ANTTI KUUSISTO

Modal Fragments of Second-Order Logic

ACADEMIC DISSERTATION
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Tampere, September 2011
Antti Kuusisto

Abstract

In this thesis we investigate various fragments of second-order logic that arise naturally in considerations related to modal logic. The focus is on questions related to expressive power. The results in the thesis are reported in four independent but related chapters (Chapters 2, 3, 4 and 5). In Chapter 2 we study *second-order propositional modal logic*, which is the system obtained by extending ordinary modal logic with second-order quantification of proposition symbols. We show that the alternation hierarchy of this logic is infinite, thereby solving an open problem from the related literature. In Chapter 3 we investigate the expressivity of a range of modal logics extended with existential prenex quantification of accessibility relations and proposition symbols. The principal result of the chapter is that the resulting extension of (a version of) *Boolean modal logic* can be effectively translated into *existential monadic second-order logic*. As a corollary we obtain decidability results for multimodal logics over various classes of frames with built-in relations. In Chapter 4 we study the equality-free fragment of existential second-order logic with second-order quantification of function symbols. We show that over directed graphs, the expressivity of the fragment is incomparable with that of first-order logic. We also show that over finite models with a unary relational vocabulary, the fragment is weaker in expressivity than first-order logic. In Chapter 5 we study the extension of polyadic modal logic with unrestricted quantification of accessibility relations and proposition symbols. We obtain a range of results related to various natural fragments of the system. Finally, we establish that this extension of modal logic exactly captures the expressivity of second-order logic.

Contents

CHAPTER 1

Introduction

Since the advent of relational semantics, modal logic has developed fast and currently the field has a wide range of applications in different disciplines ranging from computer science and artificial intelligence to economics and linguistics. *Due to its well manifest success, modal logic deserves a developed mathematical background theory.* This thesis contributes to the understanding of the model theory of very expressive extensions of modal logic. The focus is sharp; we digress very little from questions concerning expressivity of fragments of second-order logic (SO) that are directly related to modal logic. The article [21] in the Handbook of Modal Logic [3] is a relatively recent survey giving an overview the current state of the model theory of modal logic. See also [12], and see the Chapters 1–3 of [7] for background information.

In this thesis we study various fragments of second-order logic that arise naturally in considerations concerning extensions of modal logic. Understanding fragments of second-order logic can be very useful in the study of non-classical logics with constructors giving them the capacity to express properties not expressible in first-order logic (FO). A typical such non-classical logic immediately translates into a fragment of second-order logic. Armed with theorems about fragments of second-order logic, one may then immediately obtain a range of results concerning the non-classical logic under investigation. Such results can be, for example, related to decidability issues or expressivity of the logic in question.

While the principal topic of the thesis is modal logic, the investigations below can also be regarded as a study of (fragments of) second-order logic. A notably wide range of the very difficult open problems in finite model theory [44] are questions about the expressive power of fragments of SO. For example, by Fagin's theorem (see [44]) and due to the fact that PTIME is closed under complement, separating universal second-order logic and existential second-order logic in the finite immediately separates PTIME from NP. Difficult questions aside, a developed model theory of second-order logic can help in the study of a wide range of mathematical problems, for example in discrete mathematics. Since the expressive power of second-order logic is very high and related questions have proved tough, it makes sense to take rather small steps. Directing attention to fragments when developing the theory is a natural approach. While potentially directly useful, insights about fragments also elucidate the role different logical constructors (such as connectives and quantifiers) play in making the expressive power of second-order logic. In this thesis we concentrate on fragments motivated by investigations in modal logic, and from the point of view of second-order model theory such systems are obviously not the only interesting fragments. However, we believe that results about *modal fragments of second-order logic* can ideally serve two purposes. They are results about second-order logic and also potential tools for investigations in modal logic.

In Chapter 2 we study *second-order propositional modal logic* (SOPML). Second-order propositional modal logic is the system obtained by extending ordinary modal logic with propositional quantifiers $\exists P$ and $\forall P$. Informally, a formula $\exists P \varphi$ is true if there exists an interpretation of P such that φ is true. In the standard framework extending Kripke semantics, such propositional quantifiers are monadic second-order quantifiers, i.e., quantifiers ranging over subsets of the domain of a model. Johan van Benthem asks in [5] whether the alternation hierarchy of SOPML is infinite over the class of Kripke frames. (See Chapter 2 for the definition of alternation hierarchies.) The question is also posed in the article [11] of ten Cate. We show in Chapter 2 that the syntactic alternation hierarchy of SOPML induces an infinite corresponding hierarchy of definable classes of Kripke frames. The result has been published in [39].

Chapter 3 is an exercise in *arity reduction* of existential second-order quantifiers. The investigations concentrate on two systems of modal logic with existential prenex quantification of accessibility relations and proposition symbols, $\Sigma^1_1(\mathrm{ML})$ and $\Sigma^1_1(\mathrm{PBML}^=)$. The system $\Sigma^1_1(\mathrm{ML})$ is the logic obtained by extending ordinary multimodal logic with existential prenex quantification of binary accessibility relation symbols and proposition symbols. $\mathrm{PBML}^=$ is the logic obtained by extending *polyadic*[1] multimodal logic with built-in identity relations (see Subsection 3.2.1) and with operators that allow for the Boolean combination of accessibility relations; $\Sigma^1_1(\mathrm{PBML}^=)$ is the extension of $\mathrm{PBML}^=$ by existential prenex quantification of accessibility relations and proposition symbols. $\mathrm{PBML}^=$ stands for "polyadic Boolean modal logic with identity".

The principal result in Chapter 3 is that formulae of $\Sigma^1_1(\mathrm{PBML}^=)$ translate into equivalent formulae of *monadic* Σ^1_1. Recall that monadic Σ^1_1 is the extension of first-order logic with existential prenex quantification of unary relation symbols. We also establish that $\Sigma^1_1(\mathrm{ML})$ translates into *monadic* $\Sigma^1_1(\mathrm{MLE})$, which is the logic obtained by extending multimodal logic with the global modality and existential prenex quantification of proposition symbols. Both translations lead to decidability results for multimodal logics over

[1]Recall that a Kripke-style modal logic where accessibility relations are not required to be binary, but can be of a higher arity, is called polyadic. See Chapters 2 and 3 of this thesis or Chapter 1 of [7] for the related definitions.

various classes of frames. Chapter 3 is based on the article [25], which is joint work with Lauri Hella. The investigations in the chapter are related to a generalized perspective on *modal correspondence theory* and also an open problem of Grädel and Rosen [23] asking whether $\Sigma^1_1(\mathrm{FO}^2)$ is contained in monadic Σ^1_1. See the chapter for further details. In addition to [25], modal logic with quantification of binary relations has been studied for example in [13, 42, 43, 53].

In Chapter 4 we study *equalilty-free existential second-order logic with function quantification*, or $\mathrm{fESO}_{wo=}$. The system $\mathrm{fESO}_{wo=}$ is the fragment of Σ^1_1 where second-order quantification is over function symbols only, and formulae are equality-free. The original motivation for studying this fragment stems from considerations related to *Henkin quantifiers* and *independence-friendly* (IF) *logic*. Various interesting equality-free subsystems of IF logic translate into $\mathrm{fESO}_{wo=}$. In particular, systems of *independence-friendly modal logic* tend to translate into $\mathrm{fESO}_{wo=}$. For recent developments related to IF logic and IF modal logic, see for example [34, 35, 58, 65].

Despite the original motivations related to IF logic, we dwell very little on IF logic in Chapter 4. In fact, we save the reader the trouble of getting acquainted with IF logic altogether. After all, we believe that $\mathrm{fESO}_{wo=}$ is worthy of study simply because it is a relatively natural fragment of Σ^1_1. We establish that $\mathrm{fESO}_{wo=}$ admits a simple truth preserving model transformation that enables an easy access to inexpressibility results. We observe that $\mathrm{fESO}_{wo=}$ and FO are incomparable in expressive power over relational models whose vocabulary contains a binary relation symbol. The situation changes if we restrict attention to finite models whose vocabulary contains only unary relation symbols. The principal result of Chapter 4 is that over finite models with a unary relational vocabulary, $\mathrm{fESO}_{wo=}$ is weaker in expressive power than first-order logic. The result is established using an indirect argument that is, we believe, quite interesting in its own right. The argument applies to a wide range of logics, not only $\mathrm{fESO}_{wo=}$. We end the chapter by observing that $\mathrm{fESO}_{wo=}$ is strictly more expressive than equality-free first-order logic over finite models with a unary relational vocabulary. Chapter 4 is based on the articles [40, 41].

In Chapter 5 we study fragments of systems SOPMLE and SO(ML). The system SOPMLE is the logic obtained by extending SOPML with the global modality. The system SO(ML) is the logic obtained by extending polyadic modal logic with unrestricted quantification of accessibility relations (of any arity) and proposition symbols.

Consider formulae of SOPMLE of the type $\overline{\exists P}\,\overline{\forall Q}\,\varphi$, where $\overline{\exists P}$ and $\overline{\forall Q}$ are strings of existential and universal propositional quantifiers, respectively, and φ is free of propositional quantifiers. We call such formulae Σ_2 formulae of SOPMLE. We prove a range of results concerning the fragment. For example, we identify a tool for establishing inexpressibility results that apply to formulae of the fragment.

In Chapter 5 we also study the role of *arity and alternation* in second-order quantification. Consider SOPML formulae of the type $\overline{\exists P}\,\varphi$, where φ does not contain propositional quantifiers. We call the fragment of SOPML that contains these formulae the Σ_1 fragment (of SOPML). We identify a class of finite pointed models over which the expressivity of neither SOPML nor $\Sigma^1_1(\mathrm{PBML}^=)$ exceeds the expressivity of the Σ_1 fragment of SOPML, but a formula of the type $\exists R \forall P\,\varphi$—where R is binary, P unary and φ is free of second-order quantifiers—immediately takes us beyond the expressivity of the Σ_1 fragment of SOPML. Finally, we show that second-order logic is equi-expressive with SO(ML), thereby obtaining a kind of a modal normal form for second-order logic. The results in Chapter 5 are previously unpublished.

1.1 Preliminary Considerations

Technical issues required in the chapters of the thesis are developed in the chapters themselves. In spite of that, we summarize the main technical conventions here.

We denote models by M, N, M' etc. By a model we always mean a first-order structure. (See Definition 1.1 of [15] for example.) We use this notion of a model also in modal logic. The related details will be clearly developed in the chapters. We consider models with a vocabulary containing relation symbols and sometimes constant symbols. While it would be possible to incorporate function symbols into the vocabulary of models in a natural way here and there, we streamline the exposition by ignoring this possibility. (Function symbols do play a part, however, in Chapter 4, where we consider existential second-order logic with function quantification.) If M is a model, we let $Dom(M)$ denote the domain of the model. A *pointed model* is a pair (M, w), where M is a model and $w \in Dom(M)$.

In the context of predicate logic, functions that interpret first-order and second-order variable symbols in the domain of a model are called *assignments.* If f is a function with the domain S, then by $f\frac{u}{x}$ we mean a function g with the domain $S \cup \{x\}$ such that

$$g(z) = \begin{cases} u & \text{if } z = x, \\ f(z) & \text{if } z \neq x. \end{cases}$$

Note that it may or may not be the case that $x \in S$.

We denote formulae of modal and predicate logic by Greek letters mostly. We reserve the turnstile $\models$ for predicate logic and the turnstile $\Vdash$ for modal logic. If φ is a formula of predicate logic, then $M, f \models \varphi$ means that the model M satisfies φ under the assignment f. Recall that $\bigwedge \emptyset$ is interpreted to be a formula that is always true and $\bigvee \emptyset$ a formula that is always false. The symbol $\top$ denotes a formula that is always true and the symbol $\bot$ a formula that is always false.

While we have attempted to make the exposition of all results relatively rigorous and self-contained, acquaintance with logic in general and modal logic in particular is assumed. Familiarity with finite model theory is helpful, and in Chapter 5, acquaintance with the very basics of the theory of finite automata and formal languages is required.

CHAPTER 2

Modal Logic and Monadic Second-Order Alternation Hierarchies

In this chapter we establish that the quantifier alternation hierarchy of formulae of second-order propositional modal logic (SOPML) induces an infinite corresponding semantic hierarchy over the class of finite directed graphs. This solves an open question posed in [5] and [11]. We also provide modal characterizations of the expressive power of monadic second-order logic (MSO) and address a number of points that should promote the potential advantages of viewing MSO and its fragments from the modal perspective.

2.1 SOPML and Monadic Alternation Hierarchies

In this chapter we investigate the expressive power of second-order propositional modal logic (SOPML), which is the system obtained by extending ordinary modal logic by propositional quantifiers ranging over sets of domain elements. Modal logics with propositional quantifiers have been investigated by a variety of researchers, see [4, 6, 9, 11, 16, 18, 33, 36, 37, 38, 59, 60] for example.

Johan van Benthem [5] and Balder ten Cate [11] raise the question whether the prenex quantifier alternation hierarchy of SOPML formulae induces an infinitely ascending corresponding hierarchy of definable classes of Kripke frames. This is an interesting question, especially as ten Cate shows in [11] that formulae of SOPML admit a prenex normal form representation. We show that the semantic counterpart of the quantifier alternation hierarchy of SOPML formulae is infinite over the class of finite directed graphs. This automatically implies that the semantic hierarchy is infinite over the class of Kripke frames. Alternation hierarchies have received a lot of attention in finite model theory, see [51, 52, 47, 56, 57, 61] for example. As SOPML is a semantically natural fragment of MSO (see Theorem 6 in [11]), we feel that our result is relatively interesting also from the point of view of finite model theory.

Our main tool in investigating quantifier alternation in SOPML is a theorem of Schweikardt [57] which states that the alternation hierarchy of monadic second-order logic is strict over the class of *grids*. Inspired by the approach of Matz and Thomas in [52], we employ a strategy loosely based on *strong first-order reductions* in order to transfer the result of Schweikardt from the context of grids to the context of a special class of finite directed graphs that we define. Over this class the expressive power of SOPML coincides with that of MSO, and hence we easily obtain the desired result that the alternation hierarchy of SOPML is infinite over finite directed graphs. The precise definition of strong first-order reductions (found in [51]) is of no importance for the investigations in this chapter, as we give a self-contained exposition of all our results.

As a by-product of the investigations concerning alternation hierarchies, we discuss a simple, effective procedure that translates MSO sentences to equivalent formulae of second-order propositional modal logic with the global modality (SOPMLE). The procedure is based on a translation that bears a very close resemblance to a translation of ten Cate in [11], and the considerations related to the procedure are inspired by the approach in [11]. The procedure establishes that the expressive power of SOPMLE over finite/arbitrary relational structures coincides with that of MSO, and a trivial adaptation of the related argument shows that replacing the global modality with the difference modality does not change the picture. Such modal perspectives on MSO could turn out interesting from the point of view of finite model theory, for example.

The chapter is structured as follows. In Section 2.2 we fix the notation and discuss a number of preliminary issues. In Section 2.3 we show that MSO = SOPMLE with regard to expressive power. Using an approach analogous to that in Section 2.3, we then define in Section 2.4 a special class of directed graphs over which MSO and SOPML coincide in expressive power. In Section 2.5 we first work with MSO, transferring the result of Schweikardt to the context of the newly defined special class of directed graphs. Then, using the connection created in Section 2.4, we finally establish that the SOPML alternation hierarchy is infinite over directed graphs.

2.2 Preliminary Definitions

In this section we introduce technical notions that occupy a central role in the rest of the current chapter.

2.2.1 Syntax and Semantics

With a model we mean a first-order model of predicate logic, and we restrict attention to models associated with a vocabulary containing relation symbols and possibly constant symbols.

We fix countably infinite sets VAR_{FO} and VAR_{SO} of first-order and second-order variables, respectively. Naturally we assume that the sets are disjoint. We let

$$\mathrm{VAR} = \mathrm{VAR}_{FO} \cup \mathrm{VAR}_{SO}.$$

We let lower-case symbols x, y, z denote first-order variables. Upper-case symbols X, Y, Z denote second-order variables. A union f of two functions

$$f_{FO} : \mathrm{VAR}_{FO} \longrightarrow Dom(M)$$

and

$$f_{SO} : \mathrm{VAR}_{SO} \longrightarrow Pow(Dom(M)),$$

where M is a model and $Dom(M)$ its domain, is called an *assignment*. Monadic second-order logic is interpreted in terms of models and assignments, so we write $M, f \models \varphi$ when a model M satisfies an MSO formula φ under an assignment f.

Let PROP be the smallest set T such that the following conditions are satisfied.

1. If $x \in \mathrm{VAR}_{FO}$, then $P_x \in T$.

2. If $X \in \mathrm{VAR}_{SO}$, then $P_X \in T$.

The elements of PROP are proposition *variables*. Let

$$S = S_0 \cup S_1 \cup S_2 \cup S_+$$

be a vocabulary, where S_0 is a set of constant symbols, S_1 and S_2 are sets of unary and binary relation symbols respectively, and S_+ is a set of relation symbols of higher arities. We assume that S and PROP are disjoint. The language $L(S)$ of SOPML associated with the vocabulary S is the smallest set T such that the following conditions are satisfied.

1. If $c \in S_0$, then $c \in T$.

2. If $P_\# \in \mathrm{PROP}$, where $\# \in \mathrm{VAR}$, then $P_\# \in T$.

3. If $P \in S_1$, then $P \in T$.

4. If $\varphi \in T$, then $\neg\varphi \in T$.

5. If $\varphi \in T$ and $\psi \in T$, then $(\varphi \wedge \psi) \in T$.

6. If $R \in S_2$ and $\varphi \in T$, then $\langle R \rangle \varphi \in T$.

7. If $R' \in S_+$ is a k-ary relation symbol and $\varphi_i \in T$ for $i \in \{1, ..., k-1\}$, then $\langle R' \rangle (\varphi_1, ..., \varphi_{k-1}) \in T$.

8. If $P_\# \in \mathrm{PROP}$ and $\varphi \in T$, then $\exists P_\# \, \varphi \in T$.

When SOPML is extended by the global modality, we obtain *second-order modal logic with the global modality*, or SOPMLE (cf. SOEPDL in [59]). The language $L^E(S)$ of SOPMLE associated with the vocabulary

S is the smallest set T satisfying the conditions listed above when defining the language $L(S)$ of SOPML, and also the following additional condition.

$$\text{If } \varphi \in T, \text{ then } \langle E \rangle \varphi \in T.$$

Here we assume that $E \notin S$. The operator $\langle E \rangle$ is called the *global diamond.*

The elements of the sets $L(S)$ and $L^E(S)$ are called *S-formulae*, or alternatively, *formulae of the vocabulary S.* The set of symbols in S that occur in an S-formula φ is called the set of *non-logical symbols of φ*. Analogous conventions apply to formulae of predicate logic: formulae of predicate logic associated with a vocabulary S are called S-formulae or formulae of the vocabulary S, and the set of non-logical symbols in S that occur in an S-formula ψ of predicate logic is the set of non-logical symbols of ψ. For example, the MSO formulae $\exists X \forall x \big(P(x) \wedge X(y)\big)$ and $\forall x \big(x = x \vee Q(c) \vee X(x)\big)$ are both formulae of the vocabulary $\{P, Q, c\}$. Here P and Q are relation symbols, X a relation variable, c a constant symbol and x, y first-order variables. In addition to being a $\{P, Q, c\}$-formula, the first formula is also a formula of the vocabulary $\{P, Q\}$ and a $\{P, c\}$-formula, for example. The set of non-logical symbols of the first formula is $\{P\}$, and the set $\{Q, c\}$ is the set of non-logical symbols of the second formula. Notice that the identity symbol is not considered to be a non-logical symbol. The SOPMLE formula $\langle E \rangle \langle R \rangle (P \wedge P_x)$ is, for example, a $\{c, R, P, Q\}$-formula. Here R, P, Q are relation symbols, c a constant symbol and P_x a proposition variable. The set of non-logical symbols of the formula is $\{R, P\}$. Notice indeed that the symbol E associated with the global diamond $\langle E \rangle$ is not considered to be a non-logical symbol.

Formulae of SOPML and SOPMLE are interpreted with respect to pointed models. Recall that a pointed model is a pair (M, w), where M is a model and $w \in Dom(M)$. In addition to pointed models, we also need objects that interpret free occurrences of proposition variables in PROP. Any function

$$V : \text{PROP} \longrightarrow Pow(Dom(M)),$$

where M is a model, is called a *valuation.*

Let S be the vocabulary we defined above. Let M be an S-model with $w \in Dom(M) = W$. (An S-model, or a *model of the vocabulary S*, is a model M' such that the set of non-logical symbols that M' gives an interpretation to is exactly the set S.) Let V be a valuation that maps PROP to $Pow(W)$. We let $\Vdash$ denote the modal truth relation, which we now define for the model M and for S-formulae of SOPML in the following recursive fashion.

Let $c \in S_0$, $P \in S_1$ and $R \in S_2$. Let $R' \in S_+$ be a k-ary relation symbol for some integer k greater or equal to three. Let $P_\# \in \text{PROP}$, where $\#$ is a variable symbol in $\mathrm{VAR}_{FO} \cup \mathrm{VAR}_{SO}$. Let $\varphi, \psi, \varphi_1, ..., \varphi_{k-1}$ be formulae of SOPML of the vocabulary S. We define

$$
\begin{array}{lll}
(M,w), V \Vdash c & \Leftrightarrow & w = c^M, \\
(M,w), V \Vdash P & \Leftrightarrow & w \in P^M, \\
(M,w), V \Vdash P_\# & \Leftrightarrow & w \in V(P_\#), \\
(M,w), V \Vdash \neg\varphi & \Leftrightarrow & (M,w), V \nVdash \varphi, \\
(M,w), V \Vdash (\varphi \wedge \psi) & \Leftrightarrow & (M,w), V \Vdash \varphi \text{ and } (M,w), V \Vdash \psi, \\
(M,w), V \Vdash \exists P_\# \, \varphi & \Leftrightarrow & \exists U \subseteq W\big((M,w), V\frac{U}{P_\#} \Vdash \varphi\big), \\
(M,w), V \Vdash \langle R \rangle \, \varphi & \Leftrightarrow & \exists u \in W\big(wR^M u \text{ and } (M,u) \Vdash \varphi\big), \\
(M,w), V \Vdash \langle R' \rangle \, (\varphi_1, ..., \varphi_{k-1}) & \Leftrightarrow & \exists u_1 ... u_{k-1} \in W \text{ such that} \\
 & & R'^M(w, u_1, ..., u_{k-1}) \text{ and} \\
 & & (M, u_i), V \Vdash \varphi_i \text{ for each } i.
\end{array}
$$

The truth definition of SOPMLE is obtained by extending the above set of clauses by the following additional clause.

$$(M,w), V \Vdash \langle E \rangle \varphi \ \text{ iff } \ \exists u \in Dom(M)\big((M,u), V \Vdash \varphi\big).$$

As SOPML is a fragment of SOPMLE, in the remaining part of the current subsection (Subsection 2.2.1) we only refer to SOPMLE formulae when fixing conventions that apply to formulae of both SOPML and SOPMLE.

If a formula φ of SOPMLE does not contain free occurrences of proposition variables, we may drop the valuation V and write $(M,w) \Vdash \varphi$. An SOPMLE formula without free proposition variables is called a *sentence.* We extend the definition of the modal truth relation $\Vdash$ to the context of models (as opposed to pointed models) in the following standard way.

$$M \Vdash \varphi \ \text{ iff for all } w \in Dom(M), \ (M,w) \Vdash \varphi.$$

We also extend the truth relation of predicate logic to cover pointed models. Let S be a vocabulary, M an S-model and $\varphi(x)$ an S-formula of predicate logic with exactly one free variable, the first-order variable x. We define

$$(M,w) \models \varphi(x) \ \text{ iff } \ M, \frac{w}{x} \models \varphi(x),$$

where $M, \frac{w}{x} \models \varphi(x)$ means that M satisfies $\varphi(x)$ when x is interpreted as w.

Let S be a vocabulary and H_p be a class of *pointed* S-models. We say that an S-sentence φ of SOPMLE *defines the class* C of pointed models *with respect to* H_p, if

$$C = \{\, (M,w) \in H_p \mid (M,w) \Vdash \varphi \,\}.$$

We write

$$\mathrm{MOD}_{H_p}(\varphi) = C.$$

Similarly, we say that an S-formula $\psi(x)$ of MSO *defines the class* C of pointed models *with respect to* H_p if

$$C = \{(M, w) \in H_p \mid (M, w) \models \psi(x)\}.$$

The formula $\psi(x)$ is required to contain exactly one free first-order variable and no free second-order variables. We write

$$\mathrm{MOD}_{H_p}(\psi(x)) = C.$$

Let H be a class of S-models. We say that an S-sentence φ of SOPMLE *defines the class* C of models *with respect to* H if

$$C = \{M \in H \mid M \Vdash \varphi\}.$$

This mode of definability is sometimes referred to as *global definability*. We write

$$\mathrm{MOD}_H(\varphi) = C.$$

Similarly, we say that an S-sentence ψ of MSO *defines the class* C of models *with respect to* H if

$$C = \{M \in H \mid M \models \psi\}.$$

We write

$$\mathrm{MOD}_H(\psi) = C.$$

Two MSO formulae φ and ψ are called *uniformly equivalent*, if the following three conditions are satisfied.

1. The two formulae have exactly the same set of free variable symbols. That is, the subset of VAR of variables that occur free in φ is exactly the same as the subset of VAR of variables that occur free in ψ.

2. The two formulae have exactly the same set U of non-logical symbols.

3. The equivalence
$$M, f \models \varphi \;\Leftrightarrow\; M, f \models \psi$$
holds for all U-models M and all variable assignments f that map the set VAR to the set $Dom(M) \cup Pow(Dom(M))$.

For example the formulae $X(y)$ and $X(y) \vee y \neq y$ are uniformly equivalent. The formulae $P(x) \vee \neg P(x)$ and $Q(x) \vee \neg Q(x)$ are *not* uniformly equivalent since they fail to have the same set of non-logical symbols. (The set of non-logical symbols of the formula $P(x) \vee \neg P(x)$ is $\{P\}$ and that of the formula $Q(x) \vee \neg Q(x)$ is $\{Q\}$.) The formulae $x_1 = x_1$ and $x_2 = x_2$ are *not* uniformly equivalent since they fail to have the same set of free variable symbols.

Two SOPMLE formulae φ' and ψ' are uniformly equivalent, if the following three conditions are satisfied.

1. The two formulae have exactly the same set of free proposition variables. That is, the subset of PROP of proposition variables that occur free in φ' is exactly the same as the subset of PROP of proposition variables that occur free in ψ'.

2. The two formulae have exactly the same set U of non-logical symbols.

3. The equivalence

$$(M, w), V \models \varphi' \;\Leftrightarrow\; (M, w), V \models \psi'$$

holds for all pointed U-models (M, w) and all valuations that map the set PROP to the set $Pow(Dom(M))$.

Let χ be a sentence of SOPMLE and $\pi(x)$ a formula of MSO with exactly one free variable, the first-order variable x. The sentence χ and the formula $\pi(x)$ are uniformly equivalent if they have exactly the same set U of non-logical symbols, and if we have

$$(M, w) \Vdash \chi \;\Leftrightarrow\; (M, w) \models \pi(x)$$

for all pointed U-models (M, w). An SOPMLE sentence χ' and an MSO sentence π' are called *globally uniformly equivalent*, if the sentences have the same set U' of non-logical symbols, and if we have

$$M \Vdash \chi' \;\Leftrightarrow\; M \models \pi'$$

for all U'-models M.

When we informally leave out parentheses when writing formulae, the order of preference of logical connectives is such that unary connectives have the highest priority, and then come $\wedge, \vee, \rightarrow, \leftrightarrow$ in the given order.

2.2.2 Grids and Graphs

Two classes of structures have a central role in the considerations that follow.

Definition 2.1. Let $m, n \in \mathbb{N}_{\geq 1}$ and let

$$D = \{1, ..., m\} \times \{1, ..., n\}.$$

Let S_1 and S_2 be binary relation symbols. Define two binary relations S_1^{Gd} and S_2^{Gd} such that S_1^{Gd} contains exactly the pairs of the type

$$\big((i, j), (i + 1, j)\big) \quad \in \quad D \times D$$

and S_2^{Gd} exactly the pairs of the type

$$\big((i, j), (i, j + 1)\big) \quad \in \quad D \times D.$$

The structure $Gd = \big(D, S_1^{Gd}, S_2^{Gd}\big)$ is a *grid*, and the grid Gd is said to *correspond to* an $m \times n$ matrix. The element $(1,1)$ of the domain of a grid is referred to as the *top left element*. We let GRID denote the class of grids. Note that this class is not closed under isomorphism. In fact there would be no problem calling GRID the *set* of all grids.

The other class of structures we shall consider is that of (nonempty) *directed graphs*. A directed graph is a structure (W, R), where $W \neq \emptyset$ is a finite set and $R \subseteq W \times W$ a binary relation. When we refer to a graph we always mean a nonempty, finite directed graph. We let GRAPH denote the class of finite directed graphs.

2.2.3 Alternation Hierarchies

An MSO formula in *monadic prenex normal form* consists of a vector of monadic second-order quantifiers followed by a first-order part. Levels of the *monadic second-order quantifier alternation hierarchy* measure the number of alternating blocks of existential and universal second-order quantifiers of MSO formulae in monadic prenex normal form. It is natural to classify SOPML formulae in an analogous way. Below we give formal definitions of alternation hierarchies. We only define the levels containing formulae that begin with an existential quantifier, as this suffices for the purposes of our discourse.

Let S be a nonempty vocabulary not containing function symbols. Let $L_{FO}(S \cup \mathrm{VAR}_{SO})$ denote the first-order language associated with the set $S \cup \mathrm{VAR}_{SO}$. We define

$$\Sigma_0(S) = L_{FO}(S \cup \mathrm{VAR}_{SO})$$

and

$$\Sigma_{n+1}(S) = \{\exists X_1, ..., \exists X_k \neg\varphi \mid k \in \mathbb{N} \text{ and } \varphi \in \Sigma_n(S)\}.$$

The sets $\Sigma_n(S)$ are levels of the syntactic alternation hierarchy of MSO.

We write Σ_n instead of $\Sigma_n(S)$ when the vocabulary is clear from the context. With $[\Sigma_n]$ we refer to the equivalence closure of Σ_n. In other words, $[\Sigma_n]$ is the set of MSO formulae φ such that there exists some MSO formula $\varphi' \in \Sigma_n$ that is uniformly equivalent to φ.

Levels of the syntactic alternation hierarchy are associated with natural semantic counterparts. Let H be a subclass of the class of all S-structures. We define

$$\underline{\Sigma_n}(H) = \{C \in Pow(H) \mid C = \mathrm{MOD}_H(\varphi) \text{ for some sentence } \varphi \in \Sigma_n(S)\}.$$

Similarly, we let

$$\begin{aligned}&\underline{\Sigma_n}(H_p)\\ &= \{C \in Pow(H_p) \mid C = \mathrm{MOD}_{H_p}\big(\varphi(x)\big) \text{ for some formula } \varphi(x) \in \Sigma_n(S)\},\end{aligned}$$

where H_p is a class of pointed S-models.

We then deal with the quantifier alternation hierarchies of SOPML formulae of the vocabulary S. The zeroeth level of the syntactic hierarchy of SOPML contains all SOPML formulae free of propositional quantifiers, and any formula $\exists P_1 ... \exists P_k \neg\varphi$ belongs to the level $n+1$ iff φ belongs to the n-th level. We let $\Sigma_n^{ML}(S)$ denote the n-th level of this hierarchy. On the semantic side, we define

$$\underline{\Sigma_n^{ML}}(H) \\ = \{C \in Pow(H) \mid \mathrm{MOD}_H(\varphi) = C \text{ for some sentence } \varphi \in \Sigma_n^{ML}(S)\},$$

where H is a subclass of the class of S-models. Similarly, we define

$$\underline{\Sigma_n^{ML}}(H_p) \\ = \{C \in Pow(H_p) \mid \mathrm{MOD}_{H_p}(\varphi) = C \text{ for some sentence } \varphi \in \Sigma_n^{ML}(S)\},$$

where H_p is a class of pointed S-models.

If for all $n \in \mathbb{N}$ there exists a $k > n$ such that $\underline{\Sigma_n}(K) \neq \underline{\Sigma_k}(K)$, we say that *the alternation hierarchy of* MSO *is infinite over* K. Here K can be a class of models or a class of pointed models. We define infinity of SOPML alternation hierarchies analogously.

2.3 SOPMLE = MSO

In this section we show that second-order propositional modal logic with the global modality (SOPMLE) has the same expressive power as MSO. The result is closely related (for example) to the fact that the system $\mathcal{H}(\downarrow, E)$ of hybrid logic is expressively complete for first-order logic, see [3] and the references therein. In the light of the considerations in [1, 2, 11], the result is not that surprising.

In order to establish that SOPMLE is expressively complete for MSO, we define a simple translation from the set of MSO formulae into the set of SOPMLE formulae. The translation was inspired by a very similar translation defined in [11].

Let M be a model and

$$f : \mathrm{VAR} \longrightarrow Dom(M) \cup Pow\big(Dom(M)\big)$$

a related assignment. We let V_f denote the valuation mapping from the set PROP to the set $Pow\big(Dom(M)\big)$ such that the following conditions are satisfied.

1. $V_f(P_x) = \{f(x)\}$ for all $P_x \in \mathrm{PROP}$ such that $x \in \mathrm{VAR}_{FO}$.
2. $V_f(P_X) = f(X)$ for all $P_X \in \mathrm{PROP}$ such that $X \in \mathrm{VAR}_{SO}$.

Consider the formula

$$uniq(P_x) \;:=\; \langle E\rangle P_x \;\wedge\; \forall P_y\big(\langle E\rangle(P_y \wedge P_x) \to [E](P_x \to P_y)\big),$$

where $[E]$ stands for $\neg\langle E\rangle\neg$. The formula states that the proposition variable P_x is satisfied by exactly one element.

Let S be a vocabulary. Let $P \in S$ be a unary and $R \in S$ a binary relation symbol. Let $R' \in S$ be a k-ary relation symbol, where k is an integer greater or equal to three. Let $c \in S$ and $c' \in S$ be constant symbols. Let φ and ψ be MSO formulae of the vocabulary S. We define the following recursive translation Tr from the set of MSO formulae of the vocabulary S into the set of S-formulae of SOPMLE.

$$\begin{array}{lcl}
Tr(P(x)) & = & \langle E\rangle(P \wedge P_x)\\
Tr(X(y)) & = & \langle E\rangle(P_X \wedge P_y)\\
Tr(R(x,y)) & = & \langle E\rangle(P_x \wedge \langle R\rangle P_y)\\
Tr(R'(x_1,...,x_k)) & = & \langle E\rangle\big(P_{x_1} \wedge \langle R'\rangle(P_{x_2},...,P_{x_k})\big)\\
Tr(x=y) & = & \langle E\rangle(P_x \wedge P_y)\\
Tr(c=x) & = & \langle E\rangle(c \wedge P_x)\\
Tr(x=c) & = & \langle E\rangle(P_x \wedge c)\\
Tr(c=c') & = & \langle E\rangle(c \wedge c')\\
Tr(\neg\varphi) & = & \neg\, Tr(\varphi)\\
Tr(\,(\varphi \wedge \psi)\,) & = & \big(Tr(\varphi) \wedge Tr(\psi)\big)\\
Tr(\exists x\, \varphi) & = & \exists P_x\big(uniq(P_x) \wedge Tr(\varphi)\big)\\
Tr(\exists X\, \varphi) & = & \exists P_X\, Tr(\varphi)
\end{array}$$

Lemma 2.2. *Let S be a vocabulary and let (M,w) be a pointed S-model with the domain W. We have*

$$M, f \models \varphi \quad\Leftrightarrow\quad (M,w), V_f \Vdash Tr(\varphi)$$

for all MSO *formulae φ of the vocabulary S and all assignment functions $f : \mathrm{VAR} \longrightarrow W \cup Pow(W)$.*

Proof. We prove the claim by induction on the structure of S-formulae φ of MSO. The basis of the induction is established by a straightforward argument. The case where $\varphi = \neg\psi$ for some formula ψ is trivial, as is the case where φ has a conjunction as its main connective. Therefore we may proceed directly to the case where $\varphi = \exists x\, \psi$ for some formula ψ.

Assume first that $M, f \models \exists x\, \psi$. Therefore we have $M, f\frac{u}{x} \models \psi$ for some $u \in W$. Hence

$$(M,w), V_f\, \frac{\{u\}}{P_x} \;\Vdash\; Tr(\psi)$$

by the induction hypothesis. Thus

$$(M,w), V_f \;\Vdash\; \exists P_x\big(uniq(P_x) \wedge Tr(\psi)\big),$$

as required.

For the converse, assume that

$$(M, w), V_f \Vdash \exists P_x\big(uniq(P_x) \wedge Tr(\psi)\big).$$

Therefore

$$(M, w), V_f \frac{U}{P_x} \Vdash uniq(P_x) \wedge Tr(\psi)$$

for some $U \subseteq W$. As

$$(M, w), V_f \frac{U}{P_x} \Vdash uniq(P_x),$$

we have $U = \{u\}$ for some $u \in W$. Therefore

$$(M, w), V_f \frac{\{u\}}{P_x} \Vdash Tr(\psi),$$

and thus $M, f\frac{u}{x} \models \psi$ by the induction hypothesis. Therefore $M, f \models \exists x\, \psi$, as required.

Finally, the argument for the case where the formula φ is of the type $\exists X\, \psi$, is straightforward. □

We are now ready for the main results of the current section.

Theorem 2.3. *Let S be a vocabulary. A subclass K of a class C of pointed S-models is definable w.r.t. C by an* MSO *formula if and only if K is definable w.r.t. C by an* SOPMLE *sentence.*

Proof. Let φ be an arbitrary S-formula of MSO with exactly one free variable, the first-order variable x. Let (M, w) be a pointed S-model with the domain W, and let

$$f : \mathrm{VAR} \longrightarrow W \cup Pow(W)$$

be an arbitrary assignment. The following equivalence holds by Lemma 2.2.

$$M, f\tfrac{w}{x} \models \varphi \quad \Leftrightarrow \quad (M, w), V_f \tfrac{\{w\}}{P_x} \Vdash Tr(\varphi)$$

We observe that the formula $Tr(\varphi)$ has exactly one free proposition variable, P_x. We have the following equivalence.

$$\begin{aligned} & (M, w), V_f \tfrac{\{w\}}{P_x} \Vdash Tr(\varphi) \\ \Leftrightarrow \quad & \\ & (M, w) \Vdash \exists P_x\big(P_x \wedge uniq(P_x) \wedge Tr(\varphi)\big) \end{aligned}$$

By the two equivalences, it is clear that the sentence

$$\exists P_x\big(P_x \wedge uniq(P_x) \wedge Tr(\varphi)\big)$$

is an SOPMLE sentence uniformly equivalent to φ.

For the converse, we define a trivial generalization of the *standard translation* (see [7]). Let s be an injection from PROP to VAR_{SO}. If $P_\# \in$ PROP, let $X_\#$ denote the variable $s(P_\#)$. The translation operator St takes as an input a formula of SOPMLE and a first-order variable. We define the operator St recursively by the following clauses.

1. $St_x(c) \ := \ x = c$
2. $St_x(P) \ := \ P(x)$
3. $St_x(P_\#) \ := X_\#(x)$
4. $St_x(\neg\varphi) \ := \ \neg St_x(\varphi)$
5. $St_x\big((\varphi \wedge \psi)\big) \ := \ \big(St_x(\varphi) \wedge St_x(\psi)\big)$
6. $St_x(\langle R\rangle\varphi) \ := \ \exists y\big(xRy \wedge St_y(\varphi)\big)$
7. $St_x\big(\langle R'\rangle(\varphi_1, ..., \varphi_k)\big)$ $:= \ \exists y_1 ... \exists y_k\big(R'(x, y_1, ..., y_k) \wedge St_{y_1}(\varphi_1) \wedge ... \wedge St_{y_k}(\varphi_k)\big)$
8. $St_x(\langle E\rangle\varphi) \ := \ x = x \wedge \exists y\, St_y(\varphi)$
9. $St_x(\exists P_\#\, \varphi) \ := \ \exists X_\#\, St_x(\varphi)$

Here c is a constant symbol, P a unary relation symbol, $P_\#$ a relation variable in PROP, R a binary relation symbol and R' a $(k+1)$-ary relation symbol. It is easy to see that if φ is a sentence of SOPMLE, then $St_x(\varphi)$ is an MSO formula uniformly equivalent to φ. □

Theorem 2.4. *Let S be a vocabulary. A subclass K of a class C of S-models is definable w.r.t. C by an* MSO *sentence if and only if K is definable w.r.t. C by an* SOPMLE *sentence.*

Proof. Let φ be an arbitrary MSO sentence of the vocabulary S. Notice that $Tr(\varphi)$ does not contain any free proposition variables. Let $M \in K$ be a model and f a related assignment.

Assume that $M \models \varphi$. Pick an arbitrary $w \in W$. We have

$$\begin{aligned} M \models \varphi \quad &\Leftrightarrow \quad M, f \models \varphi \\ &\Leftrightarrow \quad (M, w), V_f \Vdash Tr(\varphi) \\ &\Leftrightarrow \quad (M, w) \Vdash Tr(\varphi), \end{aligned}$$

where the second equivalence follows by Lemma 2.2. Hence, as w was chosen arbitrarily, we conclude that $M \Vdash Tr(\varphi)$.

Assume then that $M \Vdash Tr(\varphi)$. Pick an arbitrary $u \in W$. We have $(M, u) \Vdash Tr(\varphi)$. Similarly to what we had above, we have

$$\begin{aligned} M \models \varphi &\Leftrightarrow M, f \models \varphi \\ &\Leftrightarrow (M, u), V_f \Vdash Tr(\varphi) \\ &\Leftrightarrow (M, u) \Vdash Tr(\varphi). \end{aligned}$$

Thus $M \models \varphi$. We conclude that

$$M \models \varphi \;\Leftrightarrow\; M \Vdash Tr(\varphi).$$

For the direction from SOPMLE to MSO, $\forall x St_x(\psi)$ is an MSO sentence globally uniformly equivalent to a sentence ψ of SOPMLE. □

It is now straightforward to observe that with regard to expressive power, SOPMLD = MSO, where SOPMLD denotes second-order propositional modal logic with the *difference modality*. The language of SOPMLD is obtained by extending the language of SOPML by a new unary operator $\langle D\rangle$—similarly to the way we obtained the language of SOPMLE. A pointed model (M, w) satisfies the formula $\langle D\rangle\varphi$ iff there exists a point $u \in Dom(M) \setminus \{w\}$ such that (M, u) satisfies φ. It is clear that SOPMLE formulae can be expressed in SOPMLD. Therefore MSO formulae can be expressed in SOPMLD. It is also clear that formulae of SOPMLD translate into MSO.

2.4 Simulating Globality

The local nature of SOPML (cf. Proposition 4 of [11]) limits its expressive power. In this section we define a class of structures over which this is not the case. The key point is to insist that each structure contains a point which connects to every point of the structure.

Definition 2.5. Let (W, R) be a structure with a binary relation R. Assume that there is a point $w \in W$ such that wRu for all $u \in W$. We call such a point w a *localizer*. Structures with a localizer are called *localized*. If $(M, w) = ((W, R), w)$ is a pointed model where w is a localizer, we say that (M, w) is *l-pointed*.

The notion of a localizer is similar to the notion of a *spypoint* applied in hybrid logic literature (see the articles [2, 8] for example).

We then prepare ourselves for the next result (Lemma 2.6) by defining local analogues of the formula $uniq(P_x)$ and the translation Tr defined in Section 2.3.

Let $uniq_R(P_x)$ be the formula

$$\langle R\rangle P_x \wedge \forall P_y\big(\langle R\rangle(P_y \wedge P_x) \rightarrow [R](P_x \rightarrow P_y)\big),$$

where $[R]$ stands for $\neg\langle R\rangle\neg$. It is easy to see that if (W,R) is a model with a localizer $w \in W$, then $\big((W,R),w\big), V\frac{U}{P_x} \Vdash uniq_R(P_x)$ if and only if $U = \{u\}$ for some $u \in W$.

We then modify the translation Tr defined in Section 2.3 to suit the context of localized $\{R\}$-models. Consider the clauses that define the translation Tr. Restrict attention to the parts that apply to $\{R\}$-models. Replace the occurrences of the global diamond $\langle E\rangle$ by the diamond $\langle R\rangle$, and also replace $uniq(P_x)$ by $uniq_R(P_x)$. We denote the obtained translation by Tr_R. In other words, the translation Tr_R is the translation defined by the following clauses.

$$
\begin{array}{lcl}
Tr_R(X(y)) & = & \langle R\rangle(P_X \wedge P_y)\\
Tr_R(R(x,y)) & = & \langle R\rangle(P_x \wedge \langle R\rangle P_y)\\
Tr_R(x=y) & = & \langle R\rangle(P_x \wedge P_y)\\
Tr_R(\neg\varphi) & = & \neg\, Tr_R(\varphi)\\
Tr_R(\,(\varphi \wedge \psi)\,) & = & \big(Tr_R(\varphi) \wedge Tr_R(\psi)\big)\\
Tr_R(\exists x\,\varphi) & = & \exists P_x\big(uniq_R(P_x) \wedge Tr_R(\varphi)\big)\\
Tr_R(\exists X\,\varphi) & = & \exists P_X\, Tr_R(\varphi)
\end{array}
$$

The following lemma is a local analogue of Lemma 2.2.

Lemma 2.6. *Let $M = (W,R)$ be a localized model. Let $w \in W$ be a localizer of M. We have*

$$M, f \models \varphi \quad \Leftrightarrow \quad (M,w), V_f \Vdash Tr_R(\varphi)$$

for all MSO *formulae φ of the vocabulary $\{R\}$ and all assignment functions $f : \mathrm{VAR} \longrightarrow W \cup Pow(W)$.*

Proof. The proof is essentially the same as that of Lemma 2.2. □

The following lemma is a local analogue of Theorem 2.3.

Lemma 2.7. *Let C be a class of l-pointed models. A class $K \subseteq C$ of l-pointed models is definable w.r.t. C by an* MSO *formula if and only if K is definable w.r.t. C by an* SOPML *sentence.*

Proof. Let the MSO formula $\varphi(x)$ define K w.r.t. C. The formula

$$\exists P_x\big(P_x \wedge uniq_R(P_x) \wedge Tr_R(\varphi)\big)$$

is an SOPML sentence corresponding to φ. The proof is essentially the same as that of Theorem 2.3. Instead of using Lemma 2.2, however, we apply the analogous lemma that applies in the context where we do not have the global modality at our disposal, i.e., Lemma 2.6. □

Let C be a class of localized $\{R\}$-models. Let φ be an $\{R\}$-sentence of SOPML such that for each model $M \in C$, there exists at least one point $w \in Dom(M)$ that satisfies φ, and furthermore, every point $u \in Dom(M)$ that satisfies φ, is a localizer. We say that φ *fixes localizers on* C.

The following lemma is a local analogue of Theorem 2.4.

Lemma 2.8. *Let C be a class of localized $\{R\}$-models and assume there exists an* SOPML *sentence φ that fixes localizers on C. A class $K \subseteq C$ of localized models is definable w.r.t. C by an* MSO *sentence if and only if K is definable w.r.t. C by an* SOPML *sentence.*

Proof. Let ψ be an MSO sentence that defines K w.r.t. C. Let $M \in C$ be a model and f a related assignment. Let $U \subseteq Dom(M)$ be the set of points $w \in Dom(M)$ such that $(M, w) \Vdash \varphi$. Notice that $Tr_R(\psi)$ does not contain free proposition variables. We have the equivalences

$$\begin{array}{rcl} M \models \psi & \Leftrightarrow & M, f \models \psi \\ & \Leftrightarrow & \forall w \in U\big(\ (M, w), V_f\ \Vdash Tr_R(\psi)\ \big) \\ & \Leftrightarrow & \forall w \in U\big(\ (M, w) \Vdash Tr_R(\psi)\ \big) \\ & \Leftrightarrow & \forall w \in U\big(\ (M, w) \Vdash \varphi \rightarrow Tr_R(\psi)\ \big) \\ & \Leftrightarrow & M \Vdash \varphi \rightarrow Tr_R(\psi), \end{array}$$

where the second equivalence follows by Lemma 2.6.

For the converse, $\forall x\, St_x(\chi)$ is an MSO sentence that corresponds to an SOPML sentence χ. □

2.5 The Alternation Hierarchy of SOPML is Infinite

In this section we prove that the alternation hierarchy of SOPML is infinite over the class of finite directed graphs. The following theorem of Schweikardt [57] is the starting point of our argument.

Theorem 2.9. *For all $n \in \mathbb{N}_{\geq 1}$, we have $\underline{\Sigma_n}(\text{GRID}) \neq \underline{\Sigma_{n+1}}(\text{GRID})$.*

While a similar result holds for directed graphs[2], on *words* and *labeled trees*, for example, the alternation hierarchy of MSO is known to collapse to the level Σ_1. See [50] for a recent survey of related results.

In Subsection 2.5.1 we show how to encode grids by *localized grid graphs*, a class of structures we shall define below (Definition 2.10). In Subsection 2.5.2 we then transfer the result of Theorem 2.9 to the context of localized

[2]See [51, 52]. In [51], the result for directed graphs is established via a reduction from the class of grids to a certain subclass of directed graphs. Let us call this class C. While we could prove Proposition 2.15 below via a reduction from the class C, we instead prove it via a direct reduction from the class of grids. The two alternative approaches are similar, but the approach via a direct reduction from the class of grids has presentational advantages.

grid graphs (Proposition 2.15) and l-pointed localized grid graphs (Proposition 2.16). The transferred results will be needed in Subsection 2.5.3, where we show that the alternation hierarchy of SOPML is infinite over pointed directed graphs (Theorem 2.17) and ordinary directed graphs (Theorem 2.18).

2.5.1 Encoding Grids by Localized Grid Graphs

In this subsection we define a map that sends each grid to a localized directed graph that encodes the structure of the grid.

Definition 2.10. Let $\alpha : \mathrm{GRID} \longrightarrow \mathrm{GRAPH}$ be a map that transforms a grid Gd to a directed graph $\alpha(Gd) = (W, R)$ such that

$$W = \big(Dom(Gd) \times \{0\}\big) \;\cup\; \big(Dom(Gd) \times \{1\}\big)$$

and

$$\begin{array}{rcl} R & = & \{\ \big((a,0),(a,0)\big) \mid a \in Dom(Gd)\ \} \\ & \cup & \{\ \big((a,0),(a,1)\big) \mid a \in Dom(Gd)\ \} \\ & \cup & \{\ \big((a,0),(b,0)\big) \mid (a,b) \in S_1^{Gd}\ \} \\ & \cup & \{\ \big((a,1),(b,0)\big) \mid (a,b) \in S_2^{Gd}\ \} \\ & \cup & \{\ \big((t,0),(a,i)\big) \mid a \in Dom(Gd),\ i \in \{0,1\}\ \} \\ & \cup & \{\ \big((t,1),(t,0)\big)\ \}, \end{array}$$

where $t = (1,1)$ is the top left element of the grid Gd. We call structures in the isomorphism closure of $\alpha(\mathrm{GRID})$ *localized grid graphs*. We let LGG denote this class of structures. We let $\mathrm{LGG_p}$ denote the corresponding class of l-pointed grid graphs. See Figure 1 for an example of a grid and the corresponding localized grid graph.

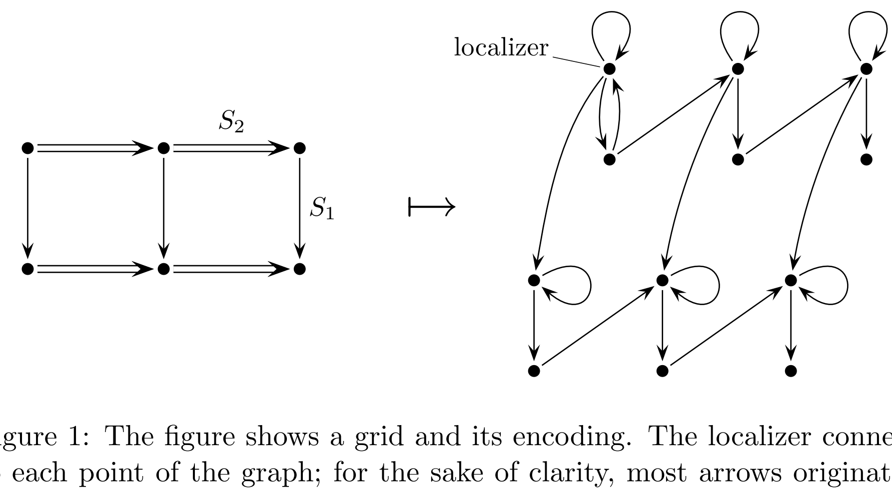

Figure 1: The figure shows a grid and its encoding. The localizer connects to each point of the graph; for the sake of clarity, most arrows originating from the localizer have not been drawn.

The point $(t,0)$ connects to every point in the graph $\alpha(Gd)$, i.e., it is a localizer. This property enables us to overcome difficulties resulting from the local nature of SOPML. We define the formula

$$\psi_{t_0}(x) \ := \ xRx \wedge \exists y(xRy \wedge yRx \wedge x \neq y).$$

The formula asserts that $x = (t, 0)$. Insisting that $(t, 1)R(t, 0)$ will help us with a number of technical issues, such as defining the formula

$$\psi_{t_1}(x) \ := \ \neg xRx \wedge \exists y(xRy \wedge yRx),$$

which asserts that $x = (t, 1)$.

We then show that the encoding $\alpha : \mathrm{GRID} \longrightarrow \mathrm{GRAPH}$ is injective in the sense that if $\alpha(Gd)$ and $\alpha(Gd')$ are isomorphic, then $Gd = Gd'$. Note that the arrows originating from the localizer of a localized grid graph make the graph in some sense irregular in comparison with the grid it corresponds to, and therefore injectivity of the encoding α is not an entirely trivial matter.

Lemma 2.11. *The encoding* $\alpha : \mathrm{GRID} \longrightarrow \mathrm{GRAPH}$ *is injective in the sense that if* $\alpha(Gd)$ *and* $\alpha(Gd')$ *are isomorphic, then* $Gd = Gd'$*.*

Proof. Let $\alpha(Gd) = (W, R) = G$ and $\alpha(Gd') = (W', R') = G'$ for some grids Gd and Gd'. Assume that $f : W \longrightarrow W'$ is an isomorphism between the graphs. Let k be the number of elements $w \in W$ with a reflexive loop. It is clear that Gd corresponds to an $m \times n$ matrix such that $m \cdot n = k$ (cf. Definition 2.1). The number of points $w' \in W'$ with a reflexive loop must also be k, as the two graphs are isomorphic. Thus the grid Gd' corresponds to an $m' \times n'$ matrix such that $m' \cdot n' = k$. To conclude the proof it suffices to show that $n = n'$.

We shall show that for each $i \in \mathbb{N}_{\geq 1}$, there is a first-order formula φ_i such that for all $M \in \mathrm{GRID}$, we have $\alpha(M) \models \varphi_i$ iff M corresponds to a $j \times i$ matrix for some j. The claim of the lemma then follows: as $G \cong G'$, they satisfy the same first-order sentences, and thus there is some i such that both G and G' satisfy the sentence φ_i, whence $n = i = n'$.

We then show how to define the formulae φ_i. We deal with the case where $i = 1$ first. We let

$$\varphi_1 \ := \ \exists x\big(\psi_{t_1}(x) \wedge \exists^{=1} y(xRy)\big),$$

where $\exists^{=1} y$ stands for "there exists exactly one y". We then consider the cases where $i \geq 2$. We first define the formulae

$$\begin{array}{lll} \pi_2(x) & := & \exists y \exists z\big(\ \psi_{t_1}(y) \wedge yRz \wedge \neg zRy \wedge zRx \wedge \neg xRx\ \big), \\ succ(x, y) & := & \exists z(xRz \wedge zRy \wedge \neg yRy). \end{array}$$

We then define φ_i (where $i \geq 2$) in the following way.

$$\varphi_i \ := \ \exists x_2 ... x_i\Big(\pi_2(x_2) \wedge \big(\bigwedge_{2 \leq r < i} succ(x_r, x_{r+1})\big) \wedge \neg \exists y(x_i R y)\Big)$$

It is relatively easy to see that formulae φ_i have the desired meaning. $\square$

2.5.2 The Alternation Hierarchy of MSO over Localized Grid Graphs

In this subsection we show that results analogous to Theorem 2.9 hold for localized grid graphs (Proposition 2.15) and l-pointed grid graphs (Proposition 2.16). We begin by showing how to transform any grid-formula $\varphi_1 \in \Sigma_n$ into a graph-formula $\varphi_2 \in \Sigma_n$ that in a sense says the same about localized grid graphs as φ_1 says about grids. In this subsection we work exclusively on formulae of MSO.

Lemma 2.12. *For every grid-formula φ_1, there exists a graph-formula φ_2 such that for all grids Gd and all assignments*

$$f : \mathrm{VAR} \to Dom(Gd) \cup Pow\big(Dom(Gd)\big),$$

we have

$$Gd, f \models \varphi_1 \quad \Leftrightarrow \quad \alpha(Gd), f' \models \varphi_2,$$

where the assignment function f' is defined such that $f'(x) = (f(x), 0)$ and $f'(X) = f(X) \times \{0\}$ for all $x, X \in \mathrm{VAR}$. Furthermore, for all $n \in \mathbb{N}$, if $\varphi_1 \in [\Sigma_n]$, then $\varphi_2 \in [\Sigma_n]$. If φ_1 is a sentence, then so is φ_2.

Proof. Consider an MSO formula χ. Assume that χ is of the form $\overline{Q}\,\chi'$, where $\overline{Q}$ is a (possibly empty) string of existential and universal monadic second-order quantifiers, and χ' is first-order. That is, χ is in monadic prenex normal form. Furthermore, assume that no second-order variable symbol occurs twice in $\overline{Q}$. Let us call such formulae *clean*. We will prove that for every clean grid-formula φ_1 there exists a clean graph-formula φ_2 with exactly the same second-order quantifier prefix as that of φ_1 such that for all grids Gd and all assignments

$$f : \mathrm{VAR} \to Dom(Gd) \cup Pow\big(Dom(Gd)\big),$$

we have

$$Gd, f \models \varphi_1 \quad \Leftrightarrow \quad \alpha(Gd), f' \models \varphi_2,$$

where f' is exactly as in the statement of the lemma.

We prove the claim by induction on the structure of clean grid-formulae φ_1. In addition to the case for atomic formulae, we will discuss the cases where the grid-formula φ_1 is of the type $\neg\pi_1,\ \pi_1 \wedge \pi_1',\ \exists x\,\pi_1,\ \exists X\,\pi_1$ and $\forall X\,\pi_1$. In the cases where φ_1 is of the type $\neg\,\pi_1,\ \pi_1 \wedge \pi_1',\ \exists x\,\pi_1$, the formulae π_1 and π_1' are first-order.

Let us then show how to define φ_2 in the case where φ_1 is atomic. If φ_1 is of the type $x = y$ or type $X(y)$, we let $\varphi_2 := \varphi_1$. If φ_1 is of the type xS_1y, we let φ_2 be the following formula.

$$\begin{array}{ll}
 & \big(\ \ \psi_{t_0}(x) \wedge \ \ \psi_{t_0}(y) \ \to\ \ \bot\,\big) \\
\wedge & \big(\ \ \psi_{t_0}(x) \wedge \neg\psi_{t_0}(y) \to\ \ \forall z\big(zRy \to (\psi_{t_0}(z) \vee z = y)\big)\,\big) \\
\wedge & \big(\neg\psi_{t_0}(x) \wedge \ \ \psi_{t_0}(y) \ \to\ \ \bot\big) \\
\wedge & \big(\neg\psi_{t_0}(x) \wedge \neg\psi_{t_0}(y) \to\ \ xRy \wedge x \neq y\,\big)
\end{array}$$

If φ_1 is of the type xS_2y, we define φ_2 to be the following formula.

$$\begin{array}{rl} & (\ \ \psi_{t_0}(x) \wedge \ \psi_{t_0}(y) \ \rightarrow \ \ \bot) \\ \wedge & (\ \ \psi_{t_0}(x) \wedge \neg\psi_{t_0}(y) \rightarrow \ \exists z\big(\psi_{t_1}(z) \wedge zRy\big)\,) \\ \wedge & \big(\neg\psi_{t_0}(x) \wedge \ \psi_{t_0}(y) \ \rightarrow \ \ \bot\big) \\ \wedge & \big(\neg\psi_{t_0}(x) \wedge \neg\psi_{t_0}(y) \rightarrow \ \exists z\big(xRz \wedge \neg zRz \wedge zRy\big)\,\big) \end{array}$$

Assume then, for the sake of induction, that $\varphi_1 = \neg\pi_1$. The formula π_1 is first-order, and by the induction hypothesis, there exists a first-order graph-formula π_2 such that

$$Gd, f \models \pi_1 \ \Leftrightarrow \ \alpha(Gd), f' \models \pi_2$$

for all grids Gd and related assignments f. Let $\varphi_2 := \neg\pi_2$. Similarly, in the case where $\varphi_1 = \pi_1 \wedge \pi_1'$, we let $\varphi_2 := \pi_2 \wedge \pi_2'$, where the graph-formulae π_2, π_2' are again chosen by the induction hypothesis. In the case where $\varphi_1 = \exists x\, \pi_1$, we let $\varphi_2 := \exists x(xRx \wedge \pi_2)$.

We then consider the case where $\varphi_1 = \exists X\, \pi_1$. Let $\pi_1 = \overline{Q}\,\chi_1$, where $\overline{Q}$ is the string of monadic second-order quantifiers in π_1. Let π_2 be the formula corresponding to π_1 obtained by the induction hypothesis. Let $\pi_2 = \overline{Q}\,\chi_2$. Define

$$\varphi_2' \ := \ \exists X\big(\forall x(X(x) \rightarrow xRx) \wedge \pi_2\big).$$

It is easy to see that we have

$$Gd, f \models \varphi_1 \ \Leftrightarrow \ \alpha(Gd), f' \models \varphi_2'$$

for all grids Gd and related assignments f, but we still need to modify the quantifier structure of φ_2'. We let φ_2 be the formula

$$\exists X\overline{Q}\big(\,\forall x(X(x) \rightarrow xRx) \wedge \chi_2\,\big),$$

which we observe to be uniformly equivalent to φ_2' and of the desired form. None of the quantifiers in $\overline{Q}$ bind the variable X, since $\varphi_1 = \exists X\,\overline{Q}\,\chi_1$ is a clean formula.

Finally, let $\varphi_1 = \forall X\, \pi_1 = \forall X\overline{Q}\,\chi_1$, where χ_1 is the first-order part of φ_1. This case is similar to the previous one. We obtain the formula $\overline{Q}\,\chi_2$ corresponding to $\overline{Q}\,\chi_1$ by the induction hypothesis, and let

$$\varphi_2 \ := \ \forall X\overline{Q}\big(\,\forall x(X(x) \rightarrow xRx) \rightarrow \chi_2\,\big).$$

The formula φ_2 has the required properties. □

Our next aim is to show that for each graph-sentence $\varphi_2 \in \Sigma_n$, there exists a grid-sentence $\varphi_1 \in \Sigma_n$ that says the same about grids as φ_2 says about localized grid graphs. In order to establish this, we first need to address a number of technical issues.

We first fix $\mathrm{VAR}_{FO} = \{x_1, x_2, x_3, ...\}$. In spite of this, we still continue using meta-variables x, y, z occasionally, for the sake of readability. We then define a new set of symbols

$$\begin{array}{rcl} \mathrm{VAR}' & = & \mathrm{VAR}_{FO} \\ & \cup & (\mathrm{VAR}_{SO} \times \{0\}) \\ & \cup & (\mathrm{VAR}_{SO} \times \{1\}) \\ & \cup & (\mathrm{VAR}_{SO} \times \{t_0\}) \\ & \cup & (\mathrm{VAR}_{SO} \times \{t_1\}). \end{array}$$

We denote the new second-order variables of the type $(X, 0)$, $(X, 1)$, (X, t_0) and (X, t_1) by X^0, X^1, X^{t_0} and X^{t_1}, respectively.

Let Gd be a grid. We partition the domain of the grid graph $\alpha(Gd)$ into the following four sets.

$$\begin{array}{rcl} V_{t_0} & = & \{ \big((1,1),0\big) \} \\ V_{t_1} & = & \{ \big((1,1),1\big) \} \\ V_0 & = & \{ \big((x,y),0\big) \in Dom(\alpha(Gd)) \mid (x,y) \neq (1,1) \} \\ V_1 & = & \{ \big((x,y),1\big) \in Dom(\alpha(Gd)) \mid (x,y) \neq (1,1) \} \end{array}$$

Let $\kappa : \mathbb{N}_{\geq 1} \longrightarrow \{ 0, 1, t_0, t_1 \}$ be a function. We say that an assignment

$$f : \mathrm{VAR} \longrightarrow Dom(\alpha(Gd)) \cup Pow\big(Dom(\alpha(Gd))\big)$$

is *of the type* κ if $f(x_i) \in V_{\kappa(i)}$ for all $i \in \mathbb{N}_{\geq 1}$. We call the function κ an *assignment type.*

Each assignment

$$f : \mathrm{VAR} \longrightarrow Dom(\alpha(Gd)) \cup Pow\big(Dom(\alpha(Gd))\big)$$

is associated with a related assignment

$$f_{Gd} : \mathrm{VAR}' \longrightarrow Dom(Gd) \cup Pow(Dom(Gd))$$

defined in a way we specify next. For first-order variables $x \in \mathrm{VAR}'$, we require that the condition

$$\forall a \in Dom(Gd)\Big(f_{Gd}(x) = a \;\Leftrightarrow\; \big(f(x) = (a,0) \text{ or } f(x) = (a,1)\big)\Big)$$

is satisfied. For second-order variables $X^0, X^1 \in \mathrm{VAR}'$ we let

$$\begin{array}{rcl} f_{Gd}(X^0) & = & \{ a \in Dom(Gd) \mid (a,0) \in f(X) \} \setminus \{ (1,1) \}, \\ f_{Gd}(X^1) & = & \{ a \in Dom(Gd) \mid (a,1) \in f(X) \} \setminus \{ (1,1) \}. \end{array}$$

Recall that $(1, 1)$ is the top left element of the grid Gd. For second-order variables X^{t_i}, where $i \in \{0, 1\}$, we let

$$f_{Gd}(X^{t_i}) = \begin{cases} \{ (1,1) \} & \text{if } \big((1,1),i\big) \in f(X), \\ \emptyset & \text{otherwise.} \end{cases}$$

We are now ready for the following lemma.

Lemma 2.13. *For every graph-formula φ_2 with its variables from* VAR *and every assignment type κ, there exists a grid-formula φ_1^κ with its variables from* VAR$'$ *such that for all grid graphs $\alpha(Gd)$ and all assignments*

$$f : \mathrm{VAR} \to \mathit{Dom}(\alpha(Gd)) \cup \mathit{Pow}\big(\mathit{Dom}(\alpha(Gd))\big)$$

of the type κ, we have

$$Gd, f_{Gd} \models \varphi_1^\kappa \;\Leftrightarrow\; \alpha(Gd), f \models \varphi_2.$$

Furthermore, for all $n \in \mathbb{N}$, if $\varphi_2 \in [\Sigma_n]$, then also $\varphi_1^\kappa \in [\Sigma_n]$. If φ_2 is a sentence, then so is φ_1^κ.

Proof. Recall the definition of clean formulae from the proof of Lemma 2.12. We will prove that for every clean graph-formula φ_2 with its variables from VAR and every assignment type κ, there exists a grid-formula φ_1^κ with its variables from VAR$'$ such that for all grid graphs $\alpha(Gd)$ and all assignments

$$f : \mathrm{VAR} \to \mathit{Dom}(\alpha(Gd)) \cup \mathit{Pow}\big(\mathit{Dom}(\alpha(Gd))\big)$$

of the type κ, we have

$$Gd, f_{Gd} \models \varphi_1^\kappa \;\Leftrightarrow\; \alpha(Gd), f \models \varphi_2,$$

and furthermore, the second-order quantifier prefix $\overline{Q}_1$ of φ_1^κ can be obtained from the second-order quantifier prefix $\overline{Q}_2$ of φ_2 by replacing each quantifier $\exists X$ in $\overline{Q}_2$ by the string $\exists X^0 \exists X^1 \exists X^{t_0} \exists X^{t_1}$, and each quantifier $\forall Y$ in $\overline{Q}_2$ by the string $\forall Y^0 \forall Y^1 \forall Y^{t_0} \forall Y^{t_1}$.

We prove the claim by induction on the structure of clean graph-formulae φ_2. In addition to the cases for atomic formulae, we will discuss the cases where the graph-formula φ_2 is of the type $\neg\pi_2$, $\pi_2 \wedge \chi_2$, $\exists x\, \pi_2$, $\exists X\, \pi_2$ and $\forall X\, \pi_2$. In the cases where φ_2 is of the type $\neg\, \pi_2$, $\pi_2 \wedge \chi_2$, $\exists x\, \pi_2$, the formulae π_2 and χ_2 are first-order.

Assume first that φ_2 is atomic. If φ_2 is $x_i = x_j$, then we let

$$\varphi_1^\kappa = \begin{cases} x_i = x_j & \text{when } \kappa(i) = \kappa(j), \\ \quad\bot & \text{when } \kappa(i) \neq \kappa(j). \end{cases}$$

Let *topleft*(z) denote the formula $\neg\exists x(xS_1z \vee xS_2z)$. If $\varphi_2 = x_iRx_j$, we define φ_1^κ according to the following table.

$(\kappa(i), \kappa(j))$	φ_1^κ
$(0,0)$	$x_i = x_j \vee x_i S_1 x_j$
$(0,1)$	$x_i = x_j$
$(1,0)$	$x_i S_2 x_j$
$(1,1)$	$\bot$
$(0,t_0)$	$\bot$
$(t_0,0)$	$\top$
$(0,t_1)$	$\bot$
$(t_1,0)$	$\exists z(\mathit{topleft}(z) \wedge z S_2 x_j)$
$(1,t_0)$	$\bot$
$(t_0,1)$	$\top$
$(1,t_1)$	$\bot$
$(t_1,1)$	$\bot$
(t_0,t_0)	$\top$
(t_0,t_1)	$\top$
(t_1,t_0)	$\top$
(t_1,t_1)	$\bot$

Finally, if $\varphi_2 = X(x_i)$, we let $\varphi_1^\kappa = X^{\kappa(i)}(x_i)$. We have now established a basis for an argument by induction.

If $\varphi_2 = \neg\pi_2$, we use π_2 and the induction hypothesis to find π_1^κ. We then let $\varphi_1^\kappa := \neg\pi_1^\kappa$. Similarly, if $\varphi_2 = \pi_2 \wedge \chi_2$, we use the induction hypothesis to find π_1^κ and χ_1^κ, and then let $\varphi_1^\kappa := \pi_1^\kappa \wedge \chi_1^\kappa$.

Let $\varphi_2 = \exists x\, \pi_2$ and let κ be an arbitrary assignment type. For the sake of readability, when $i \in \{0, 1, t_0, t_1\}$, we let $\kappa[x \mapsto i]$ denote the assignment type $\kappa\frac{i}{x}$. We apply the induction hypothesis to the formula π_2 in order to find formulae $\pi_1^{\kappa[x\mapsto i]}$, where $i \in \{0, 1, t_0, t_1\}$, such that

$$Gd, f_{Gd} \models \pi_1^{\kappa[x\mapsto i]} \;\Leftrightarrow\; \alpha(Gd), f \models \pi_2$$

holds for all grid graphs $\alpha(Gd)$ and valuations f of the type $\kappa[x \mapsto i]$. We then use these four formulae and define φ_1^κ to be the formula

$$\begin{array}{lrl}
\exists x & \Big(\mathit{topleft}(x) \;\wedge & \pi_1^{\kappa[x\mapsto t_0]} \\
\vee & \mathit{topleft}(x) \;\wedge & \pi_1^{\kappa[x\mapsto t_1]} \\
\vee & \neg\mathit{topleft}(x) \;\wedge & \pi_1^{\kappa[x\mapsto 0]} \\
\vee & \neg\mathit{topleft}(x) \;\wedge & \pi_1^{\kappa[x\mapsto 1]}\Big).
\end{array}$$

We then consider the case where $\varphi_2 = \exists X\, \pi_2$. Let κ be an assignment type and let $\pi_2 = \overline{Q}_2\, \chi_2$, where $\overline{Q}_2$ is the second-order quantifier prefix of π_2. We find a grid formula $\pi_1^\kappa = \overline{Q}_1\, \chi_1$ corresponding to the formula π_2 and the assignment type κ by the induction hypothesis. Here $\overline{Q}_1$ is the second-order quantifier prefix of π_1^κ. Consider the formula

$$\alpha_1^\kappa \;:=\; \exists X^0 \exists X^1 \exists X^{t_0} \exists X^{t_1} (\beta \wedge \pi_1^\kappa),$$

where β is the formula

$$\begin{array}{ll} & \forall x\big(X^0(x) \vee X^1(x) \rightarrow \neg \mathit{topleft}(x)\big) \\ \wedge & \\ & \forall x\big(X^{t_0}(x) \vee X^{t_1}(x) \rightarrow \mathit{topleft}(x)\big). \end{array}$$

The formula α_1^κ is almost what we need, as we have

$$Gd, f_{Gd} \models \alpha_1^\kappa \;\Leftrightarrow\; \alpha(Gd), f \models \varphi_2$$

for all grid graphs $\alpha(Gd)$ and related assignments f of the type κ. However, we still need to modify the second-order quantifier structure of α_1^κ. We define

$$\varphi_1^\kappa \;:=\; \exists X^0 \exists X^1 \exists X^{t_0} \exists X^{t_1} \overline{Q}_1 (\beta \wedge \chi_1).$$

We observe that φ_1^κ is uniformly equivalent to α_1^κ. Since the formula φ_2 is clean, none of the quantifiers in $\overline{Q}_1$ binds any of the variables X^0, X^1, X^{t_0}, X^{t_1}.

The case where $\varphi_2 = \forall X\, \pi_2$ is similar to the previous case. We find the formula $\pi_1^\kappa = \overline{Q}_1\, \chi_1$ corresponding to the formula π_2 and an assignment type κ. We define

$$\varphi_1^\kappa \;:=\; \forall X^0 \forall X^1 \forall X^{t_0} \forall X^{t_1} \overline{Q}_1 (\, \beta \rightarrow \chi_1\,),$$

where β is the same formula as in the previous case. The formula φ_1^κ has the required properties. □

Corollary 2.14. *For every monadic second-order graph-sentence φ_2 there exists a monadic second-order grid-sentence φ_1 such that for all grid graphs $\alpha(Gd)$,*

$$Gd \models \varphi_1 \;\Leftrightarrow\; \alpha(Gd) \models \varphi_2.$$

The sentence φ_1 can be chosen such that it is on the same level of the monadic second-order quantifier alternation hierarchy as φ_2.

Proof. Choose an arbitrary κ and apply Lemma 2.13. □

The next two propositions will be needed later on, but they are also interesting in their own right as they characterize the MSO alternation hierarchy with respect to *localized* graphs.

Proposition 2.15. *We have $\underline{\Sigma_n}(\mathrm{LGG}) \neq \underline{\Sigma_{n+1}}(\mathrm{LGG})$ for all $n \in \mathbb{N}_{\geq 1}$.*

Proof. Fix an arbitrary positive integer n. By Theorem 2.9 there is a class of grids

$$C \in \underline{\Sigma_{n+1}}(\mathrm{GRID}) \setminus \underline{\Sigma_n}(\mathrm{GRID}).$$

Let $\varphi_1 \in \Sigma_{n+1}$ define C w.r.t. the class GRID. We apply Lemma 2.12 to find a graph-sentence $\varphi_2 \in \Sigma_{n+1}$ such that

$$Gd \models \varphi_1 \Leftrightarrow \alpha(Gd) \models \varphi_2$$

for all grids Gd. It is clear that φ_2 defines, with respect to the class of localized grid graphs, the isomorphism closure of the class $\alpha(C)$.

We then show that there exists no graph-sentence $\psi_2 \in \Sigma_n$ that defines the isomorphism closure of the class $\alpha(C)$ w.r.t. the class LGG. Assume ad absurdum that such a ψ_2 exists. Use Corollary 2.14 to choose a related grid-sentence ψ_1. Now, since α is injective, the grid-sentence $\psi_1 \in \Sigma_n$ defines the class C w.r.t. the class of grids. This is a contradiction. □

Proposition 2.16. *We have* $\underline{\Sigma_n}(\mathrm{LGG_p}) \neq \underline{\Sigma_{n+1}}(\mathrm{LGG_p})$ *for all* $n \in \mathbb{N}_{\geq 1}$.

Proof. Fix an arbitrary $n \in \mathbb{N}_{\geq 1}$. By Proposition 2.15 there exists some sentence $\pi \in \Sigma_{n+1}$ that defines some class

$$C \in \underline{\Sigma_{n+1}}(\mathrm{LGG}) \setminus \underline{\Sigma_n}(\mathrm{LGG})$$

with respect to the class LGG. Thus the l-pointed version

$$C_p \;=\; \{\ (M, w) \mid M \in C,\ M, f\frac{w}{x} \models \psi_{t_0}(x)\ \}$$

of the class C is definable w.r.t. $\mathrm{LGG_p}$ by the formula $x = x \wedge \pi$, which is in $[\Sigma_{n+1}]$.

Assume that C_p is definable w.r.t. $\mathrm{LGG_p}$ by some formula $\varphi(x) \in \Sigma_n$. Let $\varphi(x) = \overline{Q}\,\psi(x)$, where $\psi(x)$ is the first-order matrix of $\varphi(x)$. The sentence

$$\overline{Q}\,\exists x\big(\psi_{t_0}(x) \wedge \psi(x)\big) \in \Sigma_n$$

defines the class C w.r.t. LGG. This is a contradiction. □

2.5.3 The Alternation Hierarchy of SOPML over Directed Graphs

We now prove that the alternation hierarchy of SOPML is infinite. We first show this for pointed graphs and then for graphs.

Theorem 2.17. *The alternation hierarchy of* SOPML *over the class of pointed directed graphs is infinite.*

Proof. Fix an arbitrary $n \in \mathbb{N}_{\geq 1}$. Then apply Proposition 2.16 in order to find some class

$$H_p \in \underline{\Sigma_{n+1}}(\mathrm{LGG_p}) \setminus \underline{\Sigma_n}(\mathrm{LGG_p})$$

of l-pointed grid graphs. By Lemma 2.7 there exists an SOPML sentence that defines the class H_p w.r.t. the class $\mathrm{LGG_p}$.

Now, the class H_p cannot be definable w.r.t. the class $\mathrm{LGG_p}$ by any SOPML sentence on the n-th level of the alternation hierarchy of SOPML. Assume ad absurdum that $\varphi \in \Sigma_n^{ML}$ defines H_p w.r.t. $\mathrm{LGG_p}$. Now $St_x(\varphi)$ is an MSO formula in Σ_n that defines H_p w.r.t. $\mathrm{LGG_p}$. □

Theorem 2.18. *The alternation hierarchy of* SOPML *over directed graphs is infinite.*

Proof. Fix an arbitrary $n \in \mathbb{N}$. By Proposition 2.15 there exists a class

$$H \in \underline{\Sigma_{n+3}}(\mathrm{LGG}) \setminus \underline{\Sigma_{n+2}}(\mathrm{LGG})$$

of localized grid graphs. We shall first establish that the class H is definable in SOPML w.r.t. LGG.

Consider the following SOPML sentence.

$$\psi \;:=\; \forall P_x\big(P_x \to \langle R\rangle P_x\big) \wedge \forall P_x\Big(P_x \to \exists P_y\big(\neg P_y \wedge \langle R\rangle(P_y \wedge \langle R\rangle P_x)\big)\Big)$$

In order to see that ψ fixes localizers on LGG, notice that the localizer is the only point u of a localized grid graph that satisfies the conditions

1. uRu,
2. $\exists v(v \neq u \wedge uRv \wedge vRu)$.

As the sentence ψ fixes localizers on LGG, Lemma 2.8 implies that the class H is definable w.r.t. LGG by some SOPML sentence.

Assume then, for contradiction, that $H \in \underline{\Sigma_n^{ML}}(\mathrm{LGG})$. Thus there exists an SOPML sentence $\pi \in \Sigma_n^{ML}$ that defines the class H w.r.t. LGG. Therefore the MSO sentence $\varphi := \forall x\, St_x(\pi)$ defines H w.r.t. LGG. To conclude the proof, it now suffices to show that there is an MSO sentence in Σ_{n+2} that is uniformly equivalent to φ.

We have $\pi \in \Sigma_n^{ML}$. Let $\pi = \overline{Q}\,\pi'$, where π' is the part of π that is free of propositional quantifiers. Consider the sentence

$$\forall X\, \overline{Q}'\; \forall x\Big(X(x) \wedge \forall z\big(X(z) \to x = z\big) \to\; St_x(\pi')\Big),$$

where $\overline{Q}'$ is the vector of quantified unary second-order relation variables obtained from $\overline{Q}$ by replacing the quantified proposition variables in $\overline{Q}$ by the corresponding quantified relation variables given by the injection $s : \mathrm{PROP} \longrightarrow \mathrm{VAR}_{SO}$ associated with the translation St. We assume that the variable X does not occur in $\overline{Q}'$. It is easy to see that this sentence is uniformly equivalent to φ and in $[\Sigma_{n+2}]$. □

As the class of Kripke frames is a superclass of the class of finite directed graphs, we immediately obtain the following corollary.

Corollary 2.19. *The alternation hierarchy of* SOPML *over the class of Kripke frames is infinite.*

2.6 Chapter Conclusion

We have shown that the quantifier alternation hierarchy of SOPML induces an infinite corresponding semantic hierarchy over the class of finite directed graphs (Theorem 2.18). While establishing the result, we have defined the notion of a localized structure and characterized the MSO alternation hierarchy over localized (finite directed) graphs. Theorem 2.18 answers a longstanding open problem from [5] (also addressed in [11]). The result is also relatively interesting from the point of view of finite model theory, as SOPML is a semantically natural fragment of MSO (cf. Theorem 6 in [11]).

In addition to obtaining the results related to alternation hierarchies, we have observed that with regard to expressive power,

$$\text{MSO} \;=\; \text{SOPMLE} = \;\text{SOPMLD}.$$

Connections of this kind offer an interesting modal perspective on MSO. For example, they immediately suggest alternative approaches to MSO games (see [44] for the definition).

Finally, our techniques do not directly yield *strictness* of the alternation hierarchy of SOPML. The reason for this is that conceivably, an MSO formula $\varphi \in \Sigma_n$ cannot necessarily be translated to an SOPML formula in Σ_n^{ML}, as the first-order quantifiers of φ translate to second-order quantifiers. Therefore, it remains to be investigated whether the SOPML alternation hierarchy is strict over finite directed graphs.

CHAPTER 3

Monadic Σ^1_1 and Modal Logic with Quantified Accessibility Relations

In this chapter we investigate the expressive power of a range of modal logics extended with existential prenex quantification of accessibility relations and proposition symbols. Let *polyadic Boolean modal logic with identity* ($\mathrm{PBML}^=$) be the logic obtained by extending standard polyadic multimodal logic by built-in identity modalities (see Subsection 3.2.1) and by constructors that allow for the Boolean combination of accessibility relations. Let $\Sigma^1_1(\mathrm{PBML}^=)$ be the extension of $\mathrm{PBML}^=$ with existential prenex quantification of accessibility relations and proposition symbols. The principal result of the chapter is that $\Sigma^1_1(\mathrm{PBML}^=)$ translates into *monadic* Σ^1_1. As a corollary, we obtain a variety of decidability results in multimodal logic. The result can also be seen as a step towards establishing whether every property of finite directed graphs expressible in $\Sigma^1_1(\mathrm{FO}^2)$ is also expressible in monadic Σ^1_1. This question was left open in the article [23] of Grädel and Rosen. The system $\Sigma^1_1(\mathrm{FO}^2)$ is the logic obtained by extending the two-variable fragment of first-order logic (FO^2) by existential prenex quantification of relation symbols of any arity.

3.1 Modal Fragments of Σ^1_1 and Modal Correspondence Theory

The objective of *modal correspondence theory* is to classify formulae of modal logic according to whether they define elementary classes of Kripke frames.[3] On the level of frames, modal logic can be considered a fragment of monadic Π^1_1, also known as $\forall$MSO, and therefore correspondence theory studies a special fragment of $\forall$MSO. When inspecting a modal formula from the point of view of frames, one universally quantifies the *proposition symbols* occurring in the formula; it is rather natural to ask what happens if one also quantifies *binary relation symbols* occurring in (the standard translation of) a modal formula. This question is investigated in [42], where the focus is on the expressivity of multimodal logic with universal prenex quantification of

[3] It is well known that if a class of Kripke frames is definable by a modal formula, then the class is definable by a set of FO formulae iff it is definable by a single FO formula. See [21] for example. Therefore it makes no difference *here* whether the term "elementary" is taken to mean definability by a single first-order formula or definability by a set of first-order formulae.

(not necessarily all of the) binary and unary relation symbols occurring in a formula. One question that immediately suggests itself is whether there exists any class of multimodal frames definable in this system, let us call it $\Pi^1_1(\mathrm{ML})$, but not definable in monadic second-order logic. The question can be regarded as a question of modal correspondence theory. This time, however, the correspondence language is MSO rather than FO. In addition to [42], modal logic with quantification of binary relations is studied for example in [13, 43, 53].

In the current chapter we investigate two systems of multimodal logic with existential second-order prenex quantification of accessibility relations and proposition symbols, $\Sigma^1_1(\mathrm{PBML}^=)$ and $\Sigma^1_1(\mathrm{ML})$. The logic $\Sigma^1_1(\mathrm{ML})$ is the extension of ordinary multimodal logic with existential second-order prenex quantification of binary accessibility relations and proposition symbols. We warm up by showing that $\Sigma^1_1(\mathrm{ML})$ translates into *monadic* $\Sigma^1_1(\mathrm{MLE})$, which is the extension of multimodal logic with the global modality and existential second-order prenex quantification of *only* proposition symbols. The method of proof is based on the notion of a *largest filtration* (see [7] for the definition). We then push the method and show that $\Sigma^1_1(\mathrm{PBML}^=)$ translates into monadic Σ^1_1, also known as $\exists$MSO. Note that both of these results immediately imply that $\Pi^1_1(\mathrm{ML})$ translates into $\forall$MSO, and therefore show that MSO is a somewhat dull correspondence language for correspondence theory of $\Pi^1_1(\mathrm{ML})$.

It could be argued that $\{\neg, \cup, \cap, \circ, ^*, ^\smile, E, D\}$ is, more or less, the core collection of operations on binary relations used in extensions of modal logic defined for the purposes of applications. Here $\neg$, $\cup$, $\cap$, $\circ$, *, $^\smile$ denote the complement, union, intersection, composition, transitive reflexive closure and converse operations, respectively. The symbols E and D denote the constant operations outputting the global modality and difference modality. Logics using these core operations include for example PDL [17, 24], Boolean modal logic [19, 45], description logics [31, 49], modal logic with the global modality [22] and modal logic with the difference modality [55]. One of the motivations for our study is that $\mathrm{PBML}^=$ subsumes a large number of typical extensions of modal logic. As a corollary, the translation from $\Sigma^1_1(\mathrm{PBML}^=)$ into $\exists$MSO gives a range of decidability results for extensions of multimodal logic over various classes of Kripke frames with built-in relations; see Theorem 3.17 below.

We describe a possible application of Theorem 3.17. Let $\mathcal{D}$ be a class of Kripke frames (W, R_0) and consider the class

$$\mathcal{C} = \{\ (W, \{R_i\}_{i\in\mathbb{N}}) \mid R_i \subseteq W \times W,\ (W, R_0) \in \mathcal{D}\ \}$$

of multimodal Kripke frames. Assume the $\forall$MSO theory of $\mathcal{D}$ is decidable. That is, the $\forall$MSO theory of the class of $\{R_0\}$-reducts of structures in $\mathcal{C}$ is decidable. For example, $\mathcal{C}$ could be the class of countably infinite frames

$(W, \{R_i\}_{i \in \mathbb{N}})$, where R_0 is a dense linear ordering of W without endpoints; the MSO theory of $(\mathbb{Q}, <^{\mathbb{Q}})$ is known to be decidable [54]. Assume we would like to know whether the satisfiability problem of multimodal logic—perhaps extended with, say, the difference modality—with respect to $\mathcal{C}$ is decidable. By Theorem 3.17 we *directly* see that, indeed, it is. Theorem 3.17 implies a wide range of decidability results for multimodal logic. There exists a large body of knowledge concerning structures and classes of structures with a decidable MSO (and therefore $\forall$MSO) theory (see [62] for example).

Another motivation for the investigations in this chapter is related to descriptive complexity theory [32]. Grädel and Rosen ask in [23] the question whether there exists any class of finite directed graphs that is definable in $\Sigma^1_1(\mathrm{FO}^2)$ but not in $\exists$MSO. Let $\mathrm{BML}^=$ denote ordinary Boolean modal logic with a built-in identity relation. Lutz, Sattler and Wolter show in the article [46] that $\mathrm{BML}^=$ extended with the converse operator is expressively complete for FO^2 over directed graphs. Therefore, in order to prove that $\Sigma^1_1(\mathrm{FO}^2) \leq \exists\mathrm{MSO}$ over directed graphs, one would have to modify our translation from $\Sigma^1_1(\mathrm{BML}^=)$ into $\exists$MSO such that it takes into account the possibility of using the converse operation. We have succeeded neither in this nor in identifying a $\Sigma^1_1(\mathrm{FO}^2)$ definable class of directed graphs that is not definable in $\exists$MSO. However, we find modal logic a promising framework for working on the problem.

3.2 Preliminary Definitions

In this section we discuss technical notions that occupy a central role in the rest of the chapter.

3.2.1 Syntax and Semantics of $\Sigma^1_1(\mathrm{PBML}^=)$

The semantics of $\mathrm{PBML}^=$ is obtained by combining the semantics of Boolean modal logic with the standard generalization of Kripke semantics to polyadic modal contexts.

Let V be a vocabulary containing relation symbols only. Let V_1 denote the subset of V containing exactly all the unary relation symbols in V. Let V_h be the subset of V containing all the relation symbols in V of higher arities, i.e., arities greater or equal to two. We define the set $\mathrm{MP}(V)$ of *modal parameters* over V to be the smallest set S satisfying the following conditions.

1. For each $k \in \mathbb{N}_{\geq 2}$, let id_k be a constant symbol. We assume that none of the symbols id_k is in V. We have $id_k \in S$ for all $k \in \mathbb{N}_{\geq 2}$.
2. If $R \in V_h$, then $R \in S$.
3. If $\mathcal{M} \in S$, then $\neg\mathcal{M} \in S$.

4. If $\mathcal{M} \in S$ and $\mathcal{N} \in S$, then $(\mathcal{M} \cap \mathcal{N}) \in S$.

Each modal parameter $\mathcal{M}$ is associated with an arity $Ar(\mathcal{M})$ defined as follows.

1. If $\mathcal{M} = id_k$, then $Ar(\mathcal{M}) = k$.
2. If $\mathcal{M} = R \in V_h$, then the $Ar(\mathcal{M})$ is equal to the arity of R.
3. If $\mathcal{M} = \neg\mathcal{N}$, then $Ar(\mathcal{M}) = Ar(\mathcal{N})$.
4. If $\mathcal{M} = (\mathcal{N}_1 \cap \mathcal{N}_2)$ and $Ar(\mathcal{N}_1) = Ar(\mathcal{N}_2)$, then $Ar(\mathcal{M}) = Ar(\mathcal{N}_1)$. If $Ar(\mathcal{N}_1) \neq Ar(\mathcal{N}_2)$, then $Ar(\mathcal{M}) = 2$.

The set of formulae of $\mathrm{PBML}^{=}$ of the vocabulary V is defined to be the smallest set F satisfying the following conditions.

1. If $P \in V_1$, then $P \in F$.
2. If $\varphi \in F$, then $\neg\varphi \in F$.
3. If $\varphi_1, \varphi_2 \in F$, then $(\varphi_1 \wedge \varphi_2) \in F$.
4. If $\varphi_1, ..., \varphi_k \in F$ and if $\mathcal{M} \in \mathrm{MP}(V)$ is a $(k+1)$-ary modal parameter, then $\langle\mathcal{M}\rangle(\varphi_1, ..., \varphi_k) \in F$.

Operators $\langle\mathcal{M}\rangle$ are called *diamonds*. The *modal depth* $Md(\varphi)$ of a formula φ is the maximum nesting depth of diamonds in φ.

1. $Md(P) = 0$ for $P \in V_1$.
2. $Md(\neg\varphi) = Md(\varphi)$.
3. $Md\big((\varphi_1 \wedge \varphi_2)\big) \;=\; max(\{Md(\varphi_1), Md(\varphi_2)\})$.
4. $Md\big(\langle\mathcal{M}\rangle(\varphi_1, ..., \varphi_k)\big) \;=\; 1 \;+\; max(\{Md(\varphi_1), ..., Md(\varphi_k)\})$.

Let M be a model with the domain A. The *extension* $\mathcal{M}^M$ of a modal parameter $\mathcal{M}$ over M is a relation of the arity $Ar(\mathcal{M})$ over A. The extension of $R \in V_h$ over M is simply the interpretation R^M of the symbol R. For each $k \in \mathbb{N}_{\geq 2}$, the extension id_k^M of the symbol id_k is the set

$$\{\, (w_1, ..., w_k) \in A^k \mid w_i = w_j \text{ for all } i, j \in \{1, ..., k\} \,\}.$$

Other modal parameters are interpreted recursively such that the following conditions hold.

1. If $\mathcal{M} = \neg\mathcal{N}$, then $\mathcal{M}^M \;=\; A^{Ar(\mathcal{M})} \setminus \mathcal{N}^M$.
2. If $\mathcal{M} = (\mathcal{N}_1 \cap \mathcal{N}_2)$, then $\mathcal{M}^M \;=\; \mathcal{N}_1^M \cap \mathcal{N}_2^M$.

Note that if $Ar(\mathcal{N}_1) \neq Ar(\mathcal{N}_2)$, then $(\mathcal{N}_1 \cap \mathcal{N}_2)^M = \emptyset$.

The satisfaction relation $\Vdash$ for PBML$^=$ formulae of the vocabulary V is defined with respect to pointed V-models as follows.

1. If $P \in V_1$, then
$$(M, w) \Vdash P \;\Leftrightarrow\; w \in P^M.$$

2. For other formulae, the satisfaction relation is interpreted according to the following recursive clauses.

$$\begin{array}{lcl} (M, w) \Vdash \neg\varphi & \Leftrightarrow & (M, w) \nVdash \varphi. \\ (M, w), \Vdash (\varphi_1 \wedge \varphi_2) & \Leftrightarrow & (M, w) \Vdash \varphi_1 \text{ and } (M, w) \Vdash \varphi_2. \\ (M, w) \Vdash \langle \mathcal{M} \rangle (\varphi_1, ..., \varphi_k) & \Leftrightarrow & \text{there exist } u_1, ..., u_k \in Dom(M) \\ & & \text{such that } (w, u_1, ..., u_k) \in \mathcal{M}^M \text{ and} \\ & & (M, u_i) \Vdash \varphi_i \text{ for all } i \in \{1, ..., k\}. \end{array}$$

For each V-model M and each formula φ of the vocabulary V, we let $||\varphi||^M$ denote the set
$$\{\; w \in Dom(M) \mid (M, w) \Vdash \varphi \;\}.$$
The set $||\varphi||^M$ is called the *extension* of φ over M. When φ and ψ are formulae of the vocabulary V, we write $\varphi \Vdash \psi$ if

$$(M, w) \Vdash \varphi \;\Rightarrow\; (M, w) \Vdash \psi$$

for all pointed V-models (M, w).

Let V be a vocabulary containing relation symbols only; V may be empty, and V may contain relation symbols of any finite positive arity. A formula φ of $\Sigma^1_1(\text{PBML}^=)$ of the vocabulary V is a formula of the type

$$\exists S_1 ... \exists S_n\; \psi,$$

where the variables S_i are relation symbols (of any positive arity) and ψ is a PBML$^=$ formula of the vocabulary $V \cup \{S_1, ..., S_n\}$. The sets V and $\{S_1, ..., S_n\}$ are always assumed to be disjoint. Let (M, w) be a pointed V-model. We define $(M, w) \Vdash \varphi$ if there exists an expansion

$$M' = (M, S_1^{M'}, ..., S_n^{M'})$$

of the model M such that $(M', w) \Vdash \psi$. The set of non-logical symbols of a $\Sigma^1_1(\text{PBML}^=)$ formula χ of the vocabulary V is the set of relation symbols (of any arity) that belong to V and occur in χ.

Let BML$^=$ be the fragment of PBML$^=$ where each modal parameter occurring in a formula is required to be binary. The system ML is the fragment of BML$^=$ where the modal parameters are required to be atomic binary relation symbols that belong to the vocabulary considered. Note that the modal parameter id_2 is not considered to be part of the vocabulary. The system MLE is the extension of ML with the global diamond $\langle E \rangle$, i.e., the diamond $\langle \neg(id_2 \cap \neg id_2) \rangle$. Systems $\Sigma^1_1(\mathrm{ML})$ and $\Sigma^1_1(\mathrm{MLE})$ are the fragments of $\Sigma^1_1(\mathrm{PBML}^=)$ defined by extending ML and MLE with existential prenex quantification of binary and unary relation symbols. *Monadic* $\Sigma^1_1(\mathrm{MLE})$ is the fragment of $\Sigma^1_1(\mathrm{MLE})$ where we only allow second-order quantifiers quantifying *unary* relation symbols.

The systems $\Pi^1_1(\mathrm{PBML}^=)$, $\Pi^1_1(\mathrm{ML})$ and $\Pi^1_1(\mathrm{MLE})$ are the counterparts of the systems $\Sigma^1_1(\mathrm{PBML}^=)$, $\Sigma^1_1(\mathrm{ML})$ and $\Sigma^1_1(\mathrm{MLE})$, but with universal second-order quantifiers instead of existential ones.

Let φ be a formula of $\Sigma^1_1(\mathrm{PBML}^=)$ or $\Pi^1_1(\mathrm{PBML}^=)$ of the vocabulary V. Let $\psi(x)$ be a V-formula of predicate logic with exactly one free variable, the first-order variable x. The formulae φ and $\psi(x)$ of are called V*-equivalent* if for all pointed V-models (M, w), we have

$$(M, w) \Vdash \varphi \quad \Leftrightarrow \quad M, \frac{w}{x} \models \psi(x).$$

The formulae $\psi(x)$ and φ are *uniformly equivalent* if they have the same set U of non-logical symbols and if the formulae are U-equivalent. (Recall that neither the identity symbol of predicate logic nor any of the symbols id_k is considered to be a non-logical symbol. For example the formulae $x = x \wedge \exists y R(y, y)$ and $\exists S\, \exists P\, \langle S \rangle \langle id_2 \cap R \rangle P$ are uniformly equivalent. The set of non-logical symbols of both formulae is $\{R\}$.) Two $\Sigma^1_1(\mathrm{PBML}^=)$ formulae φ_1 and φ_2 of the vocabulary V are V-equivalent if they are satisfied by exactly the same pointed V-models. The formulae φ_1 and φ_2 are uniformly equivalent if they have exactly the same set U' of non-logical symbols and if the formulae are U'-equivalent. Two V-sentences of predicate logic are uniformly equivalent if they have exactly the same set T of non-logical symbols and if they are satisfied by exactly the same T-models.

The reason we have chosen to define $\mathrm{PBML}^=$ exactly the way defined above, is relatively simple. Firstly, $\mathrm{BML}^=$ extended by the possibility of using the converse modality, is expressively complete for FO^2. We do not know whether $\Sigma^1_1(\mathrm{FO}^2)$ is contained in $\exists\mathrm{MSO}$, but we will show below that $\Sigma^1_1(\mathrm{BML}^=) \leq \exists\mathrm{MSO}$ by establishing that even the extension $\Sigma^1_1(\mathrm{PBML}^=)$ of $\Sigma^1_1(\mathrm{BML}^=)$ with polyadic modalities is indeed contained in $\exists\mathrm{MSO}$. Finally, the reason we have included the modalities id_k for $k \geq 3$ in the language of $\mathrm{PBML}^=$ is mostly due to technical presentation related issues. The reader may, indeed, think that the modalities id_k for $k \geq 3$ are not very canonical. The modalities do, however, have some interesting features. Notice for example that we can easily eliminate the use of conjunction from $\mathrm{PBML}^=$. (We shall not make any use of this feature below, however.)

3.2.2 Types

In the current subsection we define the notion of a *type* for formulae of $\mathrm{PBML}^=$.

Let V be a finite vocabulary such that $V_1 \neq \emptyset$. Let m' be the maximum arity of the modal parameters in V_h. In the case $V_h = \emptyset$, let $m' = 0$. Let m be an integer such that $2 \leq m$ and $m' \leq m$. Define the set

$$S_V \;=\; V_h \;\cup\; \{\; \neg R \;\mid\; R \in V_h \;\} \;\cup\; \{\; id_k,\; \neg id_k \mid 2 \leq k \leq m \;\}$$

of at most m-ary atomic and negated atomic modal parameters over V. Let k be an integer such that $2 \leq k \leq m$. Let $S_V(k)$ be the set that contains as elements exactly the k-ary modal parameters in S_V. Notice that $S_V(k) \neq \emptyset$. Let $T_V(k)$ denote the set whose elements are exactly the subsets $T \subseteq S_V(k)$ such that the following conditions are satisfied.

1. Exactly one of the modal parameters id_k and $\neg id_k$ is in the set T.
2. If $R \in V_h$ is k-ary, then exactly one of the modal parameters R and $\neg R$ is in the set T.

Let f be a function with the domain $T_V(k)$ that maps each $T \in T_V(k)$ to an intersection $\mathcal{N} \in \mathrm{MP}_V$ of the elements of T. (There may be several ways to choose the order of the members of T and bracketing when writing the modal parameter $\mathcal{N}$. The order and bracketing that f chooses does not matter.) The set

$$\{\; f(T) \mid T \in T_V(k) \;\}$$

of modal parameters is the set of k-ary *access types* over V. We let $\mathrm{ATP}_V(k)$ denote the set of k-ary access types over V.

Let $\mathcal{M}$ be a k-ary access type over V, and let $R \in V_h \cup \{id_k\}$ be a k-ary atomic modal parameter. We write $R \in \mathcal{M}$ if $\neg R$ does *not* occur in $\mathcal{M}$. Let $U \subseteq V$ and let $\mathcal{N}$ be a k-ary access type over U. We say that $\mathcal{N}$ is *consistent* with $\mathcal{M}$ (or alternatively, $\mathcal{M}$ is consistent with $\mathcal{N}$), if for all k-ary symbols $R \in U_h \cup \{id_k\}$, we have $R \in \mathcal{M}$ iff $R \in \mathcal{N}$.

Let (M, w) be a pointed model of the vocabulary V. We define

$$\tau^0_{(M,w),m} := \bigwedge_{\substack{P \,\in\, V_1, \\ (M,w) \,\Vdash\, P}} P \;\;\wedge \bigwedge_{\substack{Q \,\in\, V_1, \\ (M,w) \,\nVdash\, Q}} \neg Q.$$

The formula $\tau^0_{(M,w),m}$ is the *type* of (M, w) of the modal depth 0 and up to the arity m. We choose the formulae $\tau^0_{(M,w),m}$ such that if for some pointed V-models (M, w) and (N, v) the types $\tau^0_{(M,w),m}$ and $\tau^0_{(N,v),m}$ are uniformly equivalent,[4] then actually $\tau^0_{(M,w),m} = \tau^0_{(N,v),m}$. This means that the exact syntactic form of the types of pointed V-models of the modal depth 0 and up

[4] Note that the types $\tau^0_{(M,w),m}$ and $\tau_{(N,v),m}$ must have the same set of non-logical symbols (the set V_1), as the the models (M, w) and (N, v) are both V-models. Recall that the set of non-logical symbols interpreted by a V-model is exactly the set V.

to the arity m is chosen such that if two such types are uniformly equivalent, then they are in fact the one and the same formula. We let $\mathrm{TP}^0_{V,m}$ denote the set containing exactly the formulae τ such that for some pointed model (M,w) of the vocabulary V, the formula τ is the type of (M,w) of the modal depth 0 and up to the arity m. Clearly the set $\mathrm{TP}^0_{V,m}$ is finite.

Let $n \in \mathbb{N}$ and assume we have defined formulae $\tau^n_{(M,w),m}$ for all pointed models (M,w), and assume also that $\mathrm{TP}^n_{V,m}$ is a finite set containing exactly all these formulae. We define

$$
\begin{aligned}
\tau^{n+1}_{(M,w),m} \;:=\; & \tau^n_{(M,w),m} \\
& \wedge \bigwedge \{\, \langle\mathcal{M}\rangle(\sigma_1,...,\sigma_k) \mid \begin{array}[t]{l} 1 \leq k \leq m-1, \\ \mathcal{M} \in \mathrm{ATP}_V(k+1), \\ \sigma_1,...,\sigma_k \in \mathrm{TP}^n_{V,m}\,, \\ (M,w) \Vdash \langle\mathcal{M}\rangle(\sigma_1,...,\sigma_k) \,\} \end{array} \\
& \wedge \bigwedge \{\, \neg\langle\mathcal{M}\rangle(\sigma_1,...,\sigma_k) \mid \begin{array}[t]{l} 1 \leq k \leq m-1, \\ \mathcal{M} \in \mathrm{ATP}_V(k+1), \\ \sigma_1,...,\sigma_k \in \mathrm{TP}^n_{V,m}\,, \\ (M,w) \nVdash \langle\mathcal{M}\rangle(\sigma_1,...,\sigma_k) \,\}. \end{array}
\end{aligned}
$$

The formula $\tau^{n+1}_{(M,w),\,m}$ is the *type* of (M,w) of the modal depth $n+1$ and up to the arity m. Again we assume some standard ordering of the conjuncts and some standard bracketing, so that if two types $\tau^{n+1}_{(M,w),\,m}$ and $\tau^{n+1}_{(N,v),\,m}$ of pointed V-models (M,w) and (N,v) are uniformly equivalent, then the types are the same formula. We let $\mathrm{TP}^{n+1}_{V,m}$ be the set containing exactly the formulae τ such that for some pointed model (M,w) of the vocabulary V, the formula τ is the type of (M,w) of the modal depth $n+1$ and up to the arity m. We observe that the set $\mathrm{TP}^{n+1}_{V,m}$ is finite.

We then list a number of properties of types that are straightforward to prove. Let (M,w) be a pointed model of the vocabulary U, where U may be infinite. Assume that $U_1 \neq \emptyset$. Let $V \subseteq U$ be a finite vocabulary and let m be as defined above, i.e., m is at least two and greater or equal to the maximum arity of the symbols in V_h. Assume that $V_1 \neq \emptyset$. Let $n \in \mathbb{N}$. Firstly, (M,w) satisfies exactly one type in $\mathrm{TP}^n_{V,m}$. Also, for all $\tau \in \mathrm{TP}^n_{V,m}$ and all $l \leq n$, there exists exactly one type $\sigma \in \mathrm{TP}^l_{V,m}$ such that $\tau \Vdash \sigma$. Notice also that for each type $\tau \in \mathrm{TP}^n_{V,m}$, there exists some pointed V-model that satisfies τ. Let $\alpha \in \mathrm{TP}^n_{V,m}$ and let ψ be an arbitrary formula of the vocabulary V and of some modal depth $n' \leq n$. Assume that the maximum arity of the modal parameters that occur in ψ is at most m. Now either $\alpha \Vdash \psi$ or $\alpha \Vdash \neg\psi$, and thus, for all points $u,v \in ||\alpha||^M$, we have $(M,u) \Vdash \psi$ iff $(M,v) \Vdash \psi$. Finally, ψ is V-equivalent to $\bigvee\{\alpha \in \mathrm{TP}^n_{V,m} \mid \alpha \Vdash \psi\}$. Notice that $\bigvee\emptyset = \bot$, where $\bot$ is defined to be the formula $(P \wedge \neg P)$ for some $P \in V_1$.

3.3 Σ^1_1(ML) Translates into Monadic Σ^1_1(MLE)

In this subsection we show how to translate Σ^1_1(ML) formulae to uniformly equivalent formulae of monadic Σ^1_1(MLE). We begin by fixing a Σ^1_1(ML) formula φ. We will first show how to translate φ to a uniformly equivalent formula $\varphi^*(x)$ of $\exists$MSO. We will then establish that that the first-order part of $\varphi^*(x)$ translates to a uniformly equivalent formula of MLE.

Let $\varphi := \overline{Q}\,\psi$, where $\overline{Q}$ is a string of existential second-order quantifiers and ψ a formula of ML. Let V_1^ψ and V_2^ψ denote the sets of unary and binary relation symbols, respectively, that occur in ψ. Define

$$V^\psi = V_1^\psi \cup V_2^\psi.$$

Let Q_1^ψ and Q_2^ψ denote the sets of unary and binary relation symbols, respectively, that occur in $\overline{Q}$. Define

$$Q^\psi = Q_1^\psi \cup Q_2^\psi.$$

Let SUB_ψ denote the set of subformulae of the formula ψ.

We fix a unary relation symbol P_α for each formula $\alpha \in \mathrm{SUB}_\psi$. The symbols P_α are assumed not to occur in φ. We then define a collection of auxiliary formulae needed in order to define the translated formula $\varphi^*(x)$. Let

$$P',\, \neg\alpha,\, (\beta\wedge\gamma),\, \langle R\rangle\rho,\, \langle S\rangle\sigma \quad\in\quad \mathrm{SUB}_\psi,$$

where $P' \in V_1^\psi$, $R \in V_2^\psi \setminus Q_2^\psi$ and $S \in Q_2^\psi$. We define

$$\begin{array}{lcl}
\psi_{P'} & := & \forall x\Big(P_{P'}(x) \leftrightarrow P'(x)\Big),\\
\psi_{\neg\alpha} & := & \forall x\Big(P_{\neg\alpha}(x) \leftrightarrow \neg P_\alpha(x)\Big),\\
\psi_{(\beta\wedge\gamma)} & := & \forall x\Big(P_{(\beta\wedge\gamma)}(x) \leftrightarrow (P_\beta(x) \wedge P_\gamma(x))\Big),\\
\psi_{\langle R\rangle\rho} & := & \forall x\Big(P_{\langle R\rangle\rho}(x) \leftrightarrow \exists y(R(x,y) \wedge P_\rho(y))\Big),\\
\psi_{\langle S\rangle\sigma} & := & \forall x\Big(P_{\langle S\rangle\sigma}(x) \leftrightarrow \exists y(Access_S(x,y) \wedge P_\sigma(y))\Big),
\end{array}$$

where

$$Access_S(x,y) \quad := \quad \bigwedge_{\langle S\rangle\chi\ \in\ \mathrm{SUB}_\psi} \Big(\ P_\chi(y) \rightarrow P_{\langle S\rangle\chi}(x)\ \Big).$$

Finally, we define

$$\delta_\psi \quad := \quad \bigwedge_{\alpha\ \in\ \mathrm{SUB}_\psi} \psi_\alpha$$

and

$$\varphi^*(x) \ := \ \overline{Q}^*(\delta_\psi \ \wedge \ P_\psi(x)),$$

where $\overline{Q}^*$ is a string of existential quantifiers that quantify the predicate symbols $P \in Q_1^\psi$ and also the symbols P_α such that $\alpha \in \mathrm{SUB}_\psi$.

We then prove that $(M, w) \Vdash \varphi$ implies $M, \frac{w}{x} \models \varphi^*(x)$. Assume that $(M, w) \Vdash \varphi$. Therefore there exists an expansion M_2 of M by interpretations of the binary and unary symbols in Q^ψ such that we have $(M_2, w) \Vdash \psi$. We define an expansion M_1 of M by interpretations of the unary symbols occurring in $\overline{Q}^*$. For the symbols $P \in Q_1^\psi$, we let $P^{M_1} = P^{M_2}$. For the symbols P_α, where $\alpha \in \mathrm{SUB}_\psi$, we define $P_\alpha^{M_1} = ||\alpha||^{M_2}$.

Lemma 3.1. *Let $\langle S\rangle\sigma \in \mathrm{SUB}_\psi$, where $S \in Q_2^\psi$, and let $v \in Dom(M)$. Then $(M_2, v) \Vdash \langle S\rangle\sigma$ iff $M_1, \frac{v}{x} \models \exists y\big(Access_S(x, y) \wedge P_\sigma(y)\big)$.*

Proof. Assume $(M_2, v) \Vdash \langle S\rangle\sigma$. Thus $(v, u) \in S^{M_2}$ for some point

$$u \ \in \ ||\sigma||^{M_2} = P_\sigma^{M_1}.$$

To establish that

$$M_1, \frac{v}{x} \ \models \ \exists y\big(Access_S(x, y) \wedge P_\sigma(y)\big),$$

it therefore suffices to prove that for all $\langle S\rangle\chi \in \mathrm{SUB}_\psi$, if $u \in P_\chi^{M_1}$, then $v \in P_{\langle S\rangle\chi}^{M_1}$. Therefore assume that $u \in P_\chi^{M_1}$ for some formula $\langle S\rangle\chi \in \mathrm{SUB}_\psi$. As $||\chi||^{M_2} = P_\chi^{M_1}$, we have $u \in ||\chi||^{M_2}$. Since $(v, u) \in S^{M_2}$, we have $(M_2, v) \Vdash \langle S\rangle\chi$. As $||\langle S\rangle\chi||^{M_2} = P_{\langle S\rangle\chi}^{M_1}$, we must have $v \in P_{\langle S\rangle\chi}^{M_1}$, as desired.

Assume then that

$$M_1, \frac{v}{x} \ \models \ \exists y\big(Access_S(x, y) \wedge P_\sigma(y)\big).$$

Hence $M_1, \frac{v}{x}\frac{u}{y} \models Access_S(x, y)$ for some $u \in P_\sigma^{M_1} = ||\sigma||^{M_2}$. Now, by the definition of the formula $Access_S(x, y)$, we observe that $v \in P_{\langle S\rangle\sigma}^{M_1}$. As $||\langle S\rangle\sigma||^{M_2} = P_{\langle S\rangle\sigma}^{M_1}$, we have $v \in ||\langle S\rangle\sigma||^{M_2}$. Therefore $(M_2, v) \Vdash \langle S\rangle\sigma$, as desired. $\square$

Lemma 3.2. *Under the assumption $(M, w) \Vdash \varphi$, we have $M, \frac{w}{x} \models \varphi^*(x)$.*

Proof. We establish the claim of the lemma by proving that

$$M_1, \frac{w}{x} \models \delta_\psi \wedge P_\psi(x).$$

Since $(M_2, w) \Vdash \psi$ and $||\psi||^{M_2} = P_\psi^{M_1}$, we have $M_1, \frac{w}{x} \models P_\psi(x)$. The non-trivial part in establishing that $M_1 \models \delta_\psi$ is showing that $M_1 \models \psi_{\langle S\rangle\sigma}$ for each $\langle S\rangle\sigma \in \mathrm{SUB}_\psi$, where $S \in Q_2^\psi$. This follows directly by Lemma 3.1, as $P_{\langle S\rangle\sigma}^{M_1} = ||\langle S\rangle\sigma||^{M_2}$. $\square$

We then establish that $M, \frac{w}{x} \models \varphi^*(x)$ implies $(M, w) \Vdash \varphi$. Therefore we assume that $M, \frac{w}{x} \models \varphi^*(x)$. Therefore there exists an expansion M_1' of M by interpretations of the unary symbols occurring in $\overline{Q}^*$ such that $M_1', \frac{w}{x} \models \delta_\psi \wedge P_\psi(x)$. We define an expansion M_2' of M by interpretations of the binary and unary symbols that occur in $\overline{Q}$. For the symbols $P \in Q_1^\psi$, we define $P^{M_2'} = P^{M_1'}$. For the symbols $S \in Q_2^\psi$, we let $(v, u) \in S^{M_2'}$ if and only if $M_1', \frac{v}{x}\frac{u}{y} \models Access_S(x, y)$.

Lemma 3.3. *Let $\alpha \in \mathrm{SUB}_\psi$ and $v \in Dom(M)$. We have $(M_2', v) \Vdash \alpha$ iff $M_1', \frac{v}{x} \models P_\alpha(x)$.*

Proof. We establish the claim of the lemma by induction on the structure of α. Since $M_1' \models \delta_\psi$, the claim holds trivially for all atomic formulae $P \in V_1^\psi$. Also, the cases where α is of form $\neg\beta$, $(\beta \wedge \gamma)$ or $\langle R\rangle\beta$, where $R \in V_2^\psi \setminus Q_2^\psi$, are straightforward since $M_1' \models \delta_\psi$.

Assume that $(M_2', v) \Vdash \langle S\rangle\sigma$, where $S \in Q_2^\psi$ and $\langle S\rangle\sigma \in \mathrm{SUB}_\psi$. Therefore $(v, u) \in S^{M_2'}$ for some $u \in ||\sigma||^{M_2'}$. Hence $M_1, \frac{v}{x}\frac{u}{y} \models Access_S(x, y)$ by the definition of $S^{M_2'}$. We have $P_\sigma^{M_1'} = ||\sigma||^{M_2'}$ by the induction hypothesis. Therefore $u \in P_\sigma^{M_1'}$, whence we have

$$M_1', \frac{v}{x} \ \models\ \exists y\big(Access_S(x, y) \wedge P_\sigma(y)\big).$$

Therefore, as $M_1' \models \psi_{\langle S\rangle\sigma}$, we have $M_1', \frac{v}{x} \models P_{\langle S\rangle\sigma}(x)$.

For the converse, we assume that $M_1', \frac{v}{x} \models P_{\langle S\rangle\sigma}(x)$. As $M_1' \models \psi_{\langle S\rangle\sigma}$, we have

$$M_1', \frac{v}{x} \ \models\ \exists y\big(Access_S(x, y) \wedge P_\sigma(y)\big).$$

Hence there exists some element $u \in P_\sigma^{M_1'}$ such that $M_1', \frac{v}{x}\frac{u}{y} \models Access_S(x, y)$. Therefore $(v, u) \in S^{M_2'}$ by the definition of $S^{M_2'}$. Since $u \in P_\sigma^{M_1'}$ and as $||\sigma||^{M_2'} = P_\sigma^{M_1'}$ by the induction hypothesis, we may therefore conclude that $(M_2', v) \Vdash \langle S\rangle\sigma$. □

By Lemma 3.3 we immediately observe that since $M_1', \frac{w}{x} \models P_\psi(x)$, we must have $(M_2', w) \Vdash \psi$. Therefore $(M, w) \Vdash \varphi$. This, together with Lemma 3.2, justifies the following conclusion.

Theorem 3.4. *Each formula of $\Sigma_1^1(\mathrm{ML})$ translates to a uniformly equivalent formula of $\exists\mathrm{MSO}$. The translation is effective.*

We then establish that $\varphi^*(x)$ is in fact expressible in monadic $\Sigma_1^1(\mathrm{MLE})$. This is easy. Fix a symbol $S \in Q_2^\psi$ and let A be the subset of SUB_ψ that contains exactly all the formulae of the form $\langle S\rangle\alpha$. The formula

$$\exists y\big(Access_S(x, y) \wedge P_\sigma(y)\big)$$

is uniformly equivalent to the following formula of MLE.

$$\bigvee_{B \subseteq A} \Big(\bigwedge_{\langle S \rangle \chi \, \in \, B} P_{\langle S \rangle \chi} \wedge \langle E \rangle \Big(P_\sigma \wedge \bigwedge_{\langle S \rangle \chi \, \in \, B} P_\chi \wedge \bigwedge_{\langle S \rangle \chi \, \in \, A \setminus B} \neg P_\chi \Big) \Big)$$

Thus we see that for each sentence ψ_α, where $\alpha \in \mathrm{SUB}_\psi$, there exists a formula of MLE that is uniformly equivalent to the formula $x = x \wedge \psi_\alpha$. We may therefore draw the following conclusion.

Theorem 3.5. *Each formula of* $\Sigma^1_1(\mathrm{ML})$ *translates to a uniformly equivalent formula of monadic* $\Sigma^1_1(\mathrm{MLE})$*. The translation is effective.*

The following corollaries are immediate.

Corollary 3.6. *Let* $\mathcal{C}$ *be a class of unimodal Kripke frames* (W, R_0)*. Let* I *be a set of indices such that* $0 \in I$ *and call*

$$\mathcal{D} \;=\; \{\; (W, \{R_i\}_{i \in I}) \mid R_i \subseteq W \times W,\; (W, R_0) \in \mathcal{C} \;\}.$$

If the satisfiability problem of MLE *w.r.t. the class* $\mathcal{C}$ *is decidable, then the satisfiability problem of* ML *w.r.t.* $\mathcal{D}$ *is decidable.*

Corollary 3.7. *Each formula of* $\Pi^1_1(\mathrm{ML})$ *translates to a uniformly equivalent formula of monadic* $\Pi^1_1(\mathrm{MLE})$*. The translation is effective.*

3.4 $\Sigma^1_1(\mathrm{PBML}^=)$ Translates into $\exists$MSO

In this section we prove that each formula of $\Sigma^1_1(\mathrm{PBML}^=)$ can be translated to a uniformly equivalent formula of $\exists$MSO.

3.4.1 An Effective Translation

In the current subsection we define an effective translation of formulae of $\Sigma^1_1(\mathrm{PBML}^=)$ to uniformly equivalent formulae of $\exists$MSO. Let us fix a $\Sigma^1_1(\mathrm{PBML}^=)$ formula φ and show how it is translated. Let $\varphi := \overline{Q}\,\psi$, where $\overline{Q}$ is vector of existential second-order quantifiers and ψ a formula of $\mathrm{PBML}^=$. For presentation related results, assume w.l.o.g. that $Md(\psi) \geq 2$ and that each symbol in $\overline{Q}$ occurs in ψ. We let m denote the maximum arity of the modal parameters that occur in ψ. Since $Md(\psi) \geq 2$, the formula ψ must contain diamonds, and therefore m exists and $m \geq 2$.

Let V^ψ_1 denote the set of unary relation symbols that occur in ψ, and let V^ψ_h be the set of relation symbols of higher arities occurring in ψ. Let

$$V^\psi = V^\psi_1 \cup V^\psi_h.$$

Some of the relation symbols in V^ψ may occur in the quantifier prefix $\overline{Q}$ and some may not. Let Q^ψ_1 denote the set of unary relation symbols that occur in $\overline{Q}$. The set of relation symbols of higher arities occurring in $\overline{Q}$ is denoted by Q^{ψ}_{h}. Let

$$Q^{\psi} = Q^{\psi}_{1} \cup Q^{\psi}_{h}.$$

For each $k \in \mathbb{N}_{\geq 2}$, we let $\mathrm{ATP}_{\psi}(k)$ denote the set containing exactly the k-ary access types over V^{ψ}. For each $n \in \mathbb{N}$, we let TP^{n}_{ψ} denote the set $\mathrm{TP}^{n}_{V^{\psi},\,m}$ of types. We define

$$\mathrm{TP}_{\psi} \;=\; \bigcup_{i\,\leq\, Md(\psi)} \mathrm{TP}^{i}_{\psi}.$$

We then fix a set of fresh (i.e., not occurring in φ) unary predicate symbols. We fix a unique unary predicate symbol P_{τ} for each $\tau \in \mathrm{TP}_{\psi}$. We also fix a unary predicate symbol $P_{(\mathcal{M},\overline{\beta})}$ for each pair $(\mathcal{M},\,\overline{\beta})$ such that for some $k \in \{1,...,m-1\}$, we have $\mathcal{M} \in \mathrm{ATP}_{\psi}(k+1)$ and $\overline{\beta} \in (\mathrm{TP}^{Md(\psi)-1}_{\psi})^{k}$.

The translation $\varphi^{*}(x)$ of φ is the formula

$$\Big(\exists P\Big)_{P\,\in\, Q^{\psi}_{1}} \Big(\exists P_{\tau}\Big)_{\tau\,\in\,\mathrm{TP}_{\psi}} \Big(\exists P_{(\mathcal{M},\,\overline{\beta})}\Big)_{\substack{k\,\in\,\{1,...,m-1\}\\ \mathcal{M}\,\in\,\mathrm{ATP}_{\psi}(k+1),\\ \overline{\beta}\,\in\,(\mathrm{TP}^{Md(\psi)-1}_{\psi})^{k}}} \;\psi^{*}(x)\;,$$

where $\psi^{*}(x)$ is a first-order formula in one free variable, x. We let $\overline{Q}^{*}$ denote the above vector of monadic existential second-order quantifiers.

One fundamental idea in the translation we will define is that the symbols P_{τ} are used in order to encode the extensions of the types $\tau \in \mathrm{TP}_{\psi}$. This is manifest in the way the model M_{1} is defined below and also in the content of Lemma 3.13. While the symbols P_{τ} store information about extensions of types, the symbols $P_{(\mathcal{M},\,\overline{\beta}\,)}$ are used in order to encode information about the extensions of the access types $\mathcal{M} \in \mathrm{ATP}_{\psi}(Ar(\mathcal{M}))$. We use the symbols $P_{(\mathcal{M},\,\overline{\beta}\,)}$ when we define the formulae $Access_{\mathcal{M}}(x,y_{1},...,y_{k})$ below. The formulae $Access_{\mathcal{M}}(x,y_{1},...,y_{k})$ encode information about the extensions of the access types $\mathcal{M} \in \mathrm{ATP}_{\psi}(k+1)$ in a way made explicit in Lemmata 3.8 and 3.12.

Before fixing the translation $\varphi^{*}(x)$ of φ, we define a number of auxiliary formulae. The first formula we define ensures that for all $n \in \{0,...,Md(\psi)\}$, the extensions of the predicate symbols P_{τ}, where $\tau \in \mathrm{TP}^{n}_{\psi}$, always cover all of the domain of a model and never overlap each other. We define

$$\psi_{uniq} := \forall x\Big(\bigwedge_{0\,\leq\, i\,\leq\, Md(\psi)} \Big(\bigvee_{\tau\,\in\,\mathrm{TP}^{i}_{\psi}} \Big(P_{\tau}(x) \;\wedge \bigwedge_{\substack{\sigma\,\in\,\mathrm{TP}^{i}_{\psi},\\ \sigma\,\neq\,\tau}} \neg P_{\sigma}(x)\Big)\Big)\Big).$$

The next formula asserts that each symbol P_{β}, where $\beta \in \mathrm{TP}^{Md(\psi)-1}_{\psi}$, must be interpreted such that for all symbols P_{τ}, where $Md(\tau) < Md(\beta)$, the extension of P_β is either fully included in the extension of P_τ or does not overlap with it. We let

$$\psi_{pack} := \forall x \forall y \bigwedge_{\beta\, \in\, \mathrm{TP}_\psi^{Md(\psi)-1}} \Big(\big(P_\beta(x) \wedge P_\beta(y)\big) \rightarrow$$

$$\bigwedge_{\tau\, \in\, \mathrm{TP}_\psi^{<Md(\psi)-1}} \big(P_\tau(x) \leftrightarrow P_\tau(y)\big)\Big).$$

Let k be an integer such that $1 \leq k \leq m-1$ and let $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$. The next formula encodes information about the relation that the $(k+1)$-ary access type $\mathcal{M}$ defines over a V^ψ-model.

$$Access_{\mathcal{M}}(x, y_1, ..., y_k) :=$$

$$\bigvee_{(\beta_1,...,\beta_k)\, =\, \overline{\beta}\, \in\, (\mathrm{TP}_\psi^{Md(\psi)-1})^k} \Big(P_{(\mathcal{M},\overline{\beta})}(x) \wedge P_{\beta_1}(y)\ \wedge ... \wedge P_{\beta_k}(y_k)\Big).$$

We then define formulae $\chi_\tau(x)$ that recursively force the interpretations of the predicate symbols P_τ to match the extensions of the types $\tau \in \mathrm{TP}_\psi$. The content of this assertion is reflected in (the proof of) Lemma 3.13. First, let $\tau \in \mathrm{TP}_\psi^0$. We define

$$\chi_\tau(x) := \bigwedge_{\substack{P\, \in\, V_1^\psi,\\ \tau\, \Vdash\, P}} P(x) \qquad \wedge \bigwedge_{\substack{Q\, \in\, V_1^\psi,\\ \tau\, \nVdash\, Q}} \neg Q(x).$$

Now let $\tau \in \mathrm{TP}_\psi^{n+1}$, where $0 \leq n \leq Md(\psi) - 1$. We define

$$\chi_\tau^+(x) := \bigwedge_{\substack{k\, \in\, \{1,...,m-1\},\\ \mathcal{M}\, \in\, \mathrm{ATP}_\psi(k+1),\\ (\sigma_1,...,\sigma_k)\, \in\, (\mathrm{TP}_\psi^n)^k,\\ \tau\, \Vdash\, \langle\mathcal{M}\rangle(\sigma_1,...,\sigma_k)}} \exists y_1...y_k \Big(Access_{\mathcal{M}}(x, y_1, ..., y_k)$$

$$\wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k)\Big),$$

$$\chi_\tau^-(x) := \bigwedge_{\substack{k\, \in\, \{1,...,m-1\},\\ \mathcal{M}\, \in\, \mathrm{ATP}_\psi(k+1),\\ (\sigma_1,...,\sigma_k)\, \in\, (\mathrm{TP}_\psi^n)^k,\\ \tau\, \Vdash\, \neg\langle\mathcal{M}\rangle(\sigma_1,...,\sigma_k)}} \neg\exists y_1...y_k \Big(Access_{\mathcal{M}}(x, y_1, ..., y_k)$$

$$\wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k)\Big),$$

and

$$\chi_\tau(x) \ := \ P_{\tau'}(x) \ \wedge \ \chi^+_\tau(x) \ \wedge \ \chi^-_\tau(x),$$

where τ' is the unique type in TP^n_ψ such that $\tau \Vdash \tau'$.

Let $k \in \{1, ..., m-1\}$ and $A \subseteq \mathrm{ATP}_\psi(k+1)$, where $A \neq \emptyset$. Let

$$(\beta_1, ..., \beta_k) \ = \ \overline{\beta} \ \in \ (\mathrm{TP}^{Md(\psi)-1}_\psi)^k.$$

The next formula encodes information about the *set* of $(k+1)$-ary access types that connect an element of the domain of a V^ψ-model to k-tuples of elements $(u_1, ..., u_k)$ such that for all i, the element u_i satisfies the type β_i. We define

$$\psi_{(\,A,\overline{\beta}\,)}(x) := \bigwedge_{\mathcal{M}\,\in\,A} \exists y_1...y_k \Big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \\ \wedge\ P_{\beta_1}(y_1) \wedge ... \wedge P_{\beta_k}(y_k) \Big).$$

Our next aim is to define formulae ψ_{cons} and ψ'_{cons} that ensure that information about the extensions of the access types over V^ψ is consistent with the interpretation of the access types over $V^\psi \setminus Q^\psi$, i.e., the access types describing non-quantified accessibility relations.

Let k be an integer such that $1 \leq k \leq m-1$. Define a linear order on $\mathrm{ATP}_\psi(k+1)$. For each set $S \subseteq \mathrm{ATP}_\psi(k+1)$, let $S(i)$ denote the i-th member of the set S with respect to this linear order. Let $A \subseteq \mathrm{ATP}_\psi(k+1)$ be a nonempty set of access types. For each $i \in \{1, ..., |A|\}$, define a k-tuple $\overline{y}_i = (y_{i_1}, ..., y_{i_k})$ of variable symbols. Fix the collection of tuples so that no variable symbol is used twice. Let $\overline{y}_j \neq \overline{y}_l$ denote the formula

$$\bigvee_{n\,\in\,\{1,...,k\}} (\neg\, y_{j_n} = \ y_{l_n}).$$

Let $\chi_{A(i)}(x, \overline{y}_i)$ denote a first-order formula stating that the $(k+1)$-tuple $(x, \overline{y}_i)$ is connected according to the unique $(k+1)$-ary access type over $V^\psi \setminus Q^\psi$ that is consistent with the access type $A(i) \in A$. Let $\overline{\beta} = (\beta_1, ..., \beta_k)$ be a k-tuple of types in $\mathrm{TP}^{Md(\psi)-1}_\psi$. We let

$$\chi_{(A,\overline{\beta})}(x) \ := \ \exists \overline{y}_1, ..., \overline{y}_{|A|} \Big(\bigwedge_{\substack{j,l\,\in\,\{1\ ,...,\ |A|\},\\ j\,\neq\,l}} \overline{y}_j \neq \overline{y}_l \quad \wedge \\ \bigwedge_{i\,\in\,\{1\ ,...,\ |A|\}} \big(\ \chi_{A(i)}(x, \overline{y}_i) \wedge P_{\beta_1}(y_{i_1}) \wedge ... \wedge P_{\beta_k}(y_{i_k})\ \big) \Big).$$

We define

$$\psi_{cons} := \forall x\Big(\bigwedge_{\substack{k \in \{1,\ldots,m-1\}, \\ A \subseteq \mathrm{ATP}_{\psi}(k+1),\ A \neq \emptyset, \\ \overline{\beta} \in (\mathrm{TP}_{\psi}^{Md(\psi)-1})^k}} (\ \psi_{(A,\overline{\beta})}(x) \ \rightarrow\ \chi_{(A,\overline{\beta})}(x)\ )\ \Big).$$

Let $\mathcal{R} \in \mathrm{ATP}_{V^{\psi}\setminus Q^{\psi}}(k+1)$, i.e., $\mathcal{R}$ is a $(k+1)$-ary access type over $V^{\psi} \setminus Q^{\psi}$. We let $C(\mathcal{R})$ denote the set of $(k+1)$-ary access types over V^{ψ} that are consistent with $\mathcal{R}$. Let $\chi_{\mathcal{R}}(x, y_1, ..., y_k)$ denote a first-order formula stating that the $(k+1)$-tuple $(x, y_1, ..., y_k)$ is connected according to the access type $\mathcal{R}$. Let $\overline{\beta} = (\beta_1, ..., \beta_k)$ be a k-tuple of types in $\mathrm{TP}_{\psi}^{Md(\psi)-1}$. Let $\mathcal{M}$ be $(k+1)$-ary access type over V^{ψ}. We let

$$\chi_{(\mathcal{R},\overline{\beta})}(x) \ :=\ \exists y_1...y_k\big(\ \chi_{\mathcal{R}}(x, y_1, ..., y_k) \wedge P_{\beta_1}(y_1) \wedge ... \wedge P_{\beta_k}(y_k)\ \big)$$

and

$$\psi_{(\mathcal{M},\overline{\beta})}(x) \ :=\ \exists z_1...z_k\big(Access_{\mathcal{M}}(x, z_1, ..., z_k) \wedge P_{\beta_1}(z_1) \wedge ... \wedge P_{\beta_k}(z_k)\ \big).$$

We define

$$\psi'_{cons} := \forall x\Big(\bigwedge_{\substack{k \in \{1,\ldots,m-1\}, \\ \mathcal{R} \in \mathrm{ATP}_{V^{\psi}\setminus Q^{\psi}}(k+1), \\ \overline{\beta} \in (\mathrm{TP}_{\psi}^{Md(\psi)-1})^k}} \Big(\chi_{(\mathcal{R},\overline{\beta})}(x) \ \rightarrow \bigvee_{\mathcal{M} \in C(\mathcal{R})} \psi_{(\mathcal{M},\overline{\beta})}(x)\Big)\Big).$$

Finally, we define

$$\delta_{\psi} := \psi_{uniq} \wedge \psi_{pack} \wedge \psi_{cons} \wedge \psi'_{cons} \wedge \bigwedge_{\tau \in \mathrm{TP}_{\psi}} \forall x\Big(P_{\tau}(x) \leftrightarrow \chi_{\tau}(x)\Big)$$

and

$$\varphi^*(x) := \overline{Q}^*\Big(\ \ \delta_{\psi} \ \ \wedge \bigvee_{\substack{\alpha \in \mathrm{TP}_{\psi}^{Md(\psi)}, \\ \alpha \Vdash \psi}} P_{\alpha}(x) \ \ \Big).$$

We then fix an arbitrary pointed model (M, w) of the vocabulary $V^{\psi}\setminus Q^{\psi}$. In the next subsection we establish that

$$(M, w) \Vdash \varphi \ \Leftrightarrow\ M, \frac{w}{x} \models \varphi^*(x).$$

3.4.2 $\Sigma^1_1(\mathrm{PBML}^=) \leq \exists\mathrm{MSO}$

We first show that $(M, w) \Vdash \varphi$ implies $M, \frac{w}{x} \models \varphi^*(x)$. Thus we assume that $(M, w) \Vdash \varphi$. Therefore there exists some expansion M_h of M by interpretations of the symbols in Q^{ψ} such that $(M_h, w) \Vdash \psi$. The subscript "h" in M_h stands for the word "higher" and indicates that M_h is an expansion of M by interpretations of symbols of the arity one *and of higher arities.*

We then define an expansion M_1 of M by interpreting the unary symbols in Q^{ψ}_1 and also the unary symbols of the type P_τ and $P_{(\mathcal{M},\,\overline{\beta})}$, where τ is a type in TP_ψ, and where $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ and $\overline{\beta} \in (\mathrm{TP}^{Md(\psi)-1}_\psi)^k$ for some $k \in \{1, ..., m-1\}$.

For each $P \in Q^{\psi}_1$, we define $P^{M_1} = P^{M_h}$. For each $\tau \in \mathrm{TP}_\psi$, we let $P^{M_1}_\tau = ||\tau||^{M_h}$. Let $k \in \{1, ..., m-1\}$. Let $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ and

$$(\beta_1, ..., \beta_k) \ = \ \overline{\beta} \ \in \ (\mathrm{TP}^{Md(\psi)-1}_\psi)^k.$$

We define $P^{M_1}_{(\mathcal{M},\overline{\beta})}$ to be exactly the set of elements $v \in Dom(M)$ such that for some $(u_1, ..., u_k) \in (Dom(M))^k$, we have $(v, u_1, ..., u_k) \in \mathcal{M}^{M_h}$ and $u_i \in ||\beta_i||^{M_h}$ for all $i \in \{1, ..., k\}$. In other words, we define

$$P^{M_1}_{(\mathcal{M},\overline{\beta})} \ = \ ||\ \langle\mathcal{M}\rangle(\beta_1, ..., \beta_k)\ ||^{M_h}.$$

Next we discuss a number of auxiliary lemmata, and then establish that $M_1, \frac{w}{x} \models \psi^*(x)$.

The following lemma establishes that the formula $Access_{\mathcal{M}}(x, y_1, ..., y_k)$ encodes information about the action of the diamond operator $\langle\mathcal{M}\rangle$ on M_h.

Lemma 3.8. *Let n be an integer such that we have $0 \leq n < Md(\psi)$. Let $k \in \{1, ..., m-1\}$, and let $(\tau_1, ..., \tau_k)$ be a tuple of types in TP^n_ψ. Let $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ and $v \in Dom(M)$. We have*

$$(M_h, v) \Vdash \langle\mathcal{M}\rangle(\tau_1, ..., \tau_k)$$

$$\Leftrightarrow$$

$$M_1, \tfrac{v}{x} \models \exists y_1...y_k\big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k)\big).$$

Proof. Assume that $(M_h, v) \Vdash \langle\mathcal{M}\rangle(\tau_1, ..., \tau_k)$. Thus there exists some tuple

$$(u_1, ..., u_k) \in ||\tau_1||^{M_h} \times ... \times ||\tau_k||^{M_h}$$

such that $(v, u_1, ...u_k) \in \mathcal{M}^{M_h}$. Let

$$(\beta_1, ..., \beta_k) \ = \ \overline{\beta} \ \in (\mathrm{TP}^{Md(\psi)-1}_\psi)^k$$

be the k-tuple of types in $\mathrm{TP}^{Md(\psi)-1}_\psi$ such that we have $u_i \in ||\beta_i||^{M_h}$ for all $i \in \{1, ..., k\}$. Thus $v \in P^{M_1}_{(\mathcal{M},\overline{\beta})}$, and therefore

$$M_1, \frac{v}{x}\frac{u_1}{y_1}...\frac{u_k}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k).$$

As $u_i \in ||\tau_i||^{M_h} = P_{\tau_i}^{M_1}$ for all $i \in \{1, ..., k\}$, we have

$$M_1, \frac{v}{x}\frac{u_1}{y_1}...\frac{u_k}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k).$$

Therefore

$$M_1, \frac{v}{x} \models \exists y_1...y_k\big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k)\big),$$

as desired.

In order to deal with the converse direction, assume that

$$M_1, \frac{v}{x} \ \models\ \exists y_1...y_k\big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k)\big).$$

Therefore, for some tuple

$$(u_1, ..., u_k) \in P_{\tau_1}^{M_1} \times ... \times P_{\tau_k}^{M_1},$$

we have

$$M_1, \frac{v}{x}\frac{u_1}{y_1}...\frac{u_k}{y_k} \ \models\ Access_{\mathcal{M}}(x, y_1, ..., y_k).$$

Therefore $v \in P_{(\mathcal{M},\overline{\beta})}^{M_1}$ for some tuple

$$(\beta_1, ..., \beta_k) \ = \ \overline{\beta} \ \in (\mathrm{TP}_{\psi}^{Md(\psi)-1})^k$$

such that $u_i \in P_{\beta_i}^{M_1}$ for all $i \in \{1, ..., k\}$. We have $Md(\tau_i) \leq Md(\beta_i)$ for all $i \in \{1, ..., k\}$. Also, by the definition of the model M_1, we have $P_{\sigma}^{M_1} = ||\sigma||^{M_h}$ for all $\sigma \in \mathrm{TP}_{\psi}$, so each set $P_{\sigma}^{M_1}$ is the extension of the type σ. Therefore, as $u_i \in P_{\beta_i}^{M_1} \cap P_{\tau_i}^{M_1}$ for all $i \in \{1, ..., k\}$, we conclude that $||\beta_i||^{M_h} \subseteq ||\tau_i||^{M_h}$ for all $i \in \{1, ..., k\}$. Hence

$$||\beta_1||^{M_h} \times ... \times ||\beta_k||^{M_h} \ \subseteq\ ||\tau_1||^{M_h} \times ... \times ||\tau_k||^{M_h}.$$

Also, as $v \in P_{(\mathcal{M},\overline{\beta})}^{M_1}$, we have $(v, u_1', ..., u_k') \in \mathcal{M}^{M_h}$ for some tuple

$$(u_1', ..., u_k') \ \in \ ||\beta_1||^{M_h} \times ... \times ||\beta_k||^{M_h}.$$

Therefore we conclude that $(M_h, v) \Vdash \langle\mathcal{M}\rangle(\tau_1, ..., \tau_k)$, as desired. □

We then establish a link between interpretations of the formulae $\chi_\tau(x)$ and interpretations of the predicate symbols P_τ in the model M_1.

Lemma 3.9. *Let $v \in Dom(M)$ and $\tau \in \mathrm{TP}_{\psi}$. We have $M_1, \frac{v}{x} \models P_\tau(x)$ iff $M_1, \frac{v}{x} \models \chi_\tau(x)$.*

Proof. As $||P||^{M_h} = P^{M_1}$ for all $P \in V_1^{\psi}$, the claim follows directly for all $\tau \in \mathrm{TP}^0_{\psi}$. Therefore we may assume that $\tau \in \mathrm{TP}^{\geq 1}_{\psi}$. Throughout the proof, we let τ' denote the unique type in $\mathrm{TP}^{Md(\tau)-1}_{\psi}$ such that $\tau \Vdash \tau'$.

Assume that $M_1, \frac{v}{x} \models P_\tau(x)$. As $P_\tau^{M_1} = ||\tau||^{M_h}$, we have $(M_h, v) \Vdash \tau$. As $\tau \Vdash \tau'$, we have $(M_h, v) \Vdash \tau'$. Since $P_{\tau'}^{M_1} = ||\tau'||^{M_h}$, we conclude that $M_1, \frac{v}{x} \models P_{\tau'}(x)$.

We then establish that $M_1, \frac{v}{x} \models \chi_\tau^+(x) \wedge \chi_\tau^-(x)$. Let $k \in \{1, ..., m-1\}$ and assume that $\tau \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$, where we have $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ and $\sigma_i \in \mathrm{TP}^{Md(\tau)-1}_{\psi}$ for all $i \in \{1, ..., k\}$. As we have $(M_h, v) \Vdash \tau$, we also have $(M_h, v) \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$. Therefore, by Lemma 3.8,

$$M_1, \frac{v}{x} \models \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k) \big).$$

Similarly, if $\tau \Vdash \neg \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$, we conclude by Lemma 3.8 that

$$M_1, \frac{v}{x} \models \neg \exists y \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_k}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k) \big).$$

Thus $M_1, \frac{v}{x} \models \chi_\tau^+(x) \wedge \chi_\tau^-(x)$, as desired.

For the converse, assume that $M_1, \frac{v}{x} \models \chi_\tau(x)$. In order to show that $M_1, \frac{v}{x} \models P_\tau(x)$, we will establish that $(M_h, v) \Vdash \tau$. As $P_\tau^{M_1} = ||\tau||^{M_h}$, this suffices.

As $M_1, \frac{v}{x} \models P_{\tau'}(x)$ and $P_{\tau'}^{M_1} = ||\tau'||^{M_h}$, we immediately observe that $(M_h, v) \Vdash \tau'$.

Let $\tau \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$, where $\mathcal{M} \in \mathrm{ATP}_\psi$ and $\sigma_i \in \mathrm{TP}^{Md(\tau)-1}_{\psi}$ for all $i \in \{1, ..., k\}$. As $M_1, \frac{v}{x} \models \chi_\tau^+(x)$, we have

$$M_1, \frac{v}{x} \models \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k) \big),$$

and therefore $(M_h, v) \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$ by Lemma 3.8. Similarly, if we have $\tau \Vdash \neg \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$, then, as $M_1, \frac{v}{x} \models \chi_\tau^-(x)$, we conclude that

$$M_1, \frac{v}{x} \models \neg \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k) \big),$$

and therefore $(M_h, v) \Vdash \neg \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$ by Lemma 3.8. Thus $(M_h, v) \Vdash \tau$, and hence $M_1, \frac{v}{x} \models P_\tau(x)$, as desired. □

We then conclude the first direction of the proof of the claim that $(M, w) \Vdash \varphi$ iff $M, \frac{w}{x} \models \varphi^*(x)$.

Lemma 3.10. *Under the assumption $(M, w) \Vdash \varphi$, we have $M, \frac{w}{x} \models \varphi^*(x)$.*

Proof. We have assumed that $(M, w) \Vdash \varphi$ and thereby concluded that there exists a model M_h such that $(M_h, w) \Vdash \psi$. We have then defined the model

M_1, and we now establish the claim of the current lemma by proving that $M_1, \frac{w}{x} \models \psi^*(x)$. Recall that $\psi^*(x)$ is the formula

$$\delta_\psi \quad \wedge \bigvee_{\substack{\alpha \,\in\, \mathrm{TP}_\psi^{Md(\psi)},\\ \alpha \,\Vdash\, \psi}} P_\alpha(x) \ ,$$

where δ_ψ denotes the formula

$$\psi_{uniq} \wedge \psi_{pack} \wedge \psi_{cons} \wedge \psi'_{cons} \wedge \bigwedge_{\tau \,\in\, \mathrm{TP}_\psi} \forall x \Big(P_\tau(x) \leftrightarrow \chi_\tau(x) \Big).$$

Let ψ' denote a disjunction of exactly all the types $\alpha \in \mathrm{TP}_\psi^{Md(\psi)}$ such that $\alpha \Vdash \psi$. As ψ and ψ' are V^ψ-equivalent (and in fact uniformly equivalent), we have $(M_h, w) \Vdash \psi'$. Therefore $(M_h, w) \Vdash \alpha$ for some $\alpha \in \mathrm{TP}_\psi^{Md(\psi)}$ occurring in the disjunction. Hence, as $||\alpha||^{M_h} = P_\alpha^{M_1}$, we conclude that $M_1, \frac{w}{x} \models P_\alpha(x)$.

We then show that $M_1 \models \psi_{cons}$. Let $v \in Dom(M)$ and assume that $M_1, \frac{v}{x} \models \psi_{(\,A, \overline{\beta}\,)}(x)$ for some nonempty $A \subseteq \mathrm{ATP}_\psi(k+1)$ and some tuple of types

$$(\beta_1, ..., \beta_k) \ = \ \overline{\beta} \ \in \ (\mathrm{TP}_\psi^{Md(\psi)-1})^k.$$

Recall that $A(i)$ denotes the i-th access type in A with respect to the linear ordering of $\mathrm{ATP}_\psi(k+1)$ we fixed. As $M_1, \frac{v}{x} \models \psi_{(\,A, \overline{\beta}\,)}(x)$, we conclude by Lemma 3.8 that $(M_h, v) \Vdash \langle A(i) \rangle (\beta_1, ..., \beta_k)$ for each $i \in \{1, ..., |A|\}$. Thus there must exist $|A|$ distinct k-tuples

$$\overline{u}_1, \, ... \, , \, \overline{u}_{|A|} \ \in \ \ ||\beta_1||^{M_h} \times ... \times ||\beta_k||^{M_h} \ = \ \ P_{\beta_1}^{M_1} \times ... \times P_{\beta_k}^{M_1}$$

such that $(v, \overline{u}_i) \in (A(i))^{M_h}$ for each i. Let $\mathcal{R}_i$ denote the access type over $V^\psi \setminus Q^\psi$ consistent with $A(i)$. Recall that $\chi_{A(i)}(x, \overline{y}_i)$ is a first-order formula stating that the tuple $(x, \overline{y}_i)$ is connected according to the access type $\mathcal{R}_i$. We have $(v, \overline{u}_i) \in \mathcal{R}_i^{M_h} = \mathcal{R}_i^{M_1}$ for each i, and thus

$$M_1, \frac{v}{x} \frac{u_{i_1}}{y_{i_1}} ... \frac{u_{i_k}}{y_{i_k}} \ \models \ \chi_{A(i)}(x, y_{i_1}, ..., y_{i_k}) \wedge P_{\beta_1}(y_{i_1}) \wedge ... \wedge P_{\beta_k}(y_{i_k})$$

for each i.

We then establish that $M_1 \models \psi'_{cons}$. Let $k \in \{1, ..., m-1\}$ and let $\mathcal{R}$ be a $(k+1)$-ary access type over $V^\psi \setminus Q^\psi$. Let $v \in Dom(M)$ and assume that

$$M_1, \frac{v}{x} \frac{u_1}{y_1} ... \frac{u_k}{y_k} \models \chi_{\mathcal{R}}(x, y_1, ..., y_k) \wedge P_{\beta_1}(y_1) \wedge ... \wedge P_{\beta_k}(y_k)$$

for some $u_1, ..., u_k \in Dom(M)$. Let $\mathcal{M}$ be the $(k+1)$-ary access type such that $(v, u_1, ..., u_k) \in \mathcal{M}^{M_h}$. Thus $(M_h, v) \Vdash \langle \mathcal{M} \rangle (\beta_1, ..., \beta_k)$, whence by Lemma 3.8, we have

$$M_1, \frac{v}{x} \ \models \ \exists z_1 ... z_k \big(Access_{\mathcal{M}}(x, z_1, ..., z_k) \wedge P_{\beta_1}(z_1) \wedge ... \wedge P_{\beta_k}(z_k) \big).$$

Clearly $\mathcal{M}$ is consistent with $\mathcal{R}$ and hence we have $\mathcal{M} \in C(\mathcal{R})$. Therefore $M_1 \models \psi'_{cons}$.

We have $M_1 \models \psi_{uniq} \wedge \psi_{pack}$ directly by properties of types. Therefore, in order to conclude the proof, we only need to establish that for each type $\tau \in \mathrm{TP}_\psi$ and each $v \in Dom(M)$, $M_1, \frac{v}{x} \models P_\tau(x) \leftrightarrow \chi_\tau(x)$. This follows directly by Lemma 3.9. $\square$

We then show that $M, \frac{w}{x} \models \varphi^*(x)$ implies $(M, w) \Vdash \varphi$. Thus we assume that $M, \frac{w}{x} \models \varphi^*(x)$. Therefore there exists an expansion M_1' of M by interpretations of the unary symbols P_τ and $P_{(\mathcal{M},\overline{\beta})}$, and also the symbols $P \in Q_1^\psi$, such that $M_1', \frac{w}{x} \models \psi^*(x)$.

We define an expansion of M by interpreting all the relation symbols in Q^ψ. We call the resulting expansion M_h'. For each $P \in Q_1^\psi$, we define $P^{M_h'} = P^{M_1'}$. Let $v \in Dom(M)$ and $k \in \{1, ..., m-1\}$. Let

$$\overline{\beta} = (\beta_1, ..., \beta_k) \in (\mathrm{TP}_\psi^{Md(\psi)-1})^k.$$

Let $A_{k+1} \subseteq \mathrm{ATP}_\psi(k+1)$ be the set of access types $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ such that for some tuple

$$(u_1, ..., u_k) \in P_{\beta_1}^{M_1'} \times ... \times P_{\beta_k}^{M_1'},$$

we have

$$M_1', \frac{v}{x}\frac{u_1}{y_1}...\frac{u_k}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k).$$

As M_1' satisfies the formula ψ_{cons}, we see that there exists a bijection f from the set A_{k+1} to a set

$$B \subseteq P_{\beta_1}^{M_1'} \times ... \times P_{\beta_k}^{M_1'}$$

such that for all $\mathcal{M} \in A_{k+1}$, we have $\big(v, f(\mathcal{M})\big) \in \mathcal{R}_{\mathcal{M}}^{M_1'}$, where $\mathcal{R}_{\mathcal{M}}$ is the access type in $\mathrm{ATP}_{V^\psi \setminus Q^\psi}(k+1)$ consistent with $\mathcal{M}$. Let $S \in Q_h^\psi$ be a relation symbol of the arity $k+1$. We define, for each $\mathcal{M} \in A_{k+1}$,

$$\big(v, f(\mathcal{M})\big) \ \in \ S^{M_h'} \ \text{ iff } \ S \in \mathcal{M}.$$

Recall that we write $S \in \mathcal{M}$ if S occurs in the type $\mathcal{M}$ (i.e., $\neg S$ does not occur in $\mathcal{M}$). We then consider the k-tuples in the set

$$(P_{\beta_1}^{M_1'} \times ... \times P_{\beta_k}^{M_1'}) \ \setminus \ B.$$

Let the tuple $(u_1, ..., u_k)$ belong to this set. Let $\mathcal{R}$ be the access type in $\mathrm{ATP}_{V^\psi \setminus Q^\psi}(k+1)$ such that $(v, u_1, ..., u_k) \in \mathcal{R}^{M_1'}$. As M_1' satisfies ψ'_{cons}, we observe that there exists some $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$ consistent with $\mathcal{R}$ and some tuple

$$(u_1', ..., u_k') \in P_{\beta_1}^{M_1'} \times ... \times P_{\beta_k}^{M_1'}$$

such that

$$M'_1, \frac{v}{x}\frac{u'_1}{y_1}...\frac{u'_k}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k).$$

Again let $S \in Q_h^{\psi}$ be a relation symbol of the arity $k+1$. We define

$$(v, u_1, ..., u_k) \ \in \ S^{M'_h} \ \text{ iff } \ S \in \mathcal{M}.$$

For each $v \in Dom(M)$ and $k \in \{1, ..., m-1\}$, we go through each tuple $\overline{\beta} \in (\mathrm{TP}_{\psi}^{Md(\psi)-1})^k$, and construct the extensions $S^{M'_h}$ of the $(k+1)$-ary symbols $S \in Q_h^{\psi}$ in the described way. This procedure defines the expansion M'_h of M. As the model M'_1 satisfies ψ_{uniq}, the model M'_h is well defined.

Next we discuss a number of auxiliary lemmata and then establish that $(M'_h, w) \Vdash \psi$. The following lemma is a direct consequence of the way we define the extensions $S^{M'_h}$ of the relation symbols $S \in Q_h^{\psi}$.

Lemma 3.11. *Let $v \in Dom(M)$. Let $k \in \{1, ..., m-1\}$, $\mathcal{M} \in \mathrm{ATP}_{\psi}(k+1)$ and $(\beta_1, ..., \beta_k) \in (\mathrm{TP}_{\psi}^{Md(\psi)-1})^k$. Then*

$$(v, u_1, ..., u_k) \in \mathcal{M}^{M'_h}$$

for some $(u_1, ..., u_k) \in P_{\beta_1}^{M'_1} \times ... \times P_{\beta_k}^{M'_1}$ if and only if we have

$$M'_1, \frac{v}{x}\frac{u'_1}{y_1}...\frac{u'_k}{y_k} \models Access_{\mathcal{M}}(x, y_1, ...y_k)$$

for some $(u'_1, ..., u'_k) \in P_{\beta_1}^{M'_1} \times ... \times P_{\beta_k}^{M'_1}$.

The diamond $\langle \mathcal{M} \rangle$ encodes information about the relation that the formula $Access_{\mathcal{M}}(x, y_1, ..., y_k)$ defines over M'_1. The next lemma establishes this link.

Lemma 3.12. *Let n be an integer such that $0 \leq n < Md(\psi)$, and let $k \in \{1, ..., m-1\}$. Let $(\tau_1, ..., \tau_k) \in (\mathrm{TP}_{\psi}^{n})^k$ and $\mathcal{M} \in \mathrm{ATP}_{\psi}(k+1)$. Assume that $||\tau_i||^{M'_h} = P_{\tau_i}^{M'_1}$ for all $i \in \{1, ..., k\}$. Let $v \in Dom(M)$. Then*

$$(M'_h, v) \Vdash \langle \mathcal{M} \rangle(\tau_1, ..., \tau_k)$$

if and only if

$$M'_1, \frac{v}{x} \models \exists y_1...y_k\big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k)\big).$$

Proof. Assume that $(M'_h, v) \Vdash \langle \mathcal{M} \rangle(\tau_1, ..., \tau_k)$. Thus $(v, u_1, ..., u_k) \in \mathcal{M}^{M'_h}$ for some tuple

$$(u_1, ..., u_k) \in ||\tau_1||^{M'_h} \times ... \times ||\tau_k||^{M'_h} = P_{\tau_1}^{M'_1} \times ... \times P_{\tau_k}^{M'_1}.$$

As $M_1' \models \psi_{uniq}$, we observe that for each $i \in \{1, ..., k\}$, there exists exactly one type $\beta_i \in \mathrm{TP}_\psi^{Md(\psi)-1}$ such that $u_i \in P_{\beta_i}^{M_1'}$. Therefore, by Lemma 3.11, we have

$$M_1', \frac{v}{x}\frac{u_1'}{y_1}...\frac{u_k'}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k)$$

for some $(u_1', ..., u_k') \in P_{\beta_1}^{M_1'} \times ... \times P_{\beta_k}^{M_1'}$. Pick an arbitrary $j \in \{1, ..., k\}$.

1. If $n = Md(\psi) - 1$, then, as $M_1' \models \psi_{uniq}$ and $u_j \in P_{\beta_j}^{M_1'} \cap P_{\tau_j}^{M_1'}$, we have $\beta_j = \tau_j$, and thus $u_j' \in P_{\tau_j}^{M_1'}$.

2. If $n < Md(\psi) - 1$, then, since $M_1' \models \psi_{pack}$ and as $u_j \in P_{\tau_j}^{M_1'} \cap P_{\beta_j}^{M_1'}$ and $u_j' \in P_{\beta_j}^{M_1'}$, we again have $u_j' \in P_{\tau_j}^{M_1'}$.

Therefore

$$M_1', \frac{v}{x} \models \exists y_1...y_k\big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k)\big),$$

as required.

For the converse, assume that

$$M_1', \frac{v}{x} \models \exists y_1...y_k\big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\tau_1}(y_1) \wedge ... \wedge P_{\tau_k}(y_k)\big).$$

Therefore

$$M_1', \frac{v}{x}\frac{u_1}{y_1}...\frac{u_k}{y_k} \models Access_{\mathcal{M}}(x, y_1, ..., y_k)$$

for some tuple

$$(u_1, ..., u_k) \ \in \ P_{\tau_1}^{M_1'} \times ... \times P_{\tau_k}^{M_1'} \ = \ ||\tau_1||^{M_h'} \times ... \times ||\tau_k||^{M_h'}.$$

As $M_1' \models \psi_{uniq}$, we infer that for each u_i, there exists a type $\beta_i \in \mathrm{TP}_\psi^{Md(\psi)-1}$ such that $u_i \in P_{\beta_i}^{M_1'}$. By Lemma 3.11, we therefore have

$$(v, u_1', ..., u_k') \ \in \ \mathcal{M}^{M_h'}$$

for some tuple

$$(u_1', ..., u_k') \ \in \ P_{\beta_1}^{M_1'} \times ... \times P_{\beta_k}^{M_1'}.$$

Pick an arbitrary $j \in \{1, ..., k\}$. As above, we have the following cases.

1. If $n = Md(\psi) - 1$, then, as $M_1' \models \psi_{uniq}$ and $u_j \in P_{\beta_j}^{M_1'} \cap P_{\tau_j}^{M_1'}$, we have $\beta_j = \tau_j$, and thus $u_j' \in P_{\tau_j}^{M_1'}$.

2. If $n < Md(\psi) - 1$, then, since $M_1' \models \psi_{pack}$ and as $u_j \in P_{\tau_j}^{M_1'} \cap P_{\beta_j}^{M_1'}$ and $u_j' \in P_{\beta_j}^{M_1'}$, we again have $u_j' \in P_{\tau_j}^{M_1'}$.

Therefore, as we have $P_{\tau_i}^{M'_1} = ||\tau_i||^{M'_h}$ for all $i \in \{1, ..., k\}$, we conclude that $(M'_h, v) \models \langle \mathcal{M} \rangle(\tau_1, ..., \tau_k)$, as desired. $\square$

The next lemma establishes that extensions of the types $\tau \in \mathrm{TP}_\psi$ and interpretations of the predicate symbols P_τ coincide.

Lemma 3.13. *Let $\tau \in \mathrm{TP}_\psi$ and $v \in Dom(M)$. Then $(M'_h, v) \Vdash \tau$ if and only if $M'_1, \frac{v}{x} \models P_\tau(x)$.*

Proof. We prove the claim by induction on the modal depth of τ. If $\tau \in \mathrm{TP}_\psi^0$, then, as $M'_1 \models \forall x(P_\tau(x) \leftrightarrow \chi_\tau(x))$, the claim follows immediately.

Assume that $(M'_h, v) \Vdash \tau$ for some $\tau \in \mathrm{TP}_\psi^{n+1}$, where $0 \leq n < Md(\psi)$. We will show that

$$M'_1, \frac{v}{x} \models P_{\tau'}(x) \wedge \chi_\tau^+(x) \wedge \chi_\tau^-(x),$$

where τ' is the type of the modal depth n such that $\tau \Vdash \tau'$. This directly implies that $M'_1, \frac{v}{x} \models P_\tau(x)$, since $M'_1 \models \forall x(P_\tau(x) \leftrightarrow \chi_\tau(x))$.

As $\tau \Vdash \tau'$, we have $(M'_h, v) \Vdash \tau'$. Therefore $M'_1, \frac{v}{x} \models P_{\tau'}(x)$ by the induction hypothesis. In order to establish that $M'_1, \frac{v}{x} \models \chi_\tau^+(x) \wedge \chi_\tau^-(x)$, let $\tau \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$, where $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$, $k \in \{1, ..., m-1\}$ and $\sigma_i \in \mathrm{TP}_\psi^n$ for all $i \in \{1, ..., k\}$. Therefore $(M'_h, v) \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$. Since by the induction hypothesis we have $||\sigma_i||^{M'_h} = P_{\sigma_i}^{M'_1}$ for all $i \in \{1, ..., k\}$, we conclude by Lemma 3.12 that

$$M'_1, \frac{v}{x} \models \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k)\big).$$

Similarly, if $\tau \Vdash \neg\langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$, then we have

$$M'_1, \frac{v}{x} \models \neg\exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k)\big)$$

by the induction hypothesis and Lemma 3.12. Thus $M'_1, \frac{v}{x} \models \chi_\tau^+(x) \wedge \chi_\tau^-(x)$, and hence $M'_1, \frac{v}{x} \models P_\tau(x)$, as desired.

For the converse, assume that $M'_1, \frac{v}{x} \models P_\tau(x)$, where $\tau \in \mathrm{TP}_\psi^{n+1}$. Now, since $M'_1 \models \forall x(P_\tau(x) \leftrightarrow \chi_\tau(x))$, we conclude that $M'_1, \frac{v}{x} \models \chi_\tau(x)$. Therefore $M'_1, \frac{v}{x} \models P_{\tau'}(x)$, where τ' is the type of the modal depth n such that $\tau \Vdash \tau'$. Thus $(M'_h, v) \Vdash \tau'$ by the induction hypothesis.

Let $k \in \{1, ..., m-1\}$ and $\mathcal{M} \in \mathrm{ATP}_\psi(k+1)$. Assume that we have $\tau \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$ for some $\sigma_1, ..., \sigma_k \in \mathrm{TP}_\psi^n$. As $M'_1, \frac{v}{x} \models \chi_\tau(x)$, we have $M'_1, \frac{v}{x} \models \chi_\tau^+(x)$, and therefore

$$M'_1, \frac{v}{x} \models \exists y_1 ... y_k \big(Access_{\mathcal{M}}(x, y_1, ..., y_k) \wedge P_{\sigma_1}(y_1) \wedge ... \wedge P_{\sigma_k}(y_k)\big).$$

Hence, as we have $||\sigma_i||^{M'_h} = P_{\sigma_i}^{M'_1}$ for all $i \in \{1, ..., k\}$ by the induction hypothesis, we conclude that $(M'_h, v) \Vdash \langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$ by Lemma 3.12. Similarly, if $\tau \Vdash \neg\langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$, we conclude that $(M'_h, v) \Vdash \neg\langle \mathcal{M} \rangle(\sigma_1, ..., \sigma_k)$

by the induction hypothesis and Lemma 3.12. We have therefore established that $(M_h', v) \Vdash \tau$, as required. $\square$

We then conclude the proof of the claim that $M, \frac{w}{x} \models \varphi^*(x)$ if and only if $(M, w) \Vdash \varphi$.

Lemma 3.14. *Under the assumption $M, \frac{w}{x} \models \varphi^*(x)$, we have $(M, w) \Vdash \varphi$.*

Proof. We have assumed that $M, \frac{w}{x} \models \varphi^*(x)$ and thereby concluded that there exists a model M_1' such that $M_1', \frac{w}{x} \models \psi^*(x)$. We have then defined the model M_h', and we now establish the claim of current the lemma by showing that $(M_h', w) \Vdash \psi$.

As $M_1', \frac{w}{x} \models \psi^*(x)$, we have $M_1', \frac{w}{x} \models P_\alpha(x)$ for some type $\alpha \in \mathrm{TP}^{Md(\psi)}$ such that $\alpha \Vdash \psi$. Therefore $(M_h', w) \Vdash \alpha$ by Lemma 3.13. As $\alpha \Vdash \psi$, we have $(M_h', w) \Vdash \psi$, as desired. $\square$

The following theorem now follows directly by virtue of Lemmata 3.10 and 3.14.

Theorem 3.15. *Each formula of $\Sigma_1^1(\mathrm{PBML}^=)$ translates to a uniformly equivalent formula of $\exists\mathrm{MSO}$. The translation is effective.*

The following corollary is immediate.

Corollary 3.16. *Each formula of $\Pi_1^1(\mathrm{PBML}^=)$ translates to a uniformly equivalent formula of $\forall\mathrm{MSO}$. The translation is effective.*

Theorem 3.15 implies a range of decidability results.

Theorem 3.17. *Let V and $U \subseteq V$ be sets of indices. Let $\mathcal{D}$ be a class of Kripke frames $(W, \{R_j\}_{j\in U})$. Consider the class*

$$\mathcal{C} = \{\ (W, \{R_i\}_{i\in V}) \mid R_i \subseteq W \times W,\ (W, \{R_j\}_{j\in U}) \in \mathcal{D}\ \}$$

of Kripke frames. Now, if the $\forall\mathrm{MSO}$ theory of $\mathcal{D}$ is decidable, then the satisfiability problem for $\mathrm{BML}^=$ w.r.t. $\mathcal{C}$ is decidable.

Proof. Given a formula ψ of $\mathrm{BML}^=$, we existentially quantify all the relation symbols (unary and binary) occurring ψ, except for those in $\{R_j\}_{j\in U}$. We end up with a $\Sigma_1^1(\mathrm{BML}^=)$ formula φ, which we then effectively translate to a uniformly equivalent $\exists\mathrm{MSO}$ formula $\varphi^*(x)$, applying our result. We then modify this formula to an $\exists\mathrm{MSO}$ *sentence* χ, which is uniformly equivalent to the sentence $\exists x\, \varphi^*(x)$. Let χ' denote a sentence of $\forall\mathrm{MSO}$ uniformly equivalent to $\neg\chi$. Using the decision procedure for the $\forall\mathrm{MSO}$ theory of $\mathcal{D}$, we then check whether the sentence χ' is valid over $\mathcal{D}$. If it is, then ψ is not satisfiable w.r.t. $\mathcal{C}$, and if χ' is not valid over $\mathcal{D}$, then ψ is satisfiable w.r.t. $\mathcal{C}$. $\square$

3.5 Chapter Conclusion

In this chapter we have investigated the expressive power of modal logics with existential prenex quantification of accessibility relations. We have shown that $\Sigma_1^1(\mathrm{PBML}^=)$ translates into $\exists$MSO, and also that $\Sigma_1^1(\mathrm{ML})$ translates into monadic $\Sigma_1^1(\mathrm{MLE})$. These results directly imply that $\Pi_1^1(\mathrm{PBML}^=)$ translates into $\forall$MSO and $\Pi_1^1(\mathrm{ML})$ into monadic $\Pi_1^1(\mathrm{MLE})$. As corollaries of the translations, we have obtained results that can be used in order to establish decidability results for (extensions of) multimodal logics with respect to classes of frames with built-in relations.

In the future we expect to strengthen the obtained results. The main objective is to try to understand for what kinds fragments L of first-order logic the system $\Sigma_1^1(\mathrm{L})$ collapses into $\exists$MSO. The next planned step involves considering graded (polyadic) modalities. While directly interesting, investigations along these kinds of lines could elucidate the role the *arities* of existentially quantified relations play in making the expressive power of (existential) second-order logic.

It also remains to be seen whether our investigations provide a stepping stone towards answering the question about existence of a class of finite directed graphs definable in $\Sigma_1^1(\mathrm{FO}^2)$ but not definable in $\exists$MSO. To show that $\Sigma_1^1(\mathrm{FO}^2)$ is contained in $\exists$MSO, one would have to extend the translation from $\Sigma_1^1(\mathrm{BML}^=)$ into $\exists$MSO such that it takes into account the possibility of using the converse operation.

CHAPTER 4

Expressivity of Equality-Free Existential Second-Order Logic with Function Quantification

Let fESO denote the version of existential second-order logic where formulae consist of a vector of existentially quantified *function* symbols followed by a first-order part. In this chapter we investigate fESO in the equality-free setting, concentrating on questions related to expressive power. Let $\mathrm{fESO}_{wo=}$ denote the fragment of fESO where formulae are required to be equality-free. Various natural equality-free fragments of logics in the family of *independence-friendly* (IF) *logic* translate into $\mathrm{fESO}_{wo=}$ via appropriate Skolemization procedures, so insights concerning $\mathrm{fESO}_{wo=}$ can be fruitful in the study of such fragments. In particular, we believe that $\mathrm{fESO}_{wo=}$ can be more or less directly useful in investigations related to *independence-friendly modal logics* of Tulenheimo and Sevenster and others. We consider a range of questions concerning the expressivity of $\mathrm{fESO}_{wo=}$ over models with a relational vocabulary. For instance, we identify a model transformation which preserves truth of $\mathrm{fESO}_{wo=}$ formulae, thereby enabling an easy access to inexpressibility results. Our principal result—from the technical point of view—is that over finite models with a vocabulary consisting of unary relation symbols only, the fragment $\mathrm{fESO}_{wo=}$ of second-order logic is strictly less expressive than first-order logic (with equality).

4.1 Equality-Free Existential Second-order Logic with Function Quantification

The topic of this chapter is the expressivity of *equality-free existential second-order logic with function quantification*, or $\mathrm{fESO}_{wo=}$. In this system second-order quantifiers range over function symbols only. Insights about sentences of the equality-free logic $\mathrm{fESO}_{wo=}$ can be more or less directly useful for example in the study of the independence-friendly modal logics of Tulenheimo [64] and Tulenheimo and Sevenster [63] and others. Independence-friendly modal logic is part of the family of independence-friendly (IF) logics introduced by Hintikka and Sandu in [30]. See also [29] for an early exposition of the main ingredients leading to the idea of IF logic, and of course [26] for an even earlier discussion of ideas closely related to IF logic.

Results concerning $\mathrm{fESO}_{wo=}$ apply to a wide range of equality-free logics. For example, consider the delightfully exotic looking Henkin expressions of the type

$$\begin{pmatrix} \forall x_1 & \exists x_2 \\ \forall x_3 & \exists x_4 \end{pmatrix} \varphi(x_1, x_2, x_3, x_4),$$

where a finite partially ordered quantifier precedes an equality-free FO formula $\varphi(x_1, x_2, x_3, x_4)$ with exactly the variable symbols x_1, x_2, x_3, x_4 occurring free. By the original semantics of Henkin [26], each such expression is equivalent to a sentence of $\mathrm{fESO}_{wo=}$. Hence, whatever is inexpressible in $\mathrm{fESO}_{wo=}$, is automatically inexpressible with Henkin expressions of the above type. And, of course, results concerning $\mathrm{fESO}_{wo=}$ contribute to the general program of studying fragments of second-order logic.

The study of $\mathrm{fESO}_{wo=}$ presented in this chapter was originally motivated by questions related to IF logic and Henkin quantifiers, and an account of the answers to those questions can be found in [40] and [41]. The current chapter is very much based on the articles [40, 41], but the account given here discusses only the system $\mathrm{fESO}_{wo=}$. The logic $\mathrm{fESO}_{wo=}$ is worth studying in its own right, and furthermore, the reader is spared the trouble of getting acquainted with IF logic.

We begin our study by providing a very simple proof of the fact that $\mathrm{fESO}_{wo=}$ can define properties not definable in first-order logic FO (with equality), when the vocabulary under consideration contains a binary relation symbol. We then define a simple model transformation that preserves truth of $\mathrm{fESO}_{wo=}$ sentences, but not FO sentences. Therefore we observe that $\mathrm{fESO}_{wo=}$ and FO are incomparable with regard to expressive power. We then ask whether $\mathrm{fESO}_{wo=}$ and FO are also incomparable when attention is limited to nonempty vocabularies containing *only unary relation symbols.* Our principal result is that over finite models with such a vocabulary,

$$\mathrm{FO}_{wo=} < \mathrm{fESO}_{wo=} < \mathrm{FO},$$

where $\mathrm{FO}_{wo=}$ denotes first-order logic without equality. So far we have not succeeded in establishing these results without the use of somewhat involved combinatorial arguments. In addition to proving the results, we of course also wish to reflect upon and promote the proof techniques applied.

4.2 Preliminary Definitions

Let U be a vocabulary; U may contain relation symbols, function symbols and constant symbols. Recall that a U-formula of FO (or alternatively, an FO formula of the vocabulary U) is an FO formula such that the set of non-logical symbols that occur in the formula is subset of U. The equality symbol is *not* considered to be a non-logical symbol.

Let V be a vocabulary containing relation symbols only. Formulae of the vocabulary V (or V-formulae) of the logic fESO are exactly the expressions of the type $\overline{\exists f}\,\varphi$, where $\overline{\exists f}$ is a finite vector of existentially quantified function symbols and φ is an FO formula of the vocabulary

$$V \cup \{\, f \mid f \text{ occurs in } \overline{\exists f}\,\}.$$

The function symbols are allowed to be nullary, i.e., to be constant symbols. The formulae of fESO are interpreted according to the natural semantics. The set of V-formulae of the logic $\mathrm{fESO}_{wo=}$ is exactly the set of V-formulae of fESO without equality. The set of non-logical symbols of a V-formula ψ of fESO is the set $V' \subseteq V$ of symbols in V that occur in ψ.

Let V be a relational vocabulary. Recall that a V-model (or alternatively, a model of the vocabulary V) is a model such that the set of symbols interpreted by M is exactly the set V. In the current chapter a *finite V-model* is a model M of the vocabulary V such that $Dom(M)$ is finite; the vocabulary V may be infinite.

Let V be a relational vocabulary and C a class of V-models. Let φ and ψ be V-sentences of predicate logic, possibly extended with generalized quantifiers (see Section 4.5). The two sentences are *C-equivalent* if

$$M \models \varphi \;\Leftrightarrow\; M \models \psi$$

for all models $M \in C$.

Let V be a relational vocabulary and let C be a class of V-models. Let L and L' be two systems (i.e., logics) of predicate logic. Below when we say that $L \leq L'$ *over* C (or $L \leq L'$ *with respect to* C), we mean that for each V-sentence of L there exists a C-equivalent sentence of L'. We say that $L \nleq L'$ *over* C if it is not the case that $L \leq L'$ over C. We say that $L < L'$ *over* C if $L \leq L'$ over C and $L' \nleq L$ over C.

Two sentences χ and χ' of fESO are *uniformly equivalent* if they have exactly the same set S of non-logical symbols and if the sentences are C-equivalent, where C is the class of all S-models. Let φ' and ψ' be formulae of first-order logic, possibly extended with generalized quantifiers. We say that the formulae φ' and ψ' are *uniformly equivalent in the finite* if the following conditions are satisfied.

1. The formulae φ' and ψ' have exactly the same set T of free variable symbols.

2. The formulae φ' and ψ' have exactly the same set S of non-logical symbols.

3. We have
$$M, h \models \varphi' \;\Leftrightarrow\; M, h \models \psi'$$
for all finite S-models and all related variable assignments h interpreting the variable symbols in T in $Dom(M)$.

4.3 Expressivity of $\mathrm{fESO}_{wo=}$ over Models with a Relational Vocabulary

Let V be a relational vocabulary containing a binary relation symbol. We begin the section by providing a very simple proof of the fact that over the class of V-models, $\mathrm{fESO}_{wo=} \not\leq \mathrm{FO}$.

Proposition 4.1. *Let V be a relational vocabulary containing a binary relation symbol R. Then there is a class of V-models definable by a sentence of $\mathrm{fESO}_{wo=}$, but not definable by any sentence of FO. To witness this, a sentence of the form $\exists f \forall x\, \psi$, where f is a unary function symbol and ψ is quantifier-free, suffices.*

Proof. Consider the following sentence φ of fESO.

$$\exists f \forall x \Big(f\big(f(x)\big) = x \wedge\ f(x) \neq x \Big)$$

It is easy to see that this sentence is true in exactly those models whose domain has an even or an infinite cardinality.

Let φ' be the $\mathrm{fESO}_{wo=}$ sentence obtained from φ by replacing each instance of the identity symbol $=$ by the symbol R, i.e., φ' is the sentence

$$\exists f \forall x \Big(f(f(x))\, R\, x \ \wedge\ \ \neg\, f(x)\, R\, x \Big).$$

Let C be the class of finite V-models M such that

$$R^M = \{\ (a,a) \mid a \in Dom(M)\ \}.$$

It is clear that with respect to C, the sentence φ' defines the class C_{even} of models whose domain is even. A straightforward Ehrenfeucht-Fraïssé game argument shows that the class C_{even} is not definable with respect to C by any FO sentence.

Let D denote the class of all V-models. Since there is no FO sentence that defines w.r.t. C the same class of models as φ', there is no FO sentence that defines w.r.t. D exactly the same class of models as φ'. $\square$

4.3.1 Bloating Models

We now define a simple model transformation that preserves truth of $\mathrm{fESO}_{wo=}$ sentences.

Definition 4.2. Let V be a vocabulary such that each symbol in V is a relation symbol of the arity one or two. (We restrict attention to at most binary relation symbols for the sake of simplicity.) Let M be a V-model with the domain A, and let $a \in A$. Let S be any set such that $S \cap A = \emptyset$. Define a V-model N as follows.

1. The domain of N is the set $A \cup S$.
2. Let $P \in V$ be a unary relation symbol. We define P^N as follows.
   (a) For all $v \in A$, $v \in P^N$ iff $v \in P^M$.
   (b) For all $s \in S$, $s \in P^N$ iff $a \in P^M$.
3. Let $R \in V$ be a binary relation symbol. We define R^N as follows.
   (a) For all $\overline{v} \in A \times A$, $\overline{v} \in R^N$ iff $\overline{v} \in R^M$.
   (b) For all $s \in S$ and all $v \in A$, $(v, s) \in R^N$ iff $(v, a) \in R^M$.
   (c) For all $s \in S$ and all $v \in A$, $(s, v) \in R^N$ iff $(a, v) \in R^M$.
   (d) For all $s, s' \in S$, $(s, s') \in R^N$ iff $(a, a) \in R^M$.

We call the model N a *bloating* of M. Figure 2 illustrates how this model transformation affects models.

We note that the notion of a bloating is closely related for example to the notion of a *surjective strict homomorphism* (see Definition 2.1 of [10]).

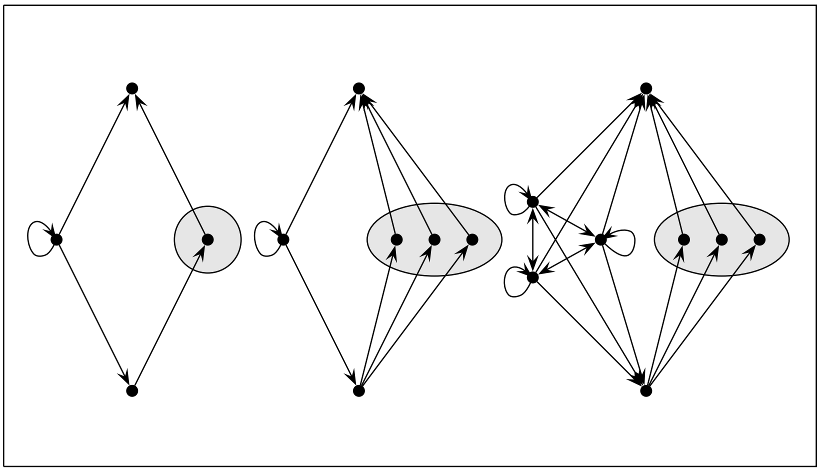

Figure 2: The figure shows three connected models of a vocabulary consisting of one binary and one unary relation symbol. The shaded areas correspond to the extensions of the unary relation symbol. The model in the middle is a bloating of the model on the left. The model in the middle is obtained from the one on the left by adding two new copies of the middle right element. The model on the right is a bloating of the model in the middle, obtained by adding two copies of the middle left element.

Theorem 4.3. *Let V be a vocabulary such that each symbol in V is a relation symbol of the arity one or two. Truth of any V-sentence of* $\mathrm{fESO}_{wo=}$ *is preserved under bloatings.*

Proof. Let φ be a V-sentence of $\mathrm{fESO}_{wo=}$. The sentence φ can be transformed into a uniformly equivalent $\mathrm{fESO}_{wo=}$ sentence $\overline{\exists f}\,\psi$, where $\overline{\exists f}$ is a finite string of existentially quantified function symbols (some of them perhaps nullary) and ψ is a first-order sentence such that the following conditions hold.

1. The sentence ψ is of the type $\overline{\forall x}\,\psi'$, where $\overline{\forall x}$ is a string of universal first-order quantifiers and ψ' is a quantifier-free formula.

2. The quantifier free part ψ' of the sentence ψ is in negation normal form, i.e., negations occur only in front of atomic formulae.

This normal form can be obtained by first transforming the first-order part of φ into prenex normal form without nested quantification of the same variable, and then Skolemizing the first-order part of the resulting sentence. The quantifier-free part of the resulting sentence can then be put into negation normal form. The freshly introduced Skolem functions are prenex quantified existentially, so the set of non-logical symbols of $\overline{\exists f}\,\psi$ is the same as that of the sentence φ. The process of transforming φ into the described normal form does not introduce equality symbols, so $\overline{\exists f}\,\psi$ is a sentence of $\mathrm{fESO}_{wo=}$.

Let M and N be as in Definition 4.2. The models there had the domains A and $A \cup S$, respectively, and we used the element $a \in A$ in order to define N. Assume that $M \models \varphi$. Therefore $M \models \overline{\exists f}\,\psi$. We expand M to a model $M' = (M, \overline{f^{M'}})$ such that $M' \models \psi$. We then expand N to a model $N' = (N, \overline{f^{N'}})$ as follows.

1. For each k-ary symbol f, we let $f^{N'} \upharpoonright A^k = f^{M'}$. Note that when $k = 0$, i.e., when f is a constant symbol, then $f^{N'} = f^{M'}$.

2. For each k-tuple $\overline{w} \in (A \cup S)^k$ containing points from the set S, we define the k-tuple $\overline{w}'$, where every co-ordinate value $s \in S$ of $\overline{w}$ is replaced by the element a. We then set $f^{N'}(\overline{w}) = f^{M'}(\overline{w}')$.

We then establish that $N' \models \psi$. The proof is a simple induction on the structure of the formula ψ. For each variable assignment h with codomain A, we let $g(h)$ denote the set of all variable assignments with codomain $A \cup S$ that can be obtained from h by allowing some subset of the variables mapping to the element a to map to elements in S. We will show that for every variable assignment h with codomain A and every subformula χ of ψ,

$$M', h \models \chi \quad \Rightarrow \quad \forall h' \in g(h)\big(N', h' \models \chi\big).$$

The cases for atomic and negated atomic formulae form the basis of the induction. The claim for these formulae follows directly with the help of the observation that $h(t) = h'(t)$ for all h and $h' \in g(h)$ and terms t that *contain function or constant symbols*, i.e., terms that are not variable symbols. We will next prove this claim by induction on the function symbol nesting depth of terms.

The basis of the induction deals with the terms whose nesting depth is one, i.e., terms of the type $f(x_1, ..., x_k)$ and c, where the symbols $x_1, ..., x_k$ are variable symbols and the symbol c is a constant symbol. It is immediate that $h(t) = h'(t)$ for all h and $h' \in g(h)$ and all such terms t of the nesting depth one.

Now let $f(t_1, ..., t_k)$ be a term of the nesting depth $n + 1$. By the induction hypothesis, we have $h(t_i) = h'(t_i)$ for each one of the terms t_i that is not a variable symbol. For the terms t_i that are variable symbols and for which $h(t_i) \neq a$, we also have $h(t_i) = h'(t_i)$. For the terms t_i that are variable symbols and for which $h(t_i) = a$, either $h'(t_i) = a$ or $h'(t_i) \in S$. We therefore observe that we obtain the tuple $(h(t_1), ..., h(t_k))$ from the tuple $(h'(t_1), ..., h'(t_k))$ by replacing the elements $u \in S$ of the tuple $((h'(t_1), ..., h'(t_k))$ by the element a. Therefore we infer, by the definition of the function $f^{N'}$, that

$$f^{N'}\big(h'(t_1), ..., h'(t_k)\big) \;=\; f^{M'}\big(h(t_1), ..., h(t_k)\big).$$

This concludes the induction on terms and thus the basis of the original induction on the structure of ψ has now been established. We return to the original induction.

The cases for connectives are trivial, and the quantifier case is relatively straightforward. We discuss the details of the quantifier case here.

Assume that $M', h \models \forall x\, \alpha(x)$. We need to show that for all $h' \in g(h)$, we have $N', h' \models \forall x\, \alpha(x)$. Assume, for the sake of contradiction, that for some $h'' \in g(h)$, we have $N', h'' \not\models \forall x\, \alpha(x)$. Hence $N', h''\frac{u}{x} \not\models \alpha(x)$ for some $u \in A \cup S$. To finish the proof, it suffices to show that $h''\frac{u}{x} \in g(h\frac{v}{x})$ for some $v \in A$. This suffices, as the assumption $M', h \models \forall x\, \alpha(x)$ first implies that $M', h\frac{v}{x} \models \alpha(x)$, which in turn then implies, by the induction hypothesis, that $N', h''\frac{u}{x} \models \alpha(x)$.

If $u \in A$, let $v = u$. Then, as $h'' \in g(h)$, we have $h''\frac{u}{x} = h''\frac{v}{x} \in g(h\frac{v}{x})$. If $u \in S$, let $v = a$. Then, as $h'' \in g(h)$, we have $h''\frac{u}{x} \in g(h\frac{a}{x}) = g(h\frac{v}{x})$. $\square$

An immediate consequence of Theorem 4.3 is that $\mathrm{FO} \not\leq \mathrm{fESO}_{wo=}$ over V-models because there exist first-order $\emptyset$-sentences (with equality) whose truth is not preserved under bloating.

4.4 Expressivity of $\mathrm{fESO}_{wo=}$ over Finite Models with a Unary Relational Vocabulary

We now turn our attention to finite models with a unary relational vocabulary. Over such finite models, the picture is quite different from the case where there is a binary relation symbol in the vocabulary. We will establish that over the class of finite V-models, where V is an arbitrary nonempty vocabulary containing only unary relation symbols, we have

$$\mathrm{FO}_{wo=} \; < \; \mathrm{fESO}_{wo=} \; < \; \mathrm{FO}.$$

We first show that $\mathrm{fESO}_{wo=} < \mathrm{FO}$, and then that $\mathrm{FO}_{wo=} < \mathrm{fESO}_{wo=}$.

4.4.1 $\mathrm{fESO}_{wo=} < \mathrm{FO}$ over Finite Models with a Unary Relational Vocabulary

Let V be an arbitrary nonempty vocabulary containing only unary relation symbols. In the current subsection we show that $\mathrm{fESO}_{wo=} < \mathrm{FO}$ over the class of finite V-models.[5] We begin by making a number of auxiliary definitions.

Let $U \subseteq V$ be a finite unary vocabulary. A *unary U-type* (with the free variable x) is a conjunction τ with $|U|$ conjuncts such that for each symbol $P \in U$, exactly one of the formulae $P(x)$ and $\neg P(x)$ is a conjunct of τ; if $U = \emptyset$, then τ is the formula $x = x$. Let $T = \{\, \tau_1, ..., \tau_{|T|} \,\}$ be the set of unary U-types.[6] The domain of each U-model M is partitioned into some number $n \leq |T|$ of sets S_i such that the elements of S_i *realize*, i.e., satisfy, the type $\tau_i \in T$. An element $a \in Dom(M)$ realizes (satisfies) the type τ_i if and only if $M, \frac{a}{x} \models \tau_i$.

Let $n \in \mathbb{N}$, and let $k = 2^n$. Any relation

$$R \;\subseteq\; \mathbb{N}^k \setminus \{0\}^k$$

is a *spectrum*. We associate sentences of FO and $\mathrm{fESO}_{wo=}$ with spectra in a way specified in the following definition.

Definition 4.4. Let φ be a V-sentence of FO or $\mathrm{fESO}_{wo=}$. Let $U \subseteq V$ be the finite set of unary relation symbols that occur in the sentence φ. Let $T = \{\, \tau_1, ..., \tau_{|T|} \,\}$ be the finite set of unary U-types. Let $\leq^T$ denote a linear ordering of the types in T defined such that $\tau_i \leq^T \tau_j$ iff $i \leq j$. Define the relation $R_\varphi \subseteq \mathbb{N}^{|T|}$ such that $(n_1, ..., n_{|T|}) \in R_\varphi$ iff there exists a finite U-model M of φ such that for all $i \in \{1, ..., |T|\}$, the number of points in the domain of M that satisfy τ_i is n_i. We call such a relation R_φ the *spectrum of* φ (with respect to the order $\leq^T$).

Notice that the class of V-models that the sentence φ defines in the finite is completely characterized by the spectrum $R_\varphi \subseteq \mathbb{N}^{|T|}$; there is a canonical one-to-one correspondence between the isomorphism classes of finite

[5] In the case $V = \emptyset$, we trivially have $\mathrm{fESO}_{wo=} < \mathrm{FO}$, since we define $\mathrm{fESO}_{wo=}$ such that there do not exists $\mathrm{fESO}_{wo=}$ formulae of the vocabulary $\emptyset$ at all.

[6] We assume that types have some standard ordering of conjuncts and bracketing, so that there exist exactly $2^{|U|}$ different unary U-types; for each subset S of U, there is exactly one unary U-type τ such that for each symbol $P \in U$, $P(x)$ is a conjunct of τ iff $P \in S$.

U-models that satisfy φ and tuples $\overline{r} \in R_{\varphi}$. See Figure 3 for an illustration of a spectrum of an FO sentence with a unary relational vocabulary.

We now define a special family of spectra and then establish that this family exactly characterizes the expressivity of FO over the class of (finite) V-models.

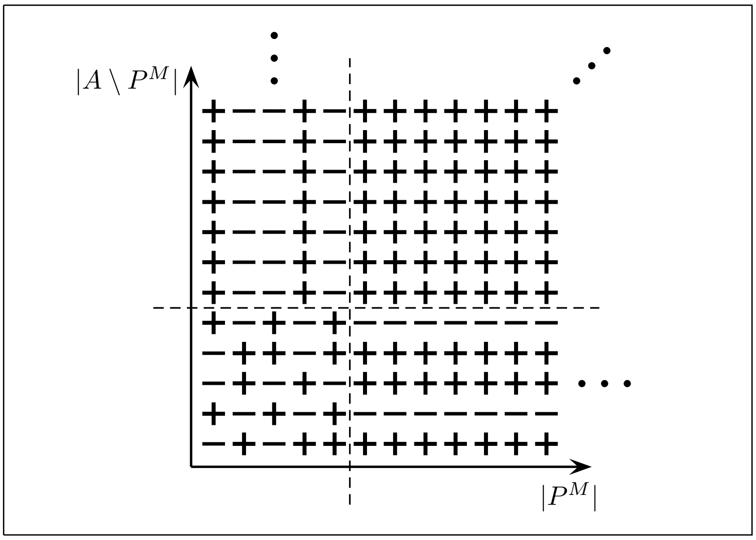

Figure 3: The figure illustrates a stabilizing spectrum (see Definition 4.5) that corresponds to some FO sentence ψ whose set of non-logical symbols is $\{P\}$, where P is a unary relation symbol. A plus symbol occurs at the position (i, j) iff there exists a $\{P\}$-model M satisfying the sentence ψ such that $|P^M| = i$ and $|A \setminus P^M| = j$, where $A = Dom(M)$. In other words, the number of points in the domain of M satisfying the type $P(x)$ is i, and the number of points satisfying the type $\neg P(x)$ is j. If φ is an FO sentence with the set $\{P\}$ of non-logical symbols, the spectrum of φ divides the xy-plane into four distinct regions. The upper right region always contains either plus symbols only or minus symbols only. In the top left region, any distribution of plus and minus symbols is possible in the horizontal direction, but in the vertical direction the distribution is uniform. The bottom right region is similar to the top left region, but with an arbitrary distribution in the vertical direction and a uniform distribution in the horizontal direction. In the bottom left region, any distribution is possible. (The point $(0, 0)$ always contains a minus symbol, however, since we do not allow for models to have an empty domain.)

Definition 4.5. Let $l = 2^{l'}$ for some $l' \in \mathbb{N}$. Let $R \subseteq \mathbb{N}^l$ be a spectrum for which there exists a number $n \in \mathbb{N}_{\geq 1}$ such that for all co-ordinate positions

$i \in \{1, ..., l\}$, all integers $k, k' > n$ and all $m_1, ..., m_{i-1}, m_{i+1}, ..., m_l \in \mathbb{N}$,

$$\begin{array}{cl} & (m_1, ..., m_{i-1}, k, m_{i+1}, ..., m_l) \in R \\ \Leftrightarrow & \\ & (m_1, ..., m_{i-1}, k', m_{i+1}, ..., m_l) \in R. \end{array}$$

Such a number n is a *stabilizer* of the spectrum R. A spectrum with a stabilizer is a *stabilizing spectrum.*

Proposition 4.6. *A spectrum R is a stabilizing spectrum if and only if R is the spectrum of some* FO *sentence.*

Proof. Assume that $R \subseteq \mathbb{N}^k$ is a stabilizing spectrum. Let $n \in \mathbb{N}_{\geq 1}$ be a stabilizer of R. Define the set $S = \{0, 1, ..., n\} \cup \{\infty\}$, where ∞ is simply a symbol. Define a function $f : \mathbb{N} \longrightarrow S$ as follows.

$$f(x) = \begin{cases} x & \text{if } x \leq n, \\ \infty & \text{if } x > n. \end{cases}$$

Define

$$R_0 = \{ \ \big(f(r_1), ..., f(r_k)\big) \mid (r_1, ..., r_k) \in R \ \}.$$

Notice that the set R_0 is finite.

Let $(s_1, ..., s_k) \in R_0$. For each $i \leq k$, define a first-order sentence χ_i such that the following conditions hold.

1. If $s_i \leq n$, then χ_i asserts that there are exactly s_i elements that satisfy the type τ_i.

2. If $s_i = \infty$, then χ_i asserts that there are at least $n + 1$ elements that satisfy the type τ_i.

Let $\psi_{(s_1,...,s_k)}$ be a conjunction of the sentences χ_i. Let φ_R be a disjunction of the sentences $\psi_{(s_1,...,s_k)}$, where $(s_1, ..., s_k) \in R_0$. The set R_0 is finite, so the disjunction is a first-order sentence. Since R is a stabilizing spectrum with a stabilizer n, we observe that the disjunction φ_R defines the spectrum R, i.e., R is the spectrum of the first-order sentence φ_R.

The fact that each spectrum of an FO sentence is stabilizing is established by a straightforward Ehrenfeucht-Fraïssé game argument. □

We next define some order theoretic notions and then prove a number of order theoretic results that are needed in the proof of the main theorem (Theorem 4.10) of the current section.

A structure $M = (A, \leq^M)$ is a *partial order* if $\leq^M \subseteq A \times A$ is a reflexive, transitive and antisymmetric binary relation. Given a partial order $M = (A, \leq^M)$, we let $<^M$ denote the irreflexive version of the order $\leq^M$. A partial order is said to be *well-founded* if no strictly decreasing infinite sequence occurs in it. That is, a partial order $M = (A, \leq^M)$ is well-founded if for each each sequence $f : \mathbb{N} \longrightarrow A$ there exist numbers $i, j \in \mathbb{N}$ such that $i < j$ and $f(j) \not<^M f(i)$. An *antichain* $S \subseteq A$ of a partial order $M = (A, \leq^M)$ is a set such that for all distinct elements $s, t \in S$, we have $s \not\leq^M t$ and $t \not\leq^M s$. In other words, the distinct elements s and t are incomparable. A well-founded partial order that does not contain an infinite antichain is a *partial well-order*, or a *pwo*.

Let $M = (A, \leq^M)$ and $N = (B, \leq^N)$ be partial orders. The *Cartesian product* $M \times N$ of the structures M and N is the partial order defined as follows.

1. The domain of $M \times N$ is the Cartesian product $A \times B$.
2. The binary relation $\leq^{M \times N} \subseteq (A \times B) \times (A \times B)$ is defined in a pointwise fashion as follows.

$$(a, b) \leq^{M \times N} (a', b') \quad \Leftrightarrow \quad \Big(a \leq^M a' \text{ and } b \leq^N b'\Big)$$

For each integer $k \in \mathbb{N}_{\geq 1}$ and each partial order $M = (A, \leq^M)$, we let $M^k = (A^k, \leq^{M^k})$ denote the partial order where the relation $\leq^{M^k} \subseteq A^k \times A^k$ is again defined in the pointwise fashion as follows.

$$(a_1, ..., a_k) \leq^{M^k} (a'_1, ..., a'_k) \quad \Leftrightarrow \quad \forall i \in \{1, ..., k\} : a_i \leq^M a'_i$$

The structure M^k is called the *k-th Cartesian power* of M. We let $(\mathbb{N}^k, \leq)$ denote the k-th Cartesian power of the linear order $(\mathbb{N}, \leq)$. When $S \subseteq \mathbb{N}^k$, we let $(S, \leq)$ denote the partial order with the domain S and with the ordering relation inherited from $(\mathbb{N}^k, \leq)$. In other words, for all $\overline{s}, \overline{t} \in S$, we have $\overline{s} \leq^{(S,\leq)} \overline{t}$ if and only if $\overline{s} \leq^{(\mathbb{N}^k,\leq)} \overline{t}$. When $\overline{u}, \overline{v} \in \mathbb{N}^k$, we simply write $\overline{u} \leq \overline{v}$ in order to assert that $\overline{u} \leq^{(\mathbb{N}^k,\leq)} \overline{v}$.

The following lemma is a paraphrase of Lemma 5 of the article [48], where the lemma is credited to Higman [28].

Lemma 4.7. *The Cartesian product of any two partial well orders is a partial well order.*

Variants of the following lemma are often attributed to Dickson [14]. The lemma follows directly from Lemma 4.7 above.

Lemma 4.8. (Dickson's Lemma variant) *Let $k \in \mathbb{N}_{\geq 1}$. The structure $(\mathbb{N}^k, \leq)$ does not contain an infinite antichain.*

Proof. The structure $(\mathbb{N}, \leq)$ is a pwo, and by Lemma 4.7, the property of being a pwo is preserved under taking finite Cartesian products. Thus the structure $(\mathbb{N}^k, \leq)$ is a pwo. By definition, a pwo does not contain infinite antichains. $\square$

Let $l \in \mathbb{N}_{\geq 1}$ and let $R \subseteq \mathbb{N}^l$ be a relation such that for all tuples $\overline{u}, \overline{v} \in \mathbb{N}^l$, if $\overline{u} \in R$ and $\overline{u} \leq \overline{v}$, then $\overline{v} \in R$. We say that the relation R is *upwards closed* with respect to $(\mathbb{N}^l, \leq)$. When the exponent l is known from the context or irrelevant, we simply say that the relation R is upwards closed.

Theorem 4.9. *If R is a spectrum that is upwards closed, then it is a stabilizing spectrum.*

Proof. Let $l' \in \mathbb{N}$ and $l = 2^{l'}$. Assume that $R \subseteq \mathbb{N}^l$ is a spectrum that is upwards closed with respect to $(\mathbb{N}^l, \leq)$. We shall establish that R is stabilizing. As $\emptyset$ is trivially a stabilizing spectrum, we may assume that $R \neq \emptyset$.

We begin the proof by defining a function f that maps each nonempty subset of the set $\{1, ..., l\}$ to a natural number. Let $I \subseteq \{1, ..., l\}$ be a nonempty set. Let $R(I)$ denote the set consisting of exactly those tuples $\overline{w} \in R$ that have a non-zero co-ordinate value at each co-ordinate position $i \in I$ and a zero co-ordinate value at each co-ordinate position $j \in \{1, ..., l\} \setminus I$. Define the value $f(I) \in \mathbb{N}$ as follows.

1. If $R(I) = \emptyset$, let $f(I) = 0$.

2. If $R(I) \neq \emptyset$, choose some tuple $\overline{w} \in R(I)$. Let $W \subseteq R(I)$ be a maximal antichain of $(R(I), \leq)$ such that $\overline{w} \in W$, i.e., let W be an antichain of $(R(I), \leq)$ such that for all $\overline{u} \in R(I) \setminus W$, there exists some $\overline{v} \in W$ such that $\overline{u} < \overline{v}$ or $\overline{v} < \overline{u}$. By Lemma 4.8, the set W is finite. Therefore there exists a maximum co-ordinate value occurring in the tuples in W. Let $f(I)$ to be equal to this value.

(Notice that we have some freedom of *choice* when defining the function f, so there need not be a unique way of defining f.)

With the function f defined, call

$$n \;=\; max\big(\{\, f(I) \mid \emptyset \neq I \subseteq \{1, ..., l\}\,\}\big).$$

We will establish that n is a stabilizer of the relation R. We assume, for the sake of contradiction, that there exist integers $k, k' > n$ and also integers $m_1, ..., m_{i-1}, m_{i+1}, ..., m_l \in \mathbb{N}$ such that the equivalence

$$\begin{gathered}(m_1, ..., m_{i-1}, k, m_{i+1}, ..., m_l) \in R \\ \Leftrightarrow \\ (m_1, ..., m_{i-1}, k', m_{i+1}, ..., m_l) \in R\end{gathered}$$

does not hold. Let $k < k'$. As the relation R is upwards closed, it must be the case that

$$\begin{gathered}(m_1, ..., m_{i-1}, k, m_{i+1}, ..., m_l) \notin R \\ \text{and} \\ (m_1, ..., m_{i-1}, k', m_{i+1}, ..., m_l) \in R.\end{gathered}$$

Otherwise we would immediately end up with a contradiction. Call

$$\begin{aligned} \overline{w}_k &= (m_1, ..., m_{i-1}, k, m_{i+1}, ..., m_l) \\ \text{and} \quad & \\ \overline{w}_{k'} &= (m_1, ..., m_{i-1}, k', m_{i+1}, ..., m_l). \end{aligned}$$

Let $I^* \subseteq \{1, ..., l\}$ be the set of co-ordinate positions where the tuple $\overline{w}_{k'}$ (and therefore also the tuple $\overline{w}_k$) has a non-zero co-ordinate value. Let $W(I^*)$ denote the domain of the maximal antichain of $(R(I^*), \leq)$ chosen when defining the value of the function f on the input I^*. The tuple $\overline{w}_{k'}$ cannot be in the set $W(I^*)$, since the co-ordinate value k' is greater than n, and therefore greater than any of the co-ordinate values of the tuples in $W(I^*)$. Hence, as $W(I^*)$ is a maximal antichain of $(R(I^*), \leq)$ and $\overline{w}_{k'} \in R(I^*)$, there exists a tuple $\overline{u} \in W(I^*)$ such that $\overline{w}_{k'} < \overline{u}$ or $\overline{u} < \overline{w}_{k'}$. Since $k' > f(I^*)$, we must have $\overline{u} < \overline{w}_{k'}$. Therefore, as also $k > f(I^*)$, we conclude that $\overline{u} < \overline{w}_k$. As R is upwards closed and $\overline{u} \in R$, we have $\overline{w}_k \in R$. This is a contradiction. □

The following theorem is the main result of the current section.

Theorem 4.10. *Let V be a vocabulary such that each symbol in V is a unary relation symbol. We have* $\mathrm{fESO}_{wo=} < \mathrm{FO}$ *over finite V-models.*

Proof. By Theorem 4.3 it is immediate that $\mathrm{FO} \not\leq \mathrm{fESO}_{wo=}$ over finite V-models. Therefore it suffices to show that $\mathrm{fESO}_{wo=} \leq \mathrm{FO}$ over finite V-models. To show this, let φ be an arbitrary V-sentence of $\mathrm{fESO}_{wo=}$. By Proposition 4.6 it suffices to establish that the spectrum R_φ of the sentence φ is stabilizing. By Theorem 4.3, the spectrum R_φ is upwards closed. Hence, by Theorem 4.9, R_φ is a stabilizing spectrum. □

Note that if the vocabulary V under consideration is finite, then Theorem 4.10 applies not only to $\mathrm{fESO}_{wo=}$ but to any logic[7] such that the definable classes of models with a unary relational vocabulary are closed under bloating. Here the restriction to models with a finite domain is required. Too see why, let L be a logic whose language consists of exactly one formula, ψ. Let the semantics of L dictate that the formula ψ is true in a model M if and only if the domain of the model M is infinite. Then truth of L formulae is preserved under bloatings, but FO and L are incomparable with regard to expressivity. Note also that our proof is nonconstructive in the sense that without additional information, the current formulation of the argument leaves it a conceivable possibility that there does not exist an *effective translation* from the system L considered into FO.

[7]The term "logic" can *here* be identified with the compound expression "class of isomorphically closed classes of V-models".

4.4.2 $\mathrm{FO}_{wo=} < \mathrm{fESO}_{wo=}$ over Finite Models with a Unary Relational Vocabulary

In this subsection we show that over the class of finite V-models, where V is a nonempty vocabulary containing only unary relation symbols, we have

$$\mathrm{FO}_{wo=} < \mathrm{fESO}_{wo=}\,.$$

Let $P \in V$ and consider the $\mathrm{fESO}_{wo=}$ sentence

$$\exists f\, \exists g\, \forall x \Big(\ P(f(x))\ \wedge\ \big(\ P(x)\ \leftrightarrow\ P(g(f(x)))\ \big)\ \Big).$$

The sentence is true in a V-model M with three points, two of which satisfy $P(x)$. The sentence is not true in a V-model N with two points, one satisfying $P(x)$ and one not. However, we will show that there exists no $\mathrm{FO}_{wo=}$ sentence φ of the vocabulary V such that exactly one of the models M and N satisfies φ. We establish this by applying a *very* simple back and forth argument. In the article [10], a characterization of the expressivity of $\mathrm{FO}_{wo=}$ is formulated in terms back and forth systems. We show that M and N satisfy exactly the same $\mathrm{FO}_{wo=}$ sentences by employing the tools defined in [10].

Definition 4.11. (cf. Definition 4.1 of [10].) Let M and N be U-models, where U contains relation symbols only. A relation

$$p\ \subseteq\ Dom(M) \times Dom(N)$$

is a *partial relativeness correspondence* if for any n-ary relation symbol $R \in U$ and any $(a_1, b_1), ..., (a_n, b_n) \in p$,

$$(a_1, ..., a_n) \in R^M\ \Leftrightarrow\ (b_1, ..., b_n) \in R^N.$$

Definition 4.12. (cf. Definition 4.2 of [10].) Let M and N be U-models, where U contains relation symbols only. Let $A = Dom(M)$ and $B = Dom(N)$. We write $M \sim_n N$, where $n \in \mathbb{N}_{\geq 1}$, if there exists a sequence $(I_k)_{k \in \{0,1,\dots,n\}}$ of sets I_k of partial relativeness correspondences $p \subseteq A \times B$ such that the following conditions hold.

1. Every I_k is a nonempty set of partial relativeness correspondences.
2. For each $i \in \{1, ..., n\}$, each $p \in I_i$ and each $a \in A$, there exists a $q \in I_{i-1}$ such that $p \subseteq q$ and $a \in Dom(q)$.
3. For each $i \in \{1, ..., n\}$, each $p \in I_i$ and each $b \in B$, there exists a $q \in I_{i-1}$ such that $p \subseteq q$ and $b \in Ran(q)$.

Proposition 4.13. (A weaker version of Proposition 4.5 of [10]) *Let M and N be U-models, where U is a finite vocabulary containing relation symbols only. Then M and N satisfy exactly the same U-sentences of* $\mathrm{FO}_{wo=}$ *of the quantifier rank $n \in \mathbb{N}_{\geq 1}$ if and only if $M \sim_n N$.*

We then prove the main result of the current subsection.

Theorem 4.14. *Let V be a nonempty vocabulary containing only unary relation symbols. We have* $\mathrm{FO}_{wo=} < \mathrm{fESO}_{wo=}$ *over the class of finite V-models.*

Proof. Let M and N be as defined in the beginning of the current subsection (subsection 4.4.2), with $Q^M = Q^N = \emptyset$ for all $Q \in V \setminus \{P\}$. We separated the models by a simple $\{P\}$-sentence of $\mathrm{fESO}_{wo=}$. To conclude the proof, it suffices to establish that for all $n \in \mathbb{N}_{\geq 1}$, all finite $U \subseteq V$ and all U-reducts $M \upharpoonright U$ and $N \upharpoonright U$ of the models M and N, we have $M \upharpoonright U \sim_n N \upharpoonright U$. Let $n \in \mathbb{N}_{\geq 1}$ and let $U \subseteq V$ be finite. Define the sets I_k of partial relativeness correspondences in the following way.

1. $I_n = \{\emptyset\}$.
2. $I_{k-1} = \{\, p \cup \{(a,b)\} \mid p \in I_k \text{ and } M, \frac{a}{x} \models P(x) \Leftrightarrow N, \frac{b}{x} \models P(x)\,\}$.

We immediately observe that the back and forth system $(I_k)_{k \in \{0,1,\dots,n\}}$ satisfies the required properties, and therefore $M \upharpoonright U \sim_n N \upharpoonright U$. □

4.5 Chapter Conclusion and a Remark on Generalized Quantifiers

In this chapter we have investigated the expressive power of $\mathrm{fESO}_{wo=}$ over models with a relational vocabulary. The results obtained can be interesting for example in the study of independence-friendly modal logics, and also other systems in the family of independence-friendly logic. In fact, our main result concerns models with a unary relational vocabulary, so the link to independence-friendly modal logics there is somewhat indirect.

We have defined the notion of a bloating and shown that truth of $\mathrm{fESO}_{wo=}$ sentences is preserved under bloating. This establishes an easy access to inexpressibility results for logics that translate into $\mathrm{fESO}_{wo=}$. We have observed that over $\{R\}$-models, where R is a a binary relation symbol, $\mathrm{fESO}_{wo=}$ is incomparable with FO with regard to expressive power. However, we have also established that when limiting attention to finite models with a nonempty unary relational vocabulary, we have

$$\mathrm{FO}_{wo=} < \mathrm{fESO}_{wo=} < \mathrm{FO}.$$

The method of proof establishing the latter inequality via Dickson's Lemma is interesting in its own right. The most important notions we defined for the purposes of the related argument are the notions of a spectrum and a stabilizer. Indeed, stabilizing spectra seem to arise in various interesting mathematical contexts. We end the chapter by demonstrating how stabilizing spectra can be used to characterize the extensions of first-order logic by *unary generalized quantifiers* that are genuinely more expressive than FO in the finite.

A *unary generalized quantifier* of a finite width is an isomorphically closed class of models $N = (W, P_1^N, ..., P_k^N)$, where each P_i is a unary relation symbol and $k \in \mathbb{N}_{\geq 1}$. The number k is the *width* of the quantifier. Let K be a unary generalized quantifier of the finite width k. The extension of FO with K is the system that extends the set of FO formulae according to the following rule.

If $\varphi_1, ..., \varphi_k$ are formulae, then also $Q_K x_{i_1}, ..., x_{i_k}\big(\varphi_1, ..., \varphi_k\big)$ is a formula.

The semantics is extended such that $M, f \models Q_K x_{i_1}, ..., x_{i_k}\big(\varphi_1, ..., \varphi_k\big)$ iff

$$M' \;=\; \big(\, Dom(M),\, P_1^{M'},\, ...\,,\, P_k^{M'}\,\big) \;\in\; K,$$

where $P_j^{M'} \;=\; \{\; a \in Dom(M) \mid M, f\frac{a}{x_{i_j}} \models \varphi_j \;\}$. If $\mathcal{Q}$ denotes a class of unary generalized quantifiers, then $\mathrm{FO}(\mathcal{Q})$ denotes the extension of first-order logic with all the quantifiers $K \in \mathcal{Q}$.

Each unary quantifier K of a finite width k can be associated with a spectrum $R_K \subseteq \mathbb{N}^{(2^k)}$. The quantifier K is a class of models of the vocabulary $U = \{P_1, ..., P_k\}$. Let $T = \{\tau_1, ..., \tau_{2^k}\}$ be the set of unary U-types with the free variable x, ordered by the relation $\leq^T$ such that $\tau_i \leq^T \tau_j$ iff $i \leq j$. We define $(n_1, ..., n_{2^k}) \in R_K$ if and only if there exists a model $M \in K$ such that for each $i \in \{1, ..., 2^k\}$, we have

$$|\,\{\; a \in Dom(M) \mid M, \frac{a}{x} \models \tau_i \;\}\,| \;=\; n_i.$$

The quantifier K is a *stabilizing quantifier* if the spectrum R_K is stabilizing. Note that even though the structure of the relation R_K depends on how the linear order $\leq^T$ is chosen, the property of being a stabilizing quantifier is independent of the choice of $\leq^T$. Also note that we have defined a spectrum to be a relation over the natural numbers, so spectra only encode the action of quantifiers over finite models. This suffices for our purposes.

Theorem 4.15. *Let V be a finite relational vocabulary containing a relation symbol of the arity at least two. Let $\mathcal{Q}$ be a class of unary generalized quantifiers of finite width. We have* $\mathrm{FO} < \mathrm{FO}(\mathcal{Q})$ *over the class of finite V-models if and only if $\mathcal{Q}$ contains a quantifier that is not stabilizing.*

Proof. Assume first that $\mathcal{Q}$ does not contain a quantifier that is not stabilizing. Let φ be an arbitrary sentence of $\mathrm{FO}(\mathcal{Q})$. Consider an arbitrary subformula

$$\psi \;:=\; Q_K x_{i_1}, ..., x_{i_k}\big(\,\psi_1, ..., \psi_k\,\big)$$

of φ, where the formulae ψ_i are first-order and K is neither the universal nor the existential quantifier. Since K is a stabilizing quantifier, there exists an

FO formula ψ' that is uniformly equivalent to ψ in the finite. This is established by a straightforward argument similar to that employed in the proof of Proposition 4.6, paying close attention at each stage to which variable symbols are to be considered free and which ones bound. Substituting ψ' for ψ in φ, we can eliminate an instance of the quantifier Q_K. Iterating the procedure, we end up with a first-order sentence that is uniformly equivalent to φ in the finite. Therefore it is *not* the case that FO $<$ FO($\mathcal{Q}$) over the class of finite V-models.

Assume then that $\mathcal{Q}$ contains a quantifier H that is not stabilizing. Let $m \in \mathbb{N}_{\geq 1}$ be the width of H. Therefore H is a class of models of the vocabulary $U = \{P_1, ..., P_m\}$. Let $T = \{\tau_1, ..., \tau_{2^m}\}$ be the set of unary U-types with the free variable x, ordered by $\leq^T$ according to the subindices. Let R_H be the spectrum of H according to the order $\leq^T$.

Let $n \in \mathbb{N}_{\geq 1}$. Since the spectrum R_H is not stabilizing, there exist some integers $k, k' > n$ and some $m_1, ...m_{i-1}, m_{i+1}, ..., m_{2^m} \in \mathbb{N}$ such that

$$(m_1, ...m_{i-1}, k, m_{i+1}, ..., m_{2^m}) \in R_H$$

and

$$(m_1, ...m_{i-1}, k', m_{i+1}, ..., m_{2^m}) \notin R_H.$$

Let us say that the *co-ordinate position i witnesses instability of R_H for n*. For each $n \in \mathbb{N}_{\geq 1}$, there exists some co-ordinate position that witnesses instability of R_H for n. Let $p \in \{1, ..., 2^m\}$ be a co-ordinate position that witnesses instability of R_H for infinitely many $n \in \mathbb{N}_{\geq 1}$. Let K_p denote the class of finite U-models where each model contains at least 2^m elements that satisfy the type τ_p.

We assume without loss of generality that there is a binary relation symbol $S \in V$. In the case there is no binary relation symbol in V, a symbol of a higher arity can be used in order to encode a binary relation. The resulting modification of the argument below is straightforward. We define a map f that encodes each model in K_p by a corresponding V-model. Let $M \in K_p$. Choose a set

$$A_M = \{a_1, ..., a_{2^m}\} \subseteq Dom(M)$$

such that $M, \frac{a_i}{x} \models \tau_p$ for each $a_i \in A_M$. For each $i \in \{1, ..., 2^m\}$, the element $a_i \in A_M$ is referred to as the *i-th element of A_M*. We let $f(M)$ be the V-model defined as follows.

1. $Dom\big(f(M)\big) = Dom(M)$.
2. For all $a_i, a_j \in A_M$, we have $(a_i, a_j) \in S^{f(M)}$ iff
   (a) $i + 1 = j$ or
   (b) $i = j$.

3. For all $a_i \in A_M$ and all $v \in Dom(M) \setminus A_M$, we have $(a_i, v) \in S^{f(M)}$ iff $M, \frac{v}{x} \models \tau_i$.

4. For all $a_i \in A_M$ and all $v \in Dom(M) \setminus A_M$, we have $(v, a_i) \notin S^{f(M)}$.

5. For all $u, v \in Dom(M) \setminus A_M$, we have $(u, v) \notin S^{f(M)}$.

6. For all relation symbols $T \in V$, $T \neq S$, we have $T^{f(M)} = \emptyset$.

Call

$$C = \{ f(M) \mid M \in K_p \}$$

and

$$H' = \{ f(M) \mid M \in K_p \cap H \}.$$

We then establish that H' is definable with respect to C by a sentence of FO($\mathcal{Q}$). We define, for each $i \in \{1, ..., 2^m\}$, a first-order $\{S\}$-formula $\chi_{\tau_i}(x)$ such that for all $f(M) \in C$ and all $u \in Dom(f(M))$, we have

$$f(M), \frac{u}{x} \models \chi_{\tau_i}(x) \Leftrightarrow M, \frac{u}{x} \models \tau_i.$$

The encoding $f(M)$ of each model $M \in C$ is constructed in such a way that this is straightforward. For $i \in \{1, ..., 2^m\} \setminus \{p\}$, the elements of $f(M)$ that should satisfy $\chi_{\tau_i}(x)$ are exactly the S-successors of the i-th element of A_M that are in $Dom(M) \setminus A_M$. The elements of $f(M)$ that should satisfy $\chi_{\tau_p}(x)$ are the elements that belong to A_M or are S-successors of the p-th element of A_M. Now, for each $j \in \{1, ..., m\}$, let $\psi_{P_j}(x)$ denote a disjunction of exactly all the formulae $\chi_{\tau_i}(x)$ such that $\tau_i \models P_j(x)$. Note that for all $f(M) \in C$ and all $u \in Dom(f(M))$, we have

$$f(M), \frac{u}{x} \models \psi_{P_j}(x) \Leftrightarrow M, \frac{u}{x} \models P_j(x).$$

The $\{S\}$-sentence

$$Q_H x_1, ..., x_m \big(\psi_{P_1}(x_1), \ldots, \psi_{P_m}(x_m) \big)$$

defines H' w.r.t. C.

We then show that the class H' is not definable w.r.t. C by any first-order V-sentence. This follows by a straightforward Ehrenfeucht-Fraïssé game argument. Let $n \in \mathbb{N}_{\geq 1}$. We will define two V-models, one in H' and the other one in $C \setminus H'$, such that the *duplicator* (see [44]) wins the n-round game played on the models.

Since the co-ordinate position p witnesses instability of R_H for infinitely many elements of $\mathbb{N}_{\geq 1}$, there exists some integers $k, k' > n + 2^m$ and $l_1, ..., l_{p-1}, l_{p+1}, ..., l_{2^m} \in \mathbb{N}$ such that

$$\overline{w}_k = \big(l_1, ..., l_{p-1}, k, l_{p+1}, ..., l_{2^m}\big) \in R_H$$

and

$$\overline{w}_{k'} \;=\; \big(l_1, ..., l_{p-1}, k', l_{p+1}, ..., l_{2^m}\big) \;\notin\; R_H.$$

Let M_k denote a U-model such that the number of elements of M_k that satisfy the type τ_p is k, and for each $i \in \{1, ..., 2^m\} \setminus \{p\}$, the number of elements of M_k that satisfy the type τ_i is l_i. Let $M_{k'}$ denote a model defined similarly, but with k replaced by k'. That is, $M_{k'}$ is a U-model such that the number of elements of $M_{k'}$ that satisfy the type τ_p is k', and for each $i \in \{1, ..., 2^m\} \setminus \{p\}$, the number of elements of $M_{k'}$ that satisfy the type τ_i is l_i. The duplicator wins the n-round game played on the V-models $f(M_k)$ and $f(M_{k'})$. The duplicator plays according to the following strategy.

1. If the *spoiler* (see [44]) chooses an element already chosen in some earlier round, the duplicator responds by choosing the corresponding earlier chosen element in the other model.

2. If the spoiler chooses an element u not chosen in any earlier round, the duplicator responds with an element u' of the other model such that the following conditions hold.

   (a) The element u' has not been chosen in any earlier round.

   (b) If the element u is chosen from the model $f(M_k)$ and u is the i-th element of the set A_{M_k}, then the duplicator chooses the i-th element of the set $A_{M_{k'}}$ of $f(M_{k'})$.

   (c) Symmetrically, if the element u is chosen from the model $f(M_{k'})$ and u is the i-th element of the set $A_{M_{k'}}$, then the duplicator responds by choosing the i-th element of the set A_{M_k} of $f(M_k)$.

   (d) If u is chosen from the model $f(M_k)$ and we have $u \notin A_{M_k}$ and $M_k, \frac{u}{x} \models \tau_i(x)$, then the duplicator chooses an element $u' \notin A_{M_{k'}}$ such that $M_{k'}, \frac{u'}{x} \models \tau_i(x)$.

   (e) Symmetrically, if u is chosen from $f(M_{k'})$ and we have $u \notin A_{M_{k'}}$ and $M_{k'}, \frac{u}{x} \models \tau_i(x)$, then the duplicator chooses an element $u' \notin A_{M_k}$ such that $M_k, \frac{u'}{x} \models \tau_i(x)$.

We observe that the duplicator can play n rounds maintaining this strategy, and the strategy is indeed a winning strategy. Since n was chosen arbitrarily, we conclude that H' is not definable with respect to C by any V-sentence of FO. We conclude that FO $<$ FO($\mathcal{Q}$) over the class of finite V-models. $\square$

The notion of a stabilizing spectrum generalizes to the context involving all infinite cardinalities in addition to finite ones, and this leads to a natural generalization of the notion of a stabilizing quantifier. However, Theorem 4.15 does not hold in the context involving infinite models in addition to finite ones. Consider, for example, the quantifier "there exists infinitely many". This is a unary stabilizing quantifier of the width one, the smallest stabilizer being $\aleph_0$. The extension of FO with this quantifier is of course stronger in expressive power over V-models than the (finitary system) FO alone.

CHAPTER 5

On Fragments of SOPMLE and SO(ML)

Let SO(ML) denote the logic obtained by extending polyadic multimodal logic by allowing for the unrestricted quantification of proposition symbols and also relation symbols associated with diamonds. In this chapter we investigate the expressivity of fragments of SO(ML) and also fragments of second-order propositional modal logic with the global modality SOPMLE. We identify a range of properties of simple fragments of the two logics. For example, we obtain a simple tool for proving inexpressibility results for the fragment of SOPMLE consisting of formulae of the type $\overline{\exists P}\,\overline{\forall Q}\,\varphi$, where φ is free of propositional quantifiers. The principal contribution of the chapter is the relatively straightforward observation that SO(ML) is equi-expressive with second-order logic SO. By showing this we identify a *modal normal form* for second-order logic. Investigating (fragments of) second-order logic from alternative perspectives—such as the ones provided by the systems studied below—can elucidate the mathematical phenomena that give rise to the expressivity of second-order logic.

5.1 SOPMLE and SO(ML)

In this chapter we investigate the expressivity of fragments of SO(ML) and SOPMLE. *We concentrate on the study of finite models*, but unless otherwise stated, the class of structures under investigation is not assumed to be finite. We shall establish that both over models and pointed models, SO(ML) = SO, and therefore the investigations below can also be regarded as investigations of fragments of MSO and SO. (Recall from Chapter 2 that SOPMLE = MSO with regard to expressive power.)

The chapter is structured as follows. In Section 5.2 we discuss a number of preliminary issues. We begin Section 5.3 by observing that the argument of ten Cate in [11]—which establishes that formulae of SOPML with diamonds corresponding to a binary relation admit a prenex normal form representation—works almost as such also in the context of SOPML and SOPMLE with polyadic modalities. We then show that already formulae of the type $\overline{\exists P}\,\overline{\forall Q}\,\varphi$ of SOPMLE, where φ is free of propositional quantifiers, can define any finite pointed directed graph up to isomorphism. We also provide an analogous result that applies to SOPML. We call the fragment of SOPMLE (SOPML), whose formulae are of the type $\overline{\exists P}\,\overline{\forall Q}\,\varphi$ specified above, the Σ_2 fragment.

Since any finite pointed directed graph is definable up to isomorphism by a Σ_2 formula of SOPMLE, there exists no model transformation that preserves truth of Σ_2 formulae of SOPMLE from a finite pointed directed graph to a non-isomorphic pointed directed graph. However, in Section 5.4 we identify a family of model transformations such that for each $k \in \mathbb{N}$, there is a non-trivial transformation that applies to formulae of the Σ_2 fragment of SOPMLE with k existential propositional quantifiers. We use this tool to prove simple hierarchy results concerning the expressivity of fragments of SOPMLE and SOPML.

In Section 5.5 we make use of a class of pointed models that we call *pointed ornamented words* in order to study fragments of SOPMLE and SO(ML). A major part of the related investigations draws its inspiration from the study culminating to Büchi's theorem in descriptive complexity theory. We observe that while the Σ_1 fragment of SOPML [8] exactly captures regular languages, neither an increase in the number of allowed *alternations* of propositional quantifiers nor an increase of the *arity* of prenex quantified existential quantifiers (i.e., a transition to the fragment of Σ_1^1 where the first-order parts of formulae are standard translations of formulae of polyadic modal logic) takes us beyond regular languages. What is needed is an increase in *both* arity *and* the number of alternations. Finally, in Section 5.6, we show that SO(ML) and second-order predicate logic SO coincide in expressive power. This result applies to pointed models as well as models.

5.2 Preliminary Definitions

We assume that the reader is familiar with the basics of the theory of finite automata and regular languages. For an introduction to the subject, see for example [27].

In the investigations below we shall make use of a version of the notion of a *bounded morphism* (see [7]). Let $k \in \mathbb{N}$, and let $V = \{R, P_1, ..., P_k\}$ be a vocabulary, where R is a binary relation symbol and P_i are unary relation symbols. A function $f : W \longrightarrow U$ from the domain W of a model

$$M = (W, R^M, P_1^M, ..., P_k^M)$$

to the domain U of a model

$$N = (U, R^N, P_1^N, ..., P_k^N)$$

is a bounded morphism if and only if the following conditions are satisfied.

1. For all $u \in W$ and all $i \in \{1, ..., k\}$, we have $u \in P_i^M$ iff $f(u) \in P_i^N$.

[8] The Σ_1 fragment of SOPML is the fragment containing exactly the formulae of the type $\overline{\exists P}\,\varphi$, where $\overline{\exists P}$ is a vector of existential propositional quantifiers and φ is free of propositional quantifiers.

2. For all $u, v \in W$, if $uR^M v$, then $f(u)R^N f(v)$.

3. For all $u \in W$ and $y \in U$, if $f(u)R^N y$, then there exists some $v \in W$ such that $uR^M v$ and $f(v) = y$.

We say that there is a bounded morphism from a pointed model (M, w) onto (N, w'), if there is a surjective bounded morphism f from the domain of M onto the domain of N such that $f(w) = w'$.

Let φ be a sentence of SOPMLE of the type $\overline{\forall P}\,\psi$, where $\overline{\forall P}$ is a vector of universal propositional quantifiers and ψ is free of propositional quantifiers. The sentences of the type of φ are called Π_1 sentences of SOPMLE. The following proposition is easy to verify and can be regarded as part of the folklore of modal logic.

Proposition 5.1. *Let $k \in \mathbb{N}$ and let $V = \{R, P_1, ..., P_k\}$ be a vocabulary where R is a binary relation symbol and P_i are unary relation symbols. Let (M, w) and (N, v) be pointed V-models and assume there is a surjective bounded morphism from (M, w) onto (N, v). Then, for all Π_1 sentences φ of* SOPMLE *of the vocabulary V, $(M, w) \Vdash \varphi$ implies $(N, v) \Vdash \varphi$.*

5.3 Basic Properties of SOPMLE and SOPML

In [11], ten Cate shows that formulae of SOPML with a binary accessibility relation admit a prenex normal form representation. Each formula of the vocabulary $\{R, P_1, ..., P_k\}$, where R is a binary relation symbol and P_i are unary relation symbols, can be written in a form that begins with a string of propositional second-order quantifiers, and this string is followed by an ordinary modal formula. The argument of ten Cate in [11] generalizes directly to the context of SOPML (and also SOPMLE) with polyadic modalities, as we shall next observe.

Similarly to what we defined in Chapter 2, we let *uniq*(P) denote the formula

$$\langle E\rangle P \wedge \forall Q\big(\langle E\rangle(Q \wedge P) \to [E](P \to Q)\big).$$

Here $\langle E\rangle$ is the global diamond. Let $k \in \mathbb{N}_{\geq 2}$ and let R be a k-ary relation symbol. Let P and Q be unary relation variables. Let $i \in \{1, ..., k-1\}$, and let $(\top, ..., P, ..., \top)$ denote the $(k-1)$-tuple, where the i-th position has the formula P, and every other position has the formula $\top := P \vee \neg P$. Similarly, let $(\top, ..., P \wedge Q, ..., \top)$ and $(\top, ..., P \wedge \neg Q, ..., \top)$ denote the $(k-1)$-tuples where the i-th positions have the formulae $P \wedge Q$ and $P \wedge \neg Q$ respectively, and all other positions have the formula $\top$. Let $uniq^i_R(P)$ denote the formula

$$\begin{aligned}&\langle R\rangle(\top, ..., P, ...\top)\\ \wedge\ &\forall Q\,\big(\,\langle R\rangle(\top, ..., P \wedge Q, ..., \top) \to \neg\langle R\rangle(\top, ..., P \wedge \neg Q, ..., \top)\,\big).\end{aligned}$$

The following uniform equivalences are immediate.

1. $\langle E\rangle\, \exists P\, \varphi \;\equiv\; \exists P\, \langle E\rangle\, \varphi$

2. $\langle E\rangle \forall P\, \varphi \;\equiv\; \exists Q \forall P\, \big(\, uniq(Q) \wedge \langle E\rangle (Q \wedge \varphi)\,\big)$

3. $\langle R\rangle\, \big(\varphi_1\,, ..., \,\varphi_{i-1},\; \exists P\, \psi(P),\; \varphi_{i+1}, ..., \,\varphi_{k-1}\big)$
   $\equiv \quad \exists P'\, \langle R\rangle\, \big(\varphi_1\,, ..., \,\varphi_{i-1},\; \psi(P'),\; \varphi_{i+1}, ..., \,\varphi_{k-1}\big)$

4. $\langle R\rangle\, \big(\varphi_1\,, ..., \,\varphi_{i-1},\; \forall P\, \psi(P),\; \varphi_{i+1}, ..., \,\varphi_{k-1}\big)$
   $\equiv\; \exists Q \forall P'\, \Big(\, uniq^i_R(Q) \;\wedge\; \langle R\rangle \big(\varphi_1\,, ..., \,\varphi_{i-1},\; Q \wedge \psi(P'),\; \varphi_{i+1}, ..., \,\varphi_{k-1}\big)\Big)$

Here $\psi(P')$ is the formula obtained from $\psi(P)$ by replacing the free occurrences of P in $\psi(P)$ by P'. We assume that the variables Q, P' do not occur free in the formulae φ, $\psi(P)$, $\varphi_1, ..., \varphi_{k-1}$.

In the light of the above uniform equivalences, it is rather easy to conclude the following.

Proposition 5.2. (cf. Proposition 3 of [11].) *Both* SOPMLE *and* SOPML *admit a prenex normal form representation of formulae. That is, for each* SOPMLE (SOPML) *formula there exists a uniformly equivalent* SOPMLE (SOPML) *formula that is of the form $\overline{Q}\,\psi$, where $\overline{Q}$ is a string of propositional quantifiers and ψ does not contain propositional quantifiers.*

5.3.1 Σ_2 Formulae and Finite Models

In this subsection we investigate the Σ_2 fragments of the logics SOPMLE and SOPML over finite models. We show that any finite pointed $\{R\}$-model, where R is a binary relation, can be characterized up to isomorphism by a Σ_2 formula of SOPMLE. We also establish an analogous result that applies to SOPML.

Let (M, w) and (M', w') be pointed models. We write $(M, w) \cong (M', w')$ if the models M and M' are isomorphic via an isomorphism f that maps w to w'. We call pointed models (M, w) and (M', w') isomorphic if and only if we have $(M, w) \cong (M', w')$.

Proposition 5.3. *Let R be a binary relation symbol. For each finite pointed $\{R\}$-model (M, w) there exists a Σ_2 sentence φ of* SOPMLE *such that for all pointed $\{R\}$-models (M', w'), we have $(M', w') \Vdash \varphi$ iff $(M, w) \cong (M', w')$.*

Proof. Let $M = (W, R^M)$. Let $|Dom(M)| = n$ and assume w.l.o.g. that $W = \{1, ..., n\}$ and $w = 1$. Let $uniq^*(P, Q)$ denote the formula

$$\langle E\rangle P \wedge \big(\langle E\rangle (P \wedge Q) \rightarrow [E](P \rightarrow Q)\big).$$

Define the following formulae.

$$
\begin{aligned}
\psi_1 &:= \bigwedge_{1\leq i\leq n} uniq^*(P_i, Q)\\
\psi_2 &:= [E] \bigvee_{1\leq i\leq n} \Big(P_i \ \wedge \bigwedge_{j\,\in\,\{1,\ldots,n\},\ j\,\neq\, i} \neg P_j \Big)\\
\psi_3 &:= [E] \bigwedge_{i,j\,\in\,\{1,\ldots,n\},\ (i,j)\,\in\, R^M} \big(P_i \rightarrow \langle R\rangle P_j \big)\\
\psi_4 &:= [E] \bigwedge_{i,j\,\in\,\{1,\ldots,n\},\ (i,j)\,\not\in\, R^M} \big(P_i \rightarrow \neg\langle R\rangle P_j \big)
\end{aligned}
$$

Let φ be the formula

$$\exists P_1 ... \exists P_n \forall Q \big(P_1 \wedge \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4 \big).$$

Let $(M', w') = ((W', R^{M'}), w')$ be a model and assume that we have $((W', R^{M'}), w') \Vdash \varphi$. Thus

$$(M^*, w') := \big((M', P_1^{M^*}, ..., P_n^{M^*}), w'\big) \Vdash P_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4 \wedge \forall Q\, \psi_1$$

for some sets

$$P_1^{M^*}, ..., P_n^{M^*} \subseteq W'.$$

The formula $\forall Q\, \psi_1$ ensures that each set $P_i^{M^*}$ is a singleton set. The formula ψ_2 makes sure that if $u \in W'$, then there is some $i \in \{1, ..., n\}$ such that $P_i^{M^*} = \{u\}$, and furthermore, $P_j^{M^*} \cap P_l^{M^*} = \emptyset$ for all $j, l \in \{1, ..., n\}$ such that $j \neq l$. Thus we observe that the sets $P_i^{M^*}$ partition W' into n singleton sets. Define a bijection $f : W \longrightarrow W'$ such that for each $i \in \{1, ..., n\}$, we have $f(i) \in P_i^{M^*}$. It is easy to see that f is an isomorphism from M to M' and $f(w) = f(w')$.

For the converse implication, it is clear that any pointed $\{R\}$-model isomorphic to (M, w) satisfies φ. $\Box$

We then establish an analogue of Proposition 5.3 that applies to SOPML. Let $(M, w) = \big((W, R^M), w\big)$ be a pointed model. Define

1. $W_w^0 = \{w\}$,
2. $W_w^{i+1} = W_w^i \cup \{\ v \in W \mid uR^M v$ for some $u \in W_w^i\}$.

For each $i \in \mathbb{N}$, define

$$R^{M_w^i} = \{\ (u, v) \in W \times W \mid u \in W_w^{i-1},\ v \in W_w^i,\ (u, v) \in R^M\ \}.$$

Here $W_w^{-1} = \emptyset$. Let $d \in \mathbb{N}$. We call the pointed model

$$(M_w^d, w) \ = \ \big((W_w^d, R^{M_w^d}), w\big)$$

the *d-generated submodel of* (M, w). A pointed model (N, v) whose d-generated submodel is the model (N, v) itself, is called a *root generated pointed model of the depth* d.

Proposition 5.4. *Let $d \in \mathbb{N}$ and let R be a binary relation symbol. For each finite $\{R\}$-model (M, w) there exists a Σ_2 sentence φ of* SOPML *of the modal depth d such that for all pointed $\{R\}$-models (N, v), we have $(N, v) \Vdash \varphi$ iff $(M^d_w, w) \cong (N^d_v, v)$.*

Proof. Define

$$\begin{aligned}
\langle R\rangle^0 \varphi &:= \varphi, \\
[R]^0 \varphi &:= \varphi, \\
\langle R\rangle^{i+1} \varphi &:= \langle R\rangle \langle R\rangle^i \varphi, \\
[R]^{i+1} \varphi &:= [R][R]^i \varphi.
\end{aligned}$$

Also define

$$\begin{aligned}
\langle R\rangle^{\leq i} \varphi &:= \bigvee_{j \,\in\, \{0,...,i\}} \langle R\rangle^j \varphi, \\
[R]^{\leq i} \varphi &:= \bigwedge_{j \,\in\, \{0,...,i\}} [R]^j \varphi.
\end{aligned}$$

Let $M = (W, R^M)$. The statement of the proposition is trivial for the case where $d = 0$, so assume that $d > 0$. Let $|Dom(M)| = n$ and assume w.l.o.g. that $W = \{1, ..., n\}$ and that $w = 1$. Let $uniq^*(P, Q)$ denote the formula

$$\langle R\rangle^{\leq d} P \;\wedge\; \big(\langle R\rangle^{\leq d}(P \wedge Q) \rightarrow [R]^{\leq d}(P \rightarrow Q)\big).$$

Define the following formulae.

$$\begin{aligned}
\psi_1 &:= \bigwedge_{1 \leq i \leq n} uniq^*(P_i, Q) \\
\psi_2 &:= [R]^{\leq d} \bigvee_{1 \leq i \leq n} \Big(\, P_i \;\wedge \bigwedge_{j \,\in\, \{1,...,n\},\ j \neq i} \neg P_j \Big) \\
\psi_3 &:= [R]^{\leq d-1} \bigwedge_{i,j \,\in\, \{1,...,n\},\ (i,j) \,\in\, R^M} \big(\, P_i \rightarrow \langle R\rangle P_j \,\big) \\
\psi_4 &:= [R]^{\leq d-1} \bigwedge_{i,j \,\in\, \{1,...,n\},\ (i,j) \,\notin\, R^M} \big(\, P_i \rightarrow \neg\langle R\rangle P_j \,\big)
\end{aligned}$$

Let φ be the formula

$$\exists P_1 ... \exists P_n \forall Q \big(\, P_1 \wedge \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4 \,\big).$$

The argument establishing that φ has the desired property is analogous to the proof of Proposition 5.3. $\square$

5.4 Σ_2 Formulae of SOPML and SOPMLE with a Bounded Number of Existential Quantifiers

By Proposition 5.3 it is impossible to design a model transformation that preserves truth of Σ_2 formulae of SOPMLE from a finite pointed $\{R\}$-model (M, w) to a pointed $\{R\}$-model (N, v) such that $(N, v) \ncong (M, w)$. Proposition 5.4 establishes an analogous result that applies to finite root generated pointed models; for an arbitrary $d \in \mathbb{N}$, there exists no model transformation that transforms a finite root generated pointed $\{R\}$-model (M, w) of the depth d to a root generated pointed $\{R\}$-model $(N, v) \ncong (M, w)$ of the depth d such that truth of all Σ_2 formulae of SOPML of the modal depth d is preserved from (M, w) to (N, v). In order to prove inexpressibility results that apply to Σ_2 formulae of SOPMLE and SOPML, we will define for each $k \in \mathbb{N}$ a model transformation that preserves truth of Σ_2 formulae of SOPMLE with at most k existential propositional quantifiers. We then use the transformations in order to prove a number of simple expressivity related hierarchy results.

We begin by defining a number of notions needed later on. If M is a model, we let $Type(M)$ denote the isomorphism type of M. When we write $M \cong_f M'$ we mean that

$$f : Dom(M) \longrightarrow Dom(M')$$

is an isomorphism from the model M to the model M'.

Definition 5.5. Let $M = (W, R^M)$ be a model with a binary relation and let $l \in \mathbb{N}_{\geq 2}$. Let $N_1, ..., N_l$ be finite disjoint submodels of M such that $N_i \cong N_j$ for all $i, j \in \{1, ..., l\}$. Assume that $N_1, ..., N_l$ are *non-adjacent*, i.e., no two points $u_i \in Dom(N_i)$ and $u_j \in Dom(N_j)$, where $i \neq j$, satisfy $u_i R^M u_j$. Let F be a set of isomorphisms

$$f_{ij} : Dom(N_i) \longrightarrow Dom(N_j),$$

one for each pair $(i, j) \in \{1, ..., l\} \times \{1, ..., l\}$, such that any automorphism obtained by composing the functions in F is an identity function. Assume that the following conditions hold.

1. If there is an $i \in \{1, ..., l\}$ such that $uR^M v_i$ for some some
$$u \in W \setminus \bigcup_{k \in \{1,...,l\}} Dom(N_k)$$
and some $v_i \in Dom(N_i)$, then $uR^M f_{ij}(v_i)$ for all $j \in \{1, ..., l\}$.
2. If there is an $i \in \{1, ..., l\}$ such that $v_i R^M u$ for some some
$$u \in W \setminus \bigcup_{k \in \{1,...,l\}} Dom(N_k)$$
and some $v_i \in Dom(N_i)$, then $f_{ij}(v_i) R^M u$ for all $j \in \{1, ..., l\}$.

We call the disjoint union

$$N \quad = \quad N_1 \biguplus ... \biguplus N_l \quad = \quad M \upharpoonright \bigcup_{k \in \{1,...,l\}} Dom(N_k)$$

an *adjacency-free isotropic sector* of M. We call $Type(N_i)$ a *sector component type* of N. Note that there may be several different sector component types associated with N, depending on what is identified as a single component of N; a sector component N_i may consist of two or more non-adjacent submodels of M. The number of components of a given type t is the *sector width* of N with respect to the type t.

We then define a model transformation that involves deleting components of an isotropic sector.

Definition 5.6. Let $M = (W, R^M)$ be a model with a binary relation R^M. Let $N = N_1 \biguplus ... \biguplus N_l$ be an isotropic sector of M and let each of the components N_i have exactly n elements. Let $f : \mathbb{N}_{\geq 1} \longrightarrow \mathbb{N}_{\geq 1}$ be a function. Assume that $0 < f(n) \leq l' < l$ and let

$$S = Dom(M) \setminus \big((Dom(N_{l'+1}) \ \cup ... \cup \ Dom(N_l)\big).$$

We say that the model $M \upharpoonright S$ is obtained from the model M by an *f-conformal deletion*. Any model that is isomorphic to a model that can be obtained from M by a finite series of f-conformal deletions, is called an *f-conformal minor* of M.

Assume that a model M' is obtained from a model M by a finite series of f-conformal deletions. Let $w \in Dom(M')$. We define that any pointed model (N, u) such that $(N, u) \cong (M', w)$, is an f-conformal minor of (M, w).

Proposition 5.7. *Let $k \in \mathbb{N}$ and let φ be a Σ_2 sentence of* SOPMLE *with at most k existential quantifiers. Assume φ is a sentence of the vocabulary $\{R\}$, where R is a binary relation symbol. Let $f : \mathbb{N}_{\geq 1} \longrightarrow \mathbb{N}_{\geq 1}$ be the function such that $f(x) = 2^{k \cdot x}$ for all $x \in \mathbb{N}_{\geq 1}$. Let (M, w) be a pointed $\{R\}$-model and (M', w') its f-conformal minor. Now, if $(M, w) \Vdash \varphi$, then $(M', w') \Vdash \varphi$.*

Proof. Since (M', w') is an f-conformal minor of (M, w), there exists some submodel M'' of M obtained from M by a series of f-conformal deletions such that $w \in Dom(M'')$ and $(M'', w) \cong (M', w')$. We assume w.l.o.g. that (M', w') is (M'', w). Furthermore, we assume w.l.o.g. that $M' = M''$ is obtained from M by a single f-conformal deletion that affects a sector

$$N = N_1 \biguplus ... \biguplus N_l$$

of the model M, deleting the components $N_{l'+1}, ...N_l$, where $l' \geq 2^{k \cdot |Dom(N_1)|}$. In other words, at least $2^{k \cdot |Dom(N_1)|}$ components remain in the sector after the deletion. We have

$$M' = M \upharpoonright (Dom(M) \setminus T),$$

where

$$T = Dom(N_{l'+1}) \cup ... \cup Dom(N_l).$$

Call

$$N' = N \upharpoonright (Dom(N) \setminus T).$$

Since N is an isotropic sector, there must exist a collection F of isomorphisms between the components N_i of N that witness this. We let f_{ij}, where $i, j \in \{1, ..., l\}$, denote the members of such a witnessing collection.

Let

$$\varphi \;=\; \exists P_1 ... \exists P_k \forall Q_1 ... \forall Q_{k'} \; \psi,$$

where $k' \in \mathbb{N}$ and ψ is free of propositional quantifiers. Now, since

$$(M, w) \Vdash \exists P_1 ... \exists P_k \forall Q_1 ... \forall Q_{k'} \; \psi,$$

there exist sets $P_1^{M^*}, ..., P_k^{M^*} \subseteq Dom(M)$ such that such that

$$(M^*, w) \;:=\; \big((M, P_1^{M^*}, ..., P_k^{M^*}), w\big) \Vdash \; \forall Q_1 ... \forall Q_{k'} \; \psi.$$

For each $i \in \{1, ..., l\}$, let N_i^* denote the model $M^* \upharpoonright Dom(N_i)$. We have

$$|\,\{\; Type(N_i^*) \mid i \in \{1, ..., l\} \;\}\,| \;\leq\; 2^{k \cdot |Dom(N_1)|}.$$

On the other hand, the sector N' of the model M' consists of the components N_1 , ..., $N_{l'}$, so the sector width of N' is $l' \geq 2^{k \cdot |Dom(N_1)|}$. Thus we can define an expansion $M'^* = (M', P_1^{M'^*}, ..., P_k^{M'^*})$ of M' such that $P_i^{M'^*}$ and $P_i^{M^*}$ agree outside N, and furthermore, the following three conditions hold.

1. For each component N_i^*, where $i \in \{1, ..., l\}$, there exists a component $M'^* \upharpoonright Dom(N_j)$, where $j \in \{1, ..., l'\}$, such that f_{ij} is an isomorphism from N_i^* to $M'^* \upharpoonright Dom(N_j)$.

2. For each component $M'^* \upharpoonright Dom(N_j)$, where $j \in \{1, ..., l'\}$, there exists a component ${N_i}^*$, where $i \in \{1, ..., l\}$, such that f_{ij} is an isomorphism from N_i^* to $M'^* \upharpoonright Dom(N_j)$.

3. For each $i \in \{1, ..., l'\}$, let $N_i'^*$ denote $M'^* \upharpoonright Dom(N_i)$. Let

   - $C = \{\; {N_i}^* \mid 1 \leq i \leq l \;\}$,
   - $C' = \{\; N_i'^* \mid 1 \leq i \leq l' \;\}$.

   Also, let

- $[N^*_m]_C \;=\; \{\; {N_i}^* \in C \mid\; {N_i}^* \cong_{f_{im}} N^*_m \;\}$,
- $[N^*_m]_{C'} \;=\; \{\; N'^*_i \in C' \mid N'^*_i \cong_{f_{im}} N^*_m \;\}$.

We have $\mid [N^*_m]_{C'} \mid \;\leq\; \mid [N^*_m]_C \mid$ for all $m \in \{\, 1, ..., l\,\}$.

The three conditions above enable us to define a surjection

$$f : Dom(N) \longrightarrow Dom(N')$$

such that for each component N^*_i, we have $f \restriction Dom(N^*_i) = f_{ij}$ for some $j \in \{\, 1, ..., l'\,\}$ such that $N'^*_j \in [N^*_i]_{C'}$. Let g be the identity function on $Dom(M) \setminus Dom(N)$. It is easy to see that the function $h = f \cup g$ is a surjective bounded morphism from (M^*, w) onto (M'^*, w). Hence, by Proposition 5.1, as

$$(M^*, w) \Vdash \forall Q_1 ... \forall Q_{k'}\, \psi,$$

also

$$(M'^*, w) \Vdash \forall Q_1 ... \forall Q_{k'}\, \psi.$$

This directly implies that

$$(M', w) \Vdash \exists P_1 ... \exists P_k \forall Q_1 ... \forall Q_{k'}\, \psi,$$

as desired. □

It is now easy to prove the following two propositions.

Proposition 5.8. *Let R be a binary relation symbol. For each $k \in \mathbb{N}$, there exists a Σ_1 sentence of* SOPML *with $k+1$ existential propositional quantifiers that defines a class of pointed $\{R\}$-models that is not definable by any Σ_2 sentence of* SOPMLE *with k existential propositional quantifiers.*

Proof. Let $M = (W, R^M)$ be a model such that the following conditions hold.

1. The domain W is finite.
2. There is an element $u \in W$, called the *centre*, such that we have $uR^M v$ for all $v \in W \setminus \{u\}$. Furthermore, if $u' R^M v'$ for some $u', v' \in W$, then $u' = u$ and $v' \neq u$.

Call such structures *stars*. Let C denote the class of all stars, and let C_p denote the class containing each pointed model (M, w) where M is a star and w is the centre of the star M. Let φ be the formula

$$\exists P_1 ... \exists P_{k+1} \Big(\bigwedge_{S \subseteq \{1,...,k+1\}} \langle R \rangle \big(\bigwedge_{i \in S} P_i \;\wedge \bigwedge_{j \in \{1,...,k+1\} \setminus S} \neg P_j \,\big) \Big).$$

It is easy to see that for all $(M, w) \in C_p$, we have $(M, w) \Vdash \varphi$ iff the centre of M connects to at least $2^{\,k+1}$ elements. By Proposition 5.7, it is immediate that there is no Σ_2 formula of SOPMLE with at most k existential propositional quantifiers that defines this property w.r.t. the class C_p. Therefore there is no $\{R\}$-formula in the Σ_2 fragment of SOPMLE with at most k existential propositional quantifiers that defines w.r.t. the class of all pointed $\{R\}$-models exactly the same class as φ. $\square$

Let R be a binary relation symbol and let $k \in \mathbb{N}_{\geq 1}$. Let $\forall P_1 ... \forall P_k\, \varphi$ be an $\{R\}$-sentence of SOPML. Let K be the class of $\{R\}$-models that the sentence $\forall P_1 ... \forall P_k\, \varphi$ defines. We say that *the class K of Kripke frames is frame definable by a formula with k types of proposition symbols*.[9] Note that here we are talking about expressivity with respect to models rather than pointed models. We make the following very simple observation.

Proposition 5.9. *For each $k \in \mathbb{N}_{\geq 1}$, there is a class of Kripke frames that is frame definable by a formula with $k+1$ types of proposition symbols, but not frame definable by a formula with k types of proposition symbols.*

Proof. Let $k \in \mathbb{N}_{\geq 1}$, and let A be the class of stars whose centre has *less than* 2^{k+1} successors. The sentence

$$\forall P_1 ... \forall P_{k+1} \Big(\langle R \rangle \top \ \rightarrow \ \neg \Big(\bigwedge_{S \subseteq \{1,...,k+1\}} \langle R \rangle \big(\bigwedge_{i \in S} P_i \ \wedge \bigwedge_{j \in \{1,...,k+1\} \setminus S} \neg P_j \big) \Big) \Big)$$

defines A with respect to the class C of all stars.

Assume ad absurdum that some $\{R\}$-sentence of SOPML

$$\forall P_1 ... \forall P_k\, \psi,$$

where ψ is free of propositional quantifiers, defines the class A with respect to the class C of stars. Therefore the second-order formula

$$\forall x\, St_x (\forall P_1 ... \forall P_k\, \psi)$$

defines A w.r.t. C. Hence the formula

$$\exists x\, St_x (\exists P_1 ... \exists P_k\, \neg \psi)$$

defines the class $B = C \setminus A$ w.r.t. C. Thus the SOPML formula

$$\chi \ := \ \exists P_1 ... \exists P_k\, \neg \psi \quad \vee \quad \langle R \rangle\, \exists P_1 ... \exists P_k\, \neg \psi$$

defines B_p w.r.t. C_p, where C_p is the class of pointed models (M, w) where M is a star and w is the centre of the star, and B_p is the class of pointed

[9] The formula $P \vee \neg P$ contains *two occurrences* (*tokens*) of proposition symbols, but only *one type* of a proposition symbol. The formula $(P \vee \neg P) \vee (Q \wedge P)$ contains two types of proposition symbols; there are three occurrences of symbols of one type (the type of P) and one occurrence of symbols of the other type (the type of Q).

models (N, v) where $N \in B$ and where v is the centre of N. The formula χ is uniformly equivalent to the Σ_1 formula

$$\chi' \;:=\; \exists P_1 ... \exists P_k \big(\neg\psi \;\vee\; \langle R \rangle \neg\psi \big)$$

of SOPML. The formula χ' therefore defines the class of B_p w.r.t. C_p, and B_p is the class of pointed models (M, w) such that the centre w of the star M has *at least* 2^{k+1} successors. By Proposition 5.7, it is immediate that there is no Σ_1 sentence of SOPML of the vocabulary $\{R\}$ with at most k existential quantifiers that defines the class B_p with respect to the class C_p. This is a contradiction. $\square$

5.5 Modal Fragments of SO and Regular Languages

Let $k \in \mathbb{N}$ and let $\{a_1, ..., a_k\}$ be a finite nonempty set of symbols. Let

$$V = \{<, Q_{a_1}, ..., Q_{a_k}\}$$

be a vocabulary where $<$ is a binary relation symbol and the symbols Q_{a_i} are unary relation symbols. Let $n \in \mathbb{N}_{\geq 1}$ and let $W = \{1, ..., n\}$. Let $<^M$ be the strict linear order of natural numbers restricted to W, i.e.,

$$<^M \;=\; \{\; (i, j) \in W \times W \mid i < j \;\}.$$

Let

$$Q^M_{a_1}, ..., Q^M_{a_k} \;\subseteq\; W$$

be unary relations such that the following conditions hold.

1. For all $i, j \in \{1, ..., k\}$ such that $i \neq j$, we have $Q^M_{a_i} \cap Q^M_{a_j} = \emptyset$.

2. For all $i \in \{1, ..., n\}$ there exists some $j \in \{1, ..., k\}$ such that $i \in Q^M_{a_j}$.

In other words, a subset of $\{Q^M_{a_1}, ..., Q^M_{a_k}\}$ partitions the set W. The model

$$M = \big(W, <^M, Q^M_{a_1}, ..., Q^M_{a_k}\big)$$

is a *word model over the letters* $\{a_1, ..., a_k\}$.

We identify finite strings over $\{a_1, ..., a_k\}$ with word models over the letters $\{a_1, ..., a_k\}$ in a one-to-one fashion. A string

$$u \;=\; a_{i_1} ... a_{i_m}$$

of the length m is identified with the word model M^u such that

$$Dom(M^u) \;=\; \{1, ..., m\}$$

and for all $j \in \{1, ..., m\}$, we have $j \in Q^{M^u}_{a_{i_j}}$. We do not allow for the domain of a model to be empty, and therefore the empty string does not have a model associated with it. Of course we could modify our encoding scheme and overcome this problem; for example we could add an isolated point to the domain of each word model, and the empty string would then be encoded by a model whose domain would simply contain this isolated point. However, for our purposes the above encoding scheme is fine.

Let λ denote the empty string and let

$$\{a_1, ..., a_k\}^+ = \{a_1, ..., a_k\}^* \setminus \{\lambda\}.$$

That is, $\{a_1, ..., a_k\}^+$ is the set that contains exactly all finite strings over the alphabet $\{a_1, ..., a_k\}$, except for the empty string. Let $L \subseteq \{a_1, ..., a_k\}^+$ be a language over the alphabet $\{a_1, ..., a_k\}$. Let φ be a sentence of predicate logic of the vocabulary $V = \{<, Q_{a_1}, ..., Q_{a_k}\}$. We say that the sentence φ *defines* L, if the sentence φ is satisfied by exactly those word models over $\{a_1, ..., a_k\}$ that are identified with a string in L. A proof of the following theorem can be found for example in [44], where the exposition allows also for empty models to exist.

Theorem 5.10. (A variant of Büchi's theorem) *Let A be a finite nonempty set of symbols. A language $L \subseteq A^+$ is regular iff L is definable by an* MSO *sentence.*

Let A be a finite nonempty set of symbols and let M be a word model over the letters A. Let N be the expansion of M by the binary relation S^N interpreted as the successor relation over the elements of the domain of M, i.e.,

$$S^N = \{\ (i, j) \in Dom(M) \times Dom(M) \mid j = i + 1\ \}.$$

Let (N, v) be the pointed model where v is the minimum element with respect to the order $<^N$, i.e., $v = 1$. The model N is a *pointed ornamented word model* over the letters A. We also call pointed ornamented word models *p-o-words.*

Pointed ornamented word models are identified with finite strings in the obvious one-to-one fashion. Let V be the vocabulary of p-o-words over the letters A. Let φ be a V-sentence of a system of modal logic, for example SOPML. We say that the sentence φ *defines the language* $L \subseteq A^+$ if and only if the set of p-o-words over the letters A that satisfy φ is exactly the set of p-o-words that are identified with a string in L.

The next proposition characterizes the expressivity of the Σ_1 fragment of SOPML over p-o-words. The article [20] discusses related results in the context of temporal logic.

Proposition 5.11. *Let A be a finite nonempty set of symbols. A language $L \subseteq A^+$ is definable by a Σ_1 formula of* SOPML *iff L is regular.*

Proof. Let $A = \{a_1, ..., a_k\}$ and let $L \subseteq A^+$ be a regular language. Therefore L is the language accepted by some deterministic finite automaton

$$N = (Q, \delta, q_s, F),$$

where $Q = \{q_1, ..., q_n\}$ is the set of states of N, δ is the transition function mapping each pair $(q, a) \in Q \times A$ to some state in Q, $q_s \in Q$ is the start state and $F \subseteq Q$ the set of accepting states.

We define a number of formulae that enable us to write a Σ_1 formula of SOPML that defines the language accepted by N with respect to the class of all p-o-words over the letters A. If ψ is a SOPML formula, let $[<]'\psi$ denote the formula $\psi \wedge [<]\psi$. Fix a proposition variable X_{q_i} for each state $q_i \in Q$. These proposition variables will correspond to states that the automaton N is in when scanning an input word. Define

$$\varphi_{part} \;:=\; [<]' \bigvee_{i \,\in\, \{1,\ldots,n\}} \big(X_{q_i} \wedge (\bigwedge_{j \,\in\, \{1,\ldots,n\},\; j \neq i} \neg X_{q_j})\big).$$

The formula φ_{part} ensures that the variables $X_{q_1}, ..., X_{q_n}$ are interpreted such that the corresponding sets always partition the domain of a p-o-word.

Define

$$\varphi_{start} \;:=\; \bigwedge_{a \,\in\, A} \Big(Q_a \rightarrow X_{\delta(q_s, a)}\Big).$$

The formula φ_{start} simulates the first state transition of the automaton.

Define

$$\varphi_{trans} \;:=\; [<]'\Big(\bigwedge_{i \,\in\, \{1,\ldots,n\},\, j \,\in\, \{1,\ldots,k\}} \big((X_{q_i} \wedge \langle S \rangle Q_{a_j}) \rightarrow \langle S \rangle X_{\delta(q_i, a_j)}\big)\Big).$$

The formula φ_{trans} simulates the state transitions of the automaton after the first transition.

Let $\perp$ be the formula $(X_{q_s} \wedge \neg X_{q_s})$ and define

$$\varphi_{end} \;:=\; [<]'\Big([<]\perp \;\rightarrow\; \bigvee_{q \,\in\, F} X_q\Big).$$

The formula φ_{end} simulates the accepting/rejecting procedure. The Σ_1 formula

$$\exists X_{q_1} ... \exists X_{q_n} (\varphi_{part} \wedge \varphi_{start} \wedge \varphi_{trans} \wedge \varphi_{end})$$

of SOPML defines the language L.

For the converse direction, assume that $L \subseteq A^+$ is definable with respect to the class of all p-o-words over the letters A by a Σ_1 formula φ of SOPML of the vocabulary $\{<, S, Q_{a_1}, ..., Q_{a_k}\}$. Let $\psi(y, z)$ denote a first-order formula of the vocabulary $\{<\}$ that defines the successor relation on any word model. Let $\chi(x)$ denote the $\exists$MSO formula obtained from $St_x(\varphi)$ by replacing each atom of the type $S(y,z)$ by the first-order formula $\psi(y,z)$. The formula $\chi(x)$ is a formula of the vocabulary $\{<, Q_{a_1}, ..., Q_{a_k}\}$ that defines the language L w.r.t. the set of all p-o-words over A. Let $\chi(x) = \overline{\exists X}\, \chi'(x)$, where $\chi'(x)$ is the first-order part of $\chi(x)$. Let $min(x)$ be a first-order $\{<\}$-formula stating in a word model that x is the minimum element. The $\exists$MSO sentence

$$\overline{\exists X}\, \exists x \left(min(x) \,\wedge\, \chi'(x) \right)$$

defines the language L with respect to the class of word models over A. By Theorem 5.10, the language L is regular. □

5.6 SO(ML) = SO

In this section we investigate the logic SO(ML), which is the system obtained by allowing for the unrestricted quantification of proposition symbols and accessibility relations in polyadic multimodal logic.

Recall the definition of the syntax of SOPML from Chapter 2. We define the syntax of SO(ML) by extending the syntax of SOPML. As in Chapter 2, let VAR_{FO} and VAR_{SO} denote the sets of first-order and *monadic* second-order variable symbols used in predicate logic, and let

$$\mathrm{PROP} \;=\; \{\; P_x \mid x \in \mathrm{VAR}_{FO} \;\} \cup \{\; P_X \mid X \in \mathrm{VAR}_{SO} \;\}$$

be the set of proposition variable symbols. Let the set

$$\mathrm{VAR}^{+}_{SO} \;=\; \{\; Y_{i,n} \mid i \in \mathbb{N}_{\geq 1},\, n \in \mathbb{N}_{\geq 2} \;\}$$

be the set of relation variable symbols of arities higher than one used in the syntax of second-order predicate logic SO. A symbol $Y_{i,n}$ is an n-ary relation symbol. We define SO without quantification of function symbols, so the set of variable symbols used in SO is

$$U_{SO} \;=\; \mathrm{VAR}_{FO} \cup \mathrm{VAR}_{SO} \cup \mathrm{VAR}^{+}_{SO},$$

where the three sets on the right hand side are of course assumed to be disjoint. The set of variable symbols used in SO(ML) is the set

$$U_{ML} \;=\; \mathrm{PROP} \cup \mathrm{VAR}^{+}_{SO}.$$

As in Chapter 2, let

$$S = S_0 \cup S_1 \cup S_2 \cup S_+$$

be a vocabulary, where S_0 is a set of constant symbols, S_1 and S_2 are sets of unary and binary relation symbols, respectively, and S_+ is a set of relation symbols of higher arities. We assume the sets S, PROP and VAR^{+}_{SO} are disjoint. The set of SO(ML) formulae of the vocabulary S is the smallest set T such that the following conditions are satisfied.

1. If $c \in S_0$, then $c \in T$.
2. If $P_\# \in \mathrm{PROP}$, then $P_\# \in T$.
3. If $P \in S_1$, then $P \in T$.
4. If $\varphi \in T$, then $\neg\varphi \in T$.
5. If $\varphi \in T$ and $\psi \in T$, then $(\varphi \wedge \psi) \in T$.
6. If $R \in S_2$ and $\varphi \in T$, then $\langle R\rangle\varphi \in T$.
7. If $Y_{i,2} \in \mathrm{VAR}^+_{SO}$ and $\varphi \in T$, then $\langle Y_{i,2}\rangle\varphi \in T$.
8. If $R' \in S_+$ is a k-ary relation symbol and $\varphi_i \in T$ for all $i \in \{1, ..., k-1\}$, then $\langle R'\rangle(\varphi_1, ..., \varphi_{k-1}) \in T$.
9. If $Y_{i,k} \in \mathrm{VAR}^+_{SO}$ is a k-ary relation variable symbol, $k \in \mathbb{N}_{\geq 3}$, and if $\varphi_j \in T$ for all $j \in \{1, ..., k-1\}$, then $\langle Y_{i,k}\rangle(\varphi_1, ..., \varphi_{k-1}) \in T$.
10. If $P_\# \in \mathrm{PROP}$ and $\varphi \in T$, then $\exists P_\#\, \varphi \in T$.
11. If $Y_{i,n} \in \mathrm{VAR}^+_{SO}$ and $\varphi \in T$, then $\exists Y_{i,n}\, \varphi \in T$.

An SO(ML) formula that does not contain free variables is called an SO(ML) sentence. The set of non-logical symbols of an SO(ML) formula φ of the vocabulary S is the set $S' \subseteq S$ of symbols that occur in S and φ.

Let $c \in S_0$, $P \in S_1$ and $R \in S_2$. Let $P_\# \in \mathrm{PROP}$ be a proposition variable. Let $Y_{i,2} \in \mathrm{VAR}^+_{SO}$ be a binary and $Y_{l,n} \in \mathrm{VAR}^+_{SO}$ an n-ary relation variable, $n \in \mathbb{N}_{\geq 2}$. Let $k \in \mathbb{N}_{\geq 3}$, and let $R' \in S_+$ be a k-ary relation symbol and $Y_{j,k} \in \mathrm{VAR}^+_{SO}$ a k-ary relation variable. Let $\varphi, \psi, \varphi_1, ..., \varphi_{k-1}$ be formulae of SOPML of the vocabulary S. Let (M, w) be a pointed model of the vocabulary S, and let $W = Dom(M)$. Let V be a valuation function that interprets the variables in U_{ML} in the model M. We define

$$
\begin{array}{lcl}
(M,w), V \Vdash c & \Leftrightarrow & w = c^M, \\
(M,w), V \Vdash P & \Leftrightarrow & w \in P^M, \\
(M,w), V \Vdash P_\# & \Leftrightarrow & w \in V(P_\#), \\
(M,w), V \Vdash \neg\varphi & \Leftrightarrow & (M,w), V \nVdash \varphi, \\
(M,w), V \Vdash (\varphi \wedge \psi) & \Leftrightarrow & (M,w), V \Vdash \varphi \text{ and } (M,w), V \Vdash \psi, \\
(M,w), V \Vdash \exists P_\#\, \varphi & \Leftrightarrow & \exists U \subseteq W\big((M,w), V\frac{U}{P_\#} \Vdash \varphi\big), \\
(M,w), V \Vdash \exists Y_{l,n}\, \varphi & \Leftrightarrow & \exists K \subseteq W^n \text{ such that } (M,w), V\frac{K}{Y_{l,n}} \Vdash \varphi, \\
(M,w), V \Vdash \langle R\rangle\, \varphi & \Leftrightarrow & \exists u \in W\big(wR^M u \text{ and } (M,u) \Vdash \varphi\big), \\
(M,w), V \Vdash \langle Y_{i,2}\rangle\, \varphi & \Leftrightarrow & \exists u \in W \text{ such that } (w,u) \in V(Y_{i,2}) \\
 & & \text{and } (M,u) \Vdash \varphi,
\end{array}
$$

$$
\begin{array}{lcl}
(M,w),V \Vdash \langle R' \rangle (\varphi_1,...,\varphi_{k-1}) & \Leftrightarrow & \exists u_1...u_{k-1} \in W \text{ such that} \\
 & & R'^M(w,u_1,...,u_{k-1}) \text{ and} \\
 & & (M,u_i),V \Vdash \varphi_i \text{ for each } i, \\
(M,w),V \Vdash \langle Y_{j,k} \rangle (\varphi_1,...,\varphi_{k-1}) & \Leftrightarrow & \exists u_1...u_{k-1} \in W \text{ such that} \\
 & & (w,u_1,...,u_{k-1}) \in V(Y_{j,k}) \text{ and} \\
 & & (M,u_i),V \Vdash \varphi_i \text{ for each } i.
\end{array}
$$

We let Σ_1SO(ML) denote the fragment of SO(ML) where formulae have a prefix consisting of a block of existential second-order quantifiers, and the prefix is followed by a formula free of second-order quantifiers. Similarly, we let Σ_2SO(ML) be the fragment of SO(ML) where formulae have a prefix consisting of a block of existential second-order quantifiers followed by a block of universal second-order quantifiers, and after this prefix there is a formula free of second-order quantifiers.

Let $A = \{a, b\}$, and let W_p denote the set of p-o-words over the letters A. By Theorems 3.15 and 5.10, it is easy to see that there exists no Σ_1SO(ML) sentence that defines w.r.t. W_p a set corresponding to a non-regular language. By Proposition 5.11 and Theorem 5.10 it is easy to see that the Σ_1 fragment of SOPML is equi-expressive with MSO over W_p. Therefore, all together, starting from the Σ_1 fragment of SOPML, *neither* increasing the number of quantifier alternations of propositional quantifiers (moving to SOPML) *nor* increasing the arity of quantifiable existential relations (moving to Σ_1SO(ML)) leads to an increase in expressivity over W_p. What is needed is an increase in *both* arity and alternation. It follows immediately from well known results that the set of palindromes in A^+ is not a regular language. However, the set of p-o-words corresponding to palindromes in A^+ is definable with respect to W_p by a sentence of Σ_2SO(ML), as the following proposition establishes.

Proposition 5.12. *Let A be a finite nonempty set of symbols. The set of p-o-words corresponding to palindromes in A^+ is definable with respect to W_p by a sentence of* Σ_2SO(ML).

Proof. Again let $[<]'\psi$ denote the formula $\psi \wedge [<]\psi$. Let X_1 be a proposition variable and let $\top$ and $\bot$ denote the formulae $X_1 \vee \neg X_1$ and $X_1 \wedge \neg X_1$, respectively. We begin by defining a number of auxiliary formulae. Let us first define

$$\varphi_{out-deg} \;:=\; [<]'\Big(\langle R\rangle\top \wedge \big(\langle R\rangle X_1 \rightarrow [R]X_1\big)\Big),$$

where R is a binary relation *variable*, to be existentially quantified later. The formula $\varphi_{out-deg}$ will ensure that the extension of R must have out-degree one everywhere.

Let us then define

$$\varphi_{min-max} \;:=\; \langle R\rangle[S]\bot.$$

Recall that S is in the vocabulary of p-o-words, corresponding to the successor relation. The formula will ensure that the extension of R connects the minimum element of the linear order of a p-o-word to the maximum element.

Define

$$\varphi_{next-prev} \ := \ [<]'\Big(\big(\neg X_1 \wedge \neg\langle S\rangle X_1 \wedge \langle R\rangle X_1\big) \to \langle S\rangle\langle R\rangle\langle S\rangle X_1\Big).$$

This formula will ensure that if uRv such that $u \neq v$ and not uSv, then $S(u)\ R\ S^{-1}(v)$, where $S(u)$ denotes the S-successor of u and $S^{-1}(v)$ the S-predecessor of v.

Finally, define

$$\varphi_{match} \ := \ [<]'\Big(\bigwedge_{a\in A}\big(Q_a \to \langle R\rangle Q_a\big)\Big).$$

This formula will ensure that if uRv, then there is a letter $a \in A$ such that u and v are both in the extension the same proposition symbol Q_a.

The formula

$$\exists R\forall X_1\big(\varphi_{out-deg} \wedge \varphi_{min-max} \wedge \varphi_{next-prev} \wedge \varphi_{match}\big)$$

defines the set of p-o-words corresponding to palindromes in A^+ with respect to the class of p-o-words over the letters A. $\square$

We then conclude the chapter by showing that SO(ML) = SO. A formula of predicate logic $\varphi(x)$ that contains exactly one free variable (the first-order variable x) is uniformly equivalent to a modal sentence φ', if $\varphi(x)$ and φ' have exactly the same set S of non-logical symbols, and if also

$$M, \frac{w}{x} \models \varphi(x) \ \Leftrightarrow \ (M, w) \Vdash \varphi'$$

for all pointed S-models (M, w). We will now show that for every second-order formula without function symbols and with exactly one free variable (which is a first-order variable), there is a uniformly equivalent sentence of SO(ML). The converse statement is obvious by (a trivial generalization of) the standard translation.

Theorem 5.13. *Let $\psi(x)$ be a formula of* SO *without function symbols and with exactly one free variable, the first-order variable x. There exists a sentence of* SO(ML) *that is uniformly equivalent to $\psi(x)$. Conversely, for each sentence of* SO(ML)*, there exists a uniformly equivalent formula of* SO *with exactly one free first-order variable and no other free variables. There exist effective translations in both directions.*

Proof. The proof of the current theorem is based on the proofs of Lemma 2.2 and Theorem 2.3.

Let $\psi(x)$ be formula of second-order logic without function symbols and with exactly one free variable, the first-order variable x. We will define a sentence ψ' of SO(ML) that is uniformly equivalent to $\psi(x)$.

Let S be the set of non-logical symbols that occur in $\psi(x)$. Let M be an S-model and let f be a variable assignment function that interprets each variable in U_{SO} in M. Let $Y_{j,2} \in \mathrm{VAR}_{SO}^{+}$ be a binary relation variable symbol that does not occur in the formula $\psi(x)$. We let V_f^M denote the valuation mapping that interprets each variable symbol in U_{ML} in M such that the following conditions hold.

1. $V_f^M(P_x) = \{f(x)\}$ for all $P_x \in \mathrm{PROP}$ such that $x \in \mathrm{VAR}_{FO}$.
2. $V_f^M(P_X) = f(X)$ for all $P_X \in \mathrm{PROP}$ such that $X \in \mathrm{VAR}_{SO}$.
3. $V_f^M(Y_{i,n}) = f(Y_{i,n})$ for all $Y_{i,n} \in \mathrm{VAR}_{SO}^{+} \setminus \{Y_{j,2}\}$.
4. $V_f^M(Y_{j,2}) = Dom(M) \times Dom(M)$.

Define

$$uniq_{Y_{j,2}}(P_x) \;:=\; \langle Y_{j,2}\rangle P_x \;\wedge\; \forall P_y\big(\langle Y_{j,2}\rangle(P_y \wedge P_x) \rightarrow [Y_{j,2}](P_x \rightarrow P_y)\big).$$

Let $X \in \mathrm{VAR}_{SO}$ and $P, R, R', c, c' \in S$. Let $Y \in \mathrm{VAR}_{SO}^{+}$ be a second-order variable of the arity two. Let $k \geq 3$, and let $Y' \in \mathrm{VAR}_{SO}^{+}$ be a second-order variable symbol of the arity k. Let $Z \in \mathrm{VAR}_{SO}^{+}$ be a symbol of an arity at least two. Let $Tr_{Y_{j,2}}$ denote the translation defined by the following clauses.

$$\begin{array}{lcl}
Tr_{Y_{j,2}}(P(x)) & = & \langle Y_{j,2}\rangle(P \wedge P_x) \\
Tr_{Y_{j,2}}(X(y)) & = & \langle Y_{j,2}\rangle(P_X \wedge P_y) \\
Tr_{Y_{j,2}}(R(x,y)) & = & \langle Y_{j,2}\rangle(P_x \wedge \langle R\rangle P_y) \\
Tr_{Y_{i,2}}(Y(x,y)) & = & \langle Y_{j,2}\rangle(P_x \wedge \langle Y\rangle P_y) \\
Tr_{Y_{j,2}}(R'(x_1,...,x_n)) & = & \langle Y_{j,2}\rangle\big(P_{x_1} \wedge \langle R'\rangle(P_{x_2},...,P_{x_n})\big) \\
Tr_{Y_{j,2}}(Y'(x_1,...,x_k)) & = & \langle Y_{j,2}\rangle\big(P_{x_1} \wedge \langle Y'\rangle(P_{x_2},...,P_{x_k})\big) \\
Tr_{Y_{j,2}}(x = y) & = & \langle Y_{j,2}\rangle(P_x \wedge P_y) \\
Tr_{Y_{j,2}}(c = x) & = & \langle Y_{j,2}\rangle(c \wedge P_x) \\
Tr_{Y_{j,2}}(x = c) & = & \langle Y_{j,2}\rangle(P_x \wedge c) \\
Tr_{Y_{j,2}}(c = c') & = & \langle Y_{j,2}\rangle(c \wedge c') \\
Tr_{Y_{j,2}}(\neg\varphi) & = & \neg\, Tr_{Y_{j,2}}(\varphi) \\
Tr_{Y_{j,2}}(\,(\varphi \wedge \psi)\,) & = & \big(Tr_{Y_{j,2}}(\varphi) \wedge Tr_{Y_{j,2}}(\psi)\big) \\
Tr_{Y_{j,2}}(\exists x\, \varphi) & = & \exists P_x\big(uniq_{Y_{j,2}}(P_x) \wedge Tr_{Y_{j,2}}(\varphi)\big) \\
Tr_{Y_{j,2}}(\exists X\, \varphi) & = & \exists P_X\, Tr_{Y_{j,2}}(\varphi) \\
Tr_{Y_{j,2}}(\exists Z\, \varphi) & = & \exists Z\, Tr_{Y_{j,2}}(\varphi)
\end{array}$$

The translation is almost identical to the translations Tr and Tr_R defined in Chapter 2. We claim that

$$M, f \models \varphi \quad \Leftrightarrow \quad (M, w), V_f^M \Vdash Tr_{Y_{j,2}}(\varphi)$$

for all S-formulae φ of SO not containing the relation symbol $Y_{j,2}$, all pointed S-models (M, w) and all assignment functions f interpreting the symbols in U_{SO} in M. The claim follows by an argument that is practically identical to the proof of Lemma 2.2, the only non-trivial case of the inductive argument being the case concerning formulae of the type $\exists x\, \alpha$. Using the claim we infer that

$$M, f\frac{w}{x} \models \psi(x) \;\Leftrightarrow\; (M, w), V_f^M \frac{\{w\}}{P_x} \Vdash Tr_{Y_{j,2}}(\psi(x))$$

for all pointed S-models (M, w) and related assignment functions f. Notice that the formula $Tr_{Y_{j,2}}(\psi(x))$ has exactly two free variables, P_x and $Y_{j,2}$. Let $m \in \mathbb{N}_{\geq 1}$, $m \neq j$. Noting that the formula

$$\forall Y_{m,2} \forall P_y \big(\langle Y_{m,2} \rangle \langle Y_{m,2} \rangle P_y \;\rightarrow\; \langle Y_{m,2} \rangle \langle Y_{j,2} \rangle P_y \big)$$

asserts in any pointed model that $Y_{j,2}$ must be interpreted as the total binary relation, it is now easy to observe that the sentence

$$\exists Y_{j,2} \exists P_x \forall Y_{m,2} \forall P_y \Big(\big(\langle Y_{m,2} \rangle \langle Y_{m,2} \rangle P_y \;\rightarrow\; \langle Y_{m,2} \rangle \langle Y_{j,2} \rangle P_y \big) \wedge$$
$$P_x \;\wedge\; uniq_{Y_{j,2}}(P_x) \;\wedge\; Tr_{Y_{j,2}}(\psi(x)) \Big)$$

is uniformly equivalent to $\psi(x)$.

The translation from SO(ML) into SO is a trivial extension of the standard translation. $\square$

Of course a similar result applies to models as well as pointed models. A sentence φ of SO(ML) is globally uniformly equivalent to a sentence of second-order predicate logic φ' if φ and φ' have exactly the same set S of non-logical symbols, and if also

$$M \models \varphi' \;\Leftrightarrow\; \forall w \in Dom(M)\big((M, w) \Vdash \varphi \big)$$

for all S-models M.

Theorem 5.14. *There exists a globally uniformly equivalent sentence of* SO(ML) *for each* SO *sentence that does not contain function symbols. Conversely, for each* SO(ML) *sentence there is a globally uniformly equivalent sentence of* SO. *There exist effective translations in both directions.*

Proof. The proof of the current theorem is a simple variation of the proof of Theorem 5.13.

Let ψ be a sentence of SO and let S be the set of non-logical symbols occurring in ψ. Assume that S does not contain function symbols. Let $Y_{j,2}$ be a relation variable symbol not occurring in ψ. As above, we have

$$M, f \models \varphi \quad \Leftrightarrow \quad (M, w), V_f^M \Vdash Tr_{Y_{j,2}}(\varphi)$$

for all S-formulae φ of SO not containing the relation variable $Y_{j,2}$, all pointed S-models (M, w) and all assignment functions f interpreting the variable symbols in U_{SO} in M. Notice that now $Tr_{Y_{j,2}}(\psi)$ has exactly one free variable, $Y_{j,2}$. Again let $m \in \mathbb{N}_{\geq 1}$, $m \neq j$. The formula

$$\exists Y_{j,2} \forall Y_{m,2} \forall P_y \Big(\big(\langle Y_{m,2} \rangle \langle Y_{m,2} \rangle P_y \;\rightarrow\; \langle Y_{m,2} \rangle \langle Y_{j,2} \rangle P_y \big) \;\wedge\; Tr_{Y_{j,2}}(\psi) \Big).$$

is globally uniformly equivalent to the sentence ψ. □

We conclude the section by observing that SO(ML) admits a prenex normal form representation of sentences. Two formulae φ and ψ of SO(ML) are uniformly equivalent, if the following conditions are satisfied.

1. Exactly the same set $U \subseteq U_{ML}$ of variables occur free in both formulae.
2. The formulae have exactly the same set S of non-logical symbols.
3. We have
$$(M, w), V \Vdash \varphi \;\Leftrightarrow\; (M, w), V \Vdash \psi.$$
for all pointed S-models M and all valuation functions V interpreting the variable symbols in U_{ML} in the model M.

Similarly, two formulae φ' and ψ' of SO are uniformly equivalent, if the following conditions are satisfied.

1. Exactly the same set $U \subseteq U_{SO}$ of variables occur free in both formulae.
2. The formulae have exactly the same set S of non-logical symbols.
3. We have
$$M, f \models \varphi' \;\Leftrightarrow\; M, f \models \psi'$$
for all S-models M and all variable assignments f interpreting the variable symbols in U_{SO} in the model M.

Theorem 5.15. *Each sentence of* SO(ML) *can be effectively transformed into a uniformly equivalent sentence of* SO(ML) *in prenex normal form, i.e., a form where formulae begin with a prefix of second-order quantifiers, and the prefix is followed by a part free of second-order quantifiers.*

Proof. Let φ be an arbitrary sentence of SO(ML). It is well known that any formula of SO can be effectively transformed into a uniformly equivalent SO formula in a form where a block of second-order quantifiers is followed by a first-order part. Let $\psi(x)$ denote a formula of SO that is uniformly equivalent to $St_x(\varphi)$ and written in a form $\overline{Q}\,\chi(x)$, where $\overline{Q}$ is a vector of second-order quantifiers and $\chi(x)$ is a first-order formula. Here St_x denotes a generalization of the standard translation operator.

Use the procedure in the proof of Theorem 5.13 to convert $\psi(x)$ to the uniformly equivalent SO(ML) sentence

$$\exists Y_{j,2} \exists P_x \forall Y_{m,2} \forall P_y \Big(\big(\langle Y_{m,2} \rangle \langle Y_{m,2} \rangle P_y \ \rightarrow \ \langle Y_{m,2} \rangle \langle Y_{j,2} \rangle P_y \big) \wedge \\ P_x \ \wedge \ uniq_{Y_{j,2}}(P_x) \ \wedge \ Tr_{Y_{j,2}}(\psi(x)) \Big).$$

Call this sentence α. Since $\psi(x)$ is of the form $\overline{Q}\,\chi(x)$, where $\chi(x)$ is first-order, we observe that $Tr_{Y_{j,2}}(\psi(x))$ is of the form $\overline{Q}'\,\beta$, where $\overline{Q}'$ is a vector of second-order quantifiers and β is *essentially* an SOPML formula; β may contain relation *variables* from VAR^{+}_{SO}, but all second-order quantifiers in β are propositional quantifiers. We know that SOPML admits a prenex normal form representation of formulae by Proposition 5.2, and by the uniform equivalences justifying Proposition 5.2, it is clear that the formula β can be transformed into prenex normal form. Hence we conclude that we can transform α into prenex normal form. $\square$

5.7 Chapter Conclusion

In this chapter we have investigated fragments of SOPMLE and SO(ML) and proved a number of straightforward expressivity-related results. Even though technically a relatively simple result, the principal discovery of the chapter is that SO(ML) is equi-expressive with SO and also that SO(ML) admits a prenex normal form representation. These results establish a *modal normal form for second-order logic.* The normal form is an example of a result that can provide alternative approaches to proving theorems about second-order logic. After all, it seems that modal logic is often a lot simpler to use than first-order logic. However, it is obvious that the normal form based on modal logic is not the only interesting normal form possible. Indeed, it would be interesting to identify even simpler normal forms for SO.

CHAPTER 6

Concluding Remarks

In the above chapters we have investigated various fragments of second-order logic, the common denominator of the fragments being that they are all directly related to extensions of modal logic. In Chapter 2 we answered an open problem from [5] and [11] by showing that the alternation hierarchy of SOPML is infinite. In Chapter 3 we established that Σ^1_1(ML) translates into monadic Σ^1_1(MLE) and $\Sigma^1_1(\mathrm{PBML}^=)$ into ∃MSO, thereby identifying fragments of Σ^1_1 that translate into ∃MSO. We showed how these observations lead to decidability results for extensions of multimodal logic over various classes of frames. In Chapter 4 we investigated the equality-free system $\mathrm{fESO}_{wo=}$, which can be useful for example in the study of independence-friendly modal logic. The main contribution of that chapter was the argument establishing that over finite models with a unary vocabulary, $\mathrm{fESO}_{wo=}$ is weaker than FO. In Chapter 5 we proved a variety of results concerning fragments of SOPLME and SO(ML). Among other things, we showed that SO(ML) is equi-expressive with second-order logic, and thereby obtained a modal normal form for second-order logic.

One of the two main open problems to be addressed in the future is the question whether the alternation hierarchy of SOPML is strict. The other one is the question of Grädel and Rosen asking whether $\Sigma^1_1(\mathrm{FO}^2)$ is contained in ∃MSO. To show this, one would have to extend the translation from $\Sigma^1_1(\mathrm{BML}^=)$ into ∃MSO developed in Chapter 3 such that it takes into account the possibility of using the converse operation. In addition to these two open problems, there are various other topics worth studying related to the investigations in this thesis. For example, the program suggested in Chapter 3 that involves classifying fragments L of FO such that Σ^1_1(L) is contained in ∃MSO, is worth mentioning here. The next planned step related to this program involves considering graded modalities.

Understanding modal fragments of second-order logic serves at least two purposes rather directly. Firstly, developing the understanding of *fragments* of second-order logic—also fragments that would not be characterized as modal—is central in second-order model theory. A developed theory of second-order logic can help solve difficult problems in finite model theory, for instance. Secondly, theorems about modal fragments of second-order logic can be used as tools in investigations of modal systems geared towards applications. Various different kinds of theorems concerning a very expressive modal logic L immediately apply to all weaker logics that translate into L. Given the success of modal logic in relation to applications, it is quite clear that modal logic deserves a developed mathematical background theory.

References

[1] C. Areces, P. Blackburn and M. Marx. Hybrid logic is the bounded fragment of first order logic. In *Proceedings of WoLLIC*, pages 33-50, 1999.

[2] C. Areces, P. Blackburn and M. Marx. Hybrid logics: characterization, interpolation and complexity. *Journal of Symbolic Logic*, 66(3):977-1010, 2001.

[3] C. Areces and B. ten Cate. Hybrid logics. In P. Blackburn, F. Wolter and J. van Benthem, editors, *Handbook of Modal Logic*. Elsevier, 2006.

[4] S. Artemov and L. Beklemishev. On propositional quantifiers in provability logic. *Notre Dame Journal of Formal Logic*, 34(3):401-419, 1993.

[5] J. van Benthem. *Modal Logic and Classical Logic*. Bibliopolis, 1983.

[6] M. Bílková. Uniform interpolation and propositional quantification in modal logics. *Studia Logica*, 85(1):1-31, 2007.

[7] P. Blackburn, M. de Rijke and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[8] P. Blackburn and J. Seligman. Hybrid Languages. *Journal of Logic, Language and Information*, 4(3):251-272, 1995.

[9] R.A. Bull. On modal logic with propositional quantifiers. *Journal of Symbolic Logic*, 34(2):257-263, 1969.

[10] E. Casanovas, P. Dellunde and R. Jansana. On elementary equivalence for equality-free logic. *Notre Dame Journal of Formal Logic*, 37(3):506-522, 1996.

[11] B. ten Cate. Expressivity of second order propositional modal logic. *Journal of Philosophical Logic*, 35(2):209-223, 2006.

[12] B. ten Cate. *Model Theory for Extended Modal Languages*. Ph.D. thesis, ILLC, University of Amsterdam, 2005.

[13] T. Costello and A. Patterson. Quantifiers and Operations on Modalities and Contexts. In *Proceedings of KR'98*, pages 270-281, 1998.

[14] L. E. Dickson. Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. *American Journal of Mathematics*, 35(4):413-422, 1913.

[15] H.-D. Ebbinghaus, J. Flum and W. Thomas. *Mathematical Logic*. 2nd edition, Springer, 1994.

[16] K. Fine. Propositional quantifiers in modal logic. *Theoria*, 36(3):336-346, 1970.

[17] M. Fischer and R. Ladner. Propositional dynamic logic of regular programs. *Journal of Computer and System Sciences*, 18(2):194-211, 1979.

[18] M. Fitting. Interpolation for first order S5. *Journal of Symbolic Logic*, 67(2):621-634, 2002.

[19] G. Gargov, S. Passy and T. Tinchev. Modal environment for Boolean speculations. In D. Skordev, editor, *Mathematical Logic and its Applications*, Plenum Press, New York, 1987.

[20] A. Gheerbrant and B. ten Cate. Craig interpolation for linear temporal languages. In *Proceedings of CSL*, pages 287-301, 2009.

[21] V. Goranko and M. Otto. Model theory of modal logic. In P. Blackburn, F. Wolter and J. van Benthem, editors, *Handbook of Modal Logic*. Elsevier, 2006.

[22] V. Goranko and S. Passy. Using the universal modality: gains and questions. *Journal of Logic and Computation*, 2(1):5-30, 1992.

[23] E. Grädel and E. Rosen. Two-variable descriptions of regularity. In *Proceedings of LICS*, pages 14-23, 1999.

[24] D. Harel, D. Kozen and J. Tiuryn. *Dynamic Logic*. MIT Press, 2000.

[25] L. Hella and A. Kuusisto. Monadic Σ^1_1 and modal logic with quantified binary relations. In *Proceedings of M4M6, Electronic Notes in Theoretical Computer Science*, vol. 262, 2010.

[26] L. Henkin. Some remarks on infinitely long formulas. In *Infinitistic Methods: Proceedings of the Symposium on Foundations of Mathematics, Warsaw, 29 September 1959*, pages 167-183, Pergamon Press, 1961.

[27] J.E. Hopcroft, R. Motwani and J.D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. 2nd edition, Addison Wesley, 2000.

[28] G. Higman. Ordering by divisibility in abstract algebras. In *Proceedings of the London Mathematical Society*, 2:326-336, 1952.

[29] J. Hintikka. Quantifiers vs. quantification theory. *Dialectica*, 27:329-358, 1973.

[30] J. Hintikka and G. Sandu. Informational independence as a semantical phenomenon. In J.E. Fenstad et al., editors, *Logic, Methodology and Philosophy of Science VIII*, pages 571-589, North-Holland, 1989.

[31] U. Hustadt and R.A. Schmidt. Issues of decidability for description logics in the framework of resolution. In R. Caferra and G. Salzer, editors, *Automated Deduction in Classical and Non-Classical Logics*, Lecture Notes in Artificial Intelligence, vol. 1761, pages 191-205, 2000.

[32] N. Immerman. *Descriptive Complexity.* Springer, 1999.

[33] D. Kaplan. S5 with quantifiable propositional variables. *Journal of Symbolic Logic*, 35(2):355, 1970.

[34] J. Kontinen, A. Kuusisto, P. Lohmann and J. Virtema. Complexity of two-variable dependence logic and IF-logic. In *Proceedings of LICS*, 2011.

[35] J. Kontinen and J. Väänänen, editors. *Dependence and Independence in Logic.* Proceedings of the ESSLLI workshop on dependence and independence in logic, 2010.

[36] P. Kremer. Defining relevant implication in a propositionally quantified S4. *Journal of Symbolic Logic*, 62(4):1057-1069, 1997.

[37] P. Kremer. On the complexity of propositional quantification in intuitionistic logic. *Journal of Symbolic Logic*, 62(2):529-544, 1997.

[38] S. Kripke. A Completeness theorem in modal logic. *Journal of Symbolic Logic*, 21(1):1-14, 1959.

[39] A. Kuusisto. A modal perspective on monadic second-order alternation hierarchies. In *Proceedings of AiML*, 2008.

[40] A. Kuusisto. Logics of imperfect information without identity. In *Proceedings of the ESSLLI Workshop on Dependence and Independence in Logic*, 2010.

[41] A. Kuusisto. Expressivity of imperfect information logics without identity. To appear in *Studia Logica.*

[42] S. Lehtinen. *Generalizing the Goldblatt-Thomason Theorem and Modal Definability.* Ph.D. thesis, Acta Universitatis Tamperensis, University of Tampere, 2008.

[43] D. Leivant. Propositional dynamic logic with program quantifiers. In *Proceedings of MFPS XXIV, Electronic Notes in Theoretical Computer Science*, vol. 218, pages 231-240, 2008.

[44] L. Libkin. *Elements of Finite Model Theory.* Springer, 2004.

[45] C. Lutz and U. Sattler. The complexity of reasoning with Boolean modal logics. In *Proceedings of AiML*, pages 329-348, 2000.

[46] C. Lutz, U. Sattler and F. Wolter. Modal logic and the two-variable fragment. In *Proceedings of CSL*, pages 247-261, 2001.

[47] J.A. Makowsky and Y.B. Pnueli. Arity and alternation in second-order logic. *Annals of Pure and Applied Logic*, 78(1-3):189-202, 1996.

[48] M. Malicki and A. Rutkowski. On operations and linear extensions of well partially ordered sets. *Order* 21(1):7-17, 2004.

[49] F. Massacci. Decision procedures for expressive description logics with intersection, composition, converse of roles and role identity. In *Proceedings of IJCAI*, pages 193-198, 2001.

[50] O. Matz and N. Schweikardt. Expressive power of monadic logics on words, trees, pictures, and graphs. In J. Flum, E. Grädel and T. Wilke, editors, *Logic and Automata: History and Perspectives*, pages 531-552, Amsterdam University Press, 2007.

[51] O. Matz, N. Schweikardt and W. Thomas. The monadic quantifier alternation hierarchy over grids and graphs. *Information and Computation*, 179(2):356-383, 2002.

[52] O. Matz and W. Thomas. The monadic quantifier alternation hierarchy over graphs is infinite. In *Proceedings of LICS*, pages 236-244, 1997

[53] R. Pesonen. *Propositional State Transition Logics*. Thesis in Minor Subject, University of Tampere, 2008.

[54] M.O. Rabin. Decidability of second-order theories and automata on infinite trees. *Transactions of the American Mathematical Society*, 141:1-35, 1969.

[55] M. de Rijke. The modal logic of inequality. *Journal of Symbolic Logic*, 57(2):566-584, 1992.

[56] N. Schweikardt. The monadic quantifier alternation hierarchy over grids and pictures. In *Proceedings of CSL*, pages 441-460, 1997.

[57] N. Schweikardt. *The Monadic Second-Order Quantifier Alternation Hierarchy over Grids and Pictures*. Diplomarbeit, Johannes Gutenberg-Universität Mainz, 1997.

[58] M. Sevenster. Decidability of independence-friendly modal logic. *The Review of Symbolic Logic*, 3:415-441, 2010.

[59] N.V. Shilov and K. Yi. On expressive and model checking power of propositional program logics. In *Ershov Memorial Conference*, pages 39-46, 2001.

[60] C. Stirling. Games and modal mu-calculus. In *Proceedings of TACAS*, pages 298-312, 1996.

[61] L.J. Stockmeyer. The polynomial-time hierarchy. *Theoretical Computer Science*, 3(1):1-22, 1976.

[62] W. Thomas. Constructing infinite graphs with a decidable MSO-theory. In *Proceedings of MFCS*, pages 113-124, 2003.

[63] T. Tulenheimo and M. Sevenster. Approaches to independence friendly modal logic. In J. van Benthem, D. Gabbay and B. Löwe, editors, *Interactive Logic. Selected Papers from the 7th Augustus de Morgan Workshop, London*, pages 247-280, 2007.

[64] T. Tulenheimo. *Independence-Friendly Modal Logic*. Ph.D. thesis, University of Helsinki, 2004.

[65] J. Väänänen. *Dependence Logic*. Cambridge University Press, 2007.