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On Definability of Functions of Several Variables

ACADEMIC DISSERTATION
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Abstract

In this research work, we focused on functions of several variables over a set A and valued in a possibly different set B. We studied function properties which can be expressed by means of functional equations and by means of relational constraints (R, S) (where R and S are relations of the same arity on A and B, respectively), taking into account the syntax of equations and the type of relations allowed in constraints.

We started by studying the Boolean case, and considered characterizations of Boolean function classes, as studied by Ekin, Foldes, Hammer and Hellerstein, but by means of functional equations written in the additive language of Boolean rings, i.e. by means of the so-called linear equations. We showed that a class of Boolean functions has a linear theory if and only if it is stable under right and left composition with the clone of constant-preserving linear functions. These classes were equivalently described within a Galois setting, more stringent than that considered by Pippenger, in which classes are defined by means of affine constraints, showing that definability by linear equations is equivalent to definability by affine constraints, since these two approaches specify exactly the same Boolean function classes. The dual question of describing the sets of affine constraints which are characterized by Boolean functions was also addressed and answered.

Then, we studied the general case, and extended Pippenger's Galois theory by removing the finiteness restriction on the underlying sets A and B. We showed that the classes of functions definable by constraints are exactly those locally closed classes which are stable under right composition with the smallest clone on A, containing only projection maps, and that the sets of relational constraints characterized by functions are essentially those locally closed sets of constraints which are closed under combining families of constraints into new constraints by means of (possibly infinitary) existential first-order schemes similar to those described by Szabó. Furthermore, we showed how these results can be used to derive the characterizations of the closed systems associated with the well-known Galois correspondence

between operations and relations.

This basic Galois correspondence between functions and constraints was specialized and generalized in several ways in this research work. We studied Galois connections arising from the restriction of the primal objects (functions) and dual objects (relational constraints) to fixed arities, and presented descriptions of the Galois closure systems by means of parametrized analogues of the closures given in the previous joint work with Stephan Foldes. Furthermore, we provided factorizations of the Galois maps in terms of operators associated with these closures.

Also, we considered the more general notion of *multivalued functions*, i.e. mappings of the form $A^n \to \mathcal{P}(B)$, where $\mathcal{P}(B)$ denotes the set of all subsets of B, which generalize the notions of total and partial functions. These different notions were studied in a unifying Galois framework arising from the natural extension of the relation "satisfies" to multivalued functions and relational constraints (whose "consequents" are still defined as relations on B, and not on $\mathcal{P}(B)$). We described the Galois closed sets by means of necessary and sufficient conditions which specialize to those given in the total single-valued case, and presented factorizations of the associated Galois maps in terms of simpler closure operators.

Moreover, we considered further Galois connections by imposing algebraic restrictions on the sets of dual objects: the relations R and S in the constraints were required to be invariant under given clones $\mathcal{C}_1$ and $\mathcal{C}_2$ on A and B, respectively. Within this framework, function classes are defined by sets of these invariant constraints, and characterized in terms of stability under left and right composition with $\mathcal{C}_2$ and $\mathcal{C}_1$. This general Galois framework subsumes, in particular, the Galois settings described by Pippenger, and those developed in previous joint work with Stephan Foldes.

The notion of functional equation was adjusted to the general case of arbitrary underlying sets. This formulation facilitated a general correspondence between definability by functional equations and by relational constraints. The proof was based on a construction which revealed general criteria by which further correspondences between equations of specific algebraic syntax, and relational constraints satisfying certain invariance conditions, can be established. As examples, we considered certain noteworthy classes of field-valued functions of Boolean variables, and proved existence and non-existence of linear characterizations of these classes, in terms of the characteristic of the underlying field. Explicit equational characterizations were also given, accordingly.

As further applications, we considered classes of affine operations on finite fields, with a bounded number of essential variables. We established a general correspondence between linear equations and affine constraints, which we used to show that these classes do not have linear theories. By making use of well-known facts from linear algebra over finite fields, we obtained characterizations of each of these classes, by means of (non-linear) functional equations.

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Some did ask
"Isn't it quite a task,
to present your thanks in a poem?"
Yes, but still I dare,
although I am aware
that the gift of Leonard Cohen
I do not share!
And even though, childish, it may be,
through rime, my acknowledgements, I choose to transmit
(in ways, hopefully sweet!)
to those who, thoughtful, were of me.
And thus I shall start,
one by one, the few,
I remember by heart.

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or even cite,
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through nights, indeed,
some of which, I had no sleep,
because of questions and problems
that I sank in deep
and, together, we fought!
But true joy they brought,
the papers we wrote,
despite the "cold showers"
in really late hours,
and the surprising shakes
because of few mistakes!
In him, a true friend,
I honestly see,
and a great supervisor too,
from the begining untill the end,
all the way through!
To me,
(and to many others, I am sure!)
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"Tudo legal!"
has a new meaning... truely special!!

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but most important,
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*As we say in Portugal,
"Espero que não levem a mal"
mas em Português, em vez de Inglês,
dedicarei, também em rima,
algumas palavras por quem tenho grande estima.*

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Sabendo o que eu então não sabia,
como matemático me viu
e o gosto, em mim nutriu...
E como gostaria
de, esta tese, mostrar,
e de, consigo, discutir,
(ao pensar, começo a sorrir!)
o que vim a alcançar!
Na verdade, sempre me fez crer,
no que, no só por palavras, expressou,
que um dia, chegaria onde estou!*

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A sua, tão peculiar, maneira,
para muitos questionável,
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Well, these have been months of some stress,
and only understanding and deep tenderness,
could have help me through the mess,
and also the fuss,
in which my mind was!
Well, Riia is her name,
with whom, this final path, I drove
and who, I unmeasurably love!
And although it is a common thought,
that in the affairs of the heart,
it is all the same,
without the tiniest shame,
I hereby say
that it doesn't matter to me,
cause she's the one I see
as the one with whom I shall be
till that very last day...
In blissful joy,
we became three,
me, her and our pearl...
a boy?!
perhaps a girl!
I do not yet know which is true!
My thoughts always had colours...
...now, I confess, mostly pink and blue!!

Also, and in bliss,
I leave a heartfull kiss,
to my extend finnish family,
Ispi, Mirja ja Karri,
and to who,
in Finland, I first met
and who I will never forget:
Liisa, Hanna, Wille ja Wilhu,
as well as Kimmo, Liina ja Tatu!

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the warmest of all,
to who I always remember,
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(my favourite one!)
and to D. Lino Goncalves,
to who I now reveal
that "as a son,
you make me feel!"
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It was four years ago,
maybe five, or so,
that it became a dream,
growing bright, not dim...
But here I stand,
(or should I say remain?)
feeling a sort of emptiness, rather than pain....
Still I do not understand,
that my thesis is finally finished...
At least now I can rest,
for all, I have acknowledged,
by the means, I thought best...

Contents

Chapter 1

Introduction

In this research work, we focused on definability of classes of functions of several variables on a set A, and valued in a possibly different set B. We considered two approaches: one by means of functional equations, the other by means of relational constraints.

The former is a more classical approach where function classes are defined by means of universally quantified first-order sentences of certain algebraic syntax, with no predicate symbols other than equality. This method was re-introduced in the Boolean (two-element) case by Ekin, Foldes, Hammer and Hellerstein in [8], who showed that the equational classes of Boolean functions are exactly those which are closed under addition of inessential variables, identification of variables and permutation of variables.

These classes appear naturally in a Galois framework developed by Pippenger in [15], in which function classes are defined by the relational constraints that the members of the class satisfy, and dually sets of relational constraints are characterized by the functions satisfying them.

As observed by Pippenger, in the Boolean case these two approaches have the same expressive power, in the sense that they define exactly the same function classes. Nonetheless, these methods appear in different settings and they give rise to questions different in nature and flavour.

In this research, we studied and developed several aspects in each of these approaches, and explored further correspondences between the two. This thesis consists of the following seven papers:

CF1 "Definability of Boolean function classes by linear equations over GF(2)", Miguel Couceiro and Stephan Foldes (in *Discrete Applied Mathematics*), CF2 “On affine constraints satisfied by Boolean functions”, Miguel Couceiro and Stephan Foldes,

CF3 “On closed sets of relational constraints and classes of functions closed under variable substitutions”, Miguel Couceiro and Stephan Foldes (in *Algebra Universalis*),

C1 “On Galois connections between external functions and relational constraints: arity restrictions and operator decompositions”, Miguel Couceiro (in *Acta Scientiarum Mathematicarum, Szeged*),

C2 “Galois connections for generalized functions and relational constraints”, Miguel Couceiro (in *Contributions to General Algebra* **16**),

CF4 “Function class composition, relational constraints and stability under compositions with clones”, Miguel Couceiro and Stephan Foldes,

CF5 “Functional equations, constraints, definability of function classes, and functions of Boolean variables”, Miguel Couceiro and Stephan Foldes.

In addition, we present in Subsection 2.8 some unpublished material concerning definability of certain classes of affine operations on finite fields, which illustrates further applications of some results presented in this thesis.

1.1 Basic notions and background

Let A, B and D be arbitrary non-empty sets, let n be a positive integer, and let $\mathbf{n}$ denote the set of positive integers less or equal to n. An *n-ary B-valued function on A* is a map $f : A^n \to B$. If $B = A$, then these (internal) functions are called *operations on A*, and if $A = B = \{0,1\}$, then they are usually referred to as *Boolean functions*. For each positive integer n, the n-ary *projections* $(a_1, \ldots, a_n) \mapsto a_i$, $i \in \mathbf{n} = \{1, \ldots, n\}$, are also called *variables* and denoted x_i^n, or simply x_i when the integer n is clear from the context. A *class* of B-valued functions on A is a subset $\mathcal{F} \subseteq \bigcup_{n \geq 1} B^{A^n}$.

For each $1 \leq i \leq n$, x_i is said to be an *essential variable* of $f : A^n \to B$ if there are $a_1, \ldots, a_{i-1}, a, b, a_{i+1}, \ldots, a_n$ in A, where $a \neq b$, such that

$$f(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

Otherwise, x_i is called a *dummy* or *inessential variable of f*. The *essential arity of f*, denoted $ess(f)$ is the number of its essential variables. Note that constant functions are the only Boolean functions whose variables are all dummy.

For any functions $g_1, \dots, g_n : A^m \to B$, and $f : B^n \to D$, their *composition* is defined as the function $f(g_1, \dots, g_n) : A^m \to D$ given by

$$f(g_1, \dots, g_n)(\mathbf{a}) = f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a}))$$

for every $\mathbf{a} \in A^m$. For $B = A$, the functions of the form $g = f(g_1, \dots, g_n)$, where $g_1, \dots, g_n$ are projections on A, are said to be obtained from f by *simple variable substitution*. This notion subsumes the Mal'cev operations of cylindrification (addition of inessential variables), diagonalization (identification of variables) and permutation of variables (see [13]). Functions obtained by simple variable substitution are referred to as *minors* in [22, 15], *identification minors* in [8], and *subfunctions* in [23]. Simple variable substitution induces a quasi-order on $\bigcup_{n \geq 1} B^{A^n}$, denoted by $\preceq$, and defined by: $g \preceq f$ if and only if g is obtained from f by simple variable substitution. Note that $g \preceq f$ implies $ess(g) \leq ess(f)$. A class $\mathcal{K}$ of B-valued functions on A is said to be *closed under simple variable substitutions* if it contains every function g for which there is $f \in \mathcal{K}$ such that $g \preceq f$. In other words, the classes closed under simple variable substitutions coincide with the "initial segments" of the quasi-ordered set $(\bigcup_{n \geq 1} B^{A^n}, \preceq)$, i.e. classes $\mathcal{K} \subseteq \bigcup_{n \geq 1} B^{A^n}$ satisfying

$$\mathcal{K} = \downarrow \mathcal{K} = \{g : g \preceq f, \text{ for some } f \in \mathcal{K}\}.$$

For a study of the quasi-order $\preceq$ on Boolean functions, see [6, 7].

The notion of composition is naturally extended to function classes by defining the *class composition* of a class $\mathcal{K}$ of D-valued functions on B, with a class $\mathcal{J}$ of B-valued functions on A, as the class

$$\mathcal{K} \circ \mathcal{J} \subseteq \bigcup_{n \geq 1} D^{A^n}$$

of all compositions $f(g_1, \dots, g_n)$ of functions f in $\mathcal{K}$ with functions $g_1, \dots, g_n$ in $\mathcal{J}$. It is not difficult to see that class composition preserves the containment relation, i.e.

$$\text{if } \mathcal{K}_1 \subseteq \mathcal{K}_2 \text{ and } \mathcal{J}_1 \subseteq \mathcal{J}_2, \text{ then } \mathcal{J}_1 \circ \mathcal{K}_1 \subseteq \mathcal{J}_2 \circ \mathcal{K}_2.$$

We say that a class $\mathcal{J}$ of B-valued functions on A is *stable under right composition* with a class $\mathcal{K}_A$ of operations on A if $\mathcal{J} \circ \mathcal{K}_A \subseteq \mathcal{J}$. Similarly, we say that $\mathcal{J}$ *stable under left composition* with a class $\mathcal{K}_B$ of operations on

B if $\mathcal{K}_B \circ \mathcal{J} \subseteq \mathcal{J}$. A *clone* on A is a class of operations on A containing the class $\mathcal{I}_A$ of all projections on A, and stable under composition with itself, i.e. idempotent with respect to class composition. Clearly, $\mathcal{I}_A$ is the smallest clone of operations on A. Note that a class $\mathcal{K}$ of B-valued functions on A is closed under simple variable substitutions if and only if it is stable under right composition with $\mathcal{I}_A$. In particular, each clone is closed under simple variable substitutions.

1.2 Definability of function classes by functional equations

For $A = B = \mathbb{B} = \{0, 1\}$, Ekin, Foldes, Hammer and Hellerstein [8] re-introduced an approach in which Boolean function classes are specified by means of "functional equations".

By a *Boolean term* we simply mean a formula of Boolean logic, i.e. a formal expression built from variable symbols by means of the Boolean connectives $\wedge, \vee, +$, the unary connective $\neg$, and the nullary connectives (constants) $\mathbf{0}$ and $\mathbf{1}$.

A (*Boolean*) *functional equation* is a formal expression

$$\begin{aligned} &H_1(\mathbf{f}(T_{11}(\mathbf{x}_1, \ldots, \mathbf{x}_p)), \ldots, \mathbf{f}(T_{1m}(\mathbf{x}_1, \ldots, \mathbf{x}_p))) = \\ &H_2(\mathbf{f}(T_{21}(\mathbf{x}_1, \ldots, \mathbf{x}_p)), \ldots, \mathbf{f}(T_{2t}(\mathbf{x}_1, \ldots, \mathbf{x}_p))) \end{aligned} \tag{1.1}$$

where $m, t, p \geq 1$, H_1, H_2 are m-ary and t-ary Boolean terms, each T_{1i} and T_{2j} is a p-ary Boolean term, $\mathbf{x}_1, \ldots, \mathbf{x}_p$ are p distinct *vector variable symbols*, and $\mathbf{f}$ is a *function symbol*. An n-ary Boolean function f is said to *satisfy* equation (1.1) if, for all $\mathbf{v}_1, \ldots, \mathbf{v}_p \in A^n$, we have

$$\begin{aligned} &H_1(f(T_{11}(\mathbf{v}_1, \ldots, \mathbf{v}_p)), \ldots, f(T_{1m}(\mathbf{v}_1, \ldots, \mathbf{v}_p))) = \\ &H_2(f(T_{21}(\mathbf{v}_1, \ldots, \mathbf{v}_p)), \ldots, f(T_{2t}(\mathbf{v}_1, \ldots, \mathbf{v}_p))) \end{aligned}$$

by interpreting the outer terms H_1 and H_2, and the inner terms T_{1i} and T_{2j}, in $\mathbb{B}$ and $\mathbb{B}^n$, respectively. In this setting, a class $\mathcal{K}$ is said to be *defined*, or *definable*, by a set $\mathcal{E}$ of functional equations, if $\mathcal{K}$ is the class of all those operations which satisfy every member of $\mathcal{E}$. We say that a class $\mathcal{K}$ is *equational* if it is definable by some set of functional equations.

To illustrate, consider the following (Boolean) functional equations

- $\mathbf{f}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{f}(\mathbf{x}_1) + \mathbf{f}(\mathbf{x}_2) + \mathbf{f}(\mathbf{0})$,
- $\mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2) \wedge \mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2)$.

These equations are known to characterize the clone of linear Boolean functions (called affine in linear algebra) and the clone of monotone (increasing) Boolean functions (also called positive functions). But clones are not the only classes which can be specified within this framework. For example, the classes of constant Boolean functions and monotone (decreasing) Boolean functions do not constitute clones but they are defined by

- $\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_2)$,
- $\mathbf{f}(\mathbf{x}_1 \wedge \mathbf{x}_2) \wedge \mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_1)$.

respectively. Further equational characterizations of several noteworthy Boolean function classes were given in [8]. In fact, Ekin, Foldes, Hammer and Hellerstein provided necessary and sufficient conditions for a class to be equationally definable, by showing that the equational classes of Boolean functions are exactly those which are closed under variable substitutions. This result motivated in part Pippenger's Galois framework for finite functions [15], where these closed classes play a fundamental role.

1.3 Galois theories for functions of several variables

An important motivation for constructing Galois theories is that they establish two-way correspondences between two different mathematical universes. These can be seen as two-way translations by which questions addressed in one universe can be dually posed in the other, where solutions are (perhaps) easier to find, and then translated back to the primal universe, providing answers to the original questions.

1.3.1 Galois connections and Galois closed sets

Let V and W be arbitrary sets, and let $\mathcal{P}(V)$ and $\mathcal{P}(W)$ be the sets of all subsets of V and W, respectively. A *Galois connection* between V and W is a pair of maps $v : \mathcal{P}(V) \to \mathcal{P}(W)$ and $w : \mathcal{P}(W) \to \mathcal{P}(V)$ such that, for $X, X' \subseteq V$ and $Y, Y' \subseteq W$

· v and w are order reversing, i.e. if $X \subseteq X'$ and $Y \subseteq Y'$, then $v(X') \subseteq v(X)$ and $w(Y') \subseteq w(Y)$, and

· $v \circ w$ and $w \circ v$ are extensive maps, i.e. $w(v(X)) \supseteq X$ and $v(w(Y)) \supseteq Y$.

From these conditions, it follows that:

· $v \circ w \circ v = v$ and $w \circ v \circ w = w$, and

· $v \circ w$ and $w \circ v$ are closure operators, i.e. extensive, monotone and idempotent.

The mappings $v \circ w$ and $w \circ v$ are referred to as *Galois operators*, and the sets $X \subseteq V$ and $Y \subseteq W$ satisfying $v \circ w(X) = X$ and $w \circ v(Y) = Y$ are referred to as (*Galois*) *closed sets* associated with v and w. If $\mathcal{V} \subseteq \mathcal{P}(V)$ and $\mathcal{W} \subseteq \mathcal{P}(W)$ are the closure systems associated with these maps, then v and w, restricted to $\mathcal{V}$ and $\mathcal{W}$, respectively, are inverse maps which establish a dual isomorphism between the lattices $\mathcal{V}$ and $\mathcal{W}$ with respect to set inclusion.

It is well known that Galois connections can be equivalently defined as mappings induced by binary relations between two sets. Let $\triangleright$ be an arbitrary binary relation between two sets V and W, and define

$$v(X) = \{b \in W : a \triangleright b, \text{ for every } a \in X\}, \text{ for } X \subseteq V, \text{ and}$$

$$w(Y) = \{a \in V : a \triangleright b, \text{ for every } b \in Y\}, \text{ for } Y \subseteq W.$$

It is not difficult to see that v and w constitute indeed a Galois connection between V and W. Conversely, if the pair of maps $v' : \mathcal{P}(V) \to \mathcal{P}(W)$ and $w' : \mathcal{P}(W) \to \mathcal{P}(V)$ is a Galois connection between V and W, then the maps v and w induced by the relation $\triangleright \subseteq V \times W$, given by

$$\triangleright = \{(a, b) \in V \times W : b \in v'(\{a\})\}, \text{ or equivalently,}$$

$$\triangleright = \{(a, b) \in V \times W : a \in w'(\{b\})\},$$

are such that $v = v'$ and $w = w'$. See [14] for further background on Galois connections. See also [9] for a later reference.

Such a Galois framework is usually used to describe certain "closure" properties of sets of given primal objects. To construct a Galois theory for these sets, one needs to provide suitable dual objects and establish a suitable binary relation between primal and dual objects, so that the the sets of primal objects fulfilling the "closure" requirements appear as Galois closed sets. Then one tries to translate these "closure" conditions into the dual universe, characterizing the dual Galois closed sets. In addition, one can explicitly describe the Galois connection established, by providing representations of the Galois operators in terms of simpler closure maps. A well-known example which illustrates the general framework just described, appears at the core of universal algebra, and based on the notion "satisfaction" of an algebraic identity by an algebra: an algebra $\mathbb{A} = \langle A, \{f : f \in \tau\}\rangle$ of "type" τ, is said to *satisfy* an algebraic identity $t_1(x_1, \ldots, x_n) \approx t_2(x_1, \ldots, x_n)$ of the same "type" τ, if for every $a_1, \ldots, a_n \in A$,

$$t_1^{\mathbb{A}}(a_1, \ldots, a_n) = t_2^{\mathbb{A}}(a_1, \ldots, a_n).$$

Let Σ be a set of algebraic identities, and let $\mathcal{A}$ be a class of algebras of the same type. Define $M(\Sigma)$ as the class of algebras satisfying each identity in Σ, and $Th(\mathcal{A})$ as the set of identities satisfied by each algebra in $\mathcal{A}$. The maps $\Sigma \mapsto M(\Sigma)$ and $\mathcal{A} \mapsto Th(\mathcal{A})$ constitute a Galois connection between algebras and algebraic identities of the same type. The characterization of the Galois closed sets is given by

- (Birkhoff:) $M(Th(\mathbb{A})) = V(\mathbb{A})$, where $V(\mathbb{A})$ denotes the smallest variety containing $\mathbb{A}$, i.e. smallest class closed under "homomorphic images", "subalgebras" and "direct products", containing $\mathbb{A}$, and
- $Th(M(\Sigma)) = D(\Sigma)$, where $D(\Sigma)$ denotes the "deductive closure" of Σ.

The description of the Galois operator $M \circ Th$ can be further refined by using Tarski's factorization theorem which states that for every class $\mathbb{A}$, $V(\mathbb{A})$ is represented as the composition $H(S(P(\mathbb{A})))$, where H, S and P denote the closures under homomorphic images, subalgebras and direct products, respectively. For a reference see [3].

In the work presented in this thesis, we were interested in function classes fulfilling stability conditions under certain class compositions. As we are going to see, these classes can be described as sets of functions which map relations "invariant" under certain operations into (possibly different) relations "invariant" under (possibly different) operations. A motivating example which deals with classes idempotent with respect to class composition (clones) is given in the following subsection.

1.3.2 Definability of function classes by relations

In [11] Geiger, and independently in [2] Bodnarchuk, Kaluzhnin, Kotov and Romov, constructed a Galois theory for classes of finite operations closed under compositions, induced by the notion of preservation of a relation by an operation.

For a positive integer m, an *m-ary relation on A* is a subset R of A^m, thought of as a class of unary A-valued maps $\mathbf{a} : i \mapsto a_i$ defined on the set $\mathbf{m} = \{1, \ldots, m\}$. An operation f on A is said to *preserve* R, and R is said to be *invariant* under f, if $fR \subseteq R$ where fR denotes the class composition

$\{f\} \circ R$. This binary relation "preserves" gives rise to the fundamental Galois connection **Pol** − **Inv** between operations and relations on a set A, defined by

- $\mathbf{Pol}(\mathcal{R}) = \{f \in \bigcup_{n\geq 1} A^{A^n} : f$ preserves every $R \in \mathcal{R}\}$
- $\mathbf{Inv}(\mathcal{F}) = \{R \in \bigcup_{m\geq 1} \mathcal{P}(A^m) : R$ is preserved by every $f \in \mathcal{F}\}$

for every $\mathcal{R} \subseteq \bigcup_{m\geq 1} \mathcal{P}(A^m)$ and $\mathcal{F} \subseteq \bigcup_{n\geq 1} A^{A^n}$. For further background and extensions to arbitray underlying sets, see e.g. [19, 20, 21].

In this way, certain classes of operations can be *defined* by means of relations that each of the class members preserve (constituting the Galois closed sets of the form $\mathbf{Pol}(\mathcal{R})$), and dually certain sets of relations can be *characterized* by means of operations preserving each relation in the set (constituting the Galois closed sets of the form $\mathbf{Inv}(\mathcal{F})$).

But concerning definability of function properties this approach has some limitations. To start with, this latter theory applies only to classes of operations, but even in the case $B = A$, there are several natural classes which cannot be described within this framework because the classes of the form $\mathbf{Pol}(\mathcal{R})$ are known to constitute clones. For example, while the class of monotone increasing operations is defined by the less-or-equal relation $\leq$, there is no relation or set of relations defining the class of monotone decreasing nor the class of constant operations because these are not clones.

In [17] and [18], Pöschel developed a Galois theory for *heterogeneous functions*, i.e. functions from a cartesian product $A_{i_1} \times \ldots \times A_{i_n}$ to A_j, where the underlying sets belong to a family $(A_i)_{i\in I}$ of pairwise disjoint finite sets. Here, classes of functions are defined by multisorted m-ary relations $R = \bigcup_{i\in I} R_i$ where $R_i \subseteq A_i^m$, in terms of the canonical extension of "preservation" to "multisorted preservation": a function $f : A_{i_1} \times \ldots \times A_{i_n} \to A_j$ is said to *preserve* a multisorted m-ary relation R, if

$$f \circ R_{i_1} \times \ldots \times R_{i_n} = \{f(\mathbf{a}_1, \ldots, \mathbf{a}_n) : \mathbf{a}_k \in R_{i_k}, k \in \mathbf{n}\} \subseteq R_j.$$

Although this framework generalizes to functions other than operations, this approach can only describe classes which are closed under arbitrary compositions, and thus it fails to capture several noteworthy function properties which are preserved under less strict closure conditions such as simple variable substitutions.

1.3.3 Definability of function classes by relational constraints

In [15], Pippenger studied a variant of Pöschel's Galois framework in which the primal objects are finite functions of the form $f : A^n \to B$. As dual objects, he considered ordered pairs (R, S) of relations on A and B, respectively, called "relational constraints", and introduced the notion of "constraint satisfaction" as the relation between functions and constraints.

Formally, an m-ary *A-to-B relational constraint*, or simply *relational constraint*, is an ordered pair (R, S) of relations $R \subseteq A^m$ and $S \subseteq B^m$, called *antecedent* and *consequent*, respectively, of the constraint. A function $f : A^n \longrightarrow B$ is said to *satisfy* the constraint (R, S) if $fR \subseteq S$. Note that an operation f preserves a relation R if and only if it satisfies the constraint (R, R). It is not difficult to see that the class of monotone decreasing and the class of constant operations can be defined in this setting as those satisfying the A-to-A constraints $(\leq, \geq)$ and $(A^2, =)$, respectively.

As mentioned, in this Galois setting, classes closed under simple variable substitutions are exactly those classes definable by relational constraints. The characterization of the Galois closed sets of constraints is analogous to the description given by Geiger [11] of the closed sets of relations with respect to the Galois connection **Pol** $-$ **Inv**. Clearly, every function satisfies the *empty constraint* $(\emptyset, \emptyset)$ and the *binary equality constraint* $(=, =)$. Also, each function satisfying a relational constraint (R, S), also satisfies its *relaxations*, i.e. constraints (R', S') where $R' \subseteq R$ and $S \subseteq S'$. Thus, each set of relational constraints characterizable by functions must contain the binary equality and the empty constraints, and must be *closed under relaxations*.

The remaining closure conditions given by Pippenger in [15], namely, of "closure under intersecting consequents" and "closure under taking simple minors", are essentially the same as those given by Geiger [11], but applied, respectively, on consequents of constraints with the same antecedent, and simultaneously on both antecedent and consequent of a given constraint. These closure conditions can be reassembled as follows.

Let $\varphi((P_i)_{i \in I}, x_1, \ldots, x_m)$ be a positive primitive first-order formula with free variables $x_1, \ldots, x_m$ and finitely many n_i-ary predicate symbols P_i, i.e. a formula of the form

$$\exists y_1 \ldots \exists y_n \bigwedge_{i \in I} P_i(x_{i1}, \ldots, x_{in_i})$$

where each y_j is among the x_{it}'s. Let $\mathbb{A} = \langle A, (R_i)_{i \in I} \rangle$ be a relational structure where each R_i has arity n_i. We denote by $\varphi_{\mathbb{A}}$ the set of realizations of φ in $\mathbb{A}$ by interpreting each symbol P_i as the relation R_i, i.e.

$$\varphi_{\mathbb{A}} = \{\mathbf{a} \in A^m \mid (\mathbb{A}, \mathbf{a}) \models \varphi((P_i)_{i\in I}, \mathbf{x})\}.$$

For further background, see e.g. [12, 16].

Let $(R_i, S_i)_{i\in I}$ be a non-empty family of A-to-B relational constraints, and consider the relational structures $\mathbb{A} = \langle A, (R_i)_{i\in I}\rangle$ and $\mathbb{B} = \langle B, (S_i)_{i\in I}\rangle$. We say that a relational constraint (R, S) is a *FO-conjunctive minor* of the family $(R_i, S_i)_{i\in I}$ if there is a positive primitive first-order formula $\varphi((P_i)_{i\in I}, \mathbf{x})$ such that $R \subseteq \varphi_{\mathbb{A}}$ and $\varphi_{\mathbb{B}} \subseteq S$. The formula φ is called a *formula scheme*. Formula schemes were used by Szabó in [21] to combine families of relations into new relations. If $R = \varphi_{\mathbb{A}}$ and $\varphi_{\mathbb{B}} = S$, then we say that (R, S) is a *tight FO-conjunctive minor* of the family $(R_i, S_i)_{i\in I}$. If I is a singleton, say $I = \{0\}$, then a tight FO-conjunctive minor of (R_0, S_0) via a formula φ of the form

$$\exists y_1 \ldots \exists y_n P_0(x_{01}, \ldots, x_{0n_0})$$

where each y_j is among the x_i's, is said to be a *simple minor* of (R_0, S_0). If φ is of the form

$$\bigwedge_{i\in I} P_i(x_{i1}, \ldots, x_{in_i})$$

and if $R_i = R$ for every $i \in I$, then a tight FO-conjunctive minor of the family $(R_i, S_i)_{i\in I}$ via φ is said to be obtained from $(R_i, S_i)_{i\in I}$ by intersecting consequents.

We say that a set $\mathcal{T}$ of relational constraints is *closed under formation of FO-conjunctive minors* if $\mathcal{T}$ contains all FO-conjunctive minors of each family $(R_i, S_i)_{i\in I}$ of relational constraints in $\mathcal{T}$.

In the case of finite underlying sets, every FO-conjunctive minor can be obtained as a combination of taking simple minors, relaxations or intersecting consequents, and Pippenger's characterization of Galois closed sets of relational constraints can be thus restated in terms of FO-conjunctive minors: A set $\mathcal{T}$ of A-to-B relational constraints is characterizable by B-valued functions on A if and only if $\mathcal{T}$ contains the binary equality and the empty constraints, and it is closed under formation of FO-conjunctive minors.

Chapter 2

Author's contribution

In this chapter, we present the main results of this thesis. Each of the following sections 2.1–2.7 summarizes a research paper having the same title as the corresponding section. The results presented in Section 2.8 constitute unpublished material, first appearing in the present manuscript.

2.1 Definability of Boolean function classes by linear equations over GF(2)

The first paper arose from considering syntactic restrictions on the defining functional equations. In particular, we asked for a characterization, in terms of necessary and sufficient conditions, of the classes of Boolean functions which are definable by *linear equations*, i.e. formal expressions build in a restricted equational language having the sum modulo 2 (also called exclusive-or) denoted $+$, and the constants $\mathbf{0}$ and $\mathbf{1}$, as the only Boolean connectives. In fact, since statements $H_1 = H_2$ hold if and only if $H_1 + H_2 = \mathbf{0}$ hold, we considered expressions of the form

$$c_1\mathbf{f}(c_{11}\mathbf{x}_1 + \ldots + c_{1m}\mathbf{x}_m + \mathbf{d}_1) + \ldots + c_q\mathbf{f}(c_{q1}\mathbf{x}_1 + \ldots + c_{qm}\mathbf{x}_m + \mathbf{d}_q) = d$$

where the subscripted c's and d, as well as the $\mathbf{d}_i$'s, are among the constants.

Essentially, we showed that the linearly definable classes of Boolean functions are exactly those which are stable under right and left composition with the Boolean clone $\mathcal{L}_c$ of constant-preserving linear functions.

The proof of "linear definability implies stability under right and left composition with $\mathcal{L}_c$" follows straightforwardly from the fact that each function in $\mathcal{L}_c$ preserves $+$ and the constants.

To prove the converse, we made use of Pippenger's results in [15]. Stability under $\mathcal{L}_c$ implies in particular closure under simple variable substitutions, and thus these "stable" classes are definable by means of relational constraints. Furthermore, stability under composition with $\mathcal{L}_c$ forces the relations in the defining constraints to be either affine subspaces of the vector space $\mathbb{B}^n$ or the empty space, because these are exactly the invariants under $\mathcal{L}_c$. From basic facts in linear algebra over the two element field, we knew that affine subspaces coincide with subsets of $\mathbb{B}^n$ definable by affine forms, which we then used to construct the desired linear equations.

In particular, from this characterization it followed that the only clones definable by linear equations are the superclones of $\mathcal{L}_c$, namely the clone of all Boolean functions, the four maximal clones of the self-dual, linear, zero-preserving and one-preserving Boolean functions, together with the six clones that can be obtained from these by taking intersections.

An important consequence of this study was the strengthnening of Pippenger's result: definability by linear equations is equivalent to definability by affine constraints, in the sense that these two approaches define exactly the same classes.

2.2 On affine constraints satisfied by Boolean functions

In this paper we addressed the question of characterizing the dual closed sets in the Galois setting introduced in the previous paper. In other words, we wanted to know which are the necessary and sufficient closure conditions on sets of affine constraints such that these have characterizations in terms of Boolean function classes, or equivalently, in terms of classes stable under left and right composition with the Boolean clone $\mathcal{L}_c$ of constant-preserving linear functions.

The key step for such a characterization was to observe that every tight FO-conjunctive minor of a non-empty family of affine constraints is still an affine constraint, and that both the binary equality and empty constraints are affine constraints. But relaxations of affine constraints are not necessarily affine. To obtain the complete description of the Galois closed sets of constraints (with respect to the induced Galois connection introduced in the previous paper) we considered the closure under "affine" relaxations induced by the restriction of relational constraints to affine constraints. The Galois closed sets of affine constraints were thus characterized as those sets of affine constraints containing the binary equality and empty constraints, and which are closed under taking tight FO-conjunctive minors and closed under affine relaxations.

2.3 On closed sets of relational constraints and classes of functions closed under variable substitutions

The connection between definability by equations and by constraints drew our attention to Pippenger's framework. In [15], Pippenger deals only with functions over finite sets, thus the natural question is to ask what happens if we remove the finiteness restriction on the underlying sets. In this paper, we answered this question and extended Pippenger's Galois theory to arbitrary, possibly infinite, underlying sets.

The characterization of the Galois closed sets of functions is essentially the same as in the finite case, but here we needed the additional condition of "local closure": a class $\mathcal{K}$ of B-valued functions on A is *locally closed* if it contains every function whose restriction to any finite subset of its domain coincides with the restriction of some member of $\mathcal{K}$ to the same finite subset. This is due to the fact that the dual objects are relational constraints of finite arities, and hence, the non-satisfaction of a constraint by a function, can only be verified in a finite restriction to the domain of the function. In the case of finite underlying sets, this condition is trivially satisfied by every function class.

To characterize the Galois closed sets of dual objects, we strengthened the closure conditions given in [15]. The Galois closed sets of constraints are required to fulfill an additional condition, analogous to the notion of local closure on function classes: a set $\mathcal{T}$ is said to be *locally closed* if it contains each constraint whose finite relaxations are all in $\mathcal{T}$. As for function classes, every set of constraints over finite underlying sets is locally closed.

The closure under formation of FO-conjunctive minors, as defined in Subsection 1.3.3, remained as a necessary condition, but it was no longer sufficient. In the case of arbitrary underlying sets, we had to consider formula schemes in a fragment of infinitary first-order logic: the number of free variables in the formula schemes remained finite, but we allowed both infinitary conjunctions and infinite existential quantifications. The combination of families of relational constraints via these formula schemes are referred to without the prefix "FO-", i.e. as *conjunctive minors* and *tight conjunctive minors.*

The comparison of the condition of closure under formation of conjunctive minors with the natural extensions to the infinite case of Pippenger's closure conditions revealed that indeed the latter are strictly subsumed by the former.

The conditions of local closure and closure under formation of conjunctive minors, together with the assumption that the sets of constraints contain both the binary equality and empty constraints, provide the characterization of the dual Galois sets. Our proof follows the strategy used by Geiger [11], and also by Pippenger [15], in which one provides for each constraint (R, S), not in a set $\mathcal{T}$ closed under the above conditions, a function separating (R, S) from $\mathcal{T}$. But instead of a pointwise construction of the separating functions, we define them at once as total functions.

In addition, we showed how these characterizations of primal and dual closed sets can be used to derive the descriptions of the Galois closed sets of operations and relations, presented by Szabó [21] and Pöschel [19, 20].

2.4 On Galois connections between external functions and relational constraints: arity restrictions and operator decompositions

In this paper we made explicit the Galois connection introduced in [15] and extended in the paper [CF3] summarized in Section 2.3. We consider the mappings **FSC** (functions satisfying constraints) and **CSF** (constraints satisfied by functions) induced by "constraint satisfaction", analogous to the mappings **Pol** and **Inv**, and provided decompositions of the Galois operators $\mathbf{FSC} \circ \mathbf{CSF}$ and $\mathbf{CSF} \circ \mathbf{FSC}$ in terms of the mappings associated with the closures given in [CF3], namely, the operators $\mathcal{K} \mapsto \mathbf{Lo}(\mathcal{K})$ (where $\mathbf{Lo}(\mathcal{K})$ denotes the smallest locally closed class containing $\mathcal{K}$), $\mathcal{K} \mapsto \mathbf{VS}(\mathcal{K})$ (where $\mathbf{VS}(\mathcal{K})$ denotes the smallest class closed under simple variable substitutions containing $\mathcal{K}$), $\mathcal{T} \mapsto \mathbf{LO}(\mathcal{T})$ (where $\mathbf{LO}(\mathcal{T})$ denotes the smallest locally closed set of constraints containing $\mathcal{T}$), and $\mathcal{T} \mapsto \mathbf{CM}(\mathcal{T})$ (where $\mathbf{CM}(\mathcal{T})$ denotes the smallest set closed under formation of conjunctive minors containing $\mathcal{T} \cup \{(\emptyset, \emptyset), (=, =)\}$). We showed that for any class of functions $\mathcal{K} \subseteq \bigcup_{n \geq 1} B^{A^n}$ and any set $\mathcal{T}$ of A-to-B relational constraints

(i) $\mathbf{FSC}(\mathbf{CSF}(\mathcal{K})) = \mathbf{Lo}(\mathbf{VS}(\mathcal{K}))$, and

(ii) $\mathbf{CSF}(\mathbf{FSC}(\mathcal{T})) = \mathbf{LO}(\mathbf{CM}(\mathcal{T}))$.

Several natural function classes are partially specified by arity conditions on their members, and similarly certain sets of constraints of interesting relational types are partially specified by arity conditions on their members. These observations led us to consider further Galois connections induced by arity restrictions on both primal and dual objects.

First, we described the induced Galois closed sets for the correspondences $\mathbf{FSC}_n - \mathbf{CSF}$ and $\mathbf{FSC} - \mathbf{CSF}_m$, where $\mathbf{FSC}_n(\mathcal{T})$ denotes the restriction of $\mathbf{FSC}(\mathcal{T})$ to the class B^{A^n} of n-ary functions, i.e. $\mathbf{FSC}_n(\mathcal{T}) = B^{A^n} \cap \mathbf{FSC}(\mathcal{T})$, and $\mathbf{CSF}_m(\mathcal{K})$ denotes the restriction of $\mathbf{CSF}(\mathcal{K})$ to the set $\mathcal{Q}_m$ of m-ary constraints, i.e. $\mathbf{CSF}_m(\mathcal{K}) = \mathcal{Q}_m \cap \mathbf{CSF}(\mathcal{K})$. Then we provided factorizations of the corresponding Galois operators by means of parametrized analogues of the closures defined above. While closure under n-ary simple variable substitutions and closure under formation of m-ary conjunctive minors, constitute restrictions of the former closures to n-ary functions and to m-ary constraints, respectively, the parametrized notions of local closure, for function classes and sets of constraints, constitute relaxations of the former ones. Here, a class $\mathcal{K}$ is said to be *m-locally closed* if it contains every function whose restriction to any finite subset of its domain, with size at most m, coincides with the restriction of some member of $\mathcal{K}$ to the same finite subset. Analogously, a set $\mathcal{T}$ of relational constraints is said to be *n-locally closed* if $\mathcal{T}$ contains every A-to-B constraint (R, S) such that the set of all its relaxations with antecedent of size at most n, is contained in $\mathcal{T}$. (Analogous Galois settings for operations were developed by Pöschel e.g. in [20].)

By combining these results, we obtained necessary and sufficient conditions for a class of n-ary functions to be definable by m-ary relational constraints, and dually for a set of m-ary relational constraints to be characterizable by n-ary functions. In the context of graph homomorphisms, i.e. where $n = 1$ and $m = 2$, these conditions reduce to:

(a) A class $\mathcal{K} \subseteq B^A$ is definable by A-to-B binary constraints if and only if $\mathcal{K}$ is 2-locally closed;

(b) A set $\mathcal{T}$ of A-to-B binary constraints is characterizable by unary B-valued functions on A if and only if $\mathcal{T}$ contains the binary equality and empty constraints, and it is closed under arbitrary unions and closed under formation of binary conjunctive minors.

Here, by a set $\mathcal{T}$ of relational constraints *closed under arbitrary unions* we mean a set $\mathcal{T}$ containing every constraint $(\bigcup_{i \in I} R_i, \bigcup_{i \in I} S_i)$, whenever $(R_i, S_i)_{i \in I}$ is a non-empty family of members of $\mathcal{T}$.

2.5 Galois connections for generalized functions and relational constraints

Here, we explored further Galois connections by considering as primal objects the more general *multivalued functions*, that is, functions of the form $A^n \to \mathcal{P}(B)$, where $\mathcal{P}(B)$ denotes the set of all subsets of B. We continued to consider relational constraints (R, S), where $R \subseteq A^m$ and $S \subseteq B^m$, as dual objects, with the notion of "constraint satisfaction" essentially as defined in the previous articles: A multivalued function $f : A^n \to \mathcal{P}(B)$ is said to *satisfy* an m-ary constraint (R, S) if $fR \subseteq S$, where the relation fR is now given by

$$fR = \bigcup \{ \prod_{i \in \mathbf{m}} f(\mathbf{a}_1, \ldots, \mathbf{a}_n)(i) : \mathbf{a}_1, \ldots, \mathbf{a}_n \in R \}$$

where $f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is the composition of f with the m-vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ (thought as A-valued maps on $\mathbf{m}$) given by

$$f(\mathbf{a}_1, \ldots, \mathbf{a}_n)(i) = f(\mathbf{a}_1(i), \ldots, \mathbf{a}_n(i))$$

for every $i \in \mathbf{m}$. In this way, the various notions of total and partial functions can be studied within a unifying setting, whose most restricted case, the total single-valued functions, correspond exactly to the Galois setting considered in [CF3]. Moreover, by taking $B = A$ and relational constraints of the form (R, R), this framework specializes to those studied e.g. by Fleischer and Rosenberg in [10] (by considering at most single-valued functions), and by Börner in [1] (by considering non-empty multivalued functions).

The conditions characterizing the Galois closed classes of multivalued functions, in addition to those given in [CF3], include closure under *subfunctions* in the sense of [1]: a class $\mathcal{M}$ is *closed under subfunctions* if it contains each function $g : A^n \to \mathcal{P}(B)$, for which there exists $f \in \mathcal{M}$ of the same arity such that $g(\mathbf{a}) \subseteq f(\mathbf{a})$, for every $\mathbf{a} \in A^n$. Also, since each empty-valued function satisfies every constraint, in the case of partial functions, the Galois closed sets contain the empty functions. Moreover, local closure was strengthened to "closure under local coverings": a class $\mathcal{M}$ of multivalued functions on A to B is *closed under local coverings* if it contains every multivalued function f on A to B such that for every finite subset $F \subseteq A^n$, there is a non-empty family $(f_i)_{i \in I}$ of members of $\mathcal{M}$ of the same arity as f, such that

$$\prod_{\mathbf{a} \in F} f(\mathbf{a}) \subseteq \bigcup_{i \in I} \prod_{\mathbf{a} \in F} f_i(\mathbf{a}).$$

The dual Galois closed sets were specified by relaxing the conditions given in [CF3]. First, we noticed that while total multivalued functions continue to satisfy conjunctive minors of families of constraints which they satisfy, this is not the case for partial functions. Here, closure under formation of conjunctive minors had to be weakened by removing existential quantification from formula schemes. Also, all multifunctions satisfy the empty constraint and the *full constraint* (A, B) but, in general, multivalued functions do not satisfy equality constraints, unless they are at most single-valued functions. Explicit descriptions of the Galois operators considered were then given in terms of closure operators associated with the conditions described above.

2.6 Function class composition, relational constraints and stability under compositions with clones

The results obtained in the research papers [CF1], [CF2] and [CF3], summarized in the Sections 2.1, 2.2 and 2.3, respectively, were placed in a general Galois setting by considering stronger closure conditions on function classes. We constructed a general Galois theory for function classes stable under both right and left composition with clones. Each pair of clones $\mathcal{C}_1, \mathcal{C}_2$ on A and B, respectively, induces a new Galois correspondence between functions and constraints, by restricting the set of dual objects to relational constraints whose antecedent and consequent are invariant under the clones $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, the so-called $(\mathcal{C}_1, \mathcal{C}_2)$-*relational constraints*.

Here, the classes defined by means of $(\mathcal{C}_1, \mathcal{C}_2)$-relational constraints are shown to be exactly those locally closed classes which are stable both under right composition with $\mathcal{C}_1$ and under left composition with $\mathcal{C}_2$. The dual Galois closed sets are essentially the restrictions to $(\mathcal{C}_1, \mathcal{C}_2)$-relational constraints, of the closed sets of constraints described in [CF3]. Both the empty and binary equality constraints are $(\mathcal{C}_1, \mathcal{C}_2)$-relational constraints. Also, each tight conjunctive minor of a non-empty family of $(\mathcal{C}_1, \mathcal{C}_2)$-relational constraints is a $(\mathcal{C}_1, \mathcal{C}_2)$-relational constraint, but not all of its relaxations are $(\mathcal{C}_1, \mathcal{C}_2)$-relational constraints. Thus, closure under conjunctive minors was weakened to "closure under formation of $(\mathcal{C}_1, \mathcal{C}_2)$-conjunctive minors": a set $\mathcal{T}$ of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints is said to be *closed under formation of $(\mathcal{C}_1, \mathcal{C}_2)$-conjunctive minors* if every $(\mathcal{C}_1, \mathcal{C}_2)$-relational constraint which is a conjunctive minor of a non-empty family of members of $\mathcal{T}$, is also in $\mathcal{T}$. Furthermore, local closure was replaced by "$(\mathcal{C}_1, \mathcal{C}_2)$-local closure": a set $\mathcal{T}$ is said to be

$(\mathcal{C}_1, \mathcal{C}_2)$*-locally closed* if the set $\mathcal{T}_r$ of all relaxations of relational constraints in $\mathcal{T}$ is locally closed. By suitable choices of clones $\mathcal{C}_1$ and $\mathcal{C}_2$, this general framework provides the descriptions of the various Galois connections studied in [15], [CF1] and [CF3].

This investigation revealed an associativity condition stated as the "Associativity Lemma", crucial in deriving the stability conditions on function classes, and whose relevance is also attested in other studies (see e.g. [5, 4]). Basically, it asserts that class composition is an associative operation on the set of classes closed under variable substitutions. As an immediate consequence, we get that the set of all equational classes of Boolean functions constitutes a monoid under class composition whose identity is the smallest clone containing only projections.

2.7 Functional equations, constraints, definability of function classes, and functions of Boolean variables

The approach to definability of function classes by means of functional equations was brought to the general case of arbitrary underlying sets, by considering a more classical notion of functional equation. Here, by a *functional equation* (for B-valued functions on A) we mean an expression

$$\begin{aligned} &h_1(\mathbf{f}(g_1(\mathbf{x}_1, \ldots, \mathbf{x}_p)), \ldots, \mathbf{f}(g_m(\mathbf{x}_1, \ldots, \mathbf{x}_p))) = \\ &h_2(\mathbf{f}(g'_1(\mathbf{x}_1, \ldots, \mathbf{x}_p)), \ldots, \mathbf{f}(g'_t(\mathbf{x}_1, \ldots, \mathbf{x}_p))) \end{aligned} \tag{2.1}$$

where $m, t, p \geq 1$, $h_1 : B^m \to B$, $h_2 : B^t \to B$, and each g_i and g'_j is a map $A^p \to A$. The symbols $\mathbf{x}_1, \ldots, \mathbf{x}_p$ are p distinct *vector variable symbols*, and $\mathbf{f}$ is a *function symbol*. The notion of "satisfaction" is given by saying that an n-ary B-valued function f on A *satisfies* equation (2.1) if, for all $\mathbf{v}_1, \ldots, \mathbf{v}_p \in A^n$, viewed as unary A-valued functions $\mathbf{v}_j : i \mapsto \mathbf{v_{ji}}$ on $\mathbf{n}$, we have

$$\begin{aligned} &h_1(f(g_1(\mathbf{v}_1, \ldots, \mathbf{v}_p)), \ldots, f(g_m(\mathbf{v}_1, \ldots, \mathbf{v}_p))) = \\ &h_2(f(g'_1(\mathbf{v}_1, \ldots, \mathbf{v}_p)), \ldots, f(g'_t(\mathbf{v}_1, \ldots, \mathbf{v}_p))). \end{aligned}$$

This formulation of functional equation facilitated the correspondence between equations and constraints in the case of arbitrary underlying sets. The complete correspondence observed by Pippenger in [15] within the Boolean universe was shown to hold in the general case of arbitrary, possibly infinite, underlying sets A and B. In fact, the proof of Theorem 1 in [CF5] shows that the following result also holds.

Corollary 1. *For every functional equation (2.1), there is a relational constraint which is satisfied by exactly the same functions as those satisfying (2.1). Conversely, for every relational constraint* (R, S)*, there is a functional equation which is satisfied by exactly the same functions as those satisfying* (R, S)*.*

The construction given in the proof of Theorem 1 in [CF5] revealed general criteria for establishing more stringent correspondences, namely, between the algebraic syntax of functional equations and invariance properties of relational constraints.

The ideas explored in [CF1] were revisited, and the results presented in the latter as well as those presented by Ekin, Foldes, Hammer and Hellerstein in [8], were placed in the general Galois framework described in [CF4]. As an immediate consequence, the equational classes of B-valued functions were characterized as those locally closed classes which are stable under right composition with the clone of projections on A, i.e. closed under simple variable substitutions.

The question of linear definability (i.e. definability by means of functional equations whose outer and inner expressions are affine forms) appears naturally in this setting, where the correspondence to definability by means of relational constraints can be once again established by considering affine constraints (see Theorem 1 in Section 2.8 below).

Applications of our results were illustrated in the context of ring-valued functions of Boolean variables. As examples of such functions, we considered the Boolean functions, and the so-called *pseudo-Boolean functions*: maps defined on the two element-set and valued in the ring of real numbers. In [CF1], it was observed that for each positive integer m, the class $\mathcal{D}^m$ of Boolean functions with (polynomial) degree at most m is linearly definable, but no explicit equational characterizations were provided.

Here, we addressed the more general question of equational characterizations in the case of field-valued functions. We showed that a class of field-valued functions of bounded degree is linearly definable if and only if the codomain field has characteristic 2, and presented linear and non-linear equational characterizations of these bounded degree classes, accordingly.

2.8 Equational definability of classes of affine operations on finite fields with bounded number of essential variables

In this section, we present further, still unpublished, results in wich we make use of ideas and tools developed in [CF1] and [CF5]. Throughout this section, we shall use p, p_1, p_2 to denote prime numbers, and we shall use q, q_1, q_2 for prime powers $p^m, p_1^{m_1}, p_2^{m_2}$, $m, m_1, m_2 \geq 1$. We consider classes $\mathcal{L}^k$ of affine operations on $GF(q)$ having at most $k \geq 1$ essential variables, i.e. functions of the form

$$f = a_1 x_{i_1} + \ldots + a_k x_{i_k} + a,$$

where $a_1, \ldots, a_k, a \in GF(q)$.

Theorem 1. *A class of $GF(q_2)$-valued functions on $GF(q_1)$ is linearly definable if and only if it is definable by means of relational constraints whose antecedents and consequents are affine subspaces over $GF(q_1)$ and $GF(q_2)$, respectively.*

Proof. First, we show that for every linear equation, say

$$\begin{aligned} &h_1(\mathbf{f}(g_1(\mathbf{x}_1, \ldots, \mathbf{x}_p)), \ldots, \mathbf{f}(g_m(\mathbf{x}_1, \ldots, \mathbf{x}_p))) = \\ &h_2(\mathbf{f}(g'_1(\mathbf{x}_1, \ldots, \mathbf{x}_p)), \ldots, \mathbf{f}(g'_t(\mathbf{x}_1, \ldots, \mathbf{x}_p))) \end{aligned} \tag{2.2}$$

where h_1, h_2 are affine operations on $GF(q_2)$, and each g_i and g'_j is an affine operation on $GF(q_1)$, there is an affine constraint (R, S) defining exactly the same class of $GF(q_2)$-valued functions on $GF(q_1)$ as defined by (2.2). By making use of basic facts from linear algebra over finite fields, it follows that both antecedent and consequent of the constraint (R, S) constructed in the proof of Theorem 1 in [CF5], are affine subspaces over $GF(q_1)$ and $GF(q_2)$, respectively. Moreover, it is easy to verify that a $GF(q_2)$-valued function on $GF(q_1)$ satisfies (2.2) if and only if it satisfies (R, S).

To show that definability by affine constraints implies linear definability, we follow the same steps as in the proof of Theorem 1 in [CF1]. Let (R, S) be a relational constraint whose antecedent and consequent is an affine subspace over $GF(q_1)$ and $GF(q_2)$, respectively. First, we observe that we may assume that R is non-empty, because constraints with empty antecedent are satisfied by every $GF(q_2)$-valued function on $GF(q_1)$, and thus they can be discarded as irrelevant.

Suppose first that S is an affine subspace of co-dimension 1, i.e. an affine hyperplane. In this case, the maps given in Fact 1 and Fact 2, in the proof of Theorem 1 in [CF5], can be chosen to be affine operations on $GF(q_1)$ and $GF(q_2)$, respectively, and hence, the functional equation thus constructed has the desired form.

Suppose now that S is an affine subspace of co-dimension greater than 1. In this case, we know that S can be represented as an intersection $\cap_{i \in I} S_i$ of (finitely many) affine hyperplanes S_i. As observed, for each constraint (R, S_i), there is a linear equation E_i which is satisfied by exactly the same functions satisfying (R, S_i). Moreover, a $GF(q_2)$-valued function on $GF(q_1)$ satisfies $(R, \cap_{i \in I} S_i)$ if and only if it satisfies each (R, S_i). Thus, for each affine constarint (R, S) there are (finitely many) linear equations which are satisfied by exactly the same functions as those satisfying (R, S). □

Note that affine relations over $GF(q)$ are precisely the relations invariant under the clone of constant-preserving affine operations on $GF(q)$. This fact is used to prove the following result.

Corollary 2. *For every positive integer k, the class $\mathcal{L}^k$ of affine operations on $GF(q)$ with at most k essential variables, is not definable by means of linear equations.*

Proof. By Theorem 1, we only have to show that, for each positive integer k, $\mathcal{L}^k$ is not definable by means of affine constraints. For that we make use of Theorem 2 in [CF5], and show that for each positive integer k, $\mathcal{L}^k$ is not stable under right composition with the clone $\mathcal{L}_c$ of constant-preserving affine operations on $GF(q)$. Indeed, $g(x, y, z) = x + y - z = x + y + (q-1)z$ is a constant-preserving affine operation on $GF(q)$, and the k-ary operation $f(x_1, \ldots, x_k) = x_1 + \ldots + x_k$ is in $\mathcal{L}^k$. But

$$f(x_1^{k+2}, \ldots, x_{k-1}^{k+2}, g(x_k^{k+2}, x_{k+1}^{k+2}, x_{k+2}^{k+2})) \in \mathcal{L}^k \circ \mathcal{L}_c$$

is an affine operation with $k+2$ essential variables, and thus it does not belong to $\mathcal{L}^k$, showing that, indeed, $\mathcal{L}^k$ is not stable under right composition with the clone $\mathcal{L}_c$ of constant-preserving affine operations. □

Still, simple variable substitutions cannot increase the number of essential variables and hence, each $\mathcal{L}^k$ is closed under simple variable substitutions. Since $GF(q)$ is finite, each of these classes constitutes an equational class. In the remainder of this section, we shall provide equational characterizations of each $\mathcal{L}^k$, $k \geq 1$. For n-vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ over $GF(q)$, let $\mathbf{c}_1\mathbf{c}_2$ denote their componentwise product over $GF(q)$.

Lemma 1. *Let f be an n-ary linear operation (i.e. 0-preserving affine operation) on $GF(q)$ with $1 \leq k \leq n$ essential variables. Suppose that $\mathbf{c}_1, \ldots, \mathbf{c}_{k+1}$ are vectors in $GF(q)^n$ such that $f(\mathbf{c}_j) = 1$, for all $j \in \mathbf{k+1}$. Then there is $j \in \mathbf{k+1}$ such that, for some $b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_{k+1} \in GF(q)$ satisfying $\sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i = 1$, we have*

$$f(\mathbf{c}_j{}^{q-1}(\sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{c}_i)) = 1.$$

Proof. Let f be an n-ary operation on $GF(q)$ with $ess(f) = k$. Let

$$I = \{i \in \mathbf{n} : x_i \text{ is an essential variable of } f\}.$$

For each $\mathbf{v} \in GF(q)^n$, let $\mathbf{v}' = (v'_1, \ldots, v'_n)$ be the n-vector such that v'_i coincides with the ith component of $\mathbf{v}$ if $i \in I$, and $v'_i = 0$ otherwise. Note that there are at most k linearly independent vectors $\mathbf{v}' \in GF(q)^n$, and for every $\mathbf{v}_1, \mathbf{v}_2 \in GF(q)^n$, $(\mathbf{v}_1 + \mathbf{v}_2)' = \mathbf{v'}_1 + \mathbf{v'}_2$ and $(\mathbf{v}_1\mathbf{v}_2)' = \mathbf{v'}_1\mathbf{v'}_2$. Moreover, for every $\mathbf{v} \in GF(q)^n$, $\mathbf{v}$ and $\mathbf{v}'$ coincide in the components corresponding to essential variables of f, and thus $f(\mathbf{v}) = f(\mathbf{v}')$.

Let $\mathbf{c}_1, \ldots, \mathbf{c}_{k+1}$ be $k+1$ vectors in $GF(q)^n$ such that $f(\mathbf{c}_j) = 1$, for all $j \in \mathbf{k+1}$. Since $f(\mathbf{0}) = 0$, and $\mathbf{c'}_1, \ldots, \mathbf{c'}_{k+1} \in GF(q)^n$ are linearly dependent, there exists $j \in \mathbf{k+1}$ such that, for some $b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_{k+1} \in GF(q)$ satisfying $\sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i = 1$, we have

$$\mathbf{c'}_j = \sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{c'}_i.$$

Since

$$\mathbf{c'}_j = \mathbf{c'}_j^q = \mathbf{c'}_j^{q-1}\mathbf{c'}_j = \mathbf{c'}_j^{q-1}(\sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{c'}_i)$$

it follows that

$$f(\mathbf{c'}_j^{q-1}(\sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{c'}_i)) = 1$$

and since

$$f(\mathbf{c'}_j^{q-1}(\sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{c'}_i)) = f(\mathbf{c}_j^{q-1}(\sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{c}_i))$$

the proof of Lemma 1 is complete. □

Let $\leq$ be the canonical ordering of the elements of the finite field $GF(q)$, i.e. $0 \leq 1 \leq \ldots \leq q-1$.

Theorem 2. *The class $\mathcal{L}_0^k$ of linear operations on $GF(q)$ with at most $k \geq 1$ essential variables, i.e., of operations of the form*

$$f = a_1 x_{i_1} + \ldots + a_k x_{i_k}$$

where $a_1, \ldots, a_k \in GF(q)$, is defined by

$$\min_{j \in \mathbf{k+1}} \prod_{\substack{l \in GF(q) \\ l \neq 1}} [\mathbf{f}(\mathbf{x}_j) - l] \prod_{j \in \mathbf{k+1}} [\min_{\substack{\sum\limits_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i = 1}} (\mathbf{f}(\mathbf{x}_j^{q-1} \sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{x}_i) - 1)] = 0 \qquad (2.3)$$

Proof. Let f be an n-ary linear operation with more than k essential variables, say

$$f = a_1 x_1 + \ldots + a_n x_n$$

where $k < n$, and $a_1, \ldots, a_n \in GF(q)$. Without loss of generality, assume that x_i is essential for $i \in \mathbf{k+1}$. Then for $\mathbf{c}_i = a_i^{-1} \mathbf{e}_i$, where $i \in \mathbf{k+1}$ and $\mathbf{e}_i$ the i-th unit n-vector, we have that for every $j \in \mathbf{k+1}$, and $b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_{k+1} \in GF(q)$

$$f(\mathbf{c}_j{}^{q-1} \sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{c}_i) = f(\mathbf{0}) = 0.$$

Hence,

$$\prod_{j \in \mathbf{k+1}} [\min_{\substack{\sum\limits_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i = 1}} (f(\mathbf{c}_j^{q-1} \sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{c}_i) - 1)] = (-1)^{k+1} \neq 0.$$

Futhermore,

$$\min_{j \in \mathbf{k+1}} \prod_{\substack{l \in GF(q) \\ l \neq 1}} [f(\mathbf{c}_j) - l] = (-1)^{k+1} \neq 0.$$

Thus f does not satisfy (2.3).

Now, suppose that f is an n-ary operation with at most k essential variables, say

$$f = a_1 x_{i_1} + \ldots + a_k x_{i_k},$$

where $a_1, \ldots, a_k \in GF(q)$. Let $\mathbf{c}_1, \ldots, \mathbf{c}_{k+1} \in GF(q)^n$. Observe that if $f(\mathbf{c}_j) \neq 1$ for some $1 \leq j \leq k+1$, then

$$\min_{j \in \mathbf{k+1}} \prod_{\substack{l \in GF(q) \\ l \neq 1}} [f(\mathbf{c}_j) - l] = 0.$$

So suppose that for every $j \in \mathbf{k+1}$, $f(\mathbf{c}_j) = 1$. By Lemma 1, it follows that there is $j \in \mathbf{k+1}$ such that, for some $b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_{k+1} \in GF(q)$ satisfying $\sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i = 1$, we have

$$f({\mathbf{c}_j}^{q-1}(\sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{c}_i)) = 1,$$

and hence

$$\prod_{j \in \mathbf{k+1}} [\min_{\substack{\sum\limits_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i = 1}} (f(\mathbf{c}_j^{q-1} \sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{c}_i) - 1)] = 0$$

which completes the proof of Theorem 2. □

Note that each affine operation $g \in \mathcal{L}^k$ is of the form $g = f + a$, for some $f \in \mathcal{L}_0^k$ and $a \in GF(q)$. Thus, for every $g \in \mathcal{L}^k$, $g - g(\mathbf{0}) \in \mathcal{L}_0^k$.

Corollary 3. *The class $\mathcal{L}^k$ of affine operations on $GF(q)$ with at most $k \geq 1$ essential variables, is defined by*

$$\min_{j \in \mathbf{k+1}} \prod_{\substack{l \in GF(q) \\ l \neq 1}} [\mathbf{f}(\mathbf{x}_j) - \mathbf{f}(\mathbf{0}) - l] \prod_{j \in \mathbf{k+1}} [\min_{\substack{\sum\limits_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i = 1}} (\mathbf{f}(\mathbf{x}_j^{q-1} \sum_{\substack{i \in \mathbf{k+1} \\ i \neq j}} b_i \mathbf{x}_i) - \mathbf{f}(\mathbf{0}) - 1)] = 0.$$

In the Boolean case $q = 2$, equivalent equational characterizations of each of these classes were recently presented in [6].

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DEFINABILITY OF BOOLEAN FUNCTION CLASSES BY LINEAR EQUATIONS OVER GF(2)

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ABSTRACT. Necessary and sufficient conditions are provided for a class of Boolean functions to be definable by a set of linear functional equations over the two-element field. The conditions are given both in terms of closure with respect to certain functional compositions and in terms of definability by relational constraints.

1. INTRODUCTION

Boolean function classes definable by algebraic equations, inequalities, other predicates or closure conditions, have been the object of a number of studies since Emil Post's classification of clones [P]. To illustrate equational definability, consider

a) the clone **M** of monotone Boolean functions, definable by equation (1) below;
b) the class of decreasing functions, definable by (2);
c) the clone **S** of self-dual functions, definable by equation (3) or (4);
d) the class of reflexive functions, definable by (5) or (6):

$$f(\mathbf{v})f(\mathbf{vw}) = f(\mathbf{vw}) \tag{1}$$

$$f(\mathbf{v})f(\mathbf{vw}) = f(\mathbf{v}) \tag{2}$$

$$f(\mathbf{v}) = \neg f(\neg \mathbf{v}) \tag{3}$$

$$f(\mathbf{v}) + f(\mathbf{v} + \mathbf{1}) = 1 \tag{4}$$

$$f(\mathbf{v}) = f(\neg \mathbf{v}) \tag{5}$$

$$f(\mathbf{v}) + f(\mathbf{v} + \mathbf{1}) = 0 \tag{6}$$

Each of these equations defines a class of Boolean functions in the sense that the class consists of those Boolean functions $f : \{0,1\}^n \to \{0,1\}$ that satisfy the equation for all choices of $\mathbf{v}$ and $\mathbf{w}$ in $\{0,1\}^n$ where $\mathbf{1}$ denotes the all-1 vector in $\{0,1\}^n$, $\neg$ denotes the interchange of 0's and 1's, $+$ denotes the

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Boolean sum, and juxtaposition denotes conjunction, i.e. Boolean product or meet (taken componentwise for vectors).

Note that neither the decreasing nor the reflexive function class forms a clone (neither is closed under functional composition). As for the form of the equations, observe that both (4) and (6) are linear, i.e. written in the additive language of the two-element field, while the other functional equations are written in the conjunction-and-negation language of the two-element Boolean lattice (juxtaposition stands for meet). These two languages are, of course, strongly related, as it is well known (see e.g. Stone [S] for an early reference). However, while the self-dual and reflexive function classes can be defined in both linear and lattice language, neither the monotone nor the deacreasing classes are linearly definable, as it will be clear from Theorem 1 in Section 3.

Necessary and sufficient conditions for a class of Boolean functions to be definable by general Boolean equations were stated and proved, with some variations in what is meant by an equation, by Ekin, Foldes, Hammer, Hellerstein [EFHH], also in [F] and by Pippenger [Pi2]; the specific class of threshold functions was examined by Hellerstein [H] and the equational characterization of clones was studied by Pogosyan [Po] and by Foldes and Pogosyan [FPo]. Essentially, the conditions for equational definability were shown to be the closure under identification of variables (diagonalization), permutation of variables and addition of inessential variables (cylindrification). Here we shall explore the more stringent notion of equational definability by linear functional equations.

2. Basic concepts

In this paper, by a *Boolean function* we mean a map $f : \mathbf{B}^n \to \mathbf{B}$, where $\mathbf{B} = \mathbf{GF}(2) = \{0, 1\}$ (the field of two elements) and $n \geq 1$. The integer n is called the *arity* of f. For a fixed arity n, the simplest Boolean functions are the n different *coordinate projection* maps $(a_1, \ldots, a_n) \mapsto a_i$, $1 \leq i \leq n$, also called *variables* and usually denoted by $x_1, \ldots, x_n$. Every n-ary Boolean function is represented by a unique multilinear polynomial in n indeterminates over $\mathbf{GF}(2)$. If f is n-ary and $g_1, \ldots, g_n$ are all m-ary Boolean functions then the *composition* $f(g_1, \ldots, g_n)$ has arity m as well, and its value on $(a_1, \ldots, a_m)$ is $f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$. Throughout this paper Boolean addition is denoted by the ordinary addition symbol $+$.

A (*Boolean*) *clone* is a set of Boolean functions closed under composition and containing all coordinate projections (variables) of all arities. (We also use the term *class* to refer any set of Boolean functions.) For classical and recent references on clones see, e.g., Davio, Deschamps and Thayse [DDTh], Mal'cev [M], Pippenger [Pi1], Pöschel and Kalužnin [PK] and Zverovich [Z].

The definability of clones by functional equations was studied by Pogosyan [Po] and Foldes and Pogosyan [FPo].

Certain natural classes of Boolean functions constitute clones, some others do not. For example, the class $\mathbf{M}$ of monotone (increasing) functions is a clone while the class of decreasing functions is not. For our purposes we wish to consider in particular the following clones:

(i) the clone $\mathbf{L}$ of affine functions, i.e. functions of the form $c_1x_1+\ldots+c_nx_n+c$, from $\mathbf{B}^n$ to $\mathbf{B}$, for some $n \geq 1$, traditionally called linear functions in the theory of Boolean functions;
(ii) the clone $\mathbf{L}_0$ of linear functions in the sense of linear algebra, i.e. members of $\mathbf{L}$ for which $f(0,\ldots,0)=0$;
(iii) the clone $\mathbf{L}_{01}$ of those members of $\mathbf{L}_0$ for which $f(1,\ldots,1)=1$.

The functions in $\mathbf{L}_{01}$ are precisely those that can be represented as a sum of an odd number of variables. If x, y and z are any three distinct variables then the single function $x+y+z$ generates the entire clone $\mathbf{L}_{01}$ (i.e. no smaller clone contains $x+y+z$).

From standard linear algebra, applied to the vector space $\mathbf{B}^n = \mathbf{GF}(2)^n$ over the two-element field we shall need the following facts about *affine varieties* (cosets of subspaces of the vector space $\mathbf{B}^n$ plus the empty space, called *proper affine variety* if the coset is not $\mathbf{B}^n$ itself):

Fact 1. *A subset R of $\mathbf{B}^n$ is an affine variety if and only if R is closed under triple sums (i.e. $\mathbf{a}+\mathbf{b}+\mathbf{c} \in R$, whenever $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in R$).*

Fact 2. *If R is any non-empty affine variety in $\mathbf{B}^n$, then there is an affine projection onto R, i.e. a map $T : \mathbf{B}^n \to \mathbf{B}^n$ with range R and whose restriction to R is the identity map, and such that for some $n \times n$ matrix M and vector $\mathbf{d} \in \mathbf{B}^n$ we have, for all $\mathbf{x} \in \mathbf{B}^n$, $T(\mathbf{x}) = M\mathbf{x}+\mathbf{d}$.*

Fact 3. *Every proper affine variety in $\mathbf{B}^n$ is the intersection of some, finitely many, affine hyperplanes (cosets of $n-1$ dimensional subspaces).*

In general, a *minor* of a Boolean function f is a composite $f(g_1,\ldots,g_n)$. Wang and Williams [WW], Wang [W] and Pippenger [Pi2] consider minors when the inner functions g_i are "monadic", i.e. for which $g_i(x_1,\ldots,x_m)=0$ or $g_i(x_1,\ldots,x_m)=1$ or, for some j, $g_i(x_1,\ldots,x_m)=x_j$ or $g_i(x_1,\ldots,x_m)=x_j+1$. The relevance of the minor concept for definability by Boolean equations was made apparent in [EFHH] and in [Pi2]. In view of the linear functional equations that we are interested in, we propose the following variant and extension of the minor concept.

We say that $f(g_1,\ldots,g_n)$ is an $\mathbf{L}_{01}$-*minor* of f if all the inner functions g_i are in $\mathbf{L}_{01}$. A class $\mathcal{K}$ of Boolean functions is said to be *closed under the formation of* $\mathbf{L}_{01}$-*minors* if every $\mathbf{L}_{01}$-minor of every function in $\mathcal{K}$ is also in $\mathcal{K}$. It is easy to see that this is the case if and only if $f(g_1,\ldots,g_n)$ is in $\mathcal{K}$ whenever $f \in \mathcal{K}$ and each g_i is the sum of three variables, $g_i = x + y + z$. Such a composition is called a *substitution of triple sums of variables for variables in* f and it subsumes cylindrification (addition of inessential variables), permutation of variables and diagonalization (identification of variables).

While the characterization of classes of Boolean functions by equations reflects an approach rooted in model theory and universal algebra, clones in particular have been known to be definable by relational constraints (see Geiger [G], Davio, Deschamps and Thayse [DDTh] and Pippenger [Pi1]).

A *Boolean relation* R of arity t is any subset of $\mathbf{B}^t$, $t \geq 1$. For a matrix M with t rows, we write $M \prec R$ if all columns of M are in R. For an n-ary Boolean function f and a $t \times n$ matrix M with row vectors $\mathbf{a}_1, \ldots, \mathbf{a}_t$, we denote by fM the vector $(f(\mathbf{a}_1), \ldots, f(\mathbf{a}_t))$. A *relational constraint* is a couple (R, S) where R and S are Boolean relations of the same arity, called the *antecedent* and *consequent*, respectively. A Boolean function is said to *satisfy* a constraint (R, S) if $M \prec R$ implies $fM \prec S$, where the number of columns of M equals the arity of f. A set $\{(R_i, S_i) : i \in I\}$ of constraints is said to *define a class* $\mathcal{K}$ of Boolean functions if $\mathcal{K}$ is the class of those functions which satisfy every (R_i, S_i).

It was shown by Geiger [G] that clones are precisely the classes definable by sets of constraints of the form (R, R). The general theory of constraints is due to Pippenger [Pi2], who established a complete correspondence between constraints and the functional equations considered by Ekin, Foldes, Hammer, Hellerstein [EFHH], thus showing that the classes of Boolean functions definable by functional equations, in the sense of [EFHH], are exactly the classes definable by constraints. (A different concept of functional equations is adopted in [F], corresponding to the universal algebraic approach. Note also a generalization of constraints by Hellerstein [H].)

For every Boolean relation $R \subseteq \mathbf{B}^t$, there is a smallest affine variety $\bar{R}$ in $\mathbf{B}^t$ that contains R, traditionally called the *affine hull of* R. Observe the following fact:

Fact 4. *For any t-ary Boolean relation R, the affine hull $\bar{R}$ of R in $\mathbf{B}^t$ is given by $\bar{R} = \{gM : M \prec R, g \in \mathbf{L}_{01}$, the arity of g equals the number of columns of $M\}$.*

We use this fact to prove:

Lemma 1. *If every $\mathbf{L}_{01}$-minor of a Boolean function f satisfies a constraint (R, S), then f also satisfies $(\bar{R}, S)$, where $\bar{R}$ is the affine hull of R.*

Proof. Let n be the arity of f and let N be a matrix with n columns such that $N \prec \bar{R}$. Every column of N is an affine combination of columns in R, thus, for a sufficiently large m, it will be true that, for some matrix $M \prec R$ with m columns and some $g_1, \ldots, g_n$ in $\mathbf{L}_{01}$, the n columns of N are $g_1M, \ldots, g_nM$. Since $f' = f(g_1, \ldots, g_n)$ is an $\mathbf{L}_{01}$-minor of f, it follows that $fN = f'M \in S$. □

3. Linear definability

By a *linear functional equation* we mean a formal expression

(7) $c_1\mathbf{f}(c_{11}\mathbf{v}_1 + \ldots + c_{1m}\mathbf{v}_m + \mathbf{d}_1) + \ldots + c_q\mathbf{f}(c_{q1}\mathbf{v}_1 + \ldots + c_{qm}\mathbf{v}_m + \mathbf{d}_q) = d,$

where the subscripted c's and d belong to $\mathbf{B}$, $\mathbf{f}$ is a variable (functional variable), the $\mathbf{v}_i$'s are variables (vector variables) and the $\mathbf{d}_i$'s are among the two symbols $\mathbf{0}$ and $\mathbf{1}$. Given an n-ary function f and vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbf{B}^n$, by interpreting $\mathbf{f}$ as f, each $\mathbf{v}_i$ as $\mathbf{a}_i$ and each $\mathbf{d}_i$ as $(0, \ldots, 0)$ or $(1, \ldots, 1)$ in $\mathbf{B}^n$, the equation becomes true or false. We say that f *satisfies* (7) if the equation becomes true with $\mathbf{f}$ interpreted as f and for all interpretaions of the $\mathbf{v}_i$'s in $\mathbf{B}^n$. Observe that for every linear functional equation (7) there is one in which $m = q$, i.e. of the form (8) below, which is satisfied by exactly the same Boolean functions:

(8) $c_1\mathbf{f}(c_{11}\mathbf{v}_1 + \ldots + c_{1t}\mathbf{v}_t + \mathbf{d}_1) + \ldots + c_t\mathbf{f}(c_{t1}\mathbf{v}_1 + \ldots + c_{tt}\mathbf{v}_t + \mathbf{d}_t) = d.$

Such an equation (8) can be constructed from (7) by taking $c_{q+1} = \ldots = c_m = 0$ if $q < m$ or $c_{i,m+1} = \ldots = c_{i,q} = 0$ for all $i = 1, \ldots, q$ if $q > m$. Note that if $\mathbf{d}_i = \mathbf{1}$ then the interpretation of $c_{i1}\mathbf{v}_1 + \ldots + c_{it}\mathbf{v}_t + \mathbf{d}_i$ as a vector of $\mathbf{B}^n$ is the Boolean complement of the interpretation of $c_{i1}\mathbf{v}_1 + \ldots + c_{it}\mathbf{v}_t$.

A set $\mathcal{E}$ of linear functional equations is said to *define a class* $\mathcal{K}$ of Boolean functions if $\mathcal{K}$ is the class of those functions which satisfy every equation in $\mathcal{E}$. For example, the class of self-dual functions is definable by

$$1\mathbf{f}(1\mathbf{v} + \mathbf{0}) + 1\mathbf{f}(1\mathbf{v} + \mathbf{1}) = 1,$$

which is essentially nothing else but (4) in form (7).

Theorem 1. *For any class $\mathcal{K}$ of Boolean functions the following conditions are equivalent:*

- (i) *$\mathcal{K}$ is definable by some set of linear functional equations;*
- (ii) *$\mathcal{K}$ is definable by some set of constraints $\{(R_i, S_i) : i \in I\}$, where each constraint (R_i, S_i) consists, for some positive integer n_i, of affine varieties R_i and S_i in $\mathbf{B}^{n_i}$;*
- (iii) *$\mathcal{K}$ is closed under substituting variable triple sums $x + y + z$ for variables and forming the triple sum $f + g + h$, where f, g and h are functions of the same arity in $\mathcal{K}$.*

Proof. We shall establish the implications $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$.

To prove $(i) \Rightarrow (iii)$, assume (i).

Let us show closure under substituting variable triple sums $x + y + z$ for variables. This closure condition is equivalent to closure under formation of $\mathbf{L}_{01}$-minors. Let us prove that if f satisfies (8) then every $\mathbf{L}_{01}$-minor of f satisfies (8) as well. So assume that f satisfies (8) and take an m-ary $\mathbf{L}_{01}$-minor $f' = f(g_1, \ldots, g_n)$ of f.

Consider the map from $\mathbf{B}^m$ to $\mathbf{B}^n$ associating to the m-vector $\mathbf{a} = (a_1, \ldots, a_m)$ the n-vector $\mathbf{a}' = (g_1(\mathbf{a}), \ldots, g_n(\mathbf{a}))$. Obviously, for all $\mathbf{a}$, $f'(\mathbf{a}) = f(\mathbf{a}')$. Also, since each g_i is in $\mathbf{L}_{01}$, the map $\mathbf{a} \mapsto \mathbf{a}'$ is a linear transformation between the vector spaces $\mathbf{B}^m$ and $\mathbf{B}^n$ which sends the zero vector of $\mathbf{B}^m$ to that of $\mathbf{B}^n$ and the all-1 vector of $\mathbf{B}^m$ to that of $\mathbf{B}^n$. Let now $\mathbf{a}_1, \ldots, \mathbf{a}_t$ be any vectors of $\mathbf{B}^m$ and denote by $\mathbf{d}_i$ the zero or the all-1 vector of $\mathbf{B}^m$ according to whether the symbol $\mathbf{d}_i$ in (8) is $\mathbf{0}$ or $\mathbf{1}$. We have:

$$\begin{aligned}
&c_1 f'(c_{11}\mathbf{a}_1 + \ldots + c_{1t}\mathbf{a}_t + \mathbf{d}_1) + \ldots + c_t f'(c_{t1}\mathbf{a}_1 + \ldots + c_{tt}\mathbf{a}_t + \mathbf{d}_t)) = \\
&= c_1 f[(c_{11}\mathbf{a}_1 + \ldots + c_{1t}\mathbf{a}_t + \mathbf{d}_1)'] + \ldots + c_t f[(c_{t1}\mathbf{a}_1 + \ldots + c_{tt}\mathbf{a}_t + \mathbf{d}_t)'] = \\
&c_1 f(c_{11}\mathbf{a}'_1 + \ldots + c_{1t}\mathbf{a}'_t + \mathbf{d}'_1) + \ldots + c_t f(c_{t1}\mathbf{a}'_1 + \ldots + c_{tt}\mathbf{a}'_t + \mathbf{d}'_t) \quad (*)
\end{aligned}$$

But (*) must be equal to the constant d appearing on the right hand side of (8) because f satisfies (8), and this shows that f' also satisfies (8) as claimed.

To conclude the proof of $(i) \Rightarrow (iii)$, one can easily see that if Boolean functions f, g and h of the same arity satisfy (8), then their sum $f + g + h$ also satisfies (8).

To prove $(iii) \Rightarrow (ii)$, assume (iii). We need to show that, for every Boolean function g not in $\mathcal{K}$, there is a constraint $(R, S) = (R_g, S_g)$ such that:

(a) every f in $\mathcal{K}$ satisfies (R, S);
(b) g does not satisfy (R, S);
(c) both R and S are affine varieties in some $\mathbf{B}^m$.

The set of various constraints (R_g, S_g), for all $g \notin \mathcal{K}$ will then define $\mathcal{K}$. This approach and construction are, essentially, due to Geiger [G] and Pippenger [Pi2], with the additional requirement that both antecedent and consequent need to be affine varieties.

So, given $g \notin \mathcal{K}$, say of arity n, let M be a $2^n \times n$ matrix whose rows are the various vectors of $\mathbf{B}^n$. Let R_0 be the set of columns of M and let $S = \{fM : f \in \mathcal{K}, f \ n\text{-ary}\}$ as in [Pi2], Theorem 2.1: every function in $\mathcal{K}$ satisfies (R_0, S) and g does not satisfy (R_0, S). As $\mathcal{K}$ is closed under triple sum of functions $f_1 + f_2 + f_3$ where $f_1, f_2, f_3 \in \mathcal{K}$, it follows, using Fact 1, that S is an affine variety in $\mathbf{B}^{2^n}$. Let now R be defined as the affine hull of R_0. By Lemma 1, the constraint (R, S) satisfies all the three conditions (a), (b) and (c) with $m = 2^n$.

Finally, to prove $(ii) \Rightarrow (i)$, assume (ii). As constraints with consequents S_i equal to the whole space $\mathbf{B}^{n_i}$ can be discarded as superfluous, we may assume that each S_i is a proper affine variety in the corresponding $\mathbf{B}^{n_i}$.

There exists then $k_i \geq 1$ affine hyperplanes in $\mathbf{B}^{n_i}$, say $H_1, \ldots, H_{k_i}$, the intersection of which is S_i. Obviously a function satisfies (R_i, S_i) if and only if it satisfies every one of the k_i constraints $(R_i, H_1), \ldots, (R_i, H_{k_i})$. Thus we can replace the set of constraints (R_i, S_i) defining $\mathcal{K}$ by a set of constraints in which each antecedent is an affine variety and each consequent is an affine hyperplane. All there remains to prove is that for every such constraint (R, H), there is a linear equation (8) that is satisfied by exactly the same Boolean functions as (R, H). Again, we need to modify a corresponding more generic construction of Pippenger [Pi2], to accommodate the linearity requirement.

So let t be the arity of (R, H). First, in $\mathbf{B}^t$, take an affine projection onto R, i.e. an affine transformation $T : \mathbf{B}^t \to \mathbf{B}^t$, whose range is R and that is idempotent, $T^2 = T$ (such an idempotent projection exists according to Fact 2). This transformation T is represented by a $t \times t$ matrix M and a vector $(d_1, \ldots, d_t)$ in R, so that, for all $(a_1, \ldots, a_t) \in \mathbf{B}^t$,

$$T \begin{pmatrix} a_1 \\ \vdots \\ a_t \end{pmatrix} = M \begin{pmatrix} a_1 \\ \vdots \\ a_t \end{pmatrix} + \begin{pmatrix} d_1 \\ \vdots \\ d_t \end{pmatrix}. \tag{9}$$

Second, consider the characteristic function Γ of the set $\mathbf{B}^t \setminus H$ in $\mathbf{B}^t$. This is a linear Boolean function $\Gamma : \mathbf{B}^t \to \mathbf{B}$, i.e. there are $c_1, \ldots, c_t, d \in \mathbf{B}$ such that, for all $(a_1, \ldots, a_t) \in \mathbf{B}^t$,

$$\Gamma \begin{pmatrix} a_1 \\ \vdots \\ a_t \end{pmatrix} = c_1 a_1 + \ldots + c_t a_t + d = \begin{cases} 1 & \text{if } (a_1, \ldots, a_t) \in \mathbf{B}^t \setminus H \\ 0 & \text{if } (a_1, \ldots, a_t) \in H. \end{cases} \tag{10}$$

In fact, $(a_1, \ldots, a_t) \mapsto c_1 a_1 + \ldots + c_t a_t + d + 1$ is the characteristic function of the hyperplane H.

Let $M = (c_{ij})_{1 \leq i,j \leq t}$. These Boolean constants c_{ij}, together with the $c_1, \ldots, c_t$ and d defined in (10), give a linear functional equation (8) if we specify, for every $j = 1, \ldots, t$, $\mathbf{d}_j = \mathbf{0}$ if $d_j = 0$, and $\mathbf{d}_j = \mathbf{1}$ if $d_j = 1$. The equation (8) thus defined is indeed a linear functional equation. Let us see that it is satisfied by the same Boolean functions that satisfy the constraint (R, H). So suppose that the n-ary function f satisfies (R, H) and take $\mathbf{a}_1, \ldots, \mathbf{a}_t \in \mathbf{B}^n$. We have to prove that

$$c_1 f(c_{11}\mathbf{a}_1 + \ldots + c_{1t}\mathbf{a}_t + \mathbf{d}_1) + \ldots + c_t f(c_{t1}\mathbf{a}_1 + \ldots + c_{tt}\mathbf{a}_t + \mathbf{d}_t) + d = 0, \tag{11}$$

where $\mathbf{d}_j, 1 \leq j \leq t$, is the vector $(0, \ldots, 0)$ or $(1, \ldots, 1)$ in $\mathbf{B}^n$ according to whether $\mathbf{d}_j$ is $\mathbf{0}$ or $\mathbf{1}$ in (8). Consider the $t \times n$ matrix N whose rows are $\mathbf{a}_1, \ldots, \mathbf{a}_t$. Let TN denote the $t \times n$ matrix obtained by applying the affine transformation T to each of the n columns of N. As $TN \prec R$, $fTN \in H$. Thus, (11) holds.

On the other hand, if g does not satisfy (R, H), then, for some $N \prec R$, $gN \notin H$. So, $\Gamma(gN) = 1$. Writing $gN = (a_1, \ldots, a_t)$, we have by (10), $\Gamma(gN) = c_1 a_1 + \ldots + c_t a_t + d = 1$ which shows that g does not satisfy (8). □

It is now easy to see which clones can be defined by linear functional equations. It is immediate that the following satisfy condition (i) of Theorem 1: the clone of all Boolean functions, the four maximal clones of the self-dual, linear, zero-preserving and one-preserving Boolean functions, together with the six clones that can be obtained from these by taking intersections. No other clone $\mathcal{K}$ satisfies condition (iii) of Theorem 1 because no other clone $\mathcal{K}$ contains $x + y + z$.

Note, however, that there are infinitely many classes of Boolean functions satisfying the conditions of Theorem 1 which are not clones. For example, for every m, the class of those Boolean functions whose polynomial representation over $\mathbf{GF}(2)$ has degree bounded by m, clearly satisfies condition (iii) of Theorem 1.

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ON AFFINE CONSTRAINTS SATISFIED BY BOOLEAN FUNCTIONS

MIGUEL COUCEIRO AND STEPHAN FOLDES

ABSTRACT. Closure under formation of relational minors is shown to characterize those sets of affine constraints which are satisfied by sets of Boolean functions

1. INTRODUCTION

In [P] Pippenger introduced the concept of a finite function satisfying a relational constraint, thereby generalizing the theory of Geiger [G] and establishing a Galois connection between finite functions and relational constraints. This includes, in particular, the case of the two-element set $\{0,1\}$ and provides a streamlined proof of the characterization of equational classes of Boolean functions given by Ekin, Foldes, Hammer, Hellerstein [EFHH]. In this note we continue to focus on Boolean functions but follow the framework of Pippenger. For definitions and background we refer to [P] as well as [CF].

The closed sets of functions and constraints of the Galois connection described in [P] are both characterized by closure under certain algebraic operations closely related to those introduced by Mal'cev in [M]. In [CF] a special type of "affine constraints" is introduced in order to characterize classes of Boolean functions definable by linear equations, where linearity refers to the field structure of $\{0,1\} = \mathbf{GF}(2)$. Such classes of Boolean functions are characterized in [CF] by Mal'cev-Pippenger type closure conditions of a linear algebraic form. To each such class obviously corresponds a set of affine constraints (the affine constraints satisfied by all members of the function class), and these classes of affine constraints constitute a closure system which is dual to the closure system of classes of Boolean functions definable by linear equations.

In this note we show that the algebraic closure conditions for affine constraints are essentially the same as Pippenger's conditions for general constraints, with an adjustement only for linearity. The proof also makes use of Theorem 2.2 in [P].

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2. Closure Conditions

All relational constraints are over $\{0,1\} = \mathbf{GF}(2)$. If a constraint (R,S) is obtained from (R',S') by restricting the antecedent or extending the consequent or both (i.e. $R \subseteq R'$ and $S \supseteq S'$) we say that (R,S) is a *relaxation* of (R',S'). Constraints in which both antecedent and consequent are affine varieties were considered in [CF]. In the sequel we shall refer to these as *affine constraints*. A relaxation (R,S) of (R',S') will be called an *affine relaxation* if (R,S) is an affine constraint. It is easy to verify that any simple minor of an affine constraint is affine. Observe also that the empty constraint and all the equality constraints are affine, and that the intersection of consequents of affine constraints is always affine. A set $\mathcal{A}$ of affine constraints is said to be *closed under taking affine relaxations* if every affine relaxation of every constraint in $\mathcal{A}$ is also in $\mathcal{A}$.

We say that a set of affine constraints is *affine minor closed* if it contains the empty constraint, the binary equality constraint, and it is closed under taking simple minors, intersecting consequents and under taking affine relaxations. Observe that these closure conditions differ from Pippenger's conditions for minor-closed sets of constraints only as far as relaxations are concerned. This adjustement is necessary since affine constraints have both affine and non-affine relaxations.

Clearly, the affine members of any minor closed sets of constraints constitute an affine minor closed set of affine constraints. The converse is also true and it will be needed in the sequel:

Lemma 1. *Let $\mathcal{T}_a$ be an affine minor closed set of affine constraints and define $\mathcal{T}$ to be the set of relaxations of the various constraints in $\mathcal{T}_a$. Then the following hold:*

(a) *$\mathcal{T}$ is minor closed;*
(b) *$\mathcal{T}_a$ is the set of affine constraints which are in $\mathcal{T}$.*

Proof. Every constraint is a relaxation of itself, thus, (b) holds.

To prove that (a) also holds, it is enough to show that $\mathcal{T}$ is closed under intersecting consequents and closed under taking simple minors. To see that $\mathcal{T}$ is indeed closed under taking simple minors, take an n-ary constraint (R',S') in $\mathcal{T}$ and suppose that the m-ary constraint (R,S) is a simple minor of (R',S'), i.e. there is p, $0 \leq p \leq n$, and $h : \{1,\ldots,n\} \to \{1,\ldots,m+p\}$ such that

$$R\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \quad \Leftrightarrow \quad \exists x_{m+1}\ldots\exists x_{m+p} \quad R'\begin{pmatrix} x_{h(1)} \\ \vdots \\ x_{h(n)} \end{pmatrix}$$

and

$$S\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \quad \Leftrightarrow \quad \exists x_{m+1}\ldots\exists x_{m+p} \quad S'\begin{pmatrix} x_{h(1)} \\ \vdots \\ x_{h(n)} \end{pmatrix}.$$

We have to prove that $(R,S) \in \mathcal{T}$.

Now, since $(R',S') \in \mathcal{T}$, there is an affine constraint $({R'}_a,{S'}_a) \in \mathcal{T}_a$ such that (R',S') is a relaxation of $({R'}_a,{S'}_a)$. So, let (R_a,S_a) be the simple minor of $({R'}_a,{S'}_a)$ defined by the map h above. $\mathcal{T}_a$ is affine minor closed, therefore, $(R_a,S_a) \in \mathcal{T}_a$.

So, let us prove that (R,S) is a relaxation of (R_a,S_a) and, thus, that $(R,S) \in \mathcal{T}$. First, take an m-ary vector $(a_1,\ldots,a_m)$ in R. By definition, there are $a_{m+1},\ldots,a_{m+p}$ such that $(a_{h(1)},\ldots,a_{h(n)})$ is in R'. Since $R' \subseteq {R'}_a$, we have that $(a_{h(1)},\ldots,a_{h(n)})$ is in ${R'}_a$. Thus, by construction, we conclude that $(a_1,\ldots,a_m)$ is in R_a.

By analogous reasoning and taking $(b_1,\ldots,b_m)$ in S_a, one can easily conclude that $(b_1,\ldots,b_m)$ is in S. Summing up, we have $R \subseteq R_a$ and $S \supseteq S_a$. In other words, (R,S) is a relaxation of (R_a,S_a) and so $(R,S) \in \mathcal{T}$.

To see that $\mathcal{T}$ is indeed closed under intersecting consequents, let $(R,S_i)_{i\in I}$ be a family of n-ary relational constraints with members in $\mathcal{T}$, i.e. for each $i \in I$ there is an m-ary affine contraint $(R_{a,i},S_{a,i}) \in \mathcal{T}_a$ such that $R \subseteq R_{a,i}$ and $S_i \supseteq S_{a,i}$. Consider the m-ary relational constraint

$$(\bigcap_{i\in I} R_{a,i}, \bigcap_{i\in I} S_{a,i})$$

Since the intersection of affine varieties is an affine variety, we have

$$(\bigcap_{i\in I} R_{a,i}, \bigcap_{i\in I} S_{a,i}) \in \mathcal{T}_a$$

Moreover,

$$R \subseteq \bigcap_{i\in I} R_{a,i} \text{ and } \bigcap_{i\in I} S_i \supseteq \bigcap_{i\in I} S_{a,i}$$

and thus $(R, \bigcap_{i\in I} S_i) \in \mathcal{T}$. In other words, $\mathcal{T}$ is closed under intersecting consequents.[1] □

We make use of Lemma 1 and Pippenger's Theorem 2.2 in [P] to prove the following:

Theorem 1. *Let $\mathcal{T}_a$ be a set of affine constraints. Then the following are equivalent:*

(i) *There is a set of Boolean functions which satisfy exactly those affine constraints that are in $\mathcal{T}_a$;*
(ii) *$\mathcal{T}_a$ is affine minor closed.*

[1]Paragraph added May 2006.

Proof. The proof of the implication $(i) \Rightarrow (ii)$ is essentially the same as the proof of Pippenger [P] that establishes minor closure for the set of all constraints satisfied by given functions. It follows immediately, from the definition of functional satisfiability of relational constraints, that:

- every Boolean function satisfies the empty and the equality constraints;
- if a Boolean function satisfies an affine constraint then it satisfies its affine relaxations;
- if a Boolean function satisfies two affine constraints (R, S_1) and (R, S_2) then it satisfies the affine constraint $(R, S_1 \cap S_2)$ obtained by intersecting the consequents.

Also, if a Boolean function satisfies an affine constraint then it clearly satisfies its simple minors. Summing up, we have that the set of affine constraints which are satisfied by a Boolean function is affine minor closed. We conclude that $(i) \Rightarrow (ii)$.

To prove the converse, first we show that, for every affine constraint (R, S) not in $\mathcal{T}_a$, there is a Boolean function g such that

1) g satisfies every constraint in $\mathcal{T}_a$, and
2) g does not satisfy (R, S).

So, let $\mathcal{T}$ be the set of relaxations of the various affine constraints in $\mathcal{T}_a$. Observe that $(R, S) \notin \mathcal{T}$ (otherwise (R, S) would be an affine relaxation of some affine constraint in $\mathcal{T}_a$, contradicting the assumption that $\mathcal{T}_a$ is affine minor closed). Now, by Lemma 1, we have that $\mathcal{T}$ is minor closed. By Theorem 2.2 in [P], there is a Boolean function g such that g does not satisfy (R, S) and g satisfies every constraint in $\mathcal{T}$ and thus, in particular, g satisfies every constraint in $\mathcal{T}_a$. $\square$

Together with [CF] the above theorem completes the description of the Galois connection between Boolean functions and affine constraints. The Galois connection is induced by the constraint satisfiability relation between functions and affine constraints. Theorem 1 of [CF] characterized the closed sets of Boolean functions and the theorem in this note characterizes the closed sets of affine constraints.

3. Generalization

Let us replace the property "affine" by an abstract "Property A" of Boolean constraints and assume that

(i) the empty and the binary equality constraints have Property A,
(ii) intersecting antecedents and intersecting consequents of constraints having Property A always yields a constraint having Property A,[2]
(iii) all simple minors of constraints having Property A have Property A.

[2]Sentence modified May 2006.

Then Lemma 1 and Theorem 1 of this note continue to hold if we replace

- "affine constraint" by "constraint having Property A",
- "affine relaxation" by "relaxation having Property A",
- "affine minor closed" by "containing the empty and equality constraints, closed under taking simple minors, intersecting consequents and under taking relaxations having Property A".

For example, Property A could be defined as both antecedent and consequent being invariant under monotone Boolean functions.[3]

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ON CLOSED SETS OF RELATIONAL CONSTRAINTS AND CLASSES OF FUNCTIONS CLOSED UNDER VARIABLE SUBSTITUTIONS

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Abstract. Pippenger's Galois theory of finite functions and relational constraints is extended to the infinite case. The functions involved are functions of several variables on a set A and taking values in a possibly different set B, where any or both of A and B may be finite or infinite.

1. Basic Concepts and Terminology

In [G] Geiger determined, by explicit closure conditions, the closed classes of endofunctions of several variables (operations) and the closed classes of relations (predicates) on a finite set A. These two dual closure systems are related in a Galois connection given by the "preservation" relation between endofunctions and relations. This Galois theory was also developed independently by Bodnarčuk, Kalužnin, Kotov and Romov in [BKKR]. Removing the finiteness restriction on the underlying set, in [Sz] Szabó characterized the closed classes of endofunctions and closed sets of relations on arbitrary sets. These characterizations involve a local closure property as well as closure under a general scheme of combining families of relations into a new relation, properly extending the schemes described by Geiger in the case of finite sets. Different approaches and formulations, as well as variant Galois theories were developed by Pöschel in [Pö2], [Pö3], and [PK] in the case of finite sets (see also [R] and [B] for further extensions).

There are many natural classes of functions that can not be defined by preservation of a single relation (or preservation of each member of a family of relations), e.g. monotone decreasing functions on an ordered set, or Boolean functions whose Zhegalkin polynomial has degree at most $m \geq 0$[1]. However such classes can often be described as consisting of those functions that "transform" one relation to another relation. Also, many natural classes are not classes of endofunctions, the sets in which the function variables are interpreted being different from the codomain of function values, e.g. rank functions of matroids. In the case of finite sets a theory of such functions of

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[1] $m \neq 1$ (footnote added May 2006)

several variables, defined as functions from a cartesian product $A_1 \times \ldots \times A_n$ of finite sets to a finite set B, was developed by Pöschel in [Pö1]: relations as ordinarily understood are replaced by tuples of relations, then the notion of preservation of relations is naturally extended to such multisorted functions and relational tuples, and the closed classes of functions and relational tuples are determined with respect to the arising Galois connection. Still in the case of finite sets, in [Pi2] Pippenger developed a particular Galois theory for functions $A^n \to B$, where the dual object role of relations is replaced by ordered pairs of relations called "constraints". In this paper we extend this latter theory by removing the finiteness restriction.

The functions of several variables we consider in this paper are defined on arbitrary sets, not necessarily finite, taking values in another, possibly different and possibly infinite set. The relations and relational constraints that we consider are also defined on arbitrary, not necessarily finite sets. Positive integers are thought of as ordinals according to the von Neumann conception, i.e. each ordinal is just the set of lesser ordinals. Thus, for a positive integer n and a set A, the *n-tuples* in A^n are formally maps from $\{0, \ldots, n-1\}$ to A. The notation $(a_t \mid t \in n)$ means the n-tuple mapping t to a_t for each $t \in n$. The notation $(b^1 \ldots b^n)$ means the n-tuple mapping t to b^{t+1} for each $t \in n$. A map (function) is always thought of as having a specific domain, codomain and graph. We need this formalism in order to streamline certain definitions and arguments in later sections of this paper.

Consider arbitrary non-empty sets A and B.

A *B-valued function of several variables on A* (or simply, *B-valued function on A*) is a map $f : A^n \to B$, where the *arity* n is a positive integer. Thus the set of all B-valued functions on A is $\cup_{n \geq 1} B^{A^n}$. We also use the term *class* for a set of functions. If $A = B$, then the B-valued functions on A are called *operations on A*. For a fixed arity n, the n different *projection maps* $\mathbf{a} = (a_t \mid t \in n) \mapsto a_i$, $i \in n$, are also called *variables*.

If l is a map from n to m then the m-ary function g defined by

$$g(\mathbf{a}) = f(\mathbf{a} \circ l)$$

for every m-tuple $\mathbf{a} \in A^m$, is said to be obtained from the n-ary function f by *simple variable substitution*. Note that this subsumes cylindrification (addition of inessential variables), permutation of variables and diagonalization (identification of variables), see e.g. [M], [Pi1], and [Pö1]. A class $\mathcal{K}$ of functions of several variables is said to be *closed under simple variable substitutions* if each function obtained from a function f in $\mathcal{K}$ by simple variable substitution is also in $\mathcal{K}$. Variable substitution plays a significant role in a number of studies of function classes and class definability (see e.g. [WW, W, EFHH, F, Pi2, Z]).

For a positive integer m, an *m-ary relation on A* is a subset R of A^m. For an m-tuple $\mathbf{a}$ we write $R(\mathbf{a})$ if $\mathbf{a} \in R$. An $m \times n$ matrix M with entries in A is thought of as an n-tuple of m-tuples, $M = (\mathbf{a}^1 \ldots \mathbf{a}^n)$. The m-tuples $\mathbf{a}^1, \ldots, \mathbf{a}^n$ are called *columns* of M. For $i \in m$, the n-tuple $(\mathbf{a}^1(i) \ldots \mathbf{a}^n(i))$ is called *row* i of M . For a matrix M with entries in A, we write $M \prec R$ if all columns of M are in R. For an n-ary function $f \in B^{A^n}$ and an $m \times n$ matrix $M = (\mathbf{a}^1 \dots \mathbf{a}^n)$, we denote by fM the m-tuple $(f(\mathbf{a}^1(i) \dots \mathbf{a}^n(i)) \mid i \in m)$ in B^m. Also, we denote by fR the m-ary relation on B given by

$$fR = \{fM : \ m \times n \text{ matrix } M \prec R\}.$$

2. Classes of Functions of Several Variables Definable by Relational Constraints

Consider arbitrary non-empty sets A and B. An m-ary *A-to-B relational constraint* (or simply, m-ary *constraint*, when the underlying sets are understood from the context) is an ordered pair (R, S) where $R \subseteq A^m$ and $S \subseteq B^m$. The relations R and S are called *antecedent* and *consequent*, respectively, of the constraint. A function of several variables $f : A^n \to B$, $n \geq 1$, is said to *satisfy* an m-ary A-to-B constraint (R, S) if $fR \subseteq S$. For general background see [Pi2].

A class $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ of B-valued functions on A is said to be *definable* by a set $\mathcal{S}$ of A-to-B constraints, if $\mathcal{K}$ is the class of all functions which satisfy every member of $\mathcal{S}$.

A class $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ of B-valued functions on A is said to be *locally closed* if for every B-valued function f of several variables on A the following holds: if every restriction of f to a finite subset of its domain A^n coincides with a restriction of some member of $\mathcal{K}$, then f belongs to $\mathcal{K}$.

Theorem 1. *Consider arbitrary non-empty sets A and B. For any class of functions $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ the following conditions are equivalent:*

(i) $\mathcal{K}$ is locally closed and it is closed under simple variable substitutions;
(ii) $\mathcal{K}$ is definable by some set of A-to-B constraints.

Proof. $(ii) \Rightarrow (i)$: As observed in the finite case by Pippenger in [Pi2], it is easy to see, also in general, that if a function f satisfies a constraint (R, S) then every function obtained from f by simple variable substitution also satisfies (R, S). Thus, any function class $\mathcal{K}$ definable by a set of constraints is closed under simple variable substitutions.

To show that $\mathcal{K}$ is locally closed, consider $f \notin \mathcal{K}$ and let (R, S) be an A-to-B constraint that is not satisfied by f but satisfied by every function g in $\mathcal{K}$. Thus for some matrix $M \prec R$, $fM \notin S$ but $gM \in S$ for every $g \in \mathcal{K}$. So there is a finite restriction of f, namely its restriction to the set of rows of M, which does not coincide with that of any member of $\mathcal{K}$.

$(i) \Rightarrow (ii)$: We need to show that, for every function g not in $\mathcal{K}$, there is an A-to-B constraint (R, S) such that:

a) every f in $\mathcal{K}$ satisfies (R, S),
b) g does not satisfy (R, S).

The case $\mathcal{K} = \emptyset$ being trivial, assume that $\mathcal{K}$ is non-empty. Suppose that g is n-ary. Since $g \notin \mathcal{K}$, there is a finite restriction g_F of g to a finite subset $F \subseteq A^n$ such that g_F disagrees with every function in $\mathcal{K}$ restricted to F.

Clearly, F is non empty. So let M be a $\mid F \mid \times n$ matrix whose rows are the various n-tuples in F. Following Geiger's strategy, also used by Pippenger, define R to be the set of columns of M and let $S = \{fM : f \in \mathcal{K}, f \ n\text{-ary}\}$. It is clear from the above construction that (R, S) is an A-to-B constraint, and, since $\mathcal{K}$ is closed under simple variable substitutions, every function in $\mathcal{K}$ satisfies (R, S). Also, g_F does not satisfy (R, S), therefore g does not satisfy (R, S) either. Thus, conditions a) and b) hold for the constraint (R, S). □

This generalizes the characterization of closed classes of functions given by Pippenger in [Pi2] by allowing both finite and infinite underlying sets.

3. Sets of Relational Constraints Characterized by Functions

The following constructions on maps will be needed.

For maps $f : A \to B$ and $g : C \to D$, the composition $g \circ f$ is defined only if $B = C$. Removing this restriction we define the *concatenation* of f and g, denoted simply gf, to be the map with domain $f^{-1}[B \cap C]$ and codomain D given by $(gf)(a) = g(f(a))$ for all $a \in f^{-1}[B \cap C]$. Clearly, if $B = C$ then $gf = g \circ f$, thus concatenation subsumes and extends functional composition. Concatenation is associative, i.e. for any maps f, g, h we have $h(gf) = (hg)f$.

Given a family $(g_i)_{i\in I}$ of maps, $g_i : A_i \to B_i$ such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$, we call *(piecewise) sum of the family* $(g_i)_{i\in I}$, denoted $\Sigma_{i\in I} g_i$, the map from $\cup_{i\in I} A_i$ to $\cup_{i\in I} B_i$ whose restriction to each A_i agrees with g_i. If I is a two-element set, say $I = \{1, 2\}$, then we write $g_1 + g_2$. Clearly, this operation is associative and commutative.

The operations of concatenation and summation are linked by distributivity, i.e. for any family $(g_i)_{i\in I}$ of maps on disjoint domains and any map f

$$(\Sigma_{i\in I} g_i)f = \Sigma_{i\in I}(g_i f) \qquad \text{and} \qquad f(\Sigma_{i\in I} g_i) = \Sigma_{i\in I}(f g_i).$$

In particular, if g and g' are maps with disjoint domains, then

$$(g + g')f = (gf) + (g'f) \qquad \text{and} \qquad f(g + g') = (fg) + (fg').$$

Let $g_1, \ldots, g_n$ be maps from A to B. The n-tuple $(g_1 \ldots g_n)$ determines a *vector-valued map* $g : A \to B^n$, given by $g(a) = (g_1(a) \ldots g_n(a))$ for every $a \in A$. If f is an n-ary C-valued function on B then the composition $f \circ g$ is a map from A to C, it is traditionally denoted by $f(g_1 \ldots g_n)$ and called the *composition of* f *with* $g_1, \ldots, g_n$. Suppose now that $A \cap A' = \emptyset$ and ${g'}_1, \ldots, {g'}_n$ are maps from A' to B. Letting g and g' be the vector-valued maps determined by $(g_1 \ldots g_n)$ and $({g'}_1 \ldots {g'}_n)$, respectively, we have that $f(g + g') = (fg) + (fg')$, i.e.

$$f((g_1 + {g'}_1) \ldots (g_n + {g'}_n)) = f(g_1 \ldots g_n) + f({g'}_1 \ldots {g'}_n).$$

For $B \subseteq A$, ι_{AB} denotes the canonical injection (inclusion map) from B to A. Thus the restriction $f\mid_B$ of any map $f : A \to C$ to the subset B is given by $f\mid_B = f\iota_{AB}$.

To discuss closed sets of constraints we need the following concepts.

We denote the binary equality relation on a set A by $=_A$. The *binary A-to-B equality constraint* is $(=_A, =_B)$. A constraint (R, S) is called the *empty constraint* if both antecedent and consequent are empty. For every $m \geq 1$, the constraints (A^m, B^m) are said to be *trivial.* Note that every B-valued function on A satisfies each of these constraints.

A constraint (R, S) is said to be obtained from (R_0, S_0) by *restricting the antecedent* if $R \subseteq R_0$ and $S = S_0$. Similarly, a constraint (R, S) is said to be obtained from (R_0, S_0) by *extending the consequent* if $S \supseteq S_0$ and $R = R_0$. If a constraint (R, S) is obtained from (R_0, S_0) by restricting the antecedent or extending the consequent or a combination of the two (i.e. $R \subseteq R_0$ and $S \supseteq S_0$) we say that (R, S) is a *relaxation* of (R_0, S_0). Given a non-empty family of constraints $(R, S_j)_{j\in J}$ of the same arity (and antecedent), the constraint $(R, \cap_{j\in J} S_j)$ is said to be obtained from $(R, S_j)_{j\in J}$ by *intersecting consequents.*

The above operations were introduced by Pippenger in [Pi2] in the context of finite sets, together with the notion of "simple minors". We propose a minor formation concept which extends and subsumes these operations. This concept is closely related to the construction of relations via the "formula schemes" of Szabó (see [Sz]) and the "general superpositions" of Pöschel (see e.g. [Pö2], [Pö3]). We shall discuss this relationship in Section 5.

Let m and n be positive integers (viewed as ordinals, i.e., $m = \{0, \ldots, m-1\}$). Let $h : n \to m \cup V$ where V is an arbitrary set of symbols disjoint from the ordinals called "*existentially quantified indeterminate indices*", or simply *indeterminates*, and $\sigma : V \to A$ any map called a *Skolem map.* Then each m-tuple $\mathbf{a} \in A^m$, being a map $\mathbf{a} : m \to A$, gives rise to an n-tuple $(\mathbf{a} + \sigma)h \in A^n$.

Let $H = (h_j)_{j\in J}$ be a non-empty family of maps $h_j : n_j \to m \cup V$, where each n_j is a positive integer (recall $n_j = \{0, \ldots, n_j - 1\}$). Then H is called a *minor formation scheme* with *target* m, *indeterminate set* V and *source family* $(n_j)_{j\in J}$. Let $(R_j)_{j\in J}$ be a family of relations (of various arities) on the same set A, each R_j of arity n_j, and let R be an m-ary relation on A. We say that R is a *restrictive conjunctive minor* of the family $(R_j)_{j\in J}$ *via* H, or simply a *restrictive conjunctive minor* of the family $(R_j)_{j\in J}$, if for every m-tuple $\mathbf{a}$ in A^m, the condition $R(\mathbf{a})$ implies that there is a Skolem map $\sigma : V \to A$ such that, for all j in J, we have $R_j[(\mathbf{a}+\sigma)h_j]$. On the other hand, if for every m-tuple $\mathbf{a}$ in A^m, the condition $R(\mathbf{a})$ holds whenever there is a Skolem map $\sigma : V \to A$ such that, for all j in J, we have $R_j[(\mathbf{a}+\sigma)h_j]$, then we say that R is an *extensive conjunctive minor* of the family $(R_j)_{j\in J}$ *via* H, or simply an *extensive conjunctive minor* of the family $(R_j)_{j\in J}$. If R is both a restrictive conjunctive minor and an extensive conjunctive minor of the family $(R_j)_{j\in J}$ via H, then R is said to be a *tight conjunctive minor* of the family $(R_j)_{j\in J}$ *via* H, or *tight conjunctive minor* of the family. Note that given a scheme H and a family $(R_j)_{j\in J}$, there is a unique tight conjunctive minor of the family $(R_j)_{j\in J}$ via H.

If $(R_j, S_j)_{j\in J}$ is a family of A-to-B constraints (of various arities) and (R, S) is an A-to-B constraint such that for a scheme H

(i) R is a restrictive conjunctive minor of $(R_j)_{j\in J}$ via H,
(ii) S is an extensive conjunctive minor of $(S_j)_{j\in J}$ via H,

then (R, S) is said to be a *conjunctive minor* of the family $(R_j, S_j)_{j\in J}$ *via* H, or simply a *conjunctive minor* of the family of constraints.

If both R and S are tight conjunctive minors of the respective families via H, the constraint (R, S) is said to be a *tight conjunctive minor* of the family $(R_j, S_j)_{j\in J}$ *via* H, or simply a *tight conjunctive minor* of the family of constraints. Note that given a scheme H and a family $(R_j, S_j)_{j\in J}$, there is a unique tight conjunctive minor of the family via the scheme H.

An important particular case of tight conjunctive minors is when the minor formation scheme $H = (h_j)_{j\in J}$ and the family $(R_j, S_j)_{j\in J}$ are indexed by a singleton $J = \{0\}$. In this case, a tight conjunctive minor (R, S) of a family containing a single constraint (R_0, S_0) is called a *simple minor* of (R_0, S_0) according to the concept introduced by Pippenger in [Pi2].

Lemma 1. *Let (R, S) be a conjunctive minor of a non-empty family $(R_j, S_j)_{j\in J}$ of A-to-B constraints. If $f : A^n \to B$ satisfies every (R_j, S_j) then f satisfies (R, S).*

Proof. Let (R, S) be an m-ary conjunctive minor of the family $(R_j, S_j)_{j\in J}$ via the scheme $H = (h_j)_{j\in J}$, $h_j : n_j \to m \cup V$. Let $M = (\mathbf{a}^1 \dots \mathbf{a}^n)$ be an $m \times n$ matrix with columns in R. We need to prove that the m-tuple fM belongs to S. Note that the m-tuple fM, being a map defined on m, is in fact the composition of f with the m-tuples $\mathbf{a}^1, \dots, \mathbf{a}^n$, i.e. $fM = f(\mathbf{a}^1 \dots \mathbf{a}^n)$. Since R is a restrictive conjunctive minor of $(R_j)_{j\in J}$ via $H = (h_j)_{j\in J}$, there are Skolem maps $\sigma_i : V \to A$, $1 \le i \le n$, such that for every j in J, for the matrix $M_j = ((\mathbf{a}^1 + \sigma_1)h_j \dots (\mathbf{a}^n + \sigma_n)h_j)$ we have $M_j \prec R_j$.

Since S is an extensive conjunctive minor of $(S_j)_{j\in J}$ via the same scheme $H = (h_j)_{j\in J}$, to prove that fM is in S, it suffices to give a Skolem map $\sigma : V \to B$ such that, for all j in J, the n_j-tuple $(fM + \sigma)h_j$ belongs to S_j. Let $\sigma = f(\sigma_1 \dots \sigma_n)$. By the rules discussed at the begining of this section, we have that for each j in J,

$$\begin{aligned}&(fM + \sigma)h_j = [f(\mathbf{a}^1 \dots \mathbf{a}^n) + f(\sigma_1 \dots \sigma_n)]h_j = \\ &= [f((\mathbf{a}^1 + \sigma_1) \dots (\mathbf{a}^n + \sigma_n))]h_j = f[(\mathbf{a}^1 + \sigma_1)h_j \dots (\mathbf{a}^n + \sigma_n)h_j] = fM_j.\end{aligned}$$

Since f satisfies (R_j, S_j), we have $fM_j \in S_j$. □

We say that a class $\mathcal{T}$ of relational constraints is *closed under formation of conjunctive minors* if whenever every member of the non-empty family $(R_j, S_j)_{j\in J}$ of constraints is in $\mathcal{T}$, all conjunctive minors of the family $(R_j, S_j)_{j\in J}$ are also in $\mathcal{T}$.

The formation of conjunctive minors subsumes the formation of simple minors as well as the operations of restricting antecedents, extending consequents and intersecting consequents. Simple minors in turn subsume permutation, identification, projection and addition of dummy arguments (see Pippenger [Pi2]).

In analogy with locally closed function classes, we say that a set $\mathcal{T}$ of relational constraints is *locally closed* if for every A-to-B constraint (R, S) the following holds: if every relaxation of (R, S) with finite antecedent coincides with some member of $\mathcal{T}$, then (R, S) belongs to $\mathcal{T}$.

A set $\mathcal{T}$ of A-to-B constraints is said to be *characterized* by a set $\mathcal{F}$ of B-valued functions on A if $\mathcal{T}$ is the set of all those constraints which are satisfied by every member of $\mathcal{F}$.

Theorem 2. *Consider arbitrary non-empty sets A and B. Let $\mathcal{T}$ be a set of A-to-B relational constraints. Then the following are equivalent:*

- (i) *$\mathcal{T}$ is locally closed and contains the binary equality constraint, the empty constraint, and it is closed under formation of conjunctive minors;*
- (ii) *$\mathcal{T}$ is characterized by some set of B-valued functions on A.*

Proof. First we prove the implication $(ii) \Rightarrow (i)$. It is clear that every function on A to B satisfies the empty and the equality constraints. It follows from Lemma 1 that if a function satisfies a non-empty family $(R_j, S_j)_{j \in J}$ of constraints then it satisfies every conjunctive minor of the family. Thus, to prove the implication $(ii) \Rightarrow (i)$ we only need to show that $\mathcal{T}$ is locally closed. For that, let (R, S) be an m-ary constraint not in $\mathcal{T}$. By (ii), there is an n-ary function f satisfying every constraint in $\mathcal{T}$ which does not satisfy (R, S). Thus, for an $m \times n$ matrix $M \prec R$, $fM \notin S$. It is easy to see that the constraint (F, S), where F is the set of columns of M, is a relaxation of (R, S) with finite antecedent such that $(F, S) \notin \mathcal{T}$. This completes the proof of implication $(ii) \Rightarrow (i)$.

To prove the implication $(i) \Rightarrow (ii)$, we need to extend the concepts of relation and constraint to infinite arities. Function arities remain finite. These extended definitions have no bearing on the Theorem itself, but are needed only as tools in its proof.

For any non-zero, possibly infinite, ordinal m, an *m-tuple* is a map defined on m. (An ordinal m is the set of lesser ordinals.) Relation and constraint arities are thus allowed to be arbitrary non-zero, possibly infinite, ordinals m, n, μ etc. In minor formation schemes, the target m and the members n_j of the source family are also allowed to be arbitrary non-zero, possibly infinite ordinals. For relations, we shall use the term *restrictive conjunctive ∞-minor* (*extensive conjunctive ∞-minor*) to indicate a restrictive conjunctive minor (extensive conjunctive minor, respectively) via a scheme whose target and source ordinals may be infinite or finite. Similarly, for constraints we shall use the term *conjunctive ∞-minor* (*simple ∞-minor*) to indicate a conjunctive minor (simple minor, respectively) via a scheme whose target and source ordinals may be infinite or finite. Thus in the sequel the use of the term "minor" without the prefix "∞-" continues to mean the respective minor via a scheme whose target and source ordinals are all finite. Matrices can also have infinitely many rows but only finitely many columns: an $m \times n$ matrix M, where n is finite but m could be infinite, is an n-tuple of m-tuples $M = (\mathbf{a}^1 \dots \mathbf{a}^n)$.

In order to discuss the formation of repeated ∞-minors, we need the following definition. Let $H = (h_j)_{j \in J}$ be an ∞-minor formation scheme with target m, indeterminate set V and source family $(n_j)_{j \in J}$, and, for each $j \in J$, let $H_j = (h_j^i)_{i \in I_j}$ be a scheme with target n_j, indeterminate set V_j and source family $(n_j^i)_{i \in I_j}$. Assume that V is disjoint from the V_j's, and that the V_j's are also pairwise disjoint. Then the *composite scheme* $H(H_j : j \in J)$ is the scheme $K = (k_j^i)_{j \in J, i \in I_j}$ defined as follows:

(i) the target of K is the target m of H,
(ii) the source family of K is $(n_j^i)_{j \in J, i \in I_j}$,
(iii) the indeterminate set of K is $U = V \cup (\cup_{j \in J} V_j)$,
(iv) $k_j^i : n_j^i \to m \cup U$ is defined by

$$k_j^i = (h_j + \iota_{UV_j}) h_j^i,$$

where ι_{UV_j} is the canonical injection (inclusion map) from V_j to U.

Claim 1. If (R, S) is a conjunctive ∞-minor of a non-empty family $(R_j, S_j)_{j \in J}$ of A-to-B constraints via the scheme H, and, for each $j \in J$, (R_j, S_j) is a conjunctive ∞-minor of a non-empty family $(R_j^i, S_j^i)_{i \in I_j}$ via the scheme H_j, then (R, S) is a conjunctive ∞-minor of the non-empty family $(R_j^i, S_j^i)_{j \in J, i \in I_j}$ via the composite scheme $K = H(H_j : j \in J)$.

Proof of Claim 1. First, we need to see that R is a restrictive conjunctive ∞-minor of the family $(R_j^i)_{j \in J, i \in I_j}$ via K. Let $\mathbf{a}$ be an m-tuple in R. This implies that there is a Skolem map $\sigma : V \to A$ such that for all j in J, we have $(\mathbf{a} + \sigma) h_j \in R_j$. In turn this implies that for every j in J there are Skolem maps $\sigma_j : V_j \to A$ such that for every i in I_j, the n_j^i-tuple $[(\mathbf{a} + \sigma) h_j + \sigma_j] h_j^i$ is in R_j^i. Define the Skolem map $\tau : U \to A$ by $\tau = \sigma + \Sigma_{l \in J} \sigma_l$. Then for every $j \in J$ and $i \in I_j$, we have $(\mathbf{a} + \tau) k_j^i \in R_j^i$ because

$$\begin{aligned}
(1) \qquad & (\mathbf{a} + \tau) k_j^i = (\mathbf{a} + \sigma + \Sigma_{l \in J} \sigma_l)(h_j + \iota_{UV_j}) h_j^i = \\
& = [(\mathbf{a} + \sigma) h_j + (\Sigma_{l \in J} \sigma_l) h_j + (\mathbf{a} + \sigma) \iota_{UV_j} + (\Sigma_{l \in J} \sigma_l) \iota_{UV_j}] h_j^i = \\
& = [(\mathbf{a} + \sigma) h_j + \sigma_j] h_j^i
\end{aligned}$$

and this n_j^i-tuple is in R_j^i.

Second, we need to see that S is an extensive conjunctive ∞-minor of the family $(S_j^i)_{j \in J, i \in I_j}$ via K. Take an m-tuple $\mathbf{b} \in B^m$ and assume that there is a Skolem map $\tau : U \to B$ such that for every $j \in J$ and $i \in I_j$, the n_j^i-tuple $(\mathbf{b} + \tau) k_j^i$ is in S_j^i. We need to show that $\mathbf{b}$ is in S. Define the Skolem maps $\sigma : V \to B$ and $\sigma_j : V_j \to B$ for every $j \in J$, by restriction of

τ, i.e. $\tau = \sigma + \Sigma_{l \in J} \sigma_l$. Similarly to (1),

$$(\mathbf{b} + \tau)k_j^i = [(\mathbf{b} + \sigma)h_j + \sigma_j]h_j^i.$$

Since S_j is an extensive conjunctive ∞-minor of the family $(S_j^i)_{j \in J, i \in I_j}$ via the scheme H_j we have $(\mathbf{b} + \sigma)h_j \in S_j$. As the condition $(\mathbf{b} + \sigma)h_j \in S_j$ holds for all j in J and S is an extensive conjunctive ∞-minor of the family $(S_j)_{j \in J}$ via H, we have that $\mathbf{b}$ is in S, which completes the proof of Claim 1.

For a set $\mathcal{T}$ of A-to-B constraints, we denote by $\mathcal{T}^\infty$ the set of those constraints which are conjunctive ∞-minors of families of members of $\mathcal{T}$. This set $\mathcal{T}^\infty$ is the smallest set of constraints containing $\mathcal{T}$ which is closed under formation of conjunctive ∞-minors and it is called the *conjunctive ∞-minor closure* of $\mathcal{T}$. In the sequel, we shall make use of the following fact:

Fact 1. Let $\mathcal{T}$ be a set of finitary A-to-B constraints and let $\mathcal{T}^\infty$ be its conjunctive ∞-minor closure. If $\mathcal{T}$ is closed under formation of conjunctive minors, then $\mathcal{T}$ is the set of all finitary constraints belonging to $\mathcal{T}^\infty$.

Claim 2. Let $\mathcal{T}$ be a locally closed set of finitary A-to-B constraints containing the binary equality constraint, the empty constraint, and closed under formation of conjunctive minors, and let $\mathcal{T}^\infty$ be its ∞-minor closure. Let (R, S) be a finitary A-to-B constraint not in $\mathcal{T}$. Then there is a B-valued function g on A such that

1) g satisfies every constraint in $\mathcal{T}^\infty$,
2) g does not satisfy (R, S).

Proof of Claim 2. We shall construct a function g which satisfies all constraints in $\mathcal{T}^\infty$ but g does not satisfy (R, S).

Note that, by Fact 1, (R, S) can not be in $\mathcal{T}^\infty$. Let m be the arity of (R, S). Since $\mathcal{T}$ is locally closed and (R, S) does not belong to $\mathcal{T}$, we know that there is a relaxation (R_1, S_1) of (R, S), where R_1 is finite, which is not in $\mathcal{T}$. Let n be the number of m-tuples in R_1. Observe that $S_1 \neq B^m$, since the constraint (A^m, B^m) is a simple minor of the binary equality constraint, and thus is in $\mathcal{T}$. Also, R_1 is non-empty, otherwise (R_1, S_1) would be a relaxation of the empty constraint, and so would belong to $\mathcal{T}$. Suppose R_1 consists of n distinct m-tuples $\mathbf{d}^1, \ldots, \mathbf{d}^n$.

Consider the $m \times n$ matrix $F = (\mathbf{d}^1 \ldots \mathbf{d}^n)$. Let $M = (\mathbf{a}^1 \ldots \mathbf{a}^n)$ be any matrix whose first m rows are the rows of F (i.e. $(\mathbf{a}^1(i) \ldots \mathbf{a}^n(i)) = (\mathbf{d}^1(i) \ldots \mathbf{d}^n(i))$ for every $i \in m$) and whose other rows are the remaining distinct n-tuples in A^n: every n-tuple in A^n is a row of M, and any repetition of rows can only occur among the first m rows of M. Let R_M be the relation whose elements are the columns of M, say of arity μ. Note that $m \leq \mu$ and that μ is infinite if and only if A is infinite. Let S_M be the μ-ary relation consisting of those μ-tuples $\mathbf{b} = (b_t \mid t \in \mu)$ in B^μ such that $(b_t \mid t \in m)$ belongs to S_1.

Observe that (R_M, S_M) can not belong to $\mathcal{T}^\infty$, because (R_1, S_1) is a simple ∞-minor of the possibly infinitary constraint (R_M, S_M), and if $(R_M, S_M) \in \mathcal{T}^\infty$ we would conclude, from Fact 1, that (R_1, S_1) is in $\mathcal{T}$. Also, there must exist a μ-tuple $\mathbf{s} = (s_t \mid t \in \mu)$ in B^μ such that $(s_t \mid t \in m)$ is not in S_1, and for which $(R_M, B^\mu \setminus \{\mathbf{s}\})$ does not belong to $\mathcal{T}^\infty$, otherwise by taking arbitrary intersections of consequents we would conclude that (R_M, S_M) belongs to $\mathcal{T}^\infty$.

Next we show that if two rows of M, say row i and j, coincide, then the corresponding components of $\mathbf{s}$ also coincide, $s_i = s_j$. For a contradiction, suppose that rows i and j coincide but $s_i \neq s_j$. Consider the μ-ary A-to-B constraint $(R^=, S^=)$ defined by

$$R^= = \{(a_t \mid t \in \mu) : a_i = a_j\} \quad \text{and} \quad S^= = \{(b_t \mid t \in \mu) : b_i = b_j\}.$$

The constraint $(R^=, S^=)$ is a simple ∞-minor of the binary equality constraint and therefore belongs to $\mathcal{T}^\infty$. On the other hand $(R_M, B^\mu \setminus \{\mathbf{s}\})$ is a relaxation of $(R^=, S^=)$ and should also belong to $\mathcal{T}^\infty$, yielding the intended contradiction.

Observe that the set of rows of M is the set all n-tuples of A^n. Also, in view of the above, we can define an n-ary function g by the condition $gM = \mathbf{s}$. By definition of $\mathbf{s}$, g does not satisfy (R_M, S_M), and so it does not satisfy (R_1, S_1). So the function g does not satisfy (R, S).

Suppose that there is a ρ-ary constraint $(R_0, S_0) \in \mathcal{T}^\infty$, possibly infinitary, which g does not satisfy. Thus, for some $\rho \times n$ matrix $M_0 = (\mathbf{c}^1 \ldots \mathbf{c}^n)$ with columns in R_0 we have $gM_0 \notin S_0$. Define $h : \rho \to \mu$ to be any map such that

$$(\mathbf{c}^1(i) \ldots \mathbf{c}^n(i)) = ((\mathbf{a}^1 h)(i) \ldots (\mathbf{a}^n h)(i))$$

for every $i \in \rho$, i.e. row i of M_0 is the same as row $h(i)$ of M, for each $i \in \rho$. Let (R_h, S_h) be the μ-ary simple ∞-minor of (R_0, S_0) via $H = \{h\}$. Note that, by Claim 1, (R_h, S_h) belongs to $\mathcal{T}^\infty$.

We claim that $R_M \subseteq R_h$. Any μ-tuple in R_M is a column $\mathbf{a}^j$ of $M = (\mathbf{a}^1 \ldots \mathbf{a}^n)$. To prove that $\mathbf{a}^j \in R_h$ we need to show that the ρ-tuple $\mathbf{a}^j h$ is in R_0. In fact, we have

$$\mathbf{a}^j h = (\mathbf{a}^j h(i) \mid i \in \rho) = (\mathbf{c}^j(i) \mid i \in \rho)$$

and this ρ-tuple is in R_0.

Next we claim that $B^\mu \setminus \{\mathbf{s}\} \supseteq S_h$, i.e. that $\mathbf{s} \notin S_h$. For that it is enough to show that $\mathbf{s}h \notin S_0$. For every $i \in \rho$ we have

$$(\mathbf{s}h)(i) = [g(\mathbf{a}^1 \ldots \mathbf{a}^n)h](i) = g[(\mathbf{a}^1 h)(i) \ldots (\mathbf{a}^n h)(i)] = g(\mathbf{c}^1(i) \ldots \mathbf{c}^n(i)).$$

Thus $\mathbf{s}h = gM_0$. Since $gM_0 \notin S_0$ we conclude that $\mathbf{s} \notin S_h$. So $(R_M, B^\mu \setminus \{\mathbf{s}\})$ is a relaxation of (R_h, S_h) and we conclude that $(R_M, B^\mu \setminus \{\mathbf{s}\})$ is in $\mathcal{T}^\infty$. By definition of $\mathbf{s}$, this is impossible. Thus we have proved Claim 2.

To see that the implication $(ii) \Rightarrow (i)$ of Theorem 2 holds, observe that, by Claim 2, for every constraint (R, S) not in $\mathcal{T}$ there is a function g which does not satisfy (R, S) but satisfies every constraint in $\mathcal{T}^\infty$, and hence satisfies every constraint in $\mathcal{T}$. Thus the set of all these "separating" functions constitutes the desired set characterizing $\mathcal{T}$. $\square$

Theorem 2 generalizes the characterization of closed classes of constraints given by Pippenger in [Pi2] by allowing both finite and infinite underlying sets and extending the closure conditions on classes of relational constraints (via the broadening of the concept of simple minors). The proof of Claim 2, being part of the proof of Theorem 2, differs from the analogous constructions of Geiger in [G] and of Pippenger in [Pi2] in that the function g separating the constraint (R, S) is not obtained by successive extensions of partial functions but it is defined at once as a total function on A^n.

Conjunctive minors are indeed strictly more general than simple minors: the fact that the former are not subsumed by the latter in the infinite case is illustrated in the following section.

4. Comparison of Closures Based on Simple and Conjunctive Minors

An n-ary *B-valued partial function on A* is a function $p : D \to B$ where $D \subseteq A^n$. The partial function p is said to be *finite* if D is a finite set. If $\mathcal{F}$ is a set of B-valued partial functions on A, $\mathcal{F}$ is called an *extensible family* if for every $p \in \mathcal{F}$, $p : D \to B$ where $D \subseteq A^n$, and every $\mathbf{y} \in A^n \setminus D$, $\mathcal{F}$ contains an extension $p' : D' \to B$ of p to the domain $D' = D \cup \{\mathbf{y}\}$.

If p is an n-ary B-valued partial function on A and (R, S) an m-ary A-to-B constraint we say that p *satisfies* (R, S) if for every $m \times n$ matrix $M \prec R$, $M = (\mathbf{a}^1 \dots \mathbf{a}^n)$, and such that every row of M belongs to the domain of p, the m-tuple $(p(\mathbf{a}^1(i) \dots \mathbf{a}^n(i)) \mid i \in m)$ belongs to S .

Proposition 1. *For any sets A and B, the set of all A-to-B constraints satisfied by an extensible family $\mathcal{F}$ of B-valued partial functions on A is locally closed and contains the binary equality constraint, the empty constraint, and it is closed under intersecting consequents and under taking simple minors*[2].

Proof. The only non-trivial claim is that the set of all A-to-B constraints satisfied by the extensible family $\mathcal{F}$ of B-valued partial functions on A is closed under taking simple minors. Suppose that every member of the extensible family $\mathcal{F}$ satisfies an n-ary constraint (R_0, S_0). Let (R, S) be an m-ary simple minor of (R_0, S_0) via $h : n \to m \cup V$.

Let $p \in \mathcal{F}$, $p : D \to B$, $D \subseteq A^t$. We need to show that p satisfies (R, S). Take an $m \times t$ matrix $M = (\mathbf{a}^1 \dots \mathbf{a}^t)$, $M \prec R$, such that every row of M is in D. We know that $\mathbf{a}^1, \dots, \mathbf{a}^t \in R$, that is, there are Skolem maps $\sigma_1, \dots, \sigma_t : V \to A$ such that $(\mathbf{a}^1 + \sigma_1)h, \dots, (\mathbf{a}^t + \sigma_t)h \in R_0$.

We claim that $(p(\mathbf{a}^1(i) \dots \mathbf{a}^t(i)) \mid i \in m)$ belongs to S. It is enough to show that for some $f : A^t \to B$ (not necessarily in $\mathcal{F}$) such that $f\mid_D = p$ we have $f(\mathbf{a}^1 \dots \mathbf{a}^t) \in S$. This latter membership in S is equivalent to the

[2]as well as closed under relaxations (footnote added May 2006)

existence of a Skolem map $\sigma : V \to B$ such that $(f(\mathbf{a}^1 \dots \mathbf{a}^t) + \sigma)h \in S_0$. We shall define such an f and σ.

Observe that the set $A_0 = \{(\sigma_1(v), \dots, \sigma_t(v)) : v \in V \cap h[n]\}$, where $h[n]$ is the range of h, is finite. By a straightfoward induction based on the definition of an extensible family, it follows that there is an extension p' of p, p' in $\mathcal{F}$, whose domain is $D' = D \cup A_0$. Let $\sigma : V \to B$ be any Skolem map such that $\sigma(v) = p'(\sigma_1(v), \dots, \sigma_t(v))$ for all $v \in V \cap h[n]$. Note that every row of the $n \times t$ matrix $N = ((\mathbf{a}^1 + \sigma_1)h \dots (\mathbf{a}^t + \sigma_t)h)$ is in the domain of p'.

Let $f : A^t \to B$ be any function (not necessarily in $\mathcal{F}$) such that $f\mid_{D'} = p'$. We show that $(f(\mathbf{a}^1 \dots \mathbf{a}^t) + \sigma)h \in S_0$. Using the rules in Section 3, we have

$$
\begin{aligned}
&[f(\mathbf{a}^1 \dots \mathbf{a}^t) + \sigma]h = [f(\mathbf{a}^1 \dots \mathbf{a}^t) + f(\sigma_1 \dots \sigma_t)]h = \\
&= [f((\mathbf{a}^1 + \sigma_1) \dots (\mathbf{a}^t + \sigma_t))]h = f[(\mathbf{a}^1 + \sigma_1)h \dots (\mathbf{a}^t + \sigma_t)h] = \\
&= (p'[(\mathbf{a}^1 + \sigma_1)h(j) \dots (\mathbf{a}^t + \sigma_t)h(j)] \mid j \in n).
\end{aligned}
\tag{2}
$$

Since $p' \in \mathcal{F}$, p' satisfies (R_0, S_0), and as the rows of N are in the domain of p', we have that (2) is in S_0. □

Let A and B be sets with different infinite cardinalities, $card(A) > card(B)$. Let $\mathcal{F}$ be the set of all injective finite B-valued partial functions of several variables on A: this $\mathcal{F}$ is an extensible family. Let Δ_A and Δ_B be the binary disequality relations on A and B, respectively, defined by

$$\Delta_A = \{(a, b) \in A^2 \mid a \neq b\}$$

$$\Delta_B = \{(c, d) \in B^2 \mid c \neq d\}.$$

It is easy to see that every member of $\mathcal{F}$ satisfies (Δ_A, Δ_B).

Let V be a set of indeterminates equipotent to A and let $\mathcal{V}$ be the set of all two-element subsets of V. Denote $\{\alpha, \beta\} \in \mathcal{V}$ by $\alpha\beta$ for short. Take any strict total ordering $<$ on V, and for $\{\alpha, \beta\} \in \mathcal{V}$ define

$$h_{\alpha\beta} : 2 \to 1 \cup V$$

by $h_{\alpha\beta}(0) = min(\alpha, \beta)$ and $h_{\alpha\beta}(1) = max(\alpha, \beta)$. (Actually, $h_{\alpha\beta}$ could be any map $2 \to 1 \cup V$ with range $\{\alpha, \beta\}$.)

Define the family $(R_{\alpha\beta}, S_{\alpha\beta})_{\alpha\beta \in \mathcal{V}}$ of constraints by $(R_{\alpha\beta}, S_{\alpha\beta}) = (\Delta_A, \Delta_B)$ for all $\alpha\beta \in \mathcal{V}$.

The tight conjunctive minor of the family $(R_{\alpha\beta})_{\alpha\beta \in \mathcal{V}}$ via the scheme $H = (h_{\alpha\beta})_{\alpha\beta \in \mathcal{V}}$ is the full unary relation A^1 on A, while the tight conjunctive minor of $(S_{\alpha\beta})_{\alpha\beta \in \mathcal{V}}$ via H is the empty unary relation. Therefore, $(A^1, \emptyset)$ is the tight conjunctive minor of the the family of constraints $(R_{\alpha\beta}, S_{\alpha\beta})_{\alpha\beta \in \mathcal{V}}$ via H, but clearly is not satisfied by the members of $\mathcal{F}$. Thus, in view of Proposition 1, we obtain the following:

Theorem 3. *Conjunctive minors subsume simple minors, relaxations and intersections of consequents, but there are conjunctive minors which can not be obtained by any combination of taking simple minors, relaxations or intersections of consequents.*

In other words, conjunctive minors properly extend the notion of simple minors, and for infinite sets Theorem 2 in the previous Section can not be strengthened by replacing "conjunctive minors" with "simple minors", as it can be in the finite case.

5. Clones of Functions and Closed Sets of Relations

In this section we make use of Theorems 1 and 2 in Sections 2 and 3, respectively, to derive variant characterizations for clones of functions (operations) and closed sets of relations on arbitrary, not necessarily finite sets. For clones of operations this general characterization is mentioned in [G], proved in [Pö2] and [Pö3], and it is implicit in [Sz]. The characterization of closed sets of relations is given below in terms of certain closure conditions which are variants of those in [G] and [PK], in the case of finite underlying sets, and in [Sz], [Pö2] and [Pö3], in the general case of arbitrary sets.

Recall that if f is an n-ary E-valued function on B and $g_1, \ldots, g_n$ are all m-ary B-valued functions on A, then the composition $f(g_1, \ldots, g_n)$ is an m-ary E-valued function on A, and its value on $\mathbf{a} \in A^m$ is $f(g_1(\mathbf{a}), \ldots, g_n(\mathbf{a}))$. In this section we are concerned with the special case $A = B = E$, and this set may be finite or infinite.

A *clone* on A is a set of operations $\mathcal{C} \subseteq \cup_{n \geq 1} A^{A^n}$ such that it contains all projections (variables) and it is closed under composition.

An operation $f \in A^{A^n}$ *preserves* a relation R on A if $fR \subseteq R$, i.e., if f satisfies the constraint (R, R). A class $\mathcal{F} \subseteq \cup_{n \geq 1} A^{A^n}$ is said to be *definable* by a set $\mathcal{R}$ of relations of various arities on A, if $\mathcal{F}$ is the class of all operations which preserve every member of $\mathcal{R}$. Similarly, a set $\mathcal{R}$ of relations of various arities on A is said to be *characterized* by a set $\mathcal{F}$ of operations on A, if $\mathcal{R}$ is the set of all relations which are preserved by every member of $\mathcal{F}$.

Theorem 4. (Pöschel) *Let A be an arbitrary non-empty set and let $\mathcal{C}$ be a set of operations on A. Then the following conditions are equivalent:*

(i) *$\mathcal{C}$ is a locally closed clone;*
(ii) *$\mathcal{C}$ is definable by some set of relations of various arities on A.*

Proof. It is easy to see that (ii) implies (i).

To see the converse, assume (i). According to Theorem 1, $\mathcal{C}$ is definable by some set $\mathcal{T}$ of A-to-A constraints. Consider any constraint (R, S) in $\mathcal{T}$. Let $\bar{R} = \cup_{f \in \mathcal{C}} fR$. Clearly $R \subseteq \bar{R} \subseteq S$. It follows that $\mathcal{C}$ is definable by $\{\bar{R} : (R, S) \in \mathcal{T}\}$. □

We say that a set $\mathcal{R}$ of relations of various arities on A is *locally closed* if for every relation R on A the following holds: if for every finite subset F of R there is a relation R' in $\mathcal{R}$ such that $F \subseteq R' \subseteq R$, then R belongs to $\mathcal{R}$.

Theorem 5. (Szabó) *Let A be an arbitrary non-empty set and let $\mathcal{R}$ be a set of relations of various arities on A. Then the following are equivalent:*

(i) *$\mathcal{R}$ is locally closed and contains the binary equality relation, the empty relation, and is closed under formation of tight conjunctive minors;*

(ii) *$\mathcal{R}$ is characterized by some set of operations on A.*

Proof. It is not difficult to see that (ii) implies (i).

To prove the converse, define the set $\mathcal{T}$ of A-to-A constraints by

$$\mathcal{T} = \{(R,S) : \text{for every finite}\, F \subseteq R\, \text{there is}\, R' \in \mathcal{R} \text{ such that } F \subseteq R' \subseteq S\}.$$

Note that $\mathcal{T} \supseteq \{(R,R) : R \in \mathcal{R}\}$, and if $R \notin \mathcal{R}$ then $(R,R) \notin \mathcal{T}$. By its definition $\mathcal{T}$ is locally closed. Also $\mathcal{T}$ contains the binary equality and the empty constraints.

Let us show that $\mathcal{T}$ is closed under formation of conjunctive minors. For that, let (R,S) be an m-ary conjunctive minor of the family $(R_j,S_j)_{j\in J}$ via a scheme $H = (h_j)_{j\in J}$, $h_j : n_j \to m \cup V$ where each (R_j,S_j) is in $\mathcal{T}$. Let F be a finite subset of R, with elements $\mathbf{a}^1,\dots,\mathbf{a}^n$. Since R is a restrictive conjunctive minor of $(R_j)_{j\in J}$ via $H = (h_j)_{j\in J}$, there are Skolem maps $\sigma_i : V \to A$, $1 \leq i \leq n$, such that, for every j in J, the finite set F_j whose elements are $(\mathbf{a}^1 + \sigma_1)h_j,\dots,(\mathbf{a}^n + \sigma_n)h_j$ is contained in R_j. By definition of $\mathcal{T}$, for every j in J there are relations R_j' in $\mathcal{R}$ such that $F_j \subseteq R_j' \subseteq S_j$. Consider the tight conjunctive minor R' of the family $(R_j')_{j\in J}$ via H. Since S is an extensive conjunctive minor of $(S_j)_{j\in J}$ via $H = (h_j)_{j\in J}$, and, for every j in J, $F_j \subseteq R_j' \subseteq S_j$, it follows that $F \subseteq R' \subseteq S$. In other words, (R,S) belongs to $\mathcal{T}$. So $\mathcal{T}$ is indeed closed under formation of conjunctive minors.

According to Theorem 2, there is a set $\mathcal{F}$ of operations on A which satisfy exactly those constraints that are in $\mathcal{T}$. Then $\mathcal{F}$ is a set of operations preserving exactly those relations which are in $\mathcal{R}$. □

As mentioned above, in the general case of arbitrary underlying sets the conditions in (i) of Theorem 5 characterizing the closed sets of relations are equivalent to those given by Szabó in [Sz] and Pöschel in [Pö2] and [Pö3]. For the equivalence between Szabó's and Pöschel's approaches see e.g. [Pö3].

The following concept was introduced by Pöschel in [Pö2]. We shall again make use of the notation introduced in Section 3. Consider arbitrary non-empty sets A and B. Let $(R_j)_{j\in J}$ be a non-empty family of relations on A where, for each $j \in J$, R_j has arity n_j. Let $m \geq 1$, $\mathbf{b} \in B^m$ and let $(\mathbf{b}_j)_{j\in J}$ be a family with $\mathbf{b}_j \in B^{n_j}$ for each $j \in J$. The m-ary relation R on A defined by

$$R = \{f\mathbf{b} \in A^m : f \in A^B \text{ and for each } j \in J, f\mathbf{b}_j \in R_j\}$$

is said to be obtained from the family $(R_j)_{j\in J}$ by *general superposition with respect to* $\mathbf{b}$ *and the family* $(\mathbf{b}_j)_{j\in J}$. (This reformulation appears e.g. in [Pö4].) Recall that $f\mathbf{b}$ is the m-tuple $(f\mathbf{b}(i) \mid i \in m)$ and, for each $j \in J$, $f\mathbf{b}_j$ is the n_j-tuple $(f\mathbf{b}_j(i) \mid i \in n_j)$. Note that general superposition subsumes formation of tight conjunctive minors of relations: indeed if R is a tight conjunctive minor of $(R_j)_{j\in J}$ via $H = (h_j)_{j\in J}$ then we can define $B = m \cup V$, $\mathbf{b} = \iota_{Bm}$ where ι_{Bm} is the canonical injection (inclusion map) from m to B, and $\mathbf{b}_j = h_j$ where $h_j : n_j \to m \cup V$. Also, it is easy to see that the binary equality constraint can be obtained from the full unary relation $R_0 = A^1$ by general superposition with singleton J and singleton B.

A set $\mathcal{R}$ of relations of various arities on A is said to be *closed under general superpositions* if whenever every member of a non-empty family $(R_j)_{j\in J}$ of relations is in $\mathcal{R}$, all relations obtained from the family $(R_j)_{j\in J}$ by general superpositions are also in $\mathcal{R}$.

The following shows that the characterization of closed sets of relations given in Theorem 5 is equivalent to that appearing in [Pö2], [Pö3] and [Pö4].

Theorem 6. *Let A be a non-empty set and $\mathcal{R}$ a locally closed set of relations of various arities on A containig the empty relation. The following conditions are equivalent:*

(i) *$\mathcal{R}$ contains the binary equality relation and is closed under formation of tight conjunctive minors;*
(ii) *$\mathcal{R}$ contains the full unary relation A^1 and is closed under general superpositions.*

Proof. The implication $(ii) \Rightarrow (i)$ follows from the observations above.

To prove that implication $(i) \Rightarrow (ii)$ holds note first that the full unary relation A^1 is a tight conjunctive minor of the binary equality relation. Let us show that every relation obtained from a family of relations $(R_j)_{j\in J}$ by general superposition can be obtained by intersecting a tight conjuctive minor of the family $(R_j)_{j\in J}$ with a tight conjunctive minor of the binary equality relation.

Let $(R_j)_{j\in J}$ be a non-empty family of relations on A where, for each $j \in J$, R_j has arity n_j, and let R be the m-ary relation on A obtained from the family $(R_j)_{j\in J}$ by general superposition with respect to $\mathbf{b} \in B^m$ and family $(\mathbf{b}_j)_{j\in J}$ with $\mathbf{b}_j \in B^{n_j}$ for each $j \in J$, where without loss of generality B is a non-empty set disjoint from the ordinals. Consider the m-ary relation $R^{=}_{\mathbf{b}}$ on A defined by

$$R^{=}_{\mathbf{b}} = \{(a_t \mid t \in m) : a_i = a_j \text{ for every } i, j \in m \text{ such that } \mathbf{b}(i) = \mathbf{b}(j)\}.$$

It is easy to see that $R^{=}_{\mathbf{b}}$ is a tight conjunctive minor of the binary equality relation.

Let V be the complement in B of the range of $\mathbf{b}$. Consider the minor formation scheme $H = (h_j)_{j\in J}$ with target m, indeterminate set V and source family $(n_j)_{j\in J}$, and where, for each $j \in J$, $h_j : n_j \to m \cup V$ is such that

$$(\mathbf{b} + \iota_{BV})h_j = \mathbf{b}_j.$$

Consider the m-ary tight conjunctive minor R' of the family $(R_j)_{j\in J}$ via H.

Let us show that $R = R' \cap R^{=}_{\mathbf{b}}$. Observe that $R \subseteq R^{=}_{\mathbf{b}}$. Let $\mathbf{a} \in R$. Then, for some function $f : B \to A$, $\mathbf{a} = f\mathbf{b}$ and, for each $j \in J$, $f\mathbf{b}_j \in R_j$. Define the Skolem map $\sigma : V \to A$ by $\sigma = f\iota_{BV}$. By definition of H and σ we have

$$(\mathbf{a}+\sigma)h_j = (f\mathbf{b}+f\iota_{BV})h_j = [f(\mathbf{b}+\iota_{BV})]h_j = f[(\mathbf{b}+\iota_{BV})h_j] = f\mathbf{b}_j$$

for each $j \in J$. Thus, for every $j \in J$, $(\mathbf{a}+\sigma)h_j \in R_j$, and we have $\mathbf{a} \in R'$. Since $R \subseteq R^{=}_{\mathbf{b}}$, we conclude $R \subseteq R' \cap R^{=}_{\mathbf{b}}$.

To show that $R' \cap R^{=}_{\mathbf{b}} \subseteq R$, let $\mathbf{a} \in R' \cap R^{=}_{\mathbf{b}}$. Since R' is the tight conjunctive minor of the family $(R_j)_{j\in J}$ via H, there is a Skolem map $\sigma : V \to A$ such that for every j in J we have $(\mathbf{a}+\sigma)h_j \in R_j$. It follows from definition of $R^{=}_{\mathbf{b}}$ that, for every i and j in m, $\mathbf{a}(i) = \mathbf{a}(j)$ if $\mathbf{b}(i) = \mathbf{b}(j)$. Let $f : B \to A$ be such that $f\mathbf{b} = \mathbf{a}$ and $f\iota_{BV} = \sigma$. We have

$$f\mathbf{b}_j = f[(\mathbf{b}+\iota_{BV})h_j] = [f(\mathbf{b}+\iota_{BV})]h_j = (f\mathbf{b}+f\iota_{BV})h_j = (\mathbf{a}+\sigma)h_j$$

and so, for each $j \in J$, $f\mathbf{b}_j \in R_j$. Thus $f\mathbf{b} \in R$, that is, $\mathbf{a} \in R$. $\square$

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ON GALOIS CONNECTIONS BETWEEN EXTERNAL OPERATIONS AND RELATIONAL CONSTRAINTS: ARITY RESTRICTIONS AND OPERATOR DECOMPOSITIONS

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Abstract. We study the basic Galois connection induced by the "satisfaction" relation between external operations $A^n \to B$ defined on a set A and valued in a possibly different set B on one hand, and ordered pairs (R, S) of relations $R \subseteq A^m$ and $S \subseteq B^m$, called relational constraints, on the other hand. We decompose the closure maps associated with this Galois connection, in terms of closure operators corresponding to simple closure conditions describing the corresponding Galois closed sets of functions and constraints. We consider further Galois correspondences by restricting the sets of primal and dual objects to fixed arities. We describe the restricted Galois closure systems by means of parametrized analogues of the simple closure conditions, and present factorizations of the corresponding Galois closure maps into simpler closure operators.

1. Introduction

In this paper we analyse the basic Galois connection implicit in [2] which extends to the infinite case the framework of Pippenger in [11], where classes of external operations (i.e. functions defined on a set A and valued in a possibly different set B) are defined by the ordered pairs of relations, called *relational constraints*, which they satisfy, and dually where sets of constraints are characterized by the functions satisfying them. As presented in [2], the results in this bi-sorted framework specialize to those concerning the fundamental Galois correspondence $\mathbf{Pol} - \mathbf{Inv}$ between operations and relations (for finite underlying sets, see [1, 8, 14], and [15, 12, 13], for arbitrary sets). In analogy with the universal algebraic setting, we consider further Galois connections arising from the restriction of the sets of functions and constraints to fixed arities (for universal algebraic analogues see e.g. [12] and [13]).

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In Section 2, we recall basic concepts and terminology, and introduce the fundamental Galois connection between external operations (functions) and relational constraints. The Galois closed sets with respect to this correspondence are described in Section 3 by means of simple closure conditions provided in [2]. Also we define operators associated with these conditions, and present factorizations of the closure maps associated with this Galois connection, analogous to those given in [13]. In Section 4, we study further Galois correspondences induced by the restriction of the sets of primal and dual objects to fixed arities. To characterize the corresponding Galois closed sets of functions and constraints, we define parametrized analogues of the simple conditions and corresponding closure operators, given in Section 3, and represent the restricted Galois closure maps as compositions of these simpler closure operators.

2. Basic Notions and Terminology

Let A, B and E be arbitrary non-empty sets. A *B-valued function on A* (or, *external operation*) is a map $f : A^n \to B$, for some positive integer n called the *arity* of f. For each positive integer n, we denote by $\mathbf{n}$ the set $\mathbf{n} = \{1, \ldots, n\}$, so that the n-tuples $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ can be thought of as unary A-valued functions $\mathbf{a} : \mathbf{n} \to A$ on $\mathbf{n}$ defined by $\mathbf{a}(i) = a_i$. A *class* of B-valued functions on A is a subset $\mathcal{F} \subseteq \cup_{n\geq 1} B^{A^n}$. For $A = B$, A-valued functions on A are usually called (*internal*) *operations on A*. For each positive integer n, the n-ary operations $(a_t \mid t \in \mathbf{n}) \mapsto a_i$, $i \in \mathbf{n}$, are called *projections*. The *composition* of an n-ary E-valued function f on B with m-ary B-valued functions $g_1, \ldots, g_n$ on A is the m-ary E-valued function $f(g_1, \ldots, g_n)$ on A, defined by

$$f(g_1, \ldots, g_n)(\mathbf{a}) = f(g_1(\mathbf{a}), \ldots, g_n(\mathbf{a}))$$

for every $\mathbf{a} \in A^m$. Composition is naturally extended to classes of functions. For $\mathcal{I} \subseteq \cup_{n\geq 1} E^{B^n}$ and $\mathcal{J} \subseteq \cup_{n\geq 1} B^{A^n}$, the *composition of $\mathcal{I}$ with $\mathcal{J}$*, denoted $\mathcal{I}\mathcal{J}$, is defined by

$$\mathcal{I}\mathcal{J} = \{f(g_1, \ldots, g_n) \mid n, m \geq 1, f \text{ } n\text{-ary in } \mathcal{I}, g_1, \ldots, g_n \text{ } m\text{-ary in } \mathcal{J}\}.$$

Note that for arbitrary non-empty sets A, B, E and G, and function classes $\mathcal{I} \subseteq \cup_{n\geq 1} G^{E^n}$, $\mathcal{J} \subseteq \cup_{n\geq 1} E^{B^n}$, and $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$, we have $(\mathcal{I}\mathcal{J})\mathcal{K} \subseteq \mathcal{I}(\mathcal{J}\mathcal{K})$. (For background on class composition see [3, 4], and [5] in the Boolean case $A = B = \{0, 1\}$.)

A *clone* on A is a class $\mathcal{C} \subseteq \cup_{n\geq 1} A^{A^n}$ of operations on A containing all projections, and satisfying $\mathcal{C}\mathcal{C} = \mathcal{C}$. We denote by $\mathcal{I}_A$ the smallest clone on A containing only projection maps.

For a positive integer m, an *m-ary relation on A* is a subset R of A^m, i.e. a class of unary A-valued functions $\mathbf{a} : \mathbf{m} \to A$ defined on $\mathbf{m}$. We use $=_A$ to denote the binary equality relation on a set A. For an n-ary function $f \in B^{A^n}$ we denote by fR the class composition

$$\{f\}R = \{f(\mathbf{a}^1, \ldots, \mathbf{a}^n) \mid \mathbf{a}^1, \ldots, \mathbf{a}^n \in R\}.$$

In the particular case $A = B$, if $fR \subseteq R$, then f is said to *preserve* R.

An m-ary *A-to-B relational constraint* (or simply, m-ary *constraint*) is an ordered pair (R, S) where $R \subseteq A^m$ and $S \subseteq B^m$ are called the *antecedent* and *consequent*, respectively, of the constraint (see [11] and [2]). A B-valued function on A, $f : A^n \to B$, $n \geq 1$, is said to *satisfy* an m-ary A-to-B constraint (R, S) if $fR \subseteq S$. In other words, the function $f : A^n \to B$ satisfies the constraint (R, S) if and only if f is a homomorphism from the relational structure $\mathbf{A}^n = \langle A^n, R^n \rangle$ to the relational structure $\mathbf{B} = \langle B, S \rangle$. Note that every B-valued function on A satisfies the *binary A-to-B equality constraint* $(=_A, =_B)$, the *empty constraint* $(\emptyset, \emptyset)$, and, for each $m \geq 1$, the *trivial constraint* (A^m, B^m).

For a set $\mathcal{T}$ of A-to-B constraints, we denote by $\mathbf{FSC}(\mathcal{T})$ the class of all B-valued functions on A satisfying every member of $\mathcal{T}$. Dually, for a class $\mathcal{K}$ of B-valued functions on A, we denote by $\mathbf{CSF}(\mathcal{K})$ the set of all A-to-B constraints satisfied by every member of $\mathcal{K}$. The notation **FSC** stands for "functions satisfying constraints", while **CSF** stands for "constraints satisfied by functions". Consider the mappings $\mathbf{FSC} : \mathcal{T} \mapsto \mathbf{FSC}(\mathcal{T})$ and $\mathbf{CSF} : \mathcal{K} \mapsto \mathbf{CSF}(\mathcal{K})$. By definition it follows that

(i) **FSC** and **CSF** are order reversing, i.e. if $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{K} \subseteq \mathcal{K}'$, then $\mathbf{FSC}(\mathcal{T}') \subseteq \mathbf{FSC}(\mathcal{T})$ and $\mathbf{CSF}(\mathcal{K}') \subseteq \mathbf{CSF}(\mathcal{K})$, and

(ii) the compositions $\mathbf{FSC} \circ \mathbf{CSF}$ and $\mathbf{CSF} \circ \mathbf{FSC}$ are extensive maps, i.e. $\mathcal{K} \subseteq \mathbf{FSC}(\mathbf{CSF}(\mathcal{K}))$ and $\mathcal{T} \subseteq \mathbf{CSF}(\mathbf{FSC}(\mathcal{T}))$.

Thus, the pair $\mathbf{FSC} - \mathbf{CSF}$ constitutes a Galois connection between external functions and relational constraints, and as a consequence we have

(a) $\mathbf{FSC} \circ \mathbf{CSF} \circ \mathbf{FSC} = \mathbf{FSC}$ and $\mathbf{CSF} \circ \mathbf{FSC} \circ \mathbf{CSF} = \mathbf{CSF}$, and

(b) $\mathbf{FSC} \circ \mathbf{CSF}$ and $\mathbf{CSF} \circ \mathbf{FSC}$ are closure operators, i.e. extensive, monotone and idempotent.

The function classes and the sets of constraints fixed by the operators in (b) are said to be (*Galois*) *closed*. (For background on Galois connections, see e.g. [9] and [10].)

3. The Galois Connection FSC – CSF

In this section we recall the basic theory in [2], and develop some factorization results for the composites $\mathbf{FSC} \circ \mathbf{CSF}$ and $\mathbf{CSF} \circ \mathbf{FSC}$.

A class $\mathcal{K} \subseteq \cup_{n \geq 1} B^{A^n}$ of B-valued functions on A is said to be *definable* (or *defined*) by a set $\mathcal{T}$ of A-to-B constraints, if $\mathcal{K} = \mathbf{FSC}(\mathcal{T})$. Dually, a set $\mathcal{T}$ of A-to-B constraints is said to be *characterized* by a set $\mathcal{K}$ of B-valued functions on A, if $\mathcal{T} = \mathbf{CSF}(\mathcal{K})$. Thus the closed sets of functions and the closed sets of relational constraints with respect to the Galois connection $\mathbf{FSC} - \mathbf{CSF}$ are precisely the classes of functions definable by constraints, and the sets of constraints characterized by functions.

In the case of finite underlying sets A and B, Pippenger determined, in [11], that the necessary and sufficient conditions for a class of functions to be definable by a set of relational constraints are essentially closure under certain functional compositions. An m-ary B-valued function g on A is said to be obtained from an n-ary B-valued function f on A by *simple variable substitution*, if there are m-ary projections $p_1, \ldots, p_n \in \mathcal{I}_A$ such that $g = f(p_1, \ldots, p_n)$. A class $\mathcal{K}$ of B-valued functions on A is said to be *closed under simple variable substitutions* if each function obtained from a function f in $\mathcal{K}$ by simple variable substitution is also in $\mathcal{K}$, i.e. if $\mathcal{K} = \mathcal{K}\mathcal{I}_A$, where $\mathcal{I}_A$ denotes the smallest clone on A containing only projections. For a class $\mathcal{K}$ of B-valued functions on A, we define the closure $\mathbf{VS}(\mathcal{K})$ of $\mathcal{K}$ under "variable substitutions" by $\mathbf{VS}(\mathcal{K}) = \mathcal{K}\mathcal{I}_A$. This is indeed the smallest class containing $\mathcal{K}$ and closed under simple variable substitutions. Clearly, the map $\mathcal{K} \mapsto \mathbf{VS}(\mathcal{K})$ is extensive and monotone, and for any class $\mathcal{K}$, we have

$$\mathbf{VS}(\mathbf{VS}(\mathcal{K})) = (\mathcal{K}\mathcal{I}_A)\mathcal{I}_A \subseteq \mathcal{K}(\mathcal{I}_A\mathcal{I}_A) = \mathcal{K}\mathcal{I}_A = \mathbf{VS}(\mathcal{K}),$$

i.e. $\mathcal{K} \mapsto \mathbf{VS}(\mathcal{K})$ is also idempotent.

Fact 1. *The operator* $\mathcal{K} \mapsto \mathbf{VS}(\mathcal{K})$ *is a closure operator on* $\cup_{n\geq 1} B^{A^n}$.

As shown in [2], in the general case of arbitrary underlying sets A and B, the above closure does not suffice to guarantee function class definability by relational constraints; "local closure" is also required on the class of functions. A class $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ is said to be *locally closed* if it contains every function for which every restriction to a finite subset of its domain A^n coincides with a restriction of some member of $\mathcal{K}$. For background on the analogous concept defined on sets of operations, see e.g. [8, 12, 13]. For any class of functions $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ we denote by $\mathbf{Lo}(\mathcal{K})$ the smallest locally closed class of functions containing $\mathcal{K}$, called the *local closure* of $\mathcal{K}$. In other words, $\mathbf{Lo}(\mathcal{K})$ is the class of functions obtained from $\mathcal{K}$ by adding all those functions whose restriction to each finite subset of its domain A^n coincides with a restriction of some member of $\mathcal{K}$.

Fact 2. *The operator* $\mathcal{K} \mapsto \mathbf{Lo}(\mathcal{K})$ *is a closure operator on* $\cup_{n\geq 1} B^{A^n}$.

Note that, if A is finite, then $\mathbf{Lo}(\mathcal{K}) = \mathcal{K}$ for every class $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$, i.e. every class $\mathcal{K}$ is locally closed.

Theorem 1. ([2]) *Consider arbitrary non-empty sets A and B. A class $\mathcal{K}$ of B-valued functions on A is definable by some set of A-to-B constraints if and only if $\mathcal{K}$ is locally closed and it is closed under simple variable substitutions.*

In other words, the closed sets of functions for the Galois connection $\mathbf{FSC} - \mathbf{CSF}$ (i.e. of the form $\mathbf{FSC}(\mathcal{T})$ for some set $\mathcal{T}$ of relational constraints) are exactly those locally closed classes which are closed under simple variable substitutions. In order to provide the characterization of the closed systems of the dual objects, i.e. relational constraints, we recall the following concepts introduced in [2].

Let A, B, C and D be arbitrary sets. For any maps $f : A \to B$ and $g : C \to D$, the *concatenation* of f and g, denoted gf, is defined to be the map with domain $f^{-1}[B \cap C]$ and codomain D given by $(gf)(a) = g(f(a))$ for all $a \in f^{-1}[B \cap C]$. Note that concatenation is associative.

Given a non-empty family $(g_i)_{i \in I}$ of maps, $g_i : A_i \to B_i$ where $(A_i)_{i \in I}$ is a family of pairwise disjoint sets, we denote by $\Sigma_{i \in I} g_i$ the map from $\cup_{i \in I} A_i$ to $\cup_{i \in I} B_i$ whose restriction to each A_i agrees with g_i, called the (*piecewise*) *sum of the family* $(g_i)_{i \in I}$. We also use $f + g$ to denote the sum of f and g. Clearly, this operation is associative and commutative, and it is not difficult to see that concatenation is distributive over sum, i.e. for any family $(g_i)_{i \in I}$ of maps on pairwise disjoint domains and any map f

$$(\Sigma_{i \in I} g_i) f = \Sigma_{i \in I} (g_i f) \qquad \text{and} \qquad f(\Sigma_{i \in I} g_i) = \Sigma_{i \in I} (f g_i).$$

Let m and n_j, $j \in J$, be positive integers, and let V be an arbitrary set disjoint from $\mathbf{m}$ and each $\mathbf{n}_j$. Any non-empty family $H = (h_j)_{j \in J}$ of maps $h_j : \mathbf{n}_j \to \mathbf{m} \cup V$ is called a *minor formation scheme* with *target* $\mathbf{m}$, *indeterminate set* V and *source family* $(\mathbf{n}_j)_{j \in J}$. Let $(R_j)_{j \in J}$ be a non-empty family of relations (of various arities) on the same set A, each R_j of arity n_j. An m-ary relation R on A is said to be a *tight conjunctive minor* of the family $(R_j)_{j \in J}$ *via the scheme* H, or simply a *tight conjunctive minor* of the family $(R_j)_{j \in J}$, if for every m-tuple $\mathbf{a}$ in A^m, the following are equivalent:

(a) $\mathbf{a} \in R$;
(b) there is a map $\sigma : V \to A$ such that, for all j in J, we have $(\mathbf{a}+\sigma)h_j \in R_j$.

The map σ is called a *Skolem map.* The n_j-tuple $(\mathbf{a} + \sigma)h_j$ denotes the concatenation of the sum $\mathbf{a}+\sigma$ and h_j. Formation of tight conjunctive minors subsumes permutation, identification, projection and addition of dummy arguments, as well as arbitrary intersection of relations of the same arity.

If for every m-tuple $\mathbf{a}$ in A^m, we have $(a) \Rightarrow (b)$, then R is said to be a *restrictive conjunctive minor* of the family $(R_j)_{j \in J}$ *via* H, or simply a *restrictive conjunctive minor* of the family $(R_j)_{j \in J}$. On the other hand, if for every m-tuple $\mathbf{a}$ in A^m, we have $(b) \Rightarrow (a)$, then we say that R is an *extensive conjunctive minor* of the family $(R_j)_{j \in J}$ *via* H, or simply an *extensive conjunctive minor* of the family $(R_j)_{j \in J}$. Thus a relation R is a tight conjunctive minor of the family $(R_j)_{j \in J}$ if it is both a restrictive conjunctive minor and an extensive conjunctive minor of the family $(R_j)_{j \in J}$.

An A-to-B constraint (R, S) is said to be a *conjunctive minor* of a non-empty family $(R_j, S_j)_{j \in J}$ of A-to-B constraints (of various arities) *via a scheme* H, (or simply a *conjunctive minor* of the family of constraints) if

(i) R is a restrictive conjunctive minor of $(R_j)_{j \in J}$ via H, and
(ii) S is an extensive conjunctive minor of $(S_j)_{j \in J}$ via H.

(For background see [2].) If the indeterminate set V of the scheme H is empty, i.e. for every j in J, the maps h_j are valued in $\mathbf{m}$, then (R, S) is called a *weak conjunctive minor* of the family $(R_j, S_j)_{j \in J}$. Observe that this operation subsumes in particular *relaxations*: (R, S) is said to be a *relaxation* of (R_0, S_0) if $R \subseteq R_0$ and $S \supseteq S_0$, and it is called a *finite relaxation*, if R is finite. If both R and S are tight conjunctive minors of the respective families $(R_j)_{j\in J}$ and $(S_j)_{j\in J}$ (on A and B, respectively) via the same scheme H, the constraint (R,S) is said to be a *tight conjunctive minor* of the family $(R_j,S_j)_{j\in J}$ *via* H, or simply a *tight conjunctive minor* of the family of constraints. In this case, if in addition $\mid J \mid= 1$, say $J = \{0\}$, then the family $(R_j,S_j)_{j\in J}$ contains a single constraint (R_0,S_0), and (R,S) is said to be a *simple minor* of (R_0,S_0) (see [11]). The following is a special case of Claim 1 in the proof of Theorem 2 in [2]:

Transitivity Lemma. *If (R,S) is a conjunctive minor of a non-empty family $(R_j,S_j)_{j\in J}$ of A-to-B constraints, and, for each $j\in J$, (R_j,S_j) is a conjunctive minor of a non-empty family $(R^i_j,S^i_j)_{i\in I_j}$, then (R,S) is a conjunctive minor of the non-empty family $(R^i_j,S^i_j)_{j\in J,i\in I_j}$.*

We say that a set $\mathcal{T}$ of relational constraints is *closed under formation of conjunctive minors* if whenever every member of a non-empty family $(R_j,S_j)_{j\in J}$ of constraints is in $\mathcal{T}$, all conjunctive minors of the family $(R_j,S_j)_{j\in J}$ are also in $\mathcal{T}$. For any set of constraints $\mathcal{T}$, we denote by $\mathbf{CM}(\mathcal{T})$ the smallest set of constraints containing $\mathcal{T}$, the binary equality constraint and the empty constraint, and closed under formation of conjunctive minors. Note that $\mathbf{CM}(\mathbf{CM}(\mathcal{T})) = \mathbf{CM}(\mathcal{T})$ and by the Transitivity Lemma it follows that $\mathbf{CM}(\mathcal{T})$ is the set of all conjunctive minors of non-empty families of A-to-B constraints in $\mathcal{T}\cup\{(=_A,=_B),(\emptyset,\emptyset)\}$.

Fact 3. *The operator $\mathcal{T}\mapsto\mathbf{CM}(\mathcal{T})$ is a closure operator on the set of all A-to-B relational constraints.*

In analogy with classes of external operations, we need to consider a further condition for the characterization of the closed sets of constraints. A set $\mathcal{T}$ of relational constraints is said to be *locally closed* if $\mathcal{T}$ contains every A-to-B constraint (R,S) such that the set of all its finite relaxations is contained in $\mathcal{T}$. The *local closure* of a set $\mathcal{T}$ of relational constraints, denoted by $\mathbf{LO}(\mathcal{T})$, is the smallest locally closed set of constraints containing $\mathcal{T}$. In other words, $\mathbf{LO}(\mathcal{T})$ is the set of constraints obtained from $\mathcal{T}$ by adding all those constraints whose finite relaxations are all in $\mathcal{T}$, and thus we have:

Fact 4. *The operator $\mathcal{T}\mapsto\mathbf{LO}(\mathcal{T})$ is a closure operator on the set of all A-to-B relational constraints.*

As in the case of function classes, if A is finite, then every set of A-to-B constraints is locally closed. The following result provides the characterization of the closed sets of constraints with respect to the Galois connection $\mathbf{FSC}-\mathbf{CSF}$:

Theorem 2. ([2]) *Consider arbitrary non-empty sets A and B. A set $\mathcal{T}$ of A-to-B relational constraints is characterized by some set of B-valued functions on A if and only if it is locally closed and contains the binary equality constraint, the empty constraint, and it is closed under formation of conjunctive minors.*

We finish this section with a description of the closure operators $\mathbf{FSC} \circ \mathbf{CSF}$ and $\mathbf{CSF} \circ \mathbf{FSC}$ as compositions of the operators $\mathbf{Lo}$ and $\mathbf{VS}$, and $\mathbf{LO}$ and $\mathbf{CM}$, respectively. The statements (a) and (b) below are analogues of (iii) in Lemma 2.5 and Proposition 3.8, respectively, in [13]:

Theorem 3. *Consider arbitrary non-empty sets A and B, and let $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ be a class of B-valued functions on A, and $\mathcal{T}$ a set of A-to-B relational constraints. The following hold:*

(a) *If $\mathbf{VS}(\mathcal{K}) = \mathcal{K}$, then $\mathbf{VS}(\mathbf{Lo}(\mathcal{K})) = \mathbf{Lo}(\mathcal{K})$.*
(b) *If $\mathbf{CM}(\mathcal{T}) = \mathcal{T}$, then $\mathbf{CM}(\mathbf{LO}(\mathcal{T})) = \mathbf{LO}(\mathcal{T})$.*

Proof. First we prove (a). Suppose that g is a t-ary function in $\mathbf{VS}(\mathbf{Lo}(\mathcal{K}))$. That is, there is an n-ary function f in $\mathbf{Lo}(\mathcal{K})$, and t-ary projections $p_1, \dots, p_n$ in $\mathcal{I}_A$ such that $g = f(p_1, \dots, p_n)$. To prove that g belongs to $\mathbf{Lo}(\mathcal{K})$, we show that, for every finite subset F of A^t, there is a t-ary function g_F in $\mathcal{K}$ such that $g(\mathbf{a}) = g_F(\mathbf{a})$ for every $\mathbf{a} \in F$. So let F be any finite subset of A^t, and consider the finite subset $F' \subseteq A^n$ defined by

$$F' = \{(p_1(\mathbf{a}), \dots, p_n(\mathbf{a})) \mid \mathbf{a} \in F\}.$$

From the fact $f \in \mathbf{Lo}(\mathcal{K})$, it follows that there is an n-ary function $f_{F'}$ in $\mathcal{K}$ such that $f(\mathbf{a}') = f_{F'}(\mathbf{a}')$, for every $\mathbf{a}' \in F'$. Consider the t-ary function g_F defined by $g_F = f_{F'}(p_1, \dots, p_n)$. Note that g_F belongs to $\mathcal{K}$, because $\mathbf{VS}(\mathcal{K}) = \mathcal{K}$. By the definition of $f_{F'}$ and g_F, we have that, for every t-tuple $\mathbf{a} \in F$,

$$g(\mathbf{a}) = f(p_1, \dots, p_n)(\mathbf{a}) = f_{F'}(p_1, \dots, p_n)(\mathbf{a}) = g_F(\mathbf{a}).$$

Since the above argument works for every finite subset F of A^t, we have that g is in $\mathbf{Lo}(\mathcal{K})$.

To prove (b), we show that every constraint in $\mathbf{CM}(\mathbf{LO}(\mathcal{T}))$ is also in $\mathbf{LO}(\mathcal{T})$. Note that the binary equality constraint and the empty constraint are in $\mathbf{LO}(\mathcal{T})$. Thus $\mathbf{CM}(\mathbf{LO}(\mathcal{T}))$ is the set of all conjunctive minors of non-empty families of A-to-B constraints in $\mathbf{LO}(\mathcal{T})$. So let (R, S) be a conjunctive minor of a non-empty family $(R_j, S_j)_{j\in J}$ of constraints in $\mathbf{LO}(\mathcal{T})$ via a scheme H with indeterminate set V. Consider the tight conjunctive minor (R_0, S_0) of the family $(R_j, S_j)_{j\in J}$ via the same scheme $H = (h_j)_{j\in J}$. Note that every relaxation of (R, S) is a relaxation of (R_0, S_0). Thus to prove that $(R, S) \in \mathbf{LO}(\mathcal{T})$, it is enough to show that every finite relaxation of (R_0, S_0) is in $\mathcal{T}$, because it follows then that every finite relaxation of (R, S) is in $\mathcal{T}$.

Let (F, S') be a finite relaxation of (R_0, S_0), say F having n distinct elements $\mathbf{a}_1, \dots, \mathbf{a}_n$. Since $F \subseteq R_0$ and R_0 is a tight conjunctive minor of the family $(R_j)_{j\in J}$ via H, we have that, for every $\mathbf{a}_i \in F$, there is a Skolem map $\sigma_i : V \to A$ such that, for all j in J, $(\mathbf{a}_i + \sigma_i)h_j \in R_j$. For each j in J, let F_j be the subset of R_j, given by

$$F_j = \{(\mathbf{a}_i + \sigma_i)h_j \mid \mathbf{a}_i \in F\}.$$

Consider the non-empty family $(F_j, S_j)_{j\in J}$ of constraints with finite antecedents F_j. Clearly, (F, S') is a conjunctive minor of the family $(F_j, S_j)_{j\in J}$, and for each j in J, (F_j, S_j) is a relaxation of (R_j, S_j). Since $\mathbf{CM}(\mathcal{T}) = \mathcal{T}$, and for each j in J, (R_j, S_j) is in $\mathbf{LO}(\mathcal{T})$, we have that every member of the family $(F_j, S_j)_{j\in J}$ belongs to $\mathcal{T}$. Hence (F, S') is a conjunctive minor of a family of members of $\mathcal{T}$, and thus (F, S') is also in $\mathcal{T}$. □

From Theorem 1, Theorem 2 and Theorem 3, we get the following factorization of the closure operators $\mathbf{FSC} \circ \mathbf{CSF}$ and $\mathbf{CSF} \circ \mathbf{FSC}$:

Theorem 4. *Consider arbitrary non-empty sets A and B. For any class of functions $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ and any set $\mathcal{T}$ of A-to-B relational constraints, we have:*

(i) $\mathbf{FSC}(\mathbf{CSF}(\mathcal{K})) = \mathbf{Lo}(\mathbf{VS}(\mathcal{K}))$, *and*
(ii) $\mathbf{CSF}(\mathbf{FSC}(\mathcal{T})) = \mathbf{LO}(\mathbf{CM}(\mathcal{T}))$.

4. Galois connections between functions and constraints with arity restrictions

Let n and m be positive integers. For any set $\mathcal{T}$ of A-to-B constraints, we denote by $\mathbf{FSC}_n(\mathcal{T})$ the class of all n-ary functions satisfying every member of $\mathcal{T}$, and for any class $\mathcal{K}$ of B-valued functions on A, we denote by $\mathbf{CSF}_m(\mathcal{K})$ the set of all m-ary constraints satisfied by every member of $\mathcal{K}$. That is,

- $\mathbf{FSC}_n(\mathcal{T}) = B^{A^n} \cap \mathbf{FSC}(\mathcal{T})$, and
- $\mathbf{CSF}_m(\mathcal{K}) = \mathcal{Q}_m \cap \mathbf{CSF}(\mathcal{K})$, where $\mathcal{Q}_m$ denotes the set of all m-ary A-to-B constraints, i.e. the cartesian product $\mathcal{P}(A^m) \times \mathcal{P}(B^m)$ of the set of all subsets of A^m and the set of all subsets of B^m.

Thus a class $\mathcal{K}_n \subseteq B^{A^n}$ of n-ary B-valued functions on A is said to be *definable within* B^{A^n} by a set $\mathcal{T}$ of A-to-B constraints, if $\mathcal{K}_n = \mathbf{FSC}_n(\mathcal{T})$, and a set $\mathcal{T}_m$ of m-ary A-to-B constraints is said to be *characterized within* $\mathcal{Q}_m$ by a set $\mathcal{K}$ of B-valued functions on A, if $\mathcal{T}_m = \mathbf{CSF}_m(\mathcal{K})$.

4.1. **Restricting function arities.** We begin with the characterization of the closed classes of functions of fixed arities definable by relational constraints, and the description of the dual closed sets characterized by functions of given arities. A class $\mathcal{K}_n$ of n-ary B-valued functions on A is said to be *closed under n-ary simple variable substitutions* if every n-ary function obtained from a member of $\mathcal{K}_n$ by simple variable substitution also belongs to $\mathcal{K}_n$, that is, if $\mathcal{K}_n = B^{A^n} \cap \mathbf{VS}(\mathcal{K}_n)$. We denote by $\mathbf{VS}_n(\mathcal{K}_n)$ the *closure under n-ary simple variable substitutions of $\mathcal{K}_n$* given by $\mathbf{VS}_n(\mathcal{K}_n) = B^{A^n} \cap \mathbf{VS}(\mathcal{K}_n)$. Note that if $\mathcal{K}$ is a locally closed class of B-valued functions on A, and closed under simple variable substitutions, and if $\mathcal{K}_n$ is the class of n-ary functions in $\mathcal{K}$, then $\mathcal{K}_n$ is locally closed and it is closed under n-ary simple variable substitutions. The following is an immediate consequence of the definitions above:

Fact 5. *Consider arbitrary non-empty sets A and B, and let n be a positive integer. For any class $\mathcal{K}_n$ of n-ary B-valued functions on A,*

$$B^{A^n} \cap \mathbf{Lo}(\mathbf{VS}(\mathcal{K}_n)) = \mathbf{Lo}(\mathbf{VS}_n(\mathcal{K}_n)).$$

We make use of Fact 5 to prove:

Theorem 5. *Consider arbitrary non-empty sets A and B, and let n be a positive integer. For any class of n-ary functions $\mathcal{K}_n \subseteq B^{A^n}$ the following conditions are equivalent:*

(i) *$\mathcal{K}_n$ is locally closed and it is closed under n-ary simple variable substitutions;*

(ii) *$\mathcal{K}_n$ is definable within B^{A^n} by some set of A-to-B constraints.*

Proof. To prove $(ii) \Rightarrow (i)$, assume (ii), i.e. $\mathcal{K}_n = \mathbf{FSC}_n(\mathcal{T})$, for some set $\mathcal{T}$ of A-to-B constraints. Let $\mathcal{K} = \mathbf{FSC}(\mathcal{T})$. By Theorem 1, we have that $\mathcal{K}$ is locally closed and it is closed under simple variable substitutions, and since $\mathcal{K}_n = B^{A^n} \cap \mathcal{K}$, it follows from the comment preceeding Fact 5 that $\mathcal{K}_n$ is locally closed and it is closed under n-ary simple variable substitutions.

To show that $(i) \Rightarrow (ii)$ holds, assume (i), and let $\mathcal{K} = \mathbf{VS}(\mathcal{K}_n)$. Since $\mathbf{Lo}(\mathcal{K})$ is closed under simple variable substitutions, it follows from Theorem 1, that $\mathbf{Lo}(\mathcal{K})$ is definable by some set $\mathcal{T}$ of A-to-B constraints. By Fact 5 $\mathcal{K}_n$ is the class of n-ary functions in $\mathbf{Lo}(\mathcal{K})$, and thus $\mathcal{K}_n$ is definable within B^{A^n} by $\mathcal{T}$. □

Note that for $n = 1$, every class $\mathcal{K} \subseteq B^A$ of unary B-valued functions on A is closed under unary simple variable substitutions. Thus, from Theorem 5, it follows:

Corollary 1. *Consider arbitrary non-empty sets A and B. A class $\mathcal{K}$ of unary B-valued functions on A is definable within B^A by some set of A-to-B constraints if and only if $\mathcal{K}$ is locally closed.*

Theorem 5 provides necessary and sufficient closure conditions for a class of external operations of fixed arity to be definable by relational constraints. To describe the closed sets of relational constraints characterized by external operations of a given arity, we need to strengthen the notion of local closure for sets of constraints.

For a positive integer n, we say that a set $\mathcal{T}$ of relational constraints is *n-locally closed* if $\mathcal{T}$ contains every A-to-B constraint (R, S) such that the set of all its relaxations with antecedent of size at most n is contained in $\mathcal{T}$. The *n-local closure* of a set $\mathcal{T}$ of relational constraints is the smallest n-locally closed set of constraints containing $\mathcal{T}$, and it is denoted by $\mathbf{LO}_n(\mathcal{T})$. Note that every n-locally closed set of constraints is in particular locally closed. In fact, for any set $\mathcal{T}$ of A-to-B relational constraints, $\mathbf{LO}(\mathcal{T}) = \cap_{m \geq 1} \mathbf{LO}_m(\mathcal{T})$. Similarly to the closure $\mathbf{LO}(\mathcal{T})$, it is easy to see that $\mathbf{LO}_n(\mathcal{T})$ is the set of constraints obtained from $\mathcal{T}$ by adding all those constraints whose finite relaxations with antecedent of size at most n are all in $\mathcal{T}$. From these observations it follows:

Fact 6. *Consider arbitrary non-empty sets A and B, and let n be a positive integer.*

(a) *The operator $\mathcal{T} \mapsto \mathbf{LO}_n(\mathcal{T})$ is a closure operator on the set of all A-to-B relational constraints.*
(b) *For any set $\mathcal{T}$ of A-to-B relational constraints, $(\mathbf{LO}_n(\mathcal{T}))_{n\geq 1}$ is a descending chain under inclusion, i.e. $\mathbf{LO}_m(\mathcal{T}) \subseteq \mathbf{LO}_n(\mathcal{T})$ whenever $m \geq n$, and its infimum is $\mathbf{LO}(\mathcal{T})$.*

The following analogue of Theorem 2 shows that, in addition, parametrized local closure guarantees the existence of characterizations of sets of constraints by classes of functions of fixed arities.

Theorem 6. *Consider arbitrary non-empty sets A and B and let n be a positive integer. Let $\mathcal{T}$ be a set of A-to-B relational constraints. Then the following are equivalent:*

(i) *$\mathcal{T}$ is n-locally closed and contains the binary equality constraint, the empty constraint, and it is closed under formation of conjunctive minors;*
(ii) *$\mathcal{T}$ is characterized by some set of n-ary B-valued functions on A.*

Proof. To show that $(ii) \Rightarrow (i)$, assume (ii). From Theorem 2, it follows that $\mathcal{T}$ contains the binary equality constraint, the empty constraint, and it is closed under formation of conjunctive minors. Thus to show that $(ii) \Rightarrow (i)$ holds, we only have to prove that $\mathcal{T}$ is n-locally closed. Let (R, S) be an m-ary constraint not in $\mathcal{T}$. From (ii), it follows that there is an n-ary function f satisfying every constraint in $\mathcal{T}$ but not (R, S), i.e. there are $\mathbf{a}^1, \ldots, \mathbf{a}^n \in R$ such that $f(\mathbf{a}^1, \ldots, \mathbf{a}^n) \notin S$. Let $F = \{\mathbf{a}^1, \ldots, \mathbf{a}^n\}$. Clearly, the constraint (F, S) is a relaxation of (R, S) with antecedent of size at most n, which is not satisfied by f. Hence (F, S) does not belong to $\mathcal{T}$.

To prove the implication $(i) \Rightarrow (ii)$, we show that for each constraint (R, S) not in $\mathcal{T}$, there is an n-ary function satisfying every constraint in $\mathcal{T}$, but not (R, S).

Suppose that (R, S) does not belong to $\mathcal{T}$. Since $\mathcal{T}$ is n-locally closed, we know that there is a relaxation (F, S') of (R, S), with finite antecedent of size $m \leq n$, which does not belong to $\mathcal{T}$. Also, by Fact 6 (b) it follows that $\mathcal{T}$ is locally closed. Since $\mathcal{T}$ also contains the binary equality constraint, the empty constraint, and it is closed under formation of conjunctive minors, it follows from Theorem 2 that $\mathcal{T}$ is characterized by some set of B-valued functions on A. Let g be a function separating (F, S') from $\mathcal{T}$, i.e. g satisfies every constraint in $\mathcal{T}$, but not (F, S'). Note that F has size $m \leq n$. Thus, by identification of variables and addition of inessential variables, we can obtain from g a separating function g' of arity n, and the proof of implication $(ii) \Rightarrow (i)$ is complete. $\square$

We say that a set $\mathcal{T}$ of relational constraints is *closed under arbitrary unions* if $(\cup_{i\in I} R_i, \cup_{i\in I} S_i)$ is in $\mathcal{T}$, whenever $(R_i, S_i)_{i\in I}$ is a non-empty family of members of $\mathcal{T}$. Closure under arbitrary unions is closely related to the notion of 1-local closure:

Proposition 1. *If $\mathcal{T}$ is a set of relational constraints closed under taking relaxations, then $\mathcal{T}$ is closed under arbitrary unions if and only if it is* 1-*locally closed.*

Proof. Clearly, every set of relational constraints closed under arbitrary unions is 1-locally closed. For the converse, let $(R_i, S_i)_{i\in I}$ be a non-empty family of members of $\mathcal{T}$. Since $\mathcal{T}$ is closed under taking relaxations, we have that $(\{r\}, \cup_{i\in I} S_i)$ belongs to $\mathcal{T}$ for every r in $\cup_{i\in I} R_i$. By 1-local closure, we conclude that $(\cup_{i\in I} R_i, \cup_{i\in I} S_i)$ is in $\mathcal{T}$. □

Using Proposition 1, we obtain as a particular case of Theorem 6 the following description of the sets of constraints characterized by unary functions.

Corollary 2. *Consider arbitrary non-empty sets A and B. Let $\mathcal{T}$ be a set of A-to-B relational constraints. Then the following are equivalent:*

(i) *$\mathcal{T}$ contains the binary equality constraint and the empty constraint, and it is closed under arbitrary unions and closed under formation of conjunctive minors;*
(ii) *$\mathcal{T}$ is characterized by some set of unary B-valued functions on A.*

The closure operators associated with the Galois connection $\mathbf{FSC}_n\mathbf{-CSF}$, have decompositions analogous to those given in Theorem 4. To establish them, one needs the following (statement (b) in Theorem 7 below is the analogue of Proposition 3.8 (ii) in [13] concerning sets of relations):

Theorem 7. *Consider arbitrary non-empty sets A and B, and for a positive integer n, let $\mathcal{K}_n \subseteq B^{A^n}$ be a class of n-ary functions, and $\mathcal{T}$ a set of A-to-B relational constraints. The following hold:*

(a) *If $\mathcal{K}_n = \mathbf{VS}_n(\mathcal{K}_n)$, then $\mathbf{VS}_n(\mathbf{Lo}(\mathcal{K}_n)) = \mathbf{Lo}(\mathcal{K}_n)$;*
(b) *If $\mathbf{CM}(\mathcal{T}) = \mathcal{T}$, then $\mathbf{CM}(\mathbf{LO}_n(\mathcal{T})) = \mathbf{LO}_n(\mathcal{T})$.*

Proof. First we prove (a). By (a) of Theorem 3 it follows that

$$\mathbf{VS}(\mathbf{Lo}(\mathbf{VS}(\mathcal{K}_n))) = \mathbf{Lo}(\mathbf{VS}(\mathcal{K}_n))$$

and therefore

$$B^{A^n} \cap \mathbf{VS}(\mathbf{Lo}(\mathbf{VS}(\mathcal{K}_n))) = B^{A^n} \cap \mathbf{Lo}(\mathbf{VS}(\mathcal{K}_n)).$$

Clearly, $\mathbf{VS}_n(\mathbf{Lo}(\mathcal{K}_n)) \subseteq B^{A^n} \cap \mathbf{VS}(\mathbf{Lo}(\mathbf{VS}(\mathcal{K}_n)))$. By Fact 5,

$$B^{A^n} \cap \mathbf{Lo}(\mathbf{VS}(\mathcal{K}_n)) = \mathbf{Lo}(\mathbf{VS}_n(\mathcal{K}_n))$$

and since $\mathcal{K}_n = \mathbf{VS}_n(\mathcal{K}_n)$, we have $\mathbf{Lo}(\mathbf{VS}_n(\mathcal{K}_n)) = \mathbf{Lo}(\mathcal{K}_n)$. Hence,

$$\mathbf{VS}_n(\mathbf{Lo}(\mathcal{K}_n)) \subseteq \mathbf{Lo}(\mathcal{K}_n) \subseteq \mathbf{VS}_n(\mathbf{Lo}(\mathcal{K}_n))$$

i.e. $\mathbf{VS}_n(\mathbf{Lo}(\mathcal{K}_n)) = \mathbf{Lo}(\mathcal{K}_n)$.[1]

A proof of (b) in Theorem 7 is obtained essentially by replacing, in the proof of (b) of Theorem 3, $\mathbf{LO}$ by $\mathbf{LO}_n$, and "finite relaxation" by "finite relaxation with antecedent of size at most n". The key observation is that $\mid F_j \mid \leq \mid F \mid \leq n$. $\square$

From Theorem 5, Theorem 6 and Theorem 7, we obtain factorizations of the closure operators $\mathbf{FSC}_n \circ \mathbf{CSF}$ and $\mathbf{CSF} \circ \mathbf{FSC}_n$, as compositions of the operators $\mathbf{Lo}$ and $\mathbf{VS}_n$, and $\mathbf{LO}_n$ and $\mathbf{CM}$, respectively:

Theorem 8. *Consider arbitrary non-empty sets A and B, and let n be a positive integer. For any class of n-ary functions $\mathcal{K}_n \subseteq B^{A^n}$ and any set $\mathcal{T}$ of A-to-B relational constraints, the following hold:*

(i) $\mathbf{FSC}_n(\mathbf{CSF}(\mathcal{K}_n)) = \mathbf{Lo}(\mathbf{VS}_n(\mathcal{K}_n))$, *and*
(ii) $\mathbf{CSF}(\mathbf{FSC}_n(\mathcal{T})) = \mathbf{LO}_n(\mathbf{CM}(\mathcal{T}))$.

4.2. **Restricting constraint arities.** We now consider arity restrictions on sets of relational constraints. First we determine necessary and sufficient closure conditions for function class definability by sets of constraints of fixed arity. The following parameterized notion of local closure corresponds to that appearing in [13], for operations on a given set. For a positive integer m, a class $\mathcal{K}$ of B-valued functions on A is said to be *m-locally closed* if for every B-valued function f on A the following holds: if every restriction of f to a finite subset $D \subseteq A^n$ of size at most m, coincides with the restriction to D of some member of $\mathcal{K}$, then f belongs to $\mathcal{K}$. (See [6] and [7] for two different but somewhat related notions of m-local closure defined on classes of *pseudo-Boolean functions*, i.e. maps of the form $\{0,1\}^n \to \mathbf{R}$, where $\mathbf{R}$ denotes the field of real numbers.) For any class of functions $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ the smallest m-locally closed class of functions containing $\mathcal{K}$, which we denote by $\mathbf{Lo}_m(\mathcal{K})$, is called the *$m$-local closure* of $\mathcal{K}$, and it is the class obtained from $\mathcal{K}$ by adding all those functions whose restriction to each subset of its domain A^n of size at most m coincides with a restriction of some member of $\mathcal{K}$. The following summarizes some immediate consequences of the definitions and the above observations.

Fact 7. *Consider arbitrary non-empty sets A and B, and let m be a positive integer.*

(a) *The operator $\mathcal{K} \mapsto \mathbf{Lo}_m(\mathcal{K})$ is a closure operator on $\cup_{n\geq 1} B^{A^n}$.*
(b) *For any class $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$, we have $\mathbf{Lo}_n(\mathcal{T}) \subseteq \mathbf{Lo}_m(\mathcal{T})$ whenever $n \geq m$, and $\mathbf{Lo}(\mathcal{K}) = \cap_{n\geq 1}\mathbf{Lo}_n(\mathcal{K})$. Thus every m-locally closed class of functions is in particular locally closed.*

As in the case of sets of relational constraints, it turns out that this parametrized notion of local closure, together with the conditions given by

[1]Alternatively, a proof of (a) in Theorem 7 is obtained by following the exact same steps as in the proof of (a) of Theorem 3 and taking $t = n$ (footnote added in May, 2006).

Theorem 1, suffices to characterize the classes of functions definable by sets of constraints of fixed arities.

Theorem 9. *Consider arbitrary non-empty sets A and B and let m be a positive integer. For a class of functions $\mathcal{K} \subseteq \cup_{n \geq 1} B^{A^n}$ the following conditions are equivalent:*

(i) *$\mathcal{K}$ is m-locally closed and it is closed under simple variable substitutions;*
(ii) *$\mathcal{K}$ is definable by some set of A-to-B m-ary constraints.*

Proof. To prove the implication $(ii) \Rightarrow (i)$, assume (ii). From Theorem 1, it follows that $\mathcal{K}$ is closed under simple variable substitutions. To see that $\mathcal{K}$ is m-locally closed, let f be an n-ary function not in $\mathcal{K}$, and let (R, S) be an A-to-B m-ary constraint satisfied by every function g in $\mathcal{K}$ but not satisfied by f. Hence, for some $\mathbf{a}^1, \ldots, \mathbf{a}^n \in R$, we have $f(\mathbf{a}^1, \ldots, \mathbf{a}^n) \notin S$, and $g(\mathbf{a}^1, \ldots, \mathbf{a}^n) \in S$, for every n-ary function g in $\mathcal{K}$. Let $F = \{(\mathbf{a}^1(i), \ldots, \mathbf{a}^n(i)) : i \in \mathbf{m}\}$. Clearly, the restriction of f to the set F, which has size at most m, does not coincide with that of any member of $\mathcal{K}$.

Now we prove the implication $(i) \Rightarrow (ii)$. If $\mathcal{K} = \emptyset$, then the single constraint $(A^m, \emptyset)$ clearly defines $\mathcal{K}$. Hence, we may assume that $\mathcal{K}$ is non-empty. Consider a function $g \notin \mathcal{K}$, say of arity n. Thus there is a restriction g_F of g to a non-empty finite subset $F \subseteq A^n$ of size $p \leq m$ which does not agree with any function in $\mathcal{K}$ restricted to F.

Let $\mathbf{a}^1, \ldots, \mathbf{a}^n$ be any m-tuples in A^m, such that $F = \{(\mathbf{a}^1(i), \ldots, \mathbf{a}^n(i)) : i \in \mathbf{m}\}$. Let (R, S) be the m-ary constraint whose antecedent is $R = \{\mathbf{a}^1, \ldots, \mathbf{a}^n\}$, and whose consequent is given by $S = \{f(\mathbf{a}^1, \ldots, \mathbf{a}^n) : f \in \mathcal{K}_n\}$, where $\mathcal{K}_n$ denotes the set of n-ary functions in $\mathcal{K}$. It follows from the definition of R and S that (R, S) is an A-to-B m-ary constraint, g does not satisfy (R, S), and, since $\mathcal{K}$ is closed under simple variable substitutions, every function in $\mathcal{K}$ satisfies (R, S). □

Now we describe the closed sets of constraints of fixed arities characterized by the functions of several variables satisfying them. Let $\mathcal{T}_m$ be a set of A-to-B m-ary relational constraints. We say that $\mathcal{T}_m$ is *closed under formation of m-ary conjunctive minors* if whenever every member of a non-empty family $(R_j, S_j)_{j \in J}$ of constraints is in $\mathcal{T}_m$, all m-ary conjunctive minors of the family are also in $\mathcal{T}_m$.

For a positive integer m, we refer to the constraint whose antecedent and consequent consists of all m-tuples with all arguments equal, as the *m-ary equality constraint*. Note that, for $2 \leq m$, the m-ary equality constraint is a tight conjunctive minor of a family of constraints with $m - 1$ binary equality constraints, and, for $m > 1$, the binary equality constraint is a tight conjunctive minor of the m-ary equality constraint. For any set $\mathcal{T}_m$ of m-ary constraints, let $\mathbf{CM}_m(\mathcal{T}_m)$ denote the smallest set of constraints containing $\mathcal{T}_m$, closed under formation of m-ary conjunctive minors, and containing the m-ary equality constraint and the empty constraint. By the Transitivity Lemma it follows that $\mathbf{CM}_m(\mathcal{T}_m) = \mathcal{Q}_m \cap \mathbf{CM}(\mathcal{T}_m)$, where $\mathcal{Q}_m$ denotes the set of all A-to-B m-ary relational constraints.

Lemma 1. *Consider arbitrary non-empty sets A and B. For any set $\mathcal{T}_m$ of m-ary constraints,*

$$\mathcal{Q}_m \cap \mathbf{LO}(\mathbf{CM}(\mathcal{T}_m)) = \mathbf{LO}(\mathbf{CM}_m(\mathcal{T}_m)),$$

where $\mathcal{Q}_m$ denotes the set of all A-to-B m-ary relational constraints.

Proof. It is easy to verify that for any set $\mathcal{T}$ of relational constraints,

$$\mathcal{Q}_m \cap \mathbf{LO}(\mathcal{T}) = \mathbf{LO}(\mathcal{Q}_m \cap \mathcal{T}).$$

By the remark preceding the lemma, it follows that for any set $\mathcal{T}_m$ of m-ary constraints,

$$\mathcal{Q}_m \cap \mathbf{LO}(\mathbf{CM}(\mathcal{T}_m)) = \mathbf{LO}(\mathcal{Q}_m \cap \mathbf{CM}(\mathcal{T}_m)) = \mathbf{LO}(\mathbf{CM}_m(\mathcal{T}_m)).$$

□

From the above definitions, one can easily verify that the following also holds:

Fact 8. *Consider arbitrary non-empty sets A and B. If $\mathcal{T}$ is a locally closed set of A-to-B relational constraints, closed under formation of conjunctive minors, and $\mathcal{T}_m$ is the set of all m-ary contraints in $\mathcal{T}$, then $\mathcal{T}_m$ is locally closed, and closed under formation of m-ary conjunctive minors.*

We use Lemma 1 and Fact 8 to prove the following, which provides necessary and sufficient conditions for a set of constraints of a given arity to be characterized by external operations:

Theorem 10. *Consider arbitrary non-empty sets A and B and let m be a positive integer. Let $\mathcal{Q}_m$ be the set of all A-to-B m-ary relational constraints, and let $\mathcal{T}_m \subseteq \mathcal{Q}_m$. Then the following are equivalent:*

(i) *$\mathcal{T}_m$ is locally closed, contains the m-ary equality constraint and the m-ary empty constraint, and it is closed under formation of m-ary conjunctive minors;*

(ii) *$\mathcal{T}_m$ is characterized within $\mathcal{Q}_m$ by some set of B-valued functions on A.*

Proof. To see that implication $(ii) \Rightarrow (i)$ holds, let $\mathcal{K} \subseteq \cup_{n \geq 1} B^{A^n}$ be a set of functions such that $\mathcal{T}_m = \mathbf{CSF}_m(\mathcal{K})$. By Theorem 2, we have that $\mathbf{CSF}(\mathcal{K})$ is locally closed, contains the binary equality constraint and the empty constraint, and it is closed under formation of conjunctive minors. Hence by Fact 8, $\mathcal{T}_m$ is locally closed, contains the m-ary equality constraint and m-ary empty constraint, and it is closed under formation of m-ary conjunctive minors.

To prove $(i) \Rightarrow (ii)$, assume (i). Let $\mathcal{T} = \mathbf{CM}(\mathcal{T}_m)$. By (b) in Theorem 3, we have that $\mathbf{LO}(\mathcal{T})$ contains the binary equality constraint, the empty constraint, and it is closed under formation of conjunctive minors. Since $\mathbf{LO}(\mathcal{T})$ is locally closed it follows from Theorem 2 that $\mathbf{LO}(\mathcal{T})$ is characterized by some set of B-valued functions of several variables on A, i.e. $\mathbf{LO}(\mathcal{T}) = \mathbf{CSF}(\mathcal{K})$ for some set $\mathcal{K}$ of B-valued functions on A. By Lemma 1, we have $\mathcal{T}_m = \mathcal{Q}_m \cap \mathbf{LO}(\mathcal{T})$. Thus $\mathcal{T}_m = \mathbf{CSF}_m(\mathcal{K})$, i.e. $\mathcal{T}_m$ is characterized within $\mathcal{Q}_m$ by some set of B-valued functions on A. □

Similarly to the Galois correspondences $\mathbf{FSC}_n - \mathbf{CSF}$, the closure operators $\mathbf{FSC} \circ \mathbf{CSF}_m$ and $\mathbf{CSF}_m \circ \mathbf{FSC}$ can be represented as compositions of $\mathbf{Lo}_m$ and $\mathbf{VS}$, and $\mathbf{LO}$ and $\mathbf{CM}_m$, respectively. To establish such factorizations, we need the following:

Theorem 11. *Consider arbitrary non-empty sets A and B, and let $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ be a class of B-valued functions on A, and $\mathcal{T}_m$ be a set of m-ary A-to-B relational constraints. The following hold:*

(a) *If $\mathbf{VS}(\mathcal{K}) = \mathcal{K}$, then $\mathbf{VS}(\mathbf{Lo}_m(\mathcal{K})) = \mathbf{Lo}_m(\mathcal{K})$;*
(b) *If $\mathbf{CM}_m(\mathcal{T}_m) = \mathcal{T}_m$, then $\mathbf{CM}_m(\mathbf{LO}(\mathcal{T}_m)) = \mathbf{LO}(\mathcal{T}_m)$.*

Proof. The proof (a) can be obtained by replacing, in the proof of (a) of Theorem 3, $\mathbf{Lo}$ by $\mathbf{Lo}_m$, and "finite subset F" by "finite subset F of size at most m".

To prove (b), we make use of (b) in Theorem 3. Let $\mathcal{Q}_m$ be the set of all A-to-B m-ary relational constraints. By Lemma 1 we have that $\mathcal{Q}_m \cap \mathbf{LO}(\mathbf{CM}(\mathcal{T}_m)) = \mathbf{LO}(\mathbf{CM}_m(\mathcal{T}_m))$, and by (b) in Theorem 3, it follows that

$$\mathcal{Q}_m \cap \mathbf{CM}(\mathbf{LO}(\mathbf{CM}(\mathcal{T}_m))) = \mathcal{Q}_m \cap \mathbf{LO}(\mathbf{CM}(\mathcal{T}_m)) = \mathbf{LO}(\mathbf{CM}_m(\mathcal{T}_m)).$$

Since $\mathbf{CM}_m(\mathbf{LO}(\mathbf{CM}(\mathcal{T}_m))) = \mathcal{Q}_m \cap \mathbf{CM}(\mathbf{LO}(\mathbf{CM}(\mathcal{T}_m)))$, we get

$$\mathbf{CM}_m(\mathbf{LO}(\mathbf{CM}(\mathcal{T}_m))) = \mathbf{LO}(\mathbf{CM}_m(\mathcal{T}_m)).$$

Observe that

$$\mathbf{CM}_m(\mathbf{LO}(\mathcal{T}_m)) \subseteq \mathbf{CM}_m(\mathbf{LO}(\mathbf{CM}(\mathcal{T}_m)))$$

and $\mathbf{LO}(\mathbf{CM}_m(\mathcal{T}_m)) = \mathbf{LO}(\mathcal{T}_m)$ because $\mathcal{T}_m = \mathbf{CM}_m(\mathcal{T}_m)$. Thus

$$\mathbf{CM}_m(\mathbf{LO}(\mathcal{T}_m)) \subseteq \mathbf{CM}_m(\mathbf{LO}(\mathbf{CM}(\mathcal{T}_m))) = \mathbf{LO}(\mathcal{T}_m).$$

Since $\mathbf{LO}(\mathcal{T}_m) \subseteq \mathbf{CM}_m(\mathbf{LO}(\mathcal{T}_m))$, we conclude that

$$\mathbf{CM}_m(\mathbf{LO}(\mathcal{T}_m)) = \mathbf{LO}(\mathcal{T}_m).[2]$$

□

Property (a) in the above Theorem, is analogous to (ii) of Lemma 2.5 in [13]. From Theorem 9, Theorem 10 and Theorem 11, we obtain the analogue of Theorem 4.

[2]Alternatively, a proof of (b) in Theorem 11 is obtained by following the exact same steps as in the proof of (b) of Theorem 3 and assuming that all constraints are m-ary (footnote added in May, 2006).

Theorem 12. *Consider arbitrary non-empty sets A and B, and let m be a positive integer. For any class of functions $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ and any set $\mathcal{T}_m$ of m-ary A-to-B relational constraints, the following hold:*

(i) $\mathbf{FSC}(\mathbf{CSF}_m(\mathcal{K})) = \mathbf{Lo}_m(\mathbf{VS}(\mathcal{K}))$, *and*
(ii) $\mathbf{CSF}_m(\mathbf{FSC}(\mathcal{T}_m)) = \mathbf{LO}(\mathbf{CM}_m(\mathcal{T}_m))$.

4.3. **Simultaneous restrictions to the arities of functions and constraints.** Let $\mathcal{K}$ be a class of B-valued functions on A, and $\mathcal{T}$ be a set of A-to-B relational constraints. It is not difficult to see that for any positive integers n and m, $B^{A^n} \cap \mathbf{Lo}_m(\mathcal{K}) = \mathbf{Lo}_m(B^{A^n} \cap \mathcal{K})$, and that $\mathcal{Q}_m \cap \mathbf{LO}_n(\mathcal{T}) = \mathbf{LO}_n(\mathcal{Q}_m \cap \mathcal{T})$, where $\mathcal{Q}_m$ denotes the set of all A-to-B m-ary constraints. Using these facts, Theorems 5 and 9, and Theorems 6 and 10 can be combined as follows:

Theorem 13. *Consider arbitrary non-empty sets A and B, and let n and m be positive integers. For a class of n-ary functions $\mathcal{K}_n \subseteq B^{A^n}$ the following conditions are equivalent:*

(i) *$\mathcal{K}_n$ is m-locally closed and it is closed under n-ary simple variable substitutions;*
(ii) *$\mathcal{K}_n$ is definable within B^{A^n} by some set of A-to-B m-ary constraints.*

Proof. $(ii) \Rightarrow (i)$: Suppose that (ii) holds, i.e. $\mathcal{K}_n = \mathbf{FSC}_n(\mathcal{T}_m)$ for some set $\mathcal{T}_m$ of m-ary constraints. Let $\mathcal{K} = \mathbf{FSC}(\mathcal{T}_m)$. By Theorem 9, $\mathcal{K}$ is m-locally closed and it is closed under simple variable substitutions. Since $\mathcal{K}_n = B^{A^n} \cap \mathcal{K}$, $\mathcal{K}_n$ is closed under n-ary simple variable substitutions, and using the fact that $B^{A^n} \cap \mathbf{Lo}_m(\mathcal{K}) = \mathbf{Lo}_m(B^{A^n} \cap \mathcal{K})$, we conclude that $\mathcal{K}_n$ is m-locally closed. Thus (i) holds.

$(i) \Rightarrow (ii)$: Suppose that (i) holds, and let $\mathcal{K} = \mathbf{Lo}_m(\mathbf{VS}(\mathcal{K}_n))$. By Lemma 1, we have that $\mathcal{K}_n = B^{A^n} \cap \mathcal{K}$, and it follows from Theorem 9 that $\mathcal{K}$ is definable by some set of A-to-B m-ary constraints, i.e. $\mathcal{K} = \mathbf{FSC}(\mathcal{T}_m)$ for some set $\mathcal{T}_m$ of m-ary constraints. Hence, $\mathcal{K}_n = B^{A^n} \cap \mathbf{FSC}(\mathcal{T}_m) = \mathbf{FSC}_n(\mathcal{T}_m)$, i.e. (ii) holds. □

Theorem 14. *Consider arbitrary non-empty sets A and B and let n and m be positive integers. Let $\mathcal{Q}_m$ be the set of all A-to-B m-ary relational constraints, and let $\mathcal{T}_m \subseteq \mathcal{Q}_m$. Then the following are equivalent:*

(i) *$\mathcal{T}_m$ is n-locally closed, contains the m-ary equality constraint and m-ary empty constraint, and it is closed under formation of m-ary conjunctive minors;*
(ii) *$\mathcal{T}_m$ is characterized within $\mathcal{Q}_m$ by some set of n-ary B-valued functions on A.*

Proof. The proof of Theorem 14 follows in complete analogy with the proof of Theorem 13, using Theorem 6 and the remarks preceding Theorem 13. □

Furthermore, combining Theorem 8 and Theorem 12 we get:

Theorem 15. *Consider arbitrary non-empty sets A and B, and let n and m be positive integers. For any class of n-ary B-valued functions on A and any set $\mathcal{T}_m$ of m-ary A-to-B relational constraints, the following hold:*

(i) $\mathbf{FSC}_n(\mathbf{CSF}_m(\mathcal{K}_n)) = \mathbf{Lo}_m(\mathbf{VS}_n(\mathcal{K}_n))$, *and*

(ii) $\mathbf{CSF}_m(\mathbf{FSC}_n(\mathcal{T}_m)) = \mathbf{LO}_n(\mathbf{CM}_m(\mathcal{T}_m))$.

Proof. By the remarks preceding Theorem 13, and Theorem 12 (i) and Theorem 8 (ii), we obtain, respectively,

(i) $\mathbf{FSC}_n(\mathbf{CSF}_m(\mathcal{K}_n)) = B^{A^n} \cap \mathbf{FSC}(\mathbf{CSF}_m(\mathcal{K}_n)) =$

$B^{A^n} \cap \mathbf{Lo}_m(\mathbf{VS}(\mathcal{K}_n)) = \mathbf{Lo}_m(B^{A^n} \cap \mathbf{VS}(\mathcal{K}_n)) = \mathbf{Lo}_m(\mathbf{VS}_n(\mathcal{K}_n))$, and for

(ii) $\mathbf{CSF}_m(\mathbf{FSC}_n(\mathcal{T}_m)) = \mathcal{Q}_m \cap \mathbf{CSF}(\mathbf{FSC}_n(\mathcal{T}_m)) =$

$\mathcal{Q}_m \cap \mathbf{LO}_n(\mathbf{CM}(\mathcal{T}_m)) = \mathbf{LO}_n(\mathcal{Q}_m \cap \mathbf{CM}(\mathcal{T}_m)) = \mathbf{LO}_n(\mathbf{CM}_m(\mathcal{T}_m))$.

□

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GALOIS CONNECTIONS FOR GENERALIZED FUNCTIONS AND RELATIONAL CONSTRAINTS

MIGUEL COUCEIRO

Abstract. In this paper we focus on functions of the form $A^n \to \mathcal{P}(B)$, for possibly different arbitrary non-empty sets A and B, where $\mathcal{P}(B)$ denotes the set of all subsets of B. These mappings are called *multivalued functions*, and they generalize total and partial functions. We study Galois connections between these generalized functions and ordered pairs (R, S) of relations on A and B, respectively, called *constraints*. We describe the Galois closed sets, and decompose the associated Galois operators, by means of necessary and sufficient conditions which specialize, in the total single-valued case, to those given in [CF].

1. Introduction

In [Pö1] and [Pö2], Pöschel developed a Galois theory for heterogeneous functions (i.e. functions from a cartesian product $A_{i_1} \times \ldots \times A_{i_n}$ to A_j, where the underlying sets belong to a family $(A_i)_{i \in I}$ of pairwise disjoint finite sets), in which the closed classes of functions are defined by invariant multisorted relations $R = \cup_{i \in I} R_i$ where $R_i \subseteq A_i^m$, and dually, the closed systems of relations are charaterized by the functions preserving them (for further background, see also [PK]). Still in the finite case, Pippenger studied in [Pi2], the particular bi-sorted case of finite functions (i.e. mappings of the form $f : A^n \to B$), and introduced a Galois framework in which the dual objects are replaced by ordered pairs (R, S) of relations on A and B, respectively, called *constraints*, and where the multisorted preservation is replaced by the more stringent notion of constraint satisfaction. This latter theory was extended in [CF] by removing the finiteness condition on the underlying sets A and B.

In this paper we study the more general notion of *multivalued functions*, that is, mappings of the form $A^n \to \mathcal{P}(B)$, where $\mathcal{P}(B)$ denotes the set of all subsets of B. We introduce the Galois connection between sets of these generalized functions and sets of constraints (R, S) (where $R \subseteq A^m$ and

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$S \subseteq B^m$), based on a more general notion of constraint satisfaction (see Section 2). Although the functions that we consider can still be treated as maps $A^n \to C$, where $C = \mathcal{P}(B)$, our approach extends the framework in [Pi2] and [CF], because we take as dual objects constraints in which the "consequent" S is a relation defined over B, and not over $C = \mathcal{P}(B)$ as it is defined in these papers.

We describe the Galois closed classes of multivalued functions (Section 3) and the Galois closed sets of constraints (Section 4), in terms of closures which essentially extend to the multivalued case the conditions presented in [CF]. We consider further Galois connections by restricting the set of primal objects to partial functions, and to total multivalued functions, i.e. mappings $A^n \to \mathcal{P}(B)$ which are non-empty-valued on every n-tuple over A. (For universal algebraic analogues, see e.g. [FR] and [B], respectively, and [Rö] for a unified approach to these extensions.) As corollaries we obtain the characterizations, given in [CF], of the closed classes of single-valued functions (see Corollary 1 (c)), and the corresponding dual closed sets of constraints (see Corollary 3). Furthermore, we present factorizations of the closure maps associated with the above-mentioned Galois connections, as compositions of simpler operators.

2. Basic notions

Throughout the paper, we shall always consider arbitrary non-empty base sets A, B, etc. Also, the integers n, m, etc., are assumed to be positive and thought of as Von Neumann ordinals, i.e. each ordinal is the non-empty set of lesser ordinals. With this formalism, n-tuples over a set A are just unary maps from $n = \{0, \dots, n-1\}$ to A. Thus an *m-ary relation R on A* (i.e. a subset $R \subseteq A^m$) is viewed as a set of unary maps $\mathbf{a} = (a_i \mid i \in m)$ from m to A. Furthermore, we shall distinguish between empty relations of different arities, and we write $\emptyset^m$ to denote the m-ary empty relation. For $m = 1$, we use $\emptyset$ (instead of $\emptyset^1$) to denote the unary empty relation. In order to present certain concepts in a unifying setting, e.g. those of total multivalued and partial functions, we shall think of functions as having specific domain, codomain and graph.

An n-ary *multivalued function on A to B* is a map $f : A^n \to \mathcal{P}(B)$, where $\mathcal{P}(B)$ denotes the set of all subsets of B. For $A = B$, these functions are called *multioperations* or *multifunctions on B*, and for $A = \mathcal{P}(B)$ the maps $f : \mathcal{P}(B)^n \to \mathcal{P}(B)$ are said to be *lifted* (see [DP]). By a *class of multivalued functions* we simply mean a set of multivalued functions of various arities. If $f : A^n \to \mathcal{P}(B)$ is non-empty-valued on every n-tuple over A, then f is said to be a *total multivalued function on A to B*. These indeed correspond to total functions in the usual sense, i.e. to each n-tuple over A, they associate at least one element of B. We denote by Θ_{AB} the class of all multivalued functions on A to B, and by Θ^t_{AB} the class of all total multivalued functions on A to B.

In this paper we also consider the following particular cases of multivalued functions. We say that a multivalued function $f : A^n \to \mathcal{P}(B)$ is a *partial function on A to B* if it is either empty or singleton-valued on every n-tuple over A, i.e. if for every $\mathbf{a}$ in A^n, we have $f(\mathbf{a}) = \emptyset$ or $f(\mathbf{a}) = \{b\}$, for some b in B. Although partial functions on A to B are usually defined as maps $p : D \to B$ where $D \subseteq A^n$ (see e.g. [BW], and for partial operations, see e.g. [R, BHP]), it is easy to establish a complete correspondence between these definitions. For each positive integer n, the n-ary partial function e_n which has empty value on every element of A^n, is called the n-ary *empty-valued function*. With Θ^p_{AB} we denote the class of all partial functions on A to B.

Observe that the functions of several variables on A to B considered in [CF], correspond to the partial functions on A to B (as formerly defined) which are, in addition, total. In other words, there is a bijection between $\Theta^s_{AB} = \Theta^t_{AB} \cap \Theta^p_{AB}$ and $\cup_{n\geq 1} B^{A^n}$. In this paper we shall refer to functions in Θ^s_{AB} as *single-valued functions on A to B*.

For a multivalued function $f : A^n \to \mathcal{P}(B)$ and m-tuples $\mathbf{a}^1, \ldots, \mathbf{a}^n$ over A, we write $f(\mathbf{a}^1 \ldots \mathbf{a}^n)$ for the m-ary relation on B, defined by

$$f(\mathbf{a}^1 \ldots \mathbf{a}^n) = \Pi_{i\in m} f((\mathbf{a}^1 \ldots \mathbf{a}^n)(i))$$

where $(\mathbf{a}^1 \ldots \mathbf{a}^n)(i) = (\mathbf{a}^1(i) \ldots \mathbf{a}^n(i))$. Note that if $f((\mathbf{a}^1 \ldots \mathbf{a}^n)(i)) = \emptyset$, for some $i \in m$, then $f(\mathbf{a}^1 \ldots \mathbf{a}^n) = \emptyset^m$. If R is an m-ary relation on A, we denote by fR the m-ary relation on B, defined by

$$fR = \cup\{f(\mathbf{a}^1 \ldots \mathbf{a}^n) : \mathbf{a}^1, \ldots, \mathbf{a}^n \in R\}.$$

An m-ary *A-to-B relational constraint* is an ordered pair (R, S) of relations $R \subseteq A^m$ and $S \subseteq B^m$, called *antecedent* and *consequent*, respectively, of the constraint. A multivalued function $f : A^n \to \mathcal{P}(B)$ is said to *satisfy* the constraint (R, S) if $fR \subseteq S$. Observe that for each $1 \leq m$, every multivalued function on A to B satisfies the m-ary *empty constraint* $(\emptyset^m, \emptyset^m)$, and the m-ary *trivial constraint* (A^m, B^m). Moreover, every partial function on A to B satisfies the *binary equality constraint* $(=_A, =_B)$, where $=_A$ and $=_B$ denote the equality relations on A and on B, respectively.

For a set $\mathcal{T}$ of A-to-B constraints, we denote by $\mathbf{mFSC}(\mathcal{T})$ the class of all multivalued functions on A to B satisfying every constraint in $\mathcal{T}$. The notation **mFSC** stands for "multivalued functions satisfying constraints". A class $\mathcal{M}$ of multivalued functions on A to B is said to be *definable* by a set $\mathcal{T}$ of A-to-B constraints, if $\mathcal{M} = \mathbf{mFSC}(\mathcal{T})$. Similarly, the classes of

(i) total functions of the form $\mathbf{tFSC}(\mathcal{T}) = \Theta^t_{AB} \cap \mathbf{mFSC}(\mathcal{T})$,
(ii) partial functions of the form $\mathbf{pFSC}(\mathcal{T}) = \Theta^p_{AB} \cap \mathbf{mFSC}(\mathcal{T})$, and
(iii) single-valued functions of the form $\mathbf{sFSC}(\mathcal{T}) = \Theta^s_{AB} \cap \mathbf{mFSC}(\mathcal{T})$

are said to be *definable within* $\Theta^t_{AB}, \Theta^p_{AB}$, and Θ^s_{AB}, respectively, by the set $\mathcal{T}$.

Dually, for a class $\mathcal{M}$ of multivalued functions on A to B, we denote by $\mathbf{CSF}(\mathcal{M})$ the set of all A-to-B constraints satisfied by every function in $\mathcal{M}$. Note that **CSF** stands for "constraints satisfied by functions". In analogy with the function case, a set $\mathcal{T}$ of A-to-B constraints is said to be *characterized* by a set $\mathcal{M}$ of multivalued functions, if $\mathcal{T} = \mathbf{CSF}(\mathcal{M})$.

Let V and W be arbitrary sets. It is well known that each binary relation $\triangleright \subseteq V \times W$ induces a *Galois connection* between V and W, determined by the pair of mappings $v : \mathcal{P}(V) \to \mathcal{P}(W)$ and $w : \mathcal{P}(W) \to \mathcal{P}(V)$, defined as follows:

$$v(X) = \{b \in W : a \triangleright b, \text{ for every } a \in X\}$$

$$w(Y) = \{a \in V : a \triangleright b, \text{ for every } b \in Y\}.$$

The associated operators $X \mapsto (w \circ v)(X)$ and $Y \mapsto (v \circ w)(Y)$ are *extensive*, *monotone* and *idempotent*, i.e. they satisfy the following conditions

E. for every $X \subseteq V$ and $Y \subseteq W$,

$$X \subseteq (w \circ v)(X) \text{ and } Y \subseteq (v \circ w)(Y)$$

M. if $X' \subseteq X$ and $Y' \subseteq Y$, then

$$(w \circ v)(X') \subseteq (w \circ v)(X) \text{ and } (v \circ w)(Y') \subseteq (v \circ w)(Y)$$

I. for every $X \subseteq W$ and $Y \subseteq V$,

$$(w \circ v)((w \circ v)(X)) = (w \circ v)(X) \text{ and } (v \circ w)((v \circ w)(Y)) = (v \circ w)(Y)$$

respectively. In other words, $w \circ v$ and $v \circ w$ are *closure operators* on V and W, respectively, and the sets X and Y satisfying $(w \circ v)(X) = X$ and $(v \circ w)(Y) = Y$ are the (*Galois*) *closed sets* associated with v and w. Moreover, $(w \circ v)(X)$ and $(v \circ w)(Y)$ are the smallest closed sets containing $X \subseteq V$ and $Y \subseteq W$, respectively, and are said to be *generated* by X and Y. (For background on Galois connections see e.g. [O], and [Pi1] for a later reference.)

Based on the relation of constraint satisfaction (between multivalued functions and constraints), we define the Galois connection $\mathbf{mFSC} - \mathbf{CSF}$ between sets of multivalued functions and sets of relational constraints. Let V be the class of all multivalued functions on A to B, and W the set of all A-to-B relational constraints. Interpreting $\triangleright$ as the binary relation "satisfies", we have that:

(a) $v(\mathcal{K}) = \mathbf{CSF}(\mathcal{K})$ for every $\mathcal{K} \subseteq V$, and
(b) $w(\mathcal{T}) = \mathbf{mFSC}(\mathcal{T})$ for every $\mathcal{T} \subseteq W$.

Similarly, we define the correspondences $\mathbf{tFSC} - \mathbf{CSF}$, $\mathbf{pFSC} - \mathbf{CSF}$, and $\mathbf{sFSC} - \mathbf{CSF}$, by restricting V to $\Theta^t_{AB}, \Theta^p_{AB}$, and Θ^s_{AB}, respectively.

With this terminology, the classes of generalized functions definable by constraints are exactly the closed sets of functions associated with the corresponding Galois connections, and the sets of relational constraints characterized by generalized functions correspond to the dual Galois closed sets.

3. Galois closed Sets of Generalized Functions

We say that an n-ary multivalued function g on A to B is a *value restriction* of an n-ary multivalued function f on A to B, if for every $\mathbf{a} \in A^n$ we have $g(\mathbf{a}) \subseteq f(\mathbf{a})$. A class $\mathcal{M}$ of multivalued functions on A to B is said to be *closed under taking value restrictions* if every value restriction of a member of $\mathcal{M}$ is also in $\mathcal{M}$. (In [B], where $A = B$ is finite, the non-empty value restrictions of a total multivalued function f are called *subfunctions* of f.)

We now introduce a key concept which extends that of *simple variable substitution* (appearing in [CF], and referred to as *minor* in [Pi2]) to multivalued functions, and subsumes value restrictions. We say that an m-ary multivalued function g from A to B is obtained from an n-ary multivalued function f from A to B by *restrictive variable substitution*, if there is a map l from n to m such that

$$g(\mathbf{a}) \subseteq f(\mathbf{a} \circ l)$$

for every m-tuple $\mathbf{a} \in A^m$. If g is non-empty valued, i.e. $g(\mathbf{a}) \neq \emptyset$ for every $\mathbf{a} \in A^m$, then we say that g is obtained from f by *non-empty restrictive variable substitution.* Note that within Θ^s_{AB}, the inclusion may be replaced by equality, and in this case we use the term "simple" instead of "restrictive" (see [CF]).

A class $\mathcal{M}$ of multivalued functions of several variables is said to be *closed under restrictive variable substitutions* if every multivalued function obtained from a function f in $\mathcal{M}$ by restrictive variable substitution is also in $\mathcal{M}$. For any class $\mathcal{M}$ of multivalued functions, we denote by $\mathbf{RVS}(\mathcal{M})$ the smallest class containing $\mathcal{M}$, and closed under "restrictive variable substitutions". Similarly, we use $\mathbf{RVS}^t(\mathcal{M})$ to denote the smallest class containing $\mathcal{M}$, and closed under non-empty restrictive variable substitutions. By the definitions above it follows:

Fact 1. *For any class* $\mathcal{M} \subseteq \Theta_{AB}$*, we have*

(i) $\mathbf{RVS}^t(\Theta^t_{AB} \cap \mathcal{M}) = \Theta^t_{AB} \cap \mathbf{RVS}(\mathcal{M})$,
(ii) $\mathbf{RVS}(\Theta^p_{AB} \cap \mathcal{M}) \subseteq \Theta^p_{AB} \cap \mathbf{RVS}(\mathcal{M})$[1]*, and*
(iii) $\mathbf{RVS}^t(\Theta^s_{AB} \cap \mathcal{M}) \subseteq \Theta^s_{AB} \cap \mathbf{RVS}^t(\mathcal{M})$[2].

It is easy to check that every member of $\mathbf{RVS}(\mathcal{M})$, and thus of $\mathbf{RVS}^t(\mathcal{M})$, satisfies every constraint in $\mathbf{CSF}(\mathcal{M})$.

Due to the fact that we consider relational constraints of finite arities, the non-satisfaction of a constraint by a multivalued function is always detected in a finite restriction to the domain of the function. For this reason, we recall the the concept of "local closure".

A class $\mathcal{M} \subseteq \Theta_{AB}$ is said to be *locally closed* if it contains every multivalued function $f : A^n \to \mathcal{P}(B)$ for which every restriction to a finite subset of

[1]Modified May, 2006
[2]Modified May, 2006

its domain A^n coincides with a restriction of some member of $\mathcal{M}$. Obviously, if A is finite, then every class $\mathcal{M} \subseteq \Theta_{AB}$ is locally closed.

It is not difficult to verify that this property is indeed a necessary condition on classes definable by constraints. But even if closure under restrictive variable substitutions is assumed, say on a class $\mathcal{M} \subseteq \Theta_{AB}$, it is not sufficient to guarantee the existence of a set of constraints defining $\mathcal{M}$.

To illustrate, let $A = B = \{0,1\}$, and let $\mathcal{M}$ be the class containing only the unary constant function $\mathbf{0} : x \mapsto \{0\}$, the unary constant function $\mathbf{1} : x \mapsto \{1\}$, and the unary "identity" $\mathbf{i} : x \mapsto \{x\}$, for every $x \in A = B$. Consider the unary multivalued function $f : A \to \mathcal{P}(B)$ defined by

$$f(x) = \mathbf{1}(x) \cup \mathbf{i}(x) \text{ i.e. } f(0) = B = \{0,1\} \text{ and } f(1) = \{1\},$$

and $g : A \to \mathcal{P}(B)$ defined by

$$g(x) = \mathbf{0}(x) \cup \mathbf{1}(x) \text{ i.e. } g(x) = B = \{0,1\}, \text{ for all } x \in A = \{0,1\}.$$

Note that

$$\Pi_{\mathbf{a}\in A} f(\mathbf{a}) \subseteq \cup_{h\in\mathcal{M}} \Pi_{\mathbf{a}\in A} h(\mathbf{a}) \subset \Pi_{\mathbf{a}\in A} g(\mathbf{a}) = B^2.$$

Thus every constraint satisfied by every function in $\mathcal{M}$ must be also satisfied by the function f, but there are constraints satisfied by every function in $\mathcal{M}$ which are not satisfied by g.

Clearly, $\mathbf{RVS}(\mathcal{M})$ is locally closed and closed under restrictive variable substitutions. Also, it is not difficult to check that f and g do not belong to $\mathbf{RVS}(\mathcal{M})$. By the fact that f satisfies every constraint satisfied by the members of $\mathbf{RVS}(\mathcal{M})$, it follows that $\mathbf{RVS}(\mathcal{M})$ is properly contained in every definable class containing $\mathcal{M}$. Furthermore, from the fact that g does not satisfy every constraint in $\mathbf{CSF}(\mathcal{M})$, we conclude that a class definable by constraints does not necessarily contain all functions which are defined as the "union" of a family of members of the class.

This example motivates the introduction of the following concept which extends local closure. We say that a class $\mathcal{M}$ of multivalued functions on A to B is *closed under local coverings* if it contains every multivalued function f on A to B such that for every finite subset $F \subseteq A^n$, there is a non-empty family $(f_i)_{i\in I}$ of members of $\mathcal{M}$ of the same arity as f, such that

$$\Pi_{\mathbf{a}\in F} f(\mathbf{a}) \subseteq \cup_{i\in I} \Pi_{\mathbf{a}\in F} f_i(\mathbf{a}) \qquad (1)$$

Clearly, if a class is closed under local coverings, then it is locally closed. Moreover, within Θ^p_{AB}, the families $(f_i)_{i\in I}$ above, all reduce to singleton families, and within Θ^s_{AB}, the inclusion relation can be replaced by equality, i.e. closure under local coverings coincides with local closure.

Note also that condition (1) is equivalent to

$$\Pi_{\mathbf{a}\in F} f(\mathbf{a}) \subseteq \cup_{g\in\mathcal{M}_n} \Pi_{\mathbf{a}\in F} g(\mathbf{a}).$$

where n denotes the arity of f, and $\mathcal{M}_n$ is the set of all n-ary multivalued functions in $\mathcal{M}$.

The smallest class of multivalued functions containing $\mathcal{M}$, and closed under "local coverings" is denoted by $\mathbf{LC}(\mathcal{M})$. It is not difficult to see that $\mathbf{LC}(\mathcal{M})$ is the class of functions obtained from $\mathcal{M}$ by adding all those functions f such that for every finite restriction F of f's domain

$$\Pi_{\mathbf{a}\in F} f(\mathbf{a}) \subseteq \cup_{g\in\mathcal{M}_f} \Pi_{\mathbf{a}\in F} g(\mathbf{a}).$$

where $\mathcal{M}_f$ denotes the set of all functions in $\mathcal{M}$ with arity equal to that of f. Moreover, we define:

(i) $\mathbf{pLC}(\mathcal{M}) = \Theta^p_{AB} \cap \mathbf{LC}(\mathcal{M})$,
(ii) $\mathbf{tLC}(\mathcal{M}) = \Theta^t_{AB} \cap \mathbf{LC}(\mathcal{M})$, and
(iii) $\mathbf{sLC}(\mathcal{M}) = \Theta^s_{AB} \cap \mathbf{LC}(\mathcal{M})$,

and we say that a class $\mathcal{M}$ is *closed under partial local coverings*, *closed under total local coverings*, or *closed under simple local coverings*, if $\mathbf{pLC}(\mathcal{M}) = \mathcal{M}$, $\mathbf{tLC}(\mathcal{M}) = \mathcal{M}$, or $\mathbf{sLC}(\mathcal{M}) = \mathcal{M}$, respectively.

Proposition 1. *Consider arbitrary non-empty sets A and B, and let $\mathcal{M}$ be a class of multivalued functions.*

(i) *The operators $\mathcal{M} \mapsto \mathbf{RVS}(\mathcal{M})$ and $\mathcal{M} \mapsto \mathbf{RVS}^t(\mathcal{M})$ are closure operators on Θ_{AB} and Θ^t_{AB}, respectively. Moreover, they are also closure operators on Θ^p_{AB} and Θ^s_{AB}, respectively.*
(ii) *The operators $\mathcal{M} \mapsto \mathbf{LC}(\mathcal{M})$, $\mathcal{M} \mapsto \mathbf{tLC}(\mathcal{M})$, $\mathcal{M} \mapsto \mathbf{pLC}(\mathcal{M})$, and $\mathcal{M} \mapsto \mathbf{sLC}(\mathcal{M})$ are closure operators on Θ_{AB}, Θ^t_{AB}, Θ^p_{AB} and Θ^s_{AB}, respectively.*
(iii) *If $\mathbf{RVS}(\mathcal{M}) = \mathcal{M}$, then $\mathbf{RVS}(\mathbf{LC}(\mathcal{M})) = \mathbf{LC}(\mathcal{M})$.*
(iv) *If $\mathbf{RVS}^t(\mathcal{M}) = \mathcal{M}$, then $\mathbf{RVS}^t(\mathbf{tLC}(\mathcal{M})) = \mathbf{tLC}(\mathcal{M})$.*

Proof. Statements (i) and (ii) follow immediately from the above definitions and Fact 1. The proof of (iii) is analogous to that of Theorem 3 (a) in [C]. We show that $\mathbf{RVS}(\mathbf{LC}(\mathcal{M})) \subseteq \mathbf{LC}(\mathcal{M})$.

Suppose that $g \in \mathbf{RVS}(\mathbf{LC}(\mathcal{M}))$, say of arity m. Thus, there is an n-ary function f in $\mathbf{LC}(\mathcal{M})$, and a map $l : n \to m$, such that

$$g(\mathbf{a}) \subseteq f(\mathbf{a} \circ l)$$

for every m-tuple $\mathbf{a} \in A^m$. Let F be a finite subset of A^m. We show that there is a non-empty family $(g_i^F)_{i\in I}$ of m-ary members of $\mathcal{M}$, such that

$$\Pi_{\mathbf{a}\in F} g(\mathbf{a}) \subseteq \cup_{i\in I} \Pi_{\mathbf{a}\in F} g_i^F(\mathbf{a}).$$

Consider the finite subset $F' \subseteq A^n$, defined by

$$F' = \{\mathbf{a} \circ l : \mathbf{a} \in F\}.$$

From the fact that $f \in \mathbf{LC}(\mathcal{M})$, it follows that there is a non-empty family $(f_i^{F'})_{i\in I}$ of n-ary members of $\mathcal{M}$, such that

$$\Pi_{\mathbf{a}'\in F'} f(\mathbf{a}') \subseteq \cup_{i\in I} \Pi_{\mathbf{a}'\in F'} f_i^{F'}(\mathbf{a}').$$

For each $i \in I$, let g_i^F be the m-ary function defined by

$$g_i^F(\mathbf{a}) = f_i^{F'}(\mathbf{a} \circ l)$$

for every m-tuple $\mathbf{a} \in A^m$. Note that $(g_i^F)_{i\in I}$ is a family of members of $\mathcal{M}$, because $\mathbf{RVS}(\mathcal{M}) = \mathcal{M}$. By the definition of $(f_i^{F'})_{i\in I}$ and $(g_i^F)_{i\in I}$, it follows that, for every m-tuple $\mathbf{a} \in F$,

$$\Pi_{\mathbf{a}\in F} g(\mathbf{a}) \subseteq \Pi_{\mathbf{a}\in F} f(\mathbf{a} \circ l) \subseteq \cup_{i\in I} \Pi_{\mathbf{a}\in F} f_i^{F'}(\mathbf{a} \circ l) = \cup_{i\in I} \Pi_{\mathbf{a}\in F} g_i^F(\mathbf{a}).$$

Since the above argument works for every finite subset F of A^m, we have that g is in $\mathbf{LC}(\mathcal{M})$. The proof of (iv) can be obtained by proceeding in analogy with the proof of (iii). $\square$

Using (i), (ii) and (iii) of Proposition 1, it is straightfoward to check that, for every class $\mathcal{M} \subseteq \Theta_{AB}$, $\mathbf{LC}(\mathbf{RVS}(\mathcal{M}))$ is the smallest class containing $\mathcal{M}$, which is closed under local coverings, and closed under restrictive variable substitutions. Similarly, using (i), (ii) and (iv) of Proposition 1, it is easy to check that, for every class $\mathcal{M} \subseteq \Theta^t_{AB}$, $\mathbf{tLC}(\mathbf{RVS}^t(\mathcal{M}))$ is the smallest class containing $\mathcal{M}$, which is closed under total local coverings, and closed under non-empty restrictive variable substitutions.

Our first main result provides necessary and sufficient conditions for a class of multivalued functions to be definable by relational constraints:

Theorem 1. *Consider arbitrary non-empty sets A and B. For any class $\mathcal{M}$ of multivalued functions on A to B, the following conditions are equivalent:*

(i) *$\mathcal{M}$ is closed under local coverings, contains the unary empty-valued function e_1, and is closed under restrictive variable substitutions;*[3]

(ii) *$\mathcal{M}$ is definable by some set of A-to-B constraints.*

Proof. First, we prove $(ii) \Rightarrow (i)$. Clearly, the unary empty-valued function satisfies every constraint and it is easy to see that if a multivalued function f satisfies a constraint (R, S), then every function obtained from f by restrictive variable substitution also satisfies (R, S). Therefore, any function class $\mathcal{M}$ definable by a set of constraints contains the unary empty-valued function, and is closed under restrictive variable substitutions.

To see that $\mathcal{M}$ is closed under local coverings, consider an n-ary multivalued function $f \notin \mathcal{M}$. From (ii) it follows that there is an m-ary constraint (R, S) which is not satisfied by f but satisfied by every function g in $\mathcal{M}$. Hence, for some $\mathbf{a}^1, \ldots, \mathbf{a}^n \in R$, we have $f(\mathbf{a}^1 \ldots \mathbf{a}^n) \not\subseteq S$, and $g(\mathbf{a}^1 \ldots \mathbf{a}^n) \subseteq S$ for every $g \in \mathcal{M}_n$, where $\mathcal{M}_n$ is the set of all n-ary multivalued functions in $\mathcal{M}$. Thus,

$$\Pi_{i\in m} f((\mathbf{a}^1 \ldots \mathbf{a}^n)(i)) \not\subseteq \cup_{g\in\mathcal{M}_n} \Pi_{i\in m} g((\mathbf{a}^1 \ldots \mathbf{a}^n)(i)).$$

To prove the implication $(i) \Rightarrow (ii)$, assume (i). We proceed as in the proof of Theorem 1 in [CF], and show that for every function $f \notin \mathcal{M}$, there

[3]As observed by the third named referee, a class is closed under local coverings and closed under restrictive variable substitutions if and only if it is closed under local coverings and closed under simple variable substitutions. Thus the descriptions of the various closed sets of multivalued functions presented in this section could have been given in terms of closure under simple variable substitutions instead of closure under restrictive variable substitutions (footnote added in May, 2006).

is a constraint (R_f, S_f) satisfied by every member of $\mathcal{M}$, but not satisfied by f. This suffices to conclude (ii) because $\mathcal{M} = \mathbf{mFSC}(\{(R_f, S_f) : f \notin \mathcal{M}\})$, i.e. the set $\{(R_f, S_f) : f \notin \mathcal{M}\}$ defines the class $\mathcal{M}$.

So suppose that $f \notin \mathcal{M}$, say of arity n. Since $\mathcal{M}$ is closed under local coverings, there is a finite subset $F \subseteq A^n$ such that

$$\Pi_{\mathbf{a}\in F} f(\mathbf{a}) \nsubseteq \cup_{i\in I} \Pi_{\mathbf{a}\in F} f_i(\mathbf{a})$$

for every non-empty family $(f_i)_{i\in I}$ of n-ary members of $\mathcal{M}$. In particular,

$$\Pi_{\mathbf{a}\in F} f(\mathbf{a}) \nsubseteq \cup_{g\in\mathcal{M}_n} \Pi_{\mathbf{a}\in F} g(\mathbf{a})$$

where $\mathcal{M}_n$ is the set of all n-ary multivalued functions in $\mathcal{M}$. Observe that F can not be empty, and that f can not be empty-valued on F. Let $\mathbf{a}^1, \dots, \mathbf{a}^n$ be tuples in $A^{|F|}$ such that $F = \{(\mathbf{a}^1 \dots \mathbf{a}^n)(i) : i \in| F |\}$, and let (R, S) be the constraint whose antecedent is $R = \{\mathbf{a}^1, \dots, \mathbf{a}^n\}$, and whose consequent is defined by $S = \cup_{g\in\mathcal{M}_n} g(\mathbf{a}^1 \dots \mathbf{a}^n)$. Clearly, f does not satisfy the A-to-B constraint (R, S), and since $\mathcal{M}$ is closed under restrictive variable substitutions, it follows that every function in $\mathcal{M}$ satisfies (R, S). Thus for every function $f \notin \mathcal{M}$, there is a constraint (R_f, S_f) satisfied by every member of $\mathcal{M}$, but not satisfied by f. $\square$

By making use of Fact 1, Proposition 1 and the definitions preceding Proposition 1, we obtain as particular cases of Theorem 1 the characterizations of classes of multivalued functions of the form $\mathbf{pFSC}(\mathcal{T})$, $\mathbf{tFSC}(\mathcal{T})$, and $\mathbf{sFSC}(\mathcal{T})$:

Corollary 1. *Consider arbitrary non-empty sets A and B.*

(a) *A class $\mathcal{M}^p$ of partial functions is definable within Θ^p_{AB} by some set of A-to-B constraints if and only if it is closed under partial local coverings, contains the unary empty-valued function, and is closed under restrictive variable substitutions.*

(b) *A class $\mathcal{M}^t$ of total multivalued functions is definable within Θ^t_{AB} by some set of A-to-B constraints if and only if it is closed under total local coverings, and is closed under non-empty restrictive variable substitutions.*

(c) ([CF]) *A class $\mathcal{M}^s$ of single-valued functions is definable within Θ^s_{AB} by some set of A-to-B constraints if and only if it is closed under simple local coverings, and is closed under simple variable substitutions.*

We finish this section with the factorizations of the Galois closure operators on Θ_{AB}, Θ^p_{AB}, Θ^t_{AB}, and Θ^s_{AB}, as compositions of the operators induced by the above closure conditions:

Proposition 2. *Consider arbitrary non-empty sets A and B. For any class of multivalued functions $\mathcal{M} \subseteq \Theta_{AB}$, the following hold:*

(i) $\mathbf{mFSC}(\mathbf{CSF}(\mathcal{M})) = \mathbf{LC}(\mathbf{RVS}(\mathcal{M} \cup \{e_1\}))$.

(ii) *If $\mathcal{M} \subseteq \Theta^p_{AB}$, then* $\mathbf{pFSC}(\mathbf{CSF}(\mathcal{M})) = \mathbf{pLC}(\mathbf{RVS}(\mathcal{M} \cup \{e_1\}))$.

(*iii*) *If* $\mathcal{M} \subseteq \Theta^t_{AB}$, *then* $\mathbf{tFSC}(\mathbf{CSF}(\mathcal{M})) = \mathbf{tLC}(\mathbf{RVS}^t(\mathcal{M}))$.
(*iv*) *If* $\mathcal{M} \subseteq \Theta^s_{AB}$, *then* $\mathbf{sFSC}(\mathbf{CSF}(\mathcal{M})) = \mathbf{sLC}(\mathbf{RVS}^t(\mathcal{M}))$.

Proof. To see that (i) holds, note first that $\mathbf{mFSC}(\mathbf{CSF}(\mathcal{M}))$ is the smallest Galois closed set of multivalued functions containing $\mathcal{M}$. Thus it follows from Theorem 1 that $\mathbf{mFSC}(\mathbf{CSF}(\mathcal{M}))$ is the is the smallest class containing $\mathcal{M} \cup \{e_1\}$, which is closed under local coverings and closed under restrictive variable substitutions. By the comments following the proof of Proposition 1, we get $\mathbf{mFSC}(\mathbf{CSF}(\mathcal{M})) = \mathbf{LC}(\mathbf{RVS}(\mathcal{M} \cup \{e_1\}))$. In other words, (i) holds. The proof of (ii), (iii) and (iv) can be obtained similarly, using Corollary 1 and Proposition 1. □

4. Galois closed Sets of Relational Constraints

In order to describe the dual closed sets, i.e. the sets of constraints characterized by multivalued functions, we need some terminology, in addition to that introduced in [CF].

Consider arbitrary sets A, B, C and D, and let $f : A \to B$ and $g : C \to D$ be maps. The *concatenation* of g with f, denoted gf, is defined to be the map with domain $f^{-1}[B \cap C]$ and codomain D given by $(gf)(a) = g(f(a))$ for all $a \in f^{-1}[B \cap C]$. Note that if $B = C$, then the concatenation gf is simply the *composition* of g with f, i.e. $(gf)(a) = g(f(a))$ for all $a \in A$. As in the particular case of composition, concatenation is associative.

If $(g_i)_{i\in I}$ is a non-empty family of maps $g_i : A_i \to B_i$, where $(A_i)_{i\in I}$ is a family of pairwise disjoint sets, then their (*piecewise*) *sum*, denoted $\Sigma_{i\in I} g_i$, is the map from $\cup_{i\in I} A_i$ to $\cup_{i\in I} B_i$ whose restriction to each A_i agrees with g_i. We also use $g_1 + g_2$ to denote the sum of g_1 and g_2. In particular, if $B_1 = B_2 = B^n$, and g_1 and g_2 are the vector-valued functions $g_1 = (g_1^1 \ldots g_1^n)$ and $g_2 = (g_2^1 \ldots g_2^n)$, where for each $1 \le j \le n$, $g_1^j : A_1 \to B$ and $g_2^j : A_2 \to B$, then their sum is defined componentwise, i.e. $g_1 + g_2$ is the vector-valued function defined by $(g_1 + g_2)(\mathbf{a}) = ((g_1^1 + g_2^1)(\mathbf{a}) \ldots (g_1^n + g_2^n)(\mathbf{a}))$, for every $\mathbf{a} \in A_1 \cup A_2$. Clearly, piecewise sum is associative and commutative, and it is not difficult to see that concatenation is distributive over sum.

Let m be a positive integer (viewed as an ordinal), $(n_j)_{j\in J}$ be a non-empty family of positive integers (also viewed as ordinals), and let V be an arbitrary set disjoint from m and each member of $(n_j)_{j\in J}$. Any non-empty family $H = (h_j)_{j\in J}$ of maps $h_j : n_j \to m \cup V$, is called a *minor formation scheme* with *target* m, *indeterminate set* V and *source family* $(n_j)_{j\in J}$. If the indeterminate set V is empty, i.e. for each $j \in J$, the maps h_j have codomain m, then we say that the minor formation scheme $H = (h_j)_{j\in J}$ is *simple.*

An m-ary A-to-B constraint (R, S) is said to be a *conjunctive minor* of a non-empty family $(R_j, S_j)_{j\in J}$ of A-to-B constraints (of various arities) *via a scheme* $H = (h_j)_{j\in J}$, if for every m-tuples $\mathbf{a} \in A^m$ and $\mathbf{b} \in B^m$,

(a) $\mathbf{a} \in R$ implies that there is a map $\sigma_A : V \to A$ such that, for all j in J, we have $(\mathbf{a} + \sigma_A)h_j \in R_j$, and

(b) if there is a map $\sigma_B : V \to B$ such that, for all j in J, we have $(\mathbf{b} + \sigma_B)h_j \in S_j$, then $\mathbf{b} \in S$.

The maps σ_A and σ_B are called *Skolem maps.* If (a) and (b) hold with "if and only if" replacing "implies" and "if", respectively, then (R, S) is called a *tight conjunctive minor* of the family $(R_j, S_j)_{j\in J}$. (See [CF] for further background.) If the minor formation scheme H is simple, then we say that (R, S) is a *weak conjunctive minor* of the family $(R_j, S_j)_{j\in J}$. Furthermore, if (R, S) is the tight conjunctive minor of the family $(R_j, S_j)_{j\in J}$ via the scheme H consisting of identity maps on m, then (R, S) is said to be obtained by *intersecting antecedents* and *intersecting consequents* of the constraints in the family $(R_j, S_j)_{j\in J}$. In addition, if $J = \{0\}$, then conditions (a) and (b) above, reduce to $R \subseteq R_0$ and $S \supseteq S_0$, respectively, and in this case (R, S) is called a *relaxation* of (R_0, S_0). We shall refer to relaxations (R, S) with finite antecedent R as *finite relaxations.*

Transitivity Lemma. *If (R, S) is a conjunctive minor of a non-empty family $(R_j, S_j)_{j\in J}$ of A-to-B constraints, and, for each $j \in J$, (R_j, S_j) is a conjunctive minor of a non-empty family $(R^i_j, S^i_j)_{i\in I_j}$, then (R, S) is a conjunctive minor of the non-empty family $(R^i_j, S^i_j)_{j\in J, i\in I_j}$.*

Proof. The proof of the Transitivity Lemma follows as the proof of Claim 1 in [CF] (see proof of Theorem 2), but the ordinals m and n_j, for each $j \in J$, are assumed to be finite.

Suppose that (R, S) is an m-ary conjunctive minor of $(R_j, S_j)_{j\in J}$ via a scheme $H = (h_j)_{j\in J}$, $h_j : n_j \to m \cup V$, and, for each $j \in J$, (R_j, S_j) is an n_j-ary conjunctive minor of $(R^i_j, S^i_j)_{i\in I_j}$ via a scheme $H_j = (h^i_j)_{i\in I_j}$, $h^i_j : n^i_j \to n_j \cup V_j$, where the V_j's are pairwise disjoint[4].

Consider the minor formation scheme $K = (k^i_j)_{j\in J, i\in I_j}$ defined as follows:

(i) the target of K is the target m of H,
(ii) the source family of K is $(n^i_j)_{j\in J, i\in I_j}$,
(iii) the indeterminate set of K is $U = V \cup (\cup_{j\in J} V_j)$,
(iv) $k^i_j : n^i_j \to m \cup U$ is defined by

$$k^i_j = (h_j + \iota_{V_j U})h^i_j$$

where $\iota_{V_j U}$ is the canonical injection (inclusion map) on V_j to U. We show that (R, S) is a conjunctive minor of the family $(R^i_j, S^i_j)_{j\in J, i\in I_j}$ via the scheme $K = (k^i_j)_{j\in J, i\in I_j}$.

If $\mathbf{a}$ is an m-tuple in R, then there is a Skolem map $\sigma : V \to A$ such that for all j in J, $(\mathbf{a}+\sigma)h_j \in R_j$. Thus, for every j in J, there are Skolem maps $\sigma_j : V_j \to A$ such that for every $i \in I_j$, we have $[(\mathbf{a} + \sigma)h_j + \sigma_j]h^i_j \in R^i_j$.

As in the proof of Claim 1 in [CF], let $\tau : U \to A$ be the Skolem map defined by $\tau = \sigma + \Sigma_{l\in J}\sigma_l$. By the fact that concatenation is associative and

[4]And each V_j is disjoint from V (footnote added in May, 2006).

distributive over sum, it follows that for every $j \in J$ and $i \in I_j$,

$$(\mathbf{a}+\tau)k_j^i = (\mathbf{a}+\sigma+\Sigma_{l\in J}\sigma_l)(h_j+\iota_{UV_j})h_j^i = [(\mathbf{a}+\sigma)h_j+\sigma_j]h_j^i.$$

Thus, for every $j \in J$ and $i \in I_j$, we have $(\mathbf{a}+\tau)k_j^i \in R_j^i$.

Now suppose that $\mathbf{b}$ is an m-tuple over B, for wich there is a Skolem map $\tau : U \to B$ such that for every $j \in J$ and $i \in I_j$, $(\mathbf{b}+\tau)k_j^i$ is in S_j^i. Consider the Skolem maps $\sigma : V \to B$ and $\sigma_j : V_j \to B$ for every $j \in J$, such that $\tau = \sigma + \Sigma_{j\in J}\sigma_j$. Again, by associativity and distributivity it follows that for every $j \in J$ and $i \in I_j$, $[(\mathbf{b}+\sigma)h_j+\sigma_j]h_j^i = (\mathbf{b}+\tau)k_j^i \in S_j^i$. Hence, for every $j \in J$, we have $(\mathbf{b}+\sigma)h_j \in S_j$, and thus we conclude $\mathbf{b} \in S$. □

Note that if H is simple and, for every $j \in J$, the schemes H_j are simple, then the scheme K defined in the proof above is also simple. Thus we get:

Transitivity Lemma for weak minors. *If (R,S) is a weak conjunctive minor of a non-empty family $(R_j,S_j)_{j\in J}$ of A-to-B constraints, and, for each $j \in J$, (R_j,S_j) is a weak conjunctive minor of a non-empty family $(R_j^i,S_j^i)_{i\in I_j}$, then (R,S) is a weak conjunctive minor of the non-empty family $(R_j^i,S_j^i)_{j\in J, i\in I_j}$.*

A set $\mathcal{T}$ of relational constraints is said to be *closed under formation of (weak) conjunctive minors* if whenever every member of a non-empty family $(R_j,S_j)_{j\in J}$ of constraints is in $\mathcal{T}$, then every (weak) conjunctive minor of the family $(R_j,S_j)_{j\in J}$ is also in $\mathcal{T}$. For any set of constraints $\mathcal{T}$, we denote by $\mathbf{CM}(\mathcal{T})$ the smallest set of constraints containing $\mathcal{T}$, and closed under formation of conjunctive minors. Similarly, we define $\mathbf{wCM}(\mathcal{T})$ to be the smallest set of constraints containing $\mathcal{T}$, and closed under formation of weak conjunctive minors.

By the Transitivity Lemma it follows that $\mathbf{CM}(\mathcal{T})$ is the set of all conjunctive minors of families of constraints in $\mathcal{T}$, and $\mathbf{CM}(\mathbf{CM}(\mathcal{T})) = \mathbf{CM}(\mathcal{T})$. Analogously, by the Transitivity Lemma for weak minors, it follows that $\mathbf{wCM}(\mathcal{T})$ is the set of all weak conjunctive minors of families of constraints in $\mathcal{T}$, and $\mathbf{wCM}(\mathbf{wCM}(\mathcal{T})) = \mathbf{wCM}(\mathcal{T})$. In other words, both $\mathcal{T} \mapsto \mathbf{CM}(\mathcal{T})$ and $\mathcal{T} \mapsto \mathbf{wCM}(\mathcal{T})$ are idempotent maps. Furthermore, both $\mathcal{T} \mapsto \mathbf{CM}(\mathcal{T})$ and $\mathcal{T} \mapsto \mathbf{wCM}(\mathcal{T})$ are monotone and extensive (in the sense of Section 2), and hence, we have:

Fact 2. *The operators $\mathcal{T} \mapsto \mathbf{CM}(\mathcal{T})$ and $\mathcal{T} \mapsto \mathbf{wCM}(\mathcal{T})$ are closure operators on the set of all A-to-B constraints.*

The following technical result shows that the sets of constraints characterized by multivalued functions, and by total multivalued functions must be closed under formation of weak conjunctive minors, and closed under formation of conjunctive minors, respectively.

Lemma 1. *Let $(R_j, S_j)_{j\in J}$ be a non-empty family of A-to-B constraints. If $f : A^n \to \mathcal{P}(B)$ satisfies every (R_j, S_j) then f satisfies every weak conjunctive minor of the family $(R_j, S_j)_{j\in J}$. Futhermore, if f is total, then f satisfies every conjunctive minor of the family $(R_j, S_j)_{j\in J}$.*

Proof. First we prove the last claim, which generalizes Lemma 1 in [CF], to total multivalued functions. Let f be a total multivalued function, say of arity n, satisfying every member of a non-empty family $(R_j, S_j)_{j\in J}$ of A-to-B constraints, and let (R, S) be an m-ary conjunctive minor of the family $(R_j, S_j)_{j\in J}$ via a scheme $H = (h_j)_{j\in J}$, $h_j : n_j \to m \cup V$. We show that for every $\mathbf{a}^1 \dots \mathbf{a}^n \in R$, the m-ary relation $f(\mathbf{a}^1 \dots \mathbf{a}^n)$ is contained in S, i.e. f satisfies (R, S). So let $\mathbf{a}^1, \dots, \mathbf{a}^n$ be any m-tuples in R. Observe that for each $1 \le i \le n$, there is a Skolem map $\sigma_i : V \to A$, such that for every j in J, $(\mathbf{a}^i + \sigma_i)h_j$ is in R_j. Since f satisfies every member of $(R_j, S_j)_{j\in J}$, we have that for every j in J, $f[(\mathbf{a}^1 + \sigma_1)h_j \dots (\mathbf{a}^n + \sigma_n)h_j] \subseteq S_j$.

Now, suppose that $\mathbf{b} \in f(\mathbf{a}^1 \dots \mathbf{a}^n)$. Since f is a total multivalued function, there is a Skolem map $\sigma : V \to B$ such that, for every $v \in V$, $\sigma(v)$ belongs to $f(\sigma_1(v) \dots \sigma_n(v))$. Fix such a Skolem map $\sigma : V \to B$. By associativity and distributivity of concatenation over sum, we have that for each j in J,

$$\begin{aligned}&(\mathbf{b} + \sigma)h_j \in [f(\mathbf{a}^1 \dots \mathbf{a}^n) + f(\sigma_1 \dots \sigma_n)]h_j = \\ &f[(\mathbf{a}^1 + \sigma_1)h_j \dots (\mathbf{a}^n + \sigma_n)h_j] \subseteq S_j.\end{aligned}$$

Since (R, S) is a conjunctive minor of $(R_j, S_j)_{j\in J}$ via the scheme $H = (h_j)_{j\in J}$, we conclude that $\mathbf{b} \in S$, which completes the proof of the last statement of Lemma 1.

To prove the first claim of Lemma 1, suppose that $f : A^n \to \mathcal{P}(B)$ is a multivalued function, not necessarily total, satisfying every member of $(R_j, S_j)_{j\in J}$, and assume that (R, S) is a weak conjunctive minor of the family $(R_j, S_j)_{j\in J}$, say via a scheme $H = (h_j)_{j\in J}$, where $h_j : n_j \to m$ for every j in J.

As before, we prove that f satisfies (R, S), by showing that for every $\mathbf{a}^1, \dots, \mathbf{a}^n$ in R, we have $f(\mathbf{a}^1 \dots \mathbf{a}^n) \subseteq S$. Clearly, if $f((\mathbf{a}^1 \dots \mathbf{a}^n)(i)) = \emptyset$, for some $i \in m$, then $f(\mathbf{a}^1 \dots \mathbf{a}^n) \subseteq S$. So we may assume that

$$f((\mathbf{a}^1 \dots \mathbf{a}^n)(i)) \neq \emptyset,$$

for every $i \in m$. As before, for each j in J, the n_j-tuples $\mathbf{a}^1 h_j, \dots, \mathbf{a}^n h_j$ belong to R_j, and since f satisfies each (R_j, S_j), we have that

$$f(\mathbf{a}^1 h_j \dots \mathbf{a}^n h_j) \subseteq S_j,$$

for every $j \in J$. By associativity, it follows that for each j in J,

$$[f(\mathbf{a}^1 \dots \mathbf{a}^n)]h_j = f[(\mathbf{a}^1 \dots \mathbf{a}^n)h_j] = f(\mathbf{a}^1 h_j \dots \mathbf{a}^n h_j) \subseteq S_j.$$

Therefore, if $\mathbf{b} \in f(\mathbf{a}^1 \dots \mathbf{a}^n)$, then for every $j \in J$, we have $\mathbf{b}h_j \in S_j$, which implies $\mathbf{b} \in S$, because (R, S) is a weak conjunctive minor of $(R_j, S_j)_{j\in J}$ via the scheme $H = (h_j)_{j\in J}$. Thus $f(\mathbf{a}^1 \dots \mathbf{a}^n)$ is indeed contained in S, and the proof of Lemma 1 is complete. □

In order to describe the Galois closed sets of constraints, we need to recall a further condition, introduced in [CF], which expresses "compactness" on the sets of these dual objects. A set $\mathcal{T}$ of relational constraints is said to be *locally closed* if $\mathcal{T}$ contains every constraint (R, S) such that the set of all its finite relaxations, is contained in $\mathcal{T}$. In analogy with Section 3, we denote by $\mathbf{LO}(\mathcal{T})$, the smallest locally closed set of constraints containing $\mathcal{T}$. Similarly to the closure $\mathbf{LC}$ defined on classes of function classes, $\mathbf{LO}(\mathcal{T})$ is the set of constraints obtained from $\mathcal{T}$ by adding all those constraints whose finite relaxations are all in $\mathcal{T}$. As an immediate consequence, we have:

Fact 3. *The operator $\mathcal{T} \mapsto \mathbf{LO}(\mathcal{T})$ is a closure operator on the set of all A-to-B constraints.*

Note that in the case of finite underlying sets A and B, the induced operator in Fact 3 is the identity map, i.e. every set of constraints is locally closed.

Theorem 2. *Consider arbitrary non-empty sets A and B. Let $\mathcal{T}$ be a set of A-to-B relational constraints. Then the following are equivalent:*

- (i) *$\mathcal{T}$ is locally closed, contains the unary empty constraint $(\emptyset, \emptyset)$ and the unary trivial constraint (A, B), and is closed under formation of weak conjunctive minors;*
- (ii) *$\mathcal{T}$ is characterized by some set of multivalued functions on A to B.*

Proof. To see that (ii) implies (i), note first that every multivalued function satisfies the empty constraint $(\emptyset, \emptyset)$, and the trivial constraint (A, B). Also, from Lemma 1, it follows that every set of constraints characterized by multivalued functions is closed under formation of weak conjunctive minors. For the remainder, let (R, S) be any constraint not in $\mathcal{T}$. By (ii) it follows that there is an n-ary multivalued function f satisfying every constraint in $\mathcal{T}$ which does not satisfy (R, S), i.e. there are $\mathbf{a}^1 \ldots \mathbf{a}^n \in R$, such that $f(\mathbf{a}^1 \ldots \mathbf{a}^n) \nsubseteq S$. Let F be the subset of R containing $\mathbf{a}^1 \ldots \mathbf{a}^n$. Clearly, the constraint (F, S) is a finite relaxation of (R, S), and $(F, S) \notin \mathcal{T}$. Since the above argument works for any constraint not in $\mathcal{T}$, we conclude that $\mathcal{T}$ is locally closed.

To prove implication (i) $\Rightarrow$ (ii), let (R, S) be any constraint not in $\mathcal{T}$, say of arity m. We show that there is a multivalued function separating (R, S) from $\mathcal{T}$. From the fact that $\mathcal{T}$ is locally closed, it follows that there is a finite relaxation (F, S_0) of (R, S), say with F of size n, which is not in $\mathcal{T}$. Observe that (F, B^m) is a weak conjunctive minor of the unary trivial constraint (A, B), and so we must have $S_0 \neq B^m$. Also, F can not be empty because $(\emptyset^m, S_0)$ is a relaxation of the m-ary empty constraint, which in turn is a weak conjunctive minor of $(\emptyset, \emptyset)$. From the fact that (F, S_0) can be obtained from the family $(F, B^m \setminus \{\mathbf{s}\})_{\mathbf{s} \notin S_0}$, by intersecting antecedents and intersecting consequents, it follows that there must exist an m-tuple $\mathbf{s} = (s_i \mid i \in m)$ in B^m which is not in S_0, and such that $(F, B^m \setminus \{\mathbf{s}\})$ does not belong to $\mathcal{T}$. Let $\mathbf{a}^1, \ldots, \mathbf{a}^n$ be the m-tuples in F.

We define a multivalued function which is not empty-valued on

$$D = \{(\mathbf{a}^1 \dots \mathbf{a}^n)(i) : i \in m\}$$

but empty-valued on the remaining n-tuples of A^n. Formally, let g be the n-ary multivalued function on A to B such that, for every $i \in m$,

$$g((\mathbf{a}^1 \dots \mathbf{a}^n)(i)) = \cup\{s_j : j \in m \text{ and } (\mathbf{a}^1 \dots \mathbf{a}^n)(j) = (\mathbf{a}^1 \dots \mathbf{a}^n)(i)\},$$

and $g(\mathbf{a}) = \emptyset$ for every $\mathbf{a} \in A^n \setminus D$. From definition of g, it follows that $\mathbf{s} \in g(\mathbf{a}^1 \dots \mathbf{a}^n)$, and thus g does not satisfy (F, S_0), and so it does not satisfy (R, S).

Now we show that g satisfies every member of $\mathcal{T}$. For a contradiction, suppose that there is an m_1-ary member (R_1, S_1) of $\mathcal{T}$, which is not satisfied by g. Thus, for some $\mathbf{c}^1, \dots, \mathbf{c}^n \in R_1$ we have $g(\mathbf{c}^1 \dots \mathbf{c}^n) \not\subseteq S_1$. Consider an m_1-tuple $\mathbf{s}_1 \in g(\mathbf{c}^1 \dots \mathbf{c}^n) \setminus S_1$, and let $h : m_1 \to m$ be any map such that

$$\mathbf{s}_1(i) = (\mathbf{s}h)(i).$$

for every $i \in m_1$. Note that for every $i \in m_1$, there is a $j \in m$ such that $(\mathbf{c}^1 \dots \mathbf{c}^n)(i) = (\mathbf{a}^1 \dots \mathbf{a}^n)(j)$, for otherwise $g(\mathbf{c}^1 \dots \mathbf{c}^n)$ would be empty and so would be contained in S_1. In fact, from definition of g and h, it follows that, for every $i \in m_1$, $(\mathbf{c}^1 \dots \mathbf{c}^n)(i) = (\mathbf{a}^1 h \dots \mathbf{a}^n h)(i)$.

Let (R_h, S_h) be the m-ary weak conjunctive minor of (R_1, S_1) via $H = \{h\}$, defined by

(a) for every $\mathbf{a} \in A^m$, $\mathbf{a} \in R_h$ if and only if $\mathbf{a}h \in R_1$, and
(b) for every $\mathbf{b} \in B^m$, $\mathbf{b} \in S_h$ if and only if $\mathbf{b}h \in S_1$.

Observe that (R_h, S_h) belongs to $\mathcal{T}$, because $\mathcal{T}$ is closed under formation of weak conjunctive minors.

Since $\mathbf{a}^1 h, \dots, \mathbf{a}^n h \in R_1$, we have $\mathbf{a}^1, \dots, \mathbf{a}^n \in R_h$, i.e. $F \subseteq R_h$. Also, $\mathbf{s} \notin S_h$ because $\mathbf{s}_1 \notin S_1$. Therefore $(F, B^m \setminus \{\mathbf{s}\})$ is a relaxation of (R_h, S_h), and we conclude that $(F, B^m \setminus \{\mathbf{s}\}) \in \mathcal{T}$, which is a contradiction. Thus g is indeed a multivalued function separating (R, S) from $\mathcal{T}$. $\square$

In Section 2, we observed that every partial function satisfies the binary equality constraint, thus any set of constraints characterized by partial functions must contain this constraint. In fact, this additional condition is also sufficient to describe the Galois closed sets of constraints associated with the correspondence **pFSC** $-$ **CSF**:

Corollary 2. *Consider arbitrary non-empty sets A and B. Let $\mathcal{T}$ be a set of A-to-B relational constraints. Then the following are equivalent:*

(i) *$\mathcal{T}$ is locally closed, contains the unary empty constraint, the unary trivial constraint and the binary equality constraint, and is closed under formation of weak conjunctive minors;*
(ii) *$\mathcal{T}$ is characterized by some set of partial functions on A to B.*

Proof. The implication $(ii) \Rightarrow (i)$, is a consequence of Theorem 2 and the observations above. The proof of the implication $(i) \Rightarrow (ii)$, follows exactly as the proof of $(i) \Rightarrow (ii)$ in Theorem 2. The key observation is that if

$\mathcal{T}$ contains the binary equality constraint $(=_A, =_B)$, and it is closed under formation of weak conjunctive minors, then for every $i, j \in m$ such that $i \neq j$, the m-ary A-to-B constraint (R_{ij}, S_{ij}) defined by

$$R_{ij} = \{(a_t \mid t \in m) : a_i = a_j\} \quad \text{and} \quad S_{ij} = \{(b_t \mid t \in m) : b_i = b_j\}$$

is in $\mathcal{T}$. From this fact, we have that in the proof of $(i) \Rightarrow (ii)$ of Theorem 2, the following holds for every $i, j \in m$:

$$\text{if } (\mathbf{a}^1 \dots \mathbf{a}^n)(j) = (\mathbf{a}^1 \dots \mathbf{a}^n)(i), \text{ then } \mathbf{s}(j) = \mathbf{s}(i).$$

Indeed, as observed in the proof of Theorem 2 in [CF], if for some $i \neq j$, we have $(\mathbf{a}^1 \dots \mathbf{a}^n)(j) = (\mathbf{a}^1 \dots \mathbf{a}^n)(i)$, but $\mathbf{s}(j) \neq \mathbf{s}(i)$, then $(F, B^m \setminus \{\mathbf{s}\})$ would be a relaxation of (R_{ij}, S_{ij}), and hence would be in $\mathcal{T}$, which is a contradiction.

Thus the separating function g defined in the proof of $(i) \Rightarrow (ii)$ of Theorem 2, is in fact a partial function, which completes the proof of Theorem 3. □

The next two results are the analogues of Theorem 2 and Corollary 2, which show that, in addition, closure under formation of conjunctive minors suffices to describe the sets of relational constraints characterized by total functions.

Theorem 3. *Consider arbitrary non-empty sets A and B. Let $\mathcal{T}$ be a set of A-to-B relational constraints. Then the following are equivalent:*

- (i) *$\mathcal{T}$ is locally closed, contains the unary empty constraint and the unary trivial constraint, and is closed under formation of conjunctive minors;*
- (ii) *$\mathcal{T}$ is characterized by some set of total multivalued functions on A to B.*

Proof. The proof of implication $(ii) \Rightarrow (i)$ follows as the proof of Theorem 2 (using the last statement in Lemma 1). To prove $(i) \Rightarrow (ii)$ we shall make use of notions and terminology, as well as few results particular to the proof of Theorem 2 in [CF]. Ordinals are allowed to be infinite, unless they denote function arities which remain finite. Thus the relations and constraints considered in this proof may be infinitary. Also, in minor formation schemes, the targets and members of the source families are allowed to be arbitrary, possibly infinite, non-zero ordinals, so that the notion of conjunctive minor is naturally extended to this more general setting. We shall use the term "conjunctive ∞-minor" to indicate a conjunctive minor which may be finitary or infinitary. As shown in [CF] (see Claim 1 in the proof of Theorem 2), the Transitivity Lemma is extended to this general setting:

Infinitary Transitivity. ([CF]) *If (R, S) is a conjunctive ∞-minor of a non-empty family $(R_j, S_j)_{j \in J}$ of A-to-B constraints, and, for each $j \in J$, (R_j, S_j) is a conjunctive ∞-minor of a non-empty family $(R_j^i, S_j^i)_{i \in I_j}$, then (R, S) is a conjunctive ∞-minor of the non-empty family $(R_j^i, S_j^i)_{j \in J, i \in I_j}$.*

A proof of the Infinitary Transitivity can be obtained by allowing infinite ordinals, in the proof of the Transitivity Lemma. We use Infinitary Transitivity to prove the analogue of Claim 2 in the proof of Theorem 2 in [CF]:

Claim. *Let $\mathcal{T}$ be a locally closed set of finitary A-to-B constraints containing the unary empty constraint and the unary trivial constraint, and closed under formation of conjunctive minors, and let $\mathcal{T}^\infty$ be its closure under formation of conjunctive ∞-minors. Let (R, S) be a finitary A-to-B constraint not in $\mathcal{T}$. Then there is a total multivalued function g on A to B such that*

(1) *g satisfies every constraint in $\mathcal{T}^\infty$, and*
(2) *g does not satisfy (R, S).*

Observe that by the Infinitary Transitivity, $\mathcal{T}$ is the set of all finitary constraints in $\mathcal{T}^\infty$.

Proof of Claim. Proceeding in analogy with the proof of Theorem 2, we construct a total multivalued function g which satisfies all constraints in $\mathcal{T}^\infty$ but g does not satisfy (R, S).

Let m be the arity of $(R, S) \notin \mathcal{T}$. By the comment following the Claim, (R, S) can not be in $\mathcal{T}^\infty$. As in the proof of $(i) \Rightarrow (ii)$ in Theorem 2, let (F, S_0) be a relaxation of (R, S) with finite antecedent, not in $\mathcal{T}$. As before, F cannot be empty, and $S_0 \neq B^m$. Let $F = \{\mathbf{d}^1, \ldots, \mathbf{d}^n\}$ of finite size $1 \leq n$.

Let $\mu =| A^n |$, and consider μ-tuples $\mathbf{a}^1, \ldots, \mathbf{a}^n \in A^\mu$ such that

$$(\mathbf{a}^1 \ldots \mathbf{a}^n)(i) = (\mathbf{d}^1 \ldots \mathbf{d}^n)(i), \text{ for every } i \in m$$

and such that $\{(\mathbf{a}^1 \ldots \mathbf{a}^n)(i) : i \in \mu\backslash m\}$ are the remaining distinct n-tuples in A^n without repetitions. Let R_F be the μ-ary relation defined by $R_F = \{\mathbf{a}^1, \ldots, \mathbf{a}^n\}$.[5]

In analogy with the proof of Theorem 2, let $\mathbf{s} = (s_t \mid t \in \mu)$ be a μ-tuple in B^μ such that $(s_t \mid t \in m)$ is not in S_0, and for which $(R_F, B^\mu \setminus \{\mathbf{s}\})$ does not belong to $\mathcal{T}^\infty$. Consider the n-ary multivalued function g on A to B, defined by

$$g((\mathbf{a}^1 \ldots \mathbf{a}^n)(i)) = \cup\{s_j : j \in \mu \text{ and } (\mathbf{a}^1 \ldots \mathbf{a}^n)(j) = (\mathbf{a}^1 \ldots \mathbf{a}^n)(i)\},$$

for every $i \in \mu$.

Note that for every $\mathbf{a} \in \{(\mathbf{a}^1 \ldots \mathbf{a}^n)(i) : i \in \mu\} = A^n$, we have $g(\mathbf{a}) \neq \emptyset$, that is, g is total. Also, $\mathbf{s} \in g(\mathbf{a}^1 \ldots \mathbf{a}^n)$, thus $(s_t \mid t \in m) \in g(\mathbf{d}^1 \ldots \mathbf{d}^n)$. Hence, g does not satisfy (F, S_0) and since (F, S_0) is a relaxation of (R, S) it follows that g does not satisfy (R, S).[6]

Now we show that g also satisfies (1). For a contradiction, suppose that there is a ρ-ary constraint $(R_1, S_1) \in \mathcal{T}^\infty$, which is not satisfied by g. That is, for some $\mathbf{c}^1, \ldots, \mathbf{c}^n$ in R_1 we have $g(\mathbf{c}^1 \ldots \mathbf{c}^n) \not\subseteq S_1$. Let $\mathbf{s}_1$ be an ρ-tuple in $g(\mathbf{c}^1 \ldots \mathbf{c}^n)$ such that $\mathbf{s}_1 \notin S_1$, and let $h : \rho \to \mu$ be any map such that, for every $i \in \rho$:

[5] Sentence modified in May, 2006.

[6] Paragraph modified in May, 2006.

(a) $\mathbf{s}_1(i) = (\mathbf{s}h)(i)$, and
(b) $(\mathbf{c}^1 \dots \mathbf{c}^n)(i) = (\mathbf{a}^1 h \dots \mathbf{a}^n h)(i)$.

Note that (b) implies that $\mathbf{c}^j = \mathbf{a}^j h$, for every $1 \leq j \leq n$.

Let (R_h, S_h) be the μ-ary tight conjunctive ∞-minor of (R_1, S_1) via $H = \{h\}$, i.e. for every μ-tuple $\mathbf{a}$ of A^μ, $\mathbf{a} \in R_h$ if and only if $\mathbf{a}h \in R_1$, and for every μ-tuple $\mathbf{b}$ of B^μ, $\mathbf{b} \in S_h$ if and only if $\mathbf{b}h \in S_1$. Clearly, $\mathbf{a}^1, \dots, \mathbf{a}^n \in R_h$, that is, $R_F \subseteq R_h$, and $\mathbf{s} \notin S_h$. Thus $(R_F, B^\mu \setminus \{\mathbf{s}\})$ is a relaxation of (R_h, S_h), and, since $\mathcal{T}^\infty$ is closed under formation of conjunctive ∞-minors, it follows from the Infinitary Transitivity that $(R_F, B^\mu \setminus \{\mathbf{s}\}) \in \mathcal{T}^\infty$, yielding the desired contradiction, and the proof of the Claim is complete.

By the Claim above, it follows that for every constraint (R, S) not in $\mathcal{T}$ there is a total multivalued function g on A to B which does not satisfy (R, S) but satisfies in particular every constraint in $\mathcal{T}$. In other words, the implication $(i) \Rightarrow (ii)$ also holds. □

Note that the unary trivial constraint (A, B), is a tight conjunctive minor of the binary equality constraint $(=_A, =_B)$.

Corollary 3. ([CF]) *Consider arbitrary non-empty sets A and B. Let $\mathcal{T}$ be a set of A-to-B relational constraints. Then the following are equivalent:*

(i) *$\mathcal{T}$ is locally closed, contains the unary empty constraint and the binary equality constraint, and it is closed under formation of conjunctive minors;*
(ii) *$\mathcal{T}$ is characterized by some set of single-valued functions on A to B.*

Proof. The implication $(ii) \Rightarrow (i)$, is a consequence of Corollary 2 and Theorem 3. The proof of $(i) \Rightarrow (ii)$ is analogous to that of Corollary 2, but following the lines in the proof of $(i) \Rightarrow (ii)$ of Theorem 3. □

In order to factorize the closure operators associated with the Galois connections for generalized functions and constraints defined in Section 2, as compositions of the operators **LO**, **wCM**, and **CM**, we shall make use of the following analogues of (iii) and (iv) in Proposition 1:

Proposition 3. *Consider arbitrary non-empty sets A and B, and let $\mathcal{T}$ be a set of A-to-B relational constraints.*

(i) *If $\mathbf{CM}(\mathcal{T}) = \mathcal{T}$, then $\mathbf{CM}(\mathbf{LO}(\mathcal{T})) = \mathbf{LO}(\mathcal{T})$.*
(ii) *If $\mathbf{wCM}(\mathcal{T}) = \mathcal{T}$, then $\mathbf{wCM}(\mathbf{LO}(\mathcal{T})) = \mathbf{LO}(\mathcal{T})$.*

Proof. We follow the strategy used in the proof of Theorem 3 (b) in [C]. By Fact 2, to prove (i) we only need to show that $\mathbf{CM}(\mathbf{LO}(\mathcal{T})) \subseteq \mathbf{LO}(\mathcal{T})$, i.e. that every conjunctive minor of a family of constraints in $\mathbf{LO}(\mathcal{T})$, is also in $\mathbf{LO}(\mathcal{T})$. So let (R, S) be a conjunctive minor of a non-empty family $(R_j, S_j)_{j \in J}$ of constraints in $\mathbf{LO}(\mathcal{T})$ via a scheme $H = (h_j)_{j \in J}$ with indeterminate set V. Consider the tight conjunctive minor (R_0, S_0) of the family $(R_j, S_j)_{j \in J}$ via the same scheme H. Since every relaxation of (R, S) is a relaxation of (R_0, S_0), in order to prove that $(R, S) \in \mathbf{LO}(\mathcal{T})$, it is enough to show that every finite relaxation of (R_0, S_0) is in $\mathcal{T}$.

Let (F, S') be a finite relaxation of (R_0, S_0), say F having n distinct elements $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Note that for every $\mathbf{a}_i \in F$, there is a Skolem map $\sigma_i : V \to A$ such that, for all j in J, we have $(\mathbf{a}_i + \sigma_i)h_j \in R_j$. For each j in J, let F_j be the subset of R_j, given by

$$F_j = \{(\mathbf{a}_i + \sigma_i)h_j : \mathbf{a}_i \in F\}.$$

Clearly, (F, S') is a conjunctive minor of the family $(F_j, S_j)_{j \in J}$, and for each j in J, (F_j, S_j) is a finite relaxation of (R_j, S_j). Since $\mathbf{CM}(\mathcal{T}) = \mathcal{T}$, and for each j in J, (R_j, S_j) is in $\mathbf{LO}(\mathcal{T})$, we have that every member of the family $(F_j, S_j)_{j \in J}$ belongs to $\mathcal{T}$. Hence (F, S') is a conjunctive minor of a family of members of $\mathcal{T}$, and thus (F, S') is also in $\mathcal{T}$.

The proof of (ii) can be easily obtained by substituting "conjunctive minor" for "weak conjunctive minor", and defining the finite subsets F_j of R_j, by $F_j = \{\mathbf{a}_i h_j : \mathbf{a}_i \in F\}$. □

In other words, $\mathbf{LO}(\mathbf{wCM}(\mathcal{T}))$ and $\mathbf{LO}(\mathbf{CM}(\mathcal{T}))$ are the smallest locally closed sets of constraints containing $\mathcal{T}$, which are closed under formation of weak conjunctive minors and closed under formation of conjunctive minors, respectively. Using the characterizations of the Galois closed sets of constraints, we obtain the following decompositions of the closure operators associated with the corresponding Galois connections:

Proposition 4. *Consider arbitrary non-empty sets A and B. For any set $\mathcal{T}$ of A-to-B relational constraints, the following hold:*

(i) $\mathbf{CSF}(\mathbf{mFSC}(\mathcal{T})) = \mathbf{LO}(\mathbf{wCM}(\mathcal{T} \cup \{(\emptyset, \emptyset), (A, B)\}))$,

(ii) $\mathbf{CSF}(\mathbf{pFSC}(\mathcal{T})) = \mathbf{LO}(\mathbf{wCM}(\mathcal{T} \cup \{(\emptyset, \emptyset), (A, B), (=_A, =_B)\}))$,

(iii) $\mathbf{CSF}(\mathbf{tFSC}(\mathcal{T})) = \mathbf{LO}(\mathbf{CM}(\mathcal{T} \cup \{(\emptyset, \emptyset), (A, B)\}))$, *and*

(iv) $\mathbf{CSF}(\mathbf{sFSC}(\mathcal{T})) = \mathbf{LO}(\mathbf{CM}(\mathcal{T} \cup \{(\emptyset, \emptyset), (=_A, =_B)\}))$.

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FUNCTION CLASS COMPOSITION, RELATIONAL CONSTRAINTS AND STABILITY UNDER COMPOSITIONS WITH CLONES

MIGUEL COUCEIRO AND STEPHAN FOLDES

Abstract. The general Galois theory for functions and relational constraints over arbitrary sets described in the authors' previous paper is refined by imposing algebraic conditions on relations.

1. Introduction

In this paper we extend the results obtained in [CF3] by considering more general closure conditions on classes of functions of several variables, and by restricting relational constraints to consist of invariant relations. In fact, the Theorems 1 and 2 in [CF3] correspond to Theorems 1 and 2 below, respectively, in the particular case $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{P}$, where $\mathcal{P}$ denotes the smallest clone containing only projections. We refer the reader to [CF3] for definitions and terminology not particular to this paper, as well as to [Pi], [CF1] and [CF2] for general background (in the case of finite underlying sets).

Let A, B, E and G be arbitrary non-empty sets.

A *function of several variables on A to B* (or simply, *function on A to B*) is a map $f : A^n \to B$, for some positive integer n called the *arity* of f. A *class* of functions on A to B is a subset $\mathcal{F} \subseteq \cup_{n\geq 1} B^{A^n}$. For a fixed arity n, the n different *projection maps* $\mathbf{a} = (a_t \mid t \in n) \mapsto a_i$, $i \in n$, are also called *variables*. For $A = B = \{0, 1\}$, a function of several variables on A to B is called a *Boolean function*.

If f is an n-ary function on B to E and $g_1, \ldots, g_n$ are all m-ary functions on A to B then the *composition* $f(g_1, \ldots, g_n)$ is an m-ary function on A to E, and its value on $(a_1, \ldots, a_m) \in A^m$ is $f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$. If $\mathcal{I} \subseteq \cup_{n\geq 1} E^{B^n}$ and $\mathcal{J} \subseteq \cup_{n\geq 1} B^{A^n}$ we define the *composition of $\mathcal{I}$ with $\mathcal{J}$*, denoted $\mathcal{I}\mathcal{J}$, by

$$\mathcal{I}\mathcal{J} = \{f(g_1, \ldots, g_n) \mid n, m \geq 1, f \text{ } n\text{-ary in } \mathcal{I}, \ g_1, \ldots, g_n \text{ } m\text{-ary in } \mathcal{J}\}.$$

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If $\mathcal{I}$ is a singleton, $\mathcal{I} = \{f\}$, then we write $f\mathcal{J}$ for $\{f\}\mathcal{J}$. We say that a class $\mathcal{I}$ of functions of several variables is *stable under right* (*left*) *composition with* $\mathcal{J}$ if, whenever the composition is well defined, $\mathcal{I}\mathcal{J} \subseteq \mathcal{I}$ ($\mathcal{J}\mathcal{I} \subseteq \mathcal{I}$, respectively). A *clone* on A is a set $\mathcal{C} \subseteq \cup_{n\geq 1} A^{A^n}$ that contains all projections and satisfies $\mathcal{C}\mathcal{C} \subseteq \mathcal{C}$ (or equivalently, $\mathcal{C}\mathcal{C} = \mathcal{C}$). Note that if $\mathcal{J}$ is a clone on A (on B) and $\mathcal{I} \subseteq \cup_{n\geq 1} B^{A^n}$, then $\mathcal{I}\mathcal{J} \subseteq \mathcal{I}$ if and only if $\mathcal{I}\mathcal{J} = \mathcal{I}$ ($\mathcal{J}\mathcal{I} \subseteq \mathcal{I}$ if and only if $\mathcal{J}\mathcal{I} = \mathcal{I}$, respectively).

Associativity Lemma. *Let A, B, E and G be arbitrary non-empty sets, and consider function classes $\mathcal{I} \subseteq \cup_{n\geq 1} G^{E^n}$, $\mathcal{J} \subseteq \cup_{n\geq 1} E^{B^n}$, and $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$. The following hold:*
 (*i*) *$(\mathcal{I}\mathcal{J})\mathcal{K} \subseteq \mathcal{I}(\mathcal{J}\mathcal{K})$;*
 (*ii*) *If $\mathcal{J}$ is stable under right composition with the clone of projections on B, then $(\mathcal{I}\mathcal{J})\mathcal{K} = \mathcal{I}(\mathcal{J}\mathcal{K})$.*

Proof. The inclusion (i) is a direct consequence of the definition of function class composition. Property (ii) asserts that the converse inclusion also holds if $\mathcal{J}$ is stable under right composition with projections. This hypothesis means in particular that all functions obtained from members of $\mathcal{J}$ by cylindrification and permutation of variables are also in $\mathcal{J}$. A typical function in $\mathcal{I}(\mathcal{J}\mathcal{K})$ is of the form

$$f(g_1(h_{11}, \ldots, h_{1m_1}), \ldots, g_n(h_{n1}, \ldots, h_{nm_n}))$$

where f is in $\mathcal{I}$, the g_i's are in $\mathcal{J}$, and the h_{ij}'s are in $\mathcal{K}$. By taking appropriate functions $g'_1, \ldots, g'_n$ obtained from $g_1, \ldots, g_n$ by cylindrification and permutation of variables, the function above can be expressed as

$$\begin{aligned} &f(g'_1(h_{11}, \ldots, h_{1m_1}, \ldots, h_{n1}, \ldots, h_{nm_n}), \ldots \\ &\ldots, g'_n(h_{11}, \ldots, h_{1m_1}, \ldots, h_{n1}, \ldots, h_{nm_n})) \end{aligned}$$

which is easily seen to be in $(\mathcal{I}\mathcal{J})\mathcal{K}$. □

Note that statement (ii) of the Associativity Lemma applies, in particular, if $\mathcal{J}$ is any clone on $E = B$.

Let $\mathcal{F}$ be a set of functions of several variables on A to B. If $\mathcal{P}$ is the clone of all projections on A, then $\mathcal{F}\mathcal{P} = \mathcal{F}$ expresses closure under taking minors in [Pi], or closure under simple variable substitutions in the terminology of [CF3]. If $A = B = \{0, 1\}$ and $\mathcal{L}_{01}$ is the clone (Post class) of constant preserving linear Boolean functions, then $\mathcal{F}\mathcal{L}_{01} = \mathcal{F}$ is equivalent to closure under substitution of triple sums $x + y + z$ for variables, while $\mathcal{L}_{01}\mathcal{F} = \mathcal{F}$ is equivalent to closure under taking triple sums of Boolean functions $f + g + h$ (see [CF1]).

An *m-ary relation on A* is a subset R of A^m. The elements of A^m are viewed as unary functions on the von Neumann ordinal $m = \{0, \ldots, m - 1\}$

to A. Thus the relation R is a class (set) of unary maps on m to A. A function f of several variables on A to A is said to *preserve* R if $fR \subseteq R$.

For a class $\mathcal{F} \subseteq \cup_{n\geq 1} A^{A^n}$ of functions on A, an m-ary relation R on A is called an *$\mathcal{F}$-invariant* if $\mathcal{F}R \subseteq R$. In other words, R is an $\mathcal{F}$-invariant if every member of $\mathcal{F}$ preserves R. If two classes of functions $\mathcal{F}$ and $\mathcal{G}$ generate the same clone, then the $\mathcal{F}$-invariants are the same as the $\mathcal{G}$-invariants. (See Pöschel [Po1] and [Po2].)

Observe that we always have $R \subseteq \mathcal{F}R$ if $\mathcal{F}$ contains the projections, but we can have $R \subseteq \mathcal{F}R$ even if $\mathcal{F}$ contains no projections. (Take the Boolean triple sum $x_1 + x_2 + x_3$ as the only member of $\mathcal{F}$.)

For a clone $\mathcal{C}$, the intersection of m-ary $\mathcal{C}$-invariants is always a $\mathcal{C}$-invariant and it is easy to see that, for an m-ary relation R, the smallest $\mathcal{C}$-invariant containing R in A^m is $\mathcal{C}R$, and it is said to be *generated by* R. (See [Po1] and [Po2], where Pöschel denotes $\mathcal{C}R$ by $\Gamma_{\mathcal{C}}(R)$.)

2. Classes of Functions Definable by Constraints Consisting of Invariant Relations

Consider arbitrary non empty sets A and B. An m-ary *A-to-B constraint* (or simply, m-ary *constraint*, when the underlying sets are understood from the context) is a couple (R, S) where $R \subseteq A^m$ and $S \subseteq B^m$. The relations R and S are called the *antecedent* and *consequent*, respectively, of the relational constraint (Pippenger [Pi]). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on A and B, respectively. If R is a $\mathcal{C}_1$-invariant and S is a $\mathcal{C}_2$-invariant, we say that (R, S) is a *$(\mathcal{C}_1, \mathcal{C}_2)$-constraint*. A function of several variables $f : A^n \longrightarrow B$, $n \geq 1$, is said to *satisfy* an m-ary A-to-B constraint (R, S) if $fR \subseteq S$.

The following result generalizes Lemma 1 in [CF1]:

Lemma 1. *Consider arbitrary non-empty sets A and B. Let f be a function of several variables on A to B and let $\mathcal{C}$ be a clone on A. If every function in $f\mathcal{C}$ satisfies an A-to-B constraint (R, S), then f satisfies $(\mathcal{C}R, S)$.*

Proof. The assumption means that $(f\mathcal{C})R \subseteq S$. By the Associativity Lemma, $(f\mathcal{C})R = f(\mathcal{C}R)$, and thus $f(\mathcal{C}R) \subseteq S$. □

A class $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ of functions of several variables on A to B is said to be *locally closed* if for every function f of several variables on A to B the following holds: if every finite restriction of f (i.e restriction to a finite subset) coincides with a finite restriction of some member of $\mathcal{K}$, then f belongs to $\mathcal{K}$.

A class $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ of functions of several variables on A to B is said to be *definable* by a set $\mathcal{T}$ of A-to-B constraints, if $\mathcal{K}$ is the class of all those functions which satisfy every constraint in $\mathcal{T}$.

Theorem 1. *Consider arbitrary non-empty sets A and B and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on A and B, respectively. For any function class $\mathcal{K} \subseteq \cup_{n\geq 1} B^{A^n}$ the following conditions are equivalent:*

- (i) *$\mathcal{K}$ is locally closed and it is stable both under right composition with $\mathcal{C}_1$ and under left composition with $\mathcal{C}_2$;*
- (ii) *$\mathcal{K}$ is definable by some set of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints.*

Proof. To show that $(ii) \Rightarrow (i)$, assume that $\mathcal{K}$ is definable by some set $\mathcal{T}$ of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints. For every (R, S) in $\mathcal{T}$, we have $\mathcal{K}R \subseteq S$. Since R is a $\mathcal{C}_1$-invariant, $\mathcal{K}R = \mathcal{K}(\mathcal{C}_1 R)$. By the Associativity Lemma, $\mathcal{K}(\mathcal{C}_1 R) = (\mathcal{K}\mathcal{C}_1)R$, and therefore $(\mathcal{K}\mathcal{C}_1)R = \mathcal{K}R \subseteq S$. Since this is true for every (R, S) in $\mathcal{T}$ we must have $\mathcal{K}\mathcal{C}_1 \subseteq \mathcal{K}$.

For every (R, S) in $\mathcal{T}$, we have $\mathcal{K}R \subseteq S$, and therefore $\mathcal{C}_2(\mathcal{K}R) \subseteq \mathcal{C}_2 S$. By the Associativity Lemma, $(\mathcal{C}_2\mathcal{K})R \subseteq \mathcal{C}_2(\mathcal{K}R) \subseteq \mathcal{C}_2 S$, and $\mathcal{C}_2 S = S$ because S is a $\mathcal{C}_2$-invariant. Thus $(\mathcal{C}_2\mathcal{K})R \subseteq S$ for every (R, S) in $\mathcal{T}$, and we must have $\mathcal{C}_2\mathcal{K} \subseteq \mathcal{K}$.

To see that $\mathcal{K}$ is locally closed, consider $f \notin \mathcal{K}$, say of arity $n \geq 1$, and let (R, S) be an m-ary $(\mathcal{C}_1, \mathcal{C}_2)$-constraint that is satisfied by every function g in $\mathcal{K}$ but not satisfied by f. Hence for some $\mathbf{a}^1, \ldots, \mathbf{a}^n$ in R, $f(\mathbf{a}^1, \ldots, \mathbf{a}^n) \notin S$ but $g(\mathbf{a}^1, \ldots, \mathbf{a}^n) \in S$, for every n-ary function g in $\mathcal{K}$. Thus the restriction of f to the finite set $\{(\mathbf{a}^1(i), \ldots, \mathbf{a}^n(i)) : i \in m\}$ does not coincide with that of any member of $\mathcal{K}$.

To prove $(i) \Rightarrow (ii)$, we show that for every function g not in $\mathcal{K}$, there is a $(\mathcal{C}_1, \mathcal{C}_2)$-constraint (R, S) which is satisfied by every member of $\mathcal{K}$ but not satisfied by g. The class $\mathcal{K}$ will then be definable by the set $\mathcal{T}$ of those $(\mathcal{C}_1, \mathcal{C}_2)$-constraints that are satisfied by all members of $\mathcal{K}$.

Note that $\mathcal{K}$ is a fortiori stable under right composition with the clone containing all projections, that is, $\mathcal{K}$ is closed under simple variable substitutions. We may assume that $\mathcal{K}$ is non-empty. Suppose that g is an n-ary function of several variables on A to B not in $\mathcal{K}$. Since $\mathcal{K}$ is locally closed, there is a finite restriction g_F of g to a finite subset $F \subseteq A^n$ such that g_F disagrees with every function in $\mathcal{K}$ restricted to F. Suppose that F has size m, and let $\mathbf{a}^1, \ldots, \mathbf{a}^n$ be m-tuples in A^m, such that $F = \{(\mathbf{a}^1(i), \ldots, \mathbf{a}^n(i)) : i \in m\}$. Define R_0 to be the set containing $\mathbf{a}^1, \ldots, \mathbf{a}^n$, and let $S = \{f(\mathbf{a}^1, \ldots, \mathbf{a}^n) : f \in \mathcal{K}, f\ n\text{-ary}\}$. Clearly, (R_0, S) is not satisfied by g, and it is not difficult to see that every member of $\mathcal{K}$ satisfies (R_0, S). As $\mathcal{K}$ is stable under left composition with $\mathcal{C}_2$, it follows that S is a $\mathcal{C}_2$-invariant. Let R be the $\mathcal{C}_1$-invariant generated by R_0, i.e. $R = \mathcal{C}_1 R_0$. By Lemma 1, the constraint (R, S) constitutes indeed the desired separating $(\mathcal{C}_1, \mathcal{C}_2)$-constraint. $\square$

This generalizes the characterizations of closed classes of functions given by Pippenger in [Pi] as well as in [CF1] and [CF3] by considering arbitrary underlying sets, possible infinite, and more general closure conditions. In the finite case, we obtain as special cases of Theorem 1 the characterizations given in Theorem 2.1 and Theorem 3.2 in [Pi], by considering $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{P}$, and $\mathcal{C}_1 = \mathcal{M}$ and $\mathcal{C}_2 = \mathcal{P}$, respectively, where $\mathcal{M}$ is a clone containing only functions having at most one essential variable, and $\mathcal{P}$ is the clone of all projections. Taking $A = B = \{0, 1\}$ and $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{L}_{01}$, we get the characterization of classes of Boolean functions definable by sets of affine constraints given in [CF1]. For arbitrary non empty underlying sets, Theorem 1 in [CF3] corresponds to the particular case $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{P}$.

3. Sets of Invariant Constraints Characterized by Functions of Several Variables

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on arbitrary non-empty sets A and B, respectively. Among all A-to-B constraints, observe that the empty constraint and the equality constraint are $(\mathcal{C}_1, \mathcal{C}_2)$-constraints.

The following lemma is essentially a restatement, in a variant form, of the closure condition given by Szabó in [Sz] on the set of relations preserved by a clone of functions. We indicate a proof via [CF3].

Lemma 2. (Szabó) *Let $\mathcal{C}$ be a clone on an arbitrary non-empty set A. If R is a tight conjunctive minor of a non-empty family $(R_j)_{j\in J}$ of $\mathcal{C}$-invariants, then R is a $\mathcal{C}$-invariant.*

Proof. Let R be a tight conjunctive minor of a non-empty family $(R_j)_{j\in J}$ of $\mathcal{C}$-invariants. We have to prove that every function in $\mathcal{C}$ preserves R or, equivalently, that every function in $\mathcal{C}$ satisfies the A-to-A constraint (R, R). Since $(R_j)_{j\in J}$ is a non-empty family of $\mathcal{C}$-invariants, every function in $\mathcal{C}$ preserves every member of the family $(R_j)_{j\in J}$, that is, every function in $\mathcal{C}$ satisfies every member of the family $(R_j, R_j)_{j\in J}$ of A-to-A constraints. From Lemma 1 in [CF3] it follows that every member of $\mathcal{C}$ satisfies (R, R), that is, R is a $\mathcal{C}$-invariant. □

Thus every tight conjunctive minor of a non-empty family $(R_j, S_j)_{j\in J}$ of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints is a $(\mathcal{C}_1, \mathcal{C}_2)$-constraint. However, not all relaxations of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints are $(\mathcal{C}_1, \mathcal{C}_2)$-constraints and so not all conjunctive minors of a non-empty family $(R_j, S_j)_{j\in J}$ of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints are $(\mathcal{C}_1, \mathcal{C}_2)$-constraints. A relaxation (R, S) of an A-to-B constraint (R_0, S_0) is called a $(\mathcal{C}_1, \mathcal{C}_2)$-*relaxation* of (R_0, S_0) if (R, S) is a $(\mathcal{C}_1, \mathcal{C}_2)$-constraint. Similarly, a conjunctive minor (R, S) of a non-empty family $(R_j, S_j)_{j\in J}$ of A-to-B constraints is called a $(\mathcal{C}_1, \mathcal{C}_2)$-*conjunctive minor* of the family $(R_j, S_j)_{j\in J}$, if (R, S) is a $(\mathcal{C}_1, \mathcal{C}_2)$-constraint.

A set $\mathcal{T}$ of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints is said to be *closed under formation of* $(\mathcal{C}_1, \mathcal{C}_2)$-*conjunctive minors* if whenever every member of the non-empty family $(R_j, S_j)_{j\in J}$ of constraints is in $\mathcal{T}$, all $(\mathcal{C}_1, \mathcal{C}_2)$-conjunctive minors of the family $(R_j, S_j)_{j\in J}$ are also in $\mathcal{T}$.

A set $\mathcal{T}$ of A-to-B constraints is said to be *locally closed* if for every A-to-B constraint (R, S) the following holds: if every relaxation of (R, S) with finite antecedent coincides with some member of $\mathcal{T}$, then (R, S) belongs to $\mathcal{T}$.

If $\mathcal{T}_0$ is a set of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints, we say that $\mathcal{T}_0$ is $(\mathcal{C}_1, \mathcal{C}_2)$-*locally closed* if the set $\mathcal{T}$ of all relaxations of the various constraints in $\mathcal{T}_0$ is locally closed.

The following result extends Lemma 1 in [CF2]:

Lemma 3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on arbitrary non-empty sets A and B, respectively. Let $\mathcal{T}_0$ be a set of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints, closed under $(\mathcal{C}_1, \mathcal{C}_2)$-relaxations. Define $\mathcal{T}$ to be the set of all relaxations of the various constraints in $\mathcal{T}_0$. Then $\mathcal{T}_0$ is the set of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints which are in $\mathcal{T}$, and the following are equivalent:*

- (a) *$\mathcal{T}_0$ is closed under formation of $(\mathcal{C}_1, \mathcal{C}_2)$-conjunctive minors;*
- (b) *$\mathcal{T}$ is closed under taking conjunctive minors.*

Proof. Clearly, the first claim holds, and it is easy to see that $(b) \Rightarrow (a)$. To prove implication $(a) \Rightarrow (b)$, assume (a). Let (R, S) be a conjunctive minor of a non-empty family $(R_j, S_j)_{j\in J}$ of A-to-B constraints in $\mathcal{T}$ via a scheme $H = (h_j)_{j\in J}$, $h_j : n_j \to m \cup V$. We have to prove that $(R, S) \in \mathcal{T}$.

Since for every j in J $(R_j, S_j) \in \mathcal{T}$, there is a non-empty family $(R_j^0, S_j^0)_{j\in J}$ of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints in $\mathcal{T}_0$ such that, for each j in J, (R_j, S_j) is a relaxation of (R_j^0, S_j^0). So let (R_0, S_0) be the tight conjunctive minor of the family $(R_j^0, S_j^0))_{j\in J}$ via the scheme H. From Lemma 2, it follows that R_0 is a $\mathcal{C}_1$-invariant and S_0 a $\mathcal{C}_2$-invariant, and since $\mathcal{T}_0$ is closed under formation of $(\mathcal{C}_1, \mathcal{C}_2)$-conjunctive minors, we have $(R_0, S_0) \in \mathcal{T}_0$.

Let us prove that (R, S) is a relaxation of (R_0, S_0) and, thus, that $(R, S) \in \mathcal{T}$. Since R is a restrictive conjunctive minor of the family $(R_j)_{j\in J}$ via the scheme $H = (h_j)_{j\in J}$, we have that for every m-tuple $\mathbf{a}$ in R there is a Skolem map $\sigma : V \to A$ such that, for all j in J, the n_j-tuple $(\mathbf{a} + \sigma)h_j$ is in R_j. Since $R_j \subseteq R_j^0$ for every j in J, it follows that $(\mathbf{a} + \sigma)h_j$ is in R_j^0 for every j in J. Thus $\mathbf{a}$ is in R_0 and we conclude $R \subseteq R_0$.

By analogous reasoning one can easily verify that $\mathbf{b}$ is in S whenever $\mathbf{b}$ is in S_0, i.e that $S \supseteq S_0$. Thus (R, S) is a relaxation of (R_0, S_0) and so $(R, S) \in \mathcal{T}$, and the proof of (a) is complete. $\square$

Theorem 2. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on arbitrary non-empty sets A and B, respectively, and let $\mathcal{T}_0$ be a set of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints. Then the following are equivalent:*

(i) *$\mathcal{T}_0$ is $(\mathcal{C}_1, \mathcal{C}_2)$-locally closed, contains the binary equality constraint, the empty constraint, and it is closed under formation of $(\mathcal{C}_1, \mathcal{C}_2)$-conjunctive minors;*

(ii) *There is a set of functions of several variables on A to B which satisfy exactly those $(\mathcal{C}_1, \mathcal{C}_2)$-constraints that are in $\mathcal{T}_0$.*

Proof. To prove implication $(ii) \Rightarrow (i)$, assume (ii). Let $\mathcal{K}$ be the set of all functions satisfying every constraint in $\mathcal{T}_0$. Note that $\mathcal{T}_0$ is closed under $(\mathcal{C}_1, \mathcal{C}_2)$-relaxations. By Theorem 1, we have $\mathcal{C}_2\mathcal{K} = \mathcal{K}$, and $\mathcal{K}\mathcal{C}_1 = \mathcal{K}$. We may assume that $\mathcal{K} \neq \emptyset$. Let $\mathcal{T}$ be the set of all those constraints (not necessarily $(\mathcal{C}_1, \mathcal{C}_2)$-constraints) satisfied by every function in $\mathcal{K}$. Observe that $\mathcal{T}_0$ is the set of all $(\mathcal{C}_1, \mathcal{C}_2)$-constraints which are in $\mathcal{T}$. We show that $\mathcal{T}$ is the set of all relaxations in $\mathcal{T}_0$.

Let (R, S) be a constraint in $\mathcal{T}$. From the definition of $\mathcal{T}$, it follows that $\mathcal{K}R \subseteq S$. Note that $\mathcal{K}$ is stable under right composition with the clone of projections on A, because $\mathcal{K}\mathcal{C}_1 = \mathcal{K}$. Thus by the Associativity Lemma it follows that $\mathcal{C}_2(\mathcal{K}R) = (\mathcal{C}_2\mathcal{K})R$. Since $\mathcal{C}_2\mathcal{K} = \mathcal{K}$, we have that $\mathcal{C}_2(\mathcal{K}R) = \mathcal{K}R$, i.e. $\mathcal{K}R$ is a $\mathcal{C}_2$-invariant. Also, again because $\mathcal{K}\mathcal{C}_1 = \mathcal{K}$, by Lemma 1 we conclude that every function in $\mathcal{K}$ satisfies $(\mathcal{C}_1 R, \mathcal{K}R)$. Clearly, $(\mathcal{C}_1 R, \mathcal{K}R)$ is a $(\mathcal{C}_1, \mathcal{C}_2)$-constraint, therefore it belongs to $\mathcal{T}_0$. Thus every constraint (R, S) in $\mathcal{T}$ is a relaxation of a member of $\mathcal{T}_0$, namely, a relaxation of $(\mathcal{C}_1 R, \mathcal{K}R)$.

By Theorem 2 in [CF3], we have that $\mathcal{T}$ is locally closed and contains the binary equality constraint, the empty constraint, and it is closed under formation of conjunctive minors. Since the binary equality constraint and the empty constraint are $(\mathcal{C}_1, \mathcal{C}_2)$-constraints, it follows from Lemma 3 that (i) holds.

To prove implication $(i) \Rightarrow (ii)$, it is enough to show that for every $(\mathcal{C}_1, \mathcal{C}_2)$-constraint (R, S) not in $\mathcal{T}_0$, there is a function g which satisfies every constraint in $\mathcal{T}_0$, but does not satisfy (R, S).

Let $\mathcal{T}$ be the set of relaxations of the various $(\mathcal{C}_1, \mathcal{C}_2)$-constraints in $\mathcal{T}_0$. Observe that $(R, S) \notin \mathcal{T}$, otherwise (R, S) would be a $(\mathcal{C}_1, \mathcal{C}_2)$-relaxation of some $(\mathcal{C}_1, \mathcal{C}_2)$-constraint in $\mathcal{T}_0$, contradicting the fact implied by (i) that $\mathcal{T}_0$ is closed under taking $(\mathcal{C}_1, \mathcal{C}_2)$-relaxations. Clearly, $\mathcal{T}$ is locally closed, contains the binary equality constraint, and the empty constraint. From Lemma 3, it follows that $\mathcal{T}$ is closed under taking conjunctive minors. By Theorem 2 in [CF3], there is a function g which does not satisfy (R, S) but satisfies every constraint in $\mathcal{T}$ and so, in particular, g satisfies every constraint in $\mathcal{T}_0$. Thus we have $(i) \Rightarrow (ii)$. □

Theorem 2 generalizes the characterizations of closed classes of constraints given in Pippenger [Pi] and also in [CF2] as well as [CF3] by considering both arbitrary, possibly infinite, underlying sets, and more general closure conditions on classes of relational constraints.

Theorems 1 and 2 may also be viewed as analogues, with constraints instead of relations, of the characterization given by Pöschel, as part of Theorem 3.2 in [Po3], of the closed sets in a class of Galois connections between operations and relations of a prescribed type on a set A.

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FUNCTIONAL EQUATIONS, CONSTRAINTS, DEFINABILITY OF FUNCTION CLASSES, AND FUNCTIONS OF BOOLEAN VARIABLES

MIGUEL COUCEIRO AND STEPHAN FOLDES

Abstract. The paper deals with classes of functions of several variables defined on an arbitrary set A and taking values in a possibly different set B. Definability of function classes by functional equations is shown to be equivalent to definability by relational constraints, generalizing a fact established by Pippenger in the case $A = B = \{0, 1\}$.

Conditions for a class of functions to be definable by constraints of a particular type are given in terms of stability under certain functional compositions. This leads to a correspondence between functional equations with particular algebraic syntax and relational constraints with certain invariance properties with respect to clones of operations on a given set.

When $A = \{0, 1\}$ and B is a commutative ring, such B-valued functions of n variables are represented by multilinear polynomials in n indeterminates in $B[X_1, \ldots, X_n]$. Functional equations are given to describe classes of field-valued functions of a specified bounded degree. Classes of Boolean and of pseudo-Boolean functions are covered as particular cases.

1. Introduction and Basic Definitions

For arbitrary sets B and C, by a *C-valued function on B* we mean a map $f : B^n \to C$, where $n \geq 1$ is called the *arity* of f. The *essential arity* of an n-ary C-valued function $f : B^n \to C$ is defined as the cardinality of the set of indices

$$I = \{1 \leq i \leq n : \text{ there are } a_1, \ldots, a_i, b_i, a_{i+1}, \ldots, a_n \text{ with } a_i \neq b_i \text{ and}$$
$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n)\}$$

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For each $i \in I$, we say that the ith *variable of f is essential*. Note that the essential arity of f is zero if and only if f is constant. If $B = C$, then a C-valued function on B is called an *operation on A*. Operations on the two-element set $B = \{0, 1\}$ are usually refered to as *Boolean functions*.

For any maps $g_1, \ldots, g_n : D \to B$, where D is again any set and $f : B^n \to C$, the *composition* $f(g_1, \ldots, g_n)$ is defined as the map from D to C given by $f(g_1, \ldots, g_n)(a) = f(g_1(a), \ldots, g_n(a))$, for every $a \in D$.

Let A, B and C be arbitrary non-empty sets, $\mathcal{I}$ a class (i.e. set) of C-valued functions on B (of various arities), and $\mathcal{J}$ a class of B-valued functions on A (of various arities). The *class composition* $\mathcal{I}\mathcal{J}$ is defined as the set

$$\mathcal{I}\mathcal{J} = \{f(g_1, \ldots, g_n) \mid n, m \geq 1, f \text{ } n\text{-ary in } \mathcal{I}, \, g_1, \ldots, g_n \text{ } m\text{-ary in } \mathcal{J}\}.$$

If $\mathcal{I}$ is a singleton, $\mathcal{I} = \{f\}$, then we write $f\mathcal{J}$ for $\{f\}\mathcal{J}$. We note that this construction underlies the various notions of subfunction and minors appearing e.g. in [13, 12, 15, 3, 8, 4].

Consider arbitrary non-empty sets A, B, and C, and let $\mathcal{I}$ be a class of C-valued functions on B and $\mathcal{J}$ a class of B-valued functions on A. We say that $\mathcal{I}$ is *stable under right composition with $\mathcal{J}$* if $\mathcal{I}\mathcal{J} \subseteq \mathcal{I}$. Similarly, we say that $\mathcal{J}$ is *stable under left composition with $\mathcal{I}$* if $\mathcal{I}\mathcal{J} \subseteq \mathcal{J}$. Note that a clone on an arbitrary set A is simply a class $\mathcal{C}$ of A-valued functions on A that contains all projections, and is stable under (left or right) composition with itself, i.e. $\mathcal{C}\mathcal{C} \subseteq \mathcal{C}$ (or equivalently, $\mathcal{C}\mathcal{C} = \mathcal{C}$).

Consider arbitrary non-empty sets A and B. A *functional equation* (for B-valued function on A) is a formal expression

$$\begin{aligned} & h_1(\mathbf{f}(g_1(\mathbf{v}_1, \ldots, \mathbf{v}_p)), \ldots, \mathbf{f}(g_m(\mathbf{v}_1, \ldots, \mathbf{v}_p))) = \\ & = h_2(\mathbf{f}(g_1^{'}(\mathbf{v}_1, \ldots, \mathbf{v}_p)), \ldots, \mathbf{f}(g_t^{'}(\mathbf{v}_1, \ldots, \mathbf{v}_p))) \end{aligned} \tag{1}$$

where $m, t, p \geq 1$, $h_1 : B^m \to C$, $h_2 : B^t \to C$, each g_i and ${g'}_j$ is a map $A^p \to A$, the $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are p distinct symbols called *vector variables*, and $\mathbf{f}$ is a distinct symbol called *function symbol*.

For $n \geq 1$, we denote by $\mathbf{n}$ the set $\mathbf{n} = \{1, \ldots, n\}$, so that an n-vector (n-tuple) v in A^n is a map $v : \mathbf{n} \to A$. In this way, if g is an p-ary operation on A and $v_1, \ldots, v_p$ are n-vectors in A^n, then $g(v_1, \ldots, v_p)$ denotes the n-vector

$$(g(v, \ldots, v_p)(1), \ldots, g(v_1, \ldots, v_p)(n)) \in A^n.$$

For an n-ary B-valued function on A, $f : A^n \to B$, we say that f *satisfies* the equation (1) if, for all $v_1, \ldots, v_p \in A^n$, we have

$$\begin{aligned} & h_1(f(g_1(v_1, \ldots, v_p)), \ldots, f(g_m(v_1, \ldots, v_p))) = \\ & = h_2(f(g_1^{'}(v_1, \ldots, v_p)), \ldots, f(g_t^{'}(v_1, \ldots, v_p))) \end{aligned}$$

A class (i.e. set) $\mathcal{K}$ of B-valued functions on A is said to be *defined*, or *definable*, by a set $\mathcal{E}$ of functional equations, if $\mathcal{K}$ is the class of all those functions which satisfy every member of $\mathcal{E}$.

To illustrate, let $A = B = C = \{0,1\}$, $m = 2$, $t = 1$, $p = 2$, and let g_1 be the projection function $(x,y) \mapsto x$, g_2 the conjunction $(x,y) \mapsto xy$, $h_1 = g_1^{'} = g_2$, and h_2 the identity $x \mapsto x$. The functional equation (1) so specified defines the clone (Post class) of monotone Boolean functions. In a more free style of notation, this equation can be displayed as

$$\mathbf{f}(\mathbf{v}_1)\mathbf{f}(\mathbf{v}_1\mathbf{v}_2) = \mathbf{f}(\mathbf{v}_1\mathbf{v}_2)$$

When the specific context is well understood, we shall present functional equations in such more informal manner.

Useful functional properties have often been advantangeously expressed by functional equations. Classical examples include the linearity of $\mathbb{F}$-valued functions on a field $\mathbb{F}$, as well as monotonicity and convexity properties traditionally expressed by functional inequalities which are obviously equivalent to functional equations in max-plus language. More contemporary examples include the submodular property of real-valued functions $\{0,1\}^n \longrightarrow \mathbb{R}$, and Post classes (clones) of Boolean functions traditionally characterized by relations. Many strong consequences of submodularity, such as the Hall-Rado theorems, follow directly from the characterizing submodular inequality which is essentially a max-plus functional equation (see Welsh [14]). For Boolean functions, equations were systematically studied in [3] and, in a variant form, by Pogosyan [9]. Also, in [5] equations were shown to provide a measure of complexity, essentially in terms of the syntax of the functional equations used to define Post classes.

2. Definability of Function Classes by Functional Equations and Relational Constraints

An *m-ary relation on A* is a subset R of A^m, and thus the relation R can be viewed as a class (set) of unary maps from $\mathbf{m}$ to A. A function $f : A^n \to A$ is said to *preserve* R, and R is said to be *invariant under* f, if $fR \subseteq R$, where fR is the class composition $\{f\}R$ as explained above. An m-ary *A-to-B constraint* (or simply, m-ary *constraint*, when the underlying sets are understood from the context) is a couple (R, S) where $R \subseteq A^m$ and $S \subseteq B^m$. The relations R and S are called the *antecedent* and *consequent*, respectively, of the relational constraint (Pippenger [8]). A B-valued function on A, $f : A^n \longrightarrow B$, $n \geq 1$, is said to *satisfy* an m-ary A-to-B constraint (R, S) if $fR \subseteq S$. A class $\mathcal{K}$ of B-valued functions on A is said to be *defined*, or *definable*, by a set $\mathcal{T}$ of A-to-B constraints, if $\mathcal{K}$ is the class of all those functions which satisfy every constraint in $\mathcal{T}$.

As an example, the already mentioned clone of monotone Boolean functions can be equivalently defined by the single constraint $(\leq, \leq)$, where $\leq$ denotes the less-or-equal relation on $\{0,1\}$.

In [8], Pippenger has shown that in the Boolean case, i.e. when $A = B = \{0, 1\}$, definability of a function class by functional equations is equivalent to definability by relational constraints. The following theorem is not contingent on the Boolean case, and not even on the finiteness of the underlying sets.

Theorem 1. *Let A be an arbitrary non-empty set, and B any set with at least two elements. For any class $\mathcal{K}$ of B-valued functions on A, the following are equivalent:*

(i) *$\mathcal{K}$ is definable by some set of functional equations;*
(ii) *$\mathcal{K}$ is definable by some set of relational constraints.*

Proof. To prove that $(i) \Rightarrow (ii)$, it is enough to show that for every functional equation (1) there is a relational constraint (R, S), such that the B-valued functions on A satisfying the equation are exactly the same as those satisfying the constraint. Indeed, we can define the constraint (R, S) by

$$R = \{(g_1(a), \ldots, g_m(a), g_1^{'}(a), \ldots, g_t^{'}(a)) : a \in A^p\},$$
$$S = \{(b_1, \ldots, b_m, b'_1, \ldots, b'_t) \in B^{m+t} : h_1(b_1, \ldots, b_m) = h_2(b'_1, \ldots, b'_t)\}.$$

Conversely, let us show that $(ii) \Rightarrow (i)$. Let $\mathcal{T}$ be a set of constraints, and let $\mathcal{K}$ be the class of B-valued functions on A defined by $\mathcal{T}$. Consider the set $\mathcal{T}'$ of constraints obtained from $\mathcal{T}$ by removing all those constraints with empty antecedent. Clearly, $\mathcal{T}$ and $\mathcal{T}'$ define the same class $\mathcal{K}$ of B-valued functions on A. Therefore, the proof will be complete if we can show that for every constraint (R, S) with $R \neq \emptyset$ there is a functional equation (1) satisfied by exactly the same functions as those satisfying (R, S).

Let m be the arity of (R, S). The construction of the equation (1) is based on the following facts.

Fact 1. *Given the non-empty relation $R \subseteq A^m$, there is a $p \geq 1$ and a map $g : A^p \to A^m$, such that the range of g is R.*

Fact 2. *Given the relation $S \subseteq B^m$, there exist maps $h_1, h_2 : B^m \to B$, such that*

$$S = \{b \in B^m : h_1(b) = h_2(b)\}.$$

Using these functions g, h_1 and h_2, the equation (1) can be defined as follows: the integer m is the arity of (R, S), $t = m$, and p is the arity of $g : A^p \to A^m$. For $1 \leq i \leq m = t$, let $g_i = g_i^{'}$ be the ith component of g, i.e. we have

$$g(a) = (g_1(a), \ldots, g_m(a))$$

for all $a \in A^p$. The maps h_1, h_2 in (1) are given by Fact 2.

$\square$

It is not difficult to see that both Fact 2 and Theorem 1 itself would fail if we allowed B to be a singleton. However, the implication $(i) \Rightarrow (ii)$ in Theorem 1 would continue to hold.

3. Definability of Function Classes by Invariant Constraints

The question of definability of Boolean function classes by constraints (R, S), when $R, S \subseteq \{0,1\}^n$ are of a special algebraic kind, was considered in [1]. Specifically, the relations R and S were required to be affine subspaces of the vector space $\{0,1\}^n$, over the two-element field $\mathbf{GF}(2)$. A subset of $\{0,1\}^n$ is an affine subspace if and only if it is closed under the triple sum operation $u+v+w$, i.e. if and only if it is invariant under the clone $\mathcal{L}_{01}$ of constant-preserving linear Boolean functions - that is, functions which are the sum of an odd number of variables. (See e.g. Godement [6].) Also it is well known that the non-empty affine subspaces can be described as ranges of affine maps, and that affine hyperplanes can be described as kernels of affine forms, i.e. as sets on which a given form agrees with the null form. As shown in [1], this accounts for the definability of certain function classes by linear equations.

In this section we consider general notions of closure for the antecedent R and the consequent S of a constraint (R, S), and we address the question of definability of classes of B-valued functions on a set A by such invariant constraints, without any restriction on the underlying sets A and B.

Associativity Lemma. *Consider arbitrary non-empty sets A, B, C and E, and let $\mathcal{I}$ be a class of E-valued functions on C, $\mathcal{J}$ a class of C-valued functions on B, and $\mathcal{K}$ a class of B-valued functions on A. The following hold:*

(i) $(\mathcal{I}\mathcal{J})\mathcal{K} \subseteq \mathcal{I}(\mathcal{J}\mathcal{K})$;

(ii) *If $\mathcal{J}$ is stable under right composition with the clone of projections on B, then $(\mathcal{I}\mathcal{J})\mathcal{K} = \mathcal{I}(\mathcal{J}\mathcal{K})$.*

Proof. The inclusion (i) is a direct consequence of the definition of function class composition. Property (ii) asserts that the converse inclusion also holds if $\mathcal{J}$ is stable under right composition with projections. A typical function in $\mathcal{I}(\mathcal{J}\mathcal{K})$ is of the form

$$f(g_1(h_{11}, \ldots, h_{1m_1}), \ldots, g_n(h_{n1}, \ldots, h_{nm_n}))$$

where f is in $\mathcal{I}$, the g_i's are in $\mathcal{J}$, and the h_{ij}'s are in $\mathcal{K}$. By taking appropriate functions $g_1', \ldots, g_n'$ obtained from $g_1, \ldots, g_n$ by addition of inessential variables and permutation of variables, the function above can be expressed as

$$f(g_1'(h_{11}, \ldots, h_{1m_1}, \ldots, h_{n1}, \ldots, h_{nm_n}), \ldots \\ \ldots, g_n'(h_{11}, \ldots, h_{1m_1}, \ldots, h_{n1}, \ldots, h_{nm_n}))$$

which is easily seen to be in $(\mathcal{I}\mathcal{J})\mathcal{K}$. $\square$

Note that statement (ii) of the Associativity Lemma applies, in particular, if $\mathcal{J}$ is any clone on $C = B$.

Let $\mathcal{F}$ be a set of B-valued functions on A. If $\mathcal{P}$ is the clone of all projections on A, then $\mathcal{F}\mathcal{P} = \mathcal{F}$ expresses closure under taking minors as in [8], or closure under simple variable substitutions in the terminology of [2].

For a class $\mathcal{F}$ of A-valued functions on A, an m-ary relation R on A is said to be *$\mathcal{F}$-invariant* if $\mathcal{F}R \subseteq R$. In other words, R is $\mathcal{F}$-invariant if every member of $\mathcal{F}$ preserves R. If two classes of functions $\mathcal{F}$ and $\mathcal{G}$ generate the same clone, then the $\mathcal{F}$-invariant relations are the same as the $\mathcal{G}$-invariant relations. (See Pöschel [10] and [11].)

Observe that we always have $R \subseteq \mathcal{F}R$ if $\mathcal{F}$ contains the projections, but we can have $R \subseteq \mathcal{F}R$ even if $\mathcal{F}$ contains no projections. (Take the Boolean triple sum $x_1 + x_2 + x_3$ as the only member of $\mathcal{F}$.)

For a clone $\mathcal{C}$, the intersection of m-ary $\mathcal{C}$-invariant relations is always $\mathcal{C}$-invariant and it is easy to see that, for an m-ary relation R, the smallest $\mathcal{C}$-invariant relation containing R in A^m is $\mathcal{C}R$, and it is said to be *generated by* R. (See [10] and [11], where Pöschel denotes $\mathcal{C}R$ by $\Gamma_{\mathcal{C}}(R)$.)

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on arbitrary non-empty sets A and B, respectively. If R is $\mathcal{C}_1$-invariant and S is $\mathcal{C}_2$-invariant, we say that (R, S) is a $(\mathcal{C}_1, \mathcal{C}_2)$-*constraint*. The following result generalizes Lemma 1 in [1]:

Lemma 1. *Consider arbitrary non-empty sets A and B. Let f be a B-valued function on A, and let $\mathcal{C}$ be a clone on A. If every function in $f\mathcal{C}$ satisfies an A-to-B constraint (R, S), then f satisfies $(\mathcal{C}R, S)$.*

Proof. The assumption means that $(f\mathcal{C})R \subseteq S$. By the Associativity Lemma, $(f\mathcal{C})R = f(\mathcal{C}R)$, and thus $f(\mathcal{C}R) \subseteq S$. □

A class $\mathcal{K}$ of B-valued functions on A is said to be *locally closed* if for every B-valued function f on A the following holds: if every finite restriction of f (i.e restriction to a finite subset) coincides with a finite restriction of some member of $\mathcal{K}$, then f belongs to $\mathcal{K}$.

Theorem 2. *Consider arbitrary non-empty sets A and B and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be clones on A and B, respectively. For any class $\mathcal{K}$ of B-valued functions on A, the following conditions are equivalent:*

(i) *$\mathcal{K}$ is locally closed and it is stable both under right composition with $\mathcal{C}_1$ and under left composition with $\mathcal{C}_2$;*
(ii) *$\mathcal{K}$ is definable by some set of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints.*

Proof. To show that $(ii) \Rightarrow (i)$, assume that $\mathcal{K}$ is definable by some set $\mathcal{T}$ of $(\mathcal{C}_1, \mathcal{C}_2)$-constraints. For every (R, S) in $\mathcal{T}$, we have $\mathcal{K}R \subseteq S$. Since R is $\mathcal{C}_1$-invariant, $\mathcal{K}R = \mathcal{K}(\mathcal{C}_1 R)$. By the Associativity Lemma, $\mathcal{K}(\mathcal{C}_1 R) = (\mathcal{K}\mathcal{C}_1)R$, and therefore $(\mathcal{K}\mathcal{C}_1)R = \mathcal{K}R \subseteq S$. Since this is true for every (R, S) in $\mathcal{T}$ we must have $\mathcal{K}\mathcal{C}_1 \subseteq \mathcal{K}$.

For every (R, S) in $\mathcal{T}$, we have $\mathcal{K}R \subseteq S$, and therefore $\mathcal{C}_2(\mathcal{K}R) \subseteq \mathcal{C}_2 S$. By the Associativity Lemma, $(\mathcal{C}_2\mathcal{K})R \subseteq \mathcal{C}_2(\mathcal{K}R) \subseteq \mathcal{C}_2 S$, and $\mathcal{C}_2 S = S$ because S is $\mathcal{C}_2$-invariant. Thus $(\mathcal{C}_2\mathcal{K})R \subseteq S$ for every (R, S) in $\mathcal{T}$, and we must have $\mathcal{C}_2\mathcal{K} \subseteq \mathcal{K}$.

To see that $\mathcal{K}$ is locally closed, consider $f \notin \mathcal{K}$, say of arity $n \geq 1$, and let (R, S) be an m-ary $(\mathcal{C}_1, \mathcal{C}_2)$-constraint that is satisfied by every function g in $\mathcal{K}$ but not satisfied by f. Hence for some $a^1, \ldots, a^n$ in R, $f(a^1, \ldots, a^n) \notin S$ but $g(a^1, \ldots, a^n) \in S$, for every n-ary function g in $\mathcal{K}$. Thus the restriction of f to the finite set $\{(a^1(i), \ldots, a^n(i)) : i \in \mathbf{m}\}$ does not coincide with that of any member of $\mathcal{K}$.

To prove $(i) \Rightarrow (ii)$, we show that for every function g not in $\mathcal{K}$, there is a $(\mathcal{C}_1, \mathcal{C}_2)$-constraint (R, S) which is satisfied by every member of $\mathcal{K}$ but not satisfied by g. The class $\mathcal{K}$ will then be definable by the set $\mathcal{T}$ of those $(\mathcal{C}_1, \mathcal{C}_2)$-constraints that are satisfied by all members of $\mathcal{K}$.

Note that $\mathcal{K}$ is a fortiori stable under right composition with the clone containing all projections, that is, $\mathcal{K}$ is closed under simple variable substitutions. We may assume that $\mathcal{K}$ is non-empty. Suppose that g is an n-ary B-valued function on A which is not in $\mathcal{K}$. Since $\mathcal{K}$ is locally closed, there is a finite restriction g_F of g to a finite subset $F \subseteq A^n$ such that g_F disagrees with every function in $\mathcal{K}$ restricted to F. Suppose that F has size m, and let $a^1, \ldots, a^n$ be m-tuples in A^m, such that $F = \{(a^1(i), \ldots, a^n(i)) : i \in \mathbf{m}\}$. Define R_0 to be the set $\{a^1, \ldots, a^n\}$, and let $S = \{f(a^1, \ldots, a^n) : f \in \mathcal{K}, f$ n-ary$\}$. Clearly, (R_0, S) is not satisfied by g, and it is not difficult to see that every member of $\mathcal{K}$ satisfies (R_0, S). As $\mathcal{K}$ is stable under left composition with $\mathcal{C}_2$, it follows that S is $\mathcal{C}_2$-invariant. Let R be the $\mathcal{C}_1$-invariant relation generated by R_0, i.e. $R = \mathcal{C}_1 R_0$. By Lemma 1, the constraint (R, S) constitutes indeed the desired separating $(\mathcal{C}_1, \mathcal{C}_2)$-constraint. $\square$

This generalizes the characterizations of closed classes of functions given by Pippenger in [8] as well as in [1] and [2] by considering arbitrary underlying sets, possible infinite, and more general closure conditions. In the finite case, we obtain as special cases of Theorem 2 the characterizations given in Theorem 2.1 and Theorem 3.2 in [8], by considering $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{P}$, and $\mathcal{C}_1 = \mathcal{U}$ and $\mathcal{C}_2 = \mathcal{P}$, respectively, where $\mathcal{U}$ is a clone containing only functions having at most one essential variable, and $\mathcal{P}$ is the clone of all projections. Taking $A = B = \{0, 1\}$ and $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{L}_{01}$, we get the characterization of classes of Boolean functions definable by sets of affine constraints given in

[1]. For arbitrary non empty underlying sets, Theorem 1 in [2] corresponds to the particular case $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{P}$. In this case, from Theorem 1 and Theorem 2 we conclude the following:

Corollary 1. *Consider arbitrary non-empty sets A and B. The equationally definable classes of B-valued functions on A are exactly those locally closed classes that are stable under right composition with the clone of projections on A.*

In certain cases, given a $(\mathcal{C}_1, \mathcal{C}_2)$-constraint (R, S), $R \subseteq A^m$, $S \subseteq B^m$, the construction of a functional equation given in the proof of Theorem 1 in the previous section can be refined to yield a functional equation with special algebraic syntax. To do this, one may seek to use, instead of arbitrary functions as given by Fact 1 and Fact 2 in the proof of Theorem 1, functions $g_1, \ldots, g_m, h_1, h_2$ of a particular kind still satisfying the conditions of these Facts. For example, in [1], the functions were chosen to be affine maps, based on the range-and-kernel theory of linear algebra. Another application of this strategy will be given in Section 4.

Also, in certain cases, given a functional equation (1) with a special algebraic syntax, if the functions $g_1, \ldots, g_m, g_1', \ldots, g_t', h_1, h_2$ appearing in the equation have particular structure-preserving properties, then it may be possible to conclude that the construction of the constraint (R, S), as given in the first part of the proof of Theorem 1, yields relations R and S invariant under certain clones $\mathcal{C}_1$ and $\mathcal{C}_2$. Thus the affine functions appearing in the "linear" functional equations defined in [1] were used to construct affine constraints. The same principle, together with Theorem 2, will be used in Section 4 to show that certain natural function classes cannot be defined by a particular type of functional equations.

4. Functions of Boolean Variables valued in a Ring

In this section we consider functions $\{0,1\}^n \to B$, where B is a commutative ring. We view $\{0,1\}$ as endowed with the two-element field structure, $\{0,1\} = \mathbf{GF}(2)$, as well as with the lattice structure where $0 < 1$. If B is also $\{0,1\} = \mathbf{GF}(2)$, then these B-valued functions are called Boolean functions. If B is the field $\mathbb{R}$ of real numbers, then the functions under consideration are called *pseudo-Boolean functions*, which provide an algebraic representation for set functions $\mathcal{P}(E) \to \mathbb{R}$ for finite E (see e.g. [4] for a recent reference).

Every Boolean function $\{0,1\}^n \to \{0,1\}$ is well known to be representable by a unique multilinear polynomial in n indeterminates over $\mathbf{GF}(2)$, i.e. a polynomial which is linear in each of its indeterminates, called its *Zhegalkin* *polynomial*, *Reed-Muller polynomial* or *ring-sum expansion*. Also, pseudo-Boolean functions can be uniquely represented by multilinear polynomials in n indeterminates over $\mathbb{R}$ (see Hammer and Rudeanu [7]).

Consider any commutative ring B with null and identity elements 0_B and 1_B, respectively. For a polynomial $p \in B[X_1, \dots, X_n]$ in n indeterminates, and an n-tuple $(a_1, \dots, a_n) \in \{0,1\}^n$, for each a_i let a_i^B denote 0_B or 1_B according to whether a_i is 0 or 1, and denote the evaluation $p(a_1^B, \dots, a_n^B)$ simply by $p(a_1, \dots, a_n)$. The B-valued function on $\{0,1\}$ given by

$$(a_1, \dots, a_n) \mapsto p(a_1, \dots, a_n)$$

is said to be *represented* by p. By a method similar to that used by Hammer and Rudeanu [7] for the case $B = \mathbb{R}$, we show in the next theorem the existence of a unique multilinear polynomial representation for any B-valued function on $\{0,1\}$, for any commutative ring B with identity. This unifies the Zhegalkin and pseudo-Boolean polynomial representations.

Theorem 3. *Consider any commutative ring B with identity. For any $n \geq 1$, every B-valued function f on $\{0,1\}$, $f : \{0,1\}^n \to B$, is represented by a unique multilinear polynomial $p \in B[X_1, \dots, X_n]$.*

Proof. The existence of representation is proved by induction on essential arity. For essential arity 0, i.e. for constant functions, representation by constant polynomials is obvious. For a function $f : \{0,1\}^n \to B$ with essential arity $m > 0$, assuming the claim proved for lesser essential arities, and taking any index i such that the ith variable of f is essential, let f_0 and f_1 be the n-ary B-valued functions given by

$$\begin{aligned} f_0(a_1, \dots, a_n) &= f(a_1, \dots a_{i-1}, 0, a_{i+1}, \dots, a_n) \\ f_1(a_1, \dots, a_n) &= f(a_1, \dots a_{i-1}, 1, a_{i+1}, \dots, a_n) \end{aligned}$$

We have

$$f(a_1, \dots, a_n) = (1 - a_i^B) f_0(a_1, \dots, a_n) + a_i^B f_1(a_1, \dots, a_n)$$

and both f_0 and f_1 have essential arity less than m. By the induction hypothesis, f_0 and f_1 are represented by polynomials p_0 and p_1, respectively. Thus f is represented by the polynomial

$$p = (1 - X_i)p_0 + X_i p_1$$

and if p had any powers of indeterminates X_j^k with $k > 1$, by replacing each such occurrence by X_j we would obtain a multilinear polynomial representing f.

Uniqueness is proved by contradiction. Suppose that f had two distinct multilinear polynomial representations p and q. Then the multilinear polynomial $p - q$ would represent the constant zero function. Let J be a set of indices of smallest possible size, such that the monomial $c \prod_{j \in J} X_j$ occurs in $p - q$ with coefficient $c \neq 0_B$: such a J must exist if $p - q$ is not the zero polynomial. But then the evaluation of $p-q$ at $(a_1,\ldots,a_n)$, where $a_j = 1_B$ if $j \in J$ and $a_j = 0_B$ otherwise, would be $c \neq 0_B$, contradicting the fact that $p-q$ represents the constant zero function. Thus $p-q$ must be the null polynomial, i.e. $p = q$. □

Let f be a B-valued function on $\{0,1\}$, $f : \{0,1\}^n \to B$, where B is a commutative ring with identity. The *degree* of f is the smallest non-negative integer d such that for every $J \subseteq \{1,\ldots,n\}$ of size $\mid J \mid > d$ the coefficient of $\prod_{j\in J} X_j$ in the multilinear polynomial representation of f is zero. Thus the functions of degree 0 are precisely the constants (including the constant zero function).

Theorem 4. *If B is any field of characteristic 2, and $k \geq 1$, then the class of B-valued functions on $\{0,1\}$ having degree less than k is defined by the following functional equation (with vector variables $\mathbf{v}_1,\ldots,\mathbf{v}_k$):*

$$\sum_{I\subseteq\{1,\ldots,k\}} \mathbf{f}(\sum_{i\in I} \mathbf{v}_i) = 0 \tag{2}$$

In 2 the inner summations refer to addition in the two-element field $\mathbf{GF}(2) = \{0,1\}$, while the outer summation refers to addition in the field B. For $I = \emptyset$, the empty sum $\sum\limits_{i\in I} \mathbf{v}_i$ represents the constant zero.

Proof. First we prove that (2) is satisfied by every B-valued function on $\{0,1\}$ having degree less than k. From the form of the equation (2), it is easy to see that the class of functions satisfying (2) is closed under linear combinations with coefficients in B. Therefore, it is sufficient to prove that, for $n \geq 1$, every n-ary B-valued function f on $\{0,1\}$ represented by a product of less than k indeterminates, i.e. of the form $\prod_{j\in J} X_j$, $\mid J \mid < k$, $J \subseteq \{1,\ldots,n\}$, satisfies (2).

Let $v_1,\ldots,v_k$ be any n-vectors in $\{0,1\}^n$. Let w^J be the characteristic vector of J in $\{0,1\}^n$, i.e. $w^J = (a_1,\ldots,a_n)$, where $a_j = 1$ if $j \in J$, and $a_j = 0$ otherwise. For every $I \subseteq \{1,\ldots,k\}$, consider the vector $w^J \cdot (\sum_{i\in I} v_i)$ in $\{0,1\}^n$, where the product $\cdot$ is defined componentwise. Observe that there are 2^k possible choices for I, yet due to the size of $\mid J \mid < k$, there at most 2^{k-1} distinct vectors of the form $w^J \cdot (\sum_{i\in I} v_i)$ in $\{0,1\}^n$. Therefore, there are distinct subsets I_1, I_2 of $\{1,\ldots,k\}$, such that

$$w^J \cdot (\sum_{i\in I_1} v_i) = w^J \cdot (\sum_{i\in I_2} v_i)$$

and for the symmetric difference D of I_1 and I_2, we have

$$w^J \cdot (\sum_{i\in D} v_i) = 0.$$

The 2^k subsets of $\{1,\ldots,k\}$, are matched into pairs $\{I, I+D\}$, where $I+D$ is the symmetric difference of I and D, and because f is represented by $\prod_{j\in J} X_j$, by the definition of w^J it follows that for each such pair we have

$$f(\sum_{i\in I} v_i) = f(w^J \cdot (\sum_{i\in I} v_i)) = f(w^J \cdot (\sum_{i\in I+D} v_i)) = f(\sum_{i\in I+D} v_i).$$

Therefore, due to the fact that the underlying field B has characteristic 2, the terms in the equation (2) cancel pairwise.

Conversely, suppose now that the n-ary function f is represented by a polynomial of degree greater than or equal to k. We show that f does not satisfy the equation (2).

Let g be the B-valued function on $\{0,1\}$ represented by the sum of those monomials in the polynomial representation of f which have degree less than k. By the first part of the proof, g satisfies (2). Working towards a contradiction, suppose that f satisfies (2). Given the form of equation (2), this is the case if and only if the n-ary function $h = f+g$, represented by the sum of all monomials in the polynomial representation of f having degree greater than or equal to k, satisfies (2).

Let J be an inclusionwise minimal subset of $\{1,\ldots,n\}$, such that the monomial $c\prod_{j\in J} X_j$ appears in the polynomial representation of h with coefficient $c \neq 0_B$. Note that $\mid J \mid \geq k$. We claim that if f (or equivalently, h) satisfies (2), then the function $h_{\mathbf{k}}$ represented by the monomial $c\prod_{j\in\mathbf{k}} X_j$ where $\mathbf{k} = \{1,\ldots,k\}$, also satisfies equation (2).

Observe that, by the construction in the proof of Theorem 1, equation (2) is equivalent to a constraint (R,S) whose antecedent R is the range of a linear map with codomain $\mathbf{GF}(2)^m$, i.e. R is a subspace of the vector space $\mathbf{GF}(2)^m$ over $\mathbf{GF}(2)$. Thus by Theorem 2 it follows that the class $\mathcal{K}$ of functions satisfying 2 is stable under right composition with the clone $\mathcal{L}_0$ of 0-preserving linear Boolean functions. In particular, $\mathcal{K}$ is closed under permutation and identification of variables, as well as under fixing variables to 0. It is not difficult to see that $h_{\mathbf{k}}$ can be obtained from h by a combination of these operations. In other words, if h satisfies the equation (2), then $h_{\mathbf{k}}$ also satisfies the equation.

Now, let $v_1,\ldots,v_k$ be the unit n-vectors $e_1,\ldots,e_k$ in $\{0,1\}^n$. We have

$$\sum_{I\subseteq\mathbf{k}} h_{\mathbf{k}}(\sum_{i\in I} v_i) = h_{\mathbf{k}}(\sum_{i\in\mathbf{k}} v_i) = c \neq 0$$

which shows that $h_{\mathbf{k}}$ does not satisfy the equation (2), and yields the desired contradiction. □

In [1] it was shown that, for any positive integer k, the class of Boolean functions whose Zhegalkin polynomial has degree less than k, can be defined by "linear" equations. Theorem 4 above explicitly gives such an equation for every $k \geq 1$. For $k = 1$, the equation (2) can be rewritten as $\mathbf{f}(\mathbf{v}) = \mathbf{f}(0)$, and for $k = 2$, as $\mathbf{f}(\mathbf{v} + \mathbf{w}) = \mathbf{f}(\mathbf{v}) + \mathbf{f}(\mathbf{w}) + \mathbf{f}(0)$.

If B is a field and $A = \{0, 1\} = \mathbf{GF}(2)$, then a functional equation (1) is called *linear* if the functions $g_1, \ldots, g_m, g_1^{'}, \ldots, g_t^{'}$ are all affine maps from the p-dimensional vector space $\mathbf{GF}(2)^p$ to $\mathbf{GF}(2)$, and h_1, h_2 are affine maps from the B-vector spaces B^m and B^t, respectively, to the scalar field B. (Recall that a function $F^n \to F$, where F is any field, is affine if and only if it is of the form $(a_1, \ldots, a_n) \mapsto c_1 a_1 + \ldots c_n a_n + c$, for fixed scalars $c_1, \ldots, c_n, c$ in F.) Obviously, the functional equation (2) in Theorem 4 is linear. Our next result shows that the requirement on the characteristic of the underlying field is indeed essential.

Theorem 5. *For any field B of characteristic different from 2, and any $k \geq 2$, the class of B-valued functions on $\{0, 1\}$ having degree less than k is not definable by any set of linear functional equations.*

Proof. As in the proof Theorem 4, if there would be a $k \geq 2$ such that the class $\mathcal{K}$ of B-valued functions on $\{0, 1\}$ having degree less than k is definable by some set of linear functional equations, then, using the construction given in the proof of Theorem 1, we would conclude that the class in question is definable by some set of constraints whose antecedents are affine subspaces of vector spaces over $\mathbf{GF}(2)$. These affine subspaces would be closed under the triple sum $u + v + w$, i.e. invariant under the clone $\mathcal{L}_{01}$ of constant-preserving linear Boolean functions. By Theorem 2, this would imply that $\mathcal{K}$ is stable under right composition with the clone $\mathcal{L}_{01}$. We show that this is not the case.

Consider the $(k-1)$-ary function f represented by the monomial

$$X_1 \ldots X_{k-1}.$$

Let τ be the $(k+1)$-ary Boolean function in $\mathcal{L}_{01}$ given by

$$(a_1, \ldots, a_{k+1}) \mapsto a_{k-1} + a_k + a_{k+1}.$$

Note that the B-valued function τ_B defined on $\{0, 1\}$ which is valued 1_B on exactly those vectors $(a_1, \ldots, a_{k+1})$ for which $\tau(a_1, \ldots, a_{k+1}) = 1$ and valued 0_B otherwise, is represented by the polynomial

$$X_{k-1} + X_k + X_{k+1} - 2X_{k-1}X_k - 2X_kX_{k+1} - 2X_{k-1}X_{k+1} + 4X_{k-1}X_kX_{k+1}$$

where $+$ and $-$ are to be interpreted in B. Thus, the composition

$$f(f_1, \ldots, f_{k-1})$$

where $f_{k-1} = \tau$ and f_i is the $(k+1)$-ary ith projection function

$$(a_1, \ldots, a_{k+1}) \mapsto a_i$$

for $k = 1, \ldots, k-2$, is represented by the polynomial in $k+1$ indeterminates

$$X_1 \ldots X_{k-2}(X_{k-1} + X_k + X_{k+1} - \\ -2X_{k-1}X_k - 2X_kX_{k+1} - 2X_{k-1}X_{k+1} + 4X_{k-1}X_kX_{k+1})$$

where $+$ and $-$ are to be interpreted in B. From the fact that B has characteristic different from 2, it follows that this polynomial has degree greater than k. $\square$

Note that for $k = 1$, the class of functions of degree less that k, i.e. the class of constants, is defined by the linear expression $\mathbf{f}(\mathbf{v}) = \mathbf{f}(0)$. In fact, from Theorem 5 above it follows that, if B is any field of characteristic different from 2, then the set of constants is the only linearly definable class of B-valued functions on $\{0, 1\}$ of bounded degree. However, Corollary 1 guarantees the existence of equational characterizations of these classes, because bounded degree classes are stable under right composition with the trivial clone $\mathcal{P}$ containing only projections. The following generalization of Corollary 3.3 in [4] provides an equation characterizing classes of bounded degree functions of Boolean variables, and whose codomain is any commutative ring with identity.

Theorem 6. *If B is any commutative ring with identity, and $k \geq 1$, then the class of B-valued functions on $\{0, 1\}$ having degree less than k is defined by the following functional equation (with vector variables $\mathbf{v}_1, \ldots, \mathbf{v}_k$):*

$$\mathbf{f}(\bigwedge_{i \in \mathbf{k}} \mathbf{v}_i) + \sum_{\substack{I \subseteq \mathbf{k} \\ I \neq \emptyset}} (-1)^{|I|} \mathbf{f}(\bigvee_{j \in I} \bigwedge_{i \in \mathbf{k} \setminus \{j\}} \mathbf{v}_i) = 0 \tag{3}$$

where $\mathbf{k} = \{1, \ldots, k\}$.

In (3) the summation refers to addition in the commutative ring B. Equation (3) was obtained in [4] as a combination of two opposite inequalities in the ordered real field $B = \mathbb{R}$. Inequalities are not available in general in a commutative ring, in particular in finite fields. However, the following direct proof, based on the principles used in establishing the functional inequality in Theorem 3.1 in [4], can still be used in the arbitrary commutative ring context.

Proof. First we show that every B-valued function on $\{0, 1\}$ of degree less than k satisfies equation (3). As in the proof of Theorem 4, it is enough to show that every monomial of degree less than k satisfies equation (3), because every linear combination (with coefficients in B) of functions satisfying (3), also satisfies the equation.

Let f be an n-ary B-valued function on $\{0, 1\}$ represented by $\prod_{j \in J} X_j$, $| J |< k$, $J \subseteq \{1, \ldots, n\}$. Let w^J be the characteristic vector of J in $\{0, 1\}^n$.

Let $v_1, \ldots, v_k$ be any n-vectors in $\{0,1\}^n$, and let u denote their conjunction $\bigwedge_{i\in\mathbf{k}} v_i$. For every $j \in \mathbf{k} = \{1, \ldots, k\}$, let

$$u_j = \bigwedge_{i\in\mathbf{k}\setminus\{j\}} v_i$$

and let the vector $z(I)$ be defined by

$$z(I) = w^J \cdot (\bigvee_{j\in I} u_j) \quad \text{for } \emptyset \neq I \subseteq \mathbf{k}, \quad \text{and} \quad z(\emptyset) = w^J \cdot u$$

where the product $\cdot$ is defined componentwise. From the fact that $k >| J |$, it follows that there is an $l \in \mathbf{k}$ such that

$$w^J \cdot u = w^J \cdot u_l.$$

Fix such an index l. It is not difficult to see that, for every $I \subseteq \mathbf{k}$, we have

$$f(\bigvee_{j\in I} u_j) = f(z(I)) \quad \text{and} \quad z(I) = z(I + \{l\})$$

and thus the terms in the sum

$$f(\bigwedge_{i\in\mathbf{k}} v_i) + \sum_{\substack{I\subseteq\mathbf{k}\\ I\neq\emptyset}} (-1)^{|I|} f(\bigvee_{j\in I} u_j)$$

cancel pairwise, i.e. the sum is zero, which shows that f satisfies (3).

In order to complete the proof of Theorem 6, we need to show that if f is an n-ary function of degree greater than or equal to k, then equation (3) is not satisfied by f.

Let g and h be the n-ary functions represented by the sum of monomials, in the polynomial representation of f, having degree less than k and greater than or equal to k, respectively. As in the proof of Theorem 4, f satisfies equation (3) if and only if h satisfies the equation. We prove that h does not satisfy (3).

Let J be an inclusionwise minimal subset of $\mathbf{n} = \{1, \ldots, n\}$, such that the monomial $c\prod_{j\in J} X_j$ appears in the polynomial representation of h, with coefficient $c \neq 0_B$. Note that $| J |\geq k$. Let J_0 be any subset of J of size k. For every $j \in J_0$, consider the n-vectors $y_j = (a_1, \ldots, a_n)$, where $a_j = 0$, $a_i = 0$ if $i \notin J$, and $a_i = 1$ if $i \in J \setminus \{j\}$. Let $v_1, \ldots, v_k$ be defined as the vectors y_j, $j \in J_0$, in any order. Let $u = \bigwedge_{i\in\mathbf{k}} v_i$, and for each $j \in \mathbf{k}$, let

$$u_j = \bigwedge_{i\in\mathbf{k}\setminus\{j\}} v_i.$$

Observe that for $I \subseteq \mathbf{k}$, all monomials in the polynomial representation of h are evaluated to zero on

$$\bigvee_{j\in I} u_j$$

except in the case $I = \mathbf{k}$, where the only monomial which has non-zero value is $c\prod_{j\in J} X_j$, because the n-vector

$$\bigvee_{j\in\mathbf{k}} u_j = (a_1, \ldots, a_n)$$

is given by $a_t = 1$ if $t \in J$, and $a_t = 0$ otherwise. Therefore, we have

$$h(\bigwedge_{i\in\mathbf{k}} v_i) + \sum_{\substack{I\subseteq\mathbf{k}\\ I\neq\emptyset}} (-1)^{|I|} h(\bigvee_{j\in I} u_j) = (-1)^k h(\bigvee_{j\in\mathbf{k}} u_j) = (-1)^k c \neq 0$$

which shows that h, and thus f, does not satisfy equation (3). □

Theorem 6 provides in particular an alternative equational characterization of classes of Boolean functions whose Zhegalkin polynomials have degree bounded by a positive integer k.

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