ARI VIRTANEN

Majorization and *k*-majorization as an Approach to Some Problems in Optimization and Eigenvalue Estimation

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To My Daughter Minna

Abstract

This thesis consists of three parts. In the first part, we survey relevant results on majorization and Schur-convexity and produce some auxiliary results needed later.

In the second part, we study the set $I_\downarrow^n$ of decreasingly ordered n-tuples of elements of a real interval I, under the elementwise (partial) order $\leq$ and the majorization (partial) order $\preceq$. We find the supremums and infimums of the set

$$\{\, \mathbf{x} \in I_\downarrow^n \mid \mathrm{S}(\mathbf{x}) = a,\, G(\mathbf{x}) = b \,\},$$

relative to these orders. Here G is a Schur-convex function and $\mathrm{S}(\mathbf{x})$ denotes the sum of the elements of $\mathbf{x}$. Besides the constraint $G(\mathbf{x}) = b$, we also consider the constraint $G(\mathbf{x}) \leq b$ as well as $G(\mathbf{x}) \geq b$. We tie this discussion to eigenvalue estimation.

In the third part, we generalize the majorization order to what we call k-majorization. We find the supremum and infimum relative to $\leq$ of the set $\{\, \mathbf{x} \in \mathbb{R}_\downarrow^n \mid \mathrm{S}(\mathbf{x}) = a,\, G(\mathbf{x}) = b,\, \mathbf{x} \preceq \mathbf{c} \,\}$, where $\mathbf{c} \in \mathbb{R}^n$. We consider the question about the extreme values of the function $f(x_k, x_\ell)$ in the set $\{\, \mathbf{x} \in I_\downarrow^n \mid \mathrm{S}(\mathbf{x}) = a,\, G(\mathbf{x}) = b \,\}$. Particularly, we solve the problem

$$\max\{\, x_k/x_\ell \mid \mathbf{x} \in \mathbb{R}_+^n,\, \sum_i x_i = a,\, \prod_i x_i = d \,\}.$$

An equivalent problem is the following: Let $\mathbf{A}$ be an $n \times n$-matrix with real eigenvalues. Find the best possible upper bound for the ratio of its kth and ℓth largest positive eigenvalues, using n, $\operatorname{tr} \mathbf{A}$, and $\det \mathbf{A}$.

In this part, we also characterize functions which are increasing relative to 3-majorization. As an application, we find the maximum and minimum of x_k subject to $\mathbf{x} \in I_\downarrow^n$, $\mathrm{S}(\mathbf{x}) = a$, $G(\mathbf{x}) = b$, and $F(\mathbf{x}) = c$, where F is increasing relative to 3-majorization in the set $\{\, \mathbf{x} \in I_\downarrow^n \mid \mathrm{S}(\mathbf{x}) = a,\, G(\mathbf{x}) = b \,\}$. As an example, we present the best possible bounds for the kth largest eigenvalue of $\mathbf{A}$, using besides n, $\operatorname{tr} \mathbf{A}$, and $\operatorname{tr} \mathbf{A}^2$, also either $\operatorname{tr} \mathbf{A}^3$ or, when the eigenvalues are nonnegative, $\operatorname{tr} \mathbf{A}^4$.

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Ari Virtanen

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Part I

Preliminaries

1 Introduction

1.1 Background

The initial inspiration for this research was provided by a 1980 result by Henry Wolkowicz and George P. H. Styan [29] concerning eigenvalue estimation. They presented the best possible upper and lower bounds for partial sums of the eigenvalues of a matrix $\mathbf{A}$ involving the traces $\operatorname{tr}\mathbf{A}$ and $\operatorname{tr}\mathbf{A}^2$. These bounds are the best possible, or, as we will also say, sharp in the following sense: Let $n \geq \ell \geq k \geq 1$ be given along with real numbers a and b such that $a^2 \leq nb$. Then there exists a matrix $\mathbf{A}$ with real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ satisfying the conditions $\operatorname{tr}\mathbf{A} = a$, $\operatorname{tr}\mathbf{A}^2 = b$ with the partial sum $\sum_{i=k}^{\ell} \lambda_i$ being equal with the bound in question.

Drazin and Haynsworth [6] presented as early as 1962 the same bounds for the largest and the smallest eigenvalue. In fact, several authors have produced comparable results. Jensen and Styan [8, 9] credit priorities to Laguerre, Samuelson, Brunk, Boyd, and Hawkins. Tarazaga [27] rediscovered some of the results of Wolkowicz and Styan. Also O. Rojo, Soto, and H. Rojo [25, 26] treated questions of this kind. Merikoski and Wolkowicz [22] discussed improving some of these bounds by using extra information. More recently, Merikoski and Virtanen [21] presented analogous bounds for the Perron root of a nonnegative matrix.

Merikoski, Styan, and Wolkowicz [15] gave the best possible upper bounds for the ratios of the eigenvalues of $\mathbf{A}$ involving n, $\operatorname{tr}\mathbf{A}$, and $\operatorname{tr}\mathbf{A}^2$. For the sums and ratios of the singular values of $\mathbf{A}$ (whose eigenvalues need not be real) Merikoski, Sarria, and Tarazaga [14] presented bounds using n, $\operatorname{tr}\mathbf{A}$, and $\operatorname{tr}\mathbf{A}^*\mathbf{A}$. These, however, are not the best possible. Merikoski and Virtanen [18, 19, 20] studied the problem of finding bounds for partial sums and partial products of the (nonnegative) eigenvalues of $\mathbf{A}$ involving n, $\operatorname{tr}\mathbf{A}$, and $\det\mathbf{A}$. The best possible bounds are included in [20]. Merikoski, Urpala, and Virtanen [16] found the best possible upper bound for the ratio of the largest and smallest eigenvalues. Merikoski, Urpala, Virtanen, Tam, and Uhlig [17] pursued this theme further by considering also the largest and smallest singular values.

There are some parallelisms between the case with n, $\operatorname{tr}\mathbf{A}$, $\operatorname{tr}\mathbf{A}^2$ and the one with n, $\operatorname{tr}\mathbf{A}$, $\det\mathbf{A}$. Denote $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathrm{S}(\mathbf{x}) = \sum_i x_i$, and $G(\mathbf{x}) = g(x_1) + g(x_2) + \cdots + g(x_n)$, where g is a given strictly convex function. Furthermore, let $\mathcal{X}$ be the set of decreasingly ordered vectors $\mathbf{x}$ satisfying $\mathrm{S}(\mathbf{x}) = a$ and $G(\mathbf{x}) = b$. To reach more general results, Merikoski, Pikhurko, and Virtanen [13] took the following approach: Let $\mathbf{x} \in \mathcal{X}$; seek for the best possible upper and lower bounds for $x_k + \cdots + x_\ell$ using only (k, ℓ,) n, a, and b. Kovačec, Merikoski, Pikhurko, and Virtanen [10] examined this problem from the point of view of the theory of convex functions.

1.2 Outline of the thesis

This thesis consists of three parts.

The first part is preliminary. After the introductory Chapter 1, we will survey majorization ordering and Schur-convex functions in Chapter 2.

In the second part we will focus on studying the problem stated at the end of Section 1.1. It turns out that there is an almost equivalent problem of finding $\sup \mathcal{X}$ and $\inf \mathcal{X}$, understood in the usual sense and in the sense of majorization.

In Chapter 3 we will make some observations on the set $\{\, \mathbf{x} \mid \mathrm{S}(\mathbf{x}) = a \,\}$. The main results of the second part will be presented in Chapter 4. We will repeat some of the results in [10] and [13]. However, instead of assuming G to be of the form $\sum_i g(x_i)$, we will generally assume only that G is strictly Schur-convex. In some cases obtaining results as strong as [10] and [13] requires an additional assumption of the quasiconvexity of G.

After illustrating our approach with a simple situation, we will show that the pertinent systems of equalities and inequalities have solutions. Thereafter it is relatively easy to find the desired supremums and infimums. Our proof technique is closer to [20] than to [13], even less so to [10]. Besides the equality $G(\mathbf{x}) = b$ we will also study the inequalities $G(\mathbf{x}) \le b$ and $G(\mathbf{x}) \ge b$. Some examples follow in Chapter 5.

In the third part we will consider the use of additional bounds. We will introduce in Chapter 6 a generalized majorization ordering called k-majorization. To the author's knowledge, this is a new concept (some authors have used the term 'k-majorization' in another sense to denote $\mathbf{x}_{1:k} \preceq_w \mathbf{y}_{1:k}$; see, for example, [5]). We will apply k-majorization in Chapter 7 to problems with an added lower or upper bound relative to majorization. It is also applied in Chapter 8 to study optimization over the set $\mathcal{X}$. We will especially consider the maximization of the ratio x_k/x_ℓ subject to $(x_1, x_2, \ldots, x_n) \in \mathcal{X}$. In Chapter 9 we will have as an additional bound the equality $F(\mathbf{x}) = c$, where F is an increasing function relative to the k-majorization ordering. We will find, for example, the sharp upper and lower bounds for x_k under the assumptions that $\mathrm{S}(\mathbf{x}) = a$, $x_1^2 + x_2^2 + \cdots + x_n^2 = b$, and $x_1^m + x_2^m + \cdots + x_n^m = c$, where $m \ge 3$ is an integer.

Our proof technique in the second part relies on the theory of majorization. In the third part we make use of k-majorization in an analogous way. We will not refer to Karush-Kuhn-Tucker theory, although many of our problems are in principle solvable by it. In practice, however, it typically involves calculation too complicated to carry out.

1.3 Notation

We use bold letters for vectors and corresponding non-bold letters with subscripts for their components: $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{M} = (M_1, M_2, \ldots, M_n)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, etc. If the dimension is not mentioned, we assume it is $n \geq 3$. The notations $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} < \mathbf{y}$ are understood elementwise. Particularly, $(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n)$ means that $x_i < y_i$ for $i = 1, 2, \ldots, n$. We denote the vector of the eigenvalues of an $n \times n$-matrix $\mathbf{A}$ by $\boldsymbol{\lambda}(\mathbf{A}) = (\lambda_1(\mathbf{A}), \lambda_2(\mathbf{A}), \ldots, \lambda_n(\mathbf{A}))$. If the eigenvalues are real, we assume them to be ordered decreasingly.

The notation $(\ldots, \langle a_i \rangle_{i=\ell}^{k}, \ldots)$ stands for $(\ldots, a_\ell, a_{\ell+1}, \ldots, a_k, \ldots)$. The notation $\langle c \rangle^k$ simply means that c is repeated $k \geq 0$ times; for example, $(\langle 1 \rangle^0, \langle 2 \rangle^3, \langle 3 \rangle^1) = (2, 2, 2, 3)$. We define $\mathbf{1} = \mathbf{1}_n = (\langle 1 \rangle^n)$ and $\mathbf{0} = \mathbf{0}_n = (\langle 0 \rangle^n)$. For $k = 1, 2, \ldots, n$, we denote the standard unit vector $(\langle 0 \rangle^{k-1}, 1, \langle 0 \rangle^{n-k})$ by $\mathbf{e}_k$.

Unless otherwise stated, the letters i, j, k, ℓ, m, n (also with subscripts) denote positive integers. We use notations such as $i \leq n$ and $k \leq j < n$ to mean that $i \in \{1, 2, 3, \ldots, n\}$ and $j \in \{k, k+1, k+2, \ldots, n-1\}$.

The sets of all n-tuples of real numbers, nonnegative real numbers, and positive real numbers are denoted by $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$, respectively. By default, I stands for a real interval $\mathbb{R}$, $\mathbb{R}_+$, or $[m, M]$, where $m < M$. For $\mathbf{x} \in \mathbb{R}^n$, we denote by $\mathbf{x}_\downarrow$ the decreasing rearrangement of $\mathbf{x}$. We define $\mathbb{R}^n_\downarrow$ to be the set $\{\, \mathbf{x}_\downarrow \mid \mathbf{x} \in \mathbb{R}^n \,\}$, i.e., the set of all n-tuples of real numbers ordered decreasingly.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $k \leq m \leq n$. We denote the subvector $(x_k, x_{k+1}, \ldots, x_m)$ of $\mathbf{x}$ by $\mathbf{x}_{k:m}$. Sometimes we denote the kth component of $\mathbf{x}$ by $\mathbf{x}_k$. For the (partial) sums of components of a vector $\mathbf{x}$ we use the following notations: $\mathrm{S}(\mathbf{x}) = \sum_{i=1}^{n} x_i$, $\mathrm{S}_{km}(\mathbf{x}) = \mathrm{S}(\mathbf{x}_{k:m})$. Analogously, we define the products $\mathrm{P}(\mathbf{x}) = x_1 x_2 \cdots x_n$ and $\mathrm{P}_{km}(\mathbf{x}) = \mathrm{P}(\mathbf{x}_{k:m})$. We denote the mean $\mathrm{S}(\mathbf{x}_{i:\ell})/(\ell - i + 1)$ of the components $x_i, x_{i+1}, \ldots, x_\ell$ by $\mathrm{M}_{i\ell}(\mathbf{x})$. Let $\mathbf{y} \in \mathbb{R}^n$. If $\mathrm{S}_{1k}(\mathbf{x}) \leq \mathrm{S}_{1k}(\mathbf{y})$ for all $k \leq n$, we write $\mathbf{x} \leq_\Sigma \mathbf{y}$.

We denote the sum of the powers $\sum_{i=1}^{n} x_i^r$ of the components of $\mathbf{x}$ by P_r. For the kth elementary symmetric function we use notation S_k (for the definition, see p. 29).

We usually denote subsets of $\mathbb{R}^n$ by calligraphic letters $\mathcal{A}$, $\mathcal{X}$, $\mathcal{Y}$, etc. We call the set $\mathcal{X} \subseteq \mathbb{R}^n$ *G-constant* if $G(\mathbf{x})$ is constant for all $\mathbf{x} \in \mathcal{X}$. In the special case when the function G is the sum S, we say that $\mathcal{X}$ is *a sum-constant set*.

Let $i_1, i_2, \ldots, i_m \geq 0$, $i_1 + i_2 + \cdots + i_m = n$, and let $a_1 \geq a_2 \geq \cdots \geq a_m$. We say that $\mathbf{x} \in \mathbb{R}^n$ is *of shape* $([i_1] \geq [i_2] \geq \cdots \geq [i_m])$ and write $\mathbf{x} \simeq$

($[i_1] \geq [i_2] \geq \cdots \geq [i_m]$) if

$$\mathbf{x} = \big(\langle a_1\rangle^{i_1}, \langle a_2\rangle^{i_2}, \ldots, \langle a_m\rangle^{i_m}\big),$$

i.e., if the first i_1, the next i_2, …, the last i_m components of $\mathbf{x}$ are equal and it is ordered decreasingly. If $i_k = 0$ for some $k < m$, an expression such as '$[i_k] \geq$' is taken to be an empty condition. If $i_m = 0$, we drop the last condition '$\geq [i_m]$'. We can substitute $\langle x\rangle^{i_k}$ for $[i_k]$ if we know that the relevant i_k components are equal to x. Instead of $\geq$, also $\leq$, $>$, $<$, or $=$ may appear.

Example 1. Let $\mathbf{x} = (3, 3, 3, 1, 1, 1, 0, 0)$. Besides saying that $\mathbf{x}$ is of shape ($[3] > [3] > [2]$), we can also say that $\mathbf{x}$ is of shape ($[3] \geq [3] > [1] = [1]$), ($[3] > [0] = [3] > [2] > [0]$), etc. We can also give more details and say, for example, that $\mathbf{x}$ is of shape ($\langle 3\rangle^3 > [3] > \langle 0\rangle^2$).

1.4 Partial orders and order-preserving functions

Let $\emptyset \neq \mathcal{A} \subseteq \mathbb{R}^n$ and $\ll$ be a partial order on $\mathcal{A}$, that is, a reflexive, transitive, and antisymmetric relation on $\mathcal{A}$. If the set $\mathcal{A}$ is not mentioned, we assume it is $\mathbb{R}^n$ or $\mathbb{R}^n_{\downarrow}$, depending on the context. By the notation '$\mathbf{x} \ll \mathbf{y}$ on $\mathcal{A}$' we mean that $\mathbf{x} \ll \mathbf{y}$ and $\mathbf{x}, \mathbf{y} \in \mathcal{A}$.

A function $F\colon \mathcal{A} \to \mathbb{R}$ is said to be *order-preserving relative to* $\ll$ if

$$\mathbf{x} \ll \mathbf{y} \Rightarrow F(\mathbf{x}) \leq F(\mathbf{y}).$$

In this case, we also call F *a $\ll$-increasing function* (*on the set* $\mathcal{A}$). If F is $\ll$-increasing and if, in addition, for all $\mathbf{x}, \mathbf{y} \in \mathcal{A}$,

$$F(\mathbf{x}) = F(\mathbf{y}) \Leftrightarrow (\mathbf{x} \ll \mathbf{y} \text{ and } \mathbf{y} \ll \mathbf{x}),$$

we call F *a strictly $\ll$-increasing function* (*on the set* $\mathcal{A}$).

For example, a (strictly) $\leq$-increasing function is simply (strictly) increasing. We will study a few partial orders with the corresponding concepts of increase. We begin by introducing majorization. An extensive presentation of the theory of majorization is provided by Marshall and Olkin [12].

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. If $\mathbf{x}_{\downarrow} \leq_{\Sigma} \mathbf{y}_{\downarrow}$, we say that $\mathbf{y}$ *weakly majorizes* $\mathbf{x}$ *from below*, or that $\mathbf{x}$ *is weakly majorized by* $\mathbf{y}$ *from below*, and denote $\mathbf{x} \preceq_w \mathbf{y}$. If, in addition, $\mathrm{S}(\mathbf{x}) = \mathrm{S}(\mathbf{y})$, we denote $\mathbf{x} \preceq \mathbf{y}$ and say that $\mathbf{y}$ *majorizes* $\mathbf{x}$, or that $\mathbf{x}$ *is majorized by* $\mathbf{y}$. If $\mathbf{x} \preceq \mathbf{y}$ but not $\mathbf{y} \preceq \mathbf{x}$, we say that $\mathbf{y}$ *strictly majorizes* $\mathbf{x}$, and denote $\mathbf{x} \prec \mathbf{y}$.

If $\mathrm{S}_{kn}(\mathbf{x}_{\downarrow}) \geq \mathrm{S}_{kn}(\mathbf{y}_{\downarrow})$ for all $k \leq n$, we say that $\mathbf{y}$ *weakly majorizes* $\mathbf{x}$ *from above*, or that $\mathbf{x}$ *is weakly majorized by* $\mathbf{y}$ *from above*, and denote $\mathbf{x} \preceq^w \mathbf{y}$.

It is easy to see that $\preceq$, $\preceq_w$, and $\preceq^w$ are partial orders on $\mathbb{R}^n_{\downarrow}$, but they are not antisymmetric relations on $\mathbb{R}^n$. It is obvious that

$$\mathbf{x}_{\downarrow} \leq_{\Sigma} \mathbf{y} \Rightarrow \mathbf{x} \preceq_w \mathbf{y} \Rightarrow \mathbf{x} \leq_{\Sigma} \mathbf{y}_{\downarrow}.$$

(Strictly) $\preceq$-increasing and $\succeq$-increasing functions are called (*strictly*) *Schur-convex* and *Schur-concave*, respectively. Note that F is strictly Schur-convex on a set $\mathcal{A}$ if and only if it preserves strict majorization, i.e.,

$$\mathbf{x} \prec \mathbf{y} \Rightarrow F(\mathbf{x}) < F(\mathbf{y}).$$

1.5 Supremum and infimum

Let $\ll$ be a partial order on a nonempty set $\mathcal{A} \subseteq \mathbb{R}^n$. For $\mathbf{m}, \mathbf{M} \in \mathcal{A}$ and $\emptyset \neq \mathcal{X} \subseteq \mathcal{A}$, we denote $\mathbf{m} \ll \mathcal{X}$ and $\mathcal{X} \ll \mathbf{M}$ to mean that for all $\mathbf{x} \in \mathcal{X}$, $\mathbf{m} \ll \mathbf{x}$ and $\mathbf{x} \ll \mathbf{M}$, respectively. If $\mathbf{m} \ll \mathcal{X}$ for some $\mathbf{m} \in \mathcal{A}$, we say that $\mathbf{m}$ is *a lower bound for* $\mathcal{X}$ and that $\mathcal{X}$ *is bounded below relative to the order* $\ll$ *on* $\mathcal{A}$, or briefly, that $\mathcal{X}$ *is* $\ll$*-bounded below on* $\mathcal{A}$. We define analogously the concept of being $\ll$*-bounded above*. By saying that $\mathcal{X}$ *is bounded relative to* $\ll$, or briefly, $\ll$*-bounded*, we mean that $\mathcal{X}$ is $\ll$-bounded both below and above. We omit the phrase 'relative to the order $\ll$' (and prefix $\ll$-), when $\ll$ is the usual elementwise partial order of vectors.

Later on, we will need the following result: if $\mathcal{X} \subseteq \mathbb{R}^n_{\downarrow}$ is both sum-equal and S_{1k}-equal for some $k \leq n-1$, then $\mathcal{X}$ is bounded. We state this in a more general form as follows:

Lemma 1. *Let $a \in \mathbb{R}$ and $\emptyset \neq \mathcal{X} \subseteq \mathbb{R}^n_{\downarrow}$. Assume that*

(1) *for some index $k < n$, the set $\{\, \mathrm{S}_{1k}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X} \,\}$ is bounded above, and $\mathrm{S}(\mathbf{x}) \geq a$ for all $\mathbf{x} \in \mathcal{X}$*

or

(2) *for some index k, $2 \leq k \leq n$, the set $\{\, \mathrm{S}_{kn}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X} \,\}$ is bounded below, and $\mathrm{S}(\mathbf{x}) \leq a$ for all $\mathbf{x} \in \mathcal{X}$.*

Then the set $\mathcal{X}$ is bounded.

Proof. For the first part, assume that $s \in \mathbb{R}$ is an upper bound for $\{\, \mathrm{S}_{1k}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X} \,\}$ and $\mathrm{S}(\mathbf{x}) \geq a$ whenever $\mathbf{x} \in \mathcal{X}$.

Let $\mathbf{x} \in \mathcal{X}$. Since $\mathbf{x} \in \mathbb{R}^n_{\downarrow}$, we have $\mathrm{S}_{k+1,n}(\mathbf{x}) \geq a - s$. Hence $x_k \geq x_{k+1} \geq (a-s)/(n-k)$ and $s \geq \mathrm{S}_{1k}(\mathbf{x}) \geq x_1 + (k-1)(a-s)/(n-k)$, which is equivalent to $x_1 \leq s - (k-1)(a-s)/(n-k)$. This means that $\mathcal{X}$ is bounded above. Likewise, we obtain

$$x_n \geq \mathrm{S}_{k+1,n}(\mathbf{x}) - (n-k-1)s/k \geq a - s - (n-k-1)s/k.$$

Hence $\mathcal{X}$ is bounded also below.

The second part is handled analogously.

Example 2. The set $\mathcal{X} = \{\, \mathbf{v} \in \mathbb{R}^4_{\downarrow} \mid \mathrm{S}(\mathbf{v}) = 3,\ \mathbf{v}_1 + \mathbf{v}_3 = 2 \,\}$ is not bounded, since $(x+2, x+1, -x, -x) \in \mathcal{X}$ for all $x \geq -\frac{1}{2}$. Hence in Lemma 1 it is necessary to assume that all first (or last) k components in the sum are considered.

We write $\max_{\ll} \mathcal{X} = \mathbf{M}$ and say that *the maximum of* $\mathcal{X}$ *relative to the order* $\ll$ exists (equivalently, is attained) and is $\mathbf{M}$ if $\mathcal{X} \ll \mathbf{M}$ and $\mathbf{M} \in \mathcal{X}$. If $\max_{\ll}\{\, \mathbf{d} \in \mathcal{A} \mid \mathbf{d} \ll \mathcal{X}\,\} = \mathbf{L}$, we say that *the infimum of* $\mathcal{X}$ *relative to the order* $\ll$ (*on* $\mathcal{A}$) exists and is $\mathbf{L}$. In this case we write $\inf_{\ll} \mathcal{X} = \mathbf{L}$. We define the concepts of *minimum* and *supremum* and the notations $\min_{\ll} \mathcal{X}$ and $\sup_{\ll} \mathcal{X}$ in an analogous way. By the notation 'inf' without subscript we mean the ordinary infimum of a nonempty subset of real numbers. The notations 'sup', 'min', and 'max' are used similarly.

Trivially,

$$\inf_{\leq} \mathcal{X} = \big(\inf_{\mathbf{x}\in\mathcal{X}} \mathbf{x}_1, \inf_{\mathbf{x}\in\mathcal{X}} \mathbf{x}_2, \dots, \inf_{\mathbf{x}\in\mathcal{X}} \mathbf{x}_n \big),$$

and if $\min_{\leq} \mathcal{X}$ is attained,

$$\min_{\leq} \mathcal{X} = \big(\min_{\mathbf{x}\in\mathcal{X}} \mathbf{x}_1, \min_{\mathbf{x}\in\mathcal{X}} \mathbf{x}_2, \dots, \min_{\mathbf{x}\in\mathcal{X}} \mathbf{x}_n \big).$$

Note that the minimum on the left-hand side is not necessarily attained even if all minimums on the right-hand side exist. Analogous results hold for the supremum and the maximum.

Consider the usual order of real numbers. Let $x \in \mathcal{X} \subset \mathbb{R}$. By saying that *$c$ is for x the sharp upper* (*lower*) *bound under the assumption that $x \in \mathcal{X}$*, we mean that $\sup \mathcal{X} = c$ ($\inf \mathcal{X} = c$). We can also say that c is the best possible bound for x using only the information that $x \in \mathcal{X}$. If the set $\mathcal{X}$ is closed, the sharp upper (lower) bound is, of course, $\max \mathcal{X}$ ($\min \mathcal{X}$).

Example 3. Let $\mathcal{X} = \{(2,2,2), (3,3,1), (4,2,2)\}$, and $\mathcal{A} = \mathbb{R}^3_{\downarrow}$. Then

$\inf_{\leq} \mathcal{X} = (2,2,1)$, $\min_{\leq} \mathcal{X}$ is not attained;

$\inf_{\preceq_w} \mathcal{X} = \min_{\preceq_w} \mathcal{X} = (2,2,2)$;

$\inf_{\preceq^w} \mathcal{X} = \min_{\preceq^w} \mathcal{X} = (4,2,2)$;

$\inf_{\preceq} \mathcal{X}$ and $\sup_{\preceq} \mathcal{X}$ are meaningless since $\mathcal{X}$ is not sum-equal;

$\sup_{\leq} \mathcal{X} = (4,3,2)$, $\max_{\leq} \mathcal{X}$ is not attained;

$\sup_{\preceq_w} \mathcal{X} = \max_{\preceq_w} \mathcal{X} = (4,2,2)$;

$\sup_{\preceq^w} \mathcal{X} = (2.5, 2.5, 1)$, $\max_{\preceq^w} \mathcal{X}$ is not attained.

It is easy to see that $\min_{\preceq} \mathcal{X} = \mathbf{m}$ if and only if $\min_{\preceq_w} \mathcal{X} = \min_{\preceq^w} \mathcal{X} = \mathbf{m}$. Analogous result holds for infimum, maximum and supremum.

Example 4. Example 3 shows that $\inf_{\leq}$, $\inf_{\preceq_w}$, and $\sup_{\preceq^w}$ (respectively $\sup_{\leq}$, $\sup_{\preceq_w}$, and $\inf_{\preceq^w}$) are not necessarily the same. If we replace inf and sup by min and max, respectively, the situation is different: Let $\mathcal{X} \subset \mathbb{R}^n_{\downarrow}$, $\min_{\leq} \mathcal{X} = \mathbf{m}$, and $\max_{\leq} \mathcal{X} = \mathbf{M}$. Then

$$\min_{\preceq_w} \mathcal{X} = \max_{\preceq^w} \mathcal{X} = \mathbf{m}$$

and

$$\max_{\preceq_w} \mathcal{X} = \min_{\preceq^w} \mathcal{X} = \mathbf{M}.$$

1.6 A lemma on optimization

Below we present a lemma which will be frequently applied in later sections. Let $\emptyset \neq \mathcal{X} \subseteq \mathbb{R}^n$ and $F, G\colon \mathcal{X} \to \mathbb{R}$. We assume $\mathcal{X}$, F, and G to be such that the following functions are defined:

$$m_F\colon G(\mathcal{X}) \to \mathbb{R},\ m_F(y) = \min_{\mathbf{x}\in\mathcal{X},\ G(\mathbf{x})=y} F(\mathbf{x}),$$

$$M_F\colon G(\mathcal{X}) \to \mathbb{R},\ M_F(y) = \max_{\mathbf{x}\in\mathcal{X},\ G(\mathbf{x})=y} F(\mathbf{x}).$$

Lemma 2. *Let $b \in G(\mathcal{X})$, $\mathbf{x}_* \in \mathcal{X}$, and $G(\mathbf{x}_*) = b$.*

(1) *If m_F is strictly increasing and $m_F(b) = F(\mathbf{x}_*) = c$, then*

$$\max_{\mathbf{x}\in\mathcal{X},\ F(\mathbf{x})=c} G(\mathbf{x}) = \max_{\mathbf{x}\in\mathcal{X},\ F(\mathbf{x})\leq c} G(\mathbf{x}) = G(\mathbf{x}_*) = b.$$

(2) *If M_F is strictly decreasing and $M_F(b) = F(\mathbf{x}_*) = c$, then*

$$\max_{\mathbf{x}\in\mathcal{X},\ F(\mathbf{x})=c} G(\mathbf{x}) = \max_{\mathbf{x}\in\mathcal{X},\ F(\mathbf{x})\geq c} G(\mathbf{x}) = G(\mathbf{x}_*) = b.$$

(3) *If m_F is strictly decreasing and $m_F(b) = F(\mathbf{x}_*) = c$, then*

$$\min_{\mathbf{x}\in\mathcal{X},\ F(\mathbf{x})=c} G(\mathbf{x}) = \min_{\mathbf{x}\in\mathcal{X},\ F(\mathbf{x})\leq c} G(\mathbf{x}) = G(\mathbf{x}_*) = b.$$

(4) *If M_F is strictly increasing and $M_F(b) = F(\mathbf{x}_*) = c$, then*

$$\min_{\mathbf{x}\in\mathcal{X},\ F(\mathbf{x})=c} G(\mathbf{x}) = \min_{\mathbf{x}\in\mathcal{X},\ F(\mathbf{x})\geq c} G(\mathbf{x}) = G(\mathbf{x}_*) = b.$$

Proof. Part (1). By the assumptions, $\mathbf{x}_* \in \mathcal{X}$, $G(\mathbf{x}_*) = b$, and $F(\mathbf{x}_*) = c$. Assume to the contrary that

$$\max_{\mathbf{x}\in\mathcal{X},\ F(\mathbf{x})=c} G(\mathbf{x})$$

is not attained or that this maximum is greater than $G(\mathbf{x}_*) = b$. Then it follows that $G(\mathbf{x}^*) > b$ for some $\mathbf{x}^* \in \mathcal{X}$ satisfying $F(\mathbf{x}^*) = c$. But since m_F is strictly increasing, $m_F(G(\mathbf{x}^*)) > m_F(b) = c$, and a contradiction $F(\mathbf{x}^*) \geq m_F(G(\mathbf{x}^*)) > c$ follows. The same proof also shows that the assumption $G(\mathbf{x}^*) > b$, $F(\mathbf{x}^*) \leq c$ leads to a contradiction. Hence

$$\max_{\mathbf{x}\in\mathcal{X},\ F(\mathbf{x})=c} G(\mathbf{x}) = \max_{\mathbf{x}\in\mathcal{X},\ F(\mathbf{x})\leq c} G(\mathbf{x}) = b = G(\mathbf{x}_*).$$

Part (2) follows from Part (1) by replacing F with $-F$. Parts (3) and (4) are proved similarly.

Example 5. Let $\mathcal{X} = \{\, (x_1, x_2) \mid x_1, x_2 \geq 1 \,\}$, $F\colon \mathcal{X} \to \mathbb{R}$, $F(x_1, x_2) = -x_1 x_2$, and $G\colon \mathcal{X} \to \mathbb{R}$, $G(x_1, x_2) = x_1 + x_2$. Then $G(\mathcal{X}) = [2, \infty[$ and $F(\mathcal{X}) =]-\infty, -1]$. Define the functions m_F and m_G as follows: $m_F\colon\ [2, \infty[\to \mathbb{R}$,

$$m_F(y) = \min_{\substack{x_1+x_2=y\\ x_1,x_2\geq 1}} (-x_1 x_2)$$

and $m_G\colon\ ]-\infty, -1] \to \mathbb{R}$,

$$m_G(z) = \min_{\substack{-x_1x_2=z\\ x_1,x_2\geq 1}} (x_1 + x_2).$$

Then $m_F(y) = -(y/2)^2$ and $m_G(z) = 2\sqrt{-z}$. Both m_F and m_G are strictly decreasing, and

$$m_F(y) = z \quad \text{if and only if} \quad m_G(z) = y.$$

Note that

$$F\biggl(\frac{y + \sqrt{y^2 + 4z}}{2}, \frac{y - \sqrt{y^2 + 4z}}{2}\biggr) = z$$

and

$$G\biggl(\frac{y + \sqrt{y^2 + 4z}}{2}, \frac{y - \sqrt{y^2 + 4z}}{2}\biggr) = y.$$

2 Majorization and Schur-convex functions

2.1 Some elementary majorization results

The definitions imply directly that

$$\mathrm{M}_{1n}(\mathbf{x})\mathbf{1}_n \preceq \mathbf{x},$$

$$\mathbf{x} \preceq_w \mathbf{y} \text{ if and only if } (\mathbf{x}_\downarrow)_{1:k} \preceq_w (\mathbf{y}_\downarrow)_{1:k} \text{ for all } k \leq n,$$

$$\mathbf{x} \preceq^w \mathbf{y} \text{ if and only if } (\mathbf{x}_\downarrow)_{k:n} \preceq^w (\mathbf{y}_\downarrow)_{k:n} \text{ for all } k \leq n,$$

$$\mathbf{x} \preceq \mathbf{y} \text{ if and only if } \mathbf{x} \preceq_w \mathbf{y} \text{ and } \mathbf{x} \preceq^w \mathbf{y},$$

$$\mathbf{x} \prec \mathbf{y} \text{ if and only if } \mathbf{x} \preceq \mathbf{y} \text{ and } \mathbf{x}_\downarrow \neq \mathbf{y}_\downarrow.$$

It is also easy to see that $\mathbf{x} \preceq \mathbf{y}$ if and only if $\mathrm{S}(\mathbf{x}) = \mathrm{S}(\mathbf{y})$ and there exists an index $k < n$ such that

$$(\mathbf{x}_\downarrow)_{1:k-1} \preceq_w (\mathbf{y}_\downarrow)_{1:k-1} \quad \text{and} \quad (\mathbf{x}_\downarrow)_{k+1,n} \preceq^w (\mathbf{y}_\downarrow)_{k+1,n}.$$

Here, of course, $k-1$ and $k+1$ can be replaced (jointly or separately) by k.

Additional simple consequences and equivalent conditions for majorization can be found in [12, p. 10–12]. It also includes the following lemma on combining majorizations.

Lemma 3 (see [12, p. 121, Proposition A.7]). *Let $\ll$ be $\preceq$, $\preceq_w$, or $\preceq^w$. If $\mathbf{x} \ll \mathbf{y}$ on $\mathbb{R}^n$ and $\mathbf{a} \ll \mathbf{b}$ on $\mathbb{R}^m$, then $(\mathbf{x}, \mathbf{a}) \ll (\mathbf{y}, \mathbf{b})$ on $\mathbb{R}^{n+m}$.*

By Lemma 3, to prove that

$$(\mathbf{z}, \mathbf{x}, \mathbf{u}) \preceq (\mathbf{z}, \mathbf{y}, \mathbf{u}),$$

it suffices to show that $\mathbf{x} \preceq \mathbf{y}$. Also the following simple lemma is useful in the proof of many majorization results.

Lemma 4. *Let $0 \leq k \leq n$, $\mathbf{x} = (x_1, \ldots, x_k, \langle x \rangle^{n-k}) \in \mathbb{R}^n$, and $\mathbf{y} \in \mathbb{R}^n_\downarrow$. Then $\mathbf{x} \leq_\Sigma \mathbf{y}$ if and only if $\mathbf{x}_{1:k} \leq_\Sigma \mathbf{y}_{1:k}$ and $\mathrm{S}(\mathbf{x}) \leq \mathrm{S}(\mathbf{y})$.*

Proof. The 'only if' -part is trivial. To prove the 'if'-part, assume to the contrary that $\mathbf{x}_{1:k} \leq_\Sigma \mathbf{y}_{1:k}$ and $\mathrm{S}(\mathbf{x}) \leq \mathrm{S}(\mathbf{y})$ but $\mathbf{x} \not\leq_\Sigma \mathbf{y}$. Let $\ell = \min\{\, i \mid \mathrm{S}_{1i}(\mathbf{x}) > \mathrm{S}_{1i}(\mathbf{y})\,\}$. Then $k < \ell < n$ and necessarily $x_\ell = x > y_\ell$. Since $\mathbf{y} \in \mathbb{R}^n_\downarrow$, a contradiction follows: $\mathrm{S}(\mathbf{x}) = \mathrm{S}_{1\ell}(\mathbf{x}) + (n-\ell)x > \mathrm{S}_{1\ell}(\mathbf{y}) + (n-\ell)y_\ell \geq \mathrm{S}(\mathbf{y})$.

In Lemma 4 we can substitute '$\preceq_w$' for '$\leq_\Sigma$', if $\mathbf{x} \in \mathbb{R}^n_\downarrow$. Since $\mathbf{x} \preceq^w \mathbf{y}$ if and only if $-\mathbf{x} \preceq_w -\mathbf{y}$, we also have

Lemma 5. *Let* $0 \leq k \leq n$, $\mathbf{x} = (\langle x\rangle^k, x_{k+1}, x_{k+2}, \ldots, x_n) \in \mathbb{R}^n_\downarrow$, *and* $\mathbf{y} \in \mathbb{R}^n_\downarrow$. *Then* $\mathbf{x} \preceq^w \mathbf{y}$ *if and only if* $\mathbf{x}_{k+1:n} \preceq^w \mathbf{y}_{k+1:n}$ *and* $\mathrm{S}(\mathbf{x}) \geq \mathrm{S}(\mathbf{y})$.

Example 6. Let

$$\mathbf{a} = \big(\langle a_1\rangle^{k_1}, \langle a_2\rangle^{k_2}, \ldots, \langle a_m\rangle^{k_m}\big) \in \mathbb{R}^n_\downarrow.$$

Define $s(i) = \sum_{j=1}^{i-1} k_j$ for $i = 1, \ldots, m+1$. Then $s(1) = 0$ and $s(m+1) = n$.

Assume $\mathbf{x} \in \mathbb{R}^n_\downarrow$ and

$$\sum_{j=1}^{i} k_j a_j \leq \mathrm{S}(\mathbf{x}_{1:s(i+1)}) \quad \text{for } i = 1, 2, \ldots, m.$$

By applying Lemma 4 repeatedly, we infer that $\mathbf{a} \preceq_w \mathbf{x}$.

The following lemmas and examples state some simple properties of majorization that we will use later on. Some of them are presented in a more general form than actually needed in our applications. The first lemma is trivial, but useful.

Lemma 6 (cf. [13, Lemma 2]). *Let* $x_1 \geq x_2$ *and* $y_1 \geq y_2$. *Then the following statements are equivalent:*

(1) $(\langle x_1\rangle^i, \langle x_2\rangle^j) \prec (\langle y_1\rangle^i, \langle y_2\rangle^j)$,

(2) $x_1 < y_1$ *and* $ix_1 + jx_2 = iy_1 + jy_2$,

(3) $x_2 > y_2$ *and* $ix_1 + jx_2 = iy_1 + jy_2$.

Example 7. Assume $\mathbf{x} \in \mathbb{R}^n_\downarrow$, $\mathrm{S}_{1k}(\mathbf{x}) = s_1$, and $\mathrm{S}_{k+1,n}(\mathbf{x}) = s_2$, in which case $(\langle s_1/k\rangle^k, \langle s_2/(n-k)\rangle^{n-k}) \preceq \mathbf{x}$. If $s_1' < s_1$ and $s_1'/k \geq (s_1+s_2)/n$, then $(\langle s_1'/k\rangle^k, \langle (s_1+s_2-s_1')/(n-k)\rangle^{n-k}) \prec (\langle s_1/k\rangle^k, \langle s_2/(n-k)\rangle^{n-k})$, implying

$$\big(\langle s_1'/k\rangle^k, \langle (s_1+s_2-s_1')/(n-k)\rangle^{n-k}\big) \prec \mathbf{x}.$$

Likewise, if $s_2' > s_2$ and $(s_1+s_2)/n \geq s_2'/(n-k)$, then

$$\big(\langle (s_1+s_2-s_2')/k\rangle^k, \langle s_2'/(n-k)\rangle^{n-k}\big) \prec \mathbf{x}.$$

Lemma 7. *Let m, M, and a be real numbers such that $m < a/(k+\ell) < M$. Denote*

$$\mathcal{V} = \{\, (\langle x\rangle^k, \langle y\rangle^\ell) \mid kx + \ell y = a,\ M \geq x \geq y \geq m \,\}.$$

Then $\min_{\preceq} \mathcal{V} = (a/(k+\ell))\mathbf{1}$ *and*

$$\max_{\preceq} \mathcal{V} = \begin{cases} \big(\langle M\rangle^k, \langle (a-kM)/\ell\rangle^\ell\big) & \text{if } kM + \ell m \leq a, \\ \big(\langle (a-\ell m)/k\rangle^k, \langle m\rangle^\ell\big) & \text{if } kM + \ell m > a. \end{cases}$$

Proof. The claim about the minimum is trivial.

By Lemma 6,

$$(\langle x\rangle^k, \langle y\rangle^\ell) \prec (\langle \tilde{x}\rangle^k, \langle \tilde{y}\rangle^\ell) \text{ on } \mathcal{V}$$

if and only if $x < \tilde{x}$, or, equivalently, $y > \tilde{y}$.

Now

$$\big(\langle M\rangle^k, \langle (a-kM)/\ell\rangle^\ell\big) \in \mathcal{V}$$

if and only if $M \geq (a-kM)/\ell \geq m$. Under the assumption that $a/(k+\ell) < M$, this is equivalent to $kM + \ell m \leq a$. Similarly,

$$\big(\langle (a-\ell m)/k\rangle^k, \langle m\rangle^\ell\big) \in \mathcal{V}$$

if and only if $kM + \ell m \geq a$. The lemma follows from this.

Lemma 8. *Let $\mathbf{x} \in \mathbb{R}^n_\downarrow$, and let $\mathbf{a}$ and $s(i)$ be as in Example 6.*

Assume that $\mathrm{S}(\mathbf{x}) = \mathrm{S}(\mathbf{a})$ and that there is $\ell \leq m$ such that $k_\ell = 1$,

$$\mathbf{x}_{s(i)+1} \leq a_i \quad \text{for } i = 1, 2, \ldots, \ell-1,$$

and

$$\mathbf{x}_{s(i+1)} \geq a_i \quad \text{for } i = \ell+1, \ell+2, \ldots, m.$$

Then $\mathbf{x} \preceq \mathbf{a}$.

Proof. From the assumptions it follows that

$$\mathbf{x}_{1:s(\ell)} \preceq_w \mathbf{a}_{1:s(\ell)}$$

and

$$\mathbf{x}_{s(\ell)+2:n} \preceq^w \mathbf{a}_{s(\ell)+2:n}.$$

This implies that $\mathbf{x} \preceq \mathbf{a}$, since $\mathbf{x} = \mathbf{x}_\downarrow$ and $\mathrm{S}(\mathbf{x}) = \mathrm{S}(\mathbf{a})$.

Example 8. Assume $a \geq b$, $c \geq d$, and $(n-1)a + b = c + (n-1)d = \mathrm{S}(\mathbf{x})$. Then $\mathbf{x} \preceq (\langle a\rangle^{n-1}, b)$ if and only if $\mathbf{x}_1 \leq a$, and $\mathbf{x} \preceq (c, \langle d\rangle^{n-1})$ if and only if $\mathbf{x}_n \geq d$.

Example 9. Let $k < \ell$, $a \geq b$, $c \geq d$, and $ka + (n-k)b = \ell c + (n-\ell)d$. Assume $a < c$. Then $b > d$ and, further,

$$\big(\langle a\rangle^k, \langle b\rangle^{n-k}\big) \prec \big(\langle c\rangle^\ell, \langle d\rangle^{n-\ell}\big).$$

Similarly, if $b < d$, then $a > c$ and

$$\big(\langle c\rangle^\ell, \langle d\rangle^{n-\ell}\big) \prec \big(\langle a\rangle^k, \langle b\rangle^{n-k}\big).$$

If G is strictly Schur-convex and $G(\mathbf{x}) = G(\mathbf{y})$, then $\mathbf{x} \not\prec \mathbf{y}$ and $\mathbf{y} \not\prec \mathbf{x}$. Hence we have

Theorem 9. *Let G be strictly Schur-convex, $k < \ell$, and $\mathcal{G} = \{\, \mathbf{v} \in \mathbb{R}^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a,\, G(\mathbf{v}) = b \,\}$. If $(\langle u_1\rangle^k, \langle v_1\rangle^{n-k})$ and $(\langle u_2\rangle^\ell, \langle v_2\rangle^{n-\ell})$ belong to $\mathcal{G}$, then $u_1 \geq u_2$ and $v_1 \geq v_2$.*

2.2 Two constructions of vectors

Let $\mathbf{x} \in \mathbb{R}^n_\downarrow$. Example 7 shows that if $s < \mathrm{S}_{1k}(\mathbf{x})$ and $s/k \geq \mathrm{S}(\mathbf{x})/n$, then there exists $\mathbf{y} \in \mathbb{R}^n_\downarrow$ such that $\mathbf{y} \prec \mathbf{x}$ and $\mathrm{S}_{1k}(\mathbf{y}) = s$. The next lemma is related:

Lemma 10. *Let $a, s \in \mathbb{R}$, $k < n$, $\mathbf{x} \in I^n_\downarrow$, $\mathrm{S}(\mathbf{x}) = a$, and $\mathrm{S}_{1k}(\mathbf{x}) = s$. Moreover, let $ka/n < s < s' \leq \min\{kM, a-(n-k)m\}$ and*

$$\mathcal{Y} = \{\, \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a,\, \mathrm{S}_{1k}(\mathbf{v}) = s' \,\}.$$

Then $\min_\preceq\{\, \mathbf{v} \in \mathcal{Y} \mid \mathbf{x} \prec \mathbf{v} \,\}$ is attained.

Proof. We may assume that $I = [m, M]$ (if $I = \mathbb{R}$ or $I = \mathbb{R}_+$, then some inequalities below are trivial). Let $\ell_1 = \min\{\, i \mid 0 \leq i \leq k-1,\, \mathrm{S}_{1i}(\mathbf{x}) + (k-i)x_{i+1} < s' \,\}$, and

$$\ell_2 = \max\{\, i \mid 1 \leq i \leq n-k,\, i x_{k+i} + \mathrm{S}_{k+i+1,n}(\mathbf{x}) > a - s' \,\}.$$

Clearly, ℓ_1 and ℓ_2 are well-defined. Let $\alpha = (s' - \mathrm{S}_{1\ell_1}(\mathbf{x}))/(k-\ell_1)$ and $\beta = (a - s' - \mathrm{S}_{k+\ell_2+1,n}(\mathbf{x}))/\ell_2$. Define the vector $\mathbf{y}$ as follows:

$$y_i = \begin{cases} x_i & \text{if } i \leq \ell_1 \text{ or } i \geq k+\ell_2+1, \\ \alpha & \text{if } \ell_1 + 1 \leq i \leq k, \\ \beta & \text{if } k+1 \leq i \leq k+\ell_2. \end{cases}$$

If $\ell_1 > 0$, then $x_{\ell_1} \geq \alpha > x_{\ell_1+1}$. If $\ell_1 = 0$, then $\alpha > x_1$, and, since $s' \leq kM$, we have $M \geq \alpha$. If $\ell_2 < n-k$, then $x_{k+\ell_2} > \beta \geq x_{k+\ell_2+1}$. If $\ell_2 = n-k$, then $x_n > \beta$, and, since $s' \leq a-(n-k)m$, we have $\beta \geq m$.

Hence $\mathbf{y} \in I^n_\downarrow$. Further, α and β are chosen so that $\mathrm{S}_{1k}(\mathbf{y}) = s'$ and $\mathrm{S}(\mathbf{y}) = a$. Therefore $\mathbf{y} \in \mathcal{Y}$. Trivially, $\mathbf{x}_{1:k} \preceq_w \mathbf{y}_{1:k}$ and $\mathbf{x}_{k+1:n} \preceq^w \mathbf{y}_{k+1:n}$. Hence $\mathbf{x} \preceq \mathbf{y}$.

Finally, assume $\mathbf{z} \in \mathcal{Y}$ and $\mathbf{x} \prec \mathbf{z}$. Since $\mathbf{y}_{1:\ell_1} = \mathbf{x}_{1:\ell_1} \leq_\Sigma \mathbf{z}_{1:\ell_1}$ and $\mathrm{S}_{1k}(\mathbf{y}) = s' = \mathrm{S}_{1k}(\mathbf{z})$, it follows from Lemma 4 that $\mathbf{y}_{1:k} \leq_\Sigma \mathbf{z}_{1:k}$. Now $\mathbf{y}_{k+\ell_2+1:n} = \mathbf{x}_{k+\ell_2+1:n} \preceq^w \mathbf{z}_{k+\ell_2+1:n}$. Hence $\mathrm{S}_{1,k+\ell_2}(\mathbf{y}) \leq \mathrm{S}_{1,k+\ell_2}(\mathbf{z})$, and reapplying Lemma 4 we have $\mathbf{y}_{1:k+\ell_2} \leq_\Sigma \mathbf{z}_{1:k+\ell_2}$. Since $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n_\downarrow$, it follows that $\mathbf{y}_{1:k+\ell_2} \preceq_w \mathbf{z}_{1:k+\ell_2}$, and, further, that $\mathbf{y} \preceq \mathbf{z}$. So we have proved that $\mathbf{y} = \min_\preceq \{\, \mathbf{v} \in \mathcal{Y} \mid \mathbf{x} \prec \mathbf{v} \,\}$.

Example 10. Let $\mathbf{x} = (10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$, $k = 3$, and $s' = 30 > \mathrm{S}_{1k}(\mathbf{x}) = 27$. Then $\mathbf{y}$ in the proof of Lemma 10 is $(10, 10, 10, 5, 5, 5, 4, 3, 2, 1)$.

Lemma 11. *Let $\mathbf{x} \in \mathbb{R}^n_\downarrow$, $\mathbf{x}_{1:\ell} > \mathbf{0}$, $k < \ell \leq n$, and $x_k/x_\ell > R \geq 1$. Then there exists $\mathbf{y} \in \mathbb{R}^n_\downarrow$ such that $y_k/y_\ell = R$ and $\mathbf{y} \prec \mathbf{x}$.*

Proof. Denote $\alpha_i = \mathrm{M}_{i\ell}(\mathbf{x})$, i.e., α_i is the mean of the components $x_i, \ldots, x_\ell$. By the assumptions, $\alpha_i > 0$ for $i \leq \ell$. We divide the proof into two parts.

First, assume $x_k/\alpha_{k+1} > R$. Choose $\beta = (\ell - k)(x_k - R\alpha_{k+1})/(\ell - k + R)$ and define the vector $\mathbf{y}$ as follows:

$$y_i = \begin{cases} x_i & \text{if } i \leq k-1, \\ x_k - \beta & \text{if } i = k, \\ \alpha_{k+1} + \beta/(\ell - k) & \text{if } k+1 \leq i \leq \ell, \\ x_i & \text{if } \ell + 1 \leq i \leq n. \end{cases}$$

Then $\mathrm{S}(\mathbf{y}) = \mathrm{S}(\mathbf{x})$ and

$$y_k/y_\ell = \frac{x_k - \beta}{\alpha_{k+1} + \beta/(\ell - k)} = R.$$

It follows from $\mathbf{x} = \mathbf{x}_\downarrow$, $\beta > 0$, and $\alpha_{k+1} \geq x_\ell$ that $x_1 \geq y_1 \geq \cdots \geq y_k$ and $y_\ell \geq y_{\ell+1} \geq \cdots \geq y_n \geq x_n$. Further, since $R \geq 1$, we have $y_k \geq y_{k+1} = \cdots = y_\ell$. Hence $\mathbf{y}$ is ordered decreasingly. As in Example 6, we infer $\mathbf{y} \preceq \mathbf{x}$, which in this case means that $\mathbf{y} \prec \mathbf{x}$.

Second, assume $x_k/\alpha_{k+1} \leq R$. Let $m = \max\{\, i \mid k+1 \leq i \leq \ell - 1, x_k/\alpha_i \leq R \,\}$. Then $x_k/\alpha_{m+1} > R$. Choose $\beta = (\ell - m)(x_k/R - \alpha_{m+1})$ (> 0) and define the vector $\mathbf{y}$ as follows:

$$y_i = \begin{cases} x_i & \text{if } i \leq m-1, \\ x_m - \beta & \text{if } i = m, \\ \alpha_{m+1} + \beta/(\ell - m) & \text{if } m+1 \leq i \leq \ell, \\ x_i & \text{if } \ell + 1 \leq i \leq n. \end{cases}$$

Also in this case $\mathrm{S}(\mathbf{y}) = \mathrm{S}(\mathbf{x})$, $x_1 \geq y_1 \geq y_n \geq x_n$, and

$$y_k/y_\ell = \frac{x_k}{\alpha_{m+1} + \beta/(\ell - m)} = R.$$

Obviously, $\mathbf{y} \leq_\Sigma \mathbf{x}$, so to finish our proof it suffices to point out that $\mathbf{y} = \mathbf{y}_\downarrow$, which follows from the inequality

$$\begin{aligned}\alpha_{m+1} + \beta/(\ell - m) + \beta &= (\ell - m + 1)x_k/R - (\ell - m)\alpha_{m+1} \\ &\leq (\ell - m + 1)\alpha_m - (\ell - m)\alpha_{m+1} \\ &= \mathrm{S}_{m\ell}(\mathbf{x}) - \mathrm{S}_{m+1,\ell}(\mathbf{x}) = x_m.\end{aligned}$$

The vector $\mathbf{y}$ constructed in Lemma 11 is maximal in the sense that if $\mathbf{y} \prec \mathbf{z} \preceq \mathbf{x}$ and $\mathbf{z} = \mathbf{z}_\downarrow$, then $\mathbf{z}_k/\mathbf{z}_\ell > R$. Our next example shows that there can be several vectors satisfying the conditions stated in Lemma 11.

Example 11. Let $\mathbf{x} = (7, 6, 3, 2, 1)$, $k = 2$, $\ell = 4$, and $R = 2 < 3 = x_2/x_4$. Then $\mathbf{y}$ constructed in the proof of Lemma 11 is $\left(7, \frac{11}{2}, \frac{11}{4}, \frac{11}{4}, 1\right)$. If $\mathbf{y} \preceq \mathbf{z} \preceq \mathbf{x}$, then $z_2 \geq 11/2$, $z_4 \leq 11/4$, and $z_2 = 11/2$, $z_4 = 11/4$ only if $\mathbf{z} = \mathbf{y}$. Hence if $\mathbf{y} \prec \mathbf{z}$, then $z_2/z_4 > 2$.

Nevertheless, even the vector $\mathbf{y}(t) = \left(7, \frac{11}{2} - 2t, \frac{11}{4} + 3t, \frac{11}{4} - t, 1\right)$ satisfies the conditions stated in Lemma 11 for all $t \in \left[0, \frac{11}{20}\right]$. Note that $\mathbf{y}(t)$ and $\mathbf{y}(t')$ are mutually incomparable relative to majorization when $t \neq t'$.

2.3 An averaging process

Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n_\downarrow$. If $\mathbf{x}$ is not ordered decreasingly, the inequality $\mathbf{x} \leq_\Sigma \mathbf{y}$ does not necessarily imply that $\mathbf{x} \preceq_w \mathbf{y}$. We will show that by averaging the components of $\mathbf{x}$ in a suitable way we obtain a vector which is majorized by $\mathbf{y}$.

Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \neq \mathbf{z}_\downarrow$. Assume $\mathbf{y} \in \mathbb{R}^n_\downarrow$ and $\mathbf{z} \leq_\Sigma \mathbf{y}$. Define the vector $\mathbf{z}^\Downarrow = (z'_1, z'_2, \ldots, z'_n)$ as follows (cf. [13, the proof of Proposition 1]): Choose indices $k < m \leq n$ such that

$$z_1 \geq \cdots \geq z_{k-1} > z_k = \cdots = z_K < z_{K+1} \leq \cdots \leq z_m > z_{m+1}.$$

(If $k = 1$, omit the first inequality, and if $m = n$, omit the last inequality.) Let $M = \mathrm{M}_{km}(\mathbf{z})$ and

$$z'_i = \begin{cases} M & \text{if } k \leq i \leq m, \\ z_i & \text{otherwise.} \end{cases}$$

If $i \leq k - 1$ or $i \geq m$, then $\mathrm{S}_{1i}(\mathbf{z}^\Downarrow) = \mathrm{S}_{1i}(\mathbf{z}) \leq \mathrm{S}_{1i}(\mathbf{y})$. Since

$$\mathrm{S}_{1m}(\mathbf{z}^\Downarrow) = \mathrm{S}_{1,k-1}(\mathbf{z}) + (m - k + 1)M = \mathrm{S}_{1m}(\mathbf{z}) \leq \mathrm{S}_{1m}(\mathbf{y}),$$

by applying Lemma 4, we obtain $\mathbf{z}^\Downarrow \leq_\Sigma \mathbf{y}$. Further, since $\mathrm{S}_{k\ell}(\mathbf{z}) \leq (\ell - k + 1)M$ for ℓ such that $k \leq \ell \leq m$, we have $\mathbf{z} \leq_\Sigma \mathbf{z}^\Downarrow$.

Now assume $\mathbf{x} \leq_\Sigma \mathbf{y}$. Define the vector $\mathbf{x}_\Downarrow$ as follows: let $\mathbf{z}_0 = \mathbf{x}$ and, for $\ell \geq 0$, let $\mathbf{z}_{\ell+1} = \mathbf{z}_\ell^\Downarrow$ if $\mathbf{z}_\ell$ is not ordered decreasingly; otherwise let $\mathbf{x}_\Downarrow = \mathbf{z}_\ell$. Clearly we reach $\mathbf{x}_\Downarrow$ by finitely many steps.

By using induction on ℓ, we infer that $\mathbf{z}_{\ell-1} \leq_\Sigma \mathbf{z}_\ell \leq_\Sigma \mathbf{y}$ for all relevant values of ℓ. Since $\mathbf{x}_\Downarrow \in \mathbb{R}^n_\downarrow$, we have proved

Lemma 12. *If $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^n_\downarrow$, and $\mathbf{x} \leq_\Sigma \mathbf{y}$, then $\mathbf{x} \leq_\Sigma \mathbf{x}_\Downarrow \preceq_w \mathbf{y}$.*

Example 12. Let $\mathbf{x} = (0, 50, 40, 34, 30)$. Then

$$\begin{aligned} \mathbf{z}_0 = \mathbf{x} \ &= (0, 50, 40, 34, 30), \\ \mathbf{z}_1 = \mathbf{z}_0^\Downarrow &= (25, 25, 40, 34, 30), \\ \mathbf{z}_2 = \mathbf{z}_1^\Downarrow &= (30, 30, 30, 34, 30), \\ \mathbf{z}_3 = \mathbf{z}_2^\Downarrow &= (31, 31, 31, 31, 30). \end{aligned}$$

Therefore $\mathbf{x}_\Downarrow = \mathbf{z}_3 = (31, 31, 31, 31, 30)$.

2.4 Schur-convex functions

We cite in this section some of the definitions and results of Schur-convexity presented in [12]. Let $\mathcal{A} \subseteq \mathbb{R}^n$, where $n \geq 1$, be a convex set, that is, let $\mathcal{A}$ satisfy the condition

$$\mathbf{x}, \mathbf{y} \in \mathcal{A},\ \lambda \in (0,1) \Rightarrow \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in \mathcal{A}.$$

A function $G\colon \mathcal{A} \to \mathbb{R}$ is (strictly) convex on $\mathcal{A}$ if

$$G(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})\ (<) \leq \lambda G(\mathbf{x}) + (1-\lambda)G(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{A}$, $\mathbf{x} \neq \mathbf{y}$, and $\lambda \in (0,1)$. A function G is (strictly) concave if $-G$ is (strictly) convex.

Let $\mathcal{A}$ be *a symmetric set* so that $\mathbf{x} \in \mathcal{A} \Rightarrow \mathbf{xP} \in \mathcal{A}$ for all permutation matrices $\mathbf{P}$. The function $G\colon \mathcal{A} \to \mathbb{R}$ is *symmetric* if $G(\mathbf{x}) = G(\mathbf{xP})$ for all permutation matrices $\mathbf{P}$. A Schur-convex function defined on a symmetric set is necessarily symmetric.

Theorem 13 (see [12, p. 67–68, Theorems C.2 and C.2.c]). *Let $\mathcal{A}$ be a symmetric set and let $G\colon \mathcal{A} \to \mathbb{R}$ be symmetric and convex. Then G is Schur-convex. If, in addition, G is strictly convex on any sum-constant convex subset of $\mathcal{A}$, then G is strictly Schur-convex.*

Note that if $G\colon \mathcal{A} \to \mathbb{R}$ is Schur-convex on a symmetric set $\mathcal{A} \subseteq \mathbb{R}^n$, its restriction to the set $\mathcal{A} \cap \mathbb{R}^n_\downarrow$ is Schur-convex on this set.

Let $g\colon I \to \mathbb{R}$. Define $G\colon I^n \to \mathbb{R}$ as follows: $G(x_1, x_2, \ldots, x_n) = \sum_i g(x_i)$. Then G is symmetric and if g is (strictly) convex on I, then G is (strictly) convex on I^n. Hence, it follows from Theorem 13 that G is (strictly) Schur-convex (see [12, p. 64]). If I is open, then g is continuous (see, e.g., [4, p. 79]), and thus also G.

It follows that if $\mathbf{x} \preceq \mathbf{y}$, then $\sum_i g(x_i) \leq \sum_i g(y_i)$ for all continuous convex functions g. The converse of this also holds, see [12, p. 108, Theorem B.1].

Example 13. Recalling $P_r(\mathbf{x}) = \sum_{i=1}^n x_i^r$ $(r \in \mathbb{R})$,

(1) if $m \geq 2$ is even, then $P_m : \mathbb{R}^n \to \mathbb{R}$ is strictly Schur-convex;

(2) if $m \geq 2$, then $P_m : \mathbb{R}_+^n \to \mathbb{R}$ is strictly Schur-convex;

(3) if $r < 0$ or $r > 1$, then $P_r : \mathbb{R}_{++}^n \to \mathbb{R}$ is strictly Schur-convex.

Let $g : I \to \mathbb{R}_{++}$. Since a positive-valued function G is Schur-convex if and only if $\log G$ is Schur-convex, the function $\mathbf{x} \mapsto \prod_i^n g(x_i)$ is (strictly) Schur-convex on I^n if and only if $\log g$ is (strictly) convex on I (see [12, p. 73, Theorem E.1]).

2.5 Quasiconvex functions

If $G : \mathcal{A} \to \mathbb{R}$ is convex, then

$(*)$ $G(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \max\{G(\mathbf{x}), G(\mathbf{y})\}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ and $\lambda \in [0, 1]$.

A function G that satisfies the condition $(*)$ is said to be *quasiconvex.* The first part of Theorem 13 remains valid if instead of convexity only quasiconvexity is assumed:

Theorem 14 (see [12, p. 69, Theorem C.3]). *A symmetric quasiconvex function is Schur-convex.*

According to the definition in [2], a function G is *strictly quasiconvex* if

$$G(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) < \max\{G(\mathbf{x}), G(\mathbf{y})\}$$

for all $\lambda \in (0, 1)$ and for all $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ with $G(\mathbf{x}) \neq G(\mathbf{y})$. This definition implies that every convex function is strictly quasiconvex. Hence strict quasiconvexity does not imply strict Schur-convexity.

Example 14. Let $G : \mathbb{R}_+^n \to \mathbb{R}$, $G(\mathbf{x}) = -\mathrm{P}(\mathbf{x})$. Then G is symmetric, quasiconvex, and, hence, Schur-convex, but not convex (see [28, p. 223, Exercise 5.4.5 and its solution p. 408–409]).

Note that the fact that the function $x \mapsto \log(x)$ is strictly concave on $\mathbb{R}_{++}$ also implies the Schur-convexity of $-\mathrm{P}$ by a continuity argument. Further, we infer directly that the function P is strictly Schur-concave on $\mathbb{R}_{++}^n$.

The product P is an elementary symmetric function of $\mathbf{x}$. *The k'th elementary symmetric function of* $\mathbf{x}$ is defined as follows:

$$S_k(\mathbf{x}) = \sum_{i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k},$$

where $0 \le k \le n$. For example, $S_0(\mathbf{x}) = 1$, $S_1(\mathbf{x}) = \mathrm{S}(\mathbf{x})$, and $S_n(\mathbf{x}) = \mathrm{P}(\mathbf{x})$. The function S_k is increasing and Schur-concave on $\mathbb{R}^n_+$. The functions S_2, S_3, …, S_n are strictly Schur-concave on $\mathbb{R}^n_{++}$. This is proved directly in [12, p. 79]. We will later need the following stronger result:

Theorem 15. *The elementary symmetric functions $-S_2$, $-S_3$, …, $-S_n$ are quasiconvex on $\mathbb{R}^n_+$.*

Proof. An equivalent condition to the quasiconvexity of G is that the level set $\{\, \mathbf{x} \mid G(\mathbf{x}) \le \alpha \,\}$ is convex for all $\alpha \in \mathbb{R}$, see e.g. [12, p. 68].

Let $2 \le k \le n$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$. Trivially, $S_k(t\mathbf{x}) = t^k S_k(\mathbf{x})$. It is also known that $S_k^{1/k}(\mathbf{x}+\mathbf{y}) \ge S_k^{1/k}(\mathbf{x}) + S_k^{1/k}(\mathbf{y})$ (see [12, p. 79–80] or [11, Theorem 2]).

Let $\lambda, \bar\lambda \ge 0$, $\lambda + \bar\lambda = 1$. Assume $-S_k(\mathbf{x}) \le \alpha$ and $-S_k(\mathbf{y}) \le \alpha$. If $\alpha \ge 0$, then trivially $-S_k(\lambda\mathbf{x} + \bar\lambda\mathbf{y}) \le \alpha$. Assume $\alpha < 0$. Then $S_k^{1/k}(\mathbf{x}) \ge (-\alpha)^{1/k}$, $S_k^{1/k}(\mathbf{y}) \ge (-\alpha)^{1/k}$, and

$$\begin{aligned} S_k^{1/k}(\lambda\mathbf{x} + \bar\lambda\mathbf{y}) &\ge S_k^{1/k}(\lambda\mathbf{x}) + S_k^{1/k}(\bar\lambda\mathbf{y}) \\ &= \lambda S_k^{1/k}(\mathbf{x}) + \bar\lambda S_k^{1/k}(\mathbf{y}) \\ &\ge (\lambda + \bar\lambda)(-\alpha)^{1/k} = (-\alpha)^{1/k}. \end{aligned}$$

Therefore $S_k(\lambda\mathbf{x} + \bar\lambda\mathbf{y}) \ge -\alpha$, i.e., $-S_k(\lambda\mathbf{x} + \bar\lambda\mathbf{y}) \le \alpha$, which concludes our proof of $-S_k$ being quasiconvex.

Example 15 (cf. [12, p. 69, Example C.3.c]). Let $n \ge 2$. Since $-S_2(\mathbf{x}) = (P_2(\mathbf{x}) - S(\mathbf{x})^2)/2$, the second elementary symmetric function is Schur-convex also on $\mathbb{R}^n$. We show that it is not quasiconvex on $\mathbb{R}^n$: Choose $\mathbf{x} = \mathbf{1}$ and $\mathbf{y} = -\mathbf{1}$; then $-S_2(\mathbf{x}) = -S_2(\mathbf{y}) = (n - n^2)/2$, but $-S_2\big(\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}\big) = -S_2(\mathbf{0}) = 0 > (n - n^2)/2$.

Part II

G-constant sets

3 Supremum and infimum of a sum-constant set

3.1 On the maximum

Let I be a closed real interval $[m, M]$. In this chapter we search for infimums and supremums of some subsets of the set $\mathcal{X} = \{\, \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a \,\}$ relative to $\leq$ or $\preceq$. We also give some lower and upper bounds relative to majorization $\preceq$. We assume throughout this chapter that $m < a/n < M$.

Trivially, $\min_\preceq \mathcal{X} = (a/n)\mathbf{1}$. The following theorem of Kemperman yields the maximum relative to majorization.

Theorem 16 (cf. [12, p. 132, Theorem C.1]). *Let* $\kappa = \min\{\, k \geq 0 \mid (k+1)M + (n-k-1)m > a \,\}$, $\theta = a - \kappa M - (n-\kappa-1)m$, *and* $\mathbf{K} = \mathbf{K}(I, n, a) = (\langle M\rangle^\kappa, \theta, \langle m\rangle^{n-\kappa-1})$. *Then* $\max_\preceq \mathcal{X} = \mathbf{K}$.

We can also write $\mathbf{K} = (\langle M\rangle^\kappa, \langle\theta\rangle^\iota, \langle m\rangle^{n-\kappa-\iota})$, where

$$\iota = \begin{cases} 1 & \text{if } \theta > m, \\ 0 & \text{if } \theta = m. \end{cases}$$

Furthermore, we stipulate that

$$\mathbf{K}(\mathbb{R}_+, n, a) = \max_\preceq\{\, \mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{x} = \mathbf{x}_\downarrow,\ S(\mathbf{x}) = a \,\},$$

i.e., $\mathbf{K}(\mathbb{R}_+, n, a) = (a, \langle 0\rangle^{n-1})$.

Using the information that $a = \kappa M + \theta + (n-\kappa-1)m$, where $M > \theta \geq m$, we easily obtain also the supremum and infimum relative to elementwise order:

$$\sup_\leq \mathcal{X} = \big(\langle M\rangle^\kappa, \langle (a-(n-k)m)/k\rangle^n_{k=\kappa+1}\big)$$

and

$$\inf_\leq \mathcal{X} = \big(\langle (a-kM)/(n-k)\rangle^\kappa_{k=0}, \langle m\rangle^{n-\kappa-1}\big).$$

Note that $\max_\leq \mathcal{X}$ and $\min_\leq \mathcal{X}$ are not generally attained.

Example 16. Let $I = [0,5]$ and $\mathcal{X} = \{\, \mathbf{v} \in I_\downarrow^6 \mid \mathrm{S}(\mathbf{v}) = 18 \,\}$. Then

$$\begin{aligned}
\mathbf{K}([0,5],6,18) &= (5,5,5,3,0,0),\\
\max\nolimits_{\preceq} \mathcal{X} &= \mathbf{K},\\
\min\nolimits_{\preceq} \mathcal{X} &= (3,3,3,3,3,3),\\
\sup\nolimits_{\leq} \mathcal{X} &= (5,5,5,18/4,18/5,3),\\
\inf\nolimits_{\leq} \mathcal{X} &= (3,13/5,2,1,0,0).
\end{aligned}$$

In the next chapter we will consider a set $\{\, \mathbf{v} \in \mathcal{X} \mid G(\mathbf{v}) = b \,\}$, where G is Schur-convex. We can now state the sufficient and necessary conditions under which a set of this kind is nonempty and infinite:

Lemma 17 (cf. [10, Theorem 7]). *Let $\mathcal{X}$ be as above, and let G be strictly Schur-convex. Denote $\mathcal{G} = \{\, \mathbf{v} \in \mathcal{X} \mid G(\mathbf{v}) = b \,\}$ and $\mathbf{K} = \mathbf{K}(I,n,a)$. Then $\mathcal{G}$ is nonempty if and only if $G((a/n)\mathbf{1}) \leq b \leq G(\mathbf{K})$. If $b = G((a/n)\mathbf{1})$, then $\mathcal{G} = \{(a/n)\mathbf{1}\}$, and if $b = G(\mathbf{K})$, then $\mathcal{G} = \{\mathbf{K}\}$. If $G((a/n)\mathbf{1}) < b < G(\mathbf{K})$, then $\mathcal{G}$ is infinite.*

3.2 On the supremum and infimum

Let $\mathcal{X} = \{\, \mathbf{v} \in I_\downarrow^n \mid \mathrm{S}(\mathbf{v}) = a \,\}$, where I is a closed interval. We have shown above that $\inf_{\leq} \mathcal{X}$, $\inf_{\preceq} \mathcal{X}$, $\sup_{\leq} \mathcal{X}$, and $\sup_{\preceq} \mathcal{X}$ are always attained. If $\emptyset \neq \mathcal{Y} \subseteq \mathcal{X}$, then the completeness of real numbers implies that $\inf_{\leq} \mathcal{Y}$ and $\sup_{\leq} \mathcal{Y}$ are attained. We cannot, however, give an immediate answer to the question about the existence of $\inf_{\preceq} \mathcal{Y}$ and $\sup_{\preceq} \mathcal{Y}$. This question can also be stated as follows: Are $\sup_{\preceq} \mathcal{Y}$ and $\inf_{\preceq} \mathcal{Y}$ attained if $\mathcal{Y}$ is bounded relative to majorization?

Now let $\mathcal{Y} \subseteq \mathbb{R}_\downarrow^n$. Even if $\min_{\preceq} \mathcal{Y}$ is attained, $\inf_{\leq} \mathcal{Y}$, for example, does not need to exist. On the other hand, if $\sup_{\preceq} \mathcal{Y} = (M_1, M_2, \ldots, M_n)$ on $\mathbb{R}_\downarrow^n$, then $\mathcal{Y} \subseteq [M_n, M_1]^n$ is a sum-constant set and it follows that $\inf_{\preceq} \mathcal{Y}$, $\inf_{\leq} \mathcal{Y}$, and $\sup_{\leq} \mathcal{Y}$ are attained.

In this section questions of this kind are looked into. Denoting $\mathbf{x}_\Sigma = (\langle \mathrm{S}_{1i}(\mathbf{x}) \rangle_{i=1}^n)$ and $\mathcal{Y}_\Sigma = \{\, \mathbf{x}_\Sigma \mid \mathbf{x} \in \mathcal{Y} \,\}$, we first present a general lemma on connections between $\sup_{\preceq} \mathcal{Y}$ and $\sup_{\leq} \mathcal{Y}_\Sigma$ as well as between $\inf_{\preceq} \mathcal{Y}$ and $\inf_{\leq} \mathcal{Y}_\Sigma$.

Lemma 18 (cf. [13, proof of Proposition 1], [10, Corollary 10], and [1, p. 62–63]). *Let $\emptyset \neq \mathcal{Y} \subseteq \mathbb{R}_\downarrow^n$. Then $\inf_{\leq} \mathcal{Y}_\Sigma$ exists on $\mathbb{R}^n$ if and only if $\inf_{\preceq_w} \mathcal{Y}$ exists on $\mathbb{R}_\downarrow^n$, and $\sup_{\leq} \mathcal{Y}_\Sigma$ exists on $\mathbb{R}^n$ if and only if $\sup_{\preceq_w} \mathcal{Y}$ exists on $\mathbb{R}_\downarrow^n$. Furthermore, if $\inf_{\leq} \mathcal{Y}_\Sigma = (m_1, m_2, \ldots, m_n) \in \mathbb{R}^n$, then*

$$\inf\nolimits_{\preceq_w} \mathcal{Y} = \big(m_1, \langle m_k - m_{k-1} \rangle_{k=2}^n\big),$$

and if $\sup_{\leq} \mathcal{Y}_\Sigma = (M_1, M_2, \ldots, M_n)$, then

$$\sup\nolimits_{\preceq_w} \mathcal{Y} = \big(M_1, \langle M_k - M_{k-1} \rangle_{k=2}^n\big)_\Downarrow.$$

Proof. If $\inf_{\preceq_w} \mathcal{Y}$ ($\sup_{\preceq_w} \mathcal{Y}$) is attained, then the set $\mathcal{Y}_\Sigma$ is bounded below (above), and thus $\inf_{\leq} \mathcal{Y}_\Sigma$ ($\sup_{\leq} \mathcal{Y}_\Sigma$) is attained.

Next, assume $\inf_{\leq} \mathcal{Y}_\Sigma = (m_1, m_2, \ldots, m_n) \in \mathbb{R}^n$. Denote $m_0 = 0$. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, where $v_i = m_i - m_{i-1}$ for $i \leq n$. Following Bapat in [1], we infer that $\inf_{\preceq_w} \mathcal{Y} = \mathbf{v}$.

Finally, assume $\sup_{\leq} \mathcal{Y}_\Sigma = (M_1, M_2, \ldots, M_n) \in \mathbb{R}^n$. Let $u_1 = M_1$, $u_i = M_i - M_{i-1}$ for $i = 2, 3, \ldots, n$, and $\mathbf{u} = (u_1, u_2, \ldots, u_n)$. Since $\mathbf{x} \leq_\Sigma \mathbf{u}$ and, by Lemma 12, $\mathbf{u} \leq_\Sigma \mathbf{u}_\Downarrow$, we have $\mathbf{x} \leq_\Sigma \mathbf{u}_\Downarrow$. Since $\mathbf{x} \in \mathbb{R}^n_\downarrow$, this implies that $\mathbf{x} \preceq_w \mathbf{u}_\Downarrow$.

If $\mathbf{x} \preceq_w \mathbf{y}$ for all $\mathbf{x} \in \mathcal{Y}$, then $M_i \leq \mathrm{S}_{1i}(\mathbf{y})$ for $i \leq n$, and hence $\mathbf{u} \leq_\Sigma \mathbf{y}$. It follows from Lemma 12 that $\mathbf{u}_\Downarrow \preceq_w \mathbf{y}$. Thus we have shown that $\sup_{\preceq_w} \mathcal{Y} = \mathbf{u}_\Downarrow$.

Example 17 (cf. [13, proof of Proposition 1]). Let $\mathcal{Y} = \{(12, 2, 2, 2), (6, 6, 6, 0)\}$. Then $\inf_{\leq} \mathcal{Y}_\Sigma = (6, 12, 16, 18)$ and $\inf_{\preceq_w} \mathcal{Y} = \inf_{\preceq} \mathcal{Y} = (6, 6, 4, 2)$. But $\sup_{\leq} \mathcal{Y}_\Sigma = (12, 14, 18, 18)$ and $(12, 14-12, 18-14, 18-18) \notin \mathbb{R}^4_\downarrow$. We have $\sup_{\preceq_w} \mathcal{Y} = \sup_{\preceq} \mathcal{Y} = (12, 2, 4, 0)_\Downarrow = (12, 3, 3, 0)$.

Assume that $\mathcal{Y} \subseteq \{\, \mathbf{v} \in \mathbb{R}^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a \,\}$ for some $a \in \mathbb{R}$. If $\mathcal{Y}$ is $\leq$-bounded below, then $\{\, \mathbf{x}_n \mid \mathbf{x} \in \mathcal{Y} \,\} \geq m$ for some $m \in \mathbb{R}$. The latter statement implies that $\{\, \mathbf{x}_1 \mid \mathbf{x} \in \mathcal{Y} \,\} \leq a - (n-1)m$, which means that $\mathcal{Y}$ is $\leq$-bounded above.

If $\mathcal{Y}$ is $\leq$-bounded above, then $\sup_{\leq} \mathcal{Y}_\Sigma$ is attained. Since $\mathcal{Y}$ is sum-constant, it follows from Lemma 18 that $\sup_{\preceq} \mathcal{Y}$ is attained.

If $\sup_{\preceq} \mathcal{Y}$ is attained, then $\mathcal{Y}$ is $\preceq$-bounded above. This implies that $\{\, \mathbf{x}_n \mid \mathbf{x} \in \mathcal{Y} \,\} \geq M$ for some $M \in \mathbb{R}$, that is, $\mathcal{Y}$ is $\leq$-bounded below.

Since $\mathcal{Y}$ is $\leq$-bounded below (above) if and only if $\inf_{\leq} \mathcal{Y}$ ($\sup_{\leq} \mathcal{Y}$) exists, we have proved

Theorem 19 (cf. [10, p. 758]). *If $\mathcal{Y} \subseteq \mathbb{R}^n_\downarrow$ is a sum-constant set, then the following conditions are equivalent:*

$\mathcal{Y}$ is $\leq$-bounded below,

$\mathcal{Y}$ is $\leq$-bounded above,

$\mathcal{Y}$ is $\preceq$-bounded above,

$\inf_{\leq} \mathcal{Y}$ *is attained,*

$\sup_{\leq} \mathcal{Y}$ *is attained,*

$\sup_{\preceq} \mathcal{Y}$ *is attained.*

Corollary 20. *If $\mathcal{Y}$ is $\preceq$-bounded, then* $\inf_{\leq} \mathcal{Y}$, $\sup_{\leq} \mathcal{Y}$, $\inf_{\preceq} \mathcal{Y}$, *and* $\sup_{\preceq} \mathcal{Y}$ *are attained.*

If $\mathcal{Y}$ is $\preceq^w$-bounded above, then $\mathcal{Y}$ is $\leq$-bounded below and both $\inf_{\leq} \mathcal{Y}$ and, by Lemma 18, $\inf_{\preceq_w} \mathcal{Y}$ are attained. The following example considers the case in which $\mathcal{Y}$ is $\preceq^w$-bounded below.

Example 18. Let $\mathcal{Y} = \{ (-x,-y) \mid 0 \le x \le y \} \cup \{ (x,-x) \mid 0 \le x \}$. Then $(0,0) \preceq^w \mathcal{Y}$, but $\inf_{\le} \mathcal{Y}$, $\sup_{\le} \mathcal{Y}$, $\inf_{\preceq_w} \mathcal{Y}$, and $\sup_{\preceq_w} \mathcal{Y}$ are not attained.

3.3 A lemma on supremums and infimums

In the next chapter we will determine the supremum and infimum of the set $\{ \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a,\ G(\mathbf{v}) = b \}$. They are derived from the solutions of certain equations. The results in this section will help us decide whether such solutions exist.

Let $\mathbf{K} = \mathbf{K}(I,n,a) = (\langle M\rangle^\kappa, \langle\theta\rangle^\iota, \langle m\rangle^{n-\kappa-\iota})$, where $I = [m,M]$. Denote

$$\mathbf{E}(k,\ell) = \mathbf{E}(k,\ell;I,n,a) = \left(\langle M\rangle^k, \left\langle\frac{a-kM-\ell m}{n-k-\ell}\right\rangle^{n-k-\ell}, \langle m\rangle^\ell\right).$$

Now $\mathbf{E}(k,\ell) \in \{ \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a \}$ if and only if $0 \le k \le \kappa$ and $0 \le \ell \le n-\kappa-\iota$. Note that $\mathbf{E}(\kappa, n-\kappa-1) = \mathbf{K}$ and $\mathbf{E}(0,0) = (a/n)\mathbf{1}$.

If $0 \le k \le \tilde{k} \le \kappa$ and $0 \le \ell \le \tilde{\ell} \le n-\kappa-\iota$, then $\mathbf{E}(k,\ell) \preceq \mathbf{E}(\tilde{k},\tilde{\ell})$. If, in addition, $k < \tilde{k}$, or $\ell < \tilde{\ell}$ and $\iota = 1$, or $\ell < \tilde{\ell}$ and $\tilde{k} \le \kappa-1$, then majorization is strict, and $G(\mathbf{E}(k,\ell)) < G(\mathbf{E}(\tilde{k},\tilde{\ell}))$ for a strictly Schur-convex function G.

Let $0 \le k_1, k_4$ and $1 \le k_2, k_3$. Define the vector $\mathbf{w} \in \mathbb{R}^n$ by

$$\mathbf{w}(x,y) = \left(\langle M\rangle^{k_1}, \langle x\rangle^{k_2}, \langle y\rangle^{k_3}, \langle m\rangle^{k_4}\right).$$

Let $\mathcal{W} = \mathcal{W}(k_1,k_2,k_3,k_4) = \{ \mathbf{w}(x,y) \mid M > x \ge y > m,\ \mathrm{S}(\mathbf{w}(x,y)) = a \}$. Since $(k_1+k_2)M + (k_3+k_4)m \le a$ if and only if $k_1+k_2 \le \kappa$, Lemma 7 implies the following:

Lemma 21. *Let* $\mathbf{K}$ *be as above,* $k_1+k_2+k_3+k_4 = n$, $0 \le k_1 \le \kappa$, *and* $0 \le k_4 \le n-\kappa-1$. *Then*

$$\inf_{\preceq} \mathcal{W} = \min_{\preceq} \mathcal{W} = \mathbf{E}(k_1,k_4)$$

and

$$\sup_{\preceq} \mathcal{W} = \begin{cases} \mathbf{E}(k_1+k_2,k_4) & \text{if } k_1+k_2 \le \kappa, \\ \mathbf{E}(k_1,k_3+k_4) & \text{if } k_1+k_2 \ge \kappa+1. \end{cases}$$

Note that $\mathbf{E}(k_1+k_2,k_4)$ and $\mathbf{E}(k_1,k_3+k_4)$ never belong to $\mathcal{W}$.

Example 19. Choose in Lemma 21 $k_2 = 1$, fix $k_4 = p \le n-\kappa-1$ and let k_1 range from 0 to κ. Denote $\mathcal{V}_{k_1} = \mathcal{W}(k_1,1,n-k_1-p-1,p)$. Then we obtain the following hierarchy:

$$\mathbf{E}(0,p) \preceq \mathcal{V}_0 \prec \mathbf{E}(1,p) \preceq \mathcal{V}_1 \prec \mathbf{E}(2,p) \preceq \cdots \prec \mathbf{E}(\kappa,p) \preceq \mathcal{V}_\kappa \preceq \mathbf{K}.$$

Similarly, fix $k_1 = p \le \kappa-1$, choose $k_3 = 1$, and let k_4 range from 0 to $n-\kappa-1$, and denote $\mathcal{U}_{k_4} = \mathcal{W}(p,n-p-k_4-1,1,k_4)$. Then

$$\mathbf{E}(p,0) \preceq \mathcal{U}_0 \prec \mathbf{E}(p,1) \preceq \mathcal{U}_1 \prec \mathbf{E}(p,2) \preceq \cdots \prec \mathbf{E}(p,n-\kappa-1) \preceq \mathcal{U}_{n-\kappa-1} \preceq \mathbf{K}.$$

4 Supremum and infimum of a sum- and G-constant set

4.1 Basic assumptions

Consider the set $\mathcal{X} = \{\, \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a,\ G(\mathbf{v}) = b \,\}$. In what follows, the conditions

$$G \text{ is strictly Schur-convex and continuous,}$$

and

$$G\big((a/n)\mathbf{1}\big) < b < G\big(\mathbf{K}(I,n,a)\big)$$

will be referred to as the basic assumptions (if $I = \mathbb{R}$, then omit the inequality '$b < G(\mathbf{K}(I,n,a))$'). They guarantee the infinity of the set $\mathcal{X}$ (see Lemma 17). When they hold, we denote the set $\mathcal{X}$ by $I^n[\mathrm{S},a;G,b]$. Note that we always assume that a vector in this set is decreasingly ordered. We also define $I^n[\mathrm{S},a] = \{\, \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a \,\}$. When using this notation, we always assume that $m < a/n < M$ when $I = [m, M]$.

Throughout, unless stated otherwise, we denote $\mathcal{G} = I^n[\mathrm{S},a;G,b]$. We also denote $\mathcal{G}^+ = \{\, \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a,\ G(\mathbf{v}) \geq b \,\}$ and $\mathcal{G}^- = \{\, \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a,\ G(\mathbf{v}) \leq b \,\}$.

In this chapter we will find the infimums and supremums of $\mathcal{G}$, $\mathcal{G}^+$, and $\mathcal{G}^-$ relative to $\leq$ and $\preceq$. We will deduce some of the same results as in [10], [13], and [20]. Nevertheless, we will not assume G to be of the form $\sum_i g(x_i)$, where g is strictly convex, as was done in [10] and [13]. In [20], the situation is much simpler, since only the case $G = -\mathrm{P}$ is studied. To illustrate our main ideas of the proofs without too much technical detail, we will first repeat certain results of [20], although not always in a similar fashion. In [13], only the cases $I = \mathbb{R}$, $I = \mathbb{R}_+$ and $I = \mathbb{R}_{++}$ are explicitly investigated. We will show in Section 4.11 that our results can also be applied to these cases.

4.2 A sum- and product-constant set

Let us consider the set $\mathcal{P} = \mathbb{R}^n_{++}[\mathrm{S},a;-\mathrm{P},d]$. It follows from the basic assumptions that $a > 0$, $d < 0$, and $-(a/n)^n < d$.

Our first task is to find the sharp lower and upper bounds for partial sums $\mathrm{S}_{k\ell}(\mathbf{x})$ assuming $\mathbf{x} \in \mathcal{P}$. We will obtain most of these bounds by applying the following:

Lemma 22 (see [20, Lemma 1]). *Let $k \leq n-1$. The system*

$$(*) \qquad \begin{aligned} kx + (n-k)y &= a, \\ -x^k y^{n-k} &= d, \\ x \geq y &> 0 \end{aligned}$$

has a unique solution x, y.

Proof. Let $f_k \colon [a/n, a/k] \to \mathbb{R}$, $f_k(x) = -x^k\big((a-kx)/(n-k)\big)^{n-k}$. Then, for $a/n < x < a/k$,

$$f_k'(x) = -(n-k)^{k-n} k x^{k-1} (a-kx)^{n-k-1}(a-nx) > 0.$$

Therefore f_k is strictly increasing on $[a/n, a/k]$.

Since f_k is continuous and strictly increasing, and

$$f_k(a/n) = -(a/n)^n < d < 0 = f_k(a/k),$$

we have $f_k(\alpha) = d$ for some unique $\alpha \in \,]a/n, a/k[$. Clearly $x = \alpha$, $y = (a-k\alpha)/(n-k)$ is the unique solution of $(*)$.

For fixed $k \leq n-1$, we denote the solution (x, y) of $(*)$ by $(\alpha_k, \bar{\alpha}_k)$. Let $k \leq n-1$ and $\mathbf{x} \in \mathcal{P}$. We show that $\mathrm{S}_{1k}(\mathbf{x}) \leq k\alpha_k$: Assume that this does not hold. Since f_k is strictly increasing, we have a contradiction

$$\begin{aligned} d &= -(x_1 \cdots x_k)(x_{k+1} \cdots x_n) \\ &\geq -\Big(\frac{x_1 + \cdots + x_k}{k}\Big)^k \Big(\frac{x_{k+1} + \cdots + x_n}{n-k}\Big)^{n-k} \\ &= f_k\Big(\frac{x_1 + \cdots + x_k}{k}\Big) > f_k(\alpha_k) = d. \end{aligned}$$

Since $(\langle\alpha_k\rangle^k, \langle\bar{\alpha}_k\rangle^{n-k}) \in \mathcal{P}$, the upper bound $k\alpha_k$ is the best possible, using only the information that $\mathbf{x} \in \mathcal{P}$. Hence we obtain the following lemma, which together with Lemma 18 yields $\sup_{\preceq} \mathcal{P}$.

Lemma 23 (cf. [20, the proof of Theorem 1]). *Let $k \leq n-1$. Then*

$$\max_{\mathbf{x} \in \mathcal{P}} \mathrm{S}_{1k}(\mathbf{x}) = k\alpha_k.$$

If $\mathbf{x}$ is ordered decreasingly and $\mathrm{S}(\mathbf{x}) = a$, then $\mathrm{S}_{\ell k}(\mathbf{x}) \leq (k - \ell + 1)\mathrm{S}_{1k}(\mathbf{x})/k$ and $\mathrm{S}_{k+1,m}(\mathbf{x}) \geq (m-k)(a - \mathrm{S}_{1k}(\mathbf{x}))/(n-k)$. Using this fact, we can easily prove

Lemma 24 (cf. [20, p. 71]). *Let $\mathcal{X} = I^n[\mathrm{S}, a]$ and $\ell \le k \le n-1$. Assume that there exist $\boldsymbol{\alpha} \in \mathcal{X}$ and $\alpha \in \mathbb{R}$ such that $\boldsymbol{\alpha}_{1:k} = \alpha\mathbf{1}$ and*

$$\max_{\mathbf{x}\in\mathcal{X}} \mathrm{S}_{1k}(\mathbf{x}) = \mathrm{S}_{1k}(\boldsymbol{\alpha}) = k\alpha.$$

Then

$$\max_{\mathbf{x}\in\mathcal{X}} \mathrm{S}_{\ell k}(\mathbf{x}) = (k-\ell+1)\alpha.$$

Analogously, if $2 \le k+1 \le m \le n$ and there exist $\boldsymbol{\beta} \in \mathcal{X}$ and $\beta \in \mathbb{R}$ such that $\boldsymbol{\beta}_{k+1:n} = \beta\mathbf{1}$ and

$$\max_{\mathbf{x}\in\mathcal{X}} \mathrm{S}_{1k}(\mathbf{x}) = \mathrm{S}_{1k}(\boldsymbol{\beta}) = a-(n-k)\beta,$$

then

$$\min_{\mathbf{x}\in\mathcal{X}} \mathrm{S}_{k+1,m}(\mathbf{x}) = (m-k)\beta.$$

Lemmas 23 and 24 yield for $x_1, \ldots, x_{n-2}$, and x_{n-1} sharp upper bounds and for $x_2, \ldots, x_{n-1}$, and x_n sharp lower bounds.

Next we show that $\alpha_{n-1} \le x_1$ and $x_n \le \bar{\alpha}_1$. Since $(\langle\alpha_{n-1}\rangle^{n-1}, \bar{\alpha}_{n-1}) \in \mathcal{P}$ and $(\alpha_1, \langle\bar{\alpha}_1\rangle^{n-1}) \in \mathcal{P}$, these bounds are sharp. Assume to the contrary that $\alpha_{n-1} > x_1$. If $x \ge y \ge t > 0$, then $xy > (x+t)(y-t)$. Applying this fact $n-1$ times, we have a contradiction

$$\begin{aligned}
-d &= x_1x_2\cdots x_n \\
&> \alpha_{n-1}x_2\cdots x_{n-1}\big(x_n - (\alpha_{n-1}-x_1)\big) \\
&\;\vdots \\
&> \alpha_{n-1}^{n-1}\big(x_n - (\alpha_{n-1}-x_1) - (\alpha_{n-1}-x_2) - \cdots - (\alpha_{n-1}-x_{n-1})\big) \\
&= \alpha_{n-1}^{n-1}\big(a-(n-1)\alpha_{n-1}\big) \\
&= \alpha_{n-1}^{n-1}\bar{\alpha}_{n-1} = -d.
\end{aligned}$$

Likewise we can show that $x_n \le \bar{\alpha}_1$.

In conclusion we state

Theorem 25 (cf. [20, Theorem 1]). *Let $\mathbf{x} = \mathbf{x}_\downarrow > \mathbf{0}$, $\mathrm{S}(\mathbf{x}) = a$, $-\mathrm{P}(\mathbf{x}) = d$, and, for $k = 1, 2, \ldots, n-1$, let $(\alpha_k, \bar{\alpha}_k)$ be the solution (x, y) of the system*

$$\begin{aligned}
kx + (n-k)y &= a, \\
-x^k y^{n-k} &= d, \\
x \ge y &> 0.
\end{aligned}$$

Then the best possible bounds for $x_1, \ldots, x_{n-1}$, and x_n, using only n, a and d, are

$$\begin{aligned}
&\alpha_{n-1} \le x_1 \le \alpha_1, \\
&\bar{\alpha}_{i-1} \le x_i \le \alpha_i \quad \text{for } i = 2, 3, \ldots, n-1, \\
&\bar{\alpha}_{n-1} \le x_n \le \bar{\alpha}_1,
\end{aligned}$$

and the best possible upper bound for $\mathbf{x}$ *relative to majorization is*

$$(x_1, x_2, \ldots, x_n) \preceq \big(\alpha_1, \langle k\alpha_k - (k-1)\alpha_{k-1}\rangle_{k=2}^{n-1}, a - (n-1)\alpha_{n-1}\big)_{\Downarrow}.$$

We have now found $\inf_{\leq} \mathcal{P}$, $\sup_{\leq} \mathcal{P}$, and $\sup_{\preceq} \mathcal{P}$. To find $\inf_{\preceq} \mathcal{P}$, we must first find sharp lower bounds for $\mathrm{S}_{1k}(\mathbf{x})$ or, equivalently, sharp upper bounds for $\mathrm{S}_{k+1,n}(\mathbf{x})$. Lemma 24 is now inapplicable and Theorem 25 gives answers to only the simplest cases $k = 1$ and $k = n - 1$. We will solve the problem of finding $\inf_{\preceq} \mathcal{P}$ in the framework of our general approach.

4.3 Generalizing Lemma 22

We now return to the task of finding the supremum and infimum of the set $\mathcal{G}$ in the general case. Recall the notation in section 3.3:

$$\mathbf{E}(k, \ell) = \left(\langle M\rangle^k, \left\langle \frac{a - kM - \ell m}{n - k - \ell}\right\rangle^{n-k-\ell}, \langle m\rangle^{\ell}\right),$$
$$\mathbf{w}(x, y) = \big(\langle M\rangle^{k_1}, \langle x\rangle^{k_2}, \langle y\rangle^{k_3}, \langle m\rangle^{k_4}\big),$$
$$\mathcal{W} = \{\, \mathbf{w}(x, y) \mid M > x \geq y > m,\ \mathrm{S}(\mathbf{w}(x, y)) = a \,\},$$
$$\mathbf{K}(I, n, a) = (\langle M\rangle^{\kappa}, \langle\theta\rangle^{\iota}, \langle m\rangle^{n-\kappa-\iota}).$$

Let k_1, k_4 be nonnegative and k_2, k_3 positive integers such that $k_1 + k_2 + k_3 + k_4 = n$. We begin by generalizing Lemma 22 by solving the system

$$k_1 M + k_2 x + k_3 y + k_4 m = a,$$
$$G(\mathbf{w}(x, y)) = b,$$
$$M > x \geq y > m,$$

i.e., by solving the equation $G(\mathbf{w}(x, y)) = b$ subject to $\mathbf{w}(x, y) \in \mathcal{W}$.

Denote $Y(x) = (a - k_1 M - k_2 x - k_4 m)/k_3$. If $\alpha < \alpha'$, then $\mathbf{w}(\alpha, Y(\alpha)) \prec \mathbf{w}(\alpha', Y(\alpha'))$ on $\mathcal{W}$. Since G is strictly Schur-convex, there is at most one $\mathbf{w}(\alpha, Y(\alpha)) \in \mathcal{W}$ such that $G(\mathbf{w}(\alpha, Y(\alpha))) = b$.

If $\mathbf{w}(\alpha, Y(\alpha)) \in \mathcal{G}$ for some $\alpha \in I$, we denote the vector $\mathbf{w}(\alpha, Y(\alpha))$ by $\mathbf{s}(\langle M\rangle^{k_1} > [k_2] \geq [k_3] > \langle m\rangle^{k_4}; \mathcal{G})$ or shortly by $\mathbf{s}(\langle M\rangle^{k_1} > [k_2] \geq [k_3] > \langle m\rangle^{k_4})$. If $k_1 = 0$ ($k_4 = 0$), then we omit the expression '$\langle M\rangle^{k_1} >$' ('$> \langle m\rangle^{k_4}$'). Note that if $\mathbf{s}(\langle M\rangle^{k_1} > [k_2] \geq [k_3] > \langle m\rangle^{k_4})$ exists, then $k_1 \leq \kappa$ and $k_4 \leq n - \kappa - \iota$.

Provided that $\mathbf{E}(k, \ell)$ exists, denote $\gamma(k, \ell) = G(\mathbf{E}(k, \ell))$. Since G is continuous, it follows from Lemma 21 that

$$G(\mathcal{W}) = \begin{cases} [\gamma(k_1, k_4), \gamma(k_1 + k_2, k_4)[& \text{if } k_1 + k_2 \leq \kappa, \\ [\gamma(k_1, k_4), \gamma(k_1, k_3 + k_4)[& \text{otherwise.} \end{cases}$$

Hence we can state the necessary and sufficient conditions for the existence of the solution as follows:

Lemma 26. *Let* $\mathbf{K}(I, n, a)$ *be as above and* $k_1 + k_2 + k_3 + k_4 = n$*, where* $k_1, k_4 \geq 0$ *and* $k_2, k_3 \geq 1$*. Then* $\mathbf{s}(\langle M\rangle^{k_1} > [k_2] \geq [k_3] > \langle m\rangle^{k_4}; \mathcal{G})$ *exists if and only if* $k_1 \leq \kappa$*,* $k_4 \leq n - \kappa - \iota$ *and either*

$$\gamma(k_1, k_4) \leq b < \gamma(k_1 + k_2, k_4) \quad \text{and} \quad k_1 + k_2 \leq \kappa$$

or

$$\gamma(k_1, k_4) \leq b < \gamma(k_1, k_3 + k_4) \quad \text{and} \quad k_1 + k_2 \geq \kappa + 1.$$

Example 20 (cf. Example 16). Let $\mathcal{P}_2(b) = \{ \mathbf{x} \in [0, 5]^6_{\downarrow} \mid \mathrm{S}(\mathbf{x}) = 18,\ P_2(\mathbf{x}) = b \}$. Since $\mathbf{K}([0, 5], 6, 18) = (5, 5, 5, 3, 0, 0)$, we have $\kappa = 3$. Consider the existence of $\mathbf{s}(\langle 5\rangle^{k_1} > [k_2] \geq [k_3] > \langle 0\rangle^{k_4}; \mathcal{P}_2(b))$, where $k_1 = 2$, $k_2 = 1$, $k_3 = 2$, and $k_4 = 1$. Now $k_1 + k_2 \leq \kappa$,

$$E(k_1, k_4) = E(2, 1) = \big(\langle 5\rangle^2, \langle (18 - 2\cdot 5)/(6 - 2 - 1)\rangle^{6-2-1}, \langle 0\rangle^1\big),$$

and $\gamma(k_1, k_4) = \gamma(2, 1) = P_2(E(2, 1)) = P_2\big(5, 5, \frac{8}{3}, \frac{8}{3}, \frac{8}{3}, 0\big) = 71\frac{1}{3}$. Analogously we obtain

$$\gamma(k_1 + k_2, k_4) = \gamma(3, 1) = P_2\big(\langle 5\rangle^3, \langle 3/2\rangle^2, \langle 0\rangle^1\big) = 79\tfrac{1}{2}.$$

We can conclude that $\mathbf{s}(\langle 5\rangle^2 > [1] \geq [2] > 0; \mathcal{P}_2(b))$ exists if and only if $71\frac{1}{3} \leq b < 79\frac{1}{2}$.

Likewise we find that $\mathbf{s}(\langle 5\rangle^2 > [2] \geq [1] > 0; \mathcal{P}_2(b))$ exists if and only if

$$P_2\big(5, 5, \tfrac{8}{3}, \tfrac{8}{3}, \tfrac{8}{3}, 0\big) = 71\tfrac{1}{3} \leq b < 82 = P_2(5, 5, 4, 4, 0, 0).$$

4.4 More on the existence of the solution s

In general, we cannot give an algebraic solution to the equation $G(\mathbf{w}(x, y)) = b$ subject to $\mathbf{w}(x, y) \in \mathcal{W}$. If the solution exists, however, it is easy to find numerically. Lemma 26, combined with the observation concerning the majorization order between vectors $\mathbf{E}(k, l)$ (see p. 36, Example 19), yields an efficient method of determining the existence of the solution.

Example 19 shows that if $\gamma(0, p) = G(\mathbf{E}(0, p)) \leq b$, then there is a unique λ such that a vector of shape $(\langle M\rangle^{\lambda} > [1] \geq [n - \lambda - 1 - p] > \langle m\rangle^p)$ belongs to $\mathcal{G}$, and if $\gamma(p, 0) \leq b$, then there is a unique μ such that a vector of shape $(\langle M\rangle^p > [n - \mu - 1 - p] \geq [1] > \langle m\rangle^{\mu})$ belongs to $\mathcal{G}$. More specifically, we can state the following two lemmas.

Lemma 27 (cf. [10, Theorem 7] and [13, Lemma 4]). *Assume that* $\mathbf{K}([m, M], n, a) = (\langle M\rangle^{\kappa}, \langle \theta\rangle^{\iota}, \langle M\rangle^{n-\kappa-\iota})$ *and* $0 \leq p \leq n - \kappa - 1$*. If* $\gamma(0, p) \leq b$*, let*

$$\lambda(p) = \max\{ k \mid 0 \leq k \leq \kappa \text{ and } \gamma(k, p) \leq b \}.$$

Then $\mathbf{s}(\langle M\rangle^k > [1] \geq [n - k - 1 - p] > \langle m\rangle^p; \mathcal{G})$ *exists if and only if* $k = \lambda(p)$*. If* $k + \ell \leq \kappa$*, then* $\mathbf{s}(\langle M\rangle^k > [\ell] \geq [n - k - \ell - p] > \langle m\rangle^p; \mathcal{G})$ *exists if and only if*

$$0 \leq k \leq \lambda(p) \leq k + \ell - 1.$$

If $\gamma(0,p) > b$*, there is no such* k *that* $\mathbf{s}(\langle M\rangle^k > [1] \geq [n-k-1-p] > \langle m\rangle^p; \mathcal{G})$ *exists.*

Proof. First, note that if $k \geq \kappa + 1$, then $\mathbf{s}(\langle M\rangle^k > [1] \geq [n-k-1-p] > \langle m\rangle^p)$ does not exist.

Assume $\gamma(0,p) \leq b$. If $\lambda(p) = \kappa$, then

$$\gamma(\kappa, p) \leq b < \gamma(\kappa, n-\kappa-1),$$

since by the basic assumptions $\gamma(\kappa, n-\kappa-1) = G(\mathbf{K}) > b$. It follows from Lemma 26 that $\mathbf{s}(\langle M\rangle^{\lambda(p)} > [1] \geq [n-\lambda(p)-1-p] > \langle m\rangle^p)$ exists in this case.

If $0 \leq k \leq \lambda(p) \leq k+\ell-1 \leq \kappa-1$, then

$$\gamma(k,p) \leq \gamma(\lambda(p),p) \leq b < \gamma(\lambda(p)+1,p) \leq \gamma(k+\ell,p),$$

which implies the existence of $\mathbf{s}(\langle M\rangle^k > [\ell] \geq [n-k-\ell-p] > \langle m\rangle^p)$. Particularly, $\mathbf{s}(\langle M\rangle^k > [1] \geq [n-k-1-p] > \langle m\rangle^p)$ exists if $k = \lambda(p)$.

Assume then that $k \geq \lambda(p)+1$. Since $b < \gamma(\lambda(p)+1,p) \leq \gamma(k,p)$, it follows that $\mathbf{s}(\langle M\rangle^k > [\ell] \geq [n-k-\ell-p] > \langle m\rangle^p)$ does not exist. Particularly, $\mathbf{s}(\langle M\rangle^k > [1] \geq [n-k-1-p] > \langle m\rangle^p)$ does not exist if $k \geq \lambda(p)+1$.

If $k \leq \lambda(p)$ but $k+\ell \leq \lambda(p)$ $(\leq \kappa)$, we have $\gamma(k+\ell,p) \leq \gamma(\lambda(p),p) \leq b$, and so $\mathbf{s}(\langle M\rangle^k > [\ell] \geq [n-k-\ell-p] > \langle m\rangle^p)$ does not exist. Particularly, $\mathbf{s}(\langle M\rangle^k > [1] \geq [n-k-1-p] > \langle m\rangle^p)$ does not exist if $k \leq \lambda(p)-1$.

Finally, assume $\gamma(0,p) > b$. Then $\gamma(k,p) > b$ for $k = 0, 1, \ldots, \kappa$, and hence there is no such k that $\mathbf{s}(\langle M\rangle^k > [1] \geq [n-k-1-p] > \langle m\rangle^p)$ exists.

The proof of the following lemma is similar.

Lemma 28 (cf. [10, Theorem 7], [13, Lemma 8, Lemma 9]). *Let* $\mathbf{K}([m,M], n, a)$ *be as in Lemma 27 and let* $0 \leq p \leq \kappa$*. If* $\gamma(p,0) \leq b$*, let*

$$\mu(p) = \max\{\,\ell \mid 0 \leq \ell \leq n-\kappa-1 \text{ and } \gamma(p,\ell) \leq b\,\}.$$

Then $\mathbf{s}(\langle M\rangle^p > [n-p-\ell-1] \geq [1] > \langle m\rangle^\ell; \mathcal{G})$ *exists if and only if* $\ell = \mu(p)$*. If* $k+\ell \leq n-\kappa-1$*, then* $\mathbf{s}(\langle M\rangle^p > [n-k-\ell-p] \geq [k] > \langle m\rangle^\ell; \mathcal{G})$ *exists if and only if*

$$0 \leq \ell \leq \mu(p) \leq k+\ell-1.$$

If $\gamma(p,0) > b$*, then there is no such* ℓ *that* $\mathbf{s}(\langle M\rangle^p > [n-p-\ell-1] \geq [1] > \langle m\rangle^\ell; \mathcal{G})$ *exists.*

We use the following notations:

$$\begin{gathered}\underline{\mathbf{s}}(p) = \mathbf{s}\big(\langle M\rangle^{\lambda(p)} > [1] \geq [n-p-1-\lambda(p)] > \langle m\rangle^p\big),\\ \overline{\mathbf{s}}(p) = \mathbf{s}\big(\langle M\rangle^p > [n-p-1-\mu(p)] \geq [1] > \langle m\rangle^{\mu(p)}\big),\\ \lambda = \lambda(0), \qquad \mu = \mu(0),\\ \underline{\mathbf{s}} = \underline{\mathbf{s}}(0), \qquad \overline{\mathbf{s}} = \overline{\mathbf{s}}(0).\end{gathered}$$

When necessary, we supplement these notations with the set $\mathcal{G}$. By the basic assumptions, $\gamma(0,0) = G((a/n)\mathbf{1}) < b$. Hence $\underline{\mathbf{s}} = \underline{\mathbf{s}}(\mathcal{G})$ and $\overline{\mathbf{s}} = \overline{\mathbf{s}}(\mathcal{G})$ always exist. More generally, $\underline{\mathbf{s}}(p, \mathcal{G})$ is defined if and only if $p \leq \mu$, and $\overline{\mathbf{s}}(p, \mathcal{G})$ is defined if and only if $p \leq \lambda$.

By Lemma 27 and Lemma 28, $\mathbf{s}([k] \geq [n-k])$ exists if and only if $\lambda + 1 \leq k \leq n - \mu - 1$. We often denote this solution by $\boldsymbol{\alpha}_k = \boldsymbol{\alpha}_k(\mathcal{G}) = (\langle \alpha_k \rangle^k, \langle \bar{\alpha}_k \rangle^{n-k})$.

Example 21 (cf. [13, Lemma 10]). By the assumptions, $M > \alpha_k > a/n > \bar{\alpha}_k > m$. Theorem 9 implies that

$$\alpha_{\lambda+1} \geq \alpha_{\lambda+2} \geq \cdots \geq \alpha_{n-\mu-1} > a/n > \bar{\alpha}_{\lambda+1} \geq \bar{\alpha}_{\lambda+2} \geq \cdots \geq \bar{\alpha}_{n-\mu-1}.$$

4.5 Upper bounds for $\mathrm{S}_{1k}(\mathbf{x})$

Let $\lambda + 1 \leq k \leq n - \mu - 1$, in which case $\boldsymbol{\alpha}_k = \mathbf{s}([k] \geq [n-k])$ exists, and let $\mathbf{x} \in I^n[\mathrm{S}, a]$. If $\mathrm{S}_{1k}(\mathbf{x}) > \mathrm{S}_{1k}(\boldsymbol{\alpha}_k)$, then $(\boldsymbol{\alpha}_k)_{1:k} \leq_\Sigma \mathbf{x}_{1:k}$, and it follows from Lemma 4 that $\boldsymbol{\alpha}_k \prec \mathbf{x}$. Since G is strictly Schur-convex, $b = G(\boldsymbol{\alpha}_k) < G(\mathbf{x})$, and hence $\mathbf{x} \notin \mathcal{G}$ and $\mathbf{x} \notin \mathcal{G}^-$. Since $\boldsymbol{\alpha}_k \in \mathcal{G} \subset \mathcal{G}^-$, we have

$$\max_{\mathbf{x} \in \mathcal{G}} \mathrm{S}_{1k}(\mathbf{x}) = \max_{\mathbf{x} \in \mathcal{G}^-} \mathrm{S}_{1k}(\mathbf{x}) = \mathrm{S}_{1k}(\boldsymbol{\alpha}_k).$$

Let $\mathbf{x} \in I^n[\mathrm{S}, a]$ and assume that $\boldsymbol{\alpha}_k$ does not exists. Since $\mathrm{S}_{1k}(\mathbf{x}) \leq \min\{kM, a - (n-k)m\}$ and both $\underline{\mathbf{s}}$ and $\overline{\mathbf{s}}$ exist, we deduce that on the sets $\mathcal{G}$ and $\mathcal{G}^-$ the maximum of $\mathrm{S}_{1k}(\mathbf{x})$ is kM if $k \leq \lambda$, and the maximum of $\mathrm{S}_{1k}(\mathbf{x})$ is $a - (n-k)m$ if $k \geq n - \mu$.

Since $\mathbf{K}(I, n, a) \in \mathcal{G}^+$, we can state

Lemma 29 (cf. [10, Theorem 8] and [13, Section 6]). *In the notation of Section 4.4,*

$$\max_{\mathbf{x} \in \mathcal{G}} \mathrm{S}_{1k}(\mathbf{x}) = \begin{cases} \mathrm{S}_{1k}(\underline{\mathbf{s}}) = kM & \text{if } k \leq \lambda, \\ \mathrm{S}_{1k}(\boldsymbol{\alpha}_k) = k\alpha_k & \text{if } \lambda + 1 \leq k \leq n - \mu - 1, \\ \mathrm{S}_{1k}(\overline{\mathbf{s}}) = a - (n-k)m & \text{if } n - \mu \leq k \leq n, \end{cases}$$

$$\max_{\mathbf{x} \in \mathcal{G}^-} \mathrm{S}_{1k}(\mathbf{x}) = \max_{\mathbf{x} \in \mathcal{G}} \mathrm{S}_{1k}(\mathbf{x}),$$

$$\max_{\mathbf{x} \in \mathcal{G}^+} \mathrm{S}_{1k}(\mathbf{x}) = \begin{cases} kM & \text{if } k \leq \kappa, \\ a - (n-k)m & \text{otherwise.} \end{cases}$$

4.6 Supremum relative to $\preceq$

We obtain from Theorem 16, Lemma 18, and Lemma 29

Theorem 30 (cf. [10, Theorem 8], [10, Corollary 10], and [10, Section 7]). *Let* α_k*,* λ*,* μ*, and* $\mathbf{K}$ *be as in Lemma 29. If* $\lambda+1 \le n-\mu-1$*, then let* $\mathbf{u}$ *be the following vector:*

$$\begin{aligned}\mathbf{u}_{1:\lambda} &= M\mathbf{1}_\lambda,\\ \mathbf{u}_{n-\mu+1:n} &= m\mathbf{1}_\mu,\\ \mathbf{u}_{\lambda+1} &= (\lambda+1)\alpha_{\lambda+1} - \lambda M,\\ \mathbf{u}_{\lambda+2:n-\mu-1} &= \left(\langle k\alpha_k - (k-1)\alpha_{k-1}\rangle_{k=\lambda+2}^{n-\mu-1}\right),\\ \mathbf{u}_{n-\mu} &= a - \mu m - (n-\mu-1)\alpha_{n-\mu-1}.\end{aligned}$$

If $\lambda+1 = n-\mu$*, then define* $\mathbf{u} = (\langle M\rangle^\lambda, a-\lambda M-\mu m, \langle m\rangle^\mu)$ $(= \mathbf{K})$.

Then

$$\sup\nolimits_{\preceq} \mathcal{G} = \sup\nolimits_{\preceq} \mathcal{G}^- = \mathbf{u}_{\Downarrow}$$

and

$$\sup\nolimits_{\preceq} \mathcal{G}^+ = \max\nolimits_{\preceq} \mathcal{G}^+ = \mathbf{K}.$$

Since $M > \alpha_{\lambda+1}$ and $a > (n-\mu-1)\alpha_{n-\mu-1} + (\mu+1)m$, the subvectors $\mathbf{u}_{1:\lambda+1}$ and $\mathbf{u}_{n-\mu:n}$ in Theorem 30 are ordered decreasingly. If $\lambda+1 \le n-\mu-1$, then we must in the general case allow for the possibility of $\mathbf{u}$ not being ordered decreasingly.

Let $g\colon I \to \mathbb{R}$ be strictly convex and let $G\colon I_\downarrow^n \to \mathbb{R}$, $G(\mathbf{x}) = \sum_{i=1}^n g(\mathbf{x}_i)$. In this case we can show, following [13, the proof of Theorem 3], that $\mathbf{u}_\Downarrow = \mathbf{u}$: G can be extended to a Schur-convex function $\tilde{G}\colon I_\downarrow^{2n} \to \mathbb{R}$, $\tilde{G}(\mathbf{z}) = \sum_{i=1}^{2n} g(z_i)$. Denote $\tilde{\mathcal{G}} = I^{2n}[\mathrm{S}, 2a; \tilde{G}, 2b]$. Assume $\mathbf{x}, \mathbf{y} \in \mathcal{G}$. Since $\tilde{G}(\mathbf{x},\mathbf{y}) = G(\mathbf{x}) + G(\mathbf{y})$, it follows that $(\mathbf{x},\mathbf{y})_\downarrow \in \tilde{\mathcal{G}}$.

Assume $\boldsymbol{\alpha}_k = \mathbf{s}([k] \ge [n-k]; \mathcal{G})$ exists. Denote $\mathbf{w} = (\boldsymbol{\alpha}_k, \boldsymbol{\alpha}_k)_\downarrow$. Then $\mathbf{w} \in \tilde{\mathcal{G}}$, and since $\mathbf{w} \simeq ([2k] \ge [2n-2k])$, we have $\mathbf{s}([2k] \ge [2(n-k)]; \tilde{\mathcal{G}}) = \mathbf{w}$. It follows that if $\mathbf{z} \in \tilde{\mathcal{G}}$ and $J \subset \{1,2,3,\dots,2n\}$ with the cardinality of $2k$, then

$$2k\alpha_k = \mathrm{S}_{1,2k}(\mathbf{w}) \ge \mathrm{S}_{1,2k}(\mathbf{z}_\downarrow) \ge \sum_{i\in J} z_i.$$

Consider the case $k = \lambda+1 \le n-\mu-2$. Since $\mathbf{s}(\langle M\rangle^\lambda > [1] \ge [n-\lambda-1]; \mathcal{G})$ and $\boldsymbol{\alpha}_{\lambda+2}$ exist (and, trivially, belong to $\mathcal{G}$), we obtain

$$2(\lambda+1)\alpha_{\lambda+1} \ge \lambda M + (\lambda+2)\alpha_{\lambda+2},$$

which means that $\mathbf{u}_{\lambda+1} \ge \mathbf{u}_{\lambda+2}$.

Next, consider the case $\lambda+2 \le k \le n-\mu-2$. Then $\boldsymbol{\alpha}_{k-1}$ and $\boldsymbol{\alpha}_{k+1}$ exist, and hence

$$2k\alpha_k \ge (k-1)\alpha_{k-1} + (k+1)\alpha_{k+1},$$

or, equivalently, $\mathbf{u}_k \ge \mathbf{u}_{k+1}$.

Last, consider the case $k = n - \mu - 1$. Since $\boldsymbol{\alpha}_{n-\mu-2}$ and $\mathbf{s}([n-1-\mu] \geq [1] > \langle m\rangle^{\mu}; \mathcal{G})$ exist, we have

$$2(n-\mu-1)\alpha_{n-\mu-1} \geq (n-\mu-2)\alpha_{n-\mu-2} + a - \mu m,$$

i.e., $\mathbf{u}_{n-\mu-1} \geq \mathbf{u}_{n-\mu}$.

Thus we have proved that in this special case $\mathbf{u} = \mathbf{u}_{\downarrow} = \mathbf{u}_{\Downarrow}$, and we obtain

Corollary 31 (cf. [13, Theorem 3]). *Let $G(\mathbf{x}) = \sum_i g(x_i)$, where $g\colon I \to \mathbb{R}$ is strictly convex. Let $\mathbf{u}$ be as in Theorem 30. Then*

$$\sup\nolimits_{\preceq} \mathcal{G} = \sup\nolimits_{\preceq} \mathcal{G}^{-} = \mathbf{u}.$$

By the basic assumptions, $\mathcal{G}$ is infinite, and we can choose such $\mathbf{x} \in \mathcal{G}$ that $\mathbf{x} \neq \mathbf{u}_{\Downarrow}$. Since $\mathcal{G} \preceq \mathbf{u}_{\Downarrow}$, we have $\mathbf{x} \prec \mathbf{u}_{\Downarrow}$. Hence $b = G(\mathbf{x}) < G(\mathbf{u}_{\Downarrow})$, and it follows that $\mathbf{u}_{\Downarrow} \notin \mathcal{G}$. So $\max_{\preceq} \mathcal{G}$ is not attained. Note that there is erroneously 'max' in [13, Theorem 3] (likewise, there is erroneously 'min' in [13, Theorem 8]).

4.7 A note on bounds for $\mathrm{S}_{k\ell}$

Let $\mathbf{x} \in \mathcal{G}$. Since $\underline{\mathbf{s}}$, $\overline{\mathbf{s}}$, and, for $\lambda + 1 \leq k \leq n - \mu - 1$, $\boldsymbol{\alpha}_k$ belongs to $\mathcal{G}$, we obtain a sharp upper bound for $\mathrm{S}_{k\ell}(\mathbf{x})$ directly from Lemma 24, when $1 \leq k \leq \ell \leq n - \mu - 1$. Likewise we obtain a sharp lower bound for $\mathrm{S}_{k\ell}(\mathbf{x})$, when $\lambda + 2 \leq k \leq \ell \leq n$.

It can be proved (see [10, Theorem 8(d)] and [13, section 10]) that if $2 \leq k \leq \ell$ and $n-\mu \leq \ell \leq n$, then the problem of finding the maximum of $\mathrm{S}_{k\ell}(\mathbf{x})$ is reduced to the problem of finding the minimum of $\mathrm{S}_{1,k-1}(\mathbf{x}_{1:\ell}, \langle m\rangle^{n-\ell})$:

$$\max_{\mathbf{x}\in\mathcal{G}} \mathrm{S}_{k\ell}(\mathbf{x}) = \mathrm{S}_{k\ell}(\hat{\mathbf{x}})$$

if and only if

$$\min_{\mathbf{x}\in\mathcal{G},\ \mathbf{x}_{\ell+1:n}=m\mathbf{1}} \mathrm{S}_{1,k-1}(\mathbf{x}) = \mathrm{S}_{1,k-1}(\hat{\mathbf{x}}).$$

Analogously, if $k \leq \lambda + 1$ and $k \leq \ell \leq n - 1$, then

$$\min_{\mathbf{x}\in\mathcal{G}} \mathrm{S}_{k\ell}(\mathbf{x}) = \mathrm{S}_{k\ell}(\hat{\mathbf{x}})$$

if and only if

$$\max_{\mathbf{x}\in\mathcal{G},\ \mathbf{x}_{1:k-1}=M\mathbf{1}} \mathrm{S}_{\ell+1,n}(\mathbf{x}) = \mathrm{S}_{\ell+1,n}(\hat{\mathbf{x}})$$

(see [10, Theorem 9(b)]).

In the next section we will prove that if $n - \mu \leq \ell \leq n$, then

$$\max_{\mathbf{x}\in\mathcal{G}} \mathbf{x}_{\ell} = \underline{\mathbf{s}}(n-\ell)_{\ell},$$

and in Section 4.9 that if $k \leq \lambda + 1$, then

$$\min_{\mathbf{x}\in\mathcal{G}} \mathbf{x}_{k} = \overline{\mathbf{s}}(k-1)_{k}.$$

Note that since $n - \ell \leq \mu$ and $k - 1 \leq \lambda$, both $\underline{\mathbf{s}}(n-\ell)$ and $\overline{\mathbf{s}}(k-1)$ exist.

4.8 Supremum relative to $\leq$

Since $\underline{\mathbf{s}}$ belongs to $\mathcal{G}$, it follows that $\max_{\mathbf{x}\in\mathcal{Y}} \mathbf{x}_\ell = M$ when $\ell \leq \lambda$ and $\mathcal{Y}$ is $\mathcal{G}^-$, $\mathcal{G}$, or $\mathcal{G}^+$. Since $\mathbf{K} = (\langle M\rangle^\kappa, \langle\theta\rangle^\iota, \langle m\rangle^{n-\kappa-\iota})$, $\max_{\mathbf{x}\in\mathcal{G}^+} \mathbf{x}_\ell$ is M also when $\lambda \leq \ell \leq \kappa$.

Let $\lambda + 1 \leq \ell \leq n - \mu - 1$. It follows from Lemmas 24 and 29 that $\max_{\mathbf{x}\in\mathcal{G}} \mathbf{x}_\ell = \max_{\mathbf{x}\in\mathcal{G}^-} \mathbf{x}_\ell = \alpha_\ell$. Assume then that $\mathbf{x} \in \mathcal{G}^+$ and $\kappa + 1 \leq \ell \leq n - \mu - 1$. Since

$$\boldsymbol{\alpha}_\ell \preceq \big(\langle (a - (n-\ell)m)/\ell\rangle^\ell, \langle m\rangle^{n-\ell}\big),$$

and, trivially, $\mathbf{x}_\ell \leq (a-(n-\ell)m)/\ell$, we have $\max_{\mathbf{x}\in\mathcal{G}^+} \mathbf{x}_\ell = (a-(n-\ell)m)/\ell$.

Let $n - \mu \leq \ell$. Denote $L = \lambda(n-\ell)$ and $\mathbf{w} = \underline{\mathbf{s}}(n-\ell)$. Then

$$\mathbf{w} \simeq \big(\langle M\rangle^L > [1] \geq [\ell - L - 1] > \langle m\rangle^{n-\ell}\big).$$

We show that $\max_{\mathbf{x}\in\mathcal{G}} \mathbf{x}_\ell = \mathbf{w}_\ell$. Since $\mathbf{w} \in \mathcal{G}$, it suffices to show that $\mathbf{x}_\ell \leq \mathbf{w}_\ell$ whenever $\mathbf{x} \in \mathcal{G}$. Assume to the contrary that $\mathbf{x} \in \mathcal{G}$ and $\mathbf{x}_\ell > \mathbf{w}_\ell = \cdots = \mathbf{w}_{L+2}$. Then $\mathbf{x} \neq \mathbf{w}$, $\mathbf{x}_{L+2:n} \preceq^w \mathbf{w}_{L+2:n}$, and, trivially, $\mathbf{x}_{1:L} \preceq_w \mathbf{w}_{1:L} = M\mathbf{1}$. It follows that $\mathbf{x} \prec \mathbf{w}$, which implies a contradiction $G(\mathbf{x}) < b$. Above we can substitute $\mathcal{G}^+$ for $\mathcal{G}$. Moreover, since

$$\big(\langle (a - (n-\ell)m)/\ell\rangle^\ell, \langle m\rangle^{n-\ell}\big) \preceq \mathbf{w},$$

we obtain $\max_{\mathbf{x}\in\mathcal{G}^-} \mathbf{x}_\ell = (a - (n-\ell)m)/\ell$.

The argument implies

Theorem 32 (cf. [10, Theorem 8] and [13, Section 10]). *Denote $\bar\beta_{n-\ell} = \underline{\mathbf{s}}(n-\ell)_\ell$. In the notation of Section 4.4,*

$$\begin{aligned}
\sup\nolimits_{\leq} \mathcal{G}^- &= \big(\langle M\rangle^\lambda, \langle \alpha_\ell\rangle_{\ell=\lambda+1}^{n-\mu-1}, \langle (a-(n-\ell)m)/\ell\rangle_{\ell=n-\mu}^n\big),\\
\sup\nolimits_{\leq} \mathcal{G} &= \big(\langle M\rangle^\lambda, \langle \alpha_\ell\rangle_{\ell=\lambda+1}^{n-\mu-1}, \langle \bar\beta_{n-\ell}\rangle_{\ell=n-\mu}^n\big),\\
\sup\nolimits_{\leq} \mathcal{G}^+ &= \big(\langle M\rangle^\kappa, \langle (a-(n-\ell)m)/\ell\rangle_{\ell=\kappa+1}^{n-\mu-1}, \langle \bar\beta_{n-\ell}\rangle_{\ell=n-\mu}^n\big).
\end{aligned}$$

As a rule, the corresponding maximums are not attained.

4.9 Infimum relative to $\leq$

Since $\mathbf{K} \in \mathcal{G}^+$, it is trivial that $\min_{\mathbf{x}\in\mathcal{G}^+} \mathbf{x}_k = m$, when $\kappa + 2 \leq k \leq n$. By Lemmas 24 and 29,

$$\min_{\mathbf{x}\in\mathcal{G}} \mathbf{x}_k = \min_{\mathbf{x}\in\mathcal{G}^-} \mathbf{x}_k = \begin{cases} \bar\alpha_{k-1} & \text{if } \lambda + 2 \leq k \leq n - \mu,\\ m & \text{if } n - \mu + 1 \leq k \leq n.\end{cases}$$

Let $k \leq \lambda+1$. Denote $L = \mu(k-1)$ and $\mathbf{w} = \bar{\mathbf{s}}(k-1)$. Assume $\mathbf{x} \in I^n[\mathrm{S}, a]$ and $\mathbf{x}_k < \mathbf{w}_k$. Since

$$\mathbf{w} \simeq \big(\langle M\rangle^{k-1} > [n-k-L] \geq [1] > \langle m\rangle^L\big),$$

we have $\mathbf{x}_{1:n-L-1} \preceq_w \mathbf{w}_{1:n-L-1}$, which implies that $\mathbf{x} \prec \mathbf{w}$, and, further, that $G(\mathbf{x}) < b$. Hence $\mathbf{w}_k$ is the sharp lower bound for $\mathbf{x}_k$ subject to $\mathbf{x} \in \mathcal{G}$ or to $\mathbf{x} \in \mathcal{G}^+$.

For $i \leq \kappa$, denote

$$\mathbf{v}(i) = \big(\langle M\rangle^i, \langle(a - iM)/(n-i)\rangle^{n-i}\big).$$

Since $\mathbf{v}(k-1) \preceq \mathbf{w}$, we have $\mathbf{v}(k-1) \in \mathcal{G}^-$ for $k \leq \lambda+1$. On the other hand, if $\mathbf{x}_k < (a-(k-1)M)/(n-k+1)$, then $\mathrm{S}(\mathbf{x}) < a$. Hence $(a-(k-1)M)/(n-k+1)$ is the sharp lower bound for $\mathbf{x}_k$ subject to $\mathbf{x} \in \mathcal{G}^-$, when $k \leq \lambda+1$.

Finally, assume that $\mathbf{x} \in \mathcal{G}^+$ and $\lambda+2 \leq k \leq \kappa+1$. Since $\lambda+1 \leq k-1 \leq \kappa \leq n-\mu-1$, we conclude that $\boldsymbol{\alpha}_{k-1}$ exists. Now $\boldsymbol{\alpha}_{k-1} \preceq \mathbf{v}(k-1)$; hence $\mathbf{v}(k-1) \in \mathcal{G}^+$. Therefore $\min_{\mathbf{x}\in\mathcal{G}^+} \mathbf{x}_k = (a-(k-1)M)/(n-k+1)$.

In conclusion we can state

Theorem 33 (cf. [10, Theorem 9] and [13, Section 11]). *Denote* $\chi_{k-1} = \bar{\mathbf{s}}(k-1)_k$. *In the notation of Section 4.4,*

$$\begin{aligned}
\inf\nolimits_{\leq} \mathcal{G}^- &= \big(\langle(a-(k-1)M)/(n-k+1)\rangle_{k=1}^{\lambda+1}, \langle\bar{\alpha}_{k-1}\rangle_{k=\lambda+2}^{n-\mu}, \langle m\rangle^{\mu}\big),\\
\inf\nolimits_{\leq} \mathcal{G} &= \big(\langle\chi_{k-1}\rangle_{k=1}^{\lambda+1}, \langle\bar{\alpha}_{k-1}\rangle_{k=\lambda+2}^{n-\mu}, \langle m\rangle^{\mu}\big),\\
\inf\nolimits_{\leq} \mathcal{G}^+ &= \big(\langle\chi_{k-1}\rangle_{k=1}^{\lambda+1}, \langle(a-(k-1)M)/(n-k+1)\rangle_{k=\lambda+2}^{\kappa+1}, \langle m\rangle^{n-\kappa-1}\big).
\end{aligned}$$

4.10 Infimum relative to $\preceq$

Trivially, $\inf_{\preceq} \mathcal{G}^- = (a/n)\mathbf{1}$. Let $\mathcal{Y} = \mathcal{G}$ or $\mathcal{Y} = \mathcal{G}^+$. To find $\inf_{\preceq} \mathcal{Y}$, we first consider the problem $\min_{\mathbf{x}\in\mathcal{Y}} \mathrm{S}_{1k}(\mathbf{x})$. Theorem 33 implies that $\min_{\mathbf{x}\in\mathcal{Y}} \mathbf{x}_1 = \bar{\mathbf{s}}_1$. From Theorem 32 we obtain $\max_{\mathbf{x}\in\mathcal{Y}} \mathbf{x}_n = \underline{\mathbf{s}}_n$, which implies

$$\min_{\mathbf{x}\in\mathcal{Y}} \mathrm{S}_{1,n-1}(\mathbf{x}) = a - \underline{\mathbf{s}}_n.$$

Let us study the case $2 \leq k \leq n-2$. Let $\mathcal{Y}_s = \{\, \mathbf{v} \in I_{\downarrow}^n \mid \mathrm{S}(\mathbf{v}) = a, \mathrm{S}_{1k}(\mathbf{v}) = s \,\}$. We assume that $\mathcal{Y}_s \neq \emptyset$, or, equivalently, that $ka/n \leq s \leq \min\{kM, a-(n-k)m\}$. We will first solve the problem $\max_{\mathbf{x}\in\mathcal{Y}_s} G(\mathbf{x})$, then show that this maximum is an increasing function of s, and, finally, apply Lemma 2 to find the minimizer for $\mathrm{S}_{1k}(\mathbf{x})$.

The set $\mathcal{Y}_s$ is closed and bounded (see Lemma 1). Therefore $\mathcal{Y}_s$ is compact, and since G is continuous, a solution $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ to the problem $\max_{\mathbf{x}\in\mathcal{Y}_s} G(\mathbf{x})$ exists.

Let $p, r \geq 0$ and $p + r \leq n-2$. Denote

$$\mathbf{w}_{pr}(x,y,z) = \big(\langle M\rangle^p, x, \langle y\rangle^{n-p-r-2}, z, \langle m\rangle^r\big)$$

and, recalling $\mathbf{K} = (\langle M\rangle^{\kappa}, \langle\theta\rangle^{\iota}, \langle m\rangle^{n-\kappa-\iota})$, further denote

$$\mathcal{W} = \{\, \mathbf{w}_{pr}(x,y,z) \mid 0 \leq p \leq \kappa-1,\ 0 \leq r \leq n-\kappa-\iota-1,\ M \geq x \geq y \geq z \geq m \,\}.$$

We show that $\hat{\mathbf{x}} \in \mathcal{Y}_s \cap \mathcal{W}$: Assume that $\mathbf{x} \in \mathcal{Y}_s$ and $\mathbf{x} \notin \mathcal{W}$. Then there exist such indices i, j, and ℓ that $i < j$, $j+1 < \ell$, and

$$M\ (= x_{i-1}) > x_i > x_j > x_{j+1} > x_\ell > (x_{\ell+1} =)\ m.$$

If $j \le k$, then choose $\varepsilon = \min\{M - x_i, x_j - x_{j+1}\}$ and $\boldsymbol{\varepsilon} = \varepsilon\mathbf{e}_i - \varepsilon\mathbf{e}_j$. If $j \ge k+1$, then choose $\varepsilon = \min\{x_j - x_{j+1}, x_\ell - m\}$ and $\boldsymbol{\varepsilon} = \varepsilon\mathbf{e}_{j+1} - \varepsilon\mathbf{e}_\ell$. Then $\tilde{\mathbf{x}} = \mathbf{x} + \boldsymbol{\varepsilon} \in \mathcal{Y}_s$ and $\mathbf{x} \prec \tilde{\mathbf{x}}$. Since G is strictly Schur-convex, $\mathbf{x}$ is not a maximizer. Thus $\hat{\mathbf{x}} = \mathbf{w}_{pr}(x, y, z)$ for some real numbers x, y, z and for some nonnegative integers p, r such that $\mathbf{w}_{pr}(x, y, z) \in \mathcal{Y}_s$.

Denote $L = ka/n$ and $U = \min\{kM, a - (n-k)m\}$. Define a function $M\colon [L, U] \to \mathbb{R}$ by $M(s) = \max_{\mathbf{x}\in\mathcal{Y}_s} G(\mathbf{x})$. Let $L \le s < s' \le U$ and $\mathbf{x} \in \mathcal{Y}_s$. By Lemma 10, there exists $\mathbf{y} \in \mathcal{Y}_{s'}$ such that $\mathbf{x} \prec \mathbf{y}$, and so $G(\mathbf{x}) < G(\mathbf{y})$. Hence the function M is strictly increasing. Therefore, by Lemma 2, if $G(\hat{\mathbf{x}}) = M(s) = b$, then

$$\min_{\mathbf{x}\in\mathcal{G}} \mathrm{S}_{1k}(\mathbf{x}) = \min_{\mathbf{x}\in\mathcal{G}^+} \mathrm{S}_{1k}(\mathbf{x}) = \mathrm{S}_{1k}(\hat{\mathbf{x}}) = s.$$

Since $\hat{\mathbf{x}} \in \mathcal{W}$, this minimum is the same as

$$\min_{\mathbf{x}\in\mathcal{G}\cap\mathcal{W}} \mathrm{S}_{1k}(\mathbf{x}).$$

From Lemma 18 we obtain

Theorem 34 (cf. [10, Theorem 9], [10, Corollary 10], and [13, Theorem 8]). *Let $\mathcal{W}$ be as above, and let*

$$s_k = \min_{\mathbf{x}\in\mathcal{G}\cap\mathcal{W}} \mathrm{S}_{1k}(\mathbf{x}) \quad \textit{for } k = 2, 3, \dots, n-2.$$

Then

$$\begin{aligned} \inf\nolimits_{\preceq} \mathcal{G}^- &= (a/n)\mathbf{1}, \\ \inf\nolimits_{\preceq} \mathcal{G} &= \big(\bar{\mathbf{s}}_1, s_2 - \bar{\mathbf{s}}_1, \langle s_k - s_{k-1}\rangle_{k=3}^{n-2}, a - \underline{\mathbf{s}}_n - s_{n-2}, \underline{\mathbf{s}}_n\big), \\ \inf\nolimits_{\preceq} \mathcal{G}^+ &= \inf\nolimits_{\preceq} \mathcal{G}. \end{aligned}$$

It seems that in the general case there is no straightforward way of finding the value of s_k in Theorem 34. We know that $\max_{\mathbf{x}\in\mathcal{Y}_s} G(\mathbf{x}) = G(\mathbf{w}_{pr}(x, y, z))$ for some $\mathbf{w}_{pr}(x, y, z) \in \mathcal{Y}_s \cap \mathcal{W}$. If $s = \min\{kM, a-(n-k)m\}$, then $\max_{\mathbf{x}\in\mathcal{Y}_s} G(\mathbf{x}) = G(\mathbf{K})$. Therefore, we assume below that $0 \le p \le \min\{k-1, \kappa-1\}$ and $0 \le r \le \min\{n-k-1, n-\kappa-\iota-1\}$.

Now $\mathbf{w}_{pr}(x, y, z) \in \mathcal{Y}_s$ if and only if $M \ge x \ge y \ge z \ge m$, $pM + x + (k-p-1)y = s$, and $(n-k-1-r)y + z + rm = a - s$. Solving x and z, we obtain

$$\begin{aligned} x &= X_{pr}(y) = S + K_1 y, \\ z &= Z_{pr}(y) = A + K_2 y, \end{aligned}$$

where $S = s - pM$, $K_1 = -(k-p-1)$, $A = a - s - rm$, and $K_2 = -(n-k-r-1)$. Below we abbreviate $Y = Y_{pr}$ and $Z = Z_{pr}$.

Next we find out the values of y for which $\mathbf{w}_{pr}(X(y), y, Z(y)) \in \mathcal{Y}_s$. Assume for now that K_1 or K_2 is nonzero. Let

$$L_{pr} = \begin{cases} A/(1-K_2) & \text{if } K_1 = 0, \\ \max\{(M-S)/K_1, A/(1-K_2)\} & \text{otherwise,} \end{cases}$$

and

$$U_{pr} = \begin{cases} S/(1-K_1) & \text{if } K_2 = 0, \\ \min\{S/(1-K_1), (m-A)/K_2\} & \text{otherwise.} \end{cases}$$

Assume that the set $\{\, y \mid M \geq X(y) \geq y \geq Z(y) \geq m \,\}$ is nonempty. Since $K_1 \leq 0$ and $K_2 \leq 0$, it follows that $X(y)$ and $Z(y)$ increase (one of the two can be constant), when y decreases. Hence

$$\min\{\, y \mid M \geq X(y) \geq y \geq Z(y) \geq m \,\} \\ = \min\{\, y \mid M \geq X(y),\ y \geq Z(y) \,\} = L_{pr},$$

and, similarly,

$$\max\{\, y \mid M \geq X(y) \geq y \geq Z(y) \geq m \,\} \\ = \max\{\, y \mid X(y) \geq y,\ Z(y) \geq m \,\} = U_{pr}.$$

Denote

$$\mathbf{L}_{pr} = \mathbf{w}_{pr}(S + K_1 L_{pr}, L_{pr}, A + K_2 L_{pr}),$$
$$\mathbf{U}_{pr} = \mathbf{w}_{pr}(S + K_1 U_{pr}, U_{pr}, A + K_2 U_{pr}).$$

Consequently, if $0 \leq p \leq \min\{k-1, \kappa-1\}$, $0 \leq r \leq \min\{n-k-1, n-\kappa-\iota-1\}$, $K_1 \neq 0$ or $K_2 \neq 0$, and $\max_{\mathbf{x} \in \mathcal{Y}_s} G(\mathbf{x}) = G(\mathbf{w}_{pr}(x,y,z))$, then $\mathbf{w}_{pr}(x,y,z) = \mathbf{w}_{pr}(S + K_1 y, y, A + K_2 y) = t\mathbf{L}_{pr} + (1-t)\mathbf{U}_{pr}$ for some $t \in [0,1]$.

If $K_1 = K_2 = 0$, i.e., if $p = k-1$ and $r = n-k-1$, then

$$\mathbf{w}_{pr}\big(X(y), y, Z(y)\big) = \big(\langle M\rangle^{k-1}, s - pM, a - s - rm, \langle m\rangle^{n-k-1}\big)$$

is independent of y.

Above we have assumed only that G is strictly Schur-convex and continuous. Assume now, in addition, that G is quasiconvex. If $\max_{\mathbf{x} \in \mathcal{Y}_s} G(\mathbf{x}) = G(t\mathbf{L}_{pr} + (1-t)\mathbf{U}_{pr})$, then $t = 0$ or $t = 1$. Since $S + K_1 L_{pr} = M$ or $L_{pr} = A + K_2 L_{pr}$, and since $S + K_1 U_{pr} = U_{pr}$ or $A + K_2 U_{pr} = m$, we conclude that the maximizer $\hat{\mathbf{x}}$ is of shape $(\langle M\rangle^{\ell} > [1] \geq [n-\ell-q-1] > \langle m\rangle^{q})$ or $(\langle M\rangle^{\ell} > [n-\ell-q-1] \geq [1] > \langle m\rangle^{q})$ for some nonnegative $\ell \leq \kappa$ and $q \leq n-\kappa-1$. (If $\ell = \kappa$ and $q = n-\kappa-1$, then $\hat{\mathbf{x}} = \mathbf{K}$.)

Let $G(\hat{\mathbf{x}}) = b$. The shape of $\hat{\mathbf{x}}$ implies that $\hat{\mathbf{x}} = \underline{\mathbf{s}}(q, \mathcal{G})$ for some $q \in \{0, 1, \ldots, \mu(0, \mathcal{S})\}$, or that $\hat{\mathbf{x}} = \overline{\mathbf{s}}(\ell, \mathcal{G})$ for some $\ell \in \{0, 1, \ldots, \lambda(0, \mathcal{G})\}$. Therefore, by applying Lemma 2, we obtain

$$\min_{\mathbf{x} \in \mathcal{G}} \mathrm{S}_{1k}(\mathbf{x}) = \min_{\mathbf{x} \in \mathcal{G}^+} \mathrm{S}_{1k}(\mathbf{x}) = \min(\mathcal{L}_k \cup \mathcal{M}_k),$$

where $\mathcal{L}_k = \{\, \mathrm{S}_{1k}(\underline{\mathbf{s}}(q, \mathcal{G})) \mid q = 0, 1, \ldots, \mu(0, \mathcal{G}) \,\}$ and $\mathcal{M}_k = \{\, \mathrm{S}_{1k}(\bar{\mathbf{s}}(\ell, \mathcal{G})) \mid \ell = 0, 1, \ldots, \lambda(0, \mathcal{G}) \,\}$. Hence we have

Theorem 35 (cf. [10, Theorem 9], [10, Corollary 10], and [13, Theorem 8]). *Let G be quasiconvex and let $\mathcal{L}_k$ and $\mathcal{M}_k$ be as above. For $k = 2, 3, \ldots, n-2$, let s_k be as in Theorem 34. Then*

$$s_k = \min_{\mathbf{x} \in \mathcal{G} \cap \mathcal{W}} \mathrm{S}_{1k}(\mathbf{x}) = \min(\mathcal{L}_k \cup \mathcal{M}_k).$$

4.11 Intervals $\mathbb{R}_+$ and $\mathbb{R}$

So far we have considered closed real intervals. In this section we will apply our results to $\mathbb{R}$ and $\mathbb{R}_+$. We will also provide some comments on $\mathbb{R}_{++}$.

If $I = \mathbb{R}_+$, then $\mathbf{x} \in I^n[\mathrm{S}, a]$ if and only if $\mathbf{x} \in [0, a]^n_\downarrow$ and $\mathrm{S}(\mathbf{x}) = a$. Noting that now $m = 0$, $M = a$, $\mathbf{K} = (a, \langle 0 \rangle^{n-1})$, $\kappa = 1$, and

$$\mu = \max\bigl\{\, \ell \bigm| 0 \le \ell \le n-2,\, G\bigl(\langle a/(n-\ell) \rangle^{n-\ell}, \langle 0 \rangle^{\ell}\bigr) \le b \,\bigr\},$$

we obtain the results concerning $\mathbb{R}_+$ directly from the results of Sections 4.6 and 4.8–4.10. If $\lambda(p)$ is defined, i.e., if $p \le \mu$, then $\lambda(p) = 0$. Hence $\bar{\beta}_0 = \underline{\mathbf{s}}_n$ in Theorem 32 is the same as $\bar{\alpha}_1$. Note that if $\mu = 0$, then $\chi_0 = \bar{\mathbf{s}}_1$ in Theorem 33 is the same as α_{n-1}.

We (re)introduce the following notations. For $k = 1, 2, \ldots, n-1-\mu$, let $(x, y) = (\alpha_k, \bar{\alpha}_k)$ be the solution of the system

$$\begin{gathered} kx + (n-k)y = a, \\ G(\langle x \rangle^k, \langle y \rangle^{n-k}) = b, \\ x \ge y > 0. \end{gathered}$$

For $i = 1, 2, \ldots, \mu$, let $(x, y) = (\beta_{1i}, \bar{\beta}_{1i})$ be the solution of the system

$$\begin{gathered} x + (n-i-1)y = a, \\ G(x, \langle y \rangle^{n-i-1}, \langle 0 \rangle^i) = b, \\ x \ge y > 0. \end{gathered}$$

Further, let $(x, y) = (\beta_{n-\mu-1,\mu}, \bar{\beta}_{n-\mu-1,\mu})$ be the solution of the system

$$\begin{gathered} (n-\mu-1)x + y = a, \\ G(\langle x \rangle^{n-\mu-1}, y, \langle 0 \rangle^{\mu}) = b, \\ x \ge y > 0. \end{gathered}$$

Finally, we define the set $\mathcal{Z}_r$ by

$$\mathcal{Z}_r = \{\, \mathbf{v} \in \mathcal{G} \mid \mathbf{v} \simeq ([1] \ge [n-r-2] \ge [1] \ge \langle 0 \rangle^r) \,\},$$

and, for $2 \le k \le n-2$, the number s_k by

$$s_k = \min\{\, \mathrm{S}_{1k}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{Z}_0 \cup \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_{\min\{n-k-1,\mu\}} \,\}.$$

Analogously with Theorem 35, we now have

Theorem 36. *If G is quasiconvex, then, for $2 \le k \le n-1-\mu$,*

$$s_k = \min(\{\, \beta_i + (k-1)\bar{\beta}_i \mid i = 1,2,\dots,\mu \,\} \cup \{\alpha_1 + (k-1)\bar{\alpha}_1, k\beta_{n-\mu-1,\mu}\}),$$

and, for $n-\mu \le k \le n-2$,

$$s_k = \min(\{\, \beta_i + (k-1)\bar{\beta}_i \mid i = 1,2,\dots,n-1-k \,\} \cup \{\alpha_1 + (k-1)\bar{\alpha}_1\}).$$

In the following four theorems we assume that μ is as above and $\mathcal{G} = \mathbb{R}^n_+[\mathrm{S}, a; G, b]$, in which case $\mathcal{G}^- = \{\, \mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{x} = \mathbf{x}_\downarrow, \mathrm{S}(\mathbf{x}) = a, G(\mathbf{x}) \le b \,\}$ and $\mathcal{G}^+ = \{\, \mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{x} = \mathbf{x}_\downarrow, \mathrm{S}(\mathbf{x}) = a, G(\mathbf{x}) \ge b \,\}$.

Theorem 37 (cf. Theorem 30 and Corollary 31). *Let α_i be as above. Then*

$$\begin{aligned}
\sup\nolimits_{\preceq} \mathcal{G}^- &= \big(\alpha_1, \langle k\alpha_k - (k-1)\alpha_{k-1}\rangle_{k=2}^{n-\mu-1}, a - (n-\mu-1)\alpha_{n-\mu-1}, \langle 0\rangle^{\mu}\big)_{\Downarrow},\\
\sup\nolimits_{\preceq} \mathcal{G} &= \sup\nolimits_{\preceq} \mathcal{G}^-,\\
\sup\nolimits_{\preceq} \mathcal{G}^+ &= \big(a, \langle 0\rangle^{n-1}\big).
\end{aligned}$$

If $G(\mathbf{x}) = \sum_i g(x_i)$, where g is strictly convex, then the subscript '$\Downarrow$' can be omitted.

Theorem 38 (cf. Theorem 32). *Let α_i, $\bar{\alpha}_i$, and $\bar{\beta}_i$ be as above. Then*

$$\begin{aligned}
\sup\nolimits_{\le} \mathcal{G}^- &= \big(\langle \alpha_\ell\rangle_{\ell=1}^{n-\mu-1}, \langle a/\ell\rangle_{\ell=n-\mu}^{n}\big),\\
\sup\nolimits_{\le} \mathcal{G} &= \big(\langle \alpha_\ell\rangle_{\ell=1}^{n-\mu-1}, \langle \bar{\beta}_{\mu-i+1}\rangle_{i=1}^{\mu}, \bar{\alpha}_1\big),\\
\sup\nolimits_{\le} \mathcal{G}^+ &= \big(\langle a/\ell\rangle_{\ell=1}^{n-\mu-1}, \langle \bar{\beta}_{\mu-i+1}\rangle_{i=1}^{\mu}, \bar{\alpha}_1\big).
\end{aligned}$$

Theorem 39 (cf. Theorem 33). *Let $\bar{\alpha}_i$ and $\beta_{n-\mu-1,\mu}$ be as above. Then*

$$\begin{aligned}
\inf\nolimits_{\le} \mathcal{G}^- &= \big(a/n, \langle \bar{\alpha}_{k-1}\rangle_{k=2}^{n-\mu}, \langle 0\rangle^{\mu}\big),\\
\inf\nolimits_{\le} \mathcal{G} &= \big(\beta_{n-\mu-1,\mu}, \langle \bar{\alpha}_{k-1}\rangle_{k=2}^{n-\mu}, \langle 0\rangle^{\mu}\big),\\
\inf\nolimits_{\le} \mathcal{G}^+ &= \big(\beta_{n-\mu-1,\mu}, \langle 0\rangle^{n-1}\big).
\end{aligned}$$

Theorem 40 (cf. Theorem 34). *Let $\beta_{n-\mu-1,\mu}$, $\bar{\alpha}_1$, and s_i be as above. Then*

$$\begin{aligned}
\inf\nolimits_{\preceq} \mathcal{G}^- &= \big(\langle a/n\rangle^{n}\big),\\
\inf\nolimits_{\preceq} \mathcal{G} &= \big(\beta_{n-\mu-1,\mu}, s_2 - \beta_{n-\mu-1,\mu}, \langle s_k - s_{k-1}\rangle_{k=3}^{n-2}, a - \bar{\alpha}_1 - s_{n-2}, \bar{\alpha}_1\big),\\
\inf\nolimits_{\preceq} \mathcal{G}^+ &= \inf\nolimits_{\preceq} \mathcal{G}.
\end{aligned}$$

In the case of $G(\mathbf{x}) = \sum_i g(x_i)$, where $g\colon \mathbb{R}_+ \to \mathbb{R}$ is strictly convex, the results of the above theorems are presented for the set $\mathcal{G}$ in [13, Theorems 2, 3, 6, and 8, and Sections 10 and 11]. (As noted before, omit 'max' in [13, Theorem 3] and 'min' in [13, Theorem 8].) It is also possible to deduce these results from [10, Theorems 8 and 9]. The notation in [13] is somewhat different; in particular, q defined in [13, p. 8] stands for $n-\mu$ if $G(\langle a/(n-\mu)\rangle^{n-\mu}, \langle 0\rangle^{\mu}) = b$, and for $n-\mu-1$ otherwise. Hence p defined in [13, p. 11] always stands for $n-\mu-1$.

Next we consider the case $\mathcal{G} = \mathbb{R}^n[\mathrm{S}, a; G, b]$. Now $\mathbf{K}$ is not defined and the basic assumptions match the condition $b > G(\langle a/n\rangle^n)$. Further, we can define $\lambda(p)$ and $\mu(p)$ when $p = 0$ by $\lambda(0) = \mu(0) = 0$. The system

$$\begin{gathered} kx + (n-k)y = a, \\ G\big(\langle x\rangle^k, \langle y\rangle^{n-k}\big) = b, \\ x \ge y, \end{gathered}$$

has a unique solution for all $k \le n-1$. As before, we denote this solution by $\boldsymbol{\alpha}_k = (\langle \alpha_k\rangle^k, \langle \bar{\alpha}_k\rangle^{n-k})$.

The following example shows that $\sup_{\le} \mathcal{G}^+$, $\sup_{\preceq} \mathcal{G}^+$, and $\inf_{\le} \mathcal{G}^+$ are not attained.

Example 22. Denote

$$\begin{aligned} \mathbf{w}(M) &= \big(\alpha_1 + (n-1)M, \langle \bar{\alpha}_1 - M\rangle^{n-1}\big), \\ \mathbf{v}(M) &= \big(\langle \alpha_{n-1} + M\rangle^{n-1}, \bar{\alpha}_{n-1} - (n-1)M\big). \end{aligned}$$

Since $\mathbf{w}(M)$ and $\mathbf{v}(M)$ belong to $\mathcal{G}^+$ for all nonnegative M, it follows that $\sup_{\mathbf{x}\in\mathcal{G}^+} \mathbf{x}_i$ is not attained for $i = 1, 2, \ldots, n-1$, and that $\inf_{\mathbf{x}\in\mathcal{G}^+} \mathbf{x}_i$ is not attained for $i = 2, 3, \ldots, n$.

The set $\mathcal{W}$ of Section 4.10 is now $\{\, \mathbf{w}_{00}(x, y, z) \mid x \ge y \ge z \,\}$, and Lemma 1 implies that $\mathcal{Y}_s = \{\, \mathbf{v} \in I^n_{\downarrow} \mid \mathrm{S}(\mathbf{v}) = a,\, \mathrm{S}_{1k}(\mathbf{v}) = s \,\}$ is bounded even if we replace the closed interval I by $\mathbb{R}$. Hence by substituting $\boldsymbol{\alpha}_1$ for $\underline{\mathbf{s}}$ in the proof of Theorem 32, and $\boldsymbol{\alpha}_{n-1}$ for $\bar{\mathbf{s}}$ in the proof of Theorem 33, we deduce the following four companions to Theorems 37–40.

Theorem 41. *Let $G\colon \mathbb{R}^n \to \mathbb{R}$ be strictly Schur-convex and continuous, and let $a \in \mathbb{R}$ and $G((a/n)\mathbf{1}) < b$. Denote $\mathcal{G} = \{\, \mathbf{v} \in \mathbb{R}^n_{\downarrow} \mid \mathrm{S}(\mathbf{v}) = a,\, G(\mathbf{v}) = b \,\}$, $\mathcal{G}^- = \{\, \mathbf{v} \in \mathbb{R}^n_{\downarrow} \mid \mathrm{S}(\mathbf{v}) = a,\, G(\mathbf{v}) \le b \,\}$, and $\mathcal{G}^+ = \{\, \mathbf{v} \in \mathbb{R}^n_{\downarrow} \mid \mathrm{S}(\mathbf{v}) = a,\, G(\mathbf{v}) \ge b \,\}$. Further, let α_k be as above. Then*

$$\begin{aligned} \sup\nolimits_{\preceq} \mathcal{G}^- &= \big(\alpha_1, \langle k\alpha_k - (k-1)\alpha_{k-1}\rangle_{k=2}^{n-1}, a - (n-1)\alpha_{n-1}\big)_{\Downarrow}, \\ \sup\nolimits_{\preceq} \mathcal{G} &= \sup\nolimits_{\preceq} \mathcal{G}^-. \end{aligned}$$

If $G(\mathbf{x}) = \sum_i g(x_i)$, where g is strictly convex, then the subscript '$\Downarrow$' can be omitted.

Theorem 42. *In the notation and with the assumptions of Theorem 41,*

$$\begin{aligned}\sup_{\leq} \mathcal{G}^- &= (\alpha_1, \alpha_2, \dots, \alpha_{n-1}, a/n),\\ \sup_{\leq} \mathcal{G} &= (\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \bar{\alpha}_1),\\ \sup_{\mathbf{x}\in\mathcal{G}^+} \mathbf{x}_n &= \bar{\alpha}_1.\end{aligned}$$

Theorem 43. *In the notation and with the assumptions of Theorem 41,*

$$\begin{aligned}\inf_{\leq} \mathcal{G}^- &= \big(a/n, \langle \bar{\alpha}_{k-1}\rangle_{k=2}^n\big),\\ \inf_{\leq} \mathcal{G} &= \big(\alpha_{n-1}, \langle \bar{\alpha}_{k-1}\rangle_{k=2}^n\big),\\ \inf_{\mathbf{x}\in\mathcal{G}^+} \mathbf{x}_1 &= \alpha_{n-1}.\end{aligned}$$

For $2 \leq k \leq n-2$, let

$$s_k = \min\{\, \mathrm{S}_{1k}(\mathbf{x}) \mid \mathrm{S}(\mathbf{x}) = a,\ G(\mathbf{x}) = b,\ \mathbf{x} \simeq ([1] \geq [n-2] \geq [1]) \,\},$$

or, when G is quasiconvex, let

$$s_k = \min\{\alpha_1 + (k-1)\bar{\alpha}_1, k\alpha_{n-1}\}.$$

Theorem 44. *In the notation and with the assumptions of Theorem 41,*

$$\begin{aligned}\inf_{\preceq} \mathcal{G}^- &= \big(\langle a/n\rangle^n\big),\\ \inf_{\preceq} \mathcal{G} &= \big(\alpha_{n-1}, s_2 - \alpha_{n-1}, \langle s_k - s_{k-1}\rangle_{k=3}^{n-2}, a - \bar{\alpha}_1 - s_{n-2}, \bar{\alpha}_1\big),\\ \inf_{\preceq} \mathcal{G}^+ &= \inf_{\preceq} \mathcal{G}.\end{aligned}$$

For the case $G(\mathbf{x}) = \sum_i g(x_i)$, where $g\colon \mathbb{R}_+ \to \mathbb{R}$ is strictly convex, these results are presented for the set $\mathcal{G}$ in [13, Theorems 1, 3, 4, 5, and 8, and Sections 10 and 11]. It is also possible to deduce these results from Theorem 8 and Theorem 9 in [10] by choosing $M = \infty$ and $m = -\infty$.

If I is a closed interval, the sets $\mathcal{G}^-$, $\mathcal{G}$, and $\mathcal{G}^+$ are bounded. By the basic assumptions, G is continuous. Hence these sets are also closed and therefore compact. The same result holds also for $I = \mathbb{R}_+$ and, by Theorems 42 and 43, for $I = \mathbb{R}$ except in the case of $\mathcal{G}^+$. Hence we have

Corollary 45. *Let $\mathcal{G} = I^n[\mathrm{S}, a; G, b]$, where I is a closed interval, $\mathbb{R}_+$, or $\mathbb{R}$. The sets $\mathcal{G}^-$ and $\mathcal{G}$ are compact.*

In [13] also the case $\{\, \mathbf{v} \in \mathbb{R}^n_{++} \mid \mathbf{v} = \mathbf{v}_\downarrow,\ \mathrm{S}(\mathbf{v}) = a,\ \sum_i g(\mathbf{v}_i) = b) \,\}$, where $g\colon \mathbb{R}_{++} \to \mathbb{R}$ is strictly convex, is considered. If $\lim_{x\to 0+} g(x)$ exists as finite, then g can be extended to a convex continuous function $\mathbb{R}_+ \to \mathbb{R}$, and this case is reduced to the case $I = \mathbb{R}_+$. Otherwise $\lim_{x\to 0+} g(x) = \infty$, and Theorems 37–40 include this case. Namely, it is easy to see that if

$$\lim_{\varepsilon\to 0+} G(x_1, x_2, \dots, x_{n-1}, \varepsilon) = \infty$$

for all $x_1 \geq x_2 \geq \cdots \geq x_{n-1} > 0$, then, defining $\mu = 0$, the conclusions of Theorems 37–40 remain valid.

5 Examples and applications

5.1 On estimating eigenvalues

As noted in the introduction, the results of Wolkowicz and Styan [29] (see also [30]) concerning eigenvalue estimation have provided an important starting point for our research. Let $\mathbf{A}$ be an $n \times n$ complex matrix with real eigenvalues $\boldsymbol{\lambda}(\mathbf{A}) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Some strictly Schur-convex functions of the eigenvalues can be expressed as a simple function of $\mathbf{A}$; for example, $P_m(\boldsymbol{\lambda}(\mathbf{A})) = \sum_i \lambda_i^m = \operatorname{tr} \mathbf{A}^m$, $-\mathrm{P}(\boldsymbol{\lambda}(\mathbf{A})) = -\prod_i \lambda_i = -\det \mathbf{A}$, and $-S_k(\boldsymbol{\lambda}(\mathbf{A})) = -\operatorname{tr} \mathbf{A}^{(k)}$, where $\mathbf{A}^{(k)}$ stands for the kth compound of $\mathbf{A}$.

We cannot generally present $\mathbf{s}([k] \geq [n-k]; \mathcal{G})$ explicitly. Perhaps the most notable exception to this is the very case $G = P_2$ considered by Wolkowicz and Styan. We will state these algebraic bounds in the next section. After that, we will give a numerical example. At the end we will touch the problem of finding algebraic bounds for eigenvalues.

5.2 The case $G = P_2$

Denote $m = a/n$ and $s = \sqrt{b/n - a^2/n^2}$, where $b \geq a^2/n$. Let $k \leq n-1$. The solution of the system

$$P_2(\mathbf{x}) = b,$$
$$\mathrm{S}(\mathbf{x}) = a,$$
$$\mathbf{x} \simeq ([k] \geq [n-k]),$$

is

$$\mathbf{x} = \left(\langle \alpha_k \rangle^k, \langle \bar{\alpha}_k \rangle^{n-k}\right),$$

where

$$\alpha_k = m + s\sqrt{\frac{n-k}{k}}$$

and

$$\bar{\alpha}_k = m - s\sqrt{\frac{k}{n-k}}.$$

We easily see that

$$\begin{aligned}
k\alpha_k - (k-1)\alpha_{k-1} &= m + s\big(\sqrt{k(n-k)} - \sqrt{(k-1)(n-k+1)}\,\big),\\
\alpha_1 + (k-1)\bar{\alpha}_1 &= km + s\big(\sqrt{n-1} - \sqrt{(k-1)^2/(n-1)}\,\big),\\
k\alpha_{n-1} &= km + s\sqrt{k^2/(n-1)},\\
\alpha_1 + (k-1)\bar{\alpha}_1 &= k\alpha_{n-1} \quad \text{when } n = 2k,
\end{aligned}$$

$$s_k = \min\{\alpha_1 + (k-1)\bar{\alpha}_1, k\alpha_{n-1}\} = \begin{cases} \alpha_1 + (k-1)\bar{\alpha}_1 & \text{if } k \geq n/2,\\ k\alpha_{n-1} & \text{if } k \leq n/2, \end{cases}$$

and

$$\begin{aligned}
&\big(\alpha_{n-1}, \langle s_k - s_{k-1}\rangle_{k=2}^{n-2}, a - \bar{\alpha}_1 - s_{n-2}, \bar{\alpha}_1\big)\\
&\qquad\qquad = \big(\langle \alpha_{n-1}\rangle^{\lfloor n/2\rfloor}, \langle m\rangle^{n-2\lfloor n/2\rfloor}, \langle \bar{\alpha}_1\rangle^{\lfloor n/2\rfloor}\big).
\end{aligned}$$

Hence, by Theorems 41–44, we have

Corollary 46 (cf. [29] p. 477–479, Theorems 2.2 and 2.3). *Let* $b > a^2/n$, $m = a/n$, $s = \sqrt{b - a^2/n}$, *and* $\mathcal{P}_2 = \mathbb{R}^n[\mathrm{S}, a; P_2, b]$. *Then*

$$\begin{aligned}
\sup\nolimits_{\preceq} \mathcal{P}_2 &= \left(\left\langle \frac{a}{n} + \left(\sqrt{\frac{k(n-k)}{n}} - \sqrt{\frac{(k-1)(n-k+1)}{n}}\right)\sqrt{b - \frac{a^2}{n}} \right\rangle_{k=1}^{n}\right),\\
\sup\nolimits_{\leq} \mathcal{P}_2 &= \left(\left\langle \frac{a}{n} + \sqrt{\frac{n-k}{nk}}\sqrt{b - \frac{a^2}{n}} \right\rangle_{k=1}^{n-1}, \frac{a}{n} - \sqrt{\frac{1}{n(n-1)}}\sqrt{b - \frac{a^2}{n}}\right),\\
\inf\nolimits_{\leq} \mathcal{P}_2 &= \left(\frac{a}{n} + \sqrt{\frac{1}{n(n-1)}}\sqrt{b - \frac{a^2}{n}}, \left\langle \frac{a}{n} - \sqrt{\frac{k}{n(n-k)}}\sqrt{b - \frac{a^2}{n}} \right\rangle_{k=1}^{n-1}\right),\\
\inf\nolimits_{\preceq} \mathcal{P}_2 &= \Bigg(\left\langle \frac{a}{n} + \sqrt{\frac{1}{n(n-1)}}\sqrt{b - \frac{a^2}{n}} \right\rangle^{\lfloor n/2\rfloor},\\
&\qquad \left\langle \frac{a}{n}\right\rangle^{n-2\lfloor n/2\rfloor}, \left\langle \frac{a}{n} - \sqrt{\frac{1}{n(n-1)}}\sqrt{b - \frac{a^2}{n}} \right\rangle^{\lfloor n/2\rfloor}\Bigg).
\end{aligned}$$

Choosing $a = \operatorname{tr}\mathbf{A}$, $b = \operatorname{tr}\mathbf{A}^2$, these results yield the best possible bounds for the eigenvalues of $\mathbf{A}$ relative to $\preceq$ and $\leq$ using only n, $\operatorname{tr}\mathbf{A}$, and $\operatorname{tr}\mathbf{A}^2$. If we know the eigenvalues to be nonnegative, we can apply Theorems 37–40 in a similar way. This produces the best possible bounds with the extra bound $\boldsymbol{\lambda}(\mathbf{A}) \geq \mathbf{0}$.

5.3 A numerical example

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{pmatrix}$$

in [29]. Now

$$\boldsymbol{\lambda}(\mathbf{A}) = (9.38, 6.42, 4.78, 1.43)$$

and $n = 4$, $p_1 = \operatorname{tr} \mathbf{A} = 22$, $p_2 = \operatorname{tr} \mathbf{A}^2 = 154$, $p_3 = \operatorname{tr} \mathbf{A}^3 = 1201$, $p_4 = \operatorname{tr} \mathbf{A}^4 = 9954$, $s_3 = \operatorname{tr} \mathbf{A}^{(3)} = (p_1^3 - 3p_1p_2 + 2p_3)/6 = 481$, $s_4 = \operatorname{tr} \mathbf{A}^{(4)} = \det \mathbf{A} = 410$.

From the bounds presented in [6] and again in [29] we deduce that $\boldsymbol{\lambda} \in [0.5, 10.5]^4$. Denoting $\mathcal{G}(G, b) = [0.5, 10.5]^4[\mathrm{S}, 22; G, b]$, we obtain by Theorem 9

$$\begin{aligned}
\sup\nolimits_{\preceq} \mathcal{G}(P_2, p_2) &= (10.47, 6.27, 4.73, 0.53), \\
\sup\nolimits_{\preceq} \mathcal{G}(P_3, p_3) &= (10.04, 6.66, 4.80, 0.50), \\
\sup\nolimits_{\preceq} \mathcal{G}(P_4, p_4) &= (\ \ 9.77, 6.98, 4.74, 0.50), \\
\sup\nolimits_{\preceq} \mathcal{G}(-S_3, -s_3) &= (10.50, 6.29, 4.31, 0.89), \\
\sup\nolimits_{\preceq} \mathcal{G}(-S_4, -s_4) &= (10.50, 6.83, 3.44, 1.24);
\end{aligned}$$

by Theorem 10

$$\begin{aligned}
\sup\nolimits_{\leq} \mathcal{G}(P_2, p_2) &= (10.47, 8.37, 7.16, 3.84), \\
\sup\nolimits_{\leq} \mathcal{G}(P_3, p_3) &= (10.04, 8.35, 6.32, 3.99), \\
\sup\nolimits_{\leq} \mathcal{G}(P_4, p_4) &= (\ \ 9.77, 8.38, 6.17, 4.08), \\
\sup\nolimits_{\leq} \mathcal{G}(-S_3, -s_3) &= (10.50, 8.40, 7.04, 2.87), \\
\sup\nolimits_{\leq} \mathcal{G}(-S_4, -s_4) &= (10.50, 8.66, 6.92, 2.42);
\end{aligned}$$

by Theorem 11

$$\begin{aligned}
\inf\nolimits_{\leq} \mathcal{G}(P_2, p_2) &= (7.16, 3.84, 2.63, 0.53), \\
\inf\nolimits_{\leq} \mathcal{G}(P_3, p_3) &= (8.05, 3.99, 2.65, 0.50), \\
\inf\nolimits_{\leq} \mathcal{G}(P_4, p_4) &= (8.27, 4.08, 2.62, 0.50), \\
\inf\nolimits_{\leq} \mathcal{G}(-S_3, -s_3) &= (7.04, 4.76, 2.60, 0.89), \\
\inf\nolimits_{\leq} \mathcal{G}(-S_4, -s_4) &= (6.92, 4.95, 2.34, 1.24);
\end{aligned}$$

and by Theorem 12

$$\begin{aligned}
\inf\nolimits_{\preceq} \mathcal{G}(P_2, p_2) &= (7.16, 7.16, 3.84, 3.84),\\
\inf\nolimits_{\preceq} \mathcal{G}(P_3, p_3) &= (8.05, 5.97, 3.99, 3.99),\\
\inf\nolimits_{\preceq} \mathcal{G}(P_4, p_4) &= (8.27, 5.58, 4.08, 4.08),\\
\inf\nolimits_{\preceq} \mathcal{G}(-S_3, -s_3) &= (7.04, 7.04, 5.06, 2.87),\\
\inf\nolimits_{\preceq} \mathcal{G}(-S_4, -s_4) &= (6.92, 6.92, 5.74, 2.42).
\end{aligned}$$

5.4 Other algebraic bounds

In [20, Sections 3 and 4] we presented algebraically expressible bounds for the eigenvalues using n, $\operatorname{tr} \mathbf{A}$, and $\det \mathbf{A}$. The only sharp bounds were the lower bound for $\lambda_{\frac{n}{2}+1}$ and the upper bound for $\lambda_{\frac{n}{2}}$ in the case where n is even. If $\operatorname{tr} \mathbf{A} = 0$, then we can give the sharp algebraic bounds for the eigenvalues $\lambda_i = \lambda_i(\mathbf{A})$ using $\operatorname{tr} \mathbf{A}^m$ for $m = 2, 4, 6, \dots$. The proof of the following lemma is elementary.

Lemma 47. *Let $\mathcal{G} = \mathbb{R}^n[\mathrm{S}, 0; P_m, b]$ and let m be even. Then*

$$\boldsymbol{\alpha}_k(\mathcal{G}) = \left(\left\langle \left(\frac{K^{m-1} b}{k K^{m-1} + k^m} \right)^{1/m} \right\rangle^k, \left\langle \frac{-k}{K} \left(\frac{K^{m-1} b}{k K^{m-1} + k^m} \right)^{1/m} \right\rangle^K \right),$$

where $K = n - k$.

The trace of the matrix $\mathbf{A} - (\operatorname{tr} \mathbf{A}/n)\mathbf{I}$, where $\mathbf{I}$ is an identity matrix, is zero. Just by knowing $\operatorname{tr} \mathbf{A}^k$ for $k = 1, 2, 3, \dots, m$, we can easily compute $\operatorname{tr}(\mathbf{A} - t\mathbf{I})^m$ (for any t). For example, denoting $a_k = \operatorname{tr} \mathbf{A}^k$ for $k \le 4$, we have

$$\operatorname{tr}\big(\mathbf{A} - (\operatorname{tr} \mathbf{A}/n)\mathbf{I}\big)^4 = a_4 - 4\frac{a_1}{n} a_3 + 6\Big(\frac{a_1}{n}\Big)^2 a_2 - 3n\Big(\frac{a_1}{n}\Big)^4.$$

Example 23. Consider the matrix

$$\mathbf{A} = \begin{pmatrix}
1 & 3 & 2 & 1 & 4 & 2 & 1\\
3 & -1 & 2 & -1 & -1 & 2 & 3\\
2 & 2 & -4 & -2 & -1 & -1 & 1\\
1 & -1 & -2 & 2 & -2 & 1 & 0\\
4 & -1 & -1 & -2 & -3 & 0 & 0\\
2 & 2 & -1 & 1 & 0 & 3 & 1\\
1 & 3 & 1 & 0 & 0 & 1 & 4
\end{pmatrix}$$

in [22, Example 5.4]. Then

$$\boldsymbol{\lambda}(\mathbf{A}) = (8.31, 4.12, 3.06, 0.92, -1.08, -5.26, -8.06),$$

$\operatorname{tr}\mathbf{A} = 2$, $\operatorname{tr}\mathbf{A}^2 = 190$, $\operatorname{tr}\mathbf{A}^3 = 2$, and $\operatorname{tr}\mathbf{A}^4 = 10134$.

Using $n = 7$, $a = \operatorname{tr}\mathbf{A}$, and $b = \operatorname{tr}\mathbf{A}^2$, we obtain

$$(2.40, -1.84, -3.01, -4.22, -5.73, -7.94, -12.46) \\ \leq \boldsymbol{\lambda}(\mathbf{A}) \leq (13.03, 8.52, 6.30, 4.80, 3.58, 2.41, -1.83),$$

whereas using $b = \operatorname{tr}\mathbf{A}^4$ instead, the bounds read

$$(2.00, -1.34, -2.95, -4.87, -6.77, -8.24, -10.01) \\ \leq \boldsymbol{\lambda}(\mathbf{A}) \leq (10.03, 8.36, 7.16, 5.58, 3.70, 2.01, -1.33).$$

Since the eigenvalues of $\mathbf{A}$ are not nonnegative, we cannot use directly $\operatorname{tr}\mathbf{A}^3$.

Denote $\mathbf{B} = \mathbf{A} - (\operatorname{tr}\mathbf{A}/n)\mathbf{I}$. Then $\operatorname{tr}\mathbf{B} = 0$ and $\operatorname{tr}\mathbf{B}^4 \approx 10225$. Using $n = 7$, $a = \operatorname{tr}\mathbf{B}$, and $b = \operatorname{tr}\mathbf{B}^4$ we obtain algebraically expressible bounds for $\boldsymbol{\lambda}(\mathbf{B})$ and for $\boldsymbol{\lambda}(\mathbf{A}) = \boldsymbol{\lambda}(\mathbf{B}) + (\operatorname{tr}\mathbf{A}/n)\mathbf{1}$. The bounds for $\boldsymbol{\lambda}(\mathbf{A})$ are the following:

$$(1.95, -1.39, -3.05, -4.97, -6.72, -8.04, -9.76) \\ \leq \boldsymbol{\lambda}(\mathbf{A}) \leq (10.33, 8.62, 7.29, 5.54, 3.62, 1.96, -1.38).$$

Note especially that both the lower and the upper bound for the smallest eigenvalue have been improved upon compared to the bounds of Wolkowicz and Styan.

Part III

k-majorization

6 Transfers

6.1 The principle of transfers

We have in Chapter 4 answered the question about the lower and upper bounds for the elements of a set $\mathcal{G} = I^n[\mathrm{S}, a; G, b]$. In the following chapters we will study how these bounds can be sharpened with additional information about the elements of $\mathcal{G}$. We will also consider some optimization problems over the set $\mathcal{G}$. Our main technique is based on a generalized majorization ordering. In order to present this concept, which we call k-majorization, we will introduce the concept of k-transfer in the next section. It is a natural generalization of transfers familiar from majorization theory (see also remark on p. 65 after Example 25).

One origin of majorization theory can be seen in the attempts to find a measure for inequality of incomes. According to [12, p. 6], H. Dalton, in 1920, described the 'principle of transfers' in the context of income distribution as follows:

> If there are only two income-receivers and a transfer of income takes place from the richer to the poorer, inequality is diminished. There is, indeed, an obvious limiting condition. The transfer must not be so large as to more than reverse the relative positions of the two income receivers [...] And, we may safely go further and say that, however great the number of income receivers and whatever the amount of their incomes, any transfer between any two of them, or, in general, any series of such transfers [satisfying the limiting condition] will diminish inequality.

As early as 1903, Muirhead discussed transfers in Dalton's sense and proved the following result:

Theorem 48 (see [12, p. 135]). *If* $\mathbf{x} \preceq \mathbf{y}$ *on* $\mathbb{Z}$*, then* $\mathbf{x}$ *can be derived from* $\mathbf{y}$ *by successive applications of a finite number of transfers of form*

$$\mathbf{z} \mapsto \mathbf{z} - \mathbf{e}_i + \mathbf{e}_j,$$

where $z_i < z_j$*.*

Let $\mathbf{P}_{ij}$ denote the permutation matrix that interchanges the coordinates i and j. Then a linear transformation with a matrix $\lambda\mathbf{I} + (1-\lambda)\mathbf{P}_{ij}$, where

$0 \leq \lambda \leq 1$, is called a T-transform (see [12, p. 21]). The concepts of a transfer and a T-transform are closely related. Namely, if $x_i - x_j \neq 0$ and if $\lambda = (x_i - x_j - \Delta)/(x_i - x_j)$, that is, $\Delta = (x_i - x_j) - (x_i - x_j)\lambda$, then

$$\mathbf{x}\big(\lambda\mathbf{I} + (1-\lambda)\mathbf{P}_{ij}\big) = \mathbf{x} - \Delta\mathbf{e}_i + \Delta\mathbf{e}_j.$$

Hardy, Littlewood, and Pólya have produced the following modification of theorem 48:

Theorem 49 (see [12, p. 21–22]). *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*. If* $\mathbf{x} \preceq \mathbf{y}$*, then* $\mathbf{x}$ *can be derived from* $\mathbf{y}$ *by successive applications of a finite number of* T*-transforms.*

The converse of Theorem 49 is trivially true. The matrices of T-transforms are doubly stochastic. Since the set of the doubly stochastic matrices is closed under matrix multiplication, we deduce that $\mathbf{x} \preceq \mathbf{y}$ if and only if $\mathbf{x} = \mathbf{yM}$ for some doubly stochastic matrix $\mathbf{M}$.

There are many ways to generalize the concept of majorization (cf. [12, Chapter 14]). Parker and Ram (see [24, definition 4.6]) replaced the semigroups of double stochastic matrices by an arbitrary semigroup of linear transformations. We will take our starting point from the definition of majorization with the help of transfers (which are affine, not linear transforms), and present yet another generalization. We will designate it k-majorization, where k is a positive integer.

6.2 k-transfers

Assume $\emptyset \neq \mathcal{X} \subseteq \mathbb{R}^n_{\downarrow}$. Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ be a nonzero vector of $\mathbb{R}^n$ and let $T_{\boldsymbol{\varepsilon}} \colon \mathcal{X} \to \mathbb{R}^n$ be a transform $T_{\boldsymbol{\varepsilon}}(\mathbf{x}) = \mathbf{x} + \boldsymbol{\varepsilon}$. If $\mathbf{x} \in \mathcal{X}$ and $T_{\boldsymbol{\varepsilon}}(\mathbf{x}) \in \mathcal{X}$, we say that $T_{\boldsymbol{\varepsilon}}$ is *an* $\mathcal{X}$*-consistent transfer of* $\mathbf{x}$. If $\mathbf{x}$ and $\mathcal{X}$ are deducible from the context, we say briefly that '$T_{\boldsymbol{\varepsilon}}$ is a transfer'.

Let the nonzero components of $\boldsymbol{\varepsilon}$ be $(\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_k})$, where the order of components is preserved. We say that $T_{\boldsymbol{\varepsilon}}$ is *of type*

$$\big(\updownarrow(\varepsilon_{i_1})i_1, \updownarrow(\varepsilon_{i_2})i_2, \ldots, \updownarrow(\varepsilon_{i_k})i_k\big),$$

where the sign '$\updownarrow(\varepsilon)$' is defined

$$\updownarrow(\varepsilon) = \begin{cases} \uparrow & \text{if } \varepsilon > 0, \\ \downarrow & \text{if } \varepsilon < 0. \end{cases}$$

If all components of $\boldsymbol{\varepsilon}$ are nonzero, then we omit the indices i_j. We also use such self-explanatory notations as $T(\mathbf{x}) = (\downarrow x_1, \downarrow x_2, x_3, \uparrow x_4, x_5)$, $T(\mathbf{x}) = (\ldots, \uparrow x_4, \ldots)$, $(\uparrow[i], \downarrow[n-i])$, etc. If the distance between $T(\mathbf{x})$ and $\mathbf{x}$ is close to zero, we say that $\mathbf{x}$ *is perturbed by* T. In this context we may call transfers *perturbations.*

Example 24. Let $a \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{X} = \mathbb{R}^n[\mathrm{S}, a]$. Then $T_{\boldsymbol{\varepsilon}}$ is an $\mathcal{X}$-consistent transfer of $\mathbf{x}$ if and only if $\mathrm{S}(\boldsymbol{\varepsilon}) = 0$ and $x_i - x_{i+1} \geq \varepsilon_{i+1} - \varepsilon_i$ for each index $i \leq n-1$. Further, if $T_{\boldsymbol{\varepsilon}}$ is a transfer, then $\mathbf{x} \prec T_{\boldsymbol{\varepsilon}}(\mathbf{x})$ if and only if $\mathbf{0} \leq_{\Sigma} \boldsymbol{\varepsilon}$.

For the purpose of this thesis, we define the following subclasses of transfers. Let the number of nonzero components of $\boldsymbol{\varepsilon}$ be k. Then we say that $T_{\boldsymbol{\varepsilon}}$ is *of size* k. If, in addition, the signs of any two consecutive nonzero components of $\boldsymbol{\varepsilon}$ are opposite, we call $T_{\boldsymbol{\varepsilon}}$ a *k-transfer*. Moreover, we define a transfer $T_{\boldsymbol{\varepsilon}}$ to be *positive* (or *negative*) if the first nonzero component of $\boldsymbol{\varepsilon}$ is positive (or negative). For example, according to these definitions, a positive 3-transfer is of type $(\uparrow i_1, \downarrow i_2, \uparrow i_3)$.

Example 25. Let $\mathcal{P}_2 = \mathbb{R}^n[\mathrm{S}, a; P_2, b]$. Assume that $\mathbf{x} \in \mathcal{P}_2$ and $x_1 \geq x_i > x_{i+1} \geq x_{j-1} > x_j$. We show that there exists a transfer $T_{\boldsymbol{\varepsilon}}$ of type $(\uparrow 1, \downarrow i, \uparrow j)$.

Assume for the time being that this kind of transfer exists. Since $\mathcal{P}_2$ is a sum-constant set, $\mathrm{S}(\boldsymbol{\varepsilon}) = \varepsilon_1 + \varepsilon_i + \varepsilon_j = 0$. Denote $t = \varepsilon_1$ and $s = s(t) = \varepsilon_j$, in which case $-t - s = \varepsilon_i$. Since $T_{\boldsymbol{\varepsilon}}$ is a positive 3-transfer, t and s are positive. We can now write $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(t) = t\mathbf{e}_1 - (t+s)\mathbf{e}_i + s\mathbf{e}_j$.

Solving the equation

$$P_2(\mathbf{x} + \boldsymbol{\varepsilon}(t)) = P_2(\mathbf{x})$$

and excluding the case $x_i - s - t < x_j + s$, we obtain

$$s(t) = \frac{-t + x_i - x_j - \sqrt{-4t(t + x_1 - x_i) + (t - x_i + x_j)^2}}{2}.$$

We can conclude that when $t > 0$ is small enough, also $s(t)$ is close to 0, $T_{\boldsymbol{\varepsilon}}(\mathbf{x})$ is decreasingly ordered, and $T_{\boldsymbol{\varepsilon}}$ is a $\mathcal{P}_2$-consistent 3-transfer of $\mathbf{x}$.

The above example points to another motivation for studying the concept of k-majorization: if we want to find $\max\{\, x_1 \mid \mathbf{x} \in \mathcal{P}_2 \,\}$, then we know that maximizer $\mathbf{x}$ must be of shape $([1] \geq [n-1])$. Otherwise there is a positive 3-transfer of $\mathbf{x}$ of type $(\uparrow 1, \downarrow i, \uparrow j)$, which increases the value of x_1. This 'perturbation technique with 3-transfers' is used in [13], [14], [15], and [22], among others, to solve optimization problems similar to those we have discussed in previous chapters. However, the use of 3-transfers does not always suffice for solving our optimization problems (see, for example, p. 89 Example 34 and Lemma 62).

6.3 k-majorization

The proof of Theorem 49 presented in [12] brings us to

Theorem 50. *Let $a \in \mathbb{R}$ and $\mathcal{X} = I^n[\mathrm{S}, a]$. Assume $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Then $\mathbf{x} \preceq \mathbf{y}$ if and only if there exist positive $\mathcal{X}$-consistent 2-transfers $T_1, T_2, \ldots, T_m$ such that $\mathbf{y} = T_1 \circ T_2 \circ \cdots \circ T_m(\mathbf{x})$.*

Theorem 50 motivates us to define the concept of *k-majorization* as follows: Let $\mathcal{X} \subseteq \mathbb{R}^n_\downarrow$ and $\mathbf{x}_\downarrow, \mathbf{y}_\downarrow \in \mathcal{X}$. Then $\mathbf{x}$ is *k-majorized by* $\mathbf{y}$ *on* $\mathcal{X}$, denoted by

$$\mathbf{x} \preceq^k_{\mathcal{X}} \mathbf{y},$$

if either $\mathbf{x}_\downarrow = \mathbf{y}_\downarrow$ or there exist transfers $T_{\boldsymbol{\varepsilon}_1}, T_{\boldsymbol{\varepsilon}_2}, \ldots, T_{\boldsymbol{\varepsilon}_m}$ such that $T_{\boldsymbol{\varepsilon}_\ell}$ is a positive $\mathcal{X}$-consistent k-transfer of $\mathbf{x}_\downarrow + \sum_{i=1}^{\ell-1} \boldsymbol{\varepsilon}_i$ for $\ell = 1, 2, \ldots, m$, and

$$\mathbf{y}_\downarrow = \mathbf{x}_\downarrow + \sum_{i=1}^{m} \boldsymbol{\varepsilon}_i.$$

If $\mathbf{x} \preceq^k_{\mathcal{X}} \mathbf{y}$, while $\mathbf{x}_\downarrow \neq \mathbf{y}_\downarrow$, then we denote $\mathbf{x} \prec^k_{\mathcal{X}} \mathbf{y}$ and say that $\mathbf{x}$ is *strictly k-majorized by* $\mathbf{y}$.

The relation $\prec^k_{\mathcal{X}}$ is reflexive and transitive, but generally not antisymmetric. Let $\mathcal{A}$ be a vector space and let $\ll$ be a reflexive and transitive relation defined on $\mathcal{A}$. If $\ll$ is compatible with the vector structure, i.e., if

$$\mathbf{x} \ll \mathbf{y} \text{ on } \mathcal{A} \Rightarrow \mathbf{x} + \mathbf{z} \ll \mathbf{y} + \mathbf{z} \text{ on } \mathcal{A} \text{ for } \mathbf{z} \in \mathcal{A},$$

$$\mathbf{x} \ll \mathbf{y} \text{ on } \mathcal{A} \Rightarrow \lambda\mathbf{x} \ll \lambda\mathbf{y} \text{ on } \mathcal{A} \text{ for } \lambda > 0,$$

then the structure $(\mathcal{A}, \ll)$ is called *a partially ordered vector space* (see, for example [31, p. 1-2]; cf. also [12, p. 424–426]). As a rule, k-majorization is not compatible with the vector structure.

Besides the usual majorization, we will mainly apply 3-majorization. In what follows, a transfer means a 3-transfer unless otherwise stated, and the superscript ‘3’ in $\preceq^3_{\mathcal{X}}$ and $\prec^3_{\mathcal{X}}$ is omitted.

According to our general definition, a function F is $\prec_{\mathcal{X}}$-increasing if $F(\mathbf{x}) \leq F(\mathbf{y})$ whenever $\mathbf{x} \prec_{\mathcal{X}} \mathbf{y}$. We derive directly from the definition of 3-majorization the following example:

Example 26. Let $\emptyset \neq \mathcal{X} \subseteq \mathbb{R}^n_\downarrow$. The functions $\mathbf{x} \mapsto x_1$, $\mathbf{x} \mapsto x_n$, and $\mathbf{x} \mapsto x_1 + x_n$ are $\prec_{\mathcal{X}}$-increasing.

Theorem 50 implies that to show the Schur-convexity of the function $F\colon \mathbb{R}^n \to \mathbb{R}$, it suffices to prove the implication

$$\text{if } \mathbf{x} \prec \mathbf{y}, \text{ then } F(\mathbf{x}) \leq F(\mathbf{y})$$

for all vectors $\mathbf{x}$ and $\mathbf{y}$ which differ in only two components (cf. [12, p. 58]). Analogously, to prove that F is $\prec_{\mathcal{X}}$-increasing, it suffices to prove that $F(\mathbf{x}) \leq F(T_{\boldsymbol{\varepsilon}}(\mathbf{x}))$ for all relevant vectors $\mathbf{x}$ and 3-transfers $T_{\boldsymbol{\varepsilon}}$.

Next we will present a sufficient condition for 3-majorization. The proof of this result will be based on the following

Lemma 51. *Let* $\mathcal{G} = I^n[\mathrm{S}, a; G, b]$. *Assume that* $T_{\boldsymbol{\varepsilon}}$ *is a* $\mathcal{G}$*-consistent transfer of type* $(\uparrow i_1, \ldots, \downarrow i_\ell, \ldots, \uparrow i_m)$ *and of size* $m \geq 4$. *Then* $T_{\boldsymbol{\varepsilon}}$ *can be presented as a composition of a transfer of size no more than* $m - 1$ *and a positive 3-transfer.*

Proof. Assume that $T_{\boldsymbol{\varepsilon}}$ is a transfer of $\mathbf{x} \in \mathcal{G}$. Denote $\mathbf{y} = T_{\boldsymbol{\varepsilon}}(\mathbf{x})$. Let

$$\begin{aligned} i &= \min\{\, \ell \mid \varepsilon_\ell > 0 \,\}, \\ j &= \max\{\, \ell \mid \varepsilon_\ell < 0 \,\}, \\ k &= \min\{\, \ell \geq j+1 \mid \varepsilon_\ell > 0 \,\}. \end{aligned}$$

The type of $T_{\boldsymbol{\varepsilon}}$ implies that all these indices are well-defined and that $i < j < k$.

Denote $\mathcal{X} = I^n[\mathrm{S}, a]$, $\boldsymbol{\delta}(s,t) = s(\mathbf{e}_i - \mathbf{e}_j) + t(-\mathbf{e}_j + \mathbf{e}_k)$, and $\mathbf{x}(s,t) = \mathbf{x} + \boldsymbol{\delta}(s,t)$. Since $T_{\boldsymbol{\varepsilon}}$ is also an $\mathcal{X}$-consistent transfer, $\mathbf{x}(s,t) \in \mathcal{X}$, provided that $0 \leq s \leq \varepsilon_i$, $0 \leq t \leq \varepsilon_k$, and $s + t \leq |\varepsilon_j|$. We begin our proof by showing that there is a $\mathcal{G}$-consistent 3-transfer $T_{\boldsymbol{\delta}}$ of $\mathbf{x}$ such that

$$T_{\boldsymbol{\delta}}(\mathbf{x})_i = y_i, \quad \text{or} \quad T_{\boldsymbol{\delta}}(\mathbf{x})_j = y_j, \quad \text{or} \quad T_{\boldsymbol{\delta}}(\mathbf{x})_k = y_k.$$

Consider the following cases:

(1) $|\varepsilon_j| = \min\{\varepsilon_i, |\varepsilon_j|, \varepsilon_k\}$,

(2) $\varepsilon_i \geq |\varepsilon_j| > \varepsilon_k$,

(3) $|\varepsilon_j| > \varepsilon_i \geq \varepsilon_k$,

(4) $\varepsilon_k \geq |\varepsilon_j| > \varepsilon_i$,

(5) $|\varepsilon_j| > \varepsilon_k > \varepsilon_i$.

Case 1. Since ε_i, ε_k are at least $|\varepsilon_j|$, both $\mathbf{x}(0, |\varepsilon_j|)$ and $\mathbf{x}(|\varepsilon_j|, 0)$ belong to $\mathcal{X}$. Now G is strictly Schur-convex and hence

$$G\big(\mathbf{x}(0, |\varepsilon_j|)\big) < G(\mathbf{x}) < G\big(\mathbf{x}(|\varepsilon_j|, 0)\big).$$

Since G is continuous, there is a $s_0 \in (0, |\varepsilon_j|)$ such that $G(\mathbf{x}(s_0, |\varepsilon_j| - s_0)) = G(\mathbf{x})$. Choose $\boldsymbol{\delta} = \boldsymbol{\delta}(s_0, |\varepsilon_j| - s_0)$. It is easy to see that $T_{\boldsymbol{\delta}}$ is a $\mathcal{G}$-consistent transfer of $\mathbf{x}$ and that $T_{\boldsymbol{\delta}}(\mathbf{x})_j = x_j - s_0 - (|\varepsilon_j| - s_0) = x_j - |\varepsilon_j| = y_j$.

Case 2. Now $|\varepsilon_j| - \varepsilon_k < \varepsilon_i$. If $G(\mathbf{x}(|\varepsilon_j| - \varepsilon_k, \varepsilon_k)) \geq G(\mathbf{x})$, then there exists an $s_0 \in (0, |\varepsilon_j| - \varepsilon_k]$ such that $G(\mathbf{x}(s_0, \varepsilon_k)) = G(\mathbf{x})$. It follows that $T_{\boldsymbol{\delta}} = T_{\boldsymbol{\delta}(s_0, \varepsilon_k)}$ is a $\mathcal{G}$-consistent transfer of $\mathbf{x}$ such that $T_{\boldsymbol{\delta}}(\mathbf{x})_k = x_k + \varepsilon_k = y_k$.

If $G(\mathbf{x}(|\varepsilon_j| - \varepsilon_k, \varepsilon_k)) < G(\mathbf{x})$, then there exists a $t_0 \in (0, \varepsilon_k)$ such that $G(\mathbf{x}(|\varepsilon_j| - t_0, t_0)) = G(\mathbf{x})$, and we can now choose a $\mathcal{G}$-consistent transfer $T_{\boldsymbol{\delta}} = T_{\boldsymbol{\delta}(|\varepsilon_j| - t_0, t_0)}$ for which $T_{\boldsymbol{\delta}}(\mathbf{x})_j = x_j - |\varepsilon_j| = y_j$.

Case 3. Assume first $|\varepsilon_j| \geq \varepsilon_i + \varepsilon_k$. If $G(\mathbf{x}(\varepsilon_i, \varepsilon_k)) \leq G(\mathbf{x})$, then there is a $t_0 \in (0, \varepsilon_k]$ such that the transfer $T_{\boldsymbol{\delta}} = T_{\boldsymbol{\delta}(\varepsilon_i, t_0)}$ satisfies $T_{\boldsymbol{\delta}}(\mathbf{x})_i = y_i$. Otherwise, we can find an $s_0 \in (0, \varepsilon_i)$ such that the transfer $T_{\boldsymbol{\delta}} = T_{\boldsymbol{\delta}(s_0, \varepsilon_k)}$ satisfies $T_{\boldsymbol{\delta}}(\mathbf{x})_k = y_k$.

Assume then that $|\varepsilon_j| < \varepsilon_i + \varepsilon_k$. Together with our initial assumption this implies that $\mathbf{v} = \mathbf{x}(\varepsilon_i, |\varepsilon_j| - \varepsilon_i)$ and $\mathbf{u} = \mathbf{x}(|\varepsilon_j| - \varepsilon_k, \varepsilon_k)$ belong to $\mathcal{X}$.

If $G(\mathbf{v}) \le G(\mathbf{x})$, then there exists a $t_0 \in (0, |\varepsilon_j| - \varepsilon_i]$ such that $G(\mathbf{x}(\varepsilon_i, t_0)) = G(\mathbf{x})$ and we can choose $T_{\boldsymbol{\delta}} = T_{\boldsymbol{\delta}(\varepsilon_i, t_0)}$. If, on the other hand, $G(\mathbf{u}) \ge G(\mathbf{x})$, then there exists a $s_0 \in (0, |\varepsilon_j| - \varepsilon_k]$ such that $G(\mathbf{x}(s_0, \varepsilon_k)) = G(\mathbf{x})$ and we can choose $T_{\boldsymbol{\delta}} = T_{\boldsymbol{\delta}(\mathbf{x}(s_0, \varepsilon_k))}$.

Consider now the possibility that $G(\mathbf{v}) > G(\mathbf{x})$ and $G(\mathbf{u}) < G(\mathbf{x})$. Then there exists a $t_0 \in (0, \varepsilon_i + \varepsilon_k - |\varepsilon_j|)$ such that $\mathbf{x}(|\varepsilon_j| - \varepsilon_k + t_0, \varepsilon_k - t_0) \in \mathcal{G}$. Choosing $\boldsymbol{\delta} = \boldsymbol{\delta}(|\varepsilon_j| - \varepsilon_k + t_0, \varepsilon_k - t_0)$, we find a $\mathcal{G}$-consistent transfer $T_{\boldsymbol{\delta}}$ for which $T_{\boldsymbol{\delta}}(\mathbf{x})_j = y_j$.

Case 4 is symmetric with Case 2, as is Case 5 with Case 3.

After the construction of the transfer $T_{\boldsymbol{\delta}}$, the rest of the proof is simple: By the assumptions, the number of nonzero components of $\boldsymbol{\varepsilon}$ is m. The vector $\boldsymbol{\delta}$ constructed above is such that $\boldsymbol{\varepsilon} - \boldsymbol{\delta}$ has at most $m-1$ nonzero components. Since $(\mathbf{x} + \boldsymbol{\delta}) + (\boldsymbol{\varepsilon} - \boldsymbol{\delta}) = \mathbf{y}$, the transform $T_{\boldsymbol{\varepsilon}-\boldsymbol{\delta}}$ is a transfer of $(\mathbf{x} + \boldsymbol{\delta})$ of size no more than $m-1$ and $T_{\boldsymbol{\varepsilon}} = T_{\boldsymbol{\varepsilon}-\boldsymbol{\delta}} \circ T_{\boldsymbol{\delta}}$.

Example 27. Let $\mathcal{P}_2 = \mathbb{R}^4_{\downarrow}[\mathrm{S}, 229; P_2, 12661]$, $\mathbf{x} = (67, 66, 46, 40, 10)$, and $\mathbf{y} = (68, 62, 55, 32, 12)$. Then $T = T_{(1,-4,9,-8,2)}$ is a transfer of type $(\uparrow, \downarrow, \uparrow, \downarrow, \uparrow)$ and $T(\mathbf{x}) = \mathbf{y}$. From the proof of Lemma 51 we obtain a composition

$$T = T_{(0,-4,9,-6,1)} \circ T_{(1,0,0,-2,1)},$$

where $T_{(1,0,0,-2,1)}$ is a positive 3-transfer of $\mathbf{x}$ and $T_{(0,-4,9,-6,1)}$ is a transfer of $(68, 66, 46, 38, 11) = T_{(1,0,0,-2,1)}(\mathbf{x})$.

Let T, $\mathbf{x}$, and $\mathbf{y}$ be as in the previous example. Now $P_5(\mathbf{x}) > P_5(\mathbf{y})$. We will prove later (see p. 91 Example 38) that this implies that $\mathbf{x} \not\prec_{\mathcal{P}_2} \mathbf{y}$, or, equivalently, that T cannot be presented as a composition of positive 3-transfers.

The sign of the components of $(1, -4, 9, -8, 2)$ changes four times. In the following theorem we show that any transfer $T_{\boldsymbol{\varepsilon}}$, where the sign of the components of $\boldsymbol{\varepsilon}$ changes only twice, can be presented as a composition of 3-transfers. By the notation $(\ldots \updownarrow_k \ldots)$ we mean that the sign '$\updownarrow$' is repeated k times.

Theorem 52. *Let $\mathcal{G} = I^n[\mathrm{S}, a; G, b]$, and let $T_{\boldsymbol{\varepsilon}}$ be a $\mathcal{G}$-consistent transfer of $\mathbf{x}$ of type $(\uparrow_{k_1}, \downarrow_{k_2}, \uparrow_{k_3})$. Then $\mathbf{x} \prec_{\mathcal{G}} T_{\boldsymbol{\varepsilon}}(\mathbf{x})$.*

Proof. By definition, if $T_{\boldsymbol{\varepsilon}}$ is of size 3, then $\mathbf{x} \prec_{\mathcal{G}} T_{\boldsymbol{\varepsilon}}(\mathbf{x})$. Hence we assume that $T_{\boldsymbol{\varepsilon}}$ is of size $m \ge 4$. By Lemma 51, $T_{\boldsymbol{\varepsilon}} = T_{\boldsymbol{\varepsilon}_1} \circ T_{\boldsymbol{\delta}_1}$, where $T_{\boldsymbol{\varepsilon}_1}$ is a transfer of size at most $m-1$ and $T_{\boldsymbol{\delta}_1}$ is a positive 3-transfer.

Since $\mathcal{G}$ is sum-constant and G-constant, the transfer $T_{\boldsymbol{\varepsilon}_1}$ must be of type $(\uparrow_{\tilde{k}_1}, \downarrow_{\tilde{k}_2}, \uparrow_{\tilde{k}_3})$ (where all indices are still positive). If $T_{\boldsymbol{\varepsilon}_1}$ is of size 3, the proof is complete. Otherwise we can continue to apply Lemma 51, until we obtain a sequence of at most $m-2$ vectors $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$, $\ldots$, $\boldsymbol{\delta}_k$ such that $T_{\boldsymbol{\delta}_i}$ is a positive 3-transfer of $T_{\boldsymbol{\delta}_{i-1}} \circ \cdots \circ T_{\boldsymbol{\delta}_2} \circ T_{\boldsymbol{\delta}_1}$ for $i = 2, 3, \ldots, k$ and $\sum_{j=1}^{k} \boldsymbol{\delta}_j = \boldsymbol{\varepsilon}$. It follows that $\mathbf{x} \prec_{\mathcal{G}} T_{\boldsymbol{\varepsilon}}(\mathbf{x})$.

Example 28. Let $\mathcal{G} = \mathbb{R}^n[\mathrm{S}, a; G, b]$. Denote $\boldsymbol{\alpha}_k = \mathbf{s}([k] \geq [n-k]; \mathcal{G})$. By the basic assumptions, $(\langle a/n \rangle^n) \notin \mathcal{G}$, and therefore $\boldsymbol{\alpha}_i \neq \boldsymbol{\alpha}_j$ when $i \neq j$. It follows from the result in Example 21 (p. 43) together with Theorem 52 that

$$\boldsymbol{\alpha}_{n-1} \prec_{\mathcal{G}} \cdots \prec_{\mathcal{G}} \boldsymbol{\alpha}_2 \prec_{\mathcal{G}} \boldsymbol{\alpha}_1.$$

In the next two chapters we will utilize to some extent transfers of type $(\uparrow_i, \downarrow_j)$ and $(\downarrow_i, \uparrow_j)$. After that, in Chapter 9, we will exploit the concept of 3-transfers in a more fundamental way.

7 Using extra bounds

7.1 An extra lower bound

Merikoski and Wolkowicz [22, Section 4] posed and solved the following problem: Let $\mathbf{A}$ be a Hermitian matrix and let $1 \le k \le n-1$. What is the best possible upper bound for $\lambda_k(\mathbf{A})$ and the best possible lower bound for $\lambda_{k+1}(\mathbf{A})$ when n, $\operatorname{tr}\mathbf{A}$, $\operatorname{tr}\mathbf{A}^2$, and the diagonal of $\mathbf{A}$ are given?

Now the information about the diagonal yields an extra majorization bound: the eigenvalues of a Hermitian matrix majorize its diagonal elements. Conversely, if $(a_1, a_2, \ldots, a_n) \preceq (\lambda_1, \lambda_2, \ldots, \lambda_n)$, then there exists a Hermitian (in fact, real symmetric) matrix $\mathbf{A}$ with the diagonal elements a_1, a_2, $\ldots$, a_n and the eigenvalues λ_1, λ_2, $\ldots$, λ_n (see, for example, [12, p. 218–220]). Hence we can restate the above problem as follows: Let $\operatorname{tr}\mathbf{A} = a$, $\operatorname{tr}\mathbf{A}^2 = b$ and let the ordered diagonal of $\mathbf{A}$ be $\mathbf{a}$. Find $\max x_k$ and $\min x_{k+1}$ subject to

$$\mathbf{x} \in \{\, \mathbf{x} \in \mathbb{R}^n_{\downarrow} \mid \mathrm{S}(\mathbf{x}) = a,\ P_2(\mathbf{x}) = b,\ \mathbf{a} \preceq \mathbf{x} \,\}.$$

The proof in [22] is easily generalized by using any strictly Schur-convex function instead of P_2. Recalling the notation

$$\boldsymbol{\alpha}_k = \big(\langle \alpha_k \rangle^k, \langle \bar{\alpha}_k \rangle^{n-k}\big) = \mathbf{s}([k] \ge [n-k]; \mathcal{G}),$$

we have

Theorem 53 (cf. [22, Theorem 4.1]). *Let $\mathcal{G} = \mathbb{R}^n[\mathrm{S}, a; G, b]$ and let $\mathbf{a} \in \mathbb{R}^n_{\downarrow}$ be such that $\mathrm{S}(\mathbf{a}) = a$ and $G(\mathbf{a}) < b$. Denote $\mathcal{G}_{\mathbf{a}} = \{\, \mathbf{v} \in \mathcal{G} \mid \mathbf{a} \preceq \mathbf{v} \,\}$.*

Then $\min\{\, x_1 \mid \mathbf{x} \in \mathcal{G}_{\mathbf{a}} \,\} = \max\{\alpha_{n-1}, a_1\}$ and $\max\{\, x_n \mid \mathbf{x} \in \mathcal{G}_{\mathbf{a}} \,\} = \min\{\bar{\alpha}_1, a_n\}$.

Let $k \le n-1$. Then there exist nonnegative indices $i \le k-1$ and $j \le n-k-1$ and real numbers α and $\bar{\alpha}$ such that the point

$$\big(a_1, \ldots, a_i, \langle \alpha \rangle^{k-i}, \langle \bar{\alpha} \rangle^{n-k-j}, a_{n-j+1}, \ldots, a_n\big)$$

belongs to $\mathcal{G}_{\mathbf{a}}$. This point is uniquely determined and

$$\max\{\, x_k \mid \mathbf{x} \in \mathcal{G}_{\mathbf{a}} \,\} = \alpha$$

and

$$\min\{\, x_{k+1} \mid \mathbf{x} \in \mathcal{G}_{\mathbf{a}} \,\} = \bar{\alpha}.$$

We will study the related question about the sets that are bounded above relative to majorization. Merikoski and Wolkowicz [22, Theorems 3.3 and 3.4] have touched upon this question by considering the extra bounds $x_n \geq a_n$ and $x_1 \leq a_1$.

7.2 An extra upper bound

Let $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n[\mathrm{S}, a]$ and let $\mathcal{G}$ be as in Theorem 53. Denote $\mathcal{G}^{\mathbf{c}} = \{\, \mathbf{v} \in \mathcal{G} \mid \mathbf{v} \preceq \mathbf{c} \,\}$. We assume that $\mathcal{G}^{\mathbf{c}}$ is nonempty and not the singleton $\{\mathbf{c}\}$, or, equivalently, that $G(\langle a/n \rangle^n) = G(\langle \mathrm{M}_{1n}(\mathbf{c}) \rangle^n) < b < G(\mathbf{c})$.

We begin by showing that there are real numbers α and $\bar{\alpha}$, and a non-negative index ℓ such that the vector

$$\hat{\mathbf{x}} = \big(c_1, c_2, \ldots, c_\ell, \alpha, \langle \bar{\alpha} \rangle^{n-\ell-1}\big)$$

belongs to $\mathcal{G}^{\mathbf{c}}$, and that $\max\{\, x_n \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \bar{\alpha}$.

Define a vector $\mathbf{v}(i)$ by

$$\mathbf{v}(i) = \big(\mathbf{c}_{1:i}, \langle \mathrm{M}_{i+1,n}(\mathbf{c}) \rangle^{n-i}\big).$$

The set

$$L = \{\, i \mid 0 \leq i \leq n-2,\ G(\mathbf{v}(i)) \leq b \,\}$$

is nonempty, and we can choose $\ell = \max L$. The choice of ℓ guarantees that $G(\mathbf{v}(\ell)) \leq b$ and $G(\mathbf{v}(\ell+1)) > b$. By the basic assumptions, G is continuous. Hence if $G(\mathbf{v}(\ell)) < b$, then there exists a positive $\mathbb{R}^n[\mathrm{S}, a]$-consistent transfer T of type $(\uparrow \ell+1, \downarrow \ell+2, \ldots, \downarrow n)$ such that $G\big(T(\mathbf{v}(\ell))\big) = b$. Moreover, $T(\mathbf{v}(\ell))_{\ell+1} < c_{\ell+1}$. Therefore, there exist uniquely determined real numbers $\alpha < c_{\ell+1}$ and $\bar{\alpha}$ such that

$$\hat{\mathbf{x}} = \big(c_1, c_2, \ldots, c_\ell, \alpha, \langle \bar{\alpha} \rangle^{n-\ell-1}\big) \in \mathcal{G}.$$

(Note that if $\alpha_1 \leq c_1$, then $\ell = 0$ and $\hat{\mathbf{x}} = \boldsymbol{\alpha}_1$.)

By Lemma 4, $\hat{\mathbf{x}} \preceq \mathbf{c}$, which implies that $\hat{\mathbf{x}} \in \mathcal{G}^{\mathbf{c}}$. Hence $\max\{\, x_n \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} \geq \bar{\alpha}$. Assume to the contrary that $x_n > \bar{\alpha}$ for some $\mathbf{x} \in \mathcal{G}^{\mathbf{c}}$. Since $\mathbf{x}_{1:\ell} \preceq_w \mathbf{c}_{1:\ell}$ and $\mathbf{x}_{\ell+2:n} \preceq^w (\langle \bar{\alpha} \rangle^{n-\ell-1})$, we arrive at a contradiction $\mathbf{x} \prec \hat{\mathbf{x}}$. Therefore

$$\max\{\, x_n \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \bar{\alpha}.$$

Next we consider the problem $\max\{\, x_k \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\}$, where $k \leq n-1$. We denote below $\beta = \mathrm{M}_{1k}(\mathbf{c})$. The condition $\mathbf{x} \preceq \mathbf{c}$ implies that $x_k \leq \beta$. If $\alpha_k \leq \beta$, then $\boldsymbol{\alpha}_k \preceq \mathbf{c}$, which implies that $\boldsymbol{\alpha}_k \in \mathcal{G}^{\mathbf{c}}$ and, further, that $\max_{\mathbf{x} \in \mathcal{G}^{\mathbf{c}}} x_k = \alpha_k$.

Assume $\alpha_k > \beta$. Since

$$\big(\langle \beta \rangle^k, \langle (a - k\beta)/(n-k) \rangle^{n-k}\big) \prec \boldsymbol{\alpha}_k,$$

we have $G(\langle\beta\rangle^k, \langle(a-k\beta)/(n-k)\rangle^{n-k}) < b$. It follows that if

$$G\big(\langle\beta\rangle^k, c_{k+1}, c_{k+2}, \dots, c_n\big) \geq b,$$

then there exists $\mathbf{y} \in \mathbb{R}^{n-k}$ such that $(\langle\beta\rangle^k, \mathbf{y}) \in \mathcal{G}^{\mathbf{c}}$ (from now on we do not explicitly refer to the transfer in the background). Hence in this case $\max_{\mathbf{x}\in\mathcal{G}^{\mathbf{c}}} x_k = \beta$.

Finally, assume that $\alpha_k > \beta$ and $G(\langle\beta\rangle^k, c_{k+1}, c_{k+2}, \dots, c_n) < b$. Since $G(\mathbf{c}) > b$, we can choose

$$\ell = \max\big\{\, i \;\big|\; 0 \leq i \leq k-2,\ G\big(\mathbf{c}_{1:i}, \langle \mathrm{M}_{i+1,k}(\mathbf{c})\rangle^{k-i}, \mathbf{c}_{k+1:n}\big) \leq b \,\big\}.$$

Now there exist real numbers $\alpha < c_{\ell+1}$ and $\bar{\alpha} \geq c_{k+1}$ such that

$$\boldsymbol{\alpha} = \big(\mathbf{c}_{1:\ell}, \alpha, \langle\bar{\alpha}\rangle^{k-\ell-1}, \mathbf{c}_{k+1:n}\big) \in \mathcal{G}.$$

Since $\boldsymbol{\alpha} \preceq \mathbf{c}$, we have $\boldsymbol{\alpha} \in \mathcal{G}^{\mathbf{c}}$.

Assume that $\mathbf{x} \preceq \mathbf{c}$ and $x_k > \bar{\alpha}$. Then $\mathbf{x}_{1:\ell} \preceq_w \boldsymbol{\alpha}_{1:\ell}$ and $\mathbf{x}_{\ell+2:n} \preceq^w \boldsymbol{\alpha}_{\ell+2:n}$, which means that $\mathbf{x} \prec \boldsymbol{\alpha}$. It follows that $G(\mathbf{x}) < b$, and we conclude that $\max_{\mathbf{x}\in\mathcal{G}^{\mathbf{c}}} x_k = \bar{\alpha}$.

We have now found $\max_{\mathbf{x}\in\mathcal{G}^{\mathbf{c}}} x_k$. Note that $\max_{\mathbf{x}\in\mathcal{G}^{\mathbf{c}}} x_1$ is $\min\{c_1, \alpha_1\}$. To summarize these results, we introduce the following notation. Denote $\tilde{G}\colon \mathbb{R}^{n-i-j} \to \mathbb{R}$, $\tilde{G}(\mathbf{y}) = G(\mathbf{c}_{1:i}, \mathbf{y}, \mathbf{c}_{n-j+1:n})$, and

$$\mathcal{S}_{ij} = \mathbb{R}^{n-i-j}[\mathrm{S}, \mathrm{S}_{i+1,n-j}(\mathbf{c}); \tilde{G}, b],$$

where i and j are nonnegative integers with the sum $i+j$ at most $n-2$. Moreover, denote

$$\big(\alpha_{ij}, \langle\bar{\alpha}_{ij}\rangle^{n-i-j-1}\big) = \mathbf{s}([1] \geq [n-i-j-1]; \mathcal{S}_{ij}).$$

Using this notation, we have

Theorem 54. *Let $\mathcal{G}$ be as in Theorem 53, $\mathbf{c} \in \mathbb{R}^n[\mathrm{S}, a]$, and $\mathcal{G}^{\mathbf{c}} = \{\, \mathbf{v} \in \mathcal{G} \mid \mathbf{v} \preceq \mathbf{c} \,\}$ with $G(\langle \mathrm{M}_{1n}(\mathbf{c})\rangle^n) < b < G(\mathbf{c})$.*

(1) *If $\alpha_1 \leq c_1$, then $\max\{\, x_n \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \bar{\alpha}_1$, which is attained at the point $(\alpha_1, \langle\bar{\alpha}_1\rangle^{n-1})$.*

(2) *If $\alpha_1 > c_1$, then $\max\{\, x_n \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \bar{\alpha}_{\ell 0}$, where ℓ is the largest non-negative index such that $G(\mathbf{c}_{1:\ell}, \langle \mathrm{M}_{\ell+1,n}(\mathbf{c})\rangle^{n-\ell}) \leq b$. The maximum is attained at the point $(c_1, \dots, c_\ell, \alpha_{\ell 0}, \langle\bar{\alpha}_{\ell 0}\rangle^{n-\ell-1})$.*

Assume below that $k \leq n-1$.

(3) *If $\alpha_k \leq \mathrm{M}_{1k}(\mathbf{c})$, then $\max\{\, x_k \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \alpha_k$, which is attained at the point $(\langle\alpha_k\rangle^k, \langle\bar{\alpha}_k\rangle^{n-k})$.*

(4) *If $\alpha_k > \mathrm{M}_{1k}(\mathbf{c})$ and $G(\langle \mathrm{M}_{1k}(\mathbf{c})\rangle^k, \mathbf{c}_{k+1:n}) \geq b$, then $\max\{\, x_k \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \mathrm{M}_{1k}(\mathbf{c})$. The maximum is attained at a point $(\langle \mathrm{M}_{1k}(\mathbf{c})\rangle^k, y_{k+1}, \dots, y_n) \in \mathcal{G}^{\mathbf{c}}$.*

(5) *If* $\alpha_k > \mathrm{M}_{1k}(\mathbf{c})$ *and* $G(\langle \mathrm{M}_{1k}(\mathbf{c})\rangle^k, \mathbf{c}_{k+1:n}) < b$, *then* $\max\{\, x_k \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \bar{\alpha}_{\ell,n-k}$, *where* ℓ *is the largest nonnegative index such that*

$$G\big(\mathbf{c}_{1:\ell}, \langle \mathrm{M}_{\ell+1,k}(\mathbf{c})\rangle^{k-\ell}, \mathbf{c}_{k+1:n}\big) \le b.$$

The maximum is attained at the point

$$\big(c_1, \dots, c_\ell, \alpha_{\ell,n-k}, \langle \bar{\alpha}_{\ell,n-k}\rangle^{k-\ell-1}, c_{k+1}, \dots, c_n\big).$$

Using analogous argumentation, and denoting

$$\big(\langle \gamma_{ij}\rangle^{n-i-j-1}, \bar{\gamma}_{ij}\big) = \mathbf{s}([n-i-j-1] \ge [1]; \mathcal{S}_{ij}),$$

we deduce the following theorem about $\min_{\mathbf{x}\in\mathcal{G}^{\mathbf{c}}} x_k$:

Theorem 55. *Let the assumptions be the same as in Theorem 54.*

(1) *If* $\bar{\alpha}_{n-1} \ge c_n$, *then* $\min\{\, x_1 \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \alpha_{n-1}$, *which is attained at the point* $(\langle \alpha_{n-1}\rangle^{n-1}, \bar{\alpha}_{n-1})$.

(2) *If* $\bar{\alpha}_{n-1} < c_n$, *then* $\min\{\, x_1 \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \gamma_{0,n-\ell-1}$, *where* ℓ *is the largest index such that* $G(\langle \mathrm{M}_{1,\ell}(\mathbf{c})\rangle^\ell, \mathbf{c}_{\ell+1:n}) > b$. *The minimum is attained at the point* $(\langle \gamma_{0,n-\ell-1}\rangle^\ell, \bar{\gamma}_{0,n-\ell-1}, c_{\ell+2}, \dots, c_n)$.

Assume below that $2 \le k \le n$.

(3) *If* $\bar{\alpha}_{k-1} \ge \mathrm{M}_{kn}(\mathbf{c})$, *then* $\min\{\, x_k \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \bar{\alpha}_{k-1}$, *which is attained at the point* $(\langle \alpha_{k-1}\rangle^{k-1}, \langle \bar{\alpha}_{k-1}\rangle^{n-k+1})$.

(4) *If* $\bar{\alpha}_{k-1} < \mathrm{M}_{kn}(\mathbf{c})$ *and* $G(\mathbf{c}_{1:k-1}, \langle \mathrm{M}_{kn}(\mathbf{c})\rangle^{n-k+1}) \ge b$, *then* $\min\{\, x_k \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \mathrm{M}_{kn}(\mathbf{c})$. *The minimum is attained at a point*

$$\big(y_1, \dots, y_{k-1}, \langle \mathrm{M}_{kn}(\mathbf{c})\rangle^{n-k+1}\big) \in \mathcal{G}^{\mathbf{c}}.$$

(5) *If* $\bar{\alpha}_{k-1} < \mathrm{M}_{kn}(\mathbf{c})$ *and* $G(\mathbf{c}_{1:k-1}, \langle \mathrm{M}_{kn}(\mathbf{c})\rangle^{n-k+1}) < b$, *then* $\min\{\, x_k \mid \mathbf{x} \in \mathcal{G}^{\mathbf{c}} \,\} = \gamma_{k-1,n-k-\ell}$, *where* ℓ *is the largest index such that*

$$G\big(\mathbf{c}_{1:k-1}, \langle \mathrm{M}_{k,k+\ell-1}(\mathbf{c})\rangle^\ell, \mathbf{c}_{k+\ell:n}\big) > b.$$

The minimum is attained at the point

$$\big(c_1, \dots, c_{k-1}, \langle \gamma_{k-1,n-k-\ell}\rangle^\ell, \bar{\gamma}_{k-1,n-k-\ell}, c_{k+\ell+1}, \dots, c_n\big).$$

7.3 Using extra bounds. An example

Consider once again the matrix $\mathbf{A}$ in Example 23 (p. 58). Recall that the eigenvalues of $\mathbf{A}$ are

$$\boldsymbol{\lambda}(\mathbf{A}) = (8.31, 4.12, 3.06, 0.92, -1.08, -5.26, -8.06),$$

$\operatorname{tr}\mathbf{A} = 2$, $\operatorname{tr}\mathbf{A}^2 = 190$, and $\operatorname{tr}\mathbf{A}^4 = 10134$.

The diagonal elements of $\mathbf{A}$ yield the extra bound

$$(4, 3, 2, 1, -1, -3, -4) \preceq \boldsymbol{\lambda}(\mathbf{A}). \tag{7.1}$$

Using the source data $(n, \operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2)$, we obtain the extra bounds

$$(2.40, 2.40, 2.40, 0.28, -1.84, -1.84, -1.84) \preceq \boldsymbol{\lambda}(\mathbf{A}) \tag{7.2}$$

and

$$\boldsymbol{\lambda}(\mathbf{A}) \preceq (13.03, 3.99, 1.86, 0.29, -1.28, -3.42, -12.46). \tag{7.3}$$

The source data $(n, \operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^4)$ yield bounds

$$(2.01, 2.01, 2.01, 0.01, -1.34, -1.34, -1.34) \preceq \boldsymbol{\lambda}(\mathbf{A}) \tag{7.4}$$

and

$$\boldsymbol{\lambda}(\mathbf{A}) \preceq (10.03, 6.69, 4.76, 0.81, -3.82, -6.46, -10.01). \tag{7.5}$$

We compute the lower and upper bounds for the eigenvalues (relative to the order $\leq$) using the source data $(n, \operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2)$ and $(n, \operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^4)$ with and without the extra bounds. Of course, it is redundant to try to enhance the $\leq$-bounds obtained from the given data with the $\preceq$-bound derived from the very same data. The upper bounds (7.3) and (7.5) are incomparable relative to majorization, hence we use both of them. Since the diagonal of $\mathbf{A}$ majorizes the lower bound (7.2), which in turn majorizes the lower bound (7.4), we use only the lower bound (7.1).

We denote these bounds as follows:

	the lower bound using	with the extra bound
L_2	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2)$	–
L_4	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^4)$	–
L_{2Ld}	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2)$	7.1
L_{4Ld}	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^4)$	7.1
L_{2U4}	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2)$	7.5
L_{4U2}	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^4)$	7.3

	the upper bound using	with the extra bound
U_2	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2)$	–
U_4	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^4)$	–
U_{2Ld}	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2)$	7.1
U_{4Ld}	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^4)$	7.1
U_{2U4}	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2)$	7.5
U_{4U2}	$(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^4)$	7.3

The bounds L_{2Ld} and U_{2Ld} were originally presented in [22] (note that there are a couple of misprints in it). The best bounds are underlined.

Lower bounds	L_2	L_4	L_{2Ld}	L_{4Ld}	L_{2U4}	L_{4U2}
$\lambda_1(\mathbf{A}) \geq$	2.40	2.01	$\underline{4.00}$	$\underline{4.00}$	3.48	2.01
$\lambda_2(\mathbf{A}) \geq$	−1.84	−1.34	−0.91	0.02	$\underline{0.28}$	−1.34
$\lambda_3(\mathbf{A}) \geq$	−3.01	−2.95	−2.66	$\underline{-2.56}$	−2.68	−2.95
$\lambda_4(\mathbf{A}) \geq$	$\underline{-4.22}$	−4.87	$\underline{-4.22}$	−4.87	$\underline{-4.22}$	$\underline{-4.22}$
$\lambda_5(\mathbf{A}) \geq$	$\underline{-5.73}$	−6.77	$\underline{-5.73}$	−6.77	$\underline{-5.73}$	$\underline{-5.73}$
$\lambda_6(\mathbf{A}) \geq$	$\underline{-7.94}$	−8.24	$\underline{-7.94}$	−8.24	$\underline{-7.94}$	$\underline{-7.94}$
$\lambda_7(\mathbf{A}) \geq$	−12.46	−10.01	−12.32	$\underline{-9.95}$	−10.01	−10.01

Upper bounds	U_2	U_4	U_{2Ld}	U_{4Ld}	U_{2U4}	U_{4U2}
$\lambda_1(\mathbf{A}) \leq$	13.03	10.03	12.71	$\underline{9.95}$	10.03	10.03
$\lambda_2(\mathbf{A}) \leq$	8.52	8.36	8.49	$\underline{8.34}$	8.36	8.36
$\lambda_3(\mathbf{A}) \leq$	$\underline{6.30}$	7.16	$\underline{6.30}$	7.16	$\underline{6.30}$	$\underline{6.30}$
$\lambda_4(\mathbf{A}) \leq$	$\underline{4.80}$	5.58	$\underline{4.80}$	5.58	$\underline{4.80}$	$\underline{4.80}$
$\lambda_5(\mathbf{A}) \leq$	3.58	3.70	$\underline{3.47}$	3.62	3.58	3.58
$\lambda_6(\mathbf{A}) \leq$	2.41	2.01	1.78	0.98	$\underline{0.53}$	2.01
$\lambda_7(\mathbf{A}) \leq$	−1.84	−1.34	$\underline{-4.00}$	$\underline{-4.00}$	−3.21	−1.33

8 Optimization over a sum- and G-constant set

8.1 Extreme values of $F(\mathbf{x}_{k:\ell})$

The set $\mathbb{R}^n_{++}[\mathrm{S}, a; \mathrm{P}, b]$ was studied in [20]. Besides bounds for the partial sums $\mathrm{S}_{k\ell}(\mathbf{x})$, also bounds for the partial products $\mathrm{P}_{k\ell}(\mathbf{x})$ were deduced. We extend this approach (for the definitions of μ and λ, see Section 4.4).

Lemma 56. *Let* $\mathcal{G} = I^n[\mathrm{S}, a; G, b]$, $\mu = \mu(0; \mathcal{G})$, $\lambda = \lambda(0; \mathcal{G})$, $1 \le k < \ell \le n$, $p = \ell - k + 1$, *and* $F \colon \mathbb{R}^{\ell-k+1} \to \mathbb{R}$. *Denote*

$$\boldsymbol{\alpha}_k = \big(\langle \alpha_k \rangle^k, \langle \bar{\alpha}_k \rangle^{n-k}\big) = \mathbf{s}([k] \ge [n-k]; \mathcal{G})$$

for $k = \lambda + 1, \ldots, n - \mu - 1$.

(1) *If* $\lambda + 1 \le \ell \le n - \mu - 1$ *and* F *is Schur-concave and increasing, then* $\max_{\mathbf{x} \in \mathcal{G}} F(\mathbf{x}_{k:\ell}) = F(\alpha_\ell \mathbf{1}_p)$;

(2) *If* $\lambda + 2 \le k \le n - \mu$ *and* F *is Schur-concave and decreasing, then* $\max_{\mathbf{x} \in \mathcal{G}} F(\mathbf{x}_{k:\ell}) = F(\bar{\alpha}_{k-1} \mathbf{1}_p)$;

(3) *If* $\lambda + 2 \le k \le n - \mu$ *and* F *is Schur-convex and increasing, then* $\min_{\mathbf{x} \in \mathcal{G}} F(\mathbf{x}_{k:\ell}) = F(\bar{\alpha}_{k-1} \mathbf{1}_p)$;

(4) *If* $\lambda + 1 \le \ell \le n - \mu - 1$ *and* F *is Schur-convex and decreasing, then* $\min_{\mathbf{x} \in \mathcal{G}} F(\mathbf{x}_{k:\ell}) = F(\alpha_\ell \mathbf{1}_p)$.

Proof. Assume that F is Schur-concave and increasing. The assumption $\lambda + 1 \le \ell \le n - \mu - 1$ implies that $\boldsymbol{\alpha}_\ell$ exists. By Lemma 29, if $\mathbf{x} \in \mathcal{G}$, then $\mathrm{S}_{k\ell}(\mathbf{x}) \le p\alpha_\ell$. Therefore

$$F(\mathbf{x}_{k:\ell}) \le F\big((\mathrm{S}_{k\ell}(\mathbf{x})/p)\mathbf{1}_p\big) \le F(\alpha_\ell \mathbf{1}_p).$$

Since $\boldsymbol{\alpha}_\ell \in \mathcal{G}$, this is a sharp upper bound.

Assume then that $\lambda + 2 \le k \le n - \mu$ and F is Schur-concave and decreasing. Now $\boldsymbol{\alpha}_{k-1}$ exists and if $\mathbf{x} \in \mathcal{G}$, then $\mathrm{S}_{k\ell}(\mathbf{x}) \ge p\bar{\alpha}_{k-1}$. Hence $\max_{\mathbf{x} \in \mathcal{Y}} F(\mathbf{x}_{k:\ell}) = F(\bar{\alpha}_{k-1} \mathbf{1}_p)$.

For the proof of parts (3) and (4), it suffices to note that F is Schur-convex and increasing (decreasing) if and only if $-F$ is Schur-concave and decreasing (increasing).

Above we have excluded a trivial case $p = 1$. If f is a monotonic function, then Theorems 32 and 33 give the sharp upper and the sharp lower bound for $f(x_k)$ under the assumption $\mathbf{x} \in \mathcal{G}$ (or $\mathbf{x} \in \mathcal{G}^-$, or $\mathbf{x} \in \mathcal{G}^+$).

Applying Lemma 56 and the bounds derived for $\mathrm{S}_{k\ell}(\mathbf{x})$, we obtain various maximum and minimum results for a function F satisfying the conditions stated in Lemma 56. We content ourselves with some examples. In these, $\boldsymbol{\alpha}_k$ is the relevant solution vector.

Example 29. Let $\mathcal{G} = \mathbb{R}^n_+[\mathrm{S}, a; G, b]$. Then $\lambda(\mathcal{G}) = 0$ and

$$\mu(\mathcal{G}) = \max\bigl\{\, \ell \le n-2 \bigm| G\bigl(\langle a/(n-\ell)\rangle^{n-\ell}, \langle 0\rangle^{\ell}\bigr) \le b \,\bigr\}.$$

By Theorem 38, α_1 is the sharp upper bound for x_1 under the assumption $\mathbf{x} \in \mathcal{G}$. Let $1 \le k < \ell \le n - \mu - 1$. Since the product $\mathrm{P}\colon \mathbb{R}^{\ell-k+1}_+ \to \mathbb{R}$ is Schur-concave and increasing, we find the following sharp upper bound

$$x_k x_{k+1} \cdots x_\ell \le \alpha_\ell^{\ell-k+1}.$$

Now, consider the specific case $b < 0$ and

$$\mathcal{G} = \mathbb{R}^n_+[\mathrm{S}, a; -\mathrm{P}, b] = \mathbb{R}^n_{++}[\mathrm{S}, a; -\mathrm{P}, b],$$

which is treated in [20]. Since in this case $\mu = 0$ and $\alpha_k^k \bar{\alpha}_k^{n-k} = -b$ for $k = 1, 2, \ldots, n-1$, we have the sharp bounds

$$\mathrm{P}_{k\ell}(\mathbf{x}) \le \alpha_\ell^{\ell-k+1}, \quad \text{when } 1 \le k \le \ell \le n-1,$$

and

$$\mathrm{P}_{k\ell}(\mathbf{x}) \ge \bar{\alpha}_{k-1}^{\ell-k+1}, \quad \text{when } 2 \le k \le \ell \le n.$$

Example 30. Let g be an increasing strictly convex function $\mathbb{R} \to \mathbb{R}$ and define $G(\mathbf{x}) = \sum_i g(x_i)$. Let $\mathcal{G} = \mathbb{R}^n[\mathrm{S}, a; G, b]$, and let $k \le \ell \le n$. In analogy with Example 29, we infer that

$$\min_{\mathbf{x}\in\mathcal{G}} \sum_{i=k}^{\ell} g(x_i) = (\ell - k + 1) g(\bar{\alpha}_{k-1}) \quad \text{for } k \ge 2$$

and that

$$\max_{\mathbf{x}\in\mathcal{G}} \sum_{i=k}^{\ell} g(x_i) = (\ell - k + 1) g(\alpha_\ell) \quad \text{for } \ell \le n-1.$$

Example 31. Let $\mathbf{c} = (\langle 0\rangle^{k-1}, c_k, c_{k+1}, \ldots, c_\ell, \langle 0\rangle^{(n-\ell)})$, where $0 < c_k \le c_{k+1} \le \cdots \le c_\ell$. The function $F\colon \mathbb{R}^{\ell-k+1}_\downarrow \to \mathbb{R}$,

$$F(y_1, y_2, \ldots, y_{\ell-k+1}) = \sum_{i=0}^{\ell-k} c_{k+i} y_{1+i}$$

is increasing and Schur-concave. The latter property can be verified, for example, by using the result in [12, p. 445]. If $k \le \ell \le n-1$, then

$$\max_{\mathbf{x}\in\mathbb{R}^n[\mathrm{S},a;G,b]} \mathbf{c}\mathbf{x}^T = \alpha_\ell \mathrm{S}(\mathbf{c}).$$

Let $\mathcal{G} = I^n[\mathrm{S}, a; G, b]$ and $F\colon \mathbb{R}^n \to \mathbb{R}$. Consider the problems

$$\max_{\mathbf{x}\in\mathcal{G}} F(\mathbf{x}) \quad \text{and} \quad \min_{\mathbf{x}\in\mathcal{G}} F(\mathbf{x}),$$

which are not covered by Lemma 56. We need, though, only the assumption that F is Schur-convex to deduce the lower and upper bounds

$$F(\inf_{\preceq} \mathcal{G}) \le F(\mathbf{x}) \le F(\sup_{\preceq} \mathcal{G}) \quad \text{for all } \mathbf{x} \in \mathcal{G}$$

from Theorems 30 and 34. These bounds, however, are sharp only in some trivial cases. We will in Chapter 9 develop a technique based on 3-majorization for treating problems of this kind.

Assume that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is, say, Schur-convex and increasing, and that $\ell = k{+}1$ and $\mathbf{x} \in \mathcal{G}$. Then we can estimate the value of $f(x_k, x_\ell)$ with the help of Lemma 56. But if k and ℓ are not consecutive indices, then Lemma 56 is not applicable. In the next section we will study the problems $\min_{\mathcal{G}} f(x_k, x_\ell)$ and $\max_{\mathcal{G}} f(x_k, x_\ell)$ without assuming that k and ℓ are consecutive indices. We will also drop the assumptions concerning the properties of f.

8.2 Extreme values of $f(x_k, x_\ell)$

Let $\mathcal{G} = I^n[\mathrm{S}, a; G, b]$, $f\colon I^2 \to \mathbb{R}$, and $c \in \mathbb{R}$. Define $F\colon I^n \to \mathbb{R}$ by $F(\mathbf{x}) = f(x_k, x_\ell)$. Denote $\mathcal{F} = \{\, \mathbf{v} \in I^n \mid \mathrm{S}(\mathbf{v}) = a,\ F(\mathbf{v}) = c \,\}$. We make the following assumptions: $1 \le k < \ell \le n$, F is continuous, and $\mathcal{F}$ is nonempty. We do not, however, assume F to be Schur-convex.

Our aim is to search for the extreme values of F in the set $\mathcal{G}$. We will apply Lemma 2, which means that we need to begin by solving

$$(*) \qquad \min_{\mathbf{x}\in\mathcal{F}} G(\mathbf{x}).$$

Choose some $\tilde{\mathbf{x}} \in \mathcal{F}$. Let $G(\tilde{\mathbf{x}}) = \tilde{b}$. By Corollary 45, $\tilde{\mathcal{G}} = \{\, \mathbf{v} \in I^n \mid \mathrm{S}(\mathbf{v}) = a,\ G(\mathbf{v}) \le \tilde{b} \,\}$ is compact. Since F is continuous, $\mathcal{F}$ is closed. Hence $\mathcal{Y} = \tilde{\mathcal{G}} \cap \mathcal{F}$ is compact and $\min_{\mathbf{x}\in\mathcal{Y}} G(\mathbf{x})$ is attained. Obviously a solution to this problem also solves $(*)$.

Let $\check{\mathbf{x}}$ be a solution to $(*)$. We show first that $\check{\mathbf{x}}$ is of shape

$$(S) \qquad [k-1] \ge [1] \ge [\ell-k-1] \ge [1] \ge [n-\ell].$$

Let $\mathbf{x} \in \mathcal{F}$. Suppose $x_i > x_j$, where $1 \le i < j \le k-1$ or $k+1 \le i < j \le \ell-1$ or $\ell+1 \le i < j \le n$. Let $p = \max\{\, p \mid x_i = x_p \,\}$. Since $x_1 \ge x_i = x_p > x_{p+1} \ge x_j \ge x_n$ and $k, \ell \notin \{p, p+1\}$, there exists a negative $\mathcal{F}$-consistent perturbation $P(\mathbf{x})$ of type $(\downarrow x_p, \uparrow x_{p+1})$. Since G is strictly Schur-convex, $G(P(\mathbf{x})) < G(\mathbf{x})$ and thus $\mathbf{x} \ne \check{\mathbf{x}}$.

Suppose then that $\mathbf{x} \in \mathcal{F}$ is of shape (S) and $x_{k-1} > x_k$. If $x_\ell > x_{\ell+1}$, we can perturb $\mathbf{x}$ by a negative $\mathcal{F}$-consistent transfer of type $(\downarrow x_{k-1}, \uparrow x_{\ell+1})$ and hence $\mathbf{x} \ne \check{\mathbf{x}}$.

We have proved

Lemma 57. *Let* $\check{\mathbf{x}}$ *be a solution to the problem* $(*)$. *Then either*

$$\check{\mathbf{x}} \simeq ([k-1] = [1] \geq [\ell-k-1] \geq [1] \geq [n-\ell])$$

or

$$\check{\mathbf{x}} \simeq ([k-1] > [1] \geq [\ell-k-1] \geq [1] = [n-\ell]).$$

Next we study some special cases of $(*)$. First we show as an example how to derive for the spread $x_1 - x_n$ an upper bound found by Mirsky [23]. Recall that $\mathcal{P}_2 = \mathbb{R}^n[\mathrm{S}, a; P_2, b]$ and that the sets ${\mathcal{P}_2}^-$ and ${\mathcal{P}_2}^+$ are defined as usual by replacing the condition $P_2(\mathbf{x}) = b$ with $P_2(\mathbf{x}) \leq b$ and $P_2(\mathbf{x}) \geq b$, respectively.

Example 32 (see [23]). Let $F\colon \mathbb{R}^n_{\downarrow} \to \mathbb{R}$, $F(x_1, x_2, \ldots, x_n) = x_1 - x_n$ and let $s \in \mathbb{R}_+$. Denote $\mathcal{F}_s = \{\, \mathbf{v} \in \mathbb{R}^n \mid \mathrm{S}(\mathbf{v}) = a,\ F(\mathbf{v}) = s \,\}$ and

$$m_{P_2}(s) = \min_{\mathbf{x} \in \mathcal{F}_s} P_2(\mathbf{x}).$$

By Lemma 57, the minimum is attained at the point of shape $([1] \geq [n-2] \geq [1])$. This implies that, for some $x \in J = [(a+s)/n, (a+(n-1)s)/n]$, the minimum point is

$$\mathbf{x}(s, x) = \big(x, \langle (a+s-2x)/(n-2) \rangle^{n-2}, x-s\big).$$

Denote $g_s(x) = P_2(\mathbf{x}(s, x))$. Then

$$g_s(x) = x^2 + (a+s-2x)^2/(n-2) + (x-s)^2.$$

Elementary calculus shows that

$$m_{P_2}(s) = \min_{x \in J} g_s(x) = a^2/n + s^2/2.$$

Since m_{P_2} is strictly increasing on $\mathbb{R}_+$ and since, under the condition that $s \in \mathbb{R}_+$ and $b \in [a^2/n, \infty[$,

$$m_{P_2}(s) = b \quad \text{if and only if} \quad s = \sqrt{2(b - a^2/n)},$$

it follows from Lemma 2 that

$$\max_{\mathbf{x} \in \mathcal{P}_2} (x_1 - x_n) = \max_{\mathbf{x} \in {\mathcal{P}_2}^-} (x_1 - x_n) = \sqrt{2(b - a^2/n)}.$$

Obviously, $\max_{\mathbf{x} \in {\mathcal{P}_2}^+}(x_1 - x_n)$ is not attained.

8.3 The ratio x_k/x_ℓ

In this subsection we consider $F\colon I_\downarrow^n \to \mathbb{R}$, $F(\mathbf{x}) = x_k/x_\ell$. Denote $\mathcal{F}_R = \{\, \mathbf{v} \in \mathbb{R}^n_{++} \mid \mathrm{S}(\mathbf{v}) = a,\ F(\mathbf{v}) = R \,\}$, and

$$m_G(R) = \min_{\mathbf{x}\in\mathcal{F}_R} G(\mathbf{x}).$$

We assume throughout this section that $1 \le k < \ell \le n$ and $R \ge 1$.

Define

$$\begin{aligned}
u &= aR/(n + -(\ell-1) + (\ell-1)R),\\
v &= aR/(n-k+kR),\\
\mathbf{u} &= \big(\langle u\rangle^{\ell-1}, \langle u/R\rangle^{n-\ell+1}\big),\\
\mathbf{v} &= \big(\langle v\rangle^{k}, \langle v/R\rangle^{n-k}\big).
\end{aligned}$$

If $R > 1$ and $\ell \ge k+2$, then $v > u > 0$. If $\ell = k+1$, then $u = v$. If $R = 1$, then $u = v = a/n$ and $\mathbf{u} = \mathbf{v} = (\langle a/n\rangle^n)$. Now $\mathrm{S}(\mathbf{u}) = \mathrm{S}(\mathbf{v}) = a$, $u_k/u_\ell = v_k/v_\ell = R$, $\mathbf{u} = \mathbf{u}_\downarrow$, and $\mathbf{v} = \mathbf{v}_\downarrow$. It follows that $\mathbf{u}$ and $\mathbf{v}$ belong to $\mathcal{F}_R$.

If $R = 1$, then we have $m_G(R) = G(\langle a/n\rangle^n)$. Assume that $R > 1$, and that $\check{\mathbf{x}}$ is a solution for the problem $\min_{\mathbf{x}\in\mathcal{F}_R} G(\mathbf{x})$. We can now strengthen Lemma 57 by showing that the minimizer

$$\check{\mathbf{x}} \simeq ([k-1] = [1] \ge [\ell-k-1] \ge [1] = [n-\ell]).$$

Assume to the contrary that

$$\check{\mathbf{x}} \simeq ([k-1] > [1] \ge [\ell-k-1] \ge [1] = [n-\ell])$$

or that

$$\check{\mathbf{x}} \simeq ([k-1] = [1] \ge [\ell-k-1] \ge [1] > [n-\ell]).$$

In the first case, define a perturbation P of type $(\downarrow[k-1], \uparrow[1], \uparrow[\ell-k-1], \uparrow[n-\ell+1])$ as follows:

$$P(\check{\mathbf{x}})_i = \begin{cases} \check{x}_i - r & \text{if } 1 \le i \le k-1,\\ \check{x}_i + t & \text{if } k \le i \le \ell-1,\\ \check{x}_i + s & \text{if } \ell \le i \le n, \end{cases}$$

where $t > 0$, $s = (\check{x}_\ell/\check{x}_k)t = t/R$, and $r = ((\ell-k)t + (n-\ell+1)s)/(k-1)$. Then P is a negative $\mathcal{F}_R$-consistent perturbation of $\check{\mathbf{x}}$ on the condition that t is a sufficiently small positive number. This means that $\check{\mathbf{x}}$ would not be a minimum point.

As to the second case, we see, as above, that there would have to exist a $\mathcal{F}_R$-consistent perturbation of type $(\downarrow[k], \downarrow[\ell-k-1], \downarrow[1], \uparrow[n-\ell])$ that would decrease the value of G.

Denote $\mathcal{F}_R^* = \{\, \mathbf{x} \mid \mathbf{x} \in \mathcal{F}_R,\, \mathbf{x} \simeq ([k] \geq [\ell - k - 1] \geq [n - \ell + 1]) \,\}$. We conclude that the minimizer $\check{\mathbf{x}} \in \mathcal{F}_R^*$. On the other hand, $\mathcal{F}_R^*$ is a convex set with extreme points $\mathbf{u}$ and $\mathbf{v}$. Assuming $\ell \geq k + 2$ denote

$$L = \frac{n + 1 - \ell + kR}{(\ell - k - 1)R}.$$

By simple calculation, we see that $u - L(v - u) = v/R$. Hence

$$\mathcal{F}_R^* = \{\, \mathbf{u}(t) \mid 0 \leq t \leq v - u \,\},$$

where

$$\mathbf{u}(t) = \Big(\langle u + t \rangle^k, \langle u - Lt \rangle^{\ell - k - 1}, \Big\langle \frac{u + t}{R} \Big\rangle^{n - \ell + 1} \Big).$$

If $\ell = k + 1$, then $\mathcal{F}_R^* = \{(\langle u \rangle^k, \langle u/R \rangle^{n-k})\}$.

It follows that if $\ell \geq k + 2$, then

$$\min_{\mathbf{x} \in \mathcal{F}_R} G(\mathbf{x}) = \min_{0 \leq t \leq v - u} G(\mathbf{u}(t)).$$

In the case $\ell = k + 1$ we have

Lemma 58. *Let $\mathcal{F}_R$ be as above, let $k \leq n - 1$, and let G be a strictly Schur-convex function $\mathbb{R}_{++}^n \to \mathbb{R}$. Then*

$$\min_{\mathbf{x} \in \mathcal{F}_R} G(\mathbf{x}) = G\bigg(\Big\langle \frac{aR}{n - k + kR} \Big\rangle^k, \Big\langle \frac{a}{n - k + kR} \Big\rangle^{n-k} \bigg).$$

Let $1 \leq R' < R$. Assume $m_G(R) = G(\mathbf{x}_R)$. By Lemma 11, there exists $\mathbf{y} \in \mathcal{F}_{R'}$ such that $\mathbf{y} \prec \mathbf{x}_R$. Since G is strictly increasing, $G(\mathbf{y}) < G(\mathbf{x}_R)$. This means that the function m_G is strictly increasing. Thus it follows from Lemma 2 that if

$$m_G(R) = b_R,$$

then

$$\max_{\mathbf{x} \in \mathbb{R}_{++}^n[\mathrm{S}, a; G, b_R]} x_k / x_\ell = \max_{\mathbf{x} \in \mathbb{R}_{++}^n[\mathrm{S}, a; G, b_R]^-} x_k / x_\ell = R.$$

In the next two subsections we will deal with the special cases $G = P_2$ and $G = \mathrm{P}$.

8.3.1 The case $G = P_2$

Let $\mathbf{A}$ be an $n \times n$-matrix with real eigenvalues and let $\operatorname{tr} \mathbf{A} \geq 0$. Denote $\gamma_{k\ell} = \lambda_k(\mathbf{A}) / \lambda_\ell(\mathbf{A})$. Merikoski, Styan, and Wolkowicz [15, Theorem 3.1, p. 114] presented the following upper bound for the ratios of the eigenvalues of the matrix $\mathbf{A}$: provided that $(\ell - 1) \operatorname{tr} \mathbf{A}^2 < (\operatorname{tr} \mathbf{A})^2$,

$$\gamma_{k\ell} \leq \frac{c + k + \sqrt{\frac{n - \ell + 1}{k}(c + k)(n - \ell + 1 - c)}}{c + k - \sqrt{\frac{k}{n - \ell + 1}(c + k)(n - \ell + 1 - c)}},$$

where

$$c = \frac{(\operatorname{tr} \mathbf{A})^2}{\operatorname{tr} \mathbf{A}^2} - (\ell - 1).$$

The condition $(\ell - 1) \operatorname{tr} \mathbf{A}^2 < (\operatorname{tr} \mathbf{A})^2$ is equivalent to the condition that $\lambda_\ell(\mathbf{B}) > 0$ whenever $\mathbf{B}$ is an $n \times n$-matrix with real eigenvalues satisfying the conditions $\operatorname{tr} \mathbf{B} = \operatorname{tr} \mathbf{A}$ and $\operatorname{tr} \mathbf{B}^2 = \operatorname{tr} \mathbf{A}^2$.

Using *Mathematica* in computations, we show that this upper bound is derivable from the results of the previous section. Denote $\mathbf{x} = \boldsymbol{\lambda}(\mathbf{A})$, $a = \operatorname{tr} \mathbf{A}$, and $b = \operatorname{tr} \mathbf{A}^2$. For simplicity, we assume that the eigenvalues of $\mathbf{A}$ are positive (which implies that $a^2 > (\ell - 1)b$).

Let u, $\mathbf{u}(t)$, and L be as above, and let $g(t) = P_2(\mathbf{u}(t))$. Now

$$g(t) = k(u+t)^2 + (\ell - k - 1)(u - Lt)^2 + (n + 1 - \ell)\left(\frac{u+t}{R}\right)^2$$

(if $\ell = k + 1$, ignore the middle term in the sum) and

$$g'(t) = \left(2k + 2(\ell - k - 1)L^2 + \frac{2(n+1-\ell)}{R^2}\right)t \\ + 2ku - 2(\ell - k - 1)Lu + \frac{2(n+1-\ell)u}{R^2}.$$

Denote

$$t^* = -\frac{kR^2 + (k+1-\ell)R^2L + n + 1 - \ell}{kR^2 + (\ell - k - 1)R^2L^2 + n + 1 - \ell}u.$$

Then $g'(t) = 0$ if and only if $t = t^*$. Since $g''(t) > 0$ for all t, we have

$$m_G(R) = g(t^*) = \frac{(-1+\ell-k)(1+L)^2(-1+\ell-n-kR^2)u^2}{-1+\ell-n-kR^2+L^2R^2-\ell L^2R^2+kL^2R^2}.$$

Substituting $(n+1-\ell+kR)/((\ell-k-1)R)$ for L and $aR/(n+-(\ell-1)+(\ell-1)R)$ for u, and solving the system

$$m_G(R) = b, \quad R \geq 1,$$

we obtain for x_k/x_ℓ a sharp upper bound

$$\frac{k(n+1-\ell)b + \sqrt{k\big(a^2 + b(k+1-\ell)\big)(n+1-\ell)(bn - a^2)}}{k(a^2 + (1-\ell)b)}.$$

This, in fact, is the same upper bound as the upper bound of Merikoski and Wolkowicz. It is sharp under the assumption that $\mathbf{x} \in \mathcal{P}_2$ (or $\mathbf{x} \in \mathcal{P}_2{}^-$).

8.3.2 The case $G = -\mathrm{P}$

Assume that $\boldsymbol{\lambda}(\mathbf{A}) \in \mathbb{R}^n_{++}$, $\operatorname{tr} \mathbf{A} = a$, and $\det \mathbf{A} = b$. An upper bound

$$\frac{1 + \sqrt{1 - (n/a)^n b}}{1 - \sqrt{1 - (n/a)^n b}}$$

was presented for the ratio $\lambda_1(\mathbf{A})/\lambda_n(\mathbf{A})$ in [16] and [17]. This is the best possible upper bound using only n, $\operatorname{tr}\mathbf{A}$, and $\det\mathbf{A}$. Next we solve the problem $\min_{\mathbf{x}\in\mathcal{F}_R} -\mathrm{P}(\mathbf{x})$. As a corollary we obtain for the ratios of the eigenvalues the best possible upper bounds using only n, $\operatorname{tr}\mathbf{A}$, and $\det\mathbf{A}$.

Let u, v, $\mathbf{u}$, and L be as above. Assume that $\ell \geq k+2$ and $R>1$. Define $g\colon [0, v-u] \to \mathbb{R}$ as follows: $g(t) = -\mathrm{P}(\mathbf{u}(t))$. Now

$$g(t) = -(1/R)^{n-\ell+1}(u+t)^{n+k-\ell+1}(u-Lt)^{\ell-k-1}$$

and

$$g'(t) = \left(\frac{1}{R}\right)^{n-\ell+1}(u+t)^{n+k-\ell}(u-Lt)^{\ell-k-2} \cdot \big(-(n+k+1-\ell+(k+1-\ell)L)u + Lnt\big).$$

If $0 \leq t \leq v-u$, then $u+t \geq u-Lt \geq (u+t)/R > 0$. Furthermore, $Ln > 0$. Let

$$t^* = \frac{n+k+1-\ell+(k+1-\ell)L}{Ln}u.$$

Then $g'(t^*) = 0$. Recalling the values of u, v, and L, we have

$$\begin{aligned} 0 < t^* &= \frac{(n+1-\ell)(\ell-k-1)(R-1)aR}{n(n+1-\ell+kR)(n+1-\ell+\ell R-R)} \\ &= \frac{-(\ell-k-1)k(R-1)}{n(n+1-\ell+kR)(n+k(R-1))}\, aR + (v-u) < v-u. \end{aligned}$$

It follows that

$$g'(t) \begin{cases} <0, & \text{if } 0 \leq t < t^*, \\ =0, & \text{if } t = t^*, \\ >0, & \text{if } t^* < t \leq v-u. \end{cases}$$

We conclude that if $R>1$ and $l>k+1$, then

$$m_G(R) = \min_{\mathbf{x}\in\mathcal{F}_R} -\mathrm{P}(\mathbf{x}) = g(t^*) = -R^k\left(\frac{n+k+1-\ell}{n+1-\ell+kR}\right)^{n+k+1-\ell}\left(\frac{a}{n}\right)^n.$$

Trivially this also holds for $R=1$. By Lemma 58, it also holds when $\ell = k+1$.

Next consider the function

$$M_F(b) = \max_{\mathbf{x}\in\mathcal{G}} x_k/x_\ell.$$

Note that now $\mathcal{G} = \mathbb{R}^n_{++}[\mathrm{S}, a; -\mathrm{P}, -b]$ with $b < (a/n)^n$ (the case $b = (a/n)^n$ is trivial). By Lemma 2, $M_F(b) = R_b$, where R_b is the unique solution of the system

$$(E) \qquad m_G(R) = -b, \quad R \geq 1.$$

We cannot in general solve the equation $m_G(R) = -b$ algebraically. There are, however, some special cases where this is possible.

In the following theorem the system (E) is formulated in an equivalent form, obtained by substituting x for $R/(n+1-\ell+kR)$ and y for $1/(n+1-\ell+kR)$.

Theorem 59. *Let $\mathcal{G}$ be as above, and let $k < \ell \leq n$. Denote*

$$B = (a/n)^{-n}(n+k+1-\ell)^{-(n+k+1-\ell)}b.$$

Let the solution of the system

$$\begin{gathered} x^k y^{n+1-\ell} = B, \\ kx + (n+1-\ell)y = 1, \\ x \geq y > 0 \end{gathered}$$

be $(x, y) = (\alpha_k, \beta_\ell)$. Then

$$\max_{\mathbf{x}\in\mathcal{G}} x_k/x_\ell = \max_{\mathbf{x}\in\mathcal{G}^-} x_k/x_\ell = \alpha_k/\beta_\ell.$$

When $k = n+1-\ell$, we can solve the system in Theorem 59 algebraically (cf. [20, p. 72]). The solution is then

$$\alpha_k = \frac{1+\sqrt{1-4k^2B^{1/k}}}{2k}, \qquad \beta_\ell = \frac{1-\sqrt{1-4k^2B^{1/k}}}{2k}.$$

By calculating the ratio α_k/β_ℓ, we arrive at

Corollary 60. *Let $b \geq (a/n)^n$, $\mathbf{x} \in \mathbb{R}^n_{++}$, $\mathrm{S}(\mathbf{x}) = a$, $\mathrm{P}(\mathbf{x}) = b$, and $k < n/2$. Under these assumptions, the sharp upper bound for x_k/x_{n+1-k} is*

$$\frac{1+\sqrt{1-\big((n/a)^n b\big)^{1/k}}}{1-\sqrt{1-\big((n/a)^n b\big)^{1/k}}}.$$

The assumption $P(\mathbf{x}) = b$ can be replaced with the assumption $P(\mathbf{x}) \geq b$.

Example 33 (cf. [15], Example 6.1). Let $n = 5$, and consider a 5×5 matrix $\mathbf{A}$ with the eigenvalues

$$\lambda_1 = 5.3, \quad \lambda_2 = 4.3, \quad \lambda_3 = 3.5, \quad \lambda_4 = 2.6, \quad \lambda_5 = 2.5.$$

Then $\operatorname{tr}\mathbf{A} = 18.2$, $\operatorname{tr}\mathbf{A}^2 = 71.84$, $\det\mathbf{A} = 518.473$, and we obtain the following bounds:

			upper bound for λ_k/λ_ℓ using n, $\operatorname{tr}\mathbf{A}$, and	
k	ℓ	λ_k/λ_ℓ	$\det \mathbf{A}$	$\operatorname{tr}\mathbf{A}^2$
1	2	1.233	1.972	1.850
1	3	1.514	2.032	1.936
1	4	2.039	2.154	2.127
1	5	2.120	2.536	2.934
2	3	1.229	1.792	1.778
2	4	1.654	1.920	1.978
2	5	1.720	2.312	2.805
3	4	1.346	1.834	1.921
3	5	1.400	2.233	2.758
4	5	1.040	2.192	2.734

All the bounds using n, $\operatorname{tr}\mathbf{A}$, and $\operatorname{tr}\mathbf{A}^2$ are algebraically expressible. For λ_1/λ_5 and λ_2/λ_4, also the bounds using n, $\operatorname{tr}\mathbf{A}$, and $\det \mathbf{A}$ are algebraically expressible.

9 3-majorization

9.1 The existence of 3-transfers

Let $\mathbf{A}$ be a positive definite Hermitian $n \times n$-matrix. The best possible bounds for the determinant of $\mathbf{A}$ using n and traces $\operatorname{tr}\mathbf{A}$ and $\operatorname{tr}\mathbf{A}^2$ were given in [3]. Let $\mathcal{P}_2 = \mathbb{R}^n[\mathrm{S}, a; P_2, b]$. Denoting

$$\boldsymbol{\alpha}_k = \big(\langle\alpha_k\rangle^k, \langle\bar{\alpha}_k\rangle^{n-k}\big) = \mathbf{s}([k] \geq [n-k]; \mathcal{P}_2),$$

we restate this result as follows: Assume $\bar{\alpha}_1 \geq 0$, i.e., $\mathcal{P}_2 \cap \mathbb{R}^n_+ \neq \emptyset$. Then

$$\max\{\, \mathrm{P}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{P}_2, \mathbf{x} \geq \mathbf{0} \,\} = \mathrm{P}(\boldsymbol{\alpha}_1)$$

and

$$\min\{\, \mathrm{P}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{P}_2, \mathbf{x} \geq \mathbf{0} \,\} = \begin{cases} \mathrm{P}(\boldsymbol{\alpha}_{n-1}) & \text{if } \alpha_{n-1} \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The best possible bounds using only n, $\operatorname{tr}\mathbf{A}$, $\operatorname{tr}\mathbf{A}^2$, and the diagonal of $\mathbf{A}$ are presented for $\det\mathbf{A}$ in [7] (see also below p. 90, Example 36). These bounds are the solutions of the problems $\max\mathrm{P}(\mathbf{x})$ and $\min\mathrm{P}(\mathbf{x})$ subject to $\mathbf{x} \geq \mathbf{0}$, $\mathrm{S}(\mathbf{x}) = a$, $P_2(\mathbf{x}) = b$, and $\mathbf{a} \preceq \mathbf{x}$.

It turns out that the product P is a $\prec_{\mathcal{P}_2}$-increasing function and that this fact implies the above results. We will generalize these results in various directions by replacing P_2 with any strictly Schur-convex function G and then studying optimization of $\prec_{\mathcal{G}}$-increasing functions on this set with the extra constraints $\mathbf{a} \preceq \mathbf{x}$, $\mathbf{x} \preceq \mathbf{b}$, $\mathbf{m} \leq \mathbf{x}$, and $\mathbf{x} \leq \mathbf{M}$, or some combination of these.

Let $\mathcal{G}$ satisfy the basic assumptions. If $\mathbf{x} \in \mathcal{G}$ and there exist indices $1 \leq i < j < k \leq n$ such that

$$x_{i-i} > x_i \geq x_j > x_{j+1} \geq x_{k-1} > x_k,$$

then there is a transfer of $\mathbf{x}$ of type $(\uparrow i, \downarrow j, \uparrow k)$ (cf. [13, Lemma 3]). Assume $\mathbf{x}, \mathbf{y} \in \mathcal{G}$, and $\mathbf{x} \neq \mathbf{y}$. Since $\mathbf{x} \not\prec \mathbf{y}$ and $\mathbf{y} \not\prec \mathbf{x}$, we can easily infer that there is a positive $\mathcal{G}$-consistent 3-transfer of $\mathbf{x}$ or $\mathbf{y}$. The next lemma shows this to be true even with some additional constraints.

Lemma 61. *Let* $\mathcal{G} = I^n[\mathrm{S}, a; G, b]$, $\mathbf{a}, \mathbf{b}, \mathbf{m}, \mathbf{M} \in \mathbb{R}^n_{\downarrow}$, *and let* $\mathcal{X} = \{\, \mathbf{v} \in \mathcal{G} \mid \mathbf{a} \prec \mathbf{v} \prec \mathbf{b},\ \mathbf{m} \leq \mathbf{v} \leq \mathbf{M} \,\}$. *Assume* $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ *and* $\mathbf{x} \neq \mathbf{y}$.

Let $t = \min\{\, i \mid x_i \neq y_i \,\}$. *If* $x_t < y_t$, *then there exist vectors* $\mathbf{u}, \mathbf{v} \in \mathcal{X}$ *such that* $\mathbf{x} \prec_{\mathcal{G}} \mathbf{u}$ *and* $\mathbf{v} \prec_{\mathcal{G}} \mathbf{y}$.

Proof. Since $x_t < y_t \le y_{t-1} = x_{t-1} = \cdots = y_1 = x_1$, we can assume without loss of generality that $x_1 < y_1$. Since $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we have $\max\{a_1, m_1\} \le x_1 < y_1 \le \min\{b_1, M_1\}$.

Since $\mathbf{x} \not\preceq \mathbf{y}$, $\mathbf{y} \not\preceq \mathbf{x}$, and $\mathrm{S}(\mathbf{x}) = \mathrm{S}(\mathbf{y})$, the following indices are well-defined:

$$\begin{aligned} i^- &= \min\{\, \ell \mid \mathrm{S}_{1\ell}(\mathbf{x}) \ge \mathrm{S}_{1\ell}(\mathbf{y}) \,\}, \\ i &= \max\{\, \ell \mid x_\ell = x_{i^-} \,\}, \\ j^+ &= \min\{\, \ell \ge i+1 \mid \mathrm{S}_{1\ell}(\mathbf{x}) \le \mathrm{S}_{1\ell}(\mathbf{y}) \,\}, \\ j &= \min\{\, \ell \mid x_\ell = x_{j^+} \,\}, \end{aligned}$$

Obviously, $x_i > y_i \ge m_i$ and

$$\mathrm{S}_{1\ell}(\mathbf{x}) < \mathrm{S}_{1\ell}(\mathbf{y}) \le \mathrm{S}_{1\ell}(\mathbf{b}) \quad \text{for } \ell = 1, 2, \ldots, i^- - 1.$$

We show that $\mathrm{S}_{1\ell}(\mathbf{x}) < \mathrm{S}_{1\ell}(\mathbf{b})$ for $\ell = i^-, i^- + 1, \ldots, i-1$: Assume to the contrary that ($i^- \ne i$ and) there exists

$$k = \min\{\, \ell \mid i^- \le \ell \le i-1,\ \mathrm{S}_{1\ell}(\mathbf{x}) = \mathrm{S}_{1\ell}(\mathbf{b}) \,\}.$$

Then $x_k > b_k$. Since $x_k = x_{k+1}$ and $\mathrm{S}_{1,k+1}(\mathbf{x}) \le \mathrm{S}_{1,k+1}(\mathbf{b})$, this leads to a contradiction $x_k \le b_{k+1} \le b_k$.

Assume now $\mathrm{S}_{1i}(\mathbf{x}) > \mathrm{S}_{1i}(\mathbf{y})$. Then $x_j < y_j$ and

$$\mathrm{S}_{1\ell}(\mathbf{x}) > \mathrm{S}_{1\ell}(\mathbf{y}) \ge \mathrm{S}_{1\ell}(\mathbf{a}) \quad \text{for } \ell = i, i+1, \ldots, j-1.$$

Choose positive real numbers ε and δ such that

$$\begin{aligned} \varepsilon &\le y_1 - x_1, \\ \varepsilon &\le \min\{\, \mathrm{S}_{1\ell}(\mathbf{b}) - \mathrm{S}_{1\ell}(\mathbf{x}) \mid \ell = 2, 3, \ldots, i-1 \,\}, \\ \varepsilon + \delta &\le \min\{x_i - y_i, x_i - x_{i+1}\}, \\ \delta &\le \min\{\, \mathrm{S}_{1\ell}(\mathbf{x}) - \mathrm{S}_{1\ell}(\mathbf{y}) \mid \ell = i, i+1, \ldots, j-1 \,\}, \\ \delta &\le \min\{y_j - x_j, x_{j-1} - x_j\}. \end{aligned}$$

Then the vector $T_{t\mathbf{e}_1 - (t+s)\mathbf{e}_i + s\mathbf{e}_j}(\mathbf{x})$ belongs to $\mathcal{X}$ whenever $0 \le t \le \varepsilon$ and $0 \le s \le \delta$. Since

$$\mathbf{x} - \delta\mathbf{e}_i + \delta\mathbf{e}_j \prec \mathbf{x} \prec \mathbf{x} + \varepsilon\mathbf{e}_1 - \varepsilon\mathbf{e}_i,$$

it follows that there exists a vector $\mathbf{u} = T_{t\mathbf{e}_1 - (t+s)\mathbf{e}_i + s\mathbf{e}_j}(\mathbf{x}) \in \mathcal{X}$ such that $t \in (0, \varepsilon]$ and $s \in (0, \delta]$ and $G(\mathbf{u}) = G(\mathbf{x}) = b$. This means that $\mathbf{x} \prec_{\mathcal{G}} \mathbf{u}$ (in fact, $\mathbf{x} \prec_{\mathcal{X}} \mathbf{u}$, too).

Assume then $\mathrm{S}_{1i}(\mathbf{x}) = \mathrm{S}_{1i}(\mathbf{y})$ and define

$$\begin{aligned} m^- &= \min\{\, \ell \mid \mathrm{S}_{1\ell}(\mathbf{x}) > \mathrm{S}_{1\ell}(\mathbf{y}) \,\}, \\ m &= \max\{\, \ell \mid x_\ell = x_{m^-} \,\}, \\ p^+ &= \min\{\, \ell \ge m+1 \mid \mathrm{S}_{1\ell}(\mathbf{x}) \le \mathrm{S}_{1\ell}(\mathbf{y}) \,\}, \\ p &= \min\{\, \ell \mid x_\ell = x_{p^+} \,\} \end{aligned}$$

(note the possibility of $p = j$). Choose $\varepsilon > 0$ as above, setting $\delta = 0$. Now $\mathbf{x} + t(\mathbf{e}_1 - \mathbf{e}_i) \in \mathcal{X}$ when $0 \le t \le \varepsilon$. If

$$0 \le s \le \min\{x_m - y_m, y_p - x_p, x_m - x_{m+1}, x_{p-1} - x_p\}$$

and

$$s \le \min\{\,\mathrm{S}_{1\ell}(\mathbf{x}) - \mathrm{S}_{1\ell}(\mathbf{y}) \mid \ell = m, m+1, \dots, p-1\,\},$$

then $\mathbf{x}+s(-\mathbf{e}_m+\mathbf{e}_p) \in \mathcal{X}$. Thus there exists a vector $\mathbf{u} = T_{t\mathbf{e}_1 - t\mathbf{e}_i - s\mathbf{e}_m + s\mathbf{e}_j}(\mathbf{x})$ such that t and s are positive, $\mathbf{u} \in \mathcal{X}$, and $G(\mathbf{u}) = b$. With the help of Theorem 52, we deduce that $\mathbf{x} \prec_{\mathcal{G}} \mathbf{u}$.

We have now proved the first claim of the lemma. Using similar reasoning, it can be shown that there exists a vector $\mathbf{v} \in \mathcal{X}$ such that $\mathbf{v} \prec_{\mathcal{G}} \mathbf{y}$.

Since we can always choose trivial bounds $\mathbf{a} = \inf_{\preceq} \mathcal{G}$, $\mathbf{b} = \sup_{\preceq} \mathcal{G}$, $\mathbf{m} = \inf_{\le} \mathcal{G}$ (or even $\mathbf{m} = -\infty\mathbf{1}$), or $\mathbf{M} = \sup_{\le} \mathcal{G}$ (or $\mathbf{M} = \infty\mathbf{1}$), Lemma 61 remains valid even if some of the extra constraints are removed. It is not, however, always true that $\mathbf{x} \prec_{\mathcal{X}} \mathbf{y}$, as shown in the following example.

Example 34. Let $P_2 \colon \mathbb{R}^4 \to \mathbb{R}$,

$$\begin{aligned}
\mathcal{P}_2 &= \{\,\mathbf{x} \in \mathbb{R}^4_{\downarrow} \mid \mathrm{S}(\mathbf{x}) = 50,\ P_2(\mathbf{x}) = 750\,\},\\
\mathbf{a} &= (20, 15, 8, 7),\\
\mathbf{b} &= (21, 14, 10, 5),\\
\mathbf{m} &= -\infty\mathbf{1}_4,\\
\mathbf{M} &= \infty\mathbf{1}_4,\\
\mathcal{X} &= \{\,\mathbf{v} \in \mathcal{P}_2 \mid \mathbf{a} \prec \mathbf{v} \prec \mathbf{b},\ \mathbf{m} \le \mathbf{v} \le \mathbf{M}\,\},\\
\mathbf{x} &= (20, 15, 10, 5),\\
\mathbf{y} &= (21, 14, 8, 7).
\end{aligned}$$

Then $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\mathbf{y} = T_{\mathbf{e}_1 - \mathbf{e}_2 - 2\mathbf{e}_3 + 2\mathbf{e}_4}(\mathbf{x})$, and, by Theorem 52, $\mathbf{x} \prec_{\mathcal{P}_2} \mathbf{y}$. Since $\mathrm{S}_{12}(\mathbf{x}) = \mathrm{S}_{12}(\mathbf{y}) = \mathrm{S}_{12}(\mathbf{a}) = \mathrm{S}_{12}(\mathbf{b})$, there is no such 3-transfer T that $T(\mathbf{x}) \in \mathcal{X}$ or $T(\mathbf{y}) \in \mathcal{X}$. This means that $\mathbf{x} \not\prec_{\mathcal{X}} \mathbf{y}$.

From Lemma 61 together with its proof we infer

Lemma 62. *Let $\mathcal{G}$ and $\mathcal{X}$ be as in Lemma 61 and let $F \colon I^n \mapsto \mathbb{R}$ be a $\prec_{\mathcal{G}}$-increasing function.*

If $\check{\mathbf{x}} \in \mathcal{X}$ is a point for which there is no transfer of type $(\uparrow\downarrow\uparrow)$ or of type $(\uparrow\downarrow\downarrow\uparrow)$, then

$$\max_{\mathbf{x} \in \mathcal{X}} F(\mathbf{x}) = F(\check{\mathbf{x}}).$$

Analogously, if $\hat{\mathbf{x}} \in \mathcal{X}$ is a point for which there is no transfer of type $(\downarrow\uparrow\downarrow)$ or of type $(\downarrow\uparrow\uparrow\downarrow)$, then

$$\min_{\mathbf{x} \in \mathcal{X}} F(\mathbf{x}) = F(\hat{\mathbf{x}}).$$

If the extra bounds relative to majorization are trivial, that is, if $\mathbf{a} \prec \mathcal{X} \prec \mathbf{b}$, then it suffices to consider 3-transfers.

Example 35. Let $\mathcal{G}$ and $\boldsymbol{\alpha}_i$ be as in Example 28 (p. 69). Assume that F is a $\prec_{\mathcal{G}}$-increasing function. By Example 28, $F(\boldsymbol{\alpha}_i) \leq F(\boldsymbol{\alpha}_j)$ whenever $i > j$.

Moreover, by Lemma 62,

$$\min_{\mathbf{x} \in \mathcal{G}} F(\mathbf{x}) = F(\boldsymbol{\alpha}_{n-1}), \qquad \max_{\mathbf{x} \in \mathcal{G}} F(\mathbf{x}) = F(\boldsymbol{\alpha}_1).$$

9.2 On $\prec_{\mathcal{X}}$-increasing functions

In [7, Lemma 3.1] it was proved that on the set $\{\, (x, y, z) \mid x \geq y \geq z \geq 0,\ x + y + z = K,\ x^2 + y^2 + z^2 = L \,\}$ the product xyz increases with x or z increasing and decreases with y increasing. This means that the product $\mathrm{P} \colon \mathbb{R}_+^n \to \mathbb{R}$ is $\prec_{\mathcal{P}_2}$-increasing.

In this section, $\mathcal{P}_+$ and $\mathcal{P}_{++}$ denote the sets $\mathbb{R}_+^n[\mathrm{S}, a; P_2, b]$ and $\mathbb{R}_{++}^n[\mathrm{S}, a; P_2, b]$, respectively.

Example 36 (cf. [7, Theorem 3.1]). Let $\mathbf{A}$ be a Hermitian positive semi-definite $n \times n$-matrix with $\operatorname{tr} \mathbf{A} = a$, $\operatorname{tr} \mathbf{A}^2 = b$, and with diagonal elements $\mathbf{a} = (a_1, a_2, \ldots, a_n)$.

There exist an index $k \geq 2$ and real numbers $\alpha \geq \bar{\alpha}$ such that $a_{k-1} \geq \bar{\alpha} \geq a_k$ and

$$\boldsymbol{\alpha} = \big(\alpha, \langle\bar{\alpha}\rangle^{k-2}, a_k, \ldots, a_n\big) \in \{\, \mathbf{v} \in \mathcal{P}_+ \mid \mathbf{a} \preceq \mathbf{v} \,\}.$$

Furthermore, there exist indices $\ell \leq n - 2$, $\ell' \leq n - \ell - 1$ and real numbers $\beta \geq \bar{\beta}$ such that $a_\ell \geq \beta \geq a_{\ell+1}$ and

$$\boldsymbol{\beta} = \big(a_1, \ldots, a_\ell, \langle\beta\rangle^{\ell'}, \bar{\beta}, \langle 0\rangle^{n-\ell-\ell'-1}\big) \in \{\, \mathbf{v} \in \mathcal{P}_+ \mid \mathbf{a} \preceq \mathbf{v} \,\}.$$

There is no transfer of $\boldsymbol{\alpha}$ of type $(\uparrow\downarrow\uparrow)$ or $(\uparrow\downarrow\downarrow\uparrow)$ and no transfer of $\boldsymbol{\beta}$ of type $(\downarrow\uparrow\downarrow)$ or $(\downarrow\uparrow\uparrow\downarrow)$. It follows from Lemma 62 that the maximum and minimum of $\det \mathbf{A}$ are achieved at the points $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively.

Assume that $f, g \colon I \to \mathbb{R}$ are differentiable and that g is strictly convex. Let $\mathcal{G} = \{\, \mathbf{x} \in I^n \mid \mathrm{S}(\mathbf{x}) = a,\ \sum_i g(x_i) = b \,\}$. We will give the necessary and sufficient conditions for the $\prec_{\mathcal{G}}$-increasingness of the function $\mathbf{x} \mapsto \sum_i f(x_i)$. It suffices to consider the case $n = 3$. We denote $F(x, y, z) = f(x) + f(y) + f(z)$ and proceed in a way parallel to [7].

Let $x > y > z$, $x + y + z = a$, and $g(x) + g(y) + g(z) = b$. Denote

$$y' = \frac{\partial y}{\partial x}, \qquad z' = \frac{\partial z}{\partial x}.$$

Implicit differentiation gives

$$1 + y' + z' = 0,$$
$$g'(x) + g'(y)y' + g'(z)z' = 0.$$

Solving these equations results in

$$y' = \frac{g'(z) - g'(x)}{g'(y) - g'(z)}, \qquad z' = \frac{g'(x) - g'(y)}{g'(y) - g'(z)}.$$

Note that since g is strictly convex, g' is strictly increasing and thus $g'(x) > g'(y) > g'(z)$.

It follows that

$$F'_x = \frac{\partial F(x,y,z)}{\partial x} = f'(x) + \frac{g'(z) - g'(x)}{g'(y) - g'(z)} f'(y) + \frac{g'(x) - g'(y)}{g'(y) - g'(z)} f'(z)$$

and, further, that $F'_x > 0$ if and only if

$$(*) \qquad t_1 f'(x) + t_2 f'(z) > f'(y),$$

where

$$t_1 = \frac{g'(y) - g'(z)}{g'(x) - g'(z)}, \qquad t_2 = \frac{g'(x) - g'(y)}{g'(x) - g'(z)}.$$

In the following theorem we assume also f' to be strictly convex and utilize the fact that $t_1 + t_2 = 1$, $t_1, t_2 > 0$.

Theorem 63. *Let f' be strictly convex on I. Denote $\mathcal{Y} = \mathcal{P}_2 \cap I^n$. Then the function $\mathbf{x} \mapsto \sum_i f(x_i)$ is strictly $\prec_{\mathcal{Y}}$-increasing.*

Proof. First note that the function $x \mapsto x^2$ is strictly convex on any interval of $\mathbb{R}$. Let $x > y > z$. The condition $(*)$ can now be written as

$$\frac{y-z}{x-z} f'(x) + \frac{x-y}{x-z} f'(z) > f'(y).$$

Since f' is strictly convex,

$$\frac{y-z}{x-z} f'(x) + \frac{x-y}{x-z} f'(z) > f'\Big(\frac{y-z}{x-z} x + \frac{x-y}{x-z} z\Big) = f'(y).$$

Hence the condition $(*)$ holds and the claim of the theorem follows.

We can now give more examples of $\prec_{\mathcal{X}}$-increasing functions.

Example 37. Let $f(x) = \ln(x)$. Since $f'(x)$ is strictly convex on $\mathbb{R}_{++}$, the function $\mathbf{x} \mapsto \sum_i \ln(x_i)$ is strictly $\prec_{\mathcal{P}_{++}}$-increasing. This, of course, is equivalent to the product P being strictly $\prec_{\mathcal{P}_{++}}$-increasing.

Example 38. Let $k \geq 3$, $f(x) = x^k$. Since f' is strictly convex on $\mathbb{R}_+$, the function P_k is strictly $\prec_{\mathcal{P}_+}$-increasing. If, in addition, k is odd, then the function P_k is strictly $\prec_{\mathcal{P}_2}$-increasing.

Since the condition

$$\frac{g'(y) - g'(z)}{g'(x) - g'(z)} f'(x) + \frac{g'(x) - g'(y)}{g'(x) - g'(z)} f'(z) > f'(y)$$

is equivalent to

$$\frac{f'(y)-f'(z)}{f'(x)-f'(z)}g'(x)+\frac{f'(x)-f'(y)}{f'(x)-f'(z)}g'(z)<g'(y),$$

we have

Theorem 64. *Let the functions* $f, g\colon I \to \mathbb{R}$ *be differentiable and strictly convex and let*

$$F(\mathbf{x})=\sum_i f(x_i), \qquad G(\mathbf{x})=\sum_i g(x_i).$$

Assume that $\mathcal{F}=I^n[\mathrm{S}, a; F, b]$ *and* $\mathcal{G}=I^n[\mathrm{S}, a; G, c]$. *Then* F *is* $\prec_{\mathcal{G}}$*-increasing if and only if* G *is* $\prec_{\mathcal{F}}$*-decreasing.*

The above theorem plus Examples 37 and 38 yield

Example 39. The function P_2 is $\prec_{\mathcal{X}}$-increasing when $\mathcal{X}=\mathbb{R}^n_{++}[\mathrm{S}, a; -\mathrm{P}, b]$ and $\prec_{\mathcal{X}}$-decreasing when $\mathcal{X}=\mathbb{R}^n_{+}[\mathrm{S}, a; P_k, b]$.

Theorem 65. *Let* $k \le n$. *The* k*th elementary symmetric function* $S_k\colon \mathbb{R}^n \to \mathbb{R}$ *is* $\prec_{\mathcal{P}_+}$*-increasing.*

Furthermore, the elementary symmetric functions S_3, S_4, $\dots$, S_n *are strictly* $\prec_{\mathcal{P}_{++}}$*-increasing.*

Proof. The case $k=1$ is trivial. Since $S_2(\mathbf{x})=(\mathrm{S}(\mathbf{x})^2-P_2(\mathbf{x}))/2$, also the function S_2 is constant on the set $\mathcal{P}_+$. By the result of [7], the product S_n is $\prec_{\mathcal{P}_+}$-increasing and strictly $\prec_{\mathcal{P}_{++}}$-increasing. So the cases remaining are $k=3, 4, 5, \dots, n-1$.

Assume that $\mathbf{x} \in \mathcal{P}_+$, and that there exist indices $1 \le i < j < m \le n$ such that

$$x_{i-1} > x_i \ge x_j > x_{j+1} \ge x_{m-1} > x_m.$$

Let T be a transfer of $\mathbf{x}$ of type $(\uparrow i, \downarrow j, \uparrow m)$ and let $\mathbf{y}=T(\mathbf{x}) \in \mathcal{P}_+$. It suffices to show that $S_k(\mathbf{x}) \le S_k(\mathbf{y})$.

Let $\mathbf{z} \in \mathbb{R}^{n-3}_+$ be a vector obtained from the vector $\mathbf{x}$ by removing the components x_i, x_j, and x_m. Then

$$S_k(\mathbf{x})=x_i x_j x_m S_{k-3}(\mathbf{z})+S_2(x_i, x_j, x_m)S_{k-2}(\mathbf{z}) \\ +(x_i+x_j+x_m)S_{k-1}(\mathbf{z})+S_k(\mathbf{z}).$$

Since the product is $\prec_{\mathcal{P}_+}$-increasing, $x_i x_j x_m < y_i y_j y_m$. The equalities $x_i + x_j + x_m = y_i + y_j + y_m$ and $x_i^2+x_j^2+x_m^2=y_i^2+y_j^2+y_m^2$ jointly imply that $S_2(x_i, x_j, x_m)=S_2(y_i, y_j, y_m)$. Hence $S_k(\mathbf{x}) \le S_k(\mathbf{y})$, which proves that S_k is $\prec_{\mathcal{P}_+}$-increasing.

If $\mathbf{x} > \mathbf{0}$, then $S_{k-3}(\mathbf{z}) > 0$, and it follows that S_k is $\prec_{\mathcal{P}_{++}}$-increasing.

Note that if $(n-1)b \le a^2$, then $\mathcal{P}_2=\mathbb{R}^n[\mathrm{S}, a; P_2, b]=\mathbb{R}^n_+[\mathrm{S}, a; P_2, b]$, and if $(n-1)b < a^2$, then $\mathcal{P}_2=\mathbb{R}^n_{++}[\mathrm{S}, a; P_2, b]$.

Example 40. With the assumptions, and in the notation, of Example 36, the maximum and minimum of $\operatorname{tr}\mathbf{A}^3$, $\operatorname{tr}\mathbf{A}^4$, $\operatorname{tr}\mathbf{A}^5$, ..., $\operatorname{tr}\mathbf{A}^{(2)}$, $\operatorname{tr}\mathbf{A}^{(3)}$, ..., $\operatorname{tr}\mathbf{A}^{(n-1)}$, and $\operatorname{tr}\mathbf{A}^{(n)}$ are achieved at the points $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively.

Note that $\operatorname{tr}\mathbf{A}^{(2)} = ((\operatorname{tr}\mathbf{A})^2 - \operatorname{tr}\mathbf{A}^2)/2 = (a^2 - b)/2$ is constant and $\operatorname{tr}\mathbf{A}^{(n)} = \det\mathbf{A}$.

9.3 A few lemmas on 3-majorization

Let $\mathcal{G} = \mathbb{R}^n[\mathrm{S}, a; G, b]$ and $\mathcal{G}_+ = \mathbb{R}^n_+[\mathrm{S}, a; G, b]$. Assume that the function F defined on the set $\mathcal{X} = \mathcal{G}$ or $\mathcal{X} = \mathcal{G}_+$ is strictly $\prec_{\mathcal{X}}$-increasing. In order to derive bounds for the components of the vectors $\mathbf{x} \in \mathcal{X}$, for which $F(\mathbf{x}) = c$, we will first search for extreme points of F, after which we will be in the position to apply Lemma 2.

The case $\mathcal{X} = \mathcal{G}$ will be dealt with in Section 9.4 and the case $\mathcal{X} = \mathcal{G}_+$ in Section 9.5. After that, in Section 9.6, we will give the solutions to the problems

$$\max\{\, x_i \mid \mathbf{x} \in \mathcal{Y} \,\} \quad \text{and} \quad \min\{\, x_i \mid \mathbf{x} \in \mathcal{Y} \,\},$$

where $i \le n$ and either $\mathcal{Y} = \{\, \mathbf{v} \in \mathcal{G} \mid F(\mathbf{v}) = c \,\}$ or $\mathcal{Y} = \{\, \mathbf{v} \in \mathcal{G}_+ \mid F(\mathbf{v}) = c \,\}$. In this section we will produce some auxiliary results to this end.

Recall Example 35, according to which

$$\min_{\mathbf{x}\in\mathcal{G}} F(\mathbf{x}) = F(\boldsymbol{\alpha}_{n-1}), \qquad \max_{\mathbf{x}\in\mathcal{G}} F(\mathbf{x}) = F(\boldsymbol{\alpha}_1).$$

Denote $\mathbf{v}(x, y, z) = (\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k)$. It is easy to see that if $\mathbf{v}(x, y, z) \in \mathcal{G}$, then

$$\boldsymbol{\alpha}_{i+j} \preceq_{\mathcal{G}} \mathbf{v}(x, y, z) \preceq_{\mathcal{G}} \boldsymbol{\alpha}_i.$$

More generally, we have

Lemma 66. *Let $\mathbf{v} = \mathbf{v}(x, y, z)$ and $\tilde{\mathbf{v}} = \mathbf{v}(\tilde{x}, \tilde{y}, \tilde{z})$ belong to $\mathcal{G}$. Then the following conditions are equivalent:*

(1) $\mathbf{v}(x, y, z) \prec_{\mathcal{G}} \mathbf{v}(\tilde{x}, \tilde{y}, \tilde{z})$,

(2) $x < \tilde{x}$,

(3) $y > \tilde{y}$,

(4) $z < \tilde{z}$.

Proof. We prove that (1) ⇔ (2). The proof of (1) ⇔ (3) and (1) ⇔ (4) is completely analogous.

Assume $x < \tilde{x}$. Since $\mathbf{v} \not\preceq \tilde{\mathbf{v}}$ and $\mathrm{S}(\mathbf{v}) = \mathrm{S}(\tilde{\mathbf{v}})$, we have $y > \tilde{y}$ and $z < \tilde{z}$. Hence it follows from Theorem 52 that $\mathbf{v} \prec_{\mathcal{G}} \tilde{\mathbf{v}}$.

For the converse, assume $\mathbf{v} \prec_{\mathcal{G}} \tilde{\mathbf{v}}$. Then $x \le \tilde{x}$, $y \ge \tilde{y}$, and $z \le \tilde{z}$. If $x = \tilde{x}$, then it would follow that $\tilde{\mathbf{v}} \prec \mathbf{v}$, which is not possible since G is strictly Schur-convex. Hence $x < \tilde{x}$.

Assume F to be strictly increasing. Consider the vectors $\mathbf{v}(x,y,z)$ belonging to $\mathcal{G}$, and regard, say, x as an independent variable on which $y = y_x$ and $z = z_x$ depend. Lemma 66 implies that the function $x \mapsto F(\mathbf{v}(x, y_x, z_x))$ is strictly increasing on its domain.

As before,

$$\mu = \mu(\mathcal{G}_+) = \max\{\, \ell \le n-2 \bigm| G\bigl(\langle a/(n-\ell)\rangle^{n-\ell}, \langle 0\rangle^{\ell}\bigr) \le b \,\}.$$

Let $j = n - i - k \ge 1$. Consider the system

$$\begin{aligned} \mathrm{S}\bigl(\langle x\rangle^i, \langle y\rangle^j, \langle 0\rangle^k\bigr) &= a,\\ G\bigl(\langle x\rangle^i, \langle y\rangle^j, \langle 0\rangle^k\bigr) &= b,\\ x \ge y &> 0. \end{aligned}$$

By Lemmas 27 and 28, the above system has a solution if and only if $1 \le i \le n-\mu-1$ and $0 \le k \le \mu$. Since G is strictly Schur-convex, the solution is unique (when it exists). We denote the solution (x,y) by $(\beta_{ik}, \bar\beta_{ik})$ and the solution vector $\mathbf{v}(\beta_{ik}, \bar\beta_{ik}, 0)$ by $\boldsymbol{\beta}_{ik}$. When $\boldsymbol{\beta}_{ik}$ exists, i.e., when i and k satisfy the conditions given above, it is a unique vector of shape $([i] \ge [j] > \langle 0\rangle^k)$ belonging to $\mathcal{G}_+$. Note that $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_{i0}$ denote the same vector.

By Lemma 62,

$$\min\{\, F(\mathbf{x}) \mid \mathbf{x} \in \mathcal{G}_+ \,\} = F(\boldsymbol{\beta}_{n-\mu-1,\mu}), \quad \max\{\, F(\mathbf{x}) \mid \mathbf{x} \in \mathcal{G}_+ \,\} = F(\boldsymbol{\beta}_{10}).$$

The following lemma orders the vectors $\boldsymbol{\beta}_{ij}$ according to $\prec_{\mathcal{G}_+}$.

Lemma 67. *Let μ be as above. If $n-\mu-1 \ge \ell \ge k$ and $0 \le m \le \mu$, then*

$$\boldsymbol{\beta}_{\ell m} \preceq_{\mathcal{G}_+} \boldsymbol{\beta}_{km}.$$

If $m \le n-\mu-1$ and $\mu \ge \ell \ge k \ge 0$, then

$$\boldsymbol{\beta}_{m\ell} \preceq_{\mathcal{G}_+} \boldsymbol{\beta}_{mk}.$$

Proof. It follows from Theorem 9 that $\beta_{km} \ge \beta_{\ell m}$ and $\bar\beta_{km} \ge \bar\beta_{\ell m}$. Since $\mathrm{S}(\boldsymbol{\beta}_{\ell m}) = \mathrm{S}(\boldsymbol{\beta}_{km})$, we further have $\bar\beta_{km} \le \beta_{\ell m}$. By Theorem 52, $\boldsymbol{\beta}_{\ell m} \prec_{\mathcal{G}_+} \boldsymbol{\beta}_{km}$.

Now consider the second part of the lemma. By the definition, $\bar\beta_{mk} > 0$. If $\beta_{mk} \le \beta_{m\ell}$, it would follow from Lemma 4 that $\boldsymbol{\beta}_{mk} \prec \boldsymbol{\beta}_{m\ell}$. Hence $\beta_{mk} > \beta_{m\ell}$. Furthermore, $\bar\beta_{mk} < \bar\beta_{m\ell}$, and Theorem 52 yields now $\boldsymbol{\beta}_{m\ell} \preceq_{\mathcal{G}_+} \boldsymbol{\beta}_{mk}$.

Lemma 66 can be trivially generalized:

Lemma 68. *Let $i \le n-\mu-1$ and $0 \le \ell \le \mu$. Denote*

$$\mathbf{w}(x,y,z) = \bigl(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k, \langle 0\rangle^{\ell}\bigr).$$

Assume $\mathbf{w}(x,y,z), \mathbf{w}(\tilde x, \tilde y, \tilde z) \in \mathcal{G}_+$. Then the following conditions are equivalent:

(1) $\mathbf{w}(x, y, z) \prec_{\mathcal{G}_+} \mathbf{w}(\tilde{x}, \tilde{y}, \tilde{z})$,

(2) $x < \tilde{x}$,

(3) $y > \tilde{y}$,

(4) $z < \tilde{z}$.

The following complement to Lemmas 66 and 68 will be needed to guarantee the existence and uniqueness of the optimum points that we will find in Sections 9.4 and 9.5.

Lemma 69. *Let $\mathcal{G}$ and $\boldsymbol{\alpha}_i$ be as above. Assume that i, j, and k are fixed and $i + j + k = n$.*

(1) *There is a (unique) point of shape $(\langle x\rangle^i \geq [j] \geq [k])$ in $\mathcal{G}$ if and only if $\alpha_{i+j} \leq x \leq \alpha_i$.*

(2) *There is a (unique) point of shape $([i] \geq \langle y\rangle^j \geq [k])$ in $\mathcal{G}$ if and only if $\bar{\alpha}_i \leq y \leq \alpha_{i+j}$.*

(3) *There is a (unique) point of shape $([i] \geq [j] \geq \langle z\rangle^k)$ in $\mathcal{G}$ if and only if $\bar{\alpha}_{i+j} \leq z \leq \bar{\alpha}_i$.*

Let $\mathcal{G}_+$, $\boldsymbol{\beta}_{ij}$, and μ be as above. Assume that i, j, k, and ℓ are fixed, $i \leq n - \mu - 1$, $0 \leq \ell \leq \mu$, and $i + j + k + \ell = n$.

(4) *There is a (unique) point of shape $(\langle x\rangle^i \geq [j] \geq [k] \geq \langle 0\rangle^\ell)$ in $\mathcal{G}_+$ if and only if $x \leq \beta_{i\ell}$, together with either $k + \ell \leq \mu$ and $\beta_{i,k+\ell} \leq x$ or $k + \ell \geq \mu + 1$ and $\beta_{i+j,\ell} \leq x$.*

(5) *There is a (unique) point of shape $([i] \geq \langle y\rangle^j \geq [k] \geq \langle 0\rangle^\ell)$ in $\mathcal{G}_+$ if and only if $y \geq \bar{\beta}_{i\ell}$, together with either $k + \ell \leq \mu$ and $\bar{\beta}_{i,k+\ell} \geq y$ or $k + \ell \geq \mu + 1$ and $\beta_{i+j,\ell} \geq y$.*

(6) *There is a (unique) point of shape $([i] \geq [j] \geq \langle z\rangle^k \geq \langle 0\rangle^\ell)$ in $\mathcal{G}_+$ if and only if $z \leq \bar{\beta}_{i\ell}$, together with either $k + \ell \leq \mu$ and $0 \leq z$ or $k + \ell \geq \mu + 1$ and $\bar{\beta}_{i+j,\ell} \leq z$.*

Proof. We prove the first and last parts. The other parts are proved similarly.

Let $\mathbf{v} = (\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k) \in \mathbb{R}^n[S, a]$. If $x < \alpha_{i+j}$, then $\mathbf{v} \prec \boldsymbol{\alpha}_{i+j}$, and if $x > \alpha_i$, then $\boldsymbol{\alpha}_i \prec \mathbf{v}$. Since G is strictly Schur-convex, it follows that $\mathbf{v} \notin \mathcal{G}$. Assume then that $x \in [\alpha_{i+j}, \alpha_i]$. Define

$$\mathbf{u} = \big(\langle x\rangle^i, \langle y_x\rangle^j, \langle z_x\rangle^k\big),$$

where

$$\big(\langle y_x\rangle^j, \langle z_x\rangle^k\big) = \mathbf{s}\big([j] \geq [k]; \mathbb{R}^{j+k}[\mathrm{S}, a - ix; \bar{G}, b]\big)$$

with $\bar{G}\colon \mathbb{R}^{j+k} \to \mathbb{R}$, $\bar{G}(\mathbf{y}) = G(\langle x\rangle^i, \mathbf{y})$. We show that $\mathbf{u} \in \mathcal{G}$.

Now $x \geq \alpha_{i+j} \geq (a - ix)/(j + k)$. Further, since $x \leq \alpha_i$, we have $(\langle x \rangle^i, \langle (a-ix)/(j+k) \rangle^{j+k}) \preceq \boldsymbol{\alpha}_i$. Hence $G(\langle x \rangle^i, \langle (a-ix)/(j+k) \rangle^{j+k}) \leq b$, which entails that $\mathbf{u}$ is well-defined. Since, trivially, $S(\mathbf{u}) = a$ and $G(\mathbf{u}) = b$, it remains to be shown that $\mathbf{u}$ is decreasingly ordered. Since $x \geq \alpha_{i+j}$, necessarily $x \geq z_x$. Since assumption $y_x > x \geq z_x$ implies a contradiction $\boldsymbol{\alpha}_{i+j} \prec \mathbf{u}$, we have $x \geq y_x$.

Let

$$\bar{\mathbf{v}} = \big(\langle x \rangle^i, \langle \bar{y} \rangle^j, \langle \bar{z} \rangle^k\big) \neq \mathbf{u}.$$

Then $\bar{y} > y_x$ or $\bar{y} < y_x$, which implies that $\mathbf{u} \prec \bar{\mathbf{v}}$ or $\bar{\mathbf{v}} \prec \mathbf{u}$ and, further, that $\bar{\mathbf{v}} \notin \mathcal{G}$. This observation guarantees the uniqueness of the vector satisfying the given requirements and concludes our proof of the first part.

As to the last part, let $\tilde{\mathbf{v}}$ be a vector of shape $([i] \geq [j] \geq \langle z \rangle^k \geq \langle 0 \rangle^\ell)$ belonging to $\mathbb{R}_+[\mathrm{S}, a]$. If $z > \bar{\beta}_{i\ell}$, then $\tilde{\mathbf{v}} \prec \boldsymbol{\beta}_{i\ell}$. This implies that $\tilde{\mathbf{v}} \notin \mathcal{G}_+$. For the rest of the proof, assume $z \leq \bar{\beta}_{i\ell}$. If $k + \ell \geq \mu + 1$, but $\bar{\beta}_{i+j,\ell} > z$, then

$$\big(\langle \bar{\beta}_{i+j,\ell} \rangle^k, \langle 0 \rangle^\ell\big) \preceq^w \big(\langle z \rangle^k, \langle 0 \rangle^\ell\big).$$

It follows from Lemma 5 that $\boldsymbol{\beta}_{i+j,\ell} \prec \tilde{\mathbf{v}}$. Thus $\tilde{\mathbf{v}} \notin \mathcal{G}_+$. If $k + \ell \leq \mu$ but $0 > z$, it follows that $\tilde{\mathbf{v}} \notin \mathcal{G}_+$.

Next, assume that $k + \ell \geq \mu + 1$ and $\bar{\beta}_{i+j,\ell} \leq z$. From the assumptions it follows that

$$\boldsymbol{\beta}_{i\ell} \preceq \big(\langle (a - (k+j)z)/i \rangle^i, \langle z \rangle^{k+j}, \langle 0 \rangle^\ell\big)$$

and

$$\big(\langle (a - kz)/(i+j) \rangle^{i+j}, \langle z \rangle^k, \langle 0 \rangle^\ell\big) \preceq \boldsymbol{\beta}_{i+j,\ell}.$$

Therefore, we can choose

$$\big(\langle x_z \rangle^i, \langle y_z \rangle^j\big) = \mathbf{s}\big([i] \geq [j]; \mathbb{R}^{i+j}[\mathrm{S}, a - kz; \tilde{G}, b]\big),$$

where $\tilde{G}\colon \mathbb{R}^{i+j} \to \mathbb{R}$, $\tilde{G}(\mathbf{y}) = G(\mathbf{y}, \langle z \rangle^k, \langle 0 \rangle^\ell)$. It is easy to see that $(\langle x_z \rangle^i, \langle y_z \rangle^j, \langle z \rangle^k, \langle 0 \rangle^\ell)$ belongs to $\mathcal{G}_+$.

Finally, assume $k + \ell \leq \mu$ and $0 \leq z$. Now

$$\boldsymbol{\beta}_{i\ell} \preceq \big(\langle (a - (k+j)z)/i \rangle^i, \langle z \rangle^{k+j}, \langle 0 \rangle^\ell\big)$$

and

$$\big(\langle (a - kz)/(i+j) \rangle^{i+j}, \langle z \rangle^k, \langle 0 \rangle^\ell\big) \preceq \boldsymbol{\beta}_{i,k+\ell}.$$

As above, we can construct a vector of shape $([i] \geq [j] \geq \langle z \rangle^k \geq \langle 0 \rangle^\ell)$ belonging to $\mathcal{G}_+$. The uniqueness of this vector is shown analogously to the proof of the first part.

After the proof of Lemma 69, the following counterpart of the result

$$\boldsymbol{\alpha}_{i+j} \preceq_{\mathcal{G}} \big(\langle x \rangle^i, \langle y \rangle^j, \langle z \rangle^k\big) \preceq_{\mathcal{G}} \boldsymbol{\alpha}_i$$

is obvious:

Example 41. Let $\mathbf{w} \in \mathcal{G}_+$ be as in Lemma 68. Then $\mathbf{w} \preceq_{\mathcal{G}_+} \boldsymbol{\beta}_{i\ell}$. Moreover, if $k + \ell \leq \mu$, then $\boldsymbol{\beta}_{i,k+\ell} \preceq_{\mathcal{G}_+} \mathbf{w}$; otherwise $\boldsymbol{\beta}_{i+j,\ell} \preceq_{\mathcal{G}_+} \mathbf{w}$.

9.4 Bounds for $F(\mathbf{x})$ subject to $\mathbf{x} \in \mathcal{G}$ with fixed x_k

Let $\mathcal{G} = \mathbb{R}^n[\mathrm{S}, a; G, b]$ and assume the function F to be strictly $\prec_{\mathcal{G}}$-increasing. For $k = 1, 2, \ldots, n$, denote

$$M_k(x) = \max\{\, F(\mathbf{x}) \mid \mathbf{x} \in \mathcal{G},\ x_k = x \,\},$$
$$m_k(x) = \min\{\, F(\mathbf{x}) \mid \mathbf{x} \in \mathcal{G},\ x_k = x \,\}.$$

Let $\alpha_{n-1} \le x \le \alpha_1$. By Lemma 69, there is a point $\check{\mathbf{x}} = (x, \langle y \rangle^{n-2}, z) \in \{\, \mathbf{v} \in \mathcal{G} \mid v_1 = x \,\}$. Since there is no negative $\{\, \mathbf{v} \in \mathcal{G} \mid v_1 = x \,\}$-consistent transfer of $\check{\mathbf{x}}$, we have, by Lemma 62, $m_1(x) = F(\check{\mathbf{x}})$. On the other hand, there is no positive $\{\, \mathbf{v} \in \mathcal{G} \mid v_n = z \,\}$-consistent transfer of $\check{\mathbf{x}}$. It follows that $F(\check{\mathbf{x}})$ is also the value of M_n at the point $z \in [\bar{\alpha}_{n-1}, \bar{\alpha}_1]$. Lemma 66 implies that the functions m_1 and M_n are strictly increasing.

Next we find the maximum of F on the set $\{\, \mathbf{v} \in \mathcal{G} \mid v_1 = x \,\}$ and the minimum on the sets $\{\, \mathbf{v} \in \mathcal{G} \mid v_n = z \,\}$. Define $K\colon [\alpha_{n-1}, \alpha_1] \to \{1, 2, \ldots, n-1\}$, $K(x) = \max\{\, \ell \le n-1 \mid x \le \alpha_\ell \,\}$. Let $x \in [\alpha_{n-1}, \alpha_1]$ and $K = K(x)$. If $K = n-1$, then $x = \alpha_{n-1}$; otherwise $\alpha_{K+1} < x \le \alpha_K$. By Lemma 69, there is a point

$$\hat{\mathbf{x}} = \big(\langle x \rangle^K, y, \langle z \rangle^{n-K-1}\big) \in \{\, \mathbf{v} \in \mathcal{G} \mid v_1 = x \,\}.$$

Since there is no $\{\, \mathbf{v} \in \mathcal{G} \mid v_1 = x \,\}$-consistent positive transfer of $\hat{\mathbf{x}}$, we have $M_1(x) = F(\hat{\mathbf{x}})$. Moreover, by symmetricity, $m_n(z) = F(\hat{\mathbf{x}})$ when $z \in [\bar{\alpha}_{K+1}, \bar{\alpha}_K]$.

Let $K \le n-1$. We infer from Lemma 66 that the function M_1 is strictly increasing on the interval $(\alpha_{K+1}, \alpha_K]$. Since, in addition,

$$\lim_{x \to \alpha_{K+1}-} M_1(x) = F(\alpha_{K+1}) = M_1(\alpha_{K+1}),$$

the function M_1 is continuous and strictly increasing on the interval $[\alpha_{n-1}, \alpha_1]$. For analogous reasons, m_n is a continuous strictly increasing function. Moreover,

$$M_1(\alpha_{n-1}) = m_1(\alpha_{n-1}) = F(\boldsymbol{\alpha}_{n-1}) = M_n(\bar{\alpha}_{n-1}) = m_n(\bar{\alpha}_{n-1})$$

and

$$M_1(\alpha_1) = m_1(\alpha_1) = F(\boldsymbol{\alpha}_1) = M_n(\bar{\alpha}_1) = m_n(\bar{\alpha}_1).$$

We have now found the extreme values of F on the set $\mathcal{G}$ under the extra constraint for x_1 and for x_n. Next assume that $2 \le k \le n-1$ and that $\bar{\alpha}_{k-1} \le x \le \alpha_k$. Using similar argumentation, we can derive the following results:

(1) If $\bar{\alpha}_{k-1} \le x \le \alpha_{n-1}$, then $m_k(x) = F(\check{\mathbf{x}}^-)$, where $\check{\mathbf{x}}^-$ is a unique point of shape $([k-1] \ge \langle x \rangle^{n-k} \ge [1])$ belonging to $\{\, \mathbf{v} \in \mathcal{G} \mid v_k = x \,\}$.

The function m_k is strictly decreasing on the interval $[\bar{\alpha}_{k-1}, \alpha_{n-1}]$.

(2) If $\alpha_{n-1} < x \le \alpha_k$, then $m_k(x) = F(\check{\mathbf{x}}^+)$, where $\check{\mathbf{x}}^+$ is a unique point of shape $(\langle x\rangle^k \ge [n-k-1] \ge [1])$ belonging to $\{\, \mathbf{v} \in \mathcal{G} \mid v_k = x \,\}$.

The function m_k is strictly increasing on the interval $[\alpha_{n-1}, \alpha_k]$.

(3) If $\bar{\alpha}_{k-1} \le x \le \bar{\alpha}_1$, then $M_k(x) = F(\hat{\mathbf{x}}^-)$, where $\hat{\mathbf{x}}^-$ is a unique point of shape $([1] \ge [k-2] \ge \langle x\rangle^{n-k+1})$ belonging to $\{\, \mathbf{v} \in \mathcal{G} \mid v_k = x \,\}$.

The function M_k is strictly increasing on the interval $[\bar{\alpha}_{k-1}, \bar{\alpha}_1]$. Furthermore, $M_k(\bar{\alpha}_{k-1}) = F(\boldsymbol{\alpha}_{k-1}) = m_k(\bar{\alpha}_{k-1})$.

(4) If $\bar{\alpha}_1 < x \le \alpha_k$, then $M_k(x) = F(\hat{\mathbf{x}}^+)$, where $\hat{\mathbf{x}}^+$ is a unique point of shape $([1] \ge \langle x\rangle^{k-1} \ge [n-k])$ belonging to $\{\, \mathbf{v} \in \mathcal{G} \mid v_k = x \,\}$.

The function M_k is strictly decreasing on the interval $[\bar{\alpha}_1, \alpha_k]$. Furthermore, $M_k(\alpha_k) = F(\boldsymbol{\alpha}_k) = m_k(\alpha_k)$.

Trivially, the functions M_2, M_3, ..., M_n, and m_1, m_2, ..., m_{n-1} are continuous.

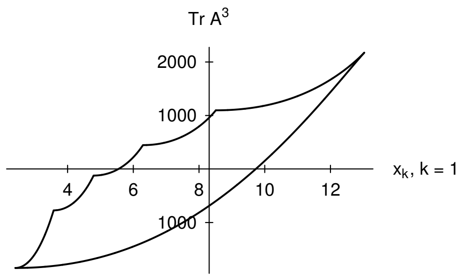
Example 42. Let $\mathbf{A}$ be the matrix in Example 23 (p. 58). Consider the set $\mathcal{P}_2 = \mathbb{R}^7[\mathrm{S}, \operatorname{tr}\mathbf{A}; P_2, \operatorname{tr}\mathbf{A}^2] = \mathbb{R}^7[\mathrm{S}, 2; P_2, 190]$. Then $\operatorname{tr}\mathbf{A}^3 = 2$,

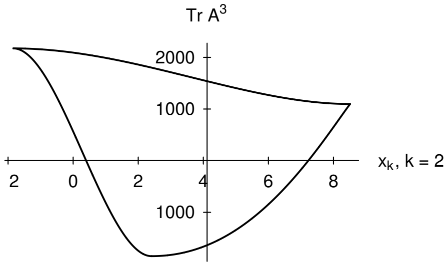
$$\min_{\mathbf{x}\in\mathcal{P}_2} P_3(\mathbf{x}) = P_3(\boldsymbol{\alpha}_6) \approx -1800$$

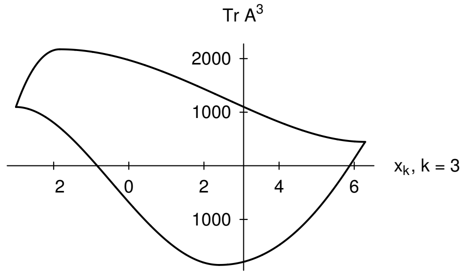
and

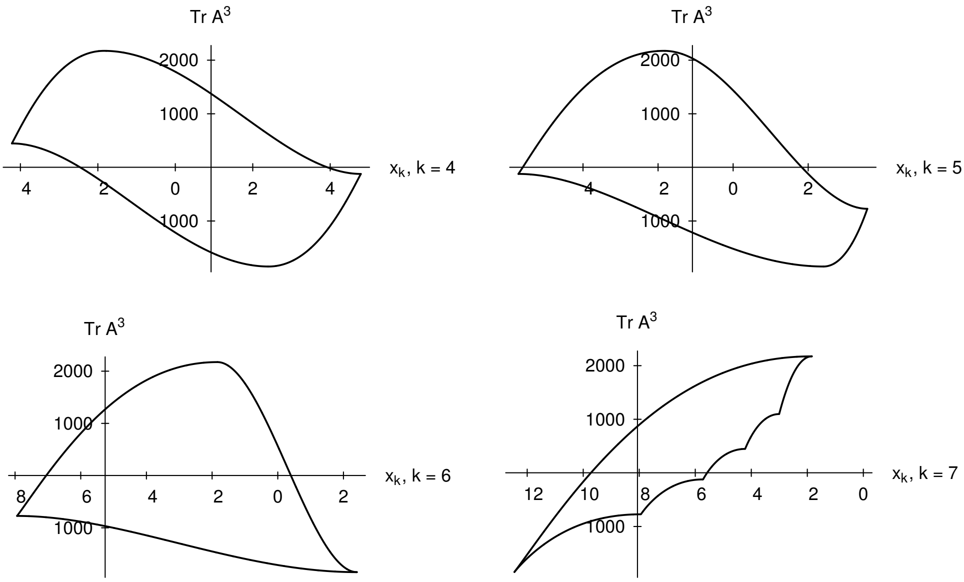
$$\max_{\mathbf{x}\in\mathcal{P}_2} P_3(\mathbf{x}) = P_3(\boldsymbol{\alpha}_1) \approx 2200.$$

The figures below represent $\max_{\mathbf{x}\in\mathcal{P}_2} P_3(\mathbf{x})$ and $\min_{\mathbf{x}\in\mathcal{P}_2} P_3(\mathbf{x})$ as functions of x_k for $k = 1, 2, \ldots, 7$. We have moved the origin to the point $(\lambda_k, \operatorname{tr}\mathbf{A}^3)$. The segment of x-axis inside the figure represents the range of the possible values of x_k, when $P_3(\mathbf{x}) = \operatorname{tr}\mathbf{A}^3$ is fixed. Similarly, the segment of y-axis inside the figure represents the range of the possible values of $P_3(\mathbf{x})$, when $x_k = \lambda_k(\mathbf{A})$ is fixed.









9.5 Bounds for $F(\mathbf{x})$ subject to $\mathbf{x} \in \mathcal{G}_+$ with fixed x_k

We now consider the set $\mathcal{G}_+$ and assume that F is strictly $\prec_{\mathcal{G}_+}$-increasing. If

$$\mu = \max\{\, \ell \mid 0 \le \ell \le n-2,\ G\big(\langle a/(n-\ell)\rangle^{n-\ell}, \langle 0\rangle^{\ell}\big) \le b \,\} = 0,$$

then, by Lemmas 27 and 28, $\mathbf{s}([k] \ge [n-k]; \mathbb{R}^n_+[\mathrm{S}, a; G, b])$ exists for $k = 1, 2, \ldots, n-1$. It follows that the bounds for F are the same with $\mathcal{G}_+$ and $\mathcal{G}$. Therefore, we assume in this section that $\mu \ge 1$.

9.5.1 x_1 or x_n fixed

We first consider the cases $k = 1$ and $k = n$. Assume $\mathbf{x} \in \mathcal{G}_+$; then $\beta_{n-\mu-1,\mu} \le x_1 \le \beta_{10}$ and $0 \le x_n \le \bar{\beta}_{10}$.

Define the function $L\colon [\beta_{n-\mu-1,\mu}, \beta_{10}] \to \{0, 1, 2, \ldots, \mu\}$ as follows: $L(x) = \max\{\, \ell \mid 0 \le \ell \le \mu,\ x \le \beta_{1\ell} \,\}$. Let $\check{\mathbf{x}} = (x, \langle y\rangle^{n-2-L(x)}, z, \langle 0\rangle^{L(x)})$. Since there is no negative $\{\, \mathbf{v} \in \mathcal{G}_+ \mid v_1 = x \,\}$-consistent transfer of $\check{\mathbf{x}}$, by Lemma 62, $m_1(x) = F(\check{\mathbf{x}})$. As in the previous subsection, but now applying Lemma 68, we can show that the function m_1 is strictly increasing and continuous.

We modify the definition of the function K given in the previous section as follows: the domain of K is now $[\beta_{n-\mu-1,\mu}, \beta_{10}]$ and

$$K(x) = \max\{\, \ell \le n-\mu-1 \mid x \le \beta_{\ell 0} \,\}.$$

Note that if $\beta_{n-\mu-1,\mu} \le x \le \beta_{n-\mu-1,0}$, then $K(x) = n-\mu-1$. Once again, we have $M_1(x) = F(\hat{\mathbf{x}})$, where $\hat{\mathbf{x}}$ is a unique point of shape $(\langle x\rangle^{K(x)} \ge [1] \ge$

$[n - K(x) - 1])$ belonging to $\mathcal{G}_+$. Obviously, M_1 is continuous and strictly increasing.

As for the minimum, the cases $k = n$ and $k = 1$ are symmetric. Define $K\colon [0, \bar{\beta}_{10}] \to \{1, 2, \ldots, n - \mu - 1\}$ as follows:

$$K(x) = \max\{\, \ell \mid \ell \le n - \mu - 1,\ x \le \bar{\beta}_{\ell 0} \,\}.$$

Now $m_n(x) = F(\hat{\mathbf{x}})$, where $\hat{\mathbf{x}}$ is a unique point of shape $([K(x)] \ge [1] \ge \langle x \rangle^{n-1-K(x)})$ belonging to $\mathcal{G}_+$. The function m_n is strictly increasing and continuous.

The function M_n is the same with $\mathcal{G}_+$ as before with $\mathcal{G}$, except for its range $[0, \bar{\beta}_{10}]$. So M_n is strictly increasing and continuous.

It is easy to verify that

$$m_1(\beta_{10}) = F(\boldsymbol{\beta}_{10}) = M_1(\beta_{10}),$$
$$m_1(\beta_{n-\mu-1,\mu}) = F(\boldsymbol{\beta}_{n-\mu-1,\mu}) = M_1(\beta_{n-\mu-1,\mu}),$$

and

$$M_n(\bar{\beta}_{10}) = F(\boldsymbol{\beta}_{10}) = m_n(\bar{\beta}_{10}),$$

but $m_n(0) = F(\boldsymbol{\beta}_{n-\mu-1,\mu})$ and $M_n(0) = F(\boldsymbol{\beta}_{11})$. Hence $M_n(0) \ge m_n(0)$, with equality only in the trivial case $n = 3$ and $\mu = 1$.

9.5.2 x_2, x_3, …, or $x_{n-\mu-1}$ fixed

Let $2 \le k \le n-1-\mu$ and $\mathbf{x} \in \mathcal{G}_+$. Then $\bar{\beta}_{k-1,0} \le x_k \le \beta_{k0}$. The function M_k is the same as with $\mathcal{G}$. Secondly, $m_k(x) = F(\check{\mathbf{x}})$, where the point $\check{\mathbf{x}}$ depends on x as follows.

Let $\bar{\beta}_{k-1,0} \le x \le \beta_{n-\mu-1,\mu}$. Define

$$L = \max\{\, \ell \mid 0 \le \ell \le \mu,\ x \ge \bar{\beta}_{k-1,\ell} \,\}.$$

Then $\check{\mathbf{x}}$ is a unique point of shape

$$\big([k-1] \ge \langle x \rangle^{n-k-L} \ge [1] > \langle 0 \rangle^{L}\big)$$

belonging to $\mathcal{G}_+$.

Let $\beta_{n-\mu-1,\mu} < x \le \beta_{k0}$. Define

$$L = \max\{\, \ell \mid 0 \le \ell \le \mu,\ x \le \beta_{k\ell} \,\}.$$

Then $\check{\mathbf{x}}$ is a unique point of shape

$$\big(\langle x \rangle^{k} \ge [n-k-1-L] \ge [1] > \langle 0 \rangle^{L}\big)$$

belonging to $\mathcal{G}_+$.

The function m_k is continuous, strictly decreasing when $x \in [\bar{\beta}_{k-1,0}, \beta_{n-\mu-1,\mu}]$, and strictly increasing when $x \in [\beta_{n-\mu-1,\mu}, \beta_{k0}]$. Obviously,

$$m_k(\bar{\beta}_{k-1,0}) = F(\boldsymbol{\beta}_{k-1,0}) = M_k(\bar{\beta}_{k-1,0})$$

and

$$m_k(\beta_{k0}) = F(\boldsymbol{\beta}_{k0}) = M_k(\beta_{k0}).$$

9.5.3 $x_{n-\mu}$ fixed

Let $k = n - \mu$. Now $\bar{\beta}_{n-\mu-1,0} \le x_{n-\mu} \le \bar{\beta}_{1\mu}$ for $\mathbf{x} \in \mathcal{G}_+$. If $\bar{\beta}_{n-\mu-1,0} \le x \le \bar{\beta}_{n-\mu-1,\mu}$, then $m_k(x)$ is the same as in the case $2 \le k \le n - \mu - 1$.

Let $\bar{\beta}_{n-\mu-1,\mu} < x \le \bar{\beta}_{1\mu}$. Define

$$K(x) = \max\{\, \ell \le n - \mu - 2 \mid x \le \bar{\beta}_{\ell\mu} \,\}.$$

Now $m_k(x) = F(\check{\mathbf{x}})$, where $\check{\mathbf{x}}$ is a unique point of shape $([K(x)] \ge [1] \ge \langle x\rangle^{n-\mu-K(x)-1} > \langle 0\rangle^{\mu})$ belonging to $\mathcal{G}_+$. The function m_k is continuous, strictly decreasing on $[\bar{\beta}_{n-\mu-1,0}, \bar{\beta}_{n-\mu-1,\mu}]$, and strictly increasing on $[\bar{\beta}_{n-\mu-1,\mu}, \bar{\beta}_{1\mu}]$.

The function $M_k(x)$ is the same as in the case $2 \le k \le n-1-\mu$ (and so the same as with $\mathcal{G}$), except that the range of M_k is limited to $[\bar{\beta}_{n-\mu-1,0}, \bar{\beta}_{1\mu}]$. Note that it follows from Lemma 67 that $\bar{\beta}_{n-1-\mu,0} < \bar{\beta}_{10} = \bar{\alpha}_1 < \bar{\beta}_{1\mu}$. Also now m_k and M_k meet at the endpoints of their domain.

9.5.4 $x_{n-\mu+1}$, $x_{n-\mu+2}$, ..., or x_{n-1} fixed

Finally, consider the case $n - \mu + 1 \le k \le n - 1$ (and $\mu \ge 2$). Then $0 \le x_k \le \bar{\beta}_{1,n-k}$ whenever $\mathbf{x} \in \mathcal{G}_+$. Define

$$K(x) = \max\{\, \ell \le n - \mu - 1 \mid x \le \bar{\beta}_{\ell,n-k} \,\}.$$

Now $m_k(x) = F(\check{\mathbf{x}})$, where $\check{\mathbf{x}}$ is a unique point of shape $([K(x)] \ge [1] \ge \langle x\rangle^{k-K(x)-1} > \langle 0\rangle^{n-k})$ belonging to $\mathcal{G}_+$. Obviously, m_k is continuous and increasing.

The function M_k is the same as with $\mathcal{G}$, except for its range, which is limited to $[0, \bar{\beta}_{1,n-k}]$ (note that $0 < \bar{\beta}_{10} < \bar{\beta}_{1,n-k}$). Now

$$m_k(\bar{\beta}_{1,n-k}) = F(\boldsymbol{\beta}_{1,n-k}) = M_k(\bar{\beta}_{1,n-k}),$$

but

$$M_k(0) = F(\boldsymbol{\beta}_{1,n-k+1}) > F(\boldsymbol{\beta}_{n-\mu-1,\mu}) = m_k(0).$$

9.5.5 An example

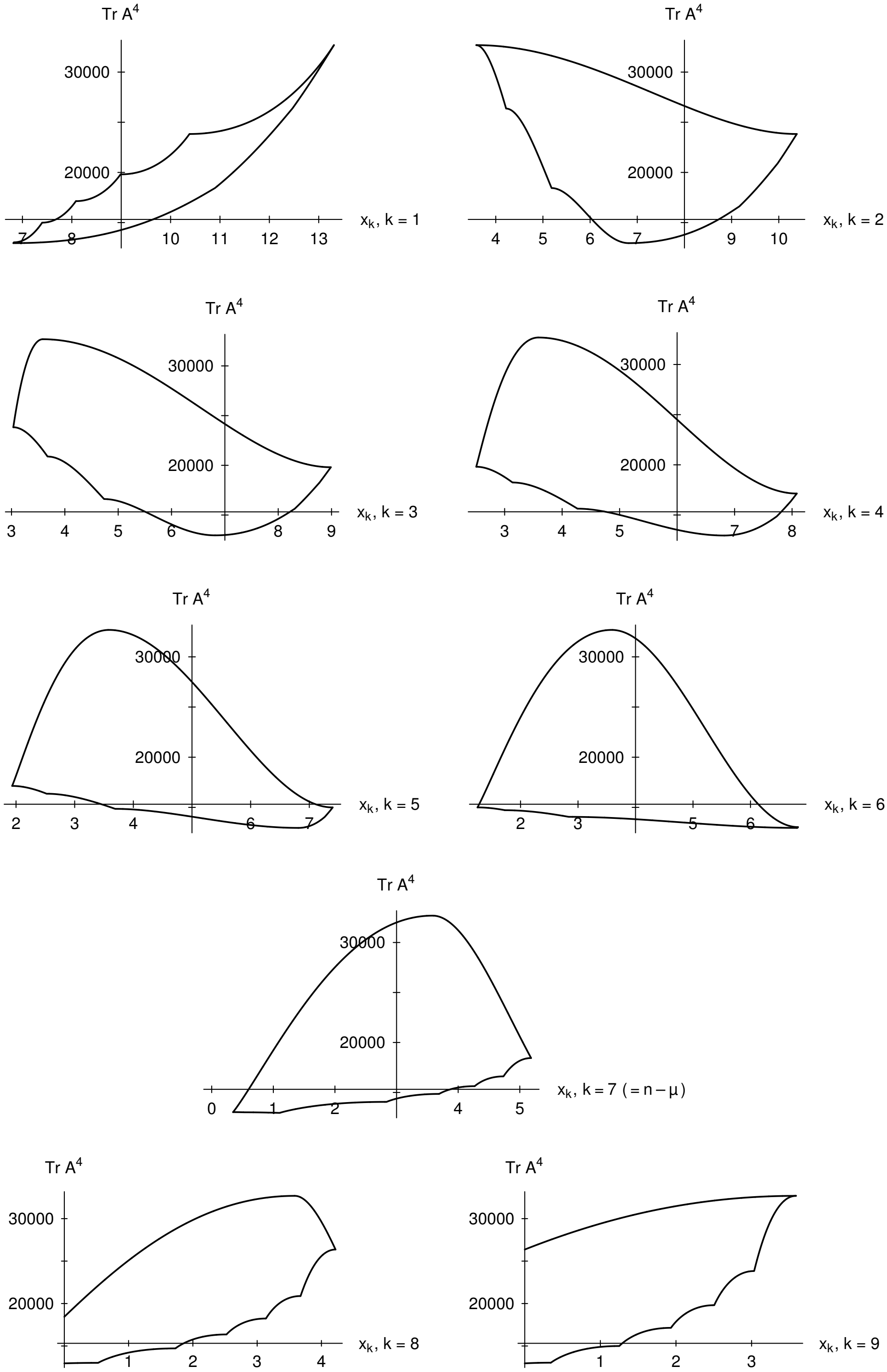
Example 43. Let $\mathbf{A}$ be the 9×9-diagonal matrix $\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_9) = \operatorname{diag}(9, 8, 7, 6, 5, 4, 3, 0, 0)$. Consider the set $\mathcal{P}_2 = \mathbb{R}^9_+[\mathrm{S}, \operatorname{tr}\mathbf{A}; P_2, \operatorname{tr}\mathbf{A}^2] = \mathbb{R}^9_+[\mathrm{S}, 42; P_2, 280]$. Then $\mu = 2$, $\operatorname{tr}\mathbf{A}^4 \approx 15000$,

$$\min_{\mathbf{x}\in\mathcal{P}_2} P_4(\mathbf{x}) = P_4(\boldsymbol{\beta}_{n-\mu-1,\mu}) \approx 13000,$$

and

$$\max_{\mathbf{x}\in\mathcal{P}_2} P_4(\mathbf{x}) = P_4(\boldsymbol{\beta}_{10}) \approx 33000.$$

The figures below represent $\max_{\mathbf{x}\in\mathcal{P}_2} P_4(\mathbf{x})$ and $\min_{\mathbf{x}\in\mathcal{P}_2} P_4(\mathbf{x})$ as functions of x_k for $k = 1, 2, \ldots, 9$. The origin is at the point $(\lambda_k, \operatorname{tr}\mathbf{A}^4)$.


Tr A^4
30000
20000
x_k, k = 1
7
8
10
11
12
13
Tr A^4
30000
20000
x_k, k = 2
4
5
6
7
9
10
Tr A^4
30000
20000
x_k, k = 3
3
4
5
6
8
9
Tr A^4
30000
20000
x_k, k = 4
3
4
5
7
8
Tr A^4
30000
20000
x_k, k = 5
2
3
4
6
7
Tr A^4
30000
20000
x_k, k = 6
2
3
5
6
Tr A^4
30000
20000
x_k, k = 7 (= n – μ)
0
1
2
4
5
Tr A^4
30000
20000
x_k, k = 8
1
2
3
4
Tr A^4
30000
20000
x_k, k = 9
1
2
3

9.6 Bounds for x_k

Assume that $\mathcal{G} = \mathbb{R}^n[\mathrm{S}, a; G, b]$, F is a strictly $\prec_{\mathcal{G}}$-increasing function and that the system

$$\begin{aligned} S\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k\big) &= a,\\ G\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k\big) &= b,\\ F\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k\big) &= c,\\ x \geq y \geq z & \end{aligned}$$

has a solution (x, y, z). By Lemma 66, the solution is unique. Denote it by $(\varphi_1(i,j,k), \varphi_2(i,j,k), \varphi_3(i,j,k))$.

Similarly, assume that the system

$$\begin{aligned} S\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k, \langle 0\rangle^\ell\big) &= a,\\ G\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k, \langle 0\rangle^\ell\big) &= b,\\ F\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k, \langle 0\rangle^\ell\big) &= c,\\ x \geq y \geq z > 0 & \end{aligned}$$

has a solution. Now the uniqueness of the solution follows from Lemma 68; we denote it by $(\psi_1(i,j,k,\ell), \psi_2(i,j,k,\ell), \psi_3(i,j,k,\ell))$.

Applying Lemma 2 to the results achieved in Sections 9.3, 9.4, and 9.5, we obtain the following two theorems:

Theorem 70. *Let $\mathcal{G} = \mathbb{R}^n[\mathrm{S}, a; G, b]$ and let F be a strictly $\prec_{\mathcal{G}}$-increasing function. Assume that $c \in \mathbb{R}$ satisfies $F(\boldsymbol{\alpha}_{n-1}) \leq c \leq F(\boldsymbol{\alpha}_1)$. Let $K_0 = \max\{\, \ell \leq n-1 \mid c \leq F(\boldsymbol{\alpha}_\ell)\,\}$. Then*

$$\begin{aligned} \varphi_1(K_0, 1, n-K_0-1) \leq x_1 &\leq \varphi_1(1, n-2, 1),\\ \varphi_3(1, n-2, 1) \leq x_n &\leq \varphi_3(K_0, 1, n-K_0-1), \end{aligned}$$

and, for $k = 2, 3, \ldots, n-1$,

$$\begin{aligned} \varphi_2(k-1, n-k, 1) \leq x_k &\leq \varphi_1(k, n-k-1, 1) && \text{if } F(\boldsymbol{\alpha}_{n-1}) \leq c < F(\boldsymbol{\alpha}_k),\\ \varphi_2(k-1, n-k, 1) \leq x_k &\leq \varphi_2(1, k-1, n-k) && \text{if } F(\boldsymbol{\alpha}_k) \leq c \leq F(\boldsymbol{\alpha}_{k-1}),\\ \varphi_3(1, k-2, n-k+1) \leq x_k &\leq \varphi_2(1, k-1, n-k) && \text{if } F(\boldsymbol{\alpha}_{k-1}) < c \leq F(\boldsymbol{\alpha}_1). \end{aligned}$$

All these bounds are sharp under the assumptions $\mathbf{x} \in \mathcal{G}$ and $F(\mathbf{x}) = c$.

Example 44. Let $\mathbf{A}$ be as in Example 42. Applying Theorem 70 to source data $(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2, \operatorname{tr} \mathbf{A}^3)$, we compute the following bounds:

$$\begin{aligned}
5.55 &\le x_1 \le 9.73, \\
0.39 &\le x_2 \le 7.23, \\
-0.85 &\le x_3 \le 5.88, \\
-2.46 &\le x_4 \le 3.93, \\
-5.61 &\le x_5 \le 1.83, \\
-7.06 &\le x_6 \le 0.40, \\
-9.73 &\le x_7 \le -5.57.
\end{aligned}$$

Theorem 71. *Let* $\mathcal{G}_+ = \mathbb{R}^n_+[\mathrm{S}, a; G, b]$ *and* $\mu = \mu(0; \mathcal{G}_+)$. *Assume that* F *is a strictly* $\prec_{\mathcal{G}_+}$*-increasing function and that* $c \in \mathbb{R}$ *satisfies* $F(\boldsymbol{\beta}_{n-\mu-1,\mu}) \le c \le F(\boldsymbol{\beta}_{10})$. *For* $i = 0, 1, \dots, \mu$, *let*

$$K_i = \max\{\, \ell \le n - \mu - 1 \mid c \le F(\boldsymbol{\beta}_{\ell i}) \,\}$$

and, for $i = 1, 2, \dots, n - \mu - 1$, *let*

$$L_i = \begin{cases} 0 & \text{if } \mu = 0, \\ \max\{\, \ell \mid 0 \le \ell \le \mu,\ c \le F(\boldsymbol{\beta}_{i\ell}) \,\} & \text{otherwise.} \end{cases}$$

Then the following bounds for x_k *are sharp under the assumptions* $\mathbf{x} \in \mathcal{G}_+$ *and* $F(\mathbf{x}) = c$:

For $k = 1$,

$$\psi_1(K_{k-1}, 1, n - K_{k-1} - 1, 0) \le x_k \le \psi_1(1, n - 2 - L_k, 1, L_k).$$

For $k = 2, 3, \dots, n - \mu - 1$,

if $c \le F(\boldsymbol{\beta}_{k0})$, *then*

$$\psi_2(k - 1, n - k - L_{k-1}, 1, L_{k-1}) \le x_k \le \psi_1(k, n - k - 1 - L_k, 1, L_k);$$

if $F(\boldsymbol{\beta}_{k0}) < c \le F(\boldsymbol{\beta}_{k-1,0})$, *then*

$$\psi_2(k - 1, n - k - L_{k-1}, 1, L_{k-1}) \le x_k \le \psi_2(1, k - 1, n - k, 0);$$

if $F(\boldsymbol{\beta}_{k-1,0}) < c$, *then*

$$\psi_3(1, k - 2, n - k + 1, 0) \le x_k \le \psi_2(1, k - 1, n - k, 0).$$

For $k = n - \mu$, *provided that* $\mu \ne 0$,

if $c \le \min\{F(\boldsymbol{\beta}_{k-1,0}), F(\boldsymbol{\beta}_{1\mu})\}$, *then*

$$\psi_2(k - 1, n - k - L_{k-1}, 1, L_{k-1}) \le x_k \le \psi_3(K_\mu, 1, n - K_\mu - \mu - 1, \mu);$$

if $F(\boldsymbol{\beta}_{k-1,0}) < c \leq F(\boldsymbol{\beta}_{1\mu})$, *then*

$$\psi_3(1, k-2, n-k+1, 0) \leq x_k \leq \psi_3(K_\mu, 1, n-K_\mu-\mu-1, \mu);$$

if $F(\boldsymbol{\beta}_{1\mu}) < c \leq F(\boldsymbol{\beta}_{k-1,0})$, *then*

$$\psi_2(k-1, n-k-L_{k-1}, 1, L_{k-1}) \leq x_k \leq \psi_2(1, k-1, n-k, 0);$$

if $\max\{F(\boldsymbol{\beta}_{k-1,0}), F(\boldsymbol{\beta}_{1\mu})\} < c$, *then*

$$\psi_3(1, k-2, n-k+1, 0) \leq x_k \leq \psi_2(1, k-1, n-k, 0).$$

For $k = n-\mu+1, n-\mu+2 \ldots, n-1$,

if $c \leq F(\boldsymbol{\beta}_{1,n-k+1})$, *then*

$$0 \leq x_k \leq \psi_3(K_{n-k}, 1, k-K_{n-k}-1, n-k);$$

if $F(\boldsymbol{\beta}_{1,n-k+1}) < c \leq F(\boldsymbol{\beta}_{1,n-k})$, *then*

$$\psi_3(1, k-2, n-k+1, 0) \leq x_k \leq \psi_3(K_{n-k}, 1, k-K_{n-k}-1, n-k);$$

if $F(\boldsymbol{\beta}_{1,n-k}) < c$, *then*

$$\psi_3(1, k-2, n-k+1, 0) \leq x_k \leq \psi_2(1, k-1, n-k, 0).$$

For $k = n$,

if $\mu \neq 0$ *and* $c \leq F(\boldsymbol{\beta}_{11})$, *then*

$$0 \leq x_n \leq \psi_3(K_0, 1, n-K_0-1, 0);$$

if $\mu = 0$ *or* $c > F(\boldsymbol{\beta}_{11})$, *then*

$$\psi_3(1, n-2, 1, 0) \leq x_n \leq \psi_3(K_0, 1, n-K_0-1, 0).$$

Since the vectors $\boldsymbol{\beta}_{n-\mu-1,0}$ and $(\boldsymbol{\beta}_{1\mu})$ are generally incomparable relative to $\preceq_{\mathcal{G}_+}$, the case $k = n-\mu$ is more complex than the other cases.

Example 45. Let $\mathbf{A} = \operatorname{diag}(9, 8, 7, 6, 5, 4, 3, 0, 0)$ as in Example 43. Applying Theorem 70 to source data $(n, \operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2, \operatorname{tr}\mathbf{A}^4)$, we compute the following bounds:

$$\begin{aligned}
7.67 &\leq x_1 \leq 9.65,\\
6.03 &\leq x_2 \leq 8.73,\\
5.51 &\leq x_3 \leq 8.22,\\
4.75 &\leq x_4 \leq 7.82,\\
3.45 &\leq x_5 \leq 7.11,\\
1.28 &\leq x_6 \leq 6.14,\\
0.60 &\leq x_7 \leq 3.88,\\
0.00 &\leq x_8 \leq 1.86,\\
0.00 &\leq x_9 \leq 1.29.
\end{aligned}$$

In Example 45,

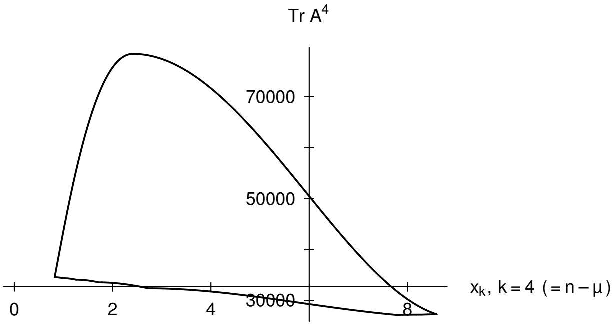
$$P_4(\boldsymbol{\beta}_{n-\mu-1,0}) \approx P_4(\langle 6.83\rangle^6, \langle 0.35\rangle^3) \approx 13000 < \operatorname{tr} \mathbf{A}^4$$
$$< 18000 \approx P_4(10.90, \langle 5.18\rangle^6, \langle 0\rangle^2) \approx P_4(\boldsymbol{\beta}_{1\mu}).$$

In the following example $P_4(\boldsymbol{\beta}_{n-\mu-1,0}) > P_4(\boldsymbol{\beta}_{1\mu})$.

Example 46. Let $\mathbf{A} = \operatorname{diag}(12, 9, 8, 6, 1, 0, 0, 0, 0)$. Then $n = 9$, $\operatorname{tr} \mathbf{A} = 36$, $\operatorname{tr} \mathbf{A}^2 = 326$, $\operatorname{tr} \mathbf{A}^4 = 32690$, $\mu = 5$, and

$$P_4(\boldsymbol{\beta}_{n-\mu-1,0}) \approx P_4(\langle 10.4\rangle^3, \langle 0.82\rangle^6) \approx 34500 > \operatorname{tr} \mathbf{A}^4 > 27300$$
$$\approx P_4(10.2, \langle 8.59\rangle^3, \langle 0\rangle^5) \approx P_4(\boldsymbol{\beta}_{1\mu}).$$

The figure below presents $\max_{\mathbf{x}\in\mathcal{P}_2} P_4(\mathbf{x})$ and $\min_{\mathbf{x}\in\mathcal{P}_2} P_4(\mathbf{x})$ as functions of $x_{n-\mu}$, i.e., of x_4. The origin is at the point $(\lambda_4, \operatorname{tr} \mathbf{A}^4)$.



Theorem 70 applied to source data $(n, \operatorname{tr} \mathbf{A}, \operatorname{tr} \mathbf{A}^2, \operatorname{tr} \mathbf{A}^4)$ yields

$$2.56 \le x_4 \le 7.68.$$

10 Directions for future research

Among the various problems raised by this research, the following appear most immediate:

finding $\inf_{\preceq}\{\, \mathbf{v} \in \mathcal{G} \mid \mathbf{a} \preceq \mathbf{v} \preceq \mathbf{b} \,\}$ and $\sup_{\preceq}\{\, \mathbf{v} \in \mathcal{G} \mid \mathbf{a} \preceq \mathbf{v} \preceq \mathbf{b} \,\}$;

describing completely the class of $\preceq^3_{\mathcal{P}_2}$-increasing functions;

finding $\inf_{\preceq}\{\, \mathbf{x} \in \mathcal{P}_2 \mid F(\mathbf{x}) = c \,\}$ and $\sup_{\preceq}\{\, \mathbf{x} \in \mathcal{P}_2 \mid F(\mathbf{x}) = c \,\}$, F being $\preceq^3_{\mathcal{P}_2}$-increasing;

giving a characterization of $\preceq^3_{\mathcal{P}_m}$-increasing functions when $m \geq 3$ (here and below $\mathcal{P}_m$ denotes $I^n[\mathrm{S}, a; P_m, b]$);

relative to $\preceq$ and to $\leq$, finding the supremums and infimums of the set $\{\, \mathbf{x} \in \mathcal{P}_m \mid F(\mathbf{x}) = c \,\}$, where $m \geq 3$ and F is $\preceq^3_{\mathcal{P}_m}$-increasing;

applying 4-majorization to find, for example, the maximum and minimum of the set $\{\, x_i \mid \mathbf{x} \in \mathbb{R}^n_+[\mathrm{S}, a; P_2, b],\ P_3(\mathbf{x}) = c,\ P_4(\mathbf{x}) = d \,\}$.

An interesting challenge is to develop a general theory of k-majorization and to compare it systematically with Karush-Kuhn-Tucker theory.

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Selected symbols

On the notation, see also Section 1.3.

I (p. 15). $\mathbb{R}$, $\mathbb{R}_+$, or a closed real interval $[m, M]$.

$\mathrm{S}_{km}(\mathbf{x})$ (p. 15). The partial sum $\sum_{i=k}^{m} x_i$; particularly, $\mathrm{S}(\mathbf{x}) = \sum_i x_i$.

$\mathrm{M}_{i\ell}(\mathbf{x})$ (p. 15). The mean $\mathrm{S}_{i\ell}(\mathbf{x})/(\ell - i + 1)$.

$\leq_\Sigma$ (p. 15). The relation defined by $\mathbf{x} \leq_\Sigma \mathbf{y} \Leftrightarrow \mathrm{S}_{1k}(\mathbf{x}) \leq \mathrm{S}_{1k}(\mathbf{y})$ for all $k \leq n$.

$\mathbf{x}_\Downarrow$ (p. 27). An 'averaged' vector obtained from $\mathbf{x}$.

$\mathbf{K}$, κ (p. 33). The vector $\mathbf{K} = \mathbf{K}(I, n, a) = (\langle M\rangle^\kappa, \theta, \langle m\rangle^{n-\kappa-1})$, where $I = [m, M]$, $\kappa = \min\{\, k \geq 0 \mid (k+1)M + (n-k-1)m > a \,\}$, and $\theta = a - \kappa M - (n-\kappa-1)m$.

$\mathbf{K}$, ι (p. 33). We may write $\mathbf{K} = (\langle M\rangle^\kappa, \langle\theta\rangle^\iota, \langle m\rangle^{n-\kappa-\iota})$, where $\iota = 1$ if $\theta > m$, and $\iota = 0$ if $\theta = m$.

$\mathbf{E}$ (p. 36). The vector $\mathbf{E}(k,\ell) = \mathbf{E}(k,\ell; I, n, a) = (\langle M\rangle^k, \langle \frac{a-kM-\ell m}{n-k-\ell} \rangle^{n-k-\ell}, \langle m\rangle^\ell)$, where $I = [m, M]$.

$I^n[\mathrm{S}, a; G, b]$ (p. 37). The set $\{\, \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a,\, G(\mathbf{v}) = b \,\}$ satisfying basic assumptions: G is strictly Schur-convex and continuous and $G((a/n)\mathbf{1}) < b < G(\mathbf{K}(I, n, a))$.

$I^n[\mathrm{S}, a]$ (p. 37). The set $\{\, \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a \,\}$, where $I = [m, M]$ with $m < a/n < M$.

$\mathcal{G}$, $\mathcal{G}^+$, $\mathcal{G}^-$ (p. 37). The sets $\mathcal{G} = I^n[\mathrm{S}, a; G, b]$, $\mathcal{G}^+ = \{\, \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a,\, G(\mathbf{v}) \geq b \,\}$, and $\mathcal{G}^- = \{\, \mathbf{v} \in I^n_\downarrow \mid \mathrm{S}(\mathbf{v}) = a,\, G(\mathbf{v}) \leq b \,\}$.

$\mathbf{s}$ (p. 40). The solution vector of the system

$$k_1 M + k_2 x + k_3 y + k_4 m = a,$$
$$G\big((\langle M\rangle^{k_1}, \langle x\rangle^{k_2}, \langle y\rangle^{k_3}, \langle m\rangle^{k_4})\big) = b,$$
$$M > x \geq y > m$$

is denoted by $\mathbf{s}(\langle M\rangle^{k_1} > [k_2] \geq [k_3] > \langle m\rangle^{k_4}; \mathcal{G})$.

γ (p. 40). The function $\gamma(k,\ell) = G(\mathbf{E}(k,\ell))$.

λ, μ. (p. 41–42). Functions: $\lambda(p) = \max\{\, k \mid 0 \le k \le \kappa \text{ and } \gamma(k,p) \le b \,\}$,
$\mu(p) = \max\{\, \ell \mid 0 \le \ell \le n - \kappa - 1 \text{ and } \gamma(p,\ell) \le b \,\}$.
Values: $\lambda = \lambda(0)$, $\mu = \mu(0)$.

$\underline{\mathbf{s}}(p)$ (p. 43). The vector $\mathbf{s}(\langle M\rangle^{\lambda(p)} > [1] \ge [n-p-1-\lambda(p)] > \langle m\rangle^{p}; \mathcal{G})$.

$\bar{\mathbf{s}}(p)$ (p. 43). The vector $\mathbf{s}(\langle M\rangle^{p} > [n-p-1-\mu(p)] \ge [1] > \langle m\rangle^{\mu(p)}; \mathcal{G})$.

$\boldsymbol{\alpha}_k$ (p. 43). The vector $(\langle \alpha_k\rangle^{k}, \langle \bar{\alpha}_k\rangle^{n-k}) = \mathbf{s}([k] \ge [n-k]; \mathcal{G})$.

$\bar{\beta}_{n-\ell}$ (p. 46). The ℓth component of $\underline{\mathbf{s}}(n-\ell)$.

χ_{k-1} (p. 47). The kth component of $\bar{\mathbf{s}}(k-1)$.

β_{1i} (p. 50). The solution (x,y) of the system

$$\begin{gathered} x + (n-i-1)y = a, \\ G\big(x, \langle y\rangle^{n-i-1}, \langle 0\rangle^{i}\big) = b, \\ x \ge y > 0 \end{gathered}$$

is denoted by $(\beta_i, \bar{\beta}_i)$.

$\beta_{n-\mu-1,\mu}$ (p. 50). The solution (x,y) of the system

$$\begin{gathered} (n-\mu-1)x + y = a, \\ G\big(\langle x\rangle^{n-\mu-1}, y, \langle 0\rangle^{\mu}\big) = b, \\ x \ge y > 0 \end{gathered}$$

is denoted by $(\beta_{n-\mu-1,1}, \bar{\beta}_{n-\mu-1,1})$.

$\mathbf{A}^{(k)}$ (p. 55). The kth compound of $\mathbf{A}$.

$\mathcal{P}_2$ (p. 56). The set $\mathbb{R}^n[\mathrm{S}, a; P_2, b]$ (where $P_2(\mathbf{x}) = \sum x_i^2$).

$\mathcal{P}_+$ (p. 90). The set $\mathbb{R}^n_+[\mathrm{S}, a; P_2, b]$.

$\mathcal{P}_{++}$ (p. 90). The set $\mathbb{R}^n_{++}[\mathrm{S}, a; P_2, b]$.

$\boldsymbol{\beta}_{ik}$ (p. 94). The solution (x,y,z) of the system

$$\begin{gathered} \mathrm{S}\big(\langle x\rangle^{i}, \langle y\rangle^{j}, \langle 0\rangle^{k}\big) = a, \\ G\big(\langle x\rangle^{i}, \langle y\rangle^{j}, \langle 0\rangle^{k}\big) = b, \\ x \ge y > 0 \end{gathered}$$

is denoted by $(\beta_{ik}, \bar{\beta}_{ik})$ and the solution vector (zeros included) by $\boldsymbol{\beta}_{ik}$.

φ (p. 103). The solution (x, y, z) of the system

$$\begin{gathered} S\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k\big) = a, \\ G\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k\big) = b, \\ F\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k\big) = c, \\ x \geq y \geq z \end{gathered}$$

is denoted by $\big(\varphi_1(i, j, k), \varphi_2(i, j, k), \varphi_3(i, j, k)\big)$.

ψ (p. 103). The solution (x, y, z) of the system

$$\begin{gathered} S\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k, \langle 0\rangle^\ell\big) = a, \\ G\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k, \langle 0\rangle^\ell\big) = b, \\ F\big(\langle x\rangle^i, \langle y\rangle^j, \langle z\rangle^k, \langle 0\rangle^\ell\big) = c, \\ x \geq y \geq z > 0 \end{gathered}$$

is denoted by $\big(\psi_1(i, j, k, \ell), \psi_2(i, j, k, \ell), \psi_3(i, j, k, \ell)\big)$.