and certain reciprocal matrices are positive definite, whereas some reciprocal matrices and nontrivial LCM matrices are indefinite. In 2006 Bhatia [10] showed once again that the usual GCD matrix is infinitely divisible and therefore positive semidefinite. Later Bhatia [11] also studied certain MIN matrices and presented six proofs for their positive definiteness (it should be noted that MIN matrices can easily be seen as special cases of meet matrices).

There are also some results for the eigenvalues of GCD-type matrices to be found in the literature. Wintner [44] published results concerning the largest eigenvalue of the $n \times n$ matrix having

$$\left(\frac{(i,j)}{[i,j]}\right)^{\alpha}$$

as its ij entry and subsequently Lindqvist and Seip [34] investigated the asymptotic behaviour of the smallest and largest eigenvalue of the same matrix. More recently Hilberdink [21] and also Berkes and Weber [8] addressed this same topic from an analytical perspective. The first paper concerning the eigenvalues of GCD matrices was by Balatoni [7] as he considered the eigenvalues of the classical Smith's GCD matrix.

One way to obtain information about the eigenvalues of GCD-type matrices is to study the norms of these matrices. The O estimates of the norms have been studied in many papers, see [2,5,17–19]. Hong and Loewy [26,27] studied the asymptotic behaviour of the eigenvalues of the matrix $(f(S_n))$ and Hong [24] gives lower bound for its eigenvalues in a case when

$$d \mid x_i \quad \text{for some } x_i \in S \Rightarrow (f * \mu)(d) > 0$$

as well as continues the research on the asymptotic behaviour of the eigenvalues. Altinisik [3] provides information about the eigenvalues of GCD matrices, a paper by Hong and Lee [25] addresses the eigenvalues of reciprocal power LCM matrices and there is also one paper about the eigenvalues of meet and join matrices by Ilmonen et al. [30].

In this paper we provide new information about the positive definiteness and the eigenvalues of meet and join matrices. The notations and most of the concepts are defined in Section 2. Section 3 contains some new characterizations and key examples of positive definite meet and join matrices. In Section 4 we make use of graph theory and study the positive definiteness of meet and join matrices from this new graph theoretic perspective. In Section 5 we provide upper bounds for all the eigenvalues of meet and join matrices in which the function f evinces certain monotonic behaviour.

2. Preliminaries

Throughout this paper $(P, \preceq)$ is an infinite but locally finite lattice, $f : P \to \mathbb{R}$ is a real-valued function on P and $(x_n)_{n=1}^{\infty}$ is an infinite sequence of distinct elements of P such that

$$x_i \preceq x_j \Rightarrow i \leq j. \tag{2.1}$$

For every $n \in \mathbb{Z}_+$, let $S_n = \{x_1, x_2, \ldots, x_n\}$. The set S_n is said to be *meet closed* if $x_i \wedge x_j \in S_n$ for all $x_i, x_j \in S_n$, in other words, the structure $(S_n, \preceq)$ is a meet semilattice. The concept of *join closed set* is defined dually.

Let $(f(S_n)) = (f(x_i \wedge x_j))$ and $(f[S_n]) = (f(x_i \vee x_j))$ denote the $n \times n$ matrices having $f(x_i \wedge x_j)$ and $f(x_i \vee x_j)$ as their (i, j)-entries, respectively. The matrices $(f(S_n))$ and $(f[S_n])$ are referred to as the *meet* and *join matrices* of the set S_n with respect to f, respectively. When $(P, \preceq) = (\mathbb{Z}_+, |)$ we have $(f(S_n)) = (f(x_i \wedge x_j)) = (f(x_i, x_j))$ and $(f[S_n]) = (f(x_i \vee x_j)) = (f[x_i, x_j])$.

Let $D_n = \{d_1, d_2, \ldots, d_{m_n}\}$ be any finite subset of P containing all the elements $x_i \wedge x_j$, where $x_i, x_j \in S_n$, and having its elements arranged so that

$$d_i \preceq d_j \Rightarrow i \leq j.$$

Next we define the function $\Psi_{D_n,f}$ on D_n inductively as

$$\Psi_{D_n,f}(d_k) = f(d_k) - \sum_{d_v \prec d_k} \Psi_{D_n,f}(d_v), \tag{2.2}$$

or equivalently

$$f(d_k) = \sum_{d_v \preceq d_k} \Psi_{D_n,f}(d_v). \tag{2.3}$$

Thus we have

$$\Psi_{D_n,f}(d_k) = \sum_{d_v \preceq d_k} f(d_v)\mu_{D_n}(d_v, d_k), \tag{2.4}$$

where μ_{D_n} is the Möbius function of the poset $(D_n, \preceq)$, see [1, Section IV.1] and [42, Proposition 3.7.1].

Let E_{D_n} denote the $n \times m_n$ matrix defined as

$$(e_{D_n})_{ij} = \begin{cases} 1 & \text{if } d_j \preceq x_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

The matrix E_{D_n} may be referred to as the incidence matrix of the set D_n with respect to the set S_n and the partial ordering $\preceq$.

The set D_n, the function $\Psi_{D_n,f}$ and the matrix E_{D_n} are needed when considering the matrix $(f(S_n))$. Next we define the dual concepts which we use in the study of the matrix $(f[S_n])$.

Let $B_n = \{b_1, b_2, \ldots, b_{l_n}\}$ be any finite subset of P containing all the elements $x_i \vee x_j$ with $x_i, x_j \in S_n$ and having its elements arranged so that

$$b_i \preceq b_j \Rightarrow i \leq j.$$

We define the function $\Phi_{B_n,f}$ on B_n inductively as

$$\Phi_{B_n,f}(b_k) = f(b_k) - \sum_{b_k \prec b_v} \Phi_{B_n,f}(b_v), \tag{2.6}$$

or equivalently

$$f(b_k) = \sum_{b_k \preceq b_v} \Phi_{B_n,f}(b_v). \tag{2.7}$$

Thus we have

$$\Phi_{B_n,f}(b_k) = \sum_{b_k \preceq b_v} f(b_v)\mu_{B_n}(b_k, b_v), \tag{2.8}$$

where μ_{B_n} is the Möbius function of the poset $(B_n, \preceq)$.

Let E_{B_n} denote the $n \times l_n$ matrix defined as

$$(e_{B_n})_{ij} = \begin{cases} 1 & \text{if } b_j \succeq x_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

We refer to the matrix E_{B_n} as the incidence matrix of the set B_n with respect to the set S_n and the partial ordering $\succeq$.

Remark 2.1. If we are only interested in the positive definiteness and eigenvalues of meet and join matrices, then the condition (2.1) is, in fact, not necessary but can still be made without restricting generality. If S_n does not satisfy the condition (2.1) and S_n' is a set obtained from S_n by rearranging its elements so that (2.1) holds, then there exists a permutation matrix P such that

$$(f(S_n')) = P(f(S_n))P^T = P(f(S_n))P^{-1}.$$

Thus the matrices $(f(S_n'))$ and $(f(S_n))$ are similar and therefore have the same eigenvalues, positive definiteness properties, etc.

It is well known (see, for example [6,38]) that adopting the above notations the matrices $(f(S_n))$ and $(f[S_n])$ can be factored as

$$(f(S_n)) = E_{D_n}\Lambda_{D_n,f}E_{D_n}^T \quad \text{and} \quad (f[S_n]) = E_{B_n}\Delta_{B_n,f}E_{B_n}^T, \tag{2.10}$$

where

$$\Lambda_{D_n,f} = \operatorname{diag}(\Psi_{D_n,f}(d_1), \Psi_{D_n,f}(d_2), \ldots, \Psi_{D_n,f}(d_{m_n}))$$

and

$$\Delta_{B_n,f} = \operatorname{diag}(\Phi_{B_n,f}(b_1), \Phi_{B_n,f}(b_2), \ldots, \Phi_{B_n,f}(b_{l_n})).$$

By using the first factorization in a case when the set S_n is meet closed, it is easy to show (see [6, Theorem 4.2]) that

$$\det(f(S_n)) = \Psi_{S_n,f}(x_1)\Psi_{S_n,f}(x_2)\cdots\Psi_{S_n,f}(x_n). \tag{2.11}$$

Similarly, when the set S_n is join closed we have

$$\det(f[S_n]) = \Phi_{S_n,f}(x_1)\Phi_{S_n,f}(x_2)\cdots\Phi_{S_n,f}(x_n) \tag{2.12}$$

(see [38, Theorem 4.2]). In the next section these determinant formulas appear also to be useful when considering the positive definiteness of meet and join matrices.

3. On the positive definiteness of meet and join matrices

We begin our study by considering the positive definiteness of the matrix $(f(S_n))$ in the case when the set S_n is meet closed. Under these circumstances we are able to give necessary and sufficient conditions for positive definiteness of the matrix $(f(S_n))$. Theorem 3.1 is also closely related to Theorem 5.1 in [4].

Theorem 3.1. *If the set S_n is meet closed, then the matrix $(f(S_n))$ is positive definite if and only if $\Psi_{S_n,f}(x_i) > 0$ for all $i = 1, 2, \ldots, n$.*

Proof. Since removing a maximal element does not affect the meet closeness of the set S_i, it follows that all the sets $S_n, S_{n-1}, \ldots, S_2, S_1$ are meet closed. In addition, the determinants of the matrices $(f(S_i))$, where $i = 1, 2, \ldots, n$, are the leading principal minors of the matrix $(f(S_n))$. By (2.11) we have

$$\begin{aligned}
&\det(f(S_1)) = \Psi_{S_n,f}(x_1)\\
&\det(f(S_2)) = \Psi_{S_n,f}(x_1)\Psi_{S_n,f}(x_2)\\
&\vdots\\
&\det(f(S_{n-1})) = \Psi_{S_n,f}(x_1)\Psi_{S_n,f}(x_2)\cdots\Psi_{S_n,f}(x_{n-1})\\
&\det(f(S_n)) = \Psi_{S_n,f}(x_1)\Psi_{S_n,f}(x_2)\cdots\Psi_{S_n,f}(x_{n-1})\Psi_{S_n,f}(x_n).
\end{aligned}$$

Now $(f(S_n))$ is positive definite if and only if $\det(f(S_i)) > 0$ for all $i = 1, 2, \ldots, n$ (see [29, Theorem 7.2.5]), and the determinants above are all positive if and only if $\Psi_{S_n,f}(x_i) > 0$ for all $i = 1, 2, \ldots, n$. □

Next we present a dual theorem for join matrices.

Theorem 3.2. *If the set S_n is join closed, then the matrix $(f[S_n])$ is positive definite if and only if $\Phi_{S_n,f}(x_i) > 0$ for all $i = 1, 2, \ldots, n$.*

Proof. Let us denote

$$S'_1 = \{x_n\}, \qquad S'_2 = \{x_{n-1}, x_n\}, \ldots, S'_{n-1} = \{x_2, \ldots, x_{n-1}, x_n\}.$$

Since the determinants of the matrices

$$(f[S'_1]),\ (f[S'_2]), \ldots, (f[S'_{n-1}]) \text{ and } (f[S_n])$$

constitute a nested sequence of n principal minors of $(f[S_n])$, the matrix $(f[S_n])$ is positive definite if and only if all of these matrices have positive determinants (again, see [29, Theorem 7.2.5]). Since all these sets are join closed, the determinants can be calculated by using (2.12). The rest of the proof is similar to the proof of Theorem 3.1. □

Example 3.1. Let S_n be a chain. Thus $x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n$. Clearly, the set S_n is both meet and join closed (the matrices $(f(S_n))$ and $(f[S_n])$ may be referred to as the MIN and MAX matrices of the chain S_n respectively). In this case we have $\Psi_{S_n,f}(x_1) = f(x_1)$ and

$$\Psi_{S_n,f}(x_i) = \sum_{x_k \preceq x_i} f(x_k)\mu_{S_n}(x_k, x_i) = f(x_i) - f(x_{i-1})$$

for all $i = 2, \ldots, n$. Now it follows from Theorem 3.1 that the matrix $(f(S_n))$ is positive definite if and only if $f(x_1) > 0$ and $f(x_i) > f(x_{i-1})$ for all $i = 2, \ldots, n$. In other words, we must have

$$0 < f(x_1) < f(x_2) < \cdots < f(x_{n-1}) < f(x_n).$$

If we set $(P, \preceq) = (\mathbb{Z}^+, \leq)$, $f(k) = k$ for all $k \in \mathbb{Z}$ and $S_n = \{1, 2, \ldots, n\}$, we obtain the MIN matrix studied recently by Bhatia [11]. Among other things, he presents six distinct proofs for the positive definiteness of this matrix. The one in this example is yet another different proof.

Similarly, by using Theorem 3.2 it is possible to show that the matrix $(f[S_n])$ is positive definite if and only if

$$0 < f(x_n) < f(x_{n-1}) < \cdots < f(x_2) < f(x_1).$$

Next we focus on the case when the set S_n is neither meet nor join closed. It turns out that by using Theorems 3.1 and 3.2 it is possible to say something about the positive definiteness of the matrices $(f(S_n))$ and $(f[S_n])$ also under these circumstances. Corollary 3.1 may be seen as a generalization of Theorem 1 (i) in [14].

Corollary 3.1. *Let D_n be any finite meet closed subset of P containing all the elements of S_n. If $\Psi_{D_n,f}(d_i) > 0$ for all $d_i \in D_n$, then the matrix $(f(S_n))$ is positive definite.*

Proof. By Theorem 3.1 the matrix $(f(D_n))$ is positive definite. Thus the matrix $(f(S_n))$ is also positive definite since it is a principal submatrix of the matrix $(f(D_n))$. □

Example 3.2. Let $(P, \preceq) = (\mathbb{Z}^+, |)$, $\alpha \in \mathbb{R}$ and $f(n) = n^\alpha$ for all $n \in \mathbb{Z}^+$. Under these assumptions the meet and join matrices become the so-called power GCD and LCM matrices, which have been studied extensively by Hong et al. [25,26]. It is well known that the matrix $(f(S_n))$ is positive definite if $\alpha > 0$ (see [13, Example 1] and [14, Example 3]). Here we give another proof for this by using the previous corollary.

Let

$$D_n = \downarrow S_n = \{d \in \mathbb{Z}^+ \mid d \mid x_i \text{ for some } x_i \in S_n\}.$$

Let $*$ denote the Dirichlet convolution and μ denote the number-theoretic Möbius function. Now for every $d_k \in D_n$ we have

$$\Psi_{D_n,f}(d_k) = \sum_{d_v | d_k} d_v^\alpha \mu\left(\frac{d_k}{d_v}\right) = (f * \mu)(d_k) = J_\alpha(d_k) = d_k^\alpha \prod_{p|d_k}\left(1 - \frac{1}{p^\alpha}\right),$$

where J_α is a generalization of the Jordan totient function. If $\alpha > 0$, then clearly $J_\alpha(d_k) > 0$ for all $d_k \in D_n$ and therefore by Corollary 3.1 the matrix $(f(S_n))$ is positive definite.

Next we present a similar corollary that concerns the matrix $(f[S_n])$. The proof is essentially the same as the proof of Corollary 3.1.

Corollary 3.2. *Let B_n be any finite join closed subset of P containing all the elements of S_n. If $\Phi_{B_n,f}(b_i) > 0$ for all $b_i \in B_n$, then the matrix $(f[S_n])$ is positive definite.*

Example 3.3. Let $(P, \preceq) = (\mathbb{Z}^+, |)$ as in Example 3.2, $\alpha \in \mathbb{R}^+$ and $f(n) = \frac{1}{n^\alpha}$ for all $n \in \mathbb{Z}^+$. Hong and Lee [25, Theorem 2.1] have shown that the matrix $(f[S_n])$ is positive definite. Here we present a different proof for this fact by using Corollary 3.2.

Let $\alpha > 0$, let $\downarrow \mathrm{lcm}\, S_n$ denote the set of divisors of $\mathrm{lcm}\, S_n$ and let $\uparrow S_n$ stand for the set $\{k \in \mathbb{Z}^+ \mid x_i \mid k \text{ for some } i = 1, \ldots, n\}$. Now let

$$B_n = \uparrow S_n \cap \downarrow \mathrm{lcm}\, S_n = \{d \in \mathbb{Z}^+ \mid x_i \mid d \text{ for some } x_i \in S_n \text{ and } d \mid \mathrm{lcm}\, S_n\}.$$

Then for every $b_k \in B_n$ we have

$$\begin{aligned}\Phi_{B_n,f}(b_k) &= \sum_{b_k | b_v | \mathrm{lcm} S_n} \frac{1}{b_v^\alpha}\, \mu\left(\frac{b_v}{b_k}\right) \\ &= \left(\frac{1}{\mathrm{lcm} S_n}\right)^\alpha \sum_{b_k | b_v | \mathrm{lcm} S_n} \left(\frac{\mathrm{lcm} S_n}{b_v}\right)^\alpha \mu\left(\frac{b_v}{b_k}\right) \\ &= \left(\frac{1}{\mathrm{lcm} S_n}\right)^\alpha \sum_{a | \frac{\mathrm{lcm} S_n}{b_k}} \left(\frac{(\mathrm{lcm} S_n)/b_k}{a}\right)^\alpha \mu(a) \\ &= \left(\frac{1}{\mathrm{lcm} S_n}\right)^\alpha J_\alpha\left(\frac{\mathrm{lcm} S_n}{b_k}\right) > 0.\end{aligned}$$

Thus by Corollary 3.2 the matrix $(f[S_n])$ is positive definite.

As seen in the above examples, there are two obvious ways to choose the sets D_n and B_n. The first is to take D_n (resp. B_n) to be the meet (resp. join) subsemilattice of P generated by the set S_n. The other is to take $D_n = \downarrow S_n$ and $B_n = \uparrow S_n$ in the case when the sets $\downarrow S_n$ and $\uparrow S_n$ are finite, and otherwise take $D_n = \downarrow S_n \cap \uparrow(\wedge S_n)$ and $B_n = \uparrow S_n \cap \downarrow(\vee S_n)$ (here $\vee S_n = x_1 \vee \cdots \vee x_n$ and $\wedge S_n = x_1 \wedge \cdots \wedge x_n$). Benefits of both choices are explained in [38].

Although Corollaries 3.1 and 3.2 can be used in many cases, their conditions are not necessary for the positive definiteness of the matrices $(f(S_n))$ and $(f[S_n])$ and thus they are not always applicable. The following example illustrates this.

Example 3.4. Let $(P, \preceq) = (\mathbb{Z}^+, |)$, $S_3 = \{6, 10, 15\}$ and

$$\begin{cases} f(1) = 0 \\ f(2) = -1 \\ f(3) = 3 \\ f(5) = -2 \\ f(6) = 5 \\ f(10) = 2 \\ f(15) = 3. \end{cases}$$

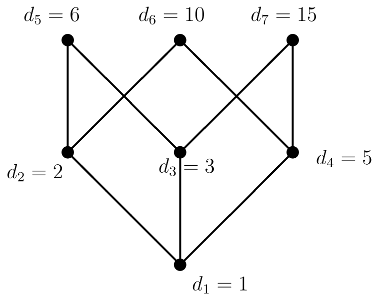

Fig. 1. The lattice $(D_3, \preceq)$ and the choices of the elements of D_3.

Then we have

$$(f(S_3)) = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & -2 \\ 3 & -2 & 3 \end{bmatrix},$$

and this matrix can easily be shown to be positive definite. However, if we choose the elements of D_3 as in Fig. 1, direct calculations show that $\Psi_{D_3,f}(d_2) = -1 < 0$ and $\Psi_{D_3,f}(d_4) = -2 < 0$. Thus the meet matrix $(f(S_3))$ is positive definite although some of the values of $\Psi_{D_3,f}$ are negative.

4. Trees, *A*-sets and positive definiteness

Next we turn our attention to the special case where the Hasse diagram of the set $\text{meetcl}(S_n)$ is a tree (when it is considered as an undirected graph). Here $\text{meetcl}(S_n)$ (resp. $\text{joincl}(S_n)$) denotes the meet subsemilattice (resp. the join subsemilattice) of P generated by the set S_n. Like in Example 3.1, also in this case a certain monotonicity property of f guarantees the positive definiteness of $(f(S_n))$ (resp. $(f[S_n])$). First we present the definitions of these properties.

Definition 4.1. The set $S_n \subseteq P$ is said to be a $\wedge$-*tree set* if the Hasse diagram of $\text{meetcl}(S_n)$ is a tree. Analogously, S_n is a $\vee$-*tree set* if the Hasse diagram of $\text{joincl}(S)$ is a tree.

There are also a couple of other characterizations for $\wedge$-tree sets and $\vee$-tree sets. We present these only for $\wedge$-tree sets, since the characterizations for $\vee$-tree sets are dual to these.

Lemma 4.1. *The following statements are equivalent:*
1. *S_n is a $\wedge$-tree set.*
2. *Every element in* $\text{meetcl}(S_n)$ *covers at most one element of* $\text{meetcl}(S_n)$.
3. *For every $x \in \text{meetcl}(S_n)$ the set*
$$(\downarrow x) \cap \text{meetcl}(S_n) = \{y \in \text{meetcl}(S_n) \mid y \preceq x\}$$
is a chain.
4. *For all $x, y, z \in \text{meetcl}(S_n)$ we have*
$$(x \preceq z \text{ and } y \preceq z) \Rightarrow (x \preceq y \text{ or } y \preceq x).$$

Proof. The proof is simple and straightforward. □

Next we define the monotonicity property for f that we mentioned earlier.

Definition 4.2. The function $f : P \to \mathbb{R}$ is *strictly order-preserving* if

$$x \prec y \Rightarrow f(x) < f(y). \tag{4.1}$$

Analogously, f is *strictly order-reversing* if

$$x \prec y \Rightarrow f(y) < f(x). \tag{4.2}$$

The function f is said to be *order-preserving* (resp. *order-reversing*) if equality is allowed on the right side of (4.1) (resp. (4.2)).

Remark 4.1. After adopting the terminology in Definition 4.2 we are able to revisit Example 3.1 and express its results in the following form: if the set S_n is a chain, then
1. $(f(S_n))$ is positive definite
$\Leftrightarrow$ f is strictly order-preserving in S_n with positive values,
2. $(f[S_n])$ is positive definite
$\Leftrightarrow$ f is strictly order-reversing in S_n with positive values.

The following theorem presents a condition for positive definiteness of $(f(S_n))$ (resp. $(f[S_n])$) in the case when the values of f are positive and f is order-preserving (resp. order-reversing).

Theorem 4.1. *Let $f(x) > 0$ for all $x \in P$. Then the following statements hold:*

1. *If S_n is a $\wedge$-tree set and f is strictly order-preserving, then $(f(S_n))$ is positive definite.*
2. *If S_n is a $\vee$-tree set and f is strictly order-reversing, then $(f[S_n])$ is positive definite.*

Proof. We prove only the first part since the proof of the second part is dual to it. Let $D_n = \text{meetcl}(S_n)$. We apply Corollary 3.1 and show that $\Psi_{D_n,f}(d_k) > 0$ for all $d_k \in D_n$. If $k = 1$, then $d_k = \min D_n$ and we have $\Psi_{D_n,f}(d_k) = f(d_k) > 0$ by assumption. Now let $k > 1$. By Lemma 4.1 d_k covers exactly one element d_w in meetcl(S_n) and by the order-preserving property we have $f(d_w) < f(d_k)$. Formula (2.3) yields

$$f(d_w) = \sum_{d_v \preceq d_w} \Psi_{D_n,f}(d_v) \quad \text{and} \quad f(d_k) = \sum_{d_v \preceq d_k} \Psi_{D_n,f}(d_v),$$

and by subtracting we obtain

$$0 < f(d_k) - f(d_w) = \sum_{d_v \preceq d_k} \Psi_{D_n,f}(d_v) - \sum_{d_v \preceq d_w} \Psi_{D_n,f}(d_v) = \Psi_{D_n,f}(d_k),$$

which completes the proof. □

As seen in Remark 4.1, sometimes it is not only sufficient but also necessary for the positive definiteness of the matrix $(f(S_n))$ that the function f is strictly order-preserving. The next theorem is an example of this. A similar statement can be made regarding join matrices.

Theorem 4.2. *If S_n is meet closed $\wedge$-tree set and the matrix $(f(S_n))$ is positive definite, then the function f is strictly order-preserving in S_n and $f(x_i) > 0$ for all $x_i \in S_n$.*

Proof. In this case the set S_n is clearly both meet closed and $\wedge$-tree set. We begin the proof by showing that if x_j covers x_i, then $f(x_i) < f(x_j)$. By Theorem 3.1 $\Psi_{S_n,f}(x_j) > 0$, and from Eq. (2.3) we obtain

$$f(x_j) = \sum_{x_k \preceq x_j} \Psi_{S_n,f}(x_k) \quad \text{and} \quad f(x_i) = \sum_{x_k \preceq x_i} \Psi_{S_n,f}(x_k).$$

Subtracting the second from the first yields

$$f(x_j) - f(x_i) = \sum_{x_k \preceq x_j} \Psi_{S_n,f}(x_k) - \sum_{x_k \preceq x_i} \Psi_{S_n,f}(x_k) = \Psi_{S_n,f}(x_j) > 0,$$

from which we obtain $f(x_i) < f(x_j)$. Then suppose that $x_i \prec x_j$ but x_j does not cover x_i for some $x_i, x_j \in S_n$. Since $(P, \preceq)$ and in particular $(S_n, \preceq)$ is locally finite, there is only a finite number of elements of S_n in the interval $[x_i, x_j]$. In fact, by item 3 in Lemma 4.1, the elements of the set $S_n \cap [x_i, x_j]$ are always comparable, and therefore the elements of this set form a chain

$$x_i \prec x_{k_1} \prec x_{k_2} \prec \cdots \prec x_{k_r} \prec x_j$$

in which the previous element is always covered by the next. This implies that

$$f(x_i) < f(x_{k_1}) < f(x_{k_2}) < \cdots < f(x_{k_r}) < f(x_j),$$

and thus we have proven the order-preservation of f in S_n in general. The second claim now follows easily. We may assume that $x_1 = \min S_n = \wedge S_n$. By Theorem 3.1 $f(x_1) = \Psi_{S_n,f}(x_1) > 0$. Further, since $x_1 \preceq x_i$ for all $x_i \in S_n$ and f is strictly order-preserving, $f(x_i) > 0$ for all $x_i \in S_n$. □

In [31] Korkee studies the meet and join matrices of an A-set, which he defines as follows.

Definition 4.3. The set S_n is an A-set if the set $A = \{x_i \wedge x_j \mid i \neq j\}$ is a chain.

Korkee derives formulas for the structure, determinant and inverse of the matrix $(f(S_n))$ in a case when S_n in an A-set. He also does the same for the matrix $(f[S_n])$ in a case when the dual of S_n is an A-set. He does not, however, consider the positive definiteness of these matrices.

It turns out that Theorem 4.1 can be applied directly to show the positive definiteness of the matrix $(f(S_n))$ when the set S_n is an A-set and the positive definiteness of the matrix $(f[S_n])$ when the dual of the set S_n is an A-set. This follows from the next theorem.

Theorem 4.3. *Every A-set S_n is also a $\wedge$-tree set and every dual of an A-set is a $\vee$-tree set.*

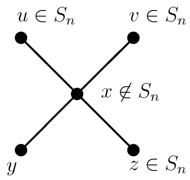

Fig. 2. Illustration of the proof of Theorem 4.3.

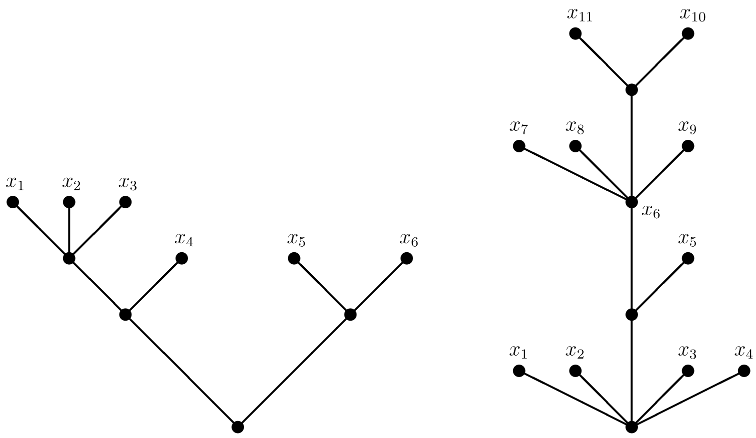

Fig. 3. The Hasse diagram on the left is an example of a set S_6 that is a $\wedge$-tree set but not an A-set. The semilattice on the right is an example of a nontrivial finite A-set S_{11}.

Proof. Again we prove only the first part of the claim, since the second part follows from it trivially. Assume that S_n is an A-set. First we need to show that meetcl$(S_n) = S_n \cup A$, where A is the set defined above. In order to do this, we only need to check that $S_n \cup A$ is meet closed. Let $x, y \in S_n \cup A$. We may assume that $x \in S_n$ and $y \in A$, since the other cases are trivial. Now $y = u \wedge v$ for some $u, v \in S_n$, and we obtain

$$x \wedge y = (x \wedge x) \wedge (u \wedge v) = \underbrace{(x \wedge u)}_{\in A} \wedge \underbrace{(x \wedge v)}_{\in A} \in A,$$

since A is a chain. Thus the first part of the proof is complete.

Next we prove that every element $x \in$ meetcl(S_n) covers at most one element of meetcl(S_n). We now suppose for a contradiction that x covers both y and z for some $x, y, z \in$ meetcl(S_n). Since y and z are incomparable, we must have $y \notin A$ or $z \notin A$ (since A is a chain). We may assume that $z \notin A$, from which it follows that $z \in S_n$. Now we must have $x \notin S_n$, since otherwise we would have $x \wedge z = z \in A$. Thus $x \in A$ and there exist elements $u, v \in S_n$ such that $x = u \wedge v$. Now, as we can see from Fig. 2, we have $v \wedge z = z \in A$, which is a contradiction. The claim now follows from Lemma 4.1. □

It is easy to see that the converse of Theorem 4.3 is not true. Fig. 3 exemplifies this. It also illustrates the structure of a typical A-set. The Hasse diagram of an A-set is always a tree whether the set S is finite or not.

5. Eigenvalue estimations

In this section we present bounds for the eigenvalues of certain meet and join matrices. In order to do this, we first need to present the following two lemmas. We here assume that f is strictly order-preserving or order-reversing and also that f is either *increasing* or *decreasing in the set* S_n with respect to the indices i of the elements x_i, i.e. $i \leq j \Rightarrow f(x_i) \leq f(x_j)$ or $i \leq j \Rightarrow f(x_i) \geq f(x_j)$. It should be noted that if $f : P \to \mathbb{R}$ is either order-preserving or order-reversing, then it is always possible to rearrange the elements of the set S_n so that f becomes increasing or decreasing with respect to the indices. And as stated in Remark 2.1, this does not affect the eigenvalues. For example, if f is order-preserving, we may list the images of the elements of the set S_n in ascending order as

$$f(x_{j_1}) \leq f(x_{j_2}) \leq \cdots \leq f(x_{j_n}),$$

and then define $x'_i = x_{j_i}$ for all $i = 1, 2, \ldots, n$. This even satisfies (2.1), since by order-preserving property we have

$$x'_i \preceq x'_j \Rightarrow f(x'_i) \leq f(x'_j) \Rightarrow i \leq j.$$

Therefore if the function f is order-preserving, assuming $i \le j \Rightarrow f(x_i) \le f(x_j)$ causes no additional restrictions in the study of eigenvalues of meet matrices.

Lemma 5.2 and Theorem 5.2 are generalizations of Hong's and Lee's results (see [25, Theorem 2.3]).

Lemma 5.1. *Let $f : P \to \mathbb{R}$ be a function with nonnegative values, and let W_k denote the k-dimensional subspace of the complex vector space $\mathbb{C}^n$ consisting of vectors that have zero entries in the coordinates at $k+1, k+2, \dots, n$ (i.e. $W_k = \text{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_k\}$). Let $\boldsymbol{y} = [y_1, \dots, y_n]^T$ be any vector in W_k (that is, $y_{k+1} = \cdots = y_n = 0$). If f is order-preserving in* meetcl(S_n), *then we have*

$$\boldsymbol{y}^*(f(S_n))\boldsymbol{y} \le k\boldsymbol{y}^*\boldsymbol{y}f(x_k), \tag{5.1}$$

where $\boldsymbol{y}^*$ is the complex conjugate transpose of $\boldsymbol{y}$.

Proof. We apply induction on k. In the case when $k = 1$ it is rather trivial that

$$\boldsymbol{y}^*(f(S_n))\boldsymbol{y} = \overline{y_1}y_1 f(x_1) = \boldsymbol{y}^*\boldsymbol{y}f(x_1),$$

where $\overline{y_1}$ denotes the complex conjugate of y_1. Our induction hypothesis is that the claim holds for all k with $1 \le k < n$, and next we show that the claim also holds for $k+1$. Let C_i denote the ith column of the matrix $(f(S_n))$, and let $\boldsymbol{y} \in W_{k+1}$. First we observe that

$$\boldsymbol{y}^*(f(S_n))\boldsymbol{y} = \boldsymbol{y}^*C_1 y_1 + \cdots + \boldsymbol{y}^*C_k y_k + \boldsymbol{y}^*C_{k+1}y_{k+1}.$$

Now let $\boldsymbol{z} \in W_k$ such that $z_i = y_i$ for all $i \ne k+1$ and $z_{k+1} = 0$. Thus the quadratic form $\boldsymbol{z}^*(f(S_n))\boldsymbol{z}$ is contained in the previous expression and it can be written as

$$\boldsymbol{y}^*(f(S_n))\boldsymbol{y} = \boldsymbol{y}^*C_{k+1}y_{k+1} + \overline{y_{k+1}}f(x_{k+1} \wedge x_1)y_1 + \cdots + \overline{y_{k+1}}f(x_{k+1} \wedge x_k)y_k + \boldsymbol{z}^*(f(S_n))\boldsymbol{z}. \tag{5.2}$$

Next we start to analyse these terms individually. First of all, the order preserving property of f yields that $0 \le f(x_{k+1} \wedge x_j) \le f(x_{k+1})$ for all $j = 1, \dots, k$. By also applying the triangle inequality and the simple fact that $|ab| \le \frac{1}{2}(|a|^2 + |b|^2)$ for all $a, b \in \mathbb{C}$ we obtain

$$\begin{aligned}
\left|\boldsymbol{y}^*C_{k+1}y_{k+1}\right| &= |y_{k+1}|\ |\overline{y_1}f(x_{k+1} \wedge x_1) + \cdots + \overline{y_k}f(x_{k+1} \wedge x_k) + \overline{y_{k+1}}f(x_{k+1})| \\
&\le |y_{k+1}|\,(|\overline{y_1}|f(x_{k+1} \wedge x_1) + \cdots + |\overline{y_k}|f(x_{k+1} \wedge x_k) + |\overline{y_{k+1}}|f(x_{k+1})) \\
&\le |y_{k+1}|\,(|\overline{y_1}| + \cdots + |\overline{y_k}| + |\overline{y_{k+1}}|)f(x_{k+1}) \\
&\le \left(|y_{k+1}|^2 + \frac{1}{2}\sum_{i=1}^{k}\left(|y_{k+1}|^2 + |y_i|^2\right)\right)f(x_{k+1}) \\
&= \frac{f(x_{k+1})}{2}\left((k+1)\,|y_{k+1}|^2 + \boldsymbol{y}^*\boldsymbol{y}\right).
\end{aligned} \tag{5.3}$$

Very similarly

$$\begin{aligned}
|\overline{y_{k+1}}f(x_{k+1} \wedge x_1)y_1 + \cdots + \overline{y_{k+1}}f(x_{k+1} \wedge x_k)y_k| &\le f(x_{k+1})\,|y_{k+1}|\,(|y_1| + \cdots + |y_k|) \\
&\le \frac{f(x_{k+1})}{2}\sum_{i=1}^{k}\left(|y_{k+1}|^2 + |y_i|^2\right) \\
&= \frac{f(x_{k+1})}{2}\left((k-1)\,|y_{k+1}|^2 + \boldsymbol{y}^*\boldsymbol{y}\right).
\end{aligned} \tag{5.4}$$

Finally, our induction hypothesis and the increase of f in the set S_n with respect to the indices i yields

$$\boldsymbol{z}^*(f(S_n))\boldsymbol{z} \le k\boldsymbol{z}^*\boldsymbol{z}f(x_k) \le k\boldsymbol{z}^*\boldsymbol{z}f(x_{k+1}). \tag{5.5}$$

Now, by combining (5.3)–(5.5) we obtain

$$\begin{aligned}
\left|\boldsymbol{y}^*(f(S_n))\boldsymbol{y}\right| &\le \frac{f(x_{k+1})}{2}\left((k+1)\,|y_{k+1}|^2 + \boldsymbol{y}^*\boldsymbol{y}\right) + \frac{f(x_{k+1})}{2}\left((k-1)\,|y_{k+1}|^2 + \boldsymbol{y}^*\boldsymbol{y}\right) + k\boldsymbol{z}^*\boldsymbol{z}f(x_{k+1}) \\
&= f(x_{k+1})\left(\boldsymbol{y}^*\boldsymbol{y} + \underbrace{k|y_{k+1}|^2 + k\boldsymbol{z}^*\boldsymbol{z}}_{=k\boldsymbol{y}^*\boldsymbol{y}}\right) = (k+1)f(x_{k+1})\boldsymbol{y}^*\boldsymbol{y}.
\end{aligned}$$

This completes the proof. □

Lemma 5.2. *Let $f : P \to \mathbb{R}$ be a function with nonnegative values, and let V_k denote the k-dimensional subspace of the complex vector space $\mathbb{C}^n$ consisting of vectors that have zero entries in the coordinates at $1, 2, \ldots, n-k$ (i.e. $V_k = \text{span}\{\boldsymbol{e}_{n-k+1}, \boldsymbol{e}_{n-k+2}, \ldots, \boldsymbol{e}_n\}$). Let $\boldsymbol{y} = [y_1, \ldots, y_n]^T$ be any vector in V_k (that is, $y_1 = \cdots = y_{n-k} = 0$). If f is order-reversing in* joincl(S_n)*, then*

$$\boldsymbol{y}^*(f[S_n])\boldsymbol{y} \le k\boldsymbol{y}^*\boldsymbol{y}f(x_{n-k+1}). \tag{5.6}$$

Proof. The proof is very similar to the proof of Lemma 5.1 and is essentially the same as Hong's and Lee's proof in [25, Theorem 2.3]. □

By applying the Courant–Fischer theorem together with Lemmas 5.1 and 5.2 we are now able to give bounds for the eigenvalues of the matrices $(f(S_n))$ and $(f[S_n])$.

Theorem 5.1. *Let $\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_n^{(n)}$, where $\lambda_1^{(n)} \le \lambda_2^{(n)} \le \cdots \le \lambda_n^{(n)}$, denote the eigenvalues of the matrix $(f(S_n))$. Under the assumptions of Lemma 5.1 we have*

$$\lambda_k^{(n)} \le kf(x_k) \tag{5.7}$$

for all $k = 1, \ldots, n$. Moreover, $f(x_n) \le \lambda_n^{(n)}$.

Proof. Let $1 \le k \le n$. By applying Lemma 5.1 and the Courant–Fischer theorem [29, Theorem 4.2.11] we obtain

$$\begin{aligned} kf(x_k) &\ge \max_{\boldsymbol{0}\ne\boldsymbol{y}\in W_k} \frac{\boldsymbol{y}^*(f(S_n))\boldsymbol{y}}{\boldsymbol{y}^*\boldsymbol{y}} = \max_{\boldsymbol{0}\ne\boldsymbol{y}\perp\boldsymbol{e}_{k+1},\ldots,\boldsymbol{e}_n} \frac{\boldsymbol{y}^*(f(S_n))\boldsymbol{y}}{\boldsymbol{y}^*\boldsymbol{y}} \\ &\ge \min_{\boldsymbol{w}_1,\boldsymbol{w}_2,\ldots,\boldsymbol{w}_{n-k}\in\mathbb{C}^n} \left(\max_{\boldsymbol{0}\ne\boldsymbol{y}\perp\boldsymbol{w}_1,\boldsymbol{w}_2,\ldots,\boldsymbol{w}_{n-k}} \frac{\boldsymbol{y}^*(f(S_n))\boldsymbol{y}}{\boldsymbol{y}^*\boldsymbol{y}} \right) = \lambda_k^{(n)}. \end{aligned}$$

The rest of the claim follows from the Rayleigh–Ritz theorem [29, Theorem 4.2.2] by setting $\boldsymbol{y} = \boldsymbol{e}_n$, since

$$\lambda_n^{(n)} = \max_{\boldsymbol{y}\ne\boldsymbol{0}} \frac{\boldsymbol{y}^*(f(S_n))\boldsymbol{y}}{\boldsymbol{y}^*\boldsymbol{y}} \ge \frac{\boldsymbol{e}_n^*(f(S_n))\boldsymbol{e}_n}{\boldsymbol{e}_n^*\boldsymbol{e}_n} = f(x_n). \quad \square$$

Theorem 5.2. *Let $\hat{\lambda}_1^{(n)}, \hat{\lambda}_2^{(n)}, \ldots, \hat{\lambda}_n^{(n)}$, where $\hat{\lambda}_1^{(n)} \le \hat{\lambda}_2^{(n)} \le \cdots \le \hat{\lambda}_n^{(n)}$, denote the eigenvalues of the matrix $(f[S_n])$. Under the assumptions of Lemma 5.2 we have*

$$\hat{\lambda}_k^{(n)} \le kf(x_{n-k+1}) \tag{5.8}$$

for all $k = 1, \ldots, n$. In addition, $f(x_1) \le \hat{\lambda}_n^{(n)}$.

Proof. The proof is similar to the proof of Theorem 5.1. □

Example 5.1. Let $\alpha \in \mathbb{R}^+$, $S_n = \{x_1, \ldots, x_n\} \subset \mathbb{Z}^+$ and $f : \mathbb{Z}^+ \to \mathbb{R}$ be the function such that $f(n) = n^\alpha$ for all $n \in \mathbb{Z}^+$. As earlier, let $(f(S_n))$ be the power GCD matrix of the set S_n with $(x_i, x_j)^\alpha$ as its ij element. In addition, let $(f((S_n)^{**}))$ denote the power GCUD matrix having $((x_i, x_j)^{**})^\alpha$, the power of the greatest common unitary divisor of x_i and x_j as its ij element (d divides x_i unitarily if $d \mid x_i$ and $(d, x_i/d) = 1$). Both these matrices fulfil the assumptions of Lemma 5.1, and therefore by Theorem 5.1 $kf(x_k) = kx_k^\alpha$ is an upper bound for the kth largest eigenvalue of both $(f(S_n))$ and $(f((S_n)^{**}))$. Moreover, $f(x_n) = x_n^\alpha$ is a lower bound for the largest eigenvalue of both $(f(S_n))$ and $(f((S_n)^{**}))$.

Example 5.2 (*[25, Theorem 2.3]*)**.** Let $\alpha \in \mathbb{R}^+$, $S_n = \{x_1, \ldots, x_n\} \subset \mathbb{Z}^+$ and $f : \mathbb{Z}^+ \to \mathbb{R}$ be the function such that $f(n) = \frac{1}{n^\alpha}$ for all $n \in \mathbb{Z}^+$. In this case the matrix $(f[S_n])$ having $\frac{1}{[x_i,x_j]^\alpha}$ as its ij element is referred to as the reciprocal power LCM matrix of the set S_n. Let $\lambda_k^{(n)}$ denote the kth largest eigenvalue of the matrix $(f[S_n])$. Thus by Theorem 5.2 we have

$$\lambda_k^{(n)} \le kf(x_{n-k+1}) = \frac{k}{x_{n-k+1}^\alpha}.$$

In addition, $f(x_1) = \frac{1}{x_1^\alpha} \le \lambda_n^{(n)}$.

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A lattice-theoretic approach to the Bourque-Ligh conjecture

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Abstract The Bourque-Ligh conjecture states that if $S = \{x_1, x_2, \ldots, x_n\}$ is a gcd-closed set of positive integers with distinct elements, then the LCM matrix $[S] = [\mathrm{lcm}(x_i, x_j)]$ is invertible. It is well known that this conjecture holds for $n \leq 7$ but does not generally hold for $n \geq 8$. In this paper we provide a lattice-theoretic explanation for this solution of the Bourque-Ligh conjecture. In fact, let $(P, \leq) = (P, \wedge, \vee)$ be a lattice, let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P and let $f : P \to \mathbb{C}$ be a function. We study under which conditions the join matrix $[S]_f = [f(x_i \vee x_j)]$ on S with respect to f is invertible on a meet closed set S (i.e., $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$).

Key words and phrases: Meet matrix, Join matrix, Semimultiplicativity, GCD matrix, LCM matrix *AMS Subject Classification:* 11C20, 15A36

1 Introduction

Let $(P, \leq) = (P, \wedge, \vee)$ be a lattice, let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P and let $f : P \to \mathbb{C}$ be a function. The meet matrix $(S)_f$ and the join matrix $[S]_f$ on S with respect to f are defined by $((S)_f)_{ij} = f(x_i \wedge x_j)$ and $([S]_f)_{ij} = f(x_i \vee x_j)$. Rajarama Bhat [23] and Haukkanen [6] introduced meet matrices and Korkee and Haukkanen [17] defined join matrices. Explicit formulae for the determinant and the inverse of meet and join matrices are presented in [6, 16, 17, 23] (see also [2, 14]). Most of these formulae are presented on meet closed sets S (i.e., $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$) and join-closed sets S (i.e., $x_i, x_j \in S \Rightarrow x_i \vee x_j \in S$). More recently Korkee and Haukkanen [18] presented a method for calculating $\det(S)_f$, $(S)_f^{-1}$, $\det[S]_f$ and $[S]_f^{-1}$ on all sets S and functions f. It is well known that $(\mathbb{Z}_+, |) = (\mathbb{Z}_+, \gcd, \mathrm{lcm})$ is a lattice, where $|$ is the usual divisibility relation and gcd and lcm stand for the greatest common divisor and the least common multiple of integers. Thus meet and join matrices are generalizations of GCD matrices $((S)_f)_{ij} = f(\gcd(x_i, x_j))$ and LCM matrices $([S]_f)_{ij} = f(\mathrm{lcm}(x_i, x_j))$, where f is an arithmetical function. If $f = N$, where $N(m) = m$ for all positive integers m, then we denote $(S)_f = (S)$ and $[S]_f = [S]$. The study of GCD and LCM matrices is considered to have begun in 1876, when Smith [27] presented his famous determinant formulae. The GCUD and LCUM matrices, which are unitary analogues of GCD and LCM matrices, are also special cases of meet and join matrices, see [7, 8, 15]. For general accounts of meet and join matrices and their number-theoretic special cases, see [9, 17, 25].

Bourque and Ligh [5] conjectured that the LCM matrix $[S]$ on any gcd-closed set is invertible. Haukkanen, Wang and Sillanpää [9] were the first to show that the conjecture does not hold (giving a counterexample with $n = 9$). Hong [11] solved the conjecture completely in the sense that it holds for $n \leq 7$ and does not hold generally for $n \geq 8$. Subsequently he also presented some conjectures on his own [12, 13, 19].

In this paper we study a lattice-theoretic generalization of the Bourque-Ligh conjecture, i.e., under which conditions the join matrix $[S]_f$ is invertible on a meet closed set S. We use the concept of covering to develop an inductive method for inserting an element to S so that the invertibility of the join matrix on the extended set is preserved. We apply this method to explain in terms of lattice theory why $n = 7$ is the greatest integer for which the original Bourque-Ligh conjecture holds.

2 Preliminaries

Let $(P, \leq)$ be a locally finite poset and let g be an incidence function of P, that is, g is a complex-valued function on $P \times P$ such that $g(x, y) = 0$ whenever $x \not\leq y$. If h is also an incidence function of P, the sum $g + h$ is defined by $(g+h)(x, y) = g(x, y) + h(x, y)$ and the convolution $g * h$ is defined by $(g * h)(x, y) = \sum_{x \leq z \leq y} g(x, z)h(z, y)$. The set of all incidence functions of P under addition and convolution forms a ring with unity, where the unity δ is defined by $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise. The zeta incidence function ζ is defined by $\zeta(x, y) = 1$ if $x \leq y$, and $\zeta(x, y) = 0$ otherwise. The Möbius function μ of P is the inverse of ζ (with respect to the convolution).

In this paper let $(P, \leq) = (P, \wedge, \vee)$ always be a lattice such that the principal order ideal $\downarrow x = \{y \in P \mid y \leq x\}$ is finite for each $x \in P$. Then P has the least element, which we denote by 0. The order ideal generated by S is $\downarrow S = \{z \in P \mid \exists x \in S : z \leq x\}$, see [4]. Let f always be a complex-valued function on P and let S be a finite subset of P, where $S = \{x_1, x_2, \ldots, x_n\}$ with $x_i < x_j \Rightarrow i < j$. We say that S is an a-set if $x_i \wedge x_j = a$ for all $i \neq j$. We say that S is lower-closed if $(x_i \in S, y \in P, y \leq x_i) \Rightarrow y \in S$. We say that S is meet closed if $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$. It is clear that a lower-closed set is always meet closed but the converse does not hold.

Definition 2.1. *We say that f is a semimultiplicative function on P if*

$$f(x)f(y) = f(x \wedge y)f(x \vee y) \tag{2.1}$$

for all $x, y \in P$.

The concept of a semimultiplicative function on P is a generalization of the concept of a semimultiplicative arithmetical function, see [24, p. 49] or [26, p. 237]. Let $f(x) \neq 0$ for all $x \in P$. Then the function $\frac{1}{f}$ on P is defined by $\left(\frac{1}{f}\right)(x) = 1/f(x)$. If g is an incidence function of P, the incidence function $\frac{1}{g}$ of P is defined similarly. One can easily show that f is semimultiplicative if and only if $\frac{1}{f}$ is semimultiplicative. We associate each $f(z)$ with incidence function value $f(0, z)$. For example, by $(f * \mu)(z)$ we mean the convolution

$$(f * \mu)(0, z) = \sum_{0 \leq w \leq z} f(0, w)\mu(w, z).$$

3 An inductive method

In this section we provide an inductive method for constructing meet closed sets S on which join matrices $[S]_f$ are nonsingular under certain conditions on f. The inductive method arises from the idea to construct meet closed sets element by element from the bottom up, see Definition 3.1.

Throughout the rest of this paper $(P, \leq) = (P, \wedge, \vee)$ is a lattice, $S = \{x_1, x_2, \ldots, x_n\}$ is a meet closed subset of P such that $x_i < x_j \Rightarrow i < j$ holds and f is a semimultiplicative function on P such that $f(x) \neq 0$ for all $x \in P$.

Now, by using the semimultiplicativity of f, we may write

$$[S]_f = \Delta_{S,f}(S)_{\frac{1}{f}}\Delta_{S,f}, \tag{3.1}$$

where $\Delta_{S,f} = \text{diag}(f(x_1), f(x_2), \ldots, f(x_n))$ (see [3, Theorem 6.1], [20, Theorem 6.1] and [17, Lemmas 5.1 and 5.2]). Since $f(x_i) \neq 0$ for all $i = 1, 2, \ldots, n$, the matrix $\Delta_{S,f}$ is clearly invertible. Therefore $[S]_f$ is invertible if and only if $(S)_{\frac{1}{f}}$ is invertible.

Let $S_i = \{x_1, x_2, \ldots, x_i\}$ for $i = 1, 2, \ldots, n$. Then $S_1 \subset S_2 \subset \cdots \subset S_n = S$ is a finite sequence of meet closed sets on $(P, \leq)$ and lower-closed sets on $(S, \leq)$. The values of the corresponding Möbius function μ_S can be easily evaluated by using the recursion

$$\begin{aligned} &\mu_S(x_i, x_i) = 1, \\ &\mu_S(x_i, x_j) = -\sum_{x_i \preceq x_k \prec x_j} \mu_S(x_i, x_k) = -\sum_{x_i \prec x_k \preceq x_j} \mu_S(x_k, x_j), \ i < j, \end{aligned} \tag{3.2}$$

see [1, p. 141] or [28, p. 116]. Note that $\mu_S = \mu_{S_i}$ on $(S_i, \leq)$ and the convolutions on $(S_i, \leq)$ and $(S, \leq)$ are equal if the arguments belong to S_i. Thus for each $i \geq 2$ we have

$$\begin{aligned} \det(S_i)_{\frac{1}{f}} &= \prod_{k=1}^{i} \left(\tfrac{1}{f} *_S \mu_S\right)(x_k) = \left(\tfrac{1}{f} *_S \mu_S\right)(x_i) \prod_{k=1}^{i-1} \left(\tfrac{1}{f} *_S \mu_S\right)(x_k) \\ &= \left(\tfrac{1}{f} *_S \mu_S\right)(x_i) \det(S_{i-1})_{\frac{1}{f}} \end{aligned} \tag{3.3}$$

(see [3, Theorem 4.2] and [6, Corollary 2]).

From (3.1) and (3.3) we see that if $[S_i]_f$ is invertible, then also $(S_i)_{\frac{1}{f}}$, $(S_{i-1})_{\frac{1}{f}}$ and $[S_{i-1}]_f$ are invertible. Conversely, let $[S_{i-1}]_f$ be invertible. We below consider which elements of P, denoted as x_i, could be added to S_{i-1} so that also $[S_i]_f$ is invertible.

Definition 3.1. *Let $S_0 = \emptyset$ and $i \geq 1$. Consider the sets S_{i-1} and $S_i = S_{i-1} \cup \{x_i\}$.*

($\mathrm{M}_{m_i,i}$) *Let m_i be the greatest integer such that $x_{i_1}, x_{i_2}, \ldots, x_{i_{m_i}} \in S_{i-1}$ are covered by x_i in S_i.*

If ($\mathrm{M}_{m_i,i}$) holds, then we say that S_i is constructed from S_{i-1} by the method ($\mathrm{M}_{m_i,i}$). Further, if

($\mathrm{C}_{m_i,i}$) $(\frac{1}{f} *_S \mu_S)(x_i) \neq 0,$

then we say that S_i is constructed from S_{i-1} by the method ($\mathrm{M}_{m_i,i}$) under the condition ($\mathrm{C}_{m_i,i}$).

Remark 3.1. *We always must have $m_1 = 0$ and $m_2 = m_3 = 1$. For example, the condition ($\mathrm{C}_{0,1}$) only states the triviality $\frac{1}{f}(x_1) \neq 0$ whereas ($\mathrm{C}_{1,2}$) means that $\frac{1}{f}(x_2) - \frac{1}{f}(x_1) \neq 0$.*

Theorem 3.1. *Let $i \geq 2$ and S_i be constructed from S_{i-1} by* $(\mathrm{M}_{m_i,i})$ *under* $(\mathrm{C}_{m_i,i})$*. Then $[S_i]_f$ is invertible if and only if $[S_{i-1}]_f$ is invertible.*

Proof. Theorem 3.1 is a direct consequence of (3.1), (3.3) and Definition 3.1. □

The method $(\mathrm{M}_{1,i})$ in Definition 3.1 allows us to add an element x_i above x_{i_1} if x_i covers x_{i_1} in S_i. The method $(\mathrm{M}_{2,i})$ allows us to join together two incomparable elements x_{i_1}, x_{i_2} with x_i if x_i covers both x_{i_1} and x_{i_2}. The method $(\mathrm{M}_{3,i})$ concerns three incomparable elements $x_{i_1}, x_{i_2}, x_{i_3}$ and so on. The condition $(\mathrm{C}_{m_i,i})$ can be written as

$$\frac{1}{f(x_i)} \neq -\sum_{k=1}^{i-1} \tfrac{1}{f}(x_k)\mu_S(x_k, x_i). \tag{3.4}$$

For $m_i = 1, 2$ using the recursive properties of μ_S, see (3.2), we easily obtain

$(\mathrm{C}_{1,i})$ $\quad f(x_i) \neq f(x_{i_1})$,

$(\mathrm{C}_{2,i})$ $\quad \frac{1}{f(x_i)} \neq \frac{1}{f}(x_{i_1}) + \frac{1}{f}(x_{i_2}) - \frac{1}{f}(x_{i_1} \wedge x_{i_2})$.

By semimultiplicativity, $(\mathrm{C}_{2,i})$ can be written without any meets as

$$f(x_{i+1}) \neq f(x_{i_1})f(x_{i_2})/[f(x_{i_1}) + f(x_{i_2}) - f(x_{i_1} \vee x_{i_2})] \tag{3.5}$$

whenever the denominator is nonzero. Each meet closed set S can be constructed inductively by a finite sequence $(\mathrm{M}_{m_1,1})$, $(\mathrm{M}_{m_2,2})$, …, $(\mathrm{M}_{m_n,n})$ (often there are multiple different ways to construct a given set S but the sequence $(m_1, m_2, \ldots, m_n)$ is in fact unique up to ordering). Thus we have the following theorem.

Theorem 3.2. *Let S be constructed inductively by a method sequence*

$$(\mathrm{M}_{m_1,1}), (\mathrm{M}_{m_2,2}), \ldots, (\mathrm{M}_{m_n,n}).$$

Then $[S]_f$ is invertible if and only if the condition sequence

$$(\mathrm{C}_{m_1,1}), (\mathrm{C}_{m_2,2}), \ldots, (\mathrm{C}_{m_n,n})$$

holds.

4 Classification of functions on the basis of the used methods

Let $\mathcal{F}$ denote the class of all semimultiplicative functions f on P such that $f(x) \neq 0$ for all $x \in P$. We divide $\mathcal{F}$ into subclasses on the basis of the numbers $m_i = 1, 2, \ldots$ in the method sequences $(\mathrm{M}_{m_1,1}), (\mathrm{M}_{m_2,2}), \ldots, (\mathrm{M}_{m_n,n})$. We introduce two kinds of subclasses $\mathcal{F}_k$ and $\mathcal{G}_{k,n}$. The classes $\mathcal{F}_k$ are smaller than the classes $\mathcal{G}_{k,n}$ and are introduced to get the presentation shorter. Let $\mathcal{S}_{k,n}$ denote the class of all meet closed subsets S of P possessing the structure as described in Figure 1. The white points in Figure 1 stand for the last added elements x_n. Note that although x_k would be the supremum of x_i and x_j in $(S, \preceq)$, it does not necessarily represent the element $x_i \vee x_j \in P$. In the notation $\mathcal{S}_{k,n}$, the number k comes from the last used method $(\mathrm{M}_{k,n})$ in constructing the set $S \in \mathcal{S}_{k,n}$ (that is, the last added element x_n covers k but no more incomparable elements $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ in S), and the letter n just indicates the number of elements in $S \in \mathcal{S}_{k,n}$. For the pair $k = 4, n = 7$ we should distinguish two distinct classes $\mathcal{S}_{4,7}^{(1)}$ and $\mathcal{S}_{4,7}^{(2)}$. We are now in a position to define the function classes $\mathcal{G}_{k,n}$.

Definition 4.1. *For each $\mathcal{S}_{k,n}$ in Figure 1 let*

$$\mathcal{G}_{k,n} = \{f \in \mathcal{F} \mid \forall S \in \mathcal{S}_{k,n} : (\tfrac{1}{f} *_S \mu_S)(x_n) \neq 0\}.$$

In addition,

$$\mathcal{G}_{4,7}^{(j)} = \{f \in \mathcal{F} \mid \forall S \in \mathcal{S}_{4,7}^{(j)} : (\tfrac{1}{f} *_S \mu_S)(x_7) \neq 0\}, \ \ j = 1, 2.$$

The condition $(\frac{1}{f} * \mu_S)(x_n) \neq 0$ means that the last condition $(\mathrm{C}_{m_n,n})$ in the condition sequence in Theorem 3.2 holds.

For each class $\mathcal{S}_{k,n} \ni S$ we have marked in Figure 1 the value of $\mu_S(x_i, x_n)$ next to each element x_i. The value of $\mu_S(x_i, x_n)$ can be easily seen by (3.2).

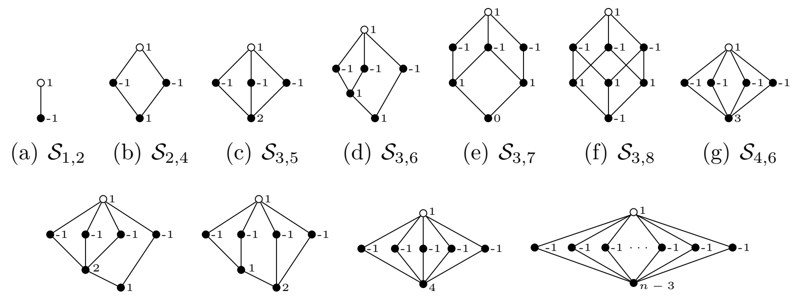
Definition 4.2. *For each $k = 1, 2, \ldots$ let $\mathcal{F}_k$ denote the set of functions $f \in \mathcal{F}$ satisfying the condition sequence $(\mathrm{C}_{m_1,1}), (\mathrm{C}_{m_2,2}), \ldots, (\mathrm{C}_{m_n,n})$ for all meet closed subsets S of P such that S can be constructed by $(\mathrm{M}_{m_1,1}), (\mathrm{M}_{m_2,2}), \ldots, (\mathrm{M}_{m_n,n})$, where $m_1, m_2, \ldots, m_n \leq k$.*

It is easy to see that $\mathcal{F} \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \mathcal{F}_3 \supseteq \cdots$ and more precisely

$$\mathcal{F}_1 = \mathcal{G}_{1,2} = \{f \mid \forall y, z \in P : y < z \Rightarrow f(y) \neq f(z)\}, \tag{4.1}$$

$$\begin{aligned}\mathcal{F}_2 = \mathcal{F}_1 \cap \mathcal{G}_{2,4} = \mathcal{F}_1 \cap \{f \mid &\forall \text{ antichains } y_1, y_2 \in P, \forall z \in P : \\ &y_1 \vee y_2 \leq z \Rightarrow \tfrac{1}{f}(z) \neq \tfrac{1}{f}(y_1) + \tfrac{1}{f}(y_2) - \tfrac{1}{f}(y_1 \wedge y_2)\},\end{aligned} \tag{4.2}$$

$$\mathcal{F}_3 = \mathcal{F}_2 \cap \mathcal{G}_{3,5} \cap \mathcal{G}_{3,6} \cap \mathcal{G}_{3,7} \cap \mathcal{G}_{3,8}. \tag{4.3}$$



(a) $\mathcal{S}_{1,2}$ (b) $\mathcal{S}_{2,4}$ (c) $\mathcal{S}_{3,5}$ (d) $\mathcal{S}_{3,6}$ (e) $\mathcal{S}_{3,7}$ (f) $\mathcal{S}_{3,8}$ (g) $\mathcal{S}_{4,6}$

(h) $\mathcal{S}_{4,7}^{(1)}$ (i) $\mathcal{S}_{4,7}^{(2)}$ (j) $\mathcal{S}_{5,7}$ (k) $\mathcal{S}_{n-2,n}$

Figure 1.

When adding the last element x_n to the set S_{n-1} the invertibility of $[S_n]_f$ depends only on the invertibility of $[S_{n-1}]_f$ and on the values $f(x_i)$ of x_i such that $\mu_S(x_i, x_n) \neq 0$. Thus when considering whether the condition $(C_{m_n,n})$ is satisfied or not we can omit all elements x_i with $\mu_S(x_i, x_n) = 0$. It will turn out that when $n \leq 7$ we can omit most of the cases and restrict ourselves to the structures presented in Figure 1.

Remark 4.1. *All the structures of S mentioned here need not appear in a fixed lattice $(P, \leq)$, and thus the structure of $(P, \leq)$ also has a bearing on the possibility of the invertibility.*

5 Chains, x_1-sets and a related class

In this section we consider invertibility of $[S]_f$ on certain sets S which we use frequently in the lattice-theoretic generalization of the Bourque-Ligh conjecture in Section 6.

Theorem 5.1. *If S is a chain, then $[S]_f$ is invertible if and only if $f(x_k) \neq f(x_{k-1})$ for $k = 2, 3, \ldots, n$. If S is an x_1-set, then $[S]_f$ is invertible if and only if $f(x_k) \neq f(x_1)$ for $k = 2, 3, \ldots, n$.*

Proof. Chains and x_1-sets are constructed using the methods $(\mathrm{M}_{1,i})$ only. By Theorem 3.2, we obtain Theorem 5.1 taking the appropriate conditions $(\mathrm{C}_{1,i})$. □

Remark 5.1. *It is easy to see that if the set S is meet closed and can be constructed by using only the methods $(\mathrm{M}_{1,i})$, then $f \in \mathcal{F}_1$ is a sufficient* *condition for the invertibility of $(S)_{\frac{1}{f}}$ and $(S)_f$ and, provided that f is semimultiplicative with nonzero values, also for the invertibility of $[S]_f$ and $[S]_{\frac{1}{f}}$. In this case the Hasse diagram of the set S considered as an undirected graph is a tree, and the positive definiteness of the matrix $(S)_f$ has an interesting connection to the properties of the function f, see [21, Theorems 4.1 and 4.2].*

Corollary 5.1. *Let $(P, \leq) = (\mathbb{Z}_+, |)$. If S is a (divisor) chain or an x_1-set, then $[S]$ is invertible.*

Proof. The arithmetical function N fulfills the conditions in Theorem 5.1. □

Note that the conditions of Theorem 5.1 also imply the invertibility of the associated meet matrix $(S)_f$, see [6, Corollary 2]. The requirement of semimultiplicativity of f in the first part of Theorem 5.1 is irrelevant, since any f is semimultiplicative on chains.

One important class of meet closed sets (termed as $\mathcal{S}_{n-2,n}$, see Figure 1) is constructed by adding an upper bound to an x_1-set.

Theorem 5.2. *Let $n \geq 3$. Let $S \in \mathcal{S}_{n-2,n}$, i.e., S_{n-1} is an x_1-set and $x_1 \vee \cdots \vee x_{n-1} \leq x_n$. Then $[S]_f$ is invertible if and only if $f(x_k) \neq f(x_1)$ for $k = 2, 3, \ldots, n-1$ and*

$$\frac{1}{f(x_n)} \neq \left(\sum_{k=2}^{n-1} \frac{1}{f(x_k)}\right) - \frac{n-3}{f(x_1)}.$$

Proof. Since S can be constructed from an x_1-set S_{n-1} by $(\mathrm{M}_{n-2,n})$, then the conditions are those mentioned in Theorem 5.1 for S_{n-1} together with condition $(\mathrm{C}_{n-2,n})$. Using (3.4) and the values $\mu_S(x_k, x_n)$ of $\mathcal{S}_{n-2,n}$ in Figure 1 we obtain

$$\frac{1}{f(x_n)} \neq \frac{1}{f(x_{n-1})} + \cdots + \frac{1}{f(x_2)} - \frac{n-3}{f(x_1)}.$$

□

Corollary 5.2. *Let $(P, \leq) = (\mathbb{Z}_+, |)$ and let $n \geq 3$. If S_{n-1} is an x_1-set and $\mathrm{lcm}(S_{n-1}) \mid x_n$, i.e., $S \in \mathcal{S}_{n-2,n}$, then $[S]$ is invertible.*

Proof. It suffices to prove that $N \in \mathcal{G}_{n-2,n}$. The case $n = 3$ follows from Corollary 5.1, so we may assume that $n \geq 4$. Now $x_1 = \gcd(x_i, x_j)$ for all $2 \leq i < j \leq n-1$. Thus for $i = 2, 3, \ldots, n-1$ we have $x_i = a_i x_1$, where a_i's are distinct and $a_i \geq 2$ for each i. Thus we have

$$\frac{1}{x_n} + \frac{n-3}{x_1} - \sum_{k=2}^{n-1} \frac{1}{x_k} = \frac{1}{x_n} + \frac{1}{x_1}\left((n-3) - \sum_{k=2}^{n-1} \frac{1}{a_k}\right) > 0,$$

since

$$\sum_{k=2}^{n-1} \frac{1}{a_k} < \sum_{k=2}^{n-1} \frac{1}{2} = \frac{n-2}{2} \le n-3.$$

Thus $N \in \mathcal{G}_{n-2,n}$. □

Remark 5.2. *Let* $(P, \le) = (\mathbb{Z}_+, |)$. *Since* $N \in \mathcal{F}_2$, *we see that if* S *is any gcd-closed set constructed by* $(\mathrm{M}_{1,i})$ *and* $(\mathrm{M}_{2,i})$ *repeatedly, then the LCM matrix* $[S]$ *is invertible, see Corollaries 5.1 and 5.2 In particular, by Corollary 5.2 we also have* $N \in \mathcal{G}_{2,4}$, $N \in \mathcal{G}_{3,5}$, $N \in \mathcal{G}_{4,6}$ *and* $N \in \mathcal{G}_{5,7}$.

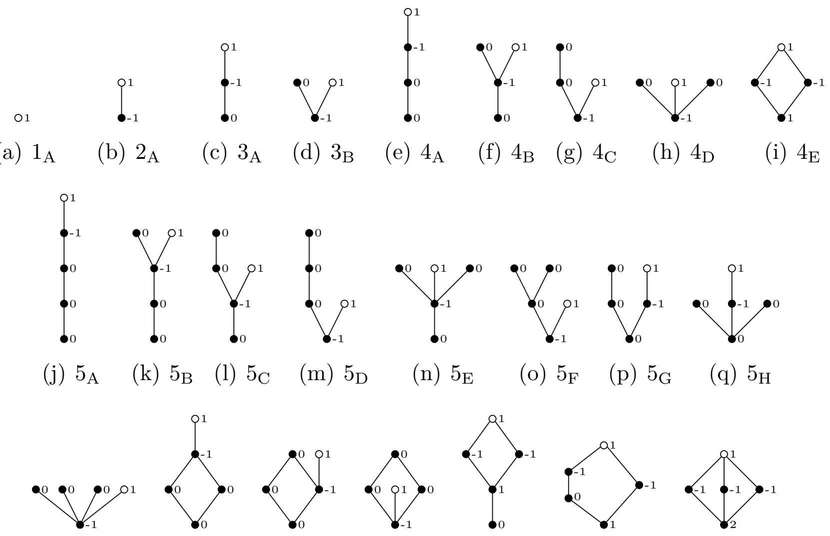
6 The Bourque-Ligh conjecture

Bourque and Ligh [5] conjectured that the LCM matrix $[S]$ is invertible on any gcd-closed set S. It is known that this conjecture holds for $n \le 7$ and does not generally hold for $n \ge 8$. A number-theoretic proof of this solution has been given in [11]. We here provide a lattice-theoretic proof. We go through all meet closed sets S (up to isomorphism) with $n = 1, 2, \ldots, 7$ elements, and applying the conditions $(\mathrm{C}_{m_i,i})$ we study the invertibility of the join matrix $[S]_f$ on S in any lattice. When we take $(P, \le) = (\mathbb{Z}_+, |)$ and $f = N$ we obtain the solution of the Bourque-Ligh conjecture given in [11]. In principle this is a simple method, since at least for small n the sets S are easy to classify on the basis of their incomparable elements and the conditions $(\mathrm{C}_{m_i,i})$ are easy to evaluate applying (3.4), the Hasse diagram of S and the recursive properties of μ_S. It would be easy to derive necessary and sufficient conditions for the invertibility of the join matrix $[S]_f$ on S in any lattice, but for the sake of brevity in we present only sufficient conditions.

6.1 Cases $n = 1, 2, 3, 4, 5$

We begin by constructing recursively all possible meet closed sets with at most 5 elements, see Figure 2. If all meet semilattices with n elements are known, then a simple but laborous way to obtain all possible meet semilattices with $n+1$ elements is first to determine all possible ways to add a maximal element to them and then to eliminate repetitions. The semilattices are then classified based on the largest m_i in the methods $(\mathrm{M}_{m_i,i})$ used to construct each semilattice. Most of them are constructed by using $(\mathrm{M}_{1,i})$ only, but for some of them also $(\mathrm{M}_{2,i})$ or even $(\mathrm{M}_{3,i})$ is needed.

In each class the white point stands for the last added element. For each class we have also marked the value of $\mu_S(x_i, x_n)$ next to each element x_i. The calculation of $\mu_S(x_i, x_n)$ bases on (3.2).



(a) 1_{A} (b) 2_{A} (c) 3_{A} (d) 3_{B} (e) 4_{A} (f) 4_{B} (g) 4_{C} (h) 4_{D} (i) 4_{E}

(j) 5_{A} (k) 5_{B} (l) 5_{C} (m) 5_{D} (n) 5_{E} (o) 5_{F} (p) 5_{G} (q) 5_{H}

(r) 5_{I} (s) 5_{J} (t) 5_{K} (u) 5_{L} (v) 5_{M} (w) 5_{N} (x) 5_{O}

Figure 2.

Theorem 6.1. *Let S be a set with at most 5 elements.*

(i) *If $S \in 1_{\mathrm{A}}$, then $[S]_f$ is always invertible (under the condition $f(x) \neq 0$ for all $x \in P$).*

(ii) *If $S \in 2_{\mathrm{A}}, 3_{\mathrm{A}}, 3_{\mathrm{B}}, 4_{\mathrm{A}}, 4_{\mathrm{B}}, 4_{\mathrm{C}}, 4_{\mathrm{D}}, 5_{\mathrm{A}}, 5_{\mathrm{B}}, \ldots, 5_{\mathrm{I}}$ and $f \in \mathcal{F}_1$, then $[S]_f$ is invertible.*

(iii) *If $S \in 4_{\mathrm{E}}, 5_{\mathrm{J}}, 5_{\mathrm{K}}, 5_{\mathrm{L}}, 5_{\mathrm{M}}, 5_{\mathrm{N}}$ and $f \in \mathcal{F}_2$, then $[S]_f$ is invertible.*

(iv) *If $S \in 5_{\mathrm{O}} = \mathcal{S}_{3,5}$ and $f \in \mathcal{F}_1 \cap \mathcal{G}_{3,5}$, then $[S]_f$ is invertible.*

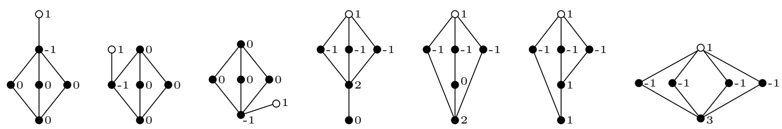
Proof. (i) The one element case is trivial. (ii) If S belongs to one of the classes mentioned in part (ii), then S can be constructed by $(\mathrm{M}_{1,i})$ only and thus $f \in \mathcal{F}_1$ is a sufficient condition for the invertibility of $[S]_f$, see Definition 4.2. (iii) If S belongs to the classes mentioned in (iii), then both $(\mathrm{M}_{1,i})$ and $(\mathrm{M}_{2,i})$ are needed and therefore $f \in \mathcal{F}_2$ is sufficient for the invertibility. (iv) If $S \in 5_{\mathrm{O}}$, then the conditions for the invertibility of $[S]_f$ follow from Theorem 5.2. □

Corollary 6.1. *If S is a meet closed set with at most 5 elements and $f \in \mathcal{F}_2 \cap \mathcal{G}_{3,5}$, then $[S]_f$ is invertible. In particular, if S is a gcd-closed set with at most 5 elements, then $[S]$ is invertible.*

Proof. The first part is a direct consequence of Theorem 6.1. For the second part we just have to recall that $N \in \mathcal{F}_2 \cap \mathcal{G}_{3,5}$ by Remark 5.2. $\square$

6.2 Case $n = 6$

For $n \geq 6$ we change our procedure slightly, since there are 53 classes of meet closed sets for $n = 6$ and 222 for $n = 7$ (see e.g. [10], the number of meet semilattices with n elements equals the number of lattices with $n + 1$ elements, since adding a maximum element to a meet semilattice results a lattice). Here we construct only the meet closed sets with 6 elements, where at least one of $m_1, \ldots, m_n$ is greater than or equal to 3. (If $m_1, \ldots, m_n \leq 2$, then the Bourque-Ligh conjecture holds by Remark 5.2.) We obtain exactly 7 different classes $6_{\mathrm{A}}, 6_{\mathrm{B}}, \ldots, 6_{\mathrm{G}}$ presented in Figure 3. In each class there can be no more than one element x_i with $m_i \geq 3$ and there exists exactly one class with $m_i = 4$. Keeping this in mind the use of mathematical programs is not necessarily needed in order to find all 7 classes, but it would be easy to do so by making suitable adjustments to the code given in Remark 6.1. The value of $\mu(x_i, x_6)$ is again marked next to each element x_i, and the white points stand for the last added element x_6.



(a) 6_{A} (b) 6_{B} (c) 6_{C} (d) 6_{D} (e) 6_{E} (f) 6_{F} (g) 6_{G}

Figure 3.

Theorem 6.2. *Let S be a meet closed set with 6 elements.*

(i) *If $S \notin 6_{\mathrm{A}}, 6_{\mathrm{B}}, \ldots, 6_{\mathrm{G}}$ and $f \in \mathcal{F}_2$, then $[S]_f$ is invertible.*

(ii) *If $S \in 6_{\mathrm{A}}, 6_{\mathrm{B}}, \ldots, 6_{\mathrm{E}}$ and $f \in \mathcal{F}_1 \cap \mathcal{G}_{3,5}$, then $[S]_f$ is invertible.*

(iii) *If $S \in 6_{\mathrm{F}} = \mathcal{S}_{3,6}$ and $f \in \mathcal{F}_1 \cap \mathcal{G}_{3,6}$, then $[S]_f$ is invertible.*

(iv) *If $S \in 6_{\mathrm{G}} = \mathcal{S}_{4,6}$ and $f \in \mathcal{F}_1 \cap \mathcal{G}_{4,6}$, then $[S]_f$ is invertible.*

Proof. (i) If $S \notin 6_{\mathrm{A}}, 6_{\mathrm{B}}, \ldots, 6_{\mathrm{G}}$, then only $(\mathrm{M}_{1,i})$ and $(\mathrm{M}_{2,i})$ have been used, and thus the condition $f \in \mathcal{F}_2$ assures that $[S]_f$ is invertible, see Definition 4.2. (ii) If $S \in 6_{\mathrm{A}}, 6_{\mathrm{B}}, 6_{\mathrm{C}}, 6_{\mathrm{D}}, 6_{\mathrm{E}}$, then S can be constructed by $(\mathrm{M}_{1,i})$ and $(\mathrm{M}_{3,i})$, and thus the assumption $f \in \mathcal{F}_1$ together with $f \in \mathcal{G}_{3,5}$ assures the fulfillment of conditions ($C_{1,i}$) and ($C_{3,i}$) and therefore the invertibility of $[S]_f$. For $S \in 6_A, 6_B, 6_C$ the condition ($C_{3,5}$) is clearly implied by f belonging to $\mathcal{G}_{3,5}$, and also for $S \in 6_D, 6_E$ the condition ($C_{3,6}$) is implied by the assumption $f \in \mathcal{G}_{3,5}$ due to the zeros of $\mu_S(x_i, x_6)$ in $6_D, 6_E$ of Figure 3. (iii) In the case when $S \in 6_F$ the semilattice S_{n-1} can be constructed by ($M_{1,i}$) and S can be constructed by ($M_{3,6}$) from S_{n-1}. In this case the assumption $f \in \mathcal{F}_1$ quarantees that the conditions ($C_{1,i}$) hold, whereas $f \in \mathcal{G}_{3,6}$ implies that ($C_{3,6}$) holds. Thus $[S]_f$ is invertible. (iv) If $S \in 6_G$, then the conditions for the invertibility of $[S]_f$ come from those in Theorem 5.2. □

Corollary 6.2. *If S is a meet closed set with 6 elements and $f \in \mathcal{F}_2 \cap \mathcal{G}_{3,5} \cap \mathcal{G}_{3,6} \cap \mathcal{G}_{4,6}$, then $[S]_f$ is invertible. In particular, if S is a gcd-closed set with 6 elements, then $[S]$ is invertible.*

Proof. The first part of this corollary is obvious, since $\mathcal{F}_2 \subseteq \mathcal{F}_1$. We only need to prove the second part. We already know that $N \in \mathcal{F}_2 \cap \mathcal{G}_{3,5} \cap \mathcal{G}_{4,6}$ (Remark 5.2 and Corollary 5.2), so it suffices to prove that $N \in \mathcal{G}_{3,6}$. Let $S \in 6_F$,

$$x_1 = \gcd(x_2, x_3) = \gcd(x_3, x_4) = \gcd(x_3, x_5),$$

$x_2 = \gcd(x_4, x_5)$ and $\operatorname{lcm}(x_3, x_4, x_5) \mid x_6$. Thus $x_2 = ax_1$, $x_3 = bx_1$, $x_4 = acx_1$, $x_5 = adx_1$, where $a, b, c, d \geq 2$ and

$$\gcd(a, b) = \gcd(b, c) = \gcd(b, d) = \gcd(c, d) = 1.$$

Therefore at least one of the numbers c and d must be greater than or equal to 3, from which it follows that $cd - c - d > 0$. Clearly we also have $b - 1 > 0$ and $x_1, x_6 > 0$ and thus we obtain

$$\begin{aligned} \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} - \frac{1}{x_3} + \frac{1}{x_2} + \frac{1}{x_1} &= \\ &= \frac{1}{x_6} + \frac{-bc - bd - acd + bcd + abcd}{abcdx_1} \\ &= \frac{1}{x_6} + \frac{acd(b-1) + b(cd - d - c)}{abcdx_1} > 0. \end{aligned} \tag{6.1}$$

This implies that $N \in \mathcal{G}_{3,6}$. □

6.3 Case $n = 7$

As in the case $n = 6$, we consider only the meet closed sets with 7 elements, where at least one of $m_1, \ldots, m_n$ is greater than or equal to 3. There are exactly 47 such semilattices, which we divide into ten categories $7_A, 7_B, \ldots, 7_I$

based on their structure, see Figures 4-8. As before, we have marked the value of $\mu_S(x_i, x_7)$ next to each element x_i, and the last added elements x_7 are denoted by white points.

Remark 6.1. *In the case $n = 6$ it is well possible to find all meet semilattices in Figure 3 without any computer calculations. As one might expect, in the case $n = 7$ the task of finding all meet semilattices with at least one $m_i \geq 3$ without any help from a computer becomes quite overwhelming. With Sage 5.10 this can easily be done by using the command*

```
P7=[p for p in Posets(7) if p.is_meet_semilattice() and
 max([len(p.lower_covers(q)) for q in p.list()]) >= 3].
```

With the command

```
for p in P7: show(p.plot())
```

it is then possible to obtain the list of Hasse diagrams of the meet semilattices in question.

Theorem 6.3. *Let S be a meet closed set with 7 elements.*

(i) *If S does not belong to any classes presented in Figures 4-8 and $f \in \mathcal{F}_2$, then $[S]_f$ is invertible.*

(ii) *If $S \in 7_{\mathrm{AA}}, 7_{\mathrm{AB}}, \ldots, 7_{\mathrm{AX}}$ and $f \in \mathcal{F}_1 \cap \mathcal{G}_{3,5}$, then $[S]_f$ is invertible.*

(iii) *If $S \in 7_{\mathrm{BA}}, 7_{\mathrm{BB}}, \ldots, 7_{\mathrm{BI}}$ and $f \in \mathcal{F}_1 \cap \mathcal{G}_{3,6}$, then $[S]_f$ is invertible.*

(iv) *If $S \in 7_{\mathrm{CA}}, 7_{\mathrm{CB}}, 7_{\mathrm{CC}}, 7_{\mathrm{CD}}, 7_{\mathrm{CE}}$ and $f \in \mathcal{F}_2 \cap \mathcal{G}_{3,5}$, then $[S]_f$ is invertible.*

(v) *If $S \in 7_{\mathrm{DA}}, 7_{\mathrm{DB}}, 7_{\mathrm{DC}}, 7_{\mathrm{DD}}, 7_{\mathrm{DE}}$ and $f \in \mathcal{F}_1 \cap \mathcal{G}_{4,6}$, then $[S]_f$ is invertible.*

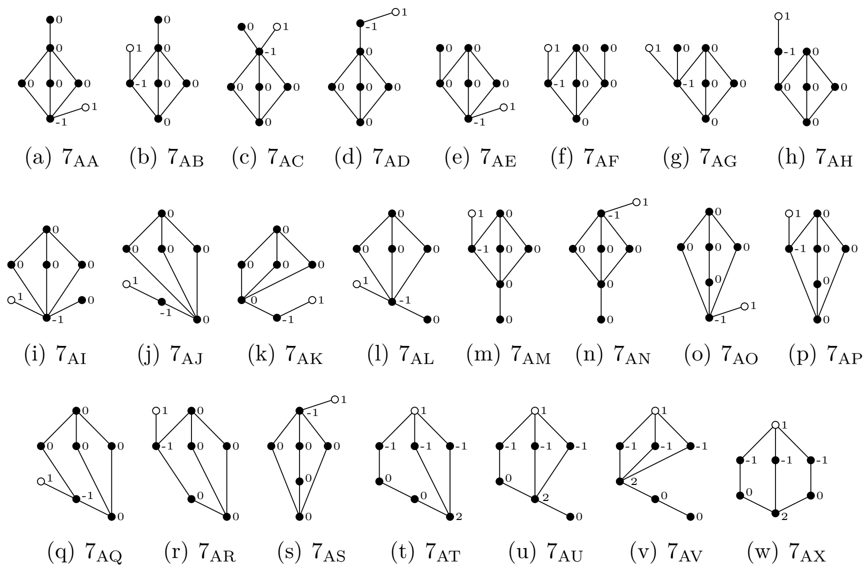
(vi) *If $S \in 7_{\mathrm{E}}$ and $f \in \mathcal{F}_2 \cap \mathcal{G}_{3,6}$, then $[S]_f$ is invertible.*

(vii) *If $S \in 7_{\mathrm{F}}$ and $f \in \mathcal{F}_2 \cap \mathcal{G}_{3,7}$, then $[S]_f$ is invertible.*

(viii) *If $S \in 7_{\mathrm{G}}$ and $f \in \mathcal{F}_1 \cap \mathcal{G}_{4,7}^{(1)}$, then $[S]_f$ is invertible.*

(ix) *If $S \in 7_{\mathrm{H}}$ and $f \in \mathcal{F}_1 \cap \mathcal{G}_{4,7}^{(2)}$, then $[S]_f$ is invertible.*

(x) *If $S \in 7_{\mathrm{I}}$ and $f \in \mathcal{F}_1 \cap \mathcal{G}_{5,7}$, then $[S]_f$ is invertible.*

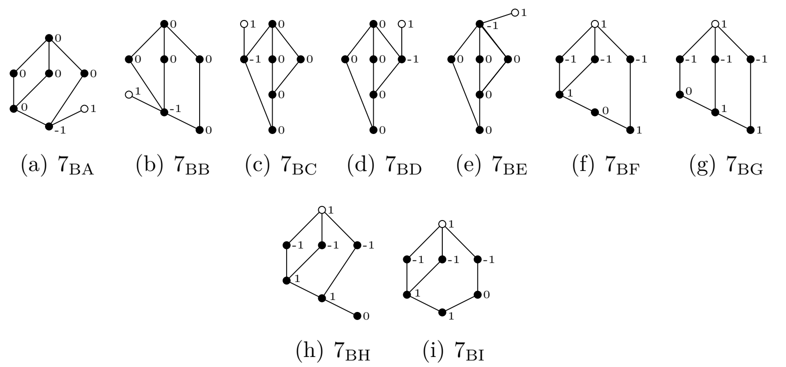


(a) 7_{AA} (b) 7_{AB} (c) 7_{AC} (d) 7_{AD} (e) 7_{AE} (f) 7_{AF} (g) 7_{AG} (h) 7_{AH}

(i) 7_{AI} (j) 7_{AJ} (k) 7_{AK} (l) 7_{AL} (m) 7_{AM} (n) 7_{AN} (o) 7_{AO} (p) 7_{AP}

(q) 7_{AQ} (r) 7_{AR} (s) 7_{AS} (t) 7_{AT} (u) 7_{AU} (v) 7_{AV} (w) 7_{AX}

Figure 4.

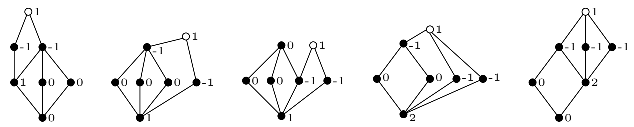


(a) 7_{BA} (b) 7_{BB} (c) 7_{BC} (d) 7_{BD} (e) 7_{BE} (f) 7_{BF} (g) 7_{BG}

(h) 7_{BH} (i) 7_{BI}

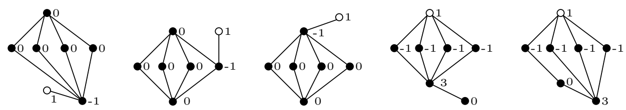
Figure 5.

Proof. (i) This case is trivial, since if S can be constructed by $(\mathrm{M}_{1,i})$ and $(\mathrm{M}_{2,i})$ only, then $f \in \mathcal{F}_2$ is a sufficient condition for the invertibility of $[S]_f$.
(ii) Let $S \in 7_{\mathrm{AA}}, 7_{\mathrm{AB}}, \ldots, 7_{\mathrm{AX}}$. Then S can be constructed by applying $(\mathrm{M}_{1,i})$ six times and $(\mathrm{M}_{3,i})$ once. Due to the zeros of the Möbius function, the condition $f \in \mathcal{G}_{3,5}$ guarantees the invertibility of $[S_i]_f$ when $(\mathrm{M}_{3,i})$ is applied. Everytime when $(\mathrm{M}_{1,i})$ is applied the invertibility follows from the condition



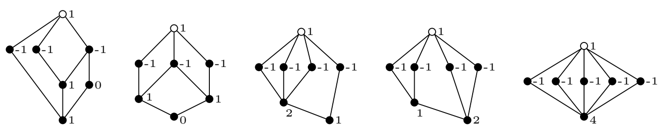
(a) 7_{CA} (b) 7_{CB} (c) 7_{CC} (d) 7_{CD} (e) 7_{CE}

Figure 6.



(f) 7_{DA} (g) 7_{DB} (h) 7_{DC} (i) 7_{DD} (j) 7_{DE}

Figure 7.



(a) 7_{E} (b) 7_{F} (c) 7_{G} (d) 7_{H} (e) 7_{I}

Figure 8.

$f \in \mathcal{F}_1$. (iii) The situation is similar to the cases $S \in 7_{\mathrm{BA}}, 7_{\mathrm{BB}}, \ldots, 7_{\mathrm{BI}}$. The only difference is that the assumption $f \in \mathcal{G}_{3,6}$ implies the invertibility of $[S_i]_f$ when $(\mathrm{M}_{3,i})$ is used. (iv) In the cases $S \in 7_{\mathrm{CA}}, 7_{\mathrm{CB}}, 7_{\mathrm{CC}}, 7_{\mathrm{CD}}, 7_{\mathrm{CE}}$ the methods $(\mathrm{M}_{1,i})$, $(\mathrm{M}_{2,i})$ and $(\mathrm{M}_{3,i})$ are all needed in the construction of the set S. In order to the matrix $[S]_f$ to be invertible, these methods require the assumptions $f \in \mathcal{F}_1$, $f \in \mathcal{F}_2$ and $f \in \mathcal{G}_{3,5}$, respectively. (v) If $S \in 7_{\mathrm{DA}}, 7_{\mathrm{DB}}, 7_{\mathrm{DC}}, 7_{\mathrm{DD}}, 7_{\mathrm{DE}}$, then $(\mathrm{M}_{1,i})$ and $(\mathrm{M}_{4,i})$ are the only used methods. Here $f \in \mathcal{G}_{4,6}$ assures the invertibility of $[S_i]_f$ when $(\mathrm{M}_{4,i})$ is applied, otherwise the invertibility of $[S_i]_f$ follows from the condition $f \in \mathcal{F}_1$. (vi) The case $S \in 7_{\mathrm{E}}$ has much recemblance to the case (iv); here we just need the condition $f \in \mathcal{G}_{3,6}$ instead of $f \in \mathcal{G}_{3,5}$ when the method $(\mathrm{M}_{3,i})$ is used. (viii)-(ix) In the cases $S \in 7_{\mathrm{G}}$ and $S \in 7_{\mathrm{H}}$ the set S_{n-1} can be constructed by $(\mathrm{M}_{1,i})$ only and S can be constructed by $(\mathrm{M}_{4,7})$ from S_{n-1}. Therefore in both cases the assumption $f \in \mathcal{F}_1$ guarantees that the matrix $[S_{n-1}]_f$ is invertible, whereas either condition $f \in \mathcal{G}_{4,7}^{(1)}$ or $f \in \mathcal{G}_{4,7}^{(2)}$ is needed to assure the invertibility of $[S]_f$ when the last element is added. (x) The case $S \in 7_{\mathrm{I}}$ is similar, here only the method $(\mathrm{M}_{5,7})$ is used instead of $(\mathrm{M}_{4,7})$ and the condition $f \in \mathcal{G}_{5,7}$ is needed instead of assuming $f \in \mathcal{G}_{4,7}^{(1)}$ or $f \in \mathcal{G}_{4,7}^{(2)}$. This last result also follows from Theorem 5.2. □

Corollary 6.3. *If S is a meet closed set with 7 elements and $f \in \mathcal{F}_2 \cap \mathcal{G}_{3,5} \cap \mathcal{G}_{3,6} \cap \mathcal{G}_{4,6} \cap \mathcal{G}_{4,7}^{(1)} \cap \mathcal{G}_{4,7}^{(2)} \cap \mathcal{G}_{5,7}$, then $[S]_f$ is invertible. In particular, if S is a gcd-closed set with 7 elements, then $[S]$ is invertible.*

Proof. The first part of this corollary is obvious, since $\mathcal{F}_1 \supseteq \mathcal{F}_2$ and the sets in parts (iii), (v) and (vi) respectively belong to classes $\mathcal{S}_{3,7}$, $\mathcal{S}_{4,7}^{(1)}$ and $\mathcal{S}_{4,7}^{(2)}$. We prove the second part of this corollary. Since by Remark 5.2, Corollary 5.2 and Corollary 6.2 $N \in \mathcal{F}_2 \cap \mathcal{G}_{3,5} \cap \mathcal{G}_{3,6} \cap \mathcal{G}_{4,6} \cap \mathcal{G}_{5,7}$, it suffices to prove that $N \in \mathcal{G}_{3,7} \cap \mathcal{G}_{4,7}^{(1)} \cap \mathcal{G}_{4,7}^{(2)}$. We prove first that $N \in \mathcal{G}_{3,7}$ ($S \in 7_\mathrm{F}$). Let $x_1 = \gcd(x_2, x_3) = \gcd(x_3, x_4) = \gcd(x_2, x_6) = \gcd(x_4, x_6)$, $x_2 = \gcd(x_4, x_5)$, $x_3 = \gcd(x_5, x_6)$ and $\mathrm{lcm}(x_4, x_5, x_6) \mid x_7$. Thus $x_2 = ax_1$, $x_3 = bx_1$, $x_4 = acx_1$, $x_5 = abdx_1$, $x_6 = bex_1$, where $a, b, c, e \geq 2$ and $d \geq 1$. Since $\gcd(c, bd) = 1$, either $c \geq 3$ or $b, d \geq 3$ and we have $(bd-1)(c-1) - 1 > 0$. In addition, $e - 1 > 0$ and $x_1, x_7 > 0$ and thus we obtain

$$\begin{aligned} &\frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} + \frac{1}{x_3} + \frac{1}{x_2} = \\ &\qquad = \frac{1}{x_7} + \frac{-acd - ce - bde + acde + bcde}{abcdex_1} \\ &\qquad = \frac{1}{x_7} + \frac{acd(e-1) + e[(bd-1)(c-1) - 1]}{abcdex_1} > 0. \end{aligned} \tag{6.2}$$

Thus $N \in \mathcal{G}_{3,7}$.

We prove second that $N \in \mathcal{G}_{4,7}^{(1)}$ ($S \in 7_\mathrm{G}$). Let $x_1 = \gcd(x_2, x_3) = \gcd(x_4, x_3) = \gcd(x_5, x_3) = \gcd(x_6, x_3)$, $x_2 = \gcd(x_4, x_5) = \gcd(x_4, x_6) = \gcd(x_5, x_6)$ and $\mathrm{lcm}(x_3, x_4, x_5, x_6) \mid x_7$. Thus $x_2 = ax_1$, $x_3 = bx_1$, $x_4 = acx_1$, $x_5 = adx_1$, $x_6 = aex_1$, where $a, b, c, d, e \geq 2$. Here $\gcd(d, e) = 1$, which implies that $d \neq e$ and either $d > 2$ or $e > 2$. Therefore $de - d - e > 0$, and since also $b - 1, c - 1 > 0$ and $x_1, x_7 > 0$, we have

$$\begin{aligned} &\frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} - \frac{1}{x_3} + \frac{2}{x_2} + \frac{1}{x_1} = \\ &\qquad = \frac{1}{x_7} + \frac{-bcd - bce - bde - acde + 2bcde + abcde}{abcdex_1} \\ &\qquad = \frac{1}{x_7} + \frac{bc(de - d - e) + bde(c-1) + acde(b-1)}{abcdex_1} > 0. \end{aligned} \tag{6.3}$$

Thus $N \in \mathcal{G}_{4,7}^{(1)}$.

We prove third that $N \in \mathcal{G}_{4,7}^{(2)}$ ($S \in 7_\mathrm{H}$). Let $x_1 \mid x_2$, $x_1 = \gcd(x_3, x_4) = \gcd(x_3, x_5) = \gcd(x_4, x_5) = \gcd(x_3, x_6) = \gcd(x_4, x_6)$, $x_2 = \gcd(x_5, x_6)$ and

16

lcm$(x_3, x_4, x_5, x_6) \mid x_7$. Thus $x_2 = ax_1$, $x_3 = bx_1$, $x_4 = cx_1$, $x_5 = adx_1$, $x_6 = aex_1$, where $a, b, c, d, e \geq 2$. Since $\gcd(e, d) = 1$, we have either $d > 2$ or $e > 2$ and further $de - d - e > 0$. In addition, since $b - 1, c - 1 > 0$ and $x_1, x_7 > 0$ we have

$$\begin{aligned}
&\frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} - \frac{1}{x_3} + \frac{1}{x_2} + \frac{2}{x_1} = \qquad (6.4)\\
&= \frac{1}{x_7} + \frac{-bcd - bce - abde - acde + bcde + 2abcde}{abcdex_1}\\
&= \frac{1}{x_7} + \frac{bc(de - d - e) + abde(c - 1) + acde(b - 1)}{abcdex_1} > 0.
\end{aligned}$$

Thus $N \in \mathcal{G}_{4,7}^{(2)}$. □

6.4 Cases $n = 8, 9, \ldots$

Haukkanen, Wang and Sillanpää [9] showed that the Bourque-Ligh conjecture is false by giving the counterexample

$$S = \{1, 2, 3, 4, 5, 6, 10, 45, 180\},$$

where $n = 9$. Hong [11] solved the conjecture completely (in a sense) showing that it holds for $n \leq 7$ and does not hold generally for $n \geq 8$. The counterexample given by Hong is

$$\begin{aligned}
S &= \{1, 2, 3, 5, 36, 230, 825, 227700\}\\
&= \{1, 2, 3, 5, 6(2 \cdot 3), 23(2 \cdot 5), 55(3 \cdot 5), (6 \cdot 23 \cdot 55)(2 \cdot 3 \cdot 5)\}.
\end{aligned}$$

For this counterexample given by Hong [11] we have $S \in \mathcal{S}_{3,8}$ with $[S]$ being singular. Thus $N \notin \mathcal{G}_{3,8}$ and, more general, $N \notin \mathcal{F}_3$. For any $n \geq 8$ we are also able to construct a gcd-closed set S possessing the structure given on the left side of Figure 9 as a subsemilattice, which makes the LCM matrix $[S]$ singular. These counterexamples together with Corollaries 6.1–6.3 serve as a lattice-theoretic solution of the Bourque-Ligh conjecture.

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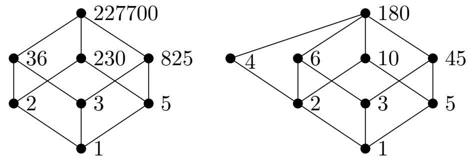

Figure 9. On the left is the counterexample for the Bourque-Ligh conjecture given by Hong. The lattice on the right is the counterexample by Haukkanen, Wang and Sillanpää.

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Publication VI

Studying the singularity of LCM-type matrices via semilattice structures and their Möbius functions

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Studying the singularity of LCM-type matrices via semilattice structures and their Möbius functions

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A B S T R A C T

The invertibility of Least Common Multiple (LCM) matrices and their Hadamard powers have been extensively studied over the years by many authors. Bourque and Ligh conjectured in 1992 that the LCM matrix $[S] = [[x_i, x_j]]$ on any GCD closed set $S = \{x_1, x_2, \ldots, x_n\}$ is invertible, but in 1997 this was proven to be false. Nevertheless, many open conjectures concerning LCM matrices and their Hadamard powers remain. In this paper we utilize lattice-theoretic structures and the Möbius function to explain the singularity of classical LCM matrices and their Hadamard powers. As a result we disprove some open conjectures of Hong. Elementary mathematical analysis is applied to prove that for most semilattice structures there exists a set $S = \{x_1, x_2, \ldots, x_n\}$ of positive integers and a real number $\alpha > 0$ such that S possesses this structure and the power LCM matrix $[[x_i, x_j]^\alpha]$ is singular.

1. Introduction

The study of GCD and LCM matrices was initiated in 1876 by the famous number theorist H.J.S. Smith [15]. Smith calculated the determinant of the basic GCD matrix with the greatest common divisor of i and j as its ij-entry. In addition, Smith derived

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determinant formulas for more general GCD and LCM matrices with (x_i, x_j) or $[x_i, x_j]$ as its ij-entry and showed that the GCD matrix (S) and the LCM matrix $[S]$ are nonsingular on factor closed sets S. He also studied GCD and LCM matrices associated with the arithmetical function f, where the ij entries are $f((x_i, x_j))$ and $f([x_i, x_j])$, respectively. Determinants of GCD-related matrices were studied in dozens of papers during the 20th century (see e.g. the references in [5]), but Bourque and Ligh [3] were the first to focus on the invertibility properties of LCM matrices through their conjecture that the LCM matrix of a GCD closed set is always invertible. Shen [14] went even further and conjectured that if the set S is GCD closed and $r \neq 0$, then the power LCM matrix $[[x_i, x_j]^r]$ is nonsingular.

Haukkanen et al. [5] soon showed that the Bourque–Ligh conjecture (and also Shen's conjecture in the case $r = 1$) is false by finding a counterexample with 9 elements. Two years later Hong [6] found a counterexample with 8 elements and proved number-theoretically that the Bourque–Ligh conjecture holds for $n \leq 7$ and does not hold in general for $n \geq 8$ (there is also a recent paper by Korkee et al. [10] which gives another, a lattice-theoretic proof for this fact). Subsequently Hong published many papers regarding power GCD and power LCM matrices (see e.g. [7–9]). Hong also presented several conjectures on his own about the nonsingularity of power GCD and power LCM matrices. For example, in [7] Hong conjectured that if S is a GCD closed set of odd integers, then every power LCM matrix of the set S with nonzero exponent is nonsingular.

In the last decade there has not been much progress on proving or disproving Hong's conjectures, and they all remain open. One of the few advances was Li's article [11], which provided some support to two of the conjectures. In this article we improve this situation by showing that some of Hong's conjectures are in fact false. This is done by using lattice-theoretic methods.

In Section 2 we introduce some key definitions and preliminary results needed in the following sections. In Section 3 we study the zeros of the Möbius function in a given meet semilattice, which gives us the leverage to analyze the product expression of the determinant of LCM-type matrices. In Section 4 we apply the mathematics software Sage [17] to show that every 8-element GCD closed set S, for which the LCM matrix $[S]$ is singular, has the same semilattice structure. We also construct a GCD closed set S of odd numbers such that the LCM matrix $[S]$ is singular. In Section 5 we prove that for most semilattice structures $(L, \preceq)$ there exists a set $S = \{x_1, x_2, \ldots, x_n\}$ of positive integers and a positive real number α such that $(S, |) \cong (L, \preceq)$ and the power LCM matrix $[S]_{N^\alpha} := [[x_i, x_j]^\alpha]$ is singular. We also point out a connection between the $\wedge$-tree structure of $(L, \preceq)$, the nonpositiveness of the nontrivial values of the Möbius function μ_L and the nonsingularity of the power LCM matrices $[S]_{N^\alpha}$ for all $(S, |) \cong (L, \preceq)$ and $\alpha > 0$. In Section 6 we discuss several conjectures by Hong and give conclusive answers to some of them.

2. Preliminaries

If $(P, \preceq)$ is a meet semilattice, f is a function $P \to \mathbb{C}$ and $S = \{x_1, \ldots, x_n\}$ is a subset of P with distinct elements arranged so that $x_i \preceq x_j \Rightarrow i \leq j$, then the *meet matrix of the set S with respect to the function f* has $f(x_i \wedge x_j)$ as its ij-entry. This matrix is usually denoted by $(S)_f$. Similarly, if $(P, \preceq)$ is a join semilattice and f and S are as above, then the *join matrix of the set S with respect to the function f* has $f(x_i \vee x_j)$ as its ij-entry. For this join matrix we use the notation $[S]_f$.

In the special case when $(P, \preceq) = (\mathbb{Z}_+, |)$ and f is an arithmetical function the meet and join matrices become the so-called *GCD and LCM matrices with respect to the arithmetical function f*, respectively. Moreover, if we set $f = N^\alpha$, where $N^\alpha(m) = m^\alpha$ for all $m \in \mathbb{Z}_+$, the matrices $(S)_f$ and $[S]_f$ become the power-GCD and power-LCM matrices with $(x_i, x_j)^\alpha$ and $[x_i, x_j]^\alpha$ as their ij-entries, respectively. And in the case when $\alpha = 1$ we denote $N^1 = N$ and obtain the usual GCD and LCM matrices with (x_i, x_j) and $[x_i, x_j]$ as their ij-entries, respectively. The usual GCD matrix of the set S is denoted by (S), and the usual LCM matrix by $[S]$.

Remark 2.1. It is often convenient to assume that $x_i \preceq x_j \Rightarrow i \leq j$ (in the case of meet and join matrices) or that $x_1 \leq x_2 \leq \cdots \leq x_n$ (in the case of GCD and LCM matrices). However, the indexing of the elements of the set S does not affect on the invertibility of the corresponding meet or join matrix, see e.g. [12, Remark 2.1]. Since in this paper we are only interested in the singular behaviour of these matrices, in most of the cases we could also do without this assumption.

We develop further the lattice-theoretic method adopted in [10], but this time we will focus solely on power-LCM and power-GCD matrices. Throughout this paper, let $S = \{x_1, \ldots, x_n\}$ be a GCD closed set of positive integers. By denoting $S_i = \{x_1, \ldots, x_i\}$ we obtain a chain of GCD closed sets $S_1 \subset S_2 \subset \cdots \subset S_n = S$. It should be noted that every set S_i is also trivially lower closed in $(S, |)$. This observation enables us to use the Möbius function μ_S of the set S, which can be given recursively as

$$\mu_S(x_i, x_i) = 1,$$
$$\mu_S(x_i, x_j) = -\sum_{\substack{x_i \,|\, x_k \,|\, x_j \\ x_k \neq x_i}} \mu_S(x_i, x_k) = -\sum_{\substack{x_i \,|\, x_k \,|\, x_j \\ x_k \neq x_j}} \mu_S(x_k, x_j).$$

Since

$$[S]_{N^\alpha} = \mathrm{diag}(x_1^\alpha, \ldots, x_n^\alpha)(S)_{\frac{1}{N^\alpha}} \mathrm{diag}(x_1^\alpha, \ldots, x_n^\alpha),$$

it follows that $[S]_{N^\alpha}$ is singular if and only if $(S)_{\frac{1}{N^\alpha}}$ is singular. Furthermore, since the set S is GCD closed, we may define the function $\Psi_{S, \frac{1}{N^\alpha}}$ on S as

$$\Psi_{S,\frac{1}{N^\alpha}}(x_i) = \sum_{x_k \,|\, x_i} \frac{\mu_S(x_k, x_i)}{x_k^\alpha} \tag{2.1}$$

(if the set S is not GCD closed we would have to define this function on an auxiliary set D such that $(x_i, x_j) \in D$ for all $x_i, x_j \in S$, as was done in [2]). By a well-known determinant formula (see e.g. [2, Theorem 4.2]) we now have

$$\det(S)_{\frac{1}{N^\alpha}} = \Psi_{S,\frac{1}{N^\alpha}}(x_1)\Psi_{S,\frac{1}{N^\alpha}}(x_2)\cdots\Psi_{S,\frac{1}{N^\alpha}}(x_n). \tag{2.2}$$

Thus we may conclude the following result.

Proposition 2.1. *The matrices $[S]_{N^\alpha}$ and $(S)_{\frac{1}{N^\alpha}}$ are both invertible if and only if $\Psi_{S,\frac{1}{N^\alpha}}(x_i) \neq 0$ for all $i = 1, \ldots, n$.*

Remark 2.2. Proposition 2.1 shows that the Möbius function plays a crucial role in invertibility of power LCM and GCD matrices of GCD closed sets. For material on the Möbius function we refer to [1,13,16].

Remark 2.3. If $(L, \preceq)$ is a poset and we are interested in the values $\mu_L(x, z)$ of the Möbius function, where $z \in [\![x, y]\!] \subseteq L$, then the recursive formula for the Möbius function implies that $\mu_L(x, z) = \mu_{[\![x,y]\!]}(x, z)$. Throughout this paper we make use of this simple fact.

Remark 2.4. By applying [2, Theorem 4.2] we can also write

$$\Psi_{S,\frac{1}{N^\alpha}}(x_i) = \sum_{\substack{z \,|\, x_i \\ z \nmid x_j \text{ for } j<i}} \sum_{w \,|\, z} \frac{1}{w^\alpha}\mu\left(\frac{z}{w}\right) = \sum_{\substack{z \,|\, x_i \\ z \nmid x_j \text{ for } j<i}} \left(\frac{1}{N^\alpha} * \mu\right)(z),$$

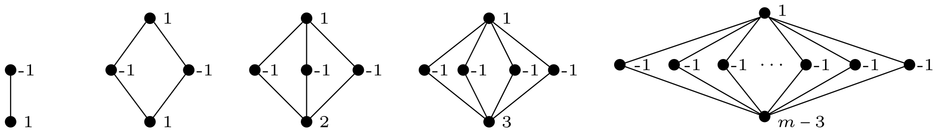
where μ is the number-theoretic Möbius function and $*$ is the Dirichlet convolution. Therefore $\Psi_{S,\frac{1}{N^\alpha}}(x_i)$ is equal to α_i (or $\alpha_i(x_1, \ldots, x_k)$), which appears in many texts by Bourque and Ligh and Hong (see e.g. [3] and [8]), but in this paper we only use a different method for calculating it.

Finally we need the following proposition.

Proposition 2.2. *Let $T = \{t_1, \ldots, t_m\}$ be any subset of S with $t_i \,|\, t_j \Rightarrow i \leq j$. If the poset $(T, |)$ belongs to one of the classes presented in Fig. 1, then*

$$\sum_{i=1}^{m} \frac{a_i}{t_i} > 0,$$

where the a_i's are the coefficients found in Fig. 1 next to each element.



(a) $\mathcal{S}_{1,2}$ (b) $\mathcal{S}_{2,4}$ (c) $\mathcal{S}_{3,5}$ (d) $\mathcal{S}_{4,6}$ (e) $\mathcal{S}_{m-2,m}$

Fig. 1. The lattice classes of Proposition 2.2. The coefficients a_i are next to each element.

Proof. Consider first Fig. 1(a). Now $(T, |) \in \mathcal{S}_{1,2}$ and $t_1 \,|\, t_2$, and thus clearly $\frac{1}{t_1} - \frac{1}{t_2} > 0$. Consider next Figs. 1(b)–1(e). Then $m \geq 4$, $t_1 \,|\, t_2, \ldots, t_{m-1}$ and $t_2, \ldots, t_{m-1} \,|\, t_m$. In this case

$$\frac{m-3}{t_1} + \frac{1}{t_m} - \sum_{k=2}^{m-1} \frac{1}{t_k} = \frac{1}{t_m} + \frac{1}{t_1}\left((m-3) - \sum_{k=2}^{m-1} \frac{t_1}{t_k}\right) > 0,$$

since

$$\sum_{k=2}^{m-1} \frac{t_1}{t_k} < \sum_{k=2}^{m-1} \frac{1}{2} = \frac{m-2}{2} \leq m-3. \qquad \square$$

3. On the zeros of the Möbius function of a meet semilattice

Before we can begin our study of singular LCM matrices we need to prove the following lemma, which tells us something important about the zeros of the Möbius function of a finite meet semilattice.

Lemma 3.1. *Let $(L, \preceq)$ be a finite meet semilattice, $x \in L$ and $C_L(x) = \{y \in L \mid x \gtrdot y\}$. Denote $\xi_L(x) = \bigwedge C_L(x)$ if $C_L(x) \neq \emptyset$ and $\xi_L(x) = x$ if $C_L(x) = \emptyset$. If*

$$z \notin [\![\xi_L(x), x]\!] := \{w \in L \mid \xi_L(x) \preceq w \preceq x\},$$

then $\mu_L(z, x) = 0$.

Proof. If $C_L(x) = \emptyset$, then $x = \min L$ and we have $\xi_L(x) = x$. Trivially $\mu_L(z, x) = 0$ for all $z \notin [\![x, x]\!]$, so we may assume that $C_L(x) \neq \emptyset$. Let m denote the number of elements in $C_L(x)$ $(m \geq 1)$. Suppose that $z \notin [\![\xi_L(x), x]\!]$. Clearly the claim is true if $z \not\preceq x$, so we may assume that $z \preceq x$. Let $\eta(z)$ denote the number of elements $y \in C_L(x)$ such that $z \prec y$, and let $y_1, y_2, \ldots, y_{\eta(z)}$ be these elements (thus $C_{[\![z,x]\!]}(x) = \{y_1, y_2, \ldots, y_{\eta(z)}\}$). In addition, $\xi_{[\![z,x]\!]}(x) = y_1 \wedge y_2 \wedge \cdots \wedge y_{\eta(z)}$ (clearly $\xi_{[\![z,x]\!]}(x) \in [\![\xi_L(x), x]\!]$). We apply double induction: first induction on the size of $C_L(x)$ and then induction on the size of the interval $[\![z, \xi_{[\![z,x]\!]}(x)[\![$.

Our base case is that the set $C_L(x)$ has one element, i.e. $m = 1$. Suppose first that there is only one element on the interval $[\![z, \xi_{[\![z,x]\!]}(x)[\![= [\![z, y_1[\![$. This element has to be z itself. In this case the interval $[\![z, x]\!]$ is equal to the chain $z \prec y_1 \prec x$, and clearly

$$\mu_L(z,x) = -\sum_{z \prec v \preceq x} \mu_L(v,x) = -(\mu_L(y_1,x) + \mu_L(x,x)) = -(-1+1) = 0.$$

Next we consider the case $m = 1$ and there are more than one elements on the interval $[\![z, \xi_{[\![z,x]\!]}(x)[\![= [\![z, y_1[\![= [\![z, \xi_L(x)[\![$. Here our secondary induction hypothesis is that if $u \notin [\![\xi_L(x), x[\![$, $m = 1$ and there are less than $k(\geq 2)$ elements on the interval $[\![u, \xi_L(x)[\![$, then $\mu_L(u,x) = 0$. Suppose that there are k elements on the interval $[\![z, \xi_L(x)[\![$. Since in this case $]\!]z, x]\!] =]\!]z, \xi_L(x)[\![\cup [\![\xi_L(x), x]\!]$, we have

$$\mu_L(z,x) = -\left(\sum_{z \prec v \prec \xi_{[\![z,x]\!]}(x)} \overbrace{\mu_L(v,x)}^{\substack{=0 \text{ by induction} \\ \text{hypothesis}}}\right) - \left(\overbrace{\sum_{\xi_{[\![z,x]\!]}(x) \preceq v \preceq x} \mu_L(v,x)}^{\substack{=\delta_L(\xi_{[\![z,x]\!]}(x),x)=0, \\ \text{since } \xi_{[\![z,x]\!]}(x) \neq x}}\right) = -0 - 0 = 0.$$

Thus our base case is complete.

Now let $m > 1$. Our primary induction hypothesis is that for all semilattices L in which x covers less than m elements we have $\mu_L(u,x) = 0$ for all $u \notin [\![\xi_L(x), x]\!]$.

Suppose first that $z \notin [\![\xi_L(x), x]\!]$ is fixed and $\eta(z) < m$. When calculating the value $\mu_L(z,x)$ we may restrict ourselves to the meet semilattice $[\![z, x]\!]$. In this structure z precedes and x covers less than m of the elements of $C_L(x)$. Thus our induction hypothesis implies that $\mu_L(z,x) = \mu_{[\![z,x]\!]}(z,x)$ can be nonzero only if $z \in [\![\xi_{[\![z,x]\!]}(x), x]\!] \subset [\![\xi_L(x), x]\!]$. Thus the claim is true for all z with $\eta(z) < m$.

Suppose then that $z \notin [\![\xi_L(x), x]\!]$ and $\eta(z) = m$. We aim to prove that $\mu_L(z,x) = 0$ by applying the formula

$$\mu_L(z,x) = -\sum_{z \prec v \preceq x} \mu_L(v,x).$$

Since in this case z is a lower bound for all the elements y_i, we must have $z \preceq \xi_L(x)$. When calculating the value $\mu_L(z,x)$ we may omit all elements $v \succ z$ such that $\mu_L(v,x) = 0$. If $\eta(v) < m$, then by the work done above we know that all the elements v with nonzero Möbius function value are located on the interval $[\![\xi_{[\![v,x]\!]}(x), x]\!] \subseteq [\![\xi_L(x), x]\!]$. All the remaining elements $v \succ z$ have $\eta(v) = m$, and therefore $v \in]\!]z, \xi_L(x)]\!]$. Thus

$$\mu_L(z,x) = -\left(\sum_{z \prec v \preceq x} \mu_L(v,x)\right) = -\left(\sum_{z \prec v \prec \xi_L(x)} \mu_L(v,x)\right) - \left(\sum_{\xi_L(x) \preceq v \preceq x} \mu_L(v,x)\right).$$

Next we use induction on the size of the interval $[\![z, \xi_L(x)[\![$ and show that $\mu_L(z,x) = 0$. If there is only one element on this interval, then

$$\mu_L(z,x) = -\left(\sum_{z\prec v\preceq x} \mu_L(v,x)\right) = -\left(\sum_{\xi_L(x)\preceq v\preceq x} \mu_L(v,x)\right) = \delta_L(\xi_L(x),x) = 0,$$

since $\xi_L(x) \neq x$. Our induction hypothesis is that if there are less than $k(\geq 2)$ elements on the interval $[\![v, \xi_L(x)[\![$, then $\mu_L(v,x) = 0$. Suppose that there are k elements on the interval $[\![z, \xi_L(x)[\![$. Now

$$\mu_L(z,x) = -\left(\sum_{z\prec v\prec \xi_{[\![z,x]\!]}(x)} \overbrace{\mu_L(v,x)}^{\substack{=0 \text{ by induction} \\ \text{hypothesis}}}\right) - \left(\overbrace{\sum_{\xi_{[\![z,x]\!]}(x)\preceq u\preceq x} \mu_L(u,x)}^{=\delta_L(\xi_L(x),x)=0}\right) = -0 - 0 = 0.$$

Thus our proof is complete. □

It is also possible to prove a stronger version of Lemma 3.1. For our purposes the original formation is mostly sufficient, but in the proof of Theorem 4.1 this stronger version is needed as well.

Lemma 3.2. *If $\mu_L(z,x) \neq 0$, then z has to be the meet of all those elements y, which are covered by x and located on the interval $[\![z,x]\!]$.*

Proof. Suppose that $\mu_L(z,x) \neq 0$. By using the same notations as in the proof of Lemma 3.1 we trivially have $z \preceq \xi_{[\![z,x]\!]}(x)$. On the other hand, Lemma 3.1 implies that $\xi_{[\![z,x]\!]}(x) \preceq z \preceq x$. Thus we must have $\xi_{[\![z,x]\!]}(x) = z$. □

4. Singularity of the usual LCM matrices

It has been known for a long time that the smallest GCD closed set S for which the LCM matrix $[S]$ is singular has 8 elements. However, the uniqueness of the structure of such a set has not been considered earlier. The next theorem addresses this.

Theorem 4.1. *If S is a GCD closed set with 8 elements and the LCM matrix $[S] = [[x_i, x_j]]$ is singular, then the semilattice $(S, |)$ always belongs to the class 8_J in Fig. 2.*

Proof. Suppose that S is a GCD closed set with 8 elements and its LCM matrix $[S]$ is singular. Thus $S_1 = \{x_1\}, S_2 = \{x_1, x_2\}, \ldots, S_8 = S$. Since $S_1, \ldots, S_7$ are all meet semilattices with less than 8 elements and all LCM matrices are invertible up to size 7×7, we know that $\Psi_{S_i,\frac{1}{N}}(x_i) = \Psi_{S,\frac{1}{N}}(x_i) \neq 0$ for all $i = 1, \ldots, 7$. Since the matrix $[S]$ is singular, we must have $\Psi_{S,\frac{1}{N}}(x_8) = 0$.

Next we should note that the last added element x_8 must cover at least three elements. Otherwise Lemma 3.2 would imply that the set of all elements $x_i \in S$ with $\mu_S(x_i, x_8) \neq 0$ belongs to either of the classes $\mathcal{S}_{1,2}$ or $\mathcal{S}_{2,4}$ in Fig. 1. In the first case we

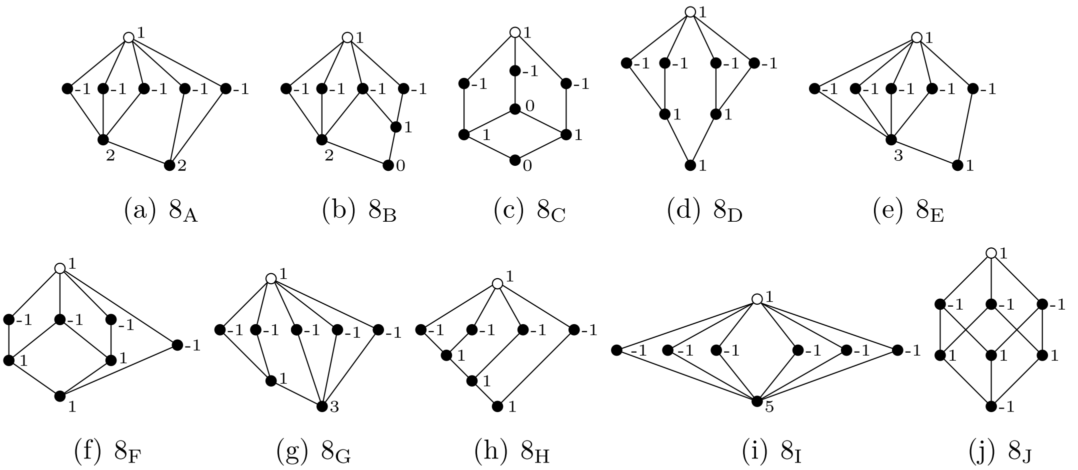


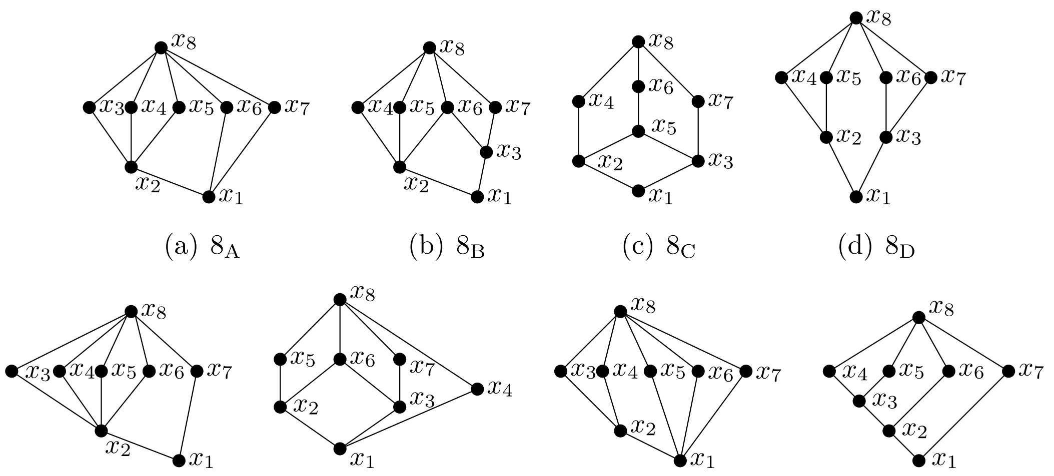
Fig. 2. The Hasse-diagrams of the meet semilattices in the proof of Theorem 4.1. For every semilattice the number next to each element is the value $\mu_S(x_i, x_8)$, where x_8 is the last added element and is denoted by the white dot.

have $\Psi_{S,\frac{1}{N}}(x_8) = \frac{1}{x_8} - \frac{1}{x_k} < 0$, where x_k is the element covered by x_8. In the second case $\Psi_{S,\frac{1}{N}}(x_8) > 0$ by Proposition 2.2. Furthermore, from this we deduce that in the Hasse diagram of $(S, |)$ every maximal element has to cover at least three elements. If this is not the case and there is a maximal element that covers at most two elements, then the set S can be constructed so that x_8 is this element. As above we obtain that $\Psi_{S,\frac{1}{N}}(x_8) \neq 0$, which is a contradiction.

There are 1078 meet semilattices with 8 elements, but the condition that every maximal element needs to cover at least three elements reduces the number of possibilities to 84 (Remark 4.1 contains the details on how the desired list of meet semilattices is obtained). By taking into account the possible zeros of the Möbius function μ_S we are able to rule out even more structures, namely those for which there exists $x_i \in S$ such that $\mu_S(x_i, x_8) = 0$, x_i covers at most one element and is covered by exactly one. Suppose for a contradiction that there exists such element x_i in S. Then $S \setminus \{x_i\}$ is a meet semilattice with 7 elements (the ordering of $S \setminus \{x_i\}$ is induced by the ordering of S), $\mu_{S\setminus\{x_i\}}(x_k, x_8) = \mu_S(x_k, x_8)$ for all $x_k \in S \setminus \{x_i\}$ and therefore

$$\Psi_{S,\frac{1}{N}}(x_8) = \Psi_{S\setminus\{x_i\},\frac{1}{N}}(x_8) \neq 0.$$

Again this means that the matrix $[S]$ is invertible, which is a contradiction. Thus S cannot contain this type of element x_i. This leaves us with the ten possible structures $8_A, \ldots, 8_J$ presented in Fig. 2. By the work by Hong [6] we already know that S may belong to the class 8_J. We only need to show that S cannot be any of the remaining types $8_A, 8_B, \ldots, 8_I$. It suffices to prove that $\Psi_{S,\frac{1}{N}}(x_8) \neq 0$ whenever $(S, |)$ belongs to any of these classes. From Proposition 2.2 we obtain directly that $\Psi_{S,\frac{1}{N}}(x_8) > 0$ when $S \in 8_I$. In order to reduce the remaining cases to this same proposition we first need to



(a) 8_A (b) 8_B (c) 8_C (d) 8_D

(e) 8_E (f) 8_F (g) 8_G (h) 8_H

Fig. 3. The numbering of elements of S in the cases when S belongs to classes $8_\mathrm{A}, 8_\mathrm{B}, \ldots, 8_\mathrm{H}$.

divide the set S into suitable blocks. Fig. 3 shows the indexing of the elements of S in each case.

(i) Let $(S, |) \in 8_\mathrm{A}$. Then $\{x_2, x_3, x_4, x_5, x_8\} \in \mathcal{S}_{3,5}$, $\{x_1, x_6\}, \{x_1, x_7\} \in \mathcal{S}_{1,2}$ and

$$\begin{aligned}\Psi_{S,\frac{1}{N}}(x_8) &= \frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} - \frac{1}{x_3} + \frac{2}{x_2} + \frac{2}{x_1} \\ &= \underbrace{\left(\frac{1}{x_8} - \frac{1}{x_5} - \frac{1}{x_4} - \frac{1}{x_3} + \frac{2}{x_2}\right)}_{>0} + \underbrace{\left(\frac{1}{x_1} - \frac{1}{x_7}\right)}_{>0} + \underbrace{\left(\frac{1}{x_1} - \frac{1}{x_6}\right)}_{>0} > 0.\end{aligned}$$

(ii) Let $(S, |) \in 8_\mathrm{B}$. Then $\{x_2, x_4, x_5, x_6, x_8\} \in \mathcal{S}_{3,5}$, $\{x_3, x_7\} \in \mathcal{S}_{1,2}$ and

$$\begin{aligned}\Psi_{S,\frac{1}{N}}(x_8) &= \frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} + \frac{1}{x_3} + \frac{2}{x_2} \\ &= \underbrace{\left(\frac{1}{x_8} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} + \frac{2}{x_2}\right)}_{>0} + \underbrace{\left(\frac{1}{x_3} - \frac{1}{x_7}\right)}_{>0} > 0.\end{aligned}$$

(iii) Let $(S, |) \in 8_\mathrm{C}$. Then $\{x_2, x_4, x_6, x_8\} \in \mathcal{S}_{2,4}$, $\{x_3, x_7\} \in \mathcal{S}_{1,2}$ and

$$\begin{aligned}\Psi_{S,\frac{1}{N}}(x_8) &= \frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_4} + \frac{1}{x_3} + \frac{1}{x_2} \\ &= \underbrace{\left(\frac{1}{x_8} - \frac{1}{x_6} - \frac{1}{x_4} + \frac{1}{x_2}\right)}_{>0} + \underbrace{\left(\frac{1}{x_3} - \frac{1}{x_7}\right)}_{>0} > 0.\end{aligned}$$

(iv) Let $(S,|) \in 8_{\mathrm{D}}$. Then $\{x_2,x_4,x_5,x_8\} \in \mathcal{S}_{2,4}$, $\{x_3,x_6\},\{x_1,x_7\} \in \mathcal{S}_{1,2}$ and

$$\Psi_{S,\frac{1}{N}}(x_8) = \frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} + \frac{1}{x_3} + \frac{1}{x_2} + \frac{1}{x_1}$$
$$= \underbrace{\left(\frac{1}{x_8} - \frac{1}{x_5} - \frac{1}{x_4} + \frac{1}{x_2}\right)}_{>0} + \underbrace{\left(\frac{1}{x_3} - \frac{1}{x_6}\right)}_{>0} + \underbrace{\left(\frac{1}{x_1} - \frac{1}{x_7}\right)}_{>0} > 0.$$

(v) Let $(S,|) \in 8_{\mathrm{E}}$. Then $\{x_2,x_3,x_4,x_5,x_6,x_8\} \in \mathcal{S}_{4,6}$, $\{x_1,x_7\} \in \mathcal{S}_{1,2}$ and

$$\Psi_{S,\frac{1}{N}}(x_8) = \frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} - \frac{1}{x_3} + \frac{3}{x_2} + \frac{1}{x_1}$$
$$= \underbrace{\left(\frac{1}{x_8} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} - \frac{1}{x_3} + \frac{3}{x_2}\right)}_{>0} + \underbrace{\left(\frac{1}{x_1} - \frac{1}{x_7}\right)}_{>0} > 0.$$

(vi) Let $(S,|) \in 8_{\mathrm{F}}$. Then $\{x_2,x_5,x_6,x_8\} \in \mathcal{S}_{2,4}$, $\{x_3,x_7\},\{x_1,x_4\} \in \mathcal{S}_{1,2}$ and

$$\Psi_{S,\frac{1}{N}}(x_8) = \frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} + \frac{1}{x_3} + \frac{1}{x_2} + \frac{1}{x_1}$$
$$= \underbrace{\left(\frac{1}{x_8} - \frac{1}{x_6} - \frac{1}{x_5} + \frac{1}{x_2}\right)}_{>0} + \underbrace{\left(\frac{1}{x_3} - \frac{1}{x_7}\right)}_{>0} + \underbrace{\left(\frac{1}{x_1} - \frac{1}{x_4}\right)}_{>0} > 0.$$

(vii) Let $(S,|) \in 8_{\mathrm{G}}$. Then $\{x_1,x_5,x_6,x_7,x_8\} \in \mathcal{S}_{3,5}$, $\{x_2,x_3\},\{x_1,x_4\} \in \mathcal{S}_{1,2}$ and

$$\Psi_{S,\frac{1}{N}}(x_8) = \frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} - \frac{1}{x_3} + \frac{1}{x_2} + \frac{3}{x_1}$$
$$= \underbrace{\left(\frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} + \frac{2}{x_1}\right)}_{>0} + \underbrace{\left(\frac{1}{x_2} - \frac{1}{x_3}\right)}_{>0} + \underbrace{\left(\frac{1}{x_1} - \frac{1}{x_4}\right)}_{>0} > 0.$$

(viii) Let $(S,|) \in 8_{\mathrm{H}}$. Then $\{x_3,x_4,x_5,x_8\} \in \mathcal{S}_{2,4}$, $\{x_2,x_6\},\{x_1,x_7\} \in \mathcal{S}_{1,2}$ and

$$\Psi_{S,\frac{1}{N}}(x_8) = \frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} - \frac{1}{x_4} + \frac{1}{x_3} + \frac{1}{x_2} + \frac{1}{x_1}$$
$$= \underbrace{\left(\frac{1}{x_8} - \frac{1}{x_5} - \frac{1}{x_4} + \frac{1}{x_3}\right)}_{>0} + \underbrace{\left(\frac{1}{x_2} - \frac{1}{x_6}\right)}_{>0} + \underbrace{\left(\frac{1}{x_1} - \frac{1}{x_7}\right)}_{>0} > 0.$$

Thus we have shown that $(S,|)$ must belong to class 8_{J} and our proof is complete. □

Remark 4.1. Since there are 1078 meet semilattices with 8 elements, one of the main challenges in the proof of Theorem 4.1 is to find all suitable meet semilattices and rule out the rest. We used Sage 6.1.1 (see [17]) in order to accomplish this. First we define the set of all meet semilattices with 8 elements by using the command

```
L8=[p for p in Posets(8) if p.is_meet_semilattice()] .
```

The command

```
L8=[l for l in L8 if not any (
 len(l.lower_covers(m))<3
 for  m in l.maximal_elements()  )]
```

then rules out all such semilattices in which some maximal element covers less than three elements. After that the command

```
L8=[l for l in L8 if not any (
 len(l.lower_covers(e))<=1 and
 len(l.upper_covers(e))==1 and
 l.mobius_function(e,7)==0
 for e in l.list()  )]
```

makes sure that in the remaining semilattices there are no elements x_i such that $\mu_S(x_i, x_8) = 0$, x_i covers at most one other element and is covered by only one. Now the command

```
for l in L8: l.show()
```

shows the Hasse diagrams of the meet semilattices $8_{\mathrm{A}}, \ldots, 8_{\mathrm{J}}$ in question.

It is easy to see that if S is an odd GCD closed set with at most 8 elements, then the LCM matrix $[S]$ is always nonsingular (an odd set is a set whose all elements are odd). The only possibility to obtain a singular LCM matrix $[S]$ would be $(S, |) \in 8_{\mathrm{J}}$, but this is impossible, since in this case

$$\Psi_{S,\frac{1}{N}}(x_8) = \frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5} + \frac{1}{x_4} + \frac{1}{x_3} + \frac{1}{x_2} - \frac{1}{x_1}$$
$$= \underbrace{\frac{1}{x_1}\left(-1 + \underbrace{\frac{x_1}{x_2}}_{\leq\frac{1}{3}} + \underbrace{\frac{x_1}{x_3}}_{\leq\frac{1}{5}} + \underbrace{\frac{x_1}{x_4}}_{\leq\frac{1}{7}}\right)}_{<0} + \underbrace{\left(\frac{1}{x_8} - \frac{1}{x_7} - \frac{1}{x_6} - \frac{1}{x_5}\right)}_{<0} < 0.$$

Hong [8] took the idea of nonsingularity of LCM matrices of odd GCD closed sets even further by presenting the following conjecture:

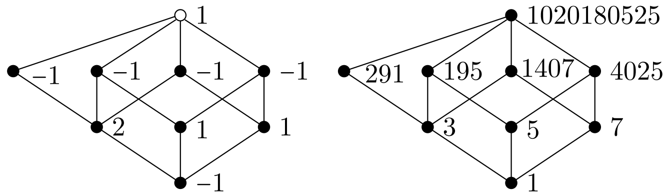


Fig. 4. The Hasse diagram of the counterexample of Theorem 4.2. The left figure shows the values $\mu_S(x_i, x_9)$, the right shows the respective elements of S.

Conjecture 4.1. *(See [8, Conjecture 4.4].) The LCM matrix $[S]$ defined on any odd GCD closed set S is nonsingular.*

However, this conjecture fails already when $n = 9$.

Theorem 4.2. *Conjecture 4.1 is false.*

Proof. Let us consider the odd set

$$\begin{aligned} S &= \{1, 3, 5, 7, 195, 291, 1407, 4025, 1\,020\,180\,525\} \\ &= \{1, 3, 5, 7, 3 \cdot 5 \cdot 13, 3 \cdot 97, 3 \cdot 7 \cdot 67, 5^2 \cdot 7 \cdot 23, 3 \cdot 5^2 \cdot 7 \cdot 13 \cdot 23 \cdot 67 \cdot 97\}. \end{aligned}$$

After calculating the values of the Möbius function (see Fig. 4) we may apply (2.1) to obtain

$$\Psi_{S,\frac{1}{N}}(1\,020\,180\,525) = \frac{1}{1\,020\,180\,525} - \frac{1}{4025} - \frac{1}{1407} - \frac{1}{291} - \frac{1}{195} + \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - 1$$

$$= \frac{1}{1\,020\,180\,525}\big(1 - 253\,461 - 725\,075 - 3\,505\,775 - 5\,231\,695 + 145\,740\,075 + 204\,036\,105 + 680\,120\,350 - 1\,020\,180\,525\big) = 0,$$

and thus it follows from Proposition 2.1 that the matrix $[S]$ is singular. □

Remark 4.2. The counterexample given in the proof of Theorem 4.2 was found by analyzing GCD closed sets S of nine elements possessing the structure presented in Fig. 4.

A positive integer x is said to be *a singular number* if there exists a GCD closed set $S = \{x_1, \ldots, x_n\}$, where $1 \leq x_1 < \cdots < x_n = x$, such that $\Psi_{S,\frac{1}{N}}(x) = 0$. Otherwise x is a *nonsingular number*. Moreover, x is a *primitive singular number* if x is singular and x' is nonsingular number for all $x' \,|\, x$, $x' \neq x$.

Hong [8] conjectured that there are infinitely many even primitive singular numbers. He has also presented the following conjecture about odd primitive singular numbers.

Conjecture 4.2. *(See [8, Conjecture 4.3].) There does not exist an odd primitive singular number.*

(a) Ex. 5.1 (b) Ex. 5.2

Fig. 5. The Hasse diagrams of the semilattices in Examples 5.1 and 5.2.

The counterexample found in the proof of Theorem 4.2 also implies that this second conjecture is false.

Corollary 4.1. *There exists an odd primitive singular number.*

Proof. By the proof of Theorem 4.2 we know that 1 020 180 525 is an odd singular number. If it is not primitive singular number itself, then it has a nontrivial factor which is an odd primitive singular number. $\square$

5. Lattice-theoretic approach to singularity of power LCM matrices with real exponent

So far we have only been studying the singularity of the usual LCM matrices. Next we consider singularity of power LCM matrices from lattice-theoretic viewpoint. The one thing that we can be sure of is that it is difficult to find singular power LCM matrices in which the exponent is an integer greater than 1. Thus it is only natural to ask how this situation changes when the exponent is allowed to be any positive *real* number. It turns out that in some cases already the semilattice structure of $(S, |)$ tells a lot about the singularity of power LCM matrices of S. We begin our study with two illustrative examples.

Example 5.1. Let $L = \{z_1, z_2, \dots, z_n\}$ be a chain with $z_1 \prec z_2 \prec \cdots \prec z_n$ (see Fig. 5(a)), let α be any positive real number and let S be any set of positive integers such that $(S, |) \cong (L, \preceq)$. Then by (2.1) we get

$$\Psi_{S,\frac{1}{N^\alpha}}(x_1) = \frac{\mu_S(x_1, x_1)}{x_1^\alpha} = \frac{1}{x_1^\alpha} > 0,$$

and for $1 < i \le n$ we have

$$\Psi_{S,\frac{1}{N^\alpha}}(x_i) = \frac{1}{x_i^\alpha} - \frac{1}{x_{i-1}^\alpha} < 0.$$

Thus the power LCM matrix $[S]_{N^\alpha} = [\mathrm{lcm}(x_i, x_j)^\alpha]$ is invertible for all $\alpha > 0$.

Example 5.2. Let $(L, \preceq)$ be the four element meet semilattice presented in Fig. 5(b). Suppose that $S = \{x_1, x_2, x_3, x_4\} = \{1, 3, 5, 45\}$. Clearly $(S, |) \cong (L, \preceq)$. Let α be any positive real number. Applying (2.1) we obtain

$$\Psi_{S,\frac{1}{N^\alpha}}(1) = 1, \quad \Psi_{S,\frac{1}{N^\alpha}}(3) = \frac{1}{3^\alpha} - 1 \quad \text{and} \quad \Psi_{S,\frac{1}{N^\alpha}}(5) = \frac{1}{5^\alpha} - 1,$$

which are all nonzero for all $\alpha > 0$. However,

$$\Psi_{S,\frac{1}{N^\alpha}}(45) = \frac{1}{45^\alpha} - \frac{1}{5^\alpha} - \frac{1}{3^\alpha} + 1,$$

which is negative for $\alpha = \frac{1}{4}$ and positive for $\alpha = 1$. Since $\Psi_{S,\frac{1}{N^\alpha}}(45)$ is a continuous function of α, this function must have zero value for some positive α_0 (this α_0 is located approximately at 0.328594). It now follows from Proposition 2.1 that the power LCM matrix $[S]_{N^{\alpha_0}} = [[x_i, x_j]^{\alpha_0}]$ is singular. This shows that our structure $(L, \preceq)$ does not possess the same property as chains were proven to have in our previous example.

Although we just found one set S that yields a singular power LCM matrix for some positive real number α, not every set of positive integers isomorphic to $(L, \preceq)$ has this property. To see this we only need to choose $S' = \{x'_1, x'_2, x'_3, x'_4\} = \{1, 3, 5, 15\}$. In this case we have

$$\Psi_{S',\frac{1}{N^\alpha}}(x'_i) = \Psi_{S,\frac{1}{N^\alpha}}(x_i) \neq 0 \quad \text{for all } \alpha > 0 \text{ and for all } i = 1, 2, 3,$$

but also

$$\Psi_{S',\frac{1}{N^\alpha}}(15) = \frac{1}{15^\alpha} - \frac{1}{5^\alpha} - \frac{1}{3^\alpha} + 1 = \frac{1}{15^\alpha}(5^\alpha - 1)(3^\alpha - 1) \neq 0$$

for all $\alpha > 0$. This means that the power LCM matrix $[S']_{N^\alpha} = [[x'_i, x'_j]^\alpha]$ is nonsingular for all $\alpha > 0$.

As we saw in Example 5.1, sometimes the lattice-theoretic structure of $(S, |)$ alone tells us that the power LCM matrix of the set S is invertible for all $\alpha > 0$. On the other hand, Example 5.2 shows that in the remaining cases the information about the structure of $(S, |)$ is inconclusive and does not reveal whether or not all the power LCM matrices of the set S are invertible. In this section our ultimate goal is to characterize all possible meet semilattices $(L, \preceq)$, whose structure is strong enough to guarantee the invertibility of the power LCM matrix for all GCD closed set $(S, |) \cong (L, \preceq)$ and for all $\alpha > 0$. By making use of Lemma 3.1 we are able to prove the following result, which brings us one step closer to achieving this goal.

Theorem 5.1. *Let $(L, \preceq)$ be a meet semilattice with n elements. Assume that there exist elements $x, y_1, \ldots, y_m$ $(m \geq 2)$ in L such that $y_1 \lessdot x$, $y_2 \lessdot x, \cdots, y_m \lessdot x$ and $\mu_L(y, x) > 0$, where $y = y_1 \wedge \cdots \wedge y_k$. Then there exists a set $S = \{x_1, x_2, \ldots, x_n\}$ of positive integers* *and a positive real number α_0 such that $(S,|) \cong (L,\preceq)$ and the power LCM matrix $[S]_{N^{\alpha_0}} = [[x_i,x_j]^{\alpha_0}]$ of the set S is singular.*

Proof. Let us denote $L = \{z_1,\ldots,z_n\}$, where $z_i \preceq z_j \Rightarrow i \le j$ (in particular, $z_1 = \min L$). We begin by constructing a GCD closed set $S' = \{x'_1, x'_2, \ldots, x'_n\}$ of positive integers such that $(S',|) \cong (L,\preceq)$. Let $p_2, p_3, \ldots, p_n$ be distinct prime numbers. We define $x'_1 = 1$ and

$$x'_i = p_i \mathrm{lcm}\{x'_j \,|\, j < i \text{ and } z_j \preceq z_i\} = \prod_{\substack{1<j\le i \\ z_j \preceq z_i}} p_j$$

for $1 < i \le n$. It is easy to see that the set S' is both GCD closed and isomorphic to L (every element of S' is either 1 or a square-free product of different primes).

Now suppose that $x'_i \in S'$ is an element such that it covers the elements $x'_{i_1}, x'_{i_2}, \ldots, x'_{i_m} \in S'$ and $\mu_{S'}(x'_k, x'_i) > 0$, where $x'_k = x'_{i_1} \wedge x'_{i_2} \wedge \cdots \wedge x'_{i_m}$. Let r be an arbitrary positive integer. Now let $S(r) = \{x_1, x_2, \ldots, x_n\}$, where

$$x_j = \begin{cases} x'_j & \text{if } x'_i \nmid x'_j, \\ p_i^r x'_j & \text{if } x'_i \,|\, x'_j. \end{cases}$$

Clearly $(S(r),|) \cong (S',|) \cong (L,\preceq)$.

Let i be as fixed above. Then $x_i = p_i^r x'_i$. Let r be sufficiently large (to be specified later). We define the function $h_{i,r} : \mathbb{R} \to \mathbb{R}$ by

$$h_{i,r}(\alpha) = \Psi_{S(r),\frac{1}{N^\alpha}}(x_i) = \sum_{j=1}^{i} \frac{\mu_{S(r)}(x_j,x_i)}{x_j^\alpha}.$$

By Lemma 3.1 we know that $\mu_{S(r)}(x_j,x_i) = 0$ for all $x_j \notin [\![x_k,x_i]\!]$. Thus the function $h_{i,r}$ comes to the form

$$h_{i,r}(\alpha) = \sum_{x_k \,|\, x_j \,|\, x_i} \frac{\mu_{S(r)}(x_j,x_i)}{x_j^\alpha} = \frac{1}{x_k^\alpha} \sum_{a \,|\, \frac{x_i}{x_k}} \frac{\mu_{S(r)}(ax_k,x_i)}{a^\alpha}$$
$$= \frac{1}{x_k^\alpha}\left(\mu_{S(r)}(x_k,x_i) + \sum_{1\ne a \,|\, \frac{x_i}{x_k}} \frac{\mu_{S(r)}(ax_k,x_i)}{a^\alpha}\right).$$

We are going to show that the factor on the right goes to zero for some α. Here we have

$$\lim_{\alpha\to\infty}(x_k^\alpha(h_{i,r}(\alpha)) = \mu_{S(r)}(x_k,x_i) + \lim_{\alpha\to\infty} \sum_{1\ne a \,|\, \frac{x_i}{x_k}} \underbrace{\frac{\mu_{S(r)}(ax_k,x_i)}{a^\alpha}}_{\to 0 \text{ as } \alpha\to\infty} = \mu_{S(r)}(x_k,x_i) > 0.$$

The definition of the Möbius function $\mu_{S(r)}$ implies that

$$x_k^0(h_{i,r}(0)) = \sum_{x_k \mid x_j \mid x_i} \mu_{S(r)}(x_j, x_i) = \delta_{S(r)}(x_k, x_i) = 0,$$

since $x_k \neq x_i$. In addition,

$$\frac{d(x_k^\alpha h_{i,r}(\alpha))}{d\alpha} = \sum_{1 \neq a \mid \frac{x_i}{x_k}} -\log(a)\frac{\mu_{S(r)}(ax_k, x_i)}{a^\alpha}$$

$$= \left(-\sum_{\substack{a \mid \frac{x_i}{x_k} \\ a \neq 1, \frac{x_i}{x_k}}} \log(a)\frac{\mu_{S(r)}(ax_k, x_i)}{a^\alpha}\right) - \left(r\log(p_i) + \log\left(\frac{x_i'}{x_k}\right)\right)\frac{\mu_{S(r)}(x_i, x_i)}{\left(\frac{x_i}{x_k}\right)^\alpha}.$$

Thus when the integer r is sufficiently large, we have

$$\frac{d(x_k^\alpha h_{i,r}(\alpha))}{d\alpha}(0) = \sum_{\substack{a \mid \frac{x_i}{x_k} \\ a \neq 1, \frac{x_i}{x_k}}} -\log(a)\mu_{S(r)}(ax_k, x_i)$$

$$- \left(r\log(p_i) + \log\left(\frac{x_i'}{x_k}\right)\right)\underbrace{\mu_{S(r)}(x_k, x_i)}_{>0} < 0.$$

Thus the function $x_k^\alpha h_{i,r}(\alpha)$ obtains negative values for some positive α. In addition, $x_k^\alpha h_{i,r}(\alpha)$ is continuous. Now it follows from Bolzano's Theorem that there exists $\alpha_0 \in]0, \infty[$ such that $x_k^{\alpha_0} h_{i,r}(\alpha_0) = 0$ and therefore $h_{i,r}(\alpha_0) = \Psi_{S(r),\frac{1}{N^\alpha}}(x_i) = 0$. Proposition 2.1 now implies that the matrix $[S(r)]_{N^{\alpha_0}}$ has to be singular. □

A subset S of a meet semilattice is said to be *a* $\wedge$*-tree set* if the Hasse diagram of the meet closure of S is a tree (when considered as an undirected graph). An alternative way of putting this is that every element of the meet closure of S covers at most one element of meetcl(S) (see [12, Lemma 4.1] for further characterizations). If the set S is meet closed, then S is a $\wedge$-tree set if and only if every element of S covers at most one element of S.

Now we are finally in a position to prove the following theorem, which gives us the desired classification of finite meet semilattices.

Theorem 5.2. *Let* $(L, \preceq)$ *be a meet semilattice with* n *elements, where* $L = \{z_1, z_2, \ldots, z_n\}$*. Then the following conditions are equivalent:*

1. *The LCM matrix* $([x_i, x_j]^\alpha)$ *is nonsingular for all* $\alpha > 0$ *and for all sets* $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{Z}^+$ *such that* $(S, |) \cong (L, \preceq)$.
2. L *is* $\wedge$*-tree set.*

3. *For all* $z_i, z_j \in L$

$$\mu_L(z_i, z_j) > 0 \Rightarrow z_i = z_j.$$

Proof. (1) ⇒ (2) First we assume Condition 1. Suppose for a contradiction that at least one element of L covers more than one element. Suppose that z_i is a minimal such element and let $z_{i_1}, \ldots, z_{i_k} \in L$ be the elements covered by z_i ($k \geq 2$). Let $z_r = z_{i_1} \wedge \cdots \wedge z_{i_k}$. If $\mu_L(z_r, z_i) > 0$, then Theorem 5.1 would imply that the matrix $([x_i, x_j]^\alpha)$ is singular for some $\alpha > 0$ and $S \subset \mathbb{Z}_+$, where $(S, |) \cong (L, \preceq)$. Thus we must have

$$\mu_L(z_r, z_i) = -\sum_{z_r \preceq z_j \prec z_i} \mu_L(z_r, z_j) \leq 0.$$

Let $z_{l_1}, \ldots, z_{l_m} \in [\![z_r, z_i]\!]$ be the elements that cover z_r. Here $m \geq 2$, since otherwise we would have $z_{l_1} \preceq z_{i_1}, \ldots, z_{i_k}$ and further $z_r \prec z_{l_1} \preceq z_{i_1} \wedge \cdots \wedge z_{i_k}$. We know that the terms $\mu_L(z_r, z_{l_1}), \ldots, \mu_L(z_r, z_{l_m})$ appear in the nonnegative sum

$$\begin{aligned} 0 &\leq \sum_{z_r \preceq z_j \prec z_i} \mu_L(z_r, z_j) \\ &= \mu_L(z_r, z_r) + \mu_L(z_r, z_{l_1}) + \cdots + \mu_L(z_r, z_{l_m}) + \sum_{z_j \in A} \mu_L(z_r, z_j) \\ &= 1 - m + \sum_{z_j \in A} \mu_L(z_r, z_j), \end{aligned}$$

where $A = \bigcup_{q=1}^{m}]\!]z_{l_q}, z_i[\![$. Therefore there exists $z_j \in A$ such that $\mu_L(z_r, z_j) > 0$. This means that z_j needs to cover more than one element even on the interval $[\![z_r, z_j]\!]$ (if z_j covered only one element, then by setting $L = [\![z_r, z_j]\!]$ and $x = z_j$ in Lemma 3.1 we would have $\mu_L(z_r, z_j) = 0$, since $z_r \prec \xi_L(x)$). This is a contradiction, since z_i was supposed to be a minimal element such that it covers at least two elements (and here $z_j \prec z_i$). Thus condition (2) must hold.

(2) ⇒ (3) Suppose then that Condition 2 holds. Let $z_i, z_j \in L$ with $\mu_L(z_i, z_j) > 0$. Here we must have $z_i \preceq z_j$. Since S is $\wedge$-tree set, the interval $[\![z_i, z_j]\!]$ is a chain (see [12, Lemma 4.1]). In addition, the interval $[\![z_i, z_j]\!]$ cannot have more than one element on it, since otherwise we would have $\mu_L(z_i, z_j) = -1$ (in the case when there are two elements on the interval) or $\mu_L(z_i, z_j) = 0$ (in the case when there are more than two elements on the interval). Thus Condition 3 is satisfied.

(3) ⇒ (1) For the last we assume Condition 3. Let S be any subset of positive integers such that $(S, |) \cong (L, \preceq)$. If $x_i = \min S$, then

$$\Psi_{S, \frac{1}{N^\alpha}}(x_i) = \sum_{j=1}^{i} \frac{\mu_S(x_j, x_i)}{x_j^\alpha} = \frac{1}{x_i^\alpha} > 0.$$

If $x_i \neq \min S$, then there is at least one element x_k that is covered by x_i and we obtain

$$\Psi_{S,\frac{1}{N^\alpha}}(x_i) = \sum_{j=1}^{i} \frac{\mu_S(x_j, x_i)}{x_j^\alpha} = \underbrace{\frac{1}{x_i^\alpha} - \frac{1}{x_k^\alpha}}_{<0} + \underbrace{\sum_{\substack{x_j \mid x_i \\ x_j \neq x_i, x_k}} \frac{\mu_S(x_j, x_i)}{x_j^\alpha}}_{\leq 0} < 0.$$

Thus the matrix $[S]_{N^\alpha} = ([x_i, x_j]^\alpha)$ is invertible for all $\alpha > 0$ and we have proven Condition 1. □

6. Notes on conjectures on singularity of power LCM matrices with real exponents

Besides those we have already discussed, Hong has also proposed several other conjectures on nonsingularity of power GCD and LCM matrices. At this point we are ready to take a closer look at them. Let us begin with the following two.

Conjecture 6.1. *(See [7, Conjecture 4.1].) Let $\alpha \neq 0$ and let $S = \{x_1, \ldots, x_n\}$ be an odd-gcd-closed set. Then the matrix $[[x_i, x_j]^\alpha]$ on S is nonsingular.*

Conjecture 6.2. *(See [7, Conjecture 4.5].) Let $\alpha \neq 0$ and let $S = \{x_1, \ldots, x_n\}$ be an odd-lcm-closed set. Then the matrix $[[x_i, x_j]^\alpha]$ on S is nonsingular.*

First we should note that every counterexample to Conjecture 6.1 generates a counterexample to Conjecture 6.2 (in fact these two conjectures are equivalent to each other). In order to see this we utilize a method similar to that presented in [4]. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a GCD closed set of odd positive integers such that $x_i \mid x_n$ for all $i = 1, \ldots, n$. Now let $S' = \{\frac{x_n}{x_1}, \frac{x_n}{x_2}, \ldots, \frac{x_n}{x_n}\}$. The elements of S' are clearly odd, and since $\gcd(x_i, x_j) \in S$ for all $i, j \in \{1, \ldots, n\}$, we have

$$\operatorname{lcm}\left(\frac{x_n}{x_i}, \frac{x_n}{x_j}\right) = \frac{x_n^2}{x_i x_j \gcd\left(\frac{x_n}{x_i}, \frac{x_n}{x_j}\right)} = \frac{x_n}{\gcd(x_j, x_i)} \in S'$$

for all $i, j = 1, \ldots, n$. Thus the set S' is LCM closed. Furthermore, if

$$\det[S]_{N^\alpha} = \det[[x_i, x_j]^\alpha] = \left(\prod_{k=1}^{n} x_k^{2\alpha}\right) \det\left[\frac{1}{(x_i, x_j)^\alpha}\right] = 0,$$

then this implies that the determinant on the right vanishes. Therefore

$$\det[S']_{N^\alpha} = \det\left[\left[\frac{x_n}{x_i}, \frac{x_n}{x_j}\right]^\alpha\right] = x_n^n \det\left[\frac{1}{(x_i, x_j)^\alpha}\right] = 0.$$

It turns out that the elements of S being odd has very little to do with the nonsingularity of the matrix $[[x_i, x_j]^\alpha]$. It follows already from Theorem 4.2 that Conjecture 6.1

does not hold for $\alpha = 1$. More counterexamples can be found by using the method presented in the proof of Theorem 5.1 (the elements of S can easily be chosen to be odd by assuming that $p_i \neq 2$ for all $i = 2, \dots, n$, as done in Example 5.2). This means that for each semilattice structure $(L, \preceq)$, where L is not a $\wedge$-tree set, there exist infinitely many counterexamples. Another consequence of Example 5.2 is that Theorem 1.5 in [7] cannot be improved as Hong suggests; the condition "$\epsilon < 0$ or $\epsilon \geq 1$" cannot be improved to "$\epsilon \neq 0$".

When applying Theorem 5.1 in practice the exponent α_0 (for which the matrix $[[x_i, x_j]^\alpha]$ is singular) is often located near zero. This leaves open the possibility that Conjecture 6.1 could be true when $\alpha > 1$. Unfortunately not even this assumption is enough to salvage Conjecture 6.1. This can be seen by modifying the counterexample in Theorem 4.2. Let us consider the set

$$S = \{1, 3, 5, 7, 195, 291, 1407, 4025q, 1\,020\,180\,525q\},$$

where $q > 1$ is an odd number. This set is clearly GCD closed, and thus we may define

$$\begin{aligned} h_{9,q}(\alpha) &= \Psi_{S,\frac{1}{N^\alpha}}(1\,020\,180\,525q) \\ &= \frac{1}{(1\,020\,180\,525q)^\alpha} - \frac{1}{(4025q)^\alpha} - \frac{1}{1407^\alpha} - \frac{1}{291^\alpha} - \frac{1}{195^\alpha} + \frac{1}{7^\alpha} + \frac{1}{5^\alpha} + \frac{2}{3^\alpha} - 1 \\ &= \frac{1}{q^\alpha} \underbrace{\left(\frac{1}{1\,020\,180\,525^\alpha} - \frac{1}{4025^\alpha} \right)}_{<0} - \frac{1}{1407^\alpha} - \frac{1}{291^\alpha} - \frac{1}{195^\alpha} + \frac{1}{7^\alpha} + \frac{1}{5^\alpha} + \frac{2}{3^\alpha} - 1. \end{aligned}$$

Now let $\alpha = 1$. By Example 4.2 we know that if also $q = 1$, then $h_{9,q}(1) = 0$. But since $q > 1$, $\frac{1}{q^\alpha} < 1$ and we must have $h_{9,q}(1) > 0$. Keeping in mind that $h_{9,q}(\alpha)$ is a continuous function of α and that in this case $\lim_{\alpha \to \infty} h_{9,q}(\alpha) = -1$, we now may conclude that there exists a real number $\alpha_0 > 1$ such that the matrix $[S]_{N^{\alpha_0}} = [[x_i, x_j]^{\alpha_0}]$ is singular.

Hong has also presented two conjectures which generalize the previous two conjectures even further. The fall of Conjectures 6.1 and 6.2 has interesting consequences to them. Since the function N^α is clearly both completely multiplicative and strictly monotonous, it is easy to see that both of the following two conjectures are false as well.

Conjecture 6.3. *(See [7, Conjecture 4.3].) Let $S = \{x_1, \dots, x_n\}$ be an odd-gcd-closed set and f a completely multiplicative function. If f is strictly monotonous function, then the matrix $[f[x_i, x_j]]$ is nonsingular.*

Conjecture 6.4. *(See [7, Conjecture 4.7].) Let $S = \{x_1, \dots, x_n\}$ be an odd-lcm-closed set and f a completely multiplicative function. If f is strictly monotonous function, then the matrix $[f[x_i, x_j]]$ is nonsingular.*

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