DEDUCTION SYSTEMS FOR MULTIMODAL LOGICS WITH OPERATIONS ON MODALITIES

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Abstract

This article is a brief, user friendly discourse on extensions of multimodal logics with operations on modalities. Such logics are considered from a general point of view with the emphasis on the theory of deduction systems. The first part of the article concentrates on developing general tools that lead to partial completeness results for logics where the algebra of modalities is effectively axiomatizable. The second part is devoted to the completeness issue of multimodal logic with union and intersection of modalities. A natural, straightforward axiomatization and a proof of completeness are given.

1 Introduction

An interesting family of logics arise when multimodal logics are generalized by allowing operations on modalities. Perhaps the most widely studied example of such a logic is Propositional Dynamic Logic (PDL), considered in [12]. Also various different kinds of extensions and variations of PDL have been considered by a large number of researchers. Boolean Modal Logic [11] is another interesting example of a multimodal logic extended with operations on modalities.

In the first part of this article (sections 2-6) we consider the family of multimodal logics with an algebraic structure over the set of modalities from a general point of view, without singling out any particular member of this family. We shall call members of the family *Multimodal Logics with Operations on Modalities*, or $MLOM$s. Such logics have been given various names, for example in [8] they are called *Modal Action Logics* (MALs).

We begin our discourse by considering the theory of deduction systems related to the whole family of $MLOM$s. We develop generally applicable tools designed for the algebraic manipulation of modalities within formal deductions. Based on this we then prove two closely related partial completeness results that apply to $MLOM$s with an effectively axiomatizable algebra of modalities. We also discuss some general properties of deduction systems related to $MLOM$s.

In the second part of the paper (sections 7-9) we apply the tools developed in the first part and give two alternative complete axiomatizations of multimodal logic with union and intersection of modalities. The related deduction systems have the advantageous property that all rules of inference involved can be applied to any $MLOM$. In other words, we do not need inference rules specifically designed to deal with $MLOM$s with only union and intersection operations. Therefore the deduction systems can be dynamically extended to deal with more complex $MLOM$s without sacrificing completeness with respect to the fragment of formulae containing union and intersection operations only. In addition to this, we believe that both axiomatizations are simple and arise in a natural way from more elementary considerations.

A number of articles have been written on completeness issues of different extensions and fragments of PDL. See [3, 1, 4, 2, 5] for example. Various interesting different proof techniques have been developed, and in some cases a particular variation has been given multiple proofs of completeness: Compare [3] and [1] for example.

A major advantage of the completeness proofs presented in this paper is that they are relatively straightforward and short: Some of the material presented in the paper deals with elementary background issues related to $MLOM$s and deduction systems in modal logic in general and can therefore be omitted by a reader familiar with the basics of the proof theory of modal logics. The material has been presented here in order to make the paper easy to read and accessible to the non-specialist.

Articles [4] and [5] deal with more general axiomatizations than those given in this paper. Also, in the light of Theorem 10.1 below, it is clear that a large number of $MLOM$s can directly be given a complete axiomatization. We believe, however, that the general deduction tools developed in this article are interesting in their own right and may prove useful for the construction of yet more general axiomatizations of $MLOM$s. In addition to this we believe that the internal simplicity of our axiomatizations of the case with union and intersection could perhaps turn out to be advantageous particularly in mathematically oriented further work.

2 Preliminary Considerations

Before introducing the syntax and semantics of $MLOM$s, we must consider a few simple notions that occupy a central role in most considerations that follow. We begin by giving a formal definition of the concept of *operation* suitable for the purposes of this discourse.

Definition 2.1. An *operation* is defined to be an object with some arity $n \in \mathbb{N}$. An n-ary operation acts on n binary relations and one additional set (the domain of the relations), returning a unique binary relation. For each operation there is a unique symbol that denotes the operation. Such a symbol is called an *operator*. If symbol Ω is an operator denoting some n-ary operation, then $\Omega(W; R_1, ..., R_n)$ denotes the relation that the operation returns after acting on relations $R_1, ..., R_n$ and set W.

An operation satisfies two further constraints: If Ω is an operator denoting some n-ary operation, then $\Omega(W; R_1, ..., R_n)$ must be defined for all binary relations $R_1, ...,$ R_n and any set W such that $R_1, ..., R_n \subseteq W \times W$. Operations are also invariant under isomorphisms. In other words, if $F : W \longrightarrow W'$ is an isomorphism between structures $\langle W, R_1, ..., R_n\rangle$ and $\langle W', R'_1, ..., R'_n\rangle$, then F is also an isomorphism between $\langle W, \Omega(W; R_1, ..., R_n)\rangle$ and $\langle W', \Omega(W'; R'_1, ..., R'_n)\rangle$.

Operations are clearly too large to be sets. Therefore we chose to define the concept of an operator. This will simplify a number of notational issues discussed later on in the article.

We shall attempt to cast light on the intuition behind Definition 2.1 by the following example:

Example 2.2. Consider the binary operation of taking the union of two sets. We let $\tilde{\cup}$ be the corresponding operator. We define $\tilde{\cup}(W; R_1, R_2) = R_1 \cup R_2$.

We observe that the outcome of the operation does not depend on set W: We have $\tilde{\cup}(W; R_1, R_2) = \tilde{\cup}(W'; R_1, R_2)$ for all sets W and W' such that $R_1, R_2 \subseteq W \times W$ and $R_1, R_2 \subseteq W' \times W'$. We call such operations *domain independent*. Another example of a domain independent operation is the binary intersection. Also composition and relative difference of relations are domain independent operations. Relation inversion is an example of a unary, domain independent operation.

We also allow for 0-ary *constant operations*. The operation always returning the empty relation $\emptyset$ is an example of such an operation. Note however, that this particular operation could be assigned any arity.

The unary operation denoted by $\sim$ and defined such that $\sim (W; R) = (W \times W) \setminus R$ is an example of a *domain dependent* operation. It denotes the complementation operation (complementation with respect to universal relation).

It is clear that when restricted to a fixed domain, an operation is nothing but a function on binary relations. This leads to the following definition:

Definition 2.3. Consider an n-ary operation Ω. We let $\Omega \upharpoonright W$ denote the n-ary function $f : (Pow(W \times W))^n \longrightarrow Pow(W \times W)$ defined such that $f(R_1, ..., R_n) = \Omega(W, R_1, ..., R_n)$ for all $R_1, ..., R_n \subseteq W \times W$.

Having discussed the notion of an operator, we are ready to turn our attention to *operator algebras*. In the context of this article, we only consider algebras over power sets of Cartesian products of non-empty sets. The following definitions capture most algebraic concepts that we shall refer to later on in the paper. (Some readers may find our notational conventions somewhat unorthodox.)

Definition 2.4. An *operator algebra* (or simply *algebra*) is a pair $\langle Pow(W \times W), \mathcal{O}\rangle$, where $Pow(W \times W)$ is the power set of the Cartesian product of a non-empty set W and $\mathcal{O}$ is a set of operators in the sense of Definition 2.1. For a fixed set $\mathcal{O}$ of operators, we let term *$\mathcal{O}$-algebra* refer to the class of operator algebras where the set of operators is $\mathcal{O}$.

Definition 2.5. Consider an operator algebra $\langle Pow(W \times W), \mathcal{O}\rangle$. Let V be a set of variable symbols such that $V \cap \mathcal{O} = \emptyset$. Let $\mathcal{T}$ be the set of terms such that

- If $v \in V$, then $v \in \mathcal{T}$.
- If $\Omega \in \mathcal{O}$ is an n-ary operator and $t_1, ..., t_n \in \mathcal{T}$, then $\Omega(t_1, ..., t_n) \in \mathcal{T}$.

Let $T_1, T_2 \in \mathcal{T}$. Let $A = \{f \mid f : V \longrightarrow Pow(W \times W)\}$ be a set of assignment functions associating the variable symbols $v \in V$ with binary relations $R \subseteq W \times W$. Interpret each operator $\Omega \in \mathcal{O}$ in terms T_1 and T_2 as the corresponding function $\Omega \upharpoonright W$. Then, if $T_1 = T_2$ for all assignments $f \in A$, we call equation $T_1 = T_2$ an *identity of* $\langle Pow(W \times W), \mathcal{O}\rangle$*-algebra*.

Definition 2.6. Consider algebras of type $\langle Pow(W \times W), \mathcal{O}\rangle$ for a fixed $\mathcal{O}$. An equation is called a *free identity of $\mathcal{O}$-algebra* if the equation is an identity of every algebra of this class of algebras. If $\mathcal{O}$ is clear from the context, we may simply refer to a *free identity*.

The following example should clarify the essence of the above definitions:

Example 2.7. Let W be a set such that $|W| = 1$. Consider algebra $\langle Pow(W \times W), \{\circ\}\rangle$ with the relation composition operator. $Pow(W \times W)$ now contains two relations, one of them the empty relation. We observe that equation $x \circ x = x$ is an identity of this algebra. However, the equation is clearly not an identity of algebra $\langle Pow(W' \times W'), \{\circ\}\rangle$, where $|W'| = 2$. Therefore the equation is not a free identity of $\{\circ\}$-algebra. Equation $x \circ (y \circ z) = (x \circ y) \circ z$ is an example of a free identity of $\{\circ\}$-algebra.

3 Syntax and Semantics

We begin the section by giving a number of central definitions that fix the syntactic properties of *MLOM*s. After this we turn to semantical issues. We finish the section by proving a few simple but rather essential results.

Definition 3.1. In the context of this discourse, a particular language is defined by fixing the following sets of symbols:

- Set Π of *proposition symbols.*
- Set $\mathcal{A}$ of symbols denoting atomic relations. We shall call members of $\mathcal{A}$ *atoms.*
- Set $\mathcal{F}$ of *syntactic operators* (alternatively, *function symbols*). These symbols are syntactic counterparts of operators.

In order to define what constitutes a set of *formulae* making up a particular language $L(\Pi, \mathcal{A}, \mathcal{F})$, we need the following auxiliary definition:

Definition 3.2. A set $\Lambda(\mathcal{A}, \mathcal{F})$ of *modal terms* is defined in the following way:

- All atoms $a \in \mathcal{A}$ are modal terms. In other words $a \in \mathcal{A} \Rightarrow a \in \Lambda(\mathcal{A}, \mathcal{F})$.
- If τ_1,...,τ_n are modal terms and $\oplus \in \mathcal{F}$ is an n-ary syntactic operator, then $\oplus(\tau_1, ..., \tau_n)$ is a modal term. In other words, if $\tau_1, ..., \tau_n \in \Lambda(\mathcal{A}, \mathcal{F})$ and $\oplus \in \mathcal{F}$, then $\oplus(\tau_1, ..., \tau_n) \in \Lambda(\mathcal{A}, \mathcal{F})$ when $\oplus$ is n-ary.

When $\mathcal{A}$ and $\mathcal{F}$ are clear from the context or not fixed, we may simply refer to Λ instead of $\Lambda(\mathcal{A}, \mathcal{F})$.

For technical purposes, it is useful to define modal terms using prefix notation only. However, below we shall take the liberty to use infix notation when denoting modal terms with binary syntactic operators.

We are now ready to define what is considered to be a formula in a particular language.

Definition 3.3. Let Π be a set of proposition symbols. Let $\mathcal{A}$ be a set of atoms and $\mathcal{F}$ a set of syntactic operators. *Language* $L(\Pi, \mathcal{A}, \mathcal{F})$ is a set of *formulae* defined in the following way:

- Symbol $\perp$ is a formula.
- Proposition symbols $p \in \Pi$ are formulae.
- If A is a formula then $(\neg A)$ is a formula.
- If A and B are formulae then $(A \wedge B)$ is a formula
- If A is a formula and $\alpha \in \Lambda$ then $([\alpha]A)$ is a formula.

We shall adopt the informal standard practice of leaving out brackets when writing formulae. We shall also make use of the standard abbreviations given below:

- $A \vee B =_{def} \neg(\neg A \wedge \neg B)$
- $A \rightarrow B =_{def} \neg A \vee B$
- $A \leftrightarrow B =_{def} (\neg A \vee B) \wedge (\neg B \vee A)$
- $\langle\alpha\rangle A =_{def} \neg[\alpha]\neg A$

where A and B are formulae and α is a modal term. Note that when brackets are informally left unwritten, the order of execution of the connectives is $\neg$,$[\alpha]$,$\langle\alpha\rangle$,$\wedge$,$\vee$,$\rightarrow$,$\leftrightarrow$.

We shall now move on to semantical issues. We begin by giving the definition of a model. Essentially, our notion of a model is a close variant of a more or less standard definition given for example in [8].

Definition 3.4. Consider some language $L(\Pi, \mathcal{A}, \mathcal{F})$. A *model* is a quadruple $\langle W, I, \tilde{R}, P\rangle$.

- $W \neq \emptyset$ is the *domain* of the model.
- I is an *interpretation mapping* that matches the syntactic operators in $\mathcal{F}$ with corresponding operators. If $\oplus \in \mathcal{F}$ is an n-ary syntactic operator, then $I(\oplus)$ is an n-ary operator in the sense of Definition 2.1.
- $\tilde{R} : \mathcal{A} \rightarrow Pow(W \times W)$ is a *relation mapping* defining a set of atomic relations.
- $P : \Pi \rightarrow Pow(W)$ is a *valuation mapping* determining for each proposition symbol p the set of points $w \in W$ satisfying p.

Note that unlike in [8], the object dealing with the interpretation of syntactic operators (I in the above definition) is a set of pairs of *symbols* and therefore independent of the domain of the model associated with it. This simplifies a number of notational issues in a way that shall become clear when we deal questions related to different modes of validity of formulae (see Definition 3.9).

Before giving the general truth definitions, we need to define an auxiliary piece of notation that shall improve readability of the elaborations below.

Definition 3.5. With each model $\langle W, I, \tilde{R}, P\rangle$ we associate a *relation interpretation* $R_{\langle W,I,\tilde{R}\rangle}$ mapping from $\Lambda(\mathcal{A}, \mathcal{F})$ to $Pow(W \times W)$. To keep the notation simple, we shall mostly write R instead of $R_{\langle W,I,\tilde{R}\rangle}$. We define $R(\alpha)$, where $\alpha \in \Lambda$, in the following way:

- If $\alpha = a \in \mathcal{A}$, then $R(\alpha) = R(a) = \tilde{R}(a)$.
- If $\alpha = \oplus(\tau_1, ..., \tau_n)$, where $\tau_1, ..., \tau_n \in \Lambda$ and $\oplus \in \mathcal{F}$, then $R(\alpha) = R(\oplus(\tau_1, ..., \tau_n)) = I(\oplus)(W; R(\tau_1), ..., R(\tau_n))$.

Example 3.6. Consider interpretation mapping $I = \{\langle^-, \sim\rangle, \langle +, \tilde{\cup}\rangle\}$ and atoms $a, b \in \mathcal{A}$. Now $R(\ a + \bar{b}\ ) = I(+)(W; R(a),\ R(\bar{b})) = \tilde{\cup}(W; R(a), R(\bar{b}))$ $= R(a) \cup R(\bar{b}) = \tilde{R}(a) \cup R(\bar{b}) = \tilde{R}(a) \cup I(^-)(R(b)) = \tilde{R}(a) \cup \sim R(b) = \tilde{R}(a) \cup ((W \times W) \setminus R(b)) = \tilde{R}(a) \cup ((W \times W) \setminus \tilde{R}(b))$.

Notational issues involving relation interpretation R can become unnecessarily complicated. We shall therefore resort to an informal, simplified version of the notation. The only particular operations we shall consider in detail are union and intersection, and therefore we do not need to include the domain set W in the expressions involving R. The reason is that union and intersection are domain independent operations. We shall also restrict our discourse to binary unions and intersections. Therefore it is possible to apply the more comfortable infix notation instead of the cumbersome prefix notation. In addition, instead of using the more exotic symbols $\tilde{\cup}$ and $\tilde{\cap}$ introduced in Example 2.2, we informally let the operators denoting union and intersection operations be $\cup$ and $\cap$ respectively.

Example 3.7. Let $I = \{\langle +, \cup\rangle, \langle \cdot, \cap\rangle\}$. Let $a, b, c \in \Lambda$. Now $R(a + (b \cdot c)) = R(a)I(+)R(b \cdot c) = R(a) \cup R(b \cdot c) = R(a) \cup (R(b)I(\cdot)R(c)) = R(a) \cup (R(b) \cap R(c)) = \tilde{R}(a) \cup (\tilde{R}(b) \cap \tilde{R}(c))$.

We are now ready to give the general truth definitions for $MLOM$s.

Definition 3.8. Fix a language $L(\Pi, \mathcal{A}, \mathcal{F})$ and a model $M = \langle W, I, \tilde{R}, P\rangle$. Assume $w \in W$ and let α be a modal term, p a proposition symbol and A and B formulae in $L(\Pi, \mathcal{A}, \mathcal{F})$. We define relation $\models$ in the following way:

- $M, w \not\models \bot$
- $M, w \models p$ iff $w \in P(p)$.
- $M, w \models \neg A$ iff $M, w \not\models A$
- $M, w \models A \wedge B$ iff ($M, w \models A$ and $M, w \models B$)
- $M, w \models [\alpha]A$ iff $\forall u \in W(wR(\alpha)u \Rightarrow M, u \models A)$

We write $\langle W, I, \tilde{R}, P\rangle \models A$ if $\langle W, I, \tilde{R}, P\rangle, w \models A$ for all $w \in W$. We write $\langle W, I, \tilde{R}\rangle \models A$ if $\langle W, I, \tilde{R}, P\rangle, w \models A$ for all $P : \Pi \longrightarrow Pow(W)$ and $w \in W$. Continuing this trend, we write $I \models A$ if $\langle W, I, \tilde{R}, P\rangle, w \models A$ for all W, $\tilde{R} : \mathcal{A} \longrightarrow Pow(W \times W)$, $P : \Pi \longrightarrow Pow(W)$ and $w \in W$. We call $\langle W, I, \tilde{R}\rangle$ a *frame*.

We shall next define a concept that is central to the rest of the discourse.

Definition 3.9. Let φ be a formula and $\mathcal{S}$ the set of syntactic operators in φ. Let I be an interpretation interpreting each syntactic operator $\oplus \in \mathcal{S}$. We say that φ is *I-valid* if it is the case that $I \models \varphi$.

We finish the section by proving three simple but interesting results. Especially the last two of these are elucidating in the sense that they explicitly reveal a link between I-validity of certain types of formulae and the nature of related relations.

Lemma 3.10. *Let* $\langle +, \cup \rangle \in I$. *Now* $I \models [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$ *for all formulae* φ *and all modal terms* $\alpha, \beta \in \Lambda$.

Proof. Choose an arbitrary model $M = \langle W, I, \tilde{R}, P \rangle$ such that $\langle +, \cup \rangle \in I$. Pick an arbitrary $w \in W$ and assume $M, w \models [\alpha + \beta]\varphi$. Therefore $\forall u \in W(wR(\alpha + \beta)u \Rightarrow M, u \models \varphi)$. Since $R(\alpha) \subseteq R(\alpha) \cup R(\beta) = R(\alpha + \beta)$, we conclude that $\forall u \in W(wR(\alpha)u \Rightarrow M, u \models \varphi)$. Therefore $M, w \models [\alpha]\varphi$. We prove $M, w \models [\beta]\varphi$ similarly. Thus we conclude $M, w \models [\alpha]\varphi \wedge [\beta]\varphi$.

For the converse, assume that $M, w \models [\alpha]\varphi \wedge [\beta]\varphi$. Hence $M, w \models [\alpha]\varphi$ and $M, w \models [\beta]\varphi$. Therefore $\forall u \in W(wR(\alpha)u \Rightarrow M, u \models \varphi)$ and $\forall u \in W(wR(\beta)u \Rightarrow M, u \models \varphi)$. Thus $\forall u \in W(w(R(\alpha) \cup R(\beta))u \Rightarrow M, u \models \varphi)$. In other words $\forall u \in W(wR(\alpha + \beta)u \Rightarrow M, u \models \varphi)$, whence $M, w \models [\alpha + \beta]\varphi$. □

Lemma 3.11. *Let* I *be an arbitrary interpretation,* α *and* β *arbitrary modal terms and* p *an arbitrary proposition symbol. Now* $I \models [\alpha]p \rightarrow [\beta]p$ *iff for all frames* $\langle W, I, \tilde{R} \rangle$ *it is the case that* $R(\beta) \subseteq R(\alpha)$.

Proof. Assume $I \models [\alpha]p \rightarrow [\beta]p$. For contradiction, assume that there exists some frame $\langle W', I, \tilde{R}' \rangle$ with $u, v \in W'$ such that $\langle u, v \rangle \in R'(\beta)$ and $\langle u, v \rangle \notin R'(\alpha)$. We can now choose P' such that $P'(p) = W' \setminus \{v\}$. Therefore $\langle W', I, \tilde{R}', P' \rangle, u \models [\alpha]p$ and $\langle W', I, \tilde{R}', P' \rangle, u \not\models [\beta]p$, which contradicts the assumption that $I \models [\alpha]p \rightarrow [\beta]p$.

Conversely, assume that for all frames $\langle W, I, \tilde{R} \rangle$ it is the case that $R(\beta) \subseteq R(\alpha)$. Choose an arbitrary model $\langle W, I, \tilde{R}, P \rangle$, let $w \in W$ and assume that $\langle W, I, \tilde{R}, P \rangle, w \models [\alpha]p$. It is therefore the case that $\forall u \in W(wR(\alpha)u \Rightarrow \langle W, I, \tilde{R}, P \rangle, u \models p)$. Since $R(\beta) \subseteq R(\alpha)$, we conclude $\forall u \in W(wR(\beta)u \Rightarrow \langle W, I, \tilde{R}, P \rangle, u \models p)$. Therefore $\langle W, I, \tilde{R}, P \rangle, w \models [\beta]p$. Hence we have shown that $\langle W, I, \tilde{R}, P \rangle, w \models [\alpha]p \rightarrow [\beta]p$. Since model $\langle W, I, \tilde{R}, P \rangle$ and point $w \in W$ were chosen arbitrarily, we conclude that $I \models [\alpha]p \rightarrow [\beta]p$. □

Corollary 3.12. *Let* I *be an arbitrary interpretation,* α *and* β *arbitrary modal terms and* p *an arbitrary proposition symbol. Now* $I \models [\alpha]p \leftrightarrow [\beta]p$ *iff for all frames* $\langle W, I, \tilde{R} \rangle$ *it is the case that* $R(\alpha) = R(\beta)$.

Proof. The assertion of the corollary follows directly from Lemma 3.11. □

4 Standard Deduction Systems

In this section we discuss some technical issues that shall be needed later on in the discourse. Some of the presented material is inherited from the elementary theory of standard deduction systems of (uni)modal logic, cf. [7, 9, 17]. We begin by making a few crucial definitions.

Definition 4.1. An *axiom schema* is an object that is constructed in a similar way as formulae are, but with the difference that modal term variables replace atoms and formula variables replace propositions. A formula is an instance of an axiom schema if it is obtained from the schema by substituting modal terms for modal term variables and formulae for formula variables. Every instance of an axiom schema is an axiom.

Example 4.2. Let $\mathcal{A} = \{a, b\}$ and $\Pi = \{p, q\}$. Let $\oplus \in \mathcal{F}$ be a binary syntactic operator. Assume $[\alpha]\varphi \leftrightarrow [\beta]\psi$ is an axiom schema. Then for example formula $[a \oplus b](p \wedge q) \leftrightarrow [a]\neg q$ is an axiom.

We shall next define the notion of *deduction system*.

Definition 4.3. A *deduction system* Σ is a pair $\langle A, \mathcal{R} \rangle$ where A is a set of axioms and $\mathcal{R}$ a set of rules of inference. We shall informally say that Σ *includes* or *contains* a particular axiom or rule. We shall also say that Σ includes or contains a particular axiom schema, when each instance of the schema is in A.

Note that in the literature, symbol Σ is often used for denoting a set of formulae deducible in some deduction system. Such sets of formulae are often called *systems*. In this article, however, we use symbol Σ to denote deduction systems in the sense of Definition 4.3.

We shall continue by considering a deduction system Σ_0 containing the following axiom schemata:

- A_1: $\varphi \to (\psi \to \varphi)$
- A_2: $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
- A_3: $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$
- A_4: $\bot \leftrightarrow (\varphi \wedge \neg\varphi)$
- K_G: $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$

and the following rules of inference:

- MP: If $\vdash A \to B$ and $\vdash A$ then $\vdash B$.
- RN_G: For all $\alpha \in \Lambda$, if $\vdash A$ then $\vdash [\alpha]A$.

The first three axiom schemata ensure together with *Modus Ponens* (MP) that any classical tautology is a theorem of the system. We shall not prove this as it is not essential from the point of view of the issues discussed in this paper. In fact, schemata A_1, A_2, A_3 and rule MP could be replaced by any finite number of axiom schemata and inference rules that constitute a complete axiomatization of propositional logic. Therefore, in the deductions that follow, we shall not refer to rule MP; we let PL denote the assertion that a formula follows directly from a subset of the preceding formulae due to the fact that our deduction system is complete with respect to classical tautologies.

Axiom schema K_G is simply a generalization of the well known axiom schema K of unimodal logic. Note that not only formula-variable φ of K_G, but also variable α can be fixed freely (within the constraints of the language under discourse) and the resulting formula is then an axiom. Similarly, we also generalize rule RN (*Rule of Necessitation*) of unimodal logic by including rule RN_G in the deduction system. The rule applies to all modal terms $\alpha \in \Lambda$.

We note that the axioms of Σ_0 are I-valid for any I. We also note that the rules of inference of Σ_0 preserve I-validity of formulae for any I. This is an important observation, since in this discourse we are interested namely in deduction systems that are complete with respect to I-validity.

System Σ_0 is a natural starting point for constructing deduction systems for different kinds of $MLOM$s. We shall move on to considering a number of results that apply to deduction systems with at least the deductive power of Σ_0. We call such deduction systems *I-elementary*:

Definition 4.4. A deduction system Σ is *I-elementary* if it satisfies the following constraints:

- Σ is complete with respect to classical tautologies.
- Rule RN_G is included in Σ.
- Axiom schema K_G is included in Σ.
- The axioms included in Σ are I-valid.
- All rules of inference in Σ preserve I-validity of formulae.

In the remaining part of the current section, we let $\vdash A$ denote the assertion that formula A is deducible in any I-elementary deduction system.

Lemma 4.5. *Let A and B be arbitrary formulae. Let $\alpha \in \Lambda$ be an arbitrary modal term. If $\vdash A \rightarrow B$, then $\vdash [\alpha]A \rightarrow [\alpha]B$.*

Proof. We assume $\vdash A \to B$ and let the deduction below prove the lemma:

1.	$A \to B$	*Assumption*
2.	$[\alpha](A \to B)$	$1, RN_G$
3.	$[\alpha](A \to B) \to ([\alpha]A \to [\alpha]B)$	K_G
4.	$[\alpha]A \to [\alpha]B$	$2, 3, PL$

Therefore $\vdash [\alpha]A \to [\alpha]B$. □

Lemma 4.6. *We have* $\vdash [\alpha](A \wedge B) \leftrightarrow ([\alpha]A \wedge [\alpha]B)$ *for all formulae* A, B *and all modal terms* $\alpha \in \Lambda$.

Proof. We prove the lemma by the following deduction:

1.	$B \to (A \to (A \wedge B))$	PL
2.	$[\alpha]B \to [\alpha](A \to (A \wedge B))$	$1, Lemma$ 4.5
3.	$[\alpha](A \to (A \wedge B)) \to ([\alpha]A \to [\alpha](A \wedge B))$	K_G
4.	$[\alpha]B \to ([\alpha]A \to [\alpha](A \wedge B))$	$2, 3, PL$
5.	$([\alpha]A \wedge [\alpha]B) \to [\alpha](A \wedge B)$	$4, PL$
6.	$(A \wedge B) \to A$	PL
7.	$[\alpha](A \wedge B) \to [\alpha]A$	$6, Lemma$ 4.5
8.	$(A \wedge B) \to B$	PL
9.	$[\alpha](A \wedge B) \to [\alpha]B$	$8, Lemma$ 4.5
10.	$[\alpha](A \wedge B) \leftrightarrow ([\alpha]A \wedge [\alpha]B)$	$5, 7, 9, PL$

Therefore the claim of the lemma holds. □

Lemma 4.7. *For all* $n \in \mathbb{N}^+$ *it is the case that if* $\vdash A_1 \wedge A_2 \wedge A_3 \wedge ... \wedge A_n \to B$, *then* $\vdash [\alpha]A_1 \wedge [\alpha]A_2 \wedge [\alpha]A_3 \wedge ... \wedge [\alpha]A_n \to [\alpha]B$, *where* $A_1, A_2, A_3, ..., A_n, B$ *are arbitrary formulae and* $\alpha \in \Lambda$ *is an arbitrary modal term.*

Proof. We prove the lemma by induction on the number of conjuncts in the premiss formula. By Lemma 4.5 we know that for all A, B and α it is the case that $\vdash A \to B \Rightarrow \vdash [\alpha]A \to [\alpha]B$. Therefore the basis for the induction is clear.

Now fix an arbitrary $\alpha \in \Lambda$ and let the induction hypothesis be that $\vdash A_1 \wedge A_2 \wedge A_3 \wedge ... \wedge A_k \to B \Rightarrow \vdash [\alpha]A_1 \wedge [\alpha]A_2 \wedge [\alpha]A_3 \wedge ... \wedge [\alpha]A_k \to [\alpha]B$ for all formulae $A_1, A_2, A_3, ..., A_k, B$. Assume then that $\vdash C_1 \wedge C_2 \wedge C_3 \wedge ... \wedge C_{k+1} \to D$, where $C_1, C_2, C_3, ..., C_{k+1}$ and D are arbitrary formulae. Since our deduction system is complete with respect to propositional logic, we can arrange the (informally unwritten) bracketing of the antecedent $C_1 \wedge C_2 \wedge C_3 \wedge ... \wedge C_{k+1}$ such that C_1 and C_2 are directly bound by the same conjunction, i.e. we can make formula $(C_1 \wedge C_2)$ part of the antecedent. Now we simply call $C' = C_1 \wedge C_2$. We therefore have $\vdash C' \wedge C_3 \wedge ... \wedge C_{k+1} \to D$, whence by the induction hypothesis we conclude $\vdash [\alpha]C' \wedge [\alpha]C_3 \wedge ... \wedge [\alpha]C_{k+1} \to [\alpha]D$. Therefore $\vdash [\alpha](C_1 \wedge C_2) \wedge [\alpha]C_3 \wedge ... \wedge [\alpha]C_{k+1} \to [\alpha]D$. By Lemma 4.6 we have $\vdash [\alpha](C_1 \wedge C_2) \leftrightarrow [\alpha]C_1 \wedge [\alpha]C_2$, whence by propositional logic we get $\vdash [\alpha]C_1 \wedge [\alpha]C_2 \wedge [\alpha]C_3 \wedge ... \wedge [\alpha]C_{k+1} \to [\alpha]D$. □

Lemma 4.8. *Let A and B be arbitrary formulae. Let $\alpha \in \Lambda$ be an arbitrary modal term. If $\vdash A \leftrightarrow B$, then $\vdash [\alpha]A \leftrightarrow [\alpha]B$.*

Proof. Assume $\vdash A \leftrightarrow B$ and consider the following deduction:

1.	$A \leftrightarrow B$	*Assumption*
2.	$A \rightarrow B$	$1, PL$
3.	$[\alpha]A \rightarrow [\alpha]B$	$2,$ *Lemma* 4.5
4.	$B \rightarrow A$	$1, PL$
5.	$[\alpha]B \rightarrow [\alpha]A$	$4,$ *Lemma* 4.5
6.	$[\alpha]A \leftrightarrow [\alpha]B$	$3, 5, PL$

Therefore $\vdash [\alpha]A \leftrightarrow [\alpha]B$. □

For the next lemma we need the following definition:

Definition 4.9. Let A, B and φ be formulae. By $A(B/\varphi)$ we refer to a new formula that can be obtained from A by replacing each subformula φ of A by B.

Lemma 4.10. *Let A, D, E and φ be arbitrary formulae. If $\vdash D \leftrightarrow E$, then $\vdash A(D/\varphi) \leftrightarrow A(E/\varphi)$.*

Proof. If $A = \varphi$, then $\vdash A(D/\varphi) \leftrightarrow A(E/\varphi)$, since $\vdash D \leftrightarrow E$. Therefore we may assume $A \neq \varphi$ throughout the proof. We proceed by induction on the structure of formula A. In order to set the basis for the induction, assume that $A = p$ for an arbitrary proposition symbol p. Since $\vdash p \leftrightarrow p$, this case is clear.

The induction hypothesis is that $\vdash B(D/\varphi) \leftrightarrow B(E/\varphi)$ and $\vdash C(D/\varphi) \leftrightarrow C(E/\varphi)$. Assume then that $A = \neg B$. This case is clear, as we can conclude $\vdash (\neg B)(D/\varphi) \leftrightarrow (\neg B)(E/\varphi)$ directly from $\vdash B(D/\varphi) \leftrightarrow B(E/\varphi)$. Assume then that $A = B \wedge C$. Since $B \wedge C = A \neq \varphi$, we see that

$$(B(D/\varphi) \leftrightarrow B(E/\varphi)) \wedge (C(D/\varphi) \leftrightarrow C(E/\varphi)) \\ \rightarrow ((B \wedge C)(D/\varphi) \leftrightarrow (B \wedge C)(E/\varphi))$$

is a tautology. Therefore we again conclude $\vdash (B \wedge C)(D/\varphi) \leftrightarrow (B \wedge C)(E/\varphi)$.

Lastly, we assume $A = [\alpha]B$, where $\alpha \in \Lambda$ is an arbitrary modal term. We first obtain $\vdash [\alpha](B(D/\varphi)) \leftrightarrow [\alpha](B(E/\varphi))$ from $\vdash B(D/\varphi) \leftrightarrow B(E/\varphi)$ by Lemma 4.8. Since $[\alpha]B = A \neq \varphi$, formulae $[\alpha](B(D/\varphi)) \leftrightarrow [\alpha](B(E/\varphi))$ and $([\alpha]B)(D/\varphi) \leftrightarrow ([\alpha]B)(E/\varphi)$ are identical. Thus we conclude $\vdash ([\alpha]B)(D/\varphi) \leftrightarrow ([\alpha]B)(E/\varphi)$. □

We add one more rule of inference to our toolbox. Among other things, this rule enables us to replace finite sets of axiom schemata by finite sets of formulae (see also Remark 7.9 below). We need the following definition in order to formulate the rule:

Definition 4.11. Let A be a formula, b an atom and β a modal term. By $A(\beta/b)$ we refer to a new formula that can be obtained from A by replacing each occurrence of atom b in A by β.

We then define a generalized version of *uniform substitution*:

Definition 4.12. Let A and B be arbitrary formulae and p any proposition symbol. The following defines inference rule US_P (*Uniform Substitution of Propositions*):

$$\frac{A}{A(B/p)}$$

In other words, from A infer $A(B/p)$.

Let β be an arbitrary modal term and b an arbitrary atom. The following defines inference rule US_A (*Uniform Substitution of Atoms*):

$$\frac{A}{A(\beta/b)}$$

In other words, from A infer $A(\beta/b)$.

Together rules US_P and US_A form an umbrella rule *Generalized Uniform Substitution* (rule US_G).

Before proving that rule US_G preserves I-validity of formulae for any I, we wish to note that a weaker version of the rule would suffice for the purposes of this discourse. Each application of rule US_G below can be dealt with by applying rule US': From $[\alpha]p \to [\beta]p$ infer $[\alpha]\varphi \to [\beta]\varphi$. The fact that this rule preserves I-validity of formulae for any I follows directly from Lemma 3.11.

Theorem 4.13. *Rule US_G preserves I-validity of formulae for any I.*

Proof. We shall begin by dealing with rule US_P. We assume $I \models A$. Therefore also $I \models A(q/p)$. This is because I is independent of any valuation mapping. For the sake of contradiction we assume that there is a model $M = \langle W, I, \tilde{R}, P \rangle$ and a point $w \in W$ such that $M, w \not\models A(B/p)$. We let q be a new proposition symbol not occurring in either A or B. We let V be the set of points $v \in W$ for whom it is the case that $M, v \models B$. We define a new valuation $P' = P \cup \{\langle q, V \rangle\}$. We have $M, w \not\models A(B/p)$, and below we shall show by a straightforward induction that therefore $\langle W, I, \tilde{R}, P' \rangle, w \not\models A(q/p)$. This is a contradiction since $I \models A(q/p)$.

Now let us show by induction on the structure of A that for all $w \in W$ we have $M, w \models A(B/p) \Leftrightarrow \langle W, I, \tilde{R}, P' \rangle, w \models A(q/p)$. Now, if $A = p$, the claim holds by the definition of valuation P'. If A is some other proposition (or if $A = \bot$), then the claim holds trivially. Therefore the basis of the induction is clear. Dealing with conjunction and negation is also trivial. Thus we assume that $A = [\alpha]C$ for some C and α in the language of model M. We have the following chain of equivalences:

$$
\begin{aligned}
& M, w \models ([\alpha]C)(B/p) \\
\Leftrightarrow\ & M, w \models [\alpha](C(B/p)) \\
\Leftrightarrow\ & \forall u \in W(wR(\alpha)u \Rightarrow M, u \models C(B/p)) \\
\Leftrightarrow\ & \forall u \in W(wR(\alpha)u \Rightarrow \langle W, I, \tilde{R}, P' \rangle, u \models C(q/p)) \\
\Leftrightarrow\ & \langle W, I, \tilde{R}, P' \rangle, w \models [\alpha](C(q/p)) \\
\Leftrightarrow\ & \langle W, I, \tilde{R}, P' \rangle, w \models ([\alpha]C)(q/p)
\end{aligned}
$$

where the third equivalence follows from the induction hypothesis.

We then deal with rule US_A. We assume that $I \models A$ and also that there exists a model $M = \langle W, I, \tilde{R}, P \rangle$ and a point $w \in W$ such that $M, w \not\models A(\beta/b)$. We let c be a new atom not occurring in A or β. Since $I \models A$ and since I is independent of any relation mapping, we have $I \models A(c/b)$. We then define a new relation mapping (recall notation from Definition 3.5): $\tilde{R}' = \tilde{R} \cup \{\langle c, R_{\langle W,I,\tilde{R}\rangle}(\beta)\}$. Since $M, w \not\models A(\beta/b)$, we have $\langle W, I, \tilde{R}', P \rangle, w \not\models A(\beta/b)$. As clearly $R_{\langle W,I,\tilde{R}'\rangle}(\beta) = R_{\langle W,I,\tilde{R}'\rangle}(c)$, we have $\langle W, I, \tilde{R}', P \rangle, w \not\models A(c/b)$. This is a contradiction since $I \models A(c/b)$. □

We finish the section by discussing *negation complete* and *consistent* sets of formulae:

Definition 4.14. A set u of formulae is *negation complete* (with respect to language L) if for all formulae $A \in L$ it is the case that $A \notin u \Rightarrow \neg A \in u$. A set u of formulae is Σ-*consistent* if it is not the case that there exists some non-empty, finite subset of $u' \subseteq u$ of formulae such that $\vdash_\Sigma \neg \bigwedge u'$.

It is easy to see that for a negation complete and Σ-consistent set u the following assertions hold:

- $A \notin u$ iff $\neg A \in u$
- $A, B \in u$ iff $A \wedge B \in u$
- ($A \in u$ or $B \in u$) iff $A \vee B \in u$
- $A \to B \in u$ iff $(A \in u \Rightarrow B \in u)$
- $A \leftrightarrow B \in u$ iff $(A \in u \Leftrightarrow B \in u)$

The following two results are needed for the completeness proof presented later on:

Lemma 4.15. *Let Σ be an I-elementary deduction system for some I. Let $\alpha \in \Lambda$ be an arbitrary modal term and A an arbitrary formula. Let S be a Σ-consistent set of formulae such that $\neg[\alpha]A \in S$. Then set $\{B \mid [\alpha]B \in S\} \cup \{\neg A\}$ is a Σ-consistent set of formulae.*

Proof. We assume for contradiction that there exists a non-empty, finite set $U \subseteq \{B \mid [\alpha]B \in S\} \cup \{\neg A\}$ such that $\vdash_\Sigma \neg \bigwedge U$. Since generally $\vdash_\Sigma \neg(C_1 \wedge C_2 \wedge ... \wedge C_n) \Rightarrow \vdash_\Sigma \neg(C_1 \wedge C_2 \wedge ... \wedge C_n \wedge \varphi)$, we may assume that $\bigwedge U = B_1 \wedge B_2 \wedge ... \wedge B_k \wedge \neg A$, where $k \geq 1$ and $B_1, B_2, ..., B_k \in \{B \mid [\alpha]B \in S\}$. Since $\vdash_\Sigma \neg(B_1 \wedge B_2 \wedge ... \wedge B_k \wedge \neg A)$, then $\vdash_\Sigma B_1 \wedge B_2 \wedge ... \wedge B_k \rightarrow A$. Hence $\vdash_\Sigma [\alpha]B_1 \wedge [\alpha]B_2 \wedge ... \wedge [\alpha]B_k \rightarrow [\alpha]A$ by Lemma 4.7. Therefore $\vdash_\Sigma \neg([\alpha]B_1 \wedge [\alpha]B_2 \wedge ... \wedge [\alpha]B_k \wedge \neg[\alpha]A)$. This implies that S is not Σ-consistent, which is a contradiction. □

Lemma 4.16 (Lindenbaum's Lemma)**.** *Let Σ be a deduction system. For any Σ-consistent set u there exists a negation complete and Σ-consistent set $v \supseteq u$.*

We omit the proof of this result as it is rather general and does not involve concepts directly linked to this discourse. The proof can be found in multiple different sources. Note that the lemma is usually proved only for countable languages. However, by the well-ordering principle, such proofs can easily be carried out in a more general setting by transfinite induction.

5 Inference Rules for Modal Term Substitution

In this section we shall develop tools that enable the algebraic manipulation of modal terms in formal deductions, i.e. deductions in equational logic. We shall show that by including a simple inference rule in an I-elementary deduction system, algebraic *identity substitution* (see Definition 5.1 below) of modal terms becomes possible. This leads naturally to partial completeness results related to such interpretations I where $I(\mathcal{F})$-algebra is effectively axiomatizable. Those results shall be the topic of the next section. We begin by defining the notion of identity substitution:

Definition 5.1. Let $T_t = S$ be an identity of some operator algebra $\langle Pow(W \times W), \mathcal{O}\rangle$, and let t be a subterm of T_t. Let also equation $t = r$ be an identity of algebra $\langle Pow(W \times W), \mathcal{O}\rangle$. Let T_r be a term obtained from T_t by replacing exactly one instance of subterm t in T_t by r. We may then infer that equation $T_r = S$ is an identity of $\langle Pow(W \times W), \mathcal{O}\rangle$. Identity $T_r = S$ is said to follow from identities $T_t = S$ and $t = r$ by the principle of *identity substitution.*

We shall now discuss a rule of inference that enables identity substitution of modal terms in deductions. We first define the rule and then prove that for any language, the rule preserves I-validity of formulae for all I. We then move on to showing that indeed the rule does enable the application of the principle of identity substitution of modal terms.

Definition 5.2. Let $p \in \Pi$ be any proposition symbol. Let $\oplus \in \mathcal{F}$ be an n-ary syntactic operator, and let $\alpha, \beta, \tau_1, \tau_2, ..., \tau_n \in \Lambda$. The following defines inference rule *RS* (*Rule of Substitution*):

$$\frac{[\alpha]p \leftrightarrow [\beta]p}{[\oplus(\tau_1, ..., \tau_i, \alpha, \tau_{i+1}, ..., \tau_n)]p \leftrightarrow [\oplus(\tau_1, ..., \tau_i, \beta, \tau_{i+1}, ..., \tau_n)]p}$$

In other words, in a deduction system containing rule *RS* it is the case that $\vdash [\alpha]p \leftrightarrow [\beta]p \ \Rightarrow \vdash [\oplus(\tau_1, ..., \tau_i, \alpha, \tau_{i+1}, ..., \tau_n)]p \leftrightarrow [\oplus(\tau_1, ..., \tau_i, \beta, \tau_{i+1}, ..., \tau_n)]p$.

Theorem 5.3. *Rule of inference RS preserves I-validity of formulae for any interpretation I.*

Proof. Let I be an arbitrary interpretation interpreting $\oplus \in \mathcal{F}$. Assume $I \models [\alpha]p \leftrightarrow [\beta]p$. Choose an arbitrary model $M = \langle W, I, \tilde{R}, P\rangle$ and an arbitrary $w \in W$. Assume $M, w \models [\oplus(\tau_1, ..., \tau_i, \alpha, \tau_{i+1}, ..., \tau_n)]p$. Therefore the following holds for all $u \in W$:

$$\langle w, u\rangle \in I(\oplus)(R(\tau_1), R(\tau_2), ..., R(\tau_i), R(\alpha), R(\tau_{i+1}), ..., R(\tau_n)) \Rightarrow M, u \models p$$

Now, since $I \models [\alpha]p \leftrightarrow [\beta]p$, we conclude that $R(\alpha) = R(\beta)$ by Corollary 3.12. Hence

$$\langle w, u\rangle \in I(\oplus)(R(\tau_1), ..., R(\tau_i), R(\beta), R(\tau_{i+1}), ..., R(\tau_n)) \Rightarrow M, u \models p$$

for all $u \in W$. Thus $M, w \models [\oplus(\tau_1, ..., \tau_i, \beta, \tau_{i+1}, ..., \tau_n)]p$, and therefore we have shown that $M, w \models [\oplus(\tau_1, ..., \tau_i, \alpha, \tau_{i+1}, ..., \tau_n)]p \rightarrow [\oplus(\tau_1, ..., \tau_i, \beta, \tau_{i+1}, ..., \tau_n)]p$. We show $M, w \models [\oplus(\tau_1, ..., \tau_i, \beta, \tau_{i+1}, ..., \tau_n)]p \rightarrow [\oplus(\tau_1, ..., \tau_i, \alpha, \tau_{i+1}, ..., \tau_n)]p$ similarly. Therefore $M, w \models [\oplus(\tau_1, ..., \tau_i, \alpha, \tau_{i+1}, ..., \tau_n)]p \leftrightarrow [\oplus(\tau_1, ..., \tau_i, \beta, \tau_{i+1}, ..., \tau_n)]p$. Since model M and $w \in W$ were chosen arbitrarily, we conclude that $I \models [\oplus(\tau_1, ..., \tau_i, \alpha, \tau_{i+1}, ..., \tau_n)]p \leftrightarrow [\oplus(\tau_1, ..., \tau_i, \beta, \tau_{i+1}, ..., \tau_n)]p$. ☐

Rule *RS* does not directly enable the identity substitution of modal terms in the general case. Note for example that the rule operates only on subterms that are directly below the main connective of a modal term: Assume $\vdash [\alpha]p \leftrightarrow [\beta]p$. We cannot directly infer that $\vdash [\oplus(\gamma, \oplus(\delta, \alpha))]p \leftrightarrow [\oplus(\gamma, \oplus(\delta, \beta))]p$. We therefore define a new rule of inference *IS* (*Identity Substitution*) that simulates identity substitution everywhere below the main connective directly. We then prove that this rule is automatically available in any deduction system with rule *RS*.

Definition 5.4. Let $p \in \Pi$ be a proposition symbol and let $\alpha, \beta \in \Lambda$. The following defines inference rule *IS* (*Identity Substitution*):

$$\frac{[\tau(\alpha)]p \leftrightarrow [\gamma]p, \ [\alpha]p \leftrightarrow [\beta]p}{[\tau(\beta)]p \leftrightarrow [\gamma]p}$$

where $\tau(\alpha) \in \Lambda$ is a term that includes $\alpha \in \Lambda$ as its subterm, and $\tau(\beta) \in \Lambda$ is a term that can be obtained from $\tau(\alpha)$ by replacing exactly one instance of term α in $\tau(\alpha)$ by β.

The following example shows how rule IS can be directly used for simulating an algebraic calculation within the constraints of a related formal deduction system.

Example 5.5. Consider the free identity $x \cap (x \cup (y \cup (y \cap z))) = x$ of $\{\cup, \cap\}$-algebra. A standard calculation proceeds as follows:

1.	$x \cap (x \cup y) = x$	*Absorption law*
2.	$y = y \cup (y \cap z)$	*Absorption law*
3.	$x \cap (x \cup (y \cup (y \cap z))) = x$	*Subst.* 2 *to* 1

Rule IS enables us to simulate this calculation. We let $I = \{\langle +, \cup \rangle, \langle \cdot, \cap \rangle\}$ and consider a deduction system with axiom schemata $[\tau_1 \cdot (\tau_1 + \tau_2)]\varphi \leftrightarrow [\tau_1]\varphi$ and $[\tau_1]\varphi \leftrightarrow [\tau_1 + (\tau_1 \cdot \tau_2)]\varphi$. These schemata correspond to the absorption laws for union and intersection. Let p be any proposition symbol and let $\alpha, \beta, \gamma \in \Lambda$. We now have the following deduction:

1.	$[\alpha \cdot (\alpha + \beta)]p \leftrightarrow [\alpha]p$	*Axiom*
2.	$[\beta]p \leftrightarrow [\beta + (\beta \cdot \gamma)]p$	*Axiom*
3.	$[\alpha \cdot (\alpha + (\beta + (\beta \cdot \gamma)))]p \leftrightarrow [\alpha]p$	$1, 2, IS$

The deduction simulates the standard calculation directly.

We shall now show that rule IS is automatically available for deductions in deduction systems including rule RS.

Lemma 5.6. *Consider deduction systems that are complete with respect to classical tautologies and contain rule RS. Rule IS is automatically available for deductions in such systems.*

Proof. Assume that we have a deduction system that includes inference rule RS. We shall first show that $\vdash [\alpha]p \leftrightarrow [\beta]p$ implies $\vdash [\tau(\alpha)]p \leftrightarrow [\tau(\beta)]p$. Therefore we assume that $\vdash [\alpha]p \leftrightarrow [\beta]p$. We proceed by induction on the number m of syntactic operators above α in the parse tree of term $\tau(\alpha)$. We note that there may be multiple instances of term α in $\tau(\alpha)$, but the induction is naturally on the number of syntactic operators above the particular single instance of α in $\tau(\alpha)$ that β replaces in $\tau(\beta)$. This instance is the only instance of α we shall refer to in the remaining part of the proof.

In order to set the basis for the induction, assume that $m = 0$. Therefore $\tau(\alpha) = \alpha$ and thus $\tau(\beta) = \beta$. therefore trivially $\vdash [\tau(\alpha)]p \leftrightarrow [\tau(\beta)]p$.

The induction hypothesis is that $\vdash [\alpha]p \leftrightarrow [\beta]p \Rightarrow \vdash [\tau(\alpha)]p \leftrightarrow [\tau(\beta)]p$ for all $\tau(\alpha)$ such that $m = k$. Now consider term $\tau(\alpha) \in \Lambda$ such that $m = k + 1$. Let $\tau(\alpha)$ have an n-ary operation symbol $\oplus \in \mathcal{F}$ as its main connective. We therefore have $\tau(\alpha) = \oplus(\tau_1, \tau_2, ..., \delta(\alpha), ..., \tau_n)$, where $\tau_1, \tau_2, ..., \tau_n \in \Lambda$, and $\delta(\alpha) \in \Lambda$ includes α as its subterm. We have $\tau(\beta) = \oplus(\tau_1, \tau_2, ..., \delta(\beta), ..., \tau_n)$. From $\vdash [\alpha]p \leftrightarrow [\beta]p$ we obtain $\vdash [\delta(\alpha)]p \leftrightarrow [\delta(\beta)]p$

by the induction hypothesis. From $\vdash [\delta(\alpha)]p \leftrightarrow [\delta(\beta)]p$, in turn, we obtain $\vdash [\oplus(\tau_1, \tau_2, ..., \delta(\alpha), ..., \tau_n)]p \leftrightarrow [\oplus(\tau_1, \tau_2, ..., \delta(\beta), ..., \tau_n)]p$ by an application of rule *RS*. This completes the induction.

We have so far established that $\vdash [\alpha]p \leftrightarrow [\beta]p$ implies $\vdash [\tau(\alpha)]p \leftrightarrow [\tau(\beta)]p$. Now assume $\vdash [\tau(\alpha)]p \leftrightarrow [\gamma]p$ and $\vdash [\alpha]p \leftrightarrow [\beta]p$. By the above we obtain $\vdash [\tau(\alpha)]p \leftrightarrow [\tau(\beta)]p$. Therefore, since $\vdash [\tau(\alpha)]p \leftrightarrow [\gamma]p$, we conclude $\vdash [\tau(\beta)]p \leftrightarrow [\gamma]p$. □

We immediately obtain the following lemma:

Lemma 5.7. *Rule IS preserves I-validity of formulae for any interpretation I.*

Proof. This follows directly from Theorem 5.3 and Lemma 5.6. □

One of the advantages of rule *RS* is that it preserves I-validity of formulae for any I. When dealing with deduction systems related to particular operators, it is possible to introduce a wide variety of *context dependent* inference rules: Assume $I_0 = \{\langle +, \cup \rangle, \langle \cdot, \cap \rangle\}$. It is then quite straightforward to prove (though we shall not bother ourselves with doing so here; see [13] for a proof) that then $I_0 \models [\alpha]\varphi \Rightarrow I_0 \models \varphi$ for any formula φ and modal term $\alpha \in \Lambda$. Therefore we could introduce a corresponding rule of inference. However, let $0 \in \mathcal{F}$ be the syntactic operator corresponding to the 0-ary operator Z defined such that $Z(W) = \emptyset$ for all W. Now if $I_1 = \{\langle +, \cup \rangle, \langle \cdot, \cap \rangle, \langle 0, Z \rangle\}$, then $I_1 \models [0]\bot$ but $I_1 \not\models \bot$. Therefore, whether this new rule preserves I-validity of formulae depends on the particular I in question. No such constraint applies to rule *RS*.

The advantage of deduction systems not containing such context dependent inference rules lies in the generality of such systems. If a deduction system contains a context dependent inference rule, it is not possible to expand the set of operators under discourse without endangering the preservation of I-validity of deducible formulae. Rule *RS* is not context dependent and therefore when generalizing a deduction system by adding inference rules or axioms or expanding the set of operators considered, it is possible to keep *RS* in the system. Moreover, the possibility of simulating identity substitution in deductions will automatically apply to the new syntactic operators as well as the old ones since *RS* is defined for all syntactic operators.

The following example should elucidate the advantage of deduction systems not containing context dependent inference rules:

Example 5.8. Consider the class of languages with $\mathcal{F} = \{+, \cdot\}$. Assume we have an I-elementary deduction system Σ_E that is complete with respect to interpretation $I = \{\langle +, \cup \rangle, \langle \cdot, \cap \rangle\}$. Assume we aim to construct a deduction system for the class of languages with $\mathcal{F} = \{+, \cdot, ^-, 0, 1\}$, and we want this system to be sound with respect to interpretation $I = \{\langle +, \cup \rangle, \langle \cdot, \cap \rangle, \langle ^-, \sim$

$\rangle, \langle 0, Z\rangle, \langle 1, U\rangle\}$, where $U(W) = W \times W$. This interpretation gives rise to a Boolean algebra of modalities.

It is now directly possible to strengthen system Σ_E with axioms and inference rules without sacrificing completeness with respect to the fragment of formulae containing no other syntactic operators than those corresponding to the union and intersection operations.

We formulated rule RS with proposition symbols instead of formula variables. Therefore care must be taken when considering anomalous languages with an empty set of proposition symbols. Replacing the proposition symbols in Definition 5.2 by formula variables results in a rule that does not preserve I-validity of formulae for all I. It is straightforward to construct counterexamples that show this; consult [13] for details. One of the many natural ways to deal with such pathological propositionless languages is to allow propositions for deductions.

We shall shortly prove the main result of the section. Before that, we make the following definition (cf. Corollary 3.12):

Definition 5.9. Let $\mathcal{F}$ be a set of syntactic operators. Assume I is an interpretation interpreting each syntactic operator in $\mathcal{F}$. Let S and T be terms of $I(\mathcal{F})$-algebra. Let σ and τ be terms obtained from S and T respectively by replacing all operators by the corresponding syntactic operators. Assume $S = T$ is an identity (equation) of $I(\mathcal{F})$-algebra. We say that axiom schema $[\sigma]\varphi \leftrightarrow [\tau]\varphi$ *corresponds* to identity (equation) $S = T$, or that $[\sigma]\varphi \leftrightarrow [\tau]\varphi$ is the *corresponding* axiom schema of identity (equation) $S = T$.

The following theorem is the main result of this section:

Theorem 5.10. *Let $\mathcal{F}$ be a set of syntactic operators and I an interpretation interpreting the syntactic operators in $\mathcal{F}$. Assume that $I(\mathcal{F})$-algebra is effectively axiomatizable by set Φ of identities. Let Σ be an I-elementary deduction system including rules US_G and RS, and also including a corresponding axiom schema for each identity in Φ. Let S and T be terms of $I(\mathcal{F})$-algebra. Let σ and τ be the syntactic counterparts of terms S and T respectively. Now $S = T$ is a free identity of $I(\mathcal{F})$-algebra iff for all formulae A it is the case that $\vdash_\Sigma [\sigma]A \leftrightarrow [\tau]A$.*

Proof. Assume first that equation $S = T$ is a free identity of $I(\mathcal{F})$-algebra. Then the equation must be obtainable by identity substitution from the axioms that axiomatize $I(\mathcal{F})$-algebra. In other words, there is a finite algebraic calculation starting from the axioms and ending up with the desired result $S = T$. On the other hand, for each of the axioms of $I(\mathcal{F})$-algebra of type $S_1 = S_2$, we have in the deduction system the corresponding axiom schema of type $[\sigma_1]\varphi \leftrightarrow [\sigma_2]\varphi$, and inference rule IS allows us to simulate identity substitution of modal terms. We can therefore simulate the calculation ending up with a deduction of formula $[\sigma]p \leftrightarrow [\tau]p$, where $p \in \Pi$. Applying rule US_G we therefore obtain a deduction of formula $[\sigma]A \leftrightarrow [\tau]A$.

Conversely, assume that $\vdash_\Sigma [\sigma]A \leftrightarrow [\tau]A$ for all formulae A. Thus $\vdash_\Sigma [\sigma]p \leftrightarrow [\tau]p$. Therefore $I \models [\sigma]p \leftrightarrow [\tau]p$. Now assume equation $S = T$ is not a free identity of $I(\mathcal{F})$-algebra. Therefore we can construct a frame $\langle W, I, \tilde{R}\rangle$ such that $R(\sigma) \neq R(\tau)$. Thus $I \not\models [\sigma]p \leftrightarrow [\tau]p$ by Lemma 3.12. This is a contradiction. □

Corollary 5.11. *Assume the situation that holds for Theorem 5.10. Let A and φ be arbitrary formulae. Let A' be any formula that can be obtained from A by replacing exactly one instance of subformula $[\sigma]\varphi$ of A by $[\tau]\varphi$. Now if equation $S = T$ is a free identity of $I(\mathcal{F})$-algebra, then $\vdash_\Sigma A \Leftrightarrow \vdash_\Sigma A'$.*

Proof. If $S = T$ is a free identity, we obtain $\vdash_\Sigma [\sigma]\varphi \leftrightarrow [\tau]\varphi$ by Theorem 5.10. By Lemma 4.10 we obtain $\vdash_\Sigma A([\sigma]\varphi/[\sigma]\varphi) \leftrightarrow A([\tau]\varphi/[\sigma]\varphi)$. In other words, $\vdash_\Sigma A \leftrightarrow A([\tau]\varphi/[\sigma]\varphi)$. We infer $\vdash_\Sigma A' \leftrightarrow A'([\tau]\varphi/[\sigma]\varphi)$ analogously. Since formulae $A([\tau]\varphi/[\sigma]\varphi)$ and $A'([\tau]\varphi/[\sigma]\varphi)$ are identical, we conclude $\vdash_\Sigma A \leftrightarrow A'$. Therefore $\vdash_\Sigma A \Leftrightarrow \vdash_\Sigma A'$. □

6 Partial Completeness Results

In this section we discuss partial completeness results related to effectively axiomatizable algebras. For example $\{\cup, \cap\}$-algebra is finitely axiomatizable (as we shall see in the next section) and so is $\{\cup, \cap, \sim, 0, 1\}$-algebra, where 0 and 1 denote the empty and universal relations respectively. We shall first present and prove the results and then discuss their significance.

Lemma 6.1. *Let p be an arbitrary proposition symbol. Assume $I(\mathcal{F})$-algebra is effectively axiomatizable by set Φ of identities and consider an I-elementary deduction system Σ that includes a corresponding axiom schema for each identity in Φ and also rules US_G and RS. Now $I \models [\alpha]p \leftrightarrow [\beta]p$ iff $\vdash_\Sigma [\alpha]p \leftrightarrow [\beta]p$.*

Proof. First assume $I \models [\alpha]p \leftrightarrow [\beta]p$. Let T_α and T_β be terms of $I(\mathcal{F})$-algebra that correspond exactly to the modal terms α and β respectively. We shall show that $T_\alpha = T_\beta$ is a free identity of $I(\mathcal{F})$-algebra. This allows us to conclude that $\vdash_\Sigma [\alpha]p \leftrightarrow [\beta]p$ by Theorem 5.10.

Let M be the set of atoms present in modal terms α and β, and let V be the set of variables in terms T_α and T_β. We let $f : M \to V$ be the bijection that maps each atom of α and β to the corresponding variable of equation $T_\alpha = T_\beta$. Assume for contradiction that there is a set W such that $T_\alpha = T_\beta$ is not an identity of algebra $\langle Pow(W \times W), I(\mathcal{F})\rangle$. Therefore we can assign members of $Pow(W \times W)$ to the variables in equation $T_\alpha = T_\beta$ such that the resulting equation is false for $\langle Pow(W \times W), I(\mathcal{F})\rangle$. Let $g : V \to Pow(W \times W)$ be the function describing the assignment. We can now define a frame $\langle W, I, \tilde{R}\rangle$ such that for each atom $a \in M$ we let $R(a) = g(f(a))$. Therefore it is now the case that $R(\alpha) \neq R(\beta)$. However, since $I \models [\alpha]p \leftrightarrow [\beta]p$, we conclude that $R(\alpha) = R(\beta)$ by Corollary 3.12. Thus we have a contradiction, whence we infer that equation $T_\alpha = T_\beta$ is a free identity of $I(\mathcal{F})$-algebra.

We then assume that $\vdash_\Sigma [\alpha]p \leftrightarrow [\beta]p$. Since all axioms of deduction system Σ are I-valid and each inference rule conserves $I-$validity of formulae, we conclude $I \models [\alpha]p \leftrightarrow [\beta]p$. □

Corollary 6.2. *Assume the situation that holds for Lemma 6.1. Now* $\vdash_\Sigma [\alpha]p \leftrightarrow [\beta]p$ *iff for all frames* $\langle W, I, \tilde{R}\rangle$ *we have* $R(\alpha) = R(\beta)$.

Proof. This follows directly from Corollary 3.12 and Lemma 6.1. □

Lemma 6.3. *Let* p *be an arbitrary proposition symbol and let* $+ \in \mathcal{F}$ *and* $\langle +, \cup\rangle \in I$. *Assume* $I(\mathcal{F})$*-algebra is effectively axiomatizable by set* Φ *of identities and consider an* I*-elementary deduction system* Σ *that includes a corresponding axiom schema for each identity in* Φ *and rules* US_G *and* RS. *Also let axiom schema* $[\sigma + \tau]\varphi \leftrightarrow [\sigma]\varphi \wedge [\tau]\varphi$ *be included in the system. Now* $I \models [\alpha]p \rightarrow [\beta]p$ *iff* $\vdash_\Sigma [\alpha]p \rightarrow [\beta]p$.

Proof. We begin by showing that $R(\beta) \subseteq R(\alpha)$ for all frames $\langle W, I, \tilde{R}\rangle$ iff $\vdash_\Sigma [\alpha + \beta]p \leftrightarrow [\alpha]p$. Therefore we assume that $R(\beta) \subseteq R(\alpha)$ for all frames $\langle W, I, \tilde{R}\rangle$. Hence it is also the case that $R(\alpha) \cup R(\beta) = R(\alpha)$ for all frames $\langle W, I, \tilde{R}\rangle$. Hence $R(\alpha + \beta) = R(\alpha)$ for all frames $\langle W, I, \tilde{R}\rangle$, whence $\vdash_\Sigma [\alpha + \beta]p \leftrightarrow [\alpha]p$ by Corollary 6.2.

Conversely, assume $\vdash_\Sigma [\alpha + \beta]p \leftrightarrow [\alpha]p$ and consider an arbitrary frame $\langle W, I, \tilde{R}\rangle$. Now $R(\alpha + \beta) = R(\alpha)$ by Corollary 6.2. Hence $R(\alpha) \cup R(\beta) = R(\alpha)$, whence $R(\beta) \subseteq R(\alpha)$.

We have established that $R(\beta) \subseteq R(\alpha)$ for all frames $\langle W, I, \tilde{R}\rangle$ iff $\vdash_\Sigma [\alpha + \beta]p \leftrightarrow [\alpha]p$. By Lemma 3.11 we conclude that $I \models [\alpha]p \rightarrow [\beta]p$ iff $\vdash_\Sigma [\alpha + \beta]p \leftrightarrow [\alpha]p$.

We then show that $\vdash_\Sigma ([\alpha + \beta]p \leftrightarrow [\alpha]p) \leftrightarrow ([\alpha]p \rightarrow [\beta]p)$. This can be established via a simple deduction:

1.	$(([\alpha]p \wedge [\beta]p) \leftrightarrow [\alpha]p) \leftrightarrow ([\alpha]p \rightarrow [\beta]p)$	PL
2.	$[\alpha + \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p$	*Axiom*
3.	$([\alpha + \beta]p \leftrightarrow [\alpha]p) \leftrightarrow ([\alpha]p \rightarrow [\beta]p)$	$1, 2, PL$

Therefore we conclude that $I \models [\alpha]p \rightarrow [\beta]p$ iff $\vdash_\Sigma [\alpha]p \rightarrow [\beta]p$. □

Corollary 6.4. *Assume the same situation that holds for Lemma 6.3. Now* $\vdash_\Sigma [\alpha]p \rightarrow [\beta]p$ *iff for all frames* $\langle W, I, \tilde{R}\rangle$ *we have* $R(\beta) \subseteq R(\alpha)$.

Proof. This follows directly from Lemmas 3.11 and 6.3. □

Much of the significance of the results presented in this section is encoded in Corollary 6.2: If $I(\mathcal{F})$-algebra is effectively axiomatizable, then we automatically know how to construct a deduction system that captures free identities of $I(\mathcal{F})$-algebra by corresponding deducible formulae. Moreover, Corollary 6.4 establishes that if $+ \in \mathcal{F}$ and $\langle +, \cup \rangle \in I$, then we can also capture all *free inequalities* (subset-inequalities that hold for each member of the class of $I(\mathcal{F})$-algebras). Deduction systems capturing all free identities and inequalities can in a sense be considered complete from the point of view of the related algebra of modal terms.

The partial completeness results can be directly applied to $MLOM$s with union and intersection operations. As we shall see in the next section, $\{\cup, \cap\}$-algebra is finitely axiomatizable by the distributive lattice axioms. One of the two complete axiomatizations presented below relies on the (finite) axiomatizability of $\{\cup, \cap\}$-algebra and the above partial completeness results.

The partial completeness results can be useful in the construction of (complete) axiomatizations of various different $MLOM$s. The results also apply for example to languages with a Boolean algebra of modalities. We shall discuss this point further at the end of the next section.

7 An Axiomatization Based on the Distributive Lattice Axioms

In this section we first show that the free identities of $\{\cup, \cap\}$-algebra can be generated from the distributive lattice axioms. This follows directly from Stone's Representation Theorem for Distributive Lattices. We then present the first one of our two complete axiomatizations of multimodal logic with union and intersection. The alternative axiomatization will be the topic of the next section. We begin this section by giving a number of essential definitions.

Definition 7.1. A *distributive lattice* is a structure $\langle D, \vee, \wedge \rangle$ with a non-empty domain D, where the *supremum operation* $\vee$ and the *infimum operation* $\wedge$ satisfy the following laws:

Assoc.	$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
Commut.	$x \vee y = y \vee x$	$x \wedge y = y \wedge x$
Absorpt.	$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$
Distrib.	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Definition 7.2. Let $\langle s, \vee, \wedge \rangle$ be a structure, where s is a set closed under finite unions and intersections. Let $\vee$ and $\wedge$ be functions such that $\forall x, y, z \in s(x \vee y = z \Leftrightarrow x \cup y = z)$ and $\forall x, y, z \in s(x \wedge y = z \Leftrightarrow x \cap y = z)$. Structure $\langle s, \vee, \wedge \rangle$ is a *lattice of sets.* If s is a set closed under finite unions and intersections, we write $\mathcal{L}(s) = \langle s, \vee, \wedge \rangle$. Note that a lattice of sets is a distributive a lattice, as union and intersection satisfy the axioms listed in Definition 7.1.

Definition 7.3. Let $\mathfrak{S}$ be an algebraic structure. We let $\mathcal{I}(\mathfrak{S})$ denote the class of algebraic identities (cf. Definition 2.5) of the structure.

Definition 7.4. We say that a particular identity $S = T$ is *generable* from set Φ of identities if there exists a finite algebraic calculation (i.e. a deduction in equational logic) that ends with identity $S = T$ and uses only identities of Φ.

We shall now establish that $\{\cup, \cap\}$-algebra is finitely axiomatizable. The following theorems lead to this realization:

Theorem 7.5 (Stone's Representation Theorem for Distributive Lattices)**.** *A lattice is a distributive lattice iff it is isomorphic to some lattice of sets.*

Theorem 7.6. *Let E be a lattice equation. There exists a lattice (a free distributive lattice for a set of generators) where E holds iff E is generable from the distributive lattice axioms.*

We omit the proofs of these theorems as they involve concepts not essential to the exposition of the results of this paper. For a proofs, see for example [6].

We are now ready to prove the following corollary:

Corollary 7.7. *The free identities of $\{\cup, \cap\}$-algebra are exactly the identities generable from the distributive lattice axioms.*

Proof. In order to improve readability of the proof, we let $C(s)$ denote the assertion that s is closed under finite unions and intersections.

Consider class $\mathcal{K}$ of identities of operator algebra $\langle s, \{\cup, \cap\}\rangle$ and class $\mathcal{I}(\mathcal{L}(s))$ of identities of $\mathcal{L}(s)$. Class $\mathcal{I}(\mathcal{L}(s))$ uses symbols $\vee$ and $\wedge$ instead of symbols $\cup$ and $\cap$ used in $\mathcal{K}$, but this distinction is irrelevant from the point of view of the proof, and shall be ignored. Therefore we can write $\mathcal{K} = \mathcal{I}(\mathcal{L}(s))$. Thus, if we let $\mathcal{S}$ denote the class of all free identities of $\{\cup, \cap\}$-algebra, we have $\mathcal{S} = \bigcap\{\ \mathcal{I}(\mathcal{L}(s)) \mid s \text{ is a set and } C(s)\}$. Similarly, we let $\mathcal{H}$ be the class of all free identities of distributive lattices. We have $\mathcal{H} = \bigcap\{\ \mathcal{I}(h) \mid h \text{ is a distributive lattice }\}$. Stone's Representation Theorem directly implies that

$$\{\ \mathcal{I}(h) \mid h \text{ is a distributive lattice }\} = \{\ \mathcal{I}(\mathcal{L}(s)) \mid s \text{ is a set and } C(s)\}$$

Therefore we obtain

$$\bigcap\{\ \mathcal{I}(h) \mid h \text{ is a distributive lattice }\} = \bigcap\{\ \mathcal{I}(\mathcal{L}(s)) \mid s \text{ is a set and } C(s)\}$$

Thus $\mathcal{S} = \mathcal{H}$. In other words, the class of free identities of $\{\cup, \cap\}$-algebra is equal to the class of free identities of distributive lattices. By Theorem 7.6 the class of free identities of distributive lattices is exactly the class of identities generable from the distributive lattice axioms. Therefore the class of free identities of $\{\cup, \cap\}$-algebra is exactly the class of identities generable from the distributive lattice axioms. $\square$

We now know that the free identities of $\{\cup, \cap\}$-algebra can be generated from the distributive lattice axioms. In other words, $\{\cup, \cap\}$-algebra is finitely axiomatizable. Therefore, keeping in mind the considerations of section 6, the following definition arises naturally:

Definition 7.8. Let $\mathcal{F} = \{+, \cdot\}$ and $I = \{\langle +, \cup \rangle, \langle \cdot, \cap \rangle\}$. We let Σ_1 denote an I-elementary deduction system containing rules US_G and RS and the following axiom schemata:

- $C_u : [\alpha + \beta]\varphi \leftrightarrow [\beta + \alpha]\varphi$
- $C_i : [\alpha \cdot \beta]\varphi \leftrightarrow [\beta \cdot \alpha]\varphi$
- $As_u : [\alpha + (\beta + \gamma)]\varphi \leftrightarrow [(\alpha + \beta) + \gamma]\varphi$
- $As_i : [\alpha \cdot (\beta \cdot \gamma)]\varphi \leftrightarrow [(\alpha \cdot \beta) \cdot \gamma]\varphi$
- $Ab_u : [\alpha + (\alpha \cdot \beta)]\varphi \leftrightarrow [\alpha]\varphi$
- $Ab_i : [\alpha \cdot (\alpha + \beta)]\varphi \leftrightarrow [\alpha]\varphi$
- $D_u : [\alpha + (\beta \cdot \gamma)]\varphi \leftrightarrow [(\alpha + \beta) \cdot (\alpha + \gamma)]\varphi$
- $D_i : [\alpha \cdot (\beta + \gamma)]\varphi \leftrightarrow [(\alpha \cdot \beta) + (\alpha \cdot \gamma)]\varphi$
- $A_u : [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$.

With the exception of the last schema, the schemata listed above correspond to the distributive lattice axioms. The list of schemata is actually redundant since it can be shown that all members of the list above can be generated from C_i, As_i, Ab_u, Ab_i, D_i and A_u by a number of formal metadeductions (i.e. deductions involving schemata rather than formulae). Hence schemata C_u and As_u and D_u are superfluous and can be done away with. The interested minimalist is invited to consult [13].

Remark 7.9. The number of axiom schemata included in system Σ_1 is finite. Since universal substitution (rule US_G) belongs to Σ_1, we could replace the set of axiom schemata by a finite set of axioms. We would have to ensure, however, that our language under discourse had a sufficient number of atoms and proposition symbols for the proper formulation of the axioms.

Stone's Representation Theorem for Boolean Algebras (see [6] or [10] for example) leads to the finite axiomatizability of $\{\cup, \cap, \sim, 0, 1\}$-algebra by an argument analogous to the proof of Corollary 7.7. We can therefore extend Σ_1 with the axiom schemata corresponding to the axioms of Boolean algebra not already included in Σ_1 (cf. Example 5.8). The partial completeness results given in section 6 automatically hold for this extension of Σ_1.

It is worth observing how one of the general conceptual symmetries between algebra and logic is here nicely exposed: If equations are conceived as formulae and identity substitution as the principal algebraic rule of inference, then a representation theorem related to a set of identities is in a very strong sense analogous to a completeness theorem.

8 An alternative Axiomatization

In this section we define a deduction system that can be seen as an alternative to the system defined in section 7.

Definition 8.1. Let $\mathcal{F} = \{+, \cdot\}$ and $I = \{\langle +, \cup \rangle, \langle \cdot, \cap \rangle\}$. We let Σ_2 denote an I-elementary deduction system containing rules US_G and RS and the following axiom schemata:

- $D_i : [\alpha \cdot (\beta + \gamma)]\varphi \leftrightarrow [(\alpha \cdot \beta) + (\alpha \cdot \gamma)]\varphi$
- $I_i : [\alpha \cdot \alpha]\varphi \leftrightarrow [\alpha]\varphi$
- $As_i : [\alpha \cdot (\beta \cdot \gamma)]\varphi \leftrightarrow [(\alpha \cdot \beta) \cdot \gamma]\varphi$
- $C_i : [\alpha \cdot \beta]\varphi \leftrightarrow [\beta \cdot \alpha]\varphi$
- $A_i : [\alpha]\varphi \rightarrow [\alpha \cdot \beta]\varphi$
- $A_u : [\alpha + \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$.

We shall prove two simple properties of this system. For this purpose, we need the following auxiliary result:

Lemma 8.2. *Every term T of $\{\cup, \cap\}$-algebra has an equivalent term in disjunctive normal form. Such a term can be obtained from the original term by the a finite number of applications of the distributive law $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.*

We omit the proof as the it is straightforward and widely known. A proof can be found in numerous texts on lattice theory.

With the help of Lemma 8.2, we can now prove the following result:

Lemma 8.3. *Let $\tau \in \Lambda$ be an arbitrary modal term. Now $\vdash_{\Sigma_2} [\tau]p \leftrightarrow [\delta]p$, where term $\delta \in \Lambda$ is equivalent to τ and in disjunctive normal form.*

Proof. Let T be the term of $\{\cup, \cap\}$-algebra corresponding to syntactic term τ. By Lemma 8.2 we know that there is a finite calculation that uses the distributive law $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$ and leads to an identity $T = D$, where D is equivalent to T and in disjunctive normal form. Since schema $[\alpha \cdot (\beta + \gamma)]\varphi \leftrightarrow [(\alpha \cdot \beta) + (\alpha \cdot \gamma)]\varphi$ is included in Σ_2, we can use rule IS in order to simulate this calculation. Therefore $\vdash_{\Sigma_2} [\tau]p \leftrightarrow [\delta]p$, where $\delta \in \Lambda$ is the syntactic term corresponding to term D. $\square$

Another important property of deduction system Σ_2 is given by the following lemma:

Lemma 8.4. *Let α and β be modal terms not containing instances of symbol $+$. If β contains at least one instance of each atom symbol in α, then $\vdash_{\Sigma_2} [\alpha]p \to [\beta]p$.*

Proof. As in the proof of Lemma 8.3, we shall again rely on the fact that rule RS allows us to simulate algebraic calculations.

Term β has at least one instance of each atom in α. However, there might be atoms that have more repetitions in α than in β. We obtain a new modal term β' by replacing the leftmost occurrence of each such atom b in β by a term of type $b \cdot b \cdot \ldots \cdot b$, so that the number of atoms b in β' is equal to the number of occurrences of b in α. Using schema I_i and rule IS, we infer $\vdash_{\Sigma_2} [\beta]p \leftrightarrow [\beta']p$. Now, by associativity and commutativity provided by schemata As_i and C_i, we obtain $\vdash_{\Sigma_2} [\beta']p \leftrightarrow [\alpha \cdot \gamma]p$, where term γ consists of the atoms occurring in β' but not in α. We then conclude $\vdash_{\Sigma_2} [\beta]p \leftrightarrow [\alpha \cdot \gamma]p$ by propositional logic. By schema A_i we have $\vdash_{\Sigma_2} [\alpha]p \to [\alpha \cdot \gamma]p$. Therefore we conclude $\vdash_{\Sigma_2} [\alpha]p \to [\beta]p$ by propositional logic. □

It is possible to show deductively that Σ_1 and Σ_2 are equivalent (consult [13]). However, in the next section we shall prove completeness of Σ_1 and Σ_2 independently. We shall do this because on one hand we want to demonstrate the applicability of the partial completeness results of section 6 to the process of proving full completeness results, and on the other hand we also want to establish completeness without appealing to Stone's Representation Theorem (or any other result of sections 6 and 7).

9 Completeness

In this section we prove that deduction systems Σ_1 and Σ_2 (defined in sections 7 and 8 respectively) are complete with respect to interpretation $I = \{\langle +, \cup \rangle, \langle \cdot, \cap \rangle\}$. In other words, assuming this interpretation I we show that $I \models \varphi \Leftrightarrow \vdash_{\Sigma_i} \varphi$ for each case $i = 1$ and $i = 2$. Throughout the section we let Σ refer to both Σ_1 and Σ_2: Any statement made of Σ means that the statement holds for both Σ_1 and Σ_2.

Completeness proofs in modal logic are mostly based on the construction of a *canonical model*. For example, a trivial generalization of the canonical model of system K of unimodal logic enables the proof of completeness for the case with $I = \{\langle +, \cup \rangle\}$ (see [13] for details). However, it can be shown that for $I = \{\langle +, \cup \rangle, \langle \cdot, \cap \rangle\}$ this trivial generalization does not work (a proof due to Ari Virtanen can be found in [13]). Below we shall define a canonical model M_Σ that enables the proof of completeness for $I = \{\langle +, \cup \rangle, \langle \cdot, \cap \rangle\}$.

We begin by making a number of definitions whose purpose is to improve readability of the elaborations that follow.

Definition 9.1. Let N be an ordered tuple. By $N[n]$ we refer to the n:th member of the tuple. Therefore for example $\langle x, y\rangle[2] = y$.

Definition 9.2. Let $\gamma \in \Lambda$ be a term that does not include instances of symbol $+$. We call γ an *atom cluster*, or simply a *cluster*.

Definition 9.3. We let $\mathcal{M}_\Sigma$ denote the set of all negation complete and Σ-consistent sets of formulae of the language under discourse.

Definition 9.4. Let $\alpha, \beta \in \Lambda$ be modal terms. We write $\alpha \leq \beta$, when equation $\alpha + \beta = \beta$ is a syntactic counterpart of a free identity of $\{\cup, \cap\}$-algebra. In other words, we write $\alpha \leq \beta$ if this inequality is a syntactic counterpart of a free inequality of $\{\cup, \cap\}$-algebra.

The essence of the above definition should be clarified by the example below:

Example 9.5. Consider a model with two relations $R(a) = R(b) = \{\langle w_1, w_2\rangle\}$. In this fixed model it happens to be the case that $R(a) \subseteq R(a) \cap R(b)$. However, equation $a \subseteq a \cap b$ is not a free inequality of $\{\cup, \cap\}$-algebra. Therefore $a \nleq a \cdot b$. On the other hand, for example $a \cdot b \leq a$, as $a \cap b \subseteq a$ holds for any sets a, b.

We are ready to give the definition of canonical model M_Σ:

Definition 9.6. Canonical model M_Σ is a quadruple $\langle W_\Sigma, I_\Sigma, \tilde{R}_\Sigma, P_\Sigma\rangle$. We define domain W_Σ of the model in the following way:

$$W_\Sigma = \{\langle \Gamma, \gamma\rangle \mid \Gamma \in \mathcal{M}_\Sigma \text{ and } \gamma \in \Lambda \text{ is an atom cluster}\}$$

It is instructive to think of the members of domain W_Σ as negation complete and Σ-consistent sets *named* or *tagged* by atom clusters.

We define interpretation mapping I_Σ in the natural way:

$$I_\Sigma = \{\langle +, \cup\rangle, \langle \cdot, \cap\rangle\}$$

Let $a_i \in \mathcal{A}$ be an arbitrary atom and $w, u \in W_\Sigma$ arbitrary points of the domain. We define relation $R_\Sigma(a_i)$ in the following way:

$$\begin{aligned} & w\ R_\Sigma(a_i)\ u \\ \Leftrightarrow & \\ & \{C \mid [\delta]C \in w[1]\} \subseteq u[1] \text{ and } \delta \leq a_i, \text{ where } u[2] = \delta \end{aligned}$$

It is worth noticing that here $\delta \leq a_i$ means exactly that atom a_i is one of the atoms of atom cluster δ.

We define map P_Σ in the following way:

$$w \in P_\Sigma(p_i) \Leftrightarrow p_i \in w[1]$$

This completes the definition of the canonical model M_Σ.

In order to prove completeness of the system under discourse, we first need to establish two auxiliary results.

Lemma 9.7 (DNF-Truth Lemma). *Let M_Σ be the canonical model defined above. Let w be an arbitrary member of the domain W_Σ of M_Σ. Let A be an arbitrary formula whose subformulae of type $[\alpha]\varphi$ are such that α is in disjunctive normal form. Then it is the case that $A \in w[1]$ iff $M_\Sigma, w \models A$.*

Proof. The proof is by induction on the structure of formula A. In order to establish the basis of the induction, we assume that $A = p$, where p is an arbitrary proposition symbol. The claim

$$p \in w[1] \text{ iff } M_\Sigma, w \models p$$

follows trivially from the definition of valuation mapping P_Σ given in Definition 9.6. The induction hypothesis is that claims

$$\psi \in w[1] \text{ iff } M_\Sigma, w \models \psi$$

and

$$\psi' \in w[1] \text{ iff } M_\Sigma, w \models \psi'$$

hold. Assume then that $A = \neg\psi$. We get

$$\begin{aligned} & M_\Sigma, w \models \neg\psi \\ \Leftrightarrow\ & M_\Sigma, w \not\models \psi \\ \Leftrightarrow\ & \psi \notin w[1] \\ \Leftrightarrow\ & \neg\psi \in w[1] \end{aligned}$$

where the second equivalence follows from the induction hypothesis, and the third equivalence from the fact that $w[1]$ is a negation complete and Σ-consistent set of formulae.

Next assume $A = \psi \wedge \psi'$. We get

$$\begin{aligned} & M_\Sigma, w \models \psi \wedge \psi' \\ \Leftrightarrow\ & M_\Sigma, w \models \psi \text{ and } M_\Sigma, w \models \psi' \\ \Leftrightarrow\ & \psi \in w[1] \text{ and } \psi' \in w[1] \\ \Leftrightarrow\ & \psi \wedge \psi' \in w[1] \end{aligned}$$

where again the second equivalence follows from the induction hypothesis and the third equivalence from the fact that $w[1]$ is a negation complete and Σ-consistent set of formulae.

We then move on to the non-trivial case where $A = [\alpha]\psi$. This case requires a sub-induction on the structure of α. We know that α is in disjunctive normal form. As the basis of the sub-induction, we consider the case where $A = [\tau]\psi$ such that τ is an arbitrary atom cluster. Naturally this case also covers the situation where τ is an individual atom. The principal requirement is that τ does not contain any instances of symbol $+$. We also note that τ must be bracketed according to the binary nature of the (syntactic counterpart) of the intersection operator. However, it shall become clear that the details related to the bracketing bear absolutely no significance to the argument below. Therefore we may simply write that $\tau = a_{i_1} \cdot a_{i_2} \cdot a_{i_3} \cdot \ldots \cdot a_{i_n}$.

Let us first assume that $[\tau]\psi \in w[1]$. We consider an arbitrary point $u \in W_\Sigma$ such that $w\ R_\Sigma(\tau)\ u$. We let $u[2] = \delta$. Since $\tau = a_{i_1} \cdot a_{i_2} \cdot a_{i_3} \cdot \ldots \cdot a_{i_n}$ and $w\ R_\Sigma(\tau)\ u$, we conclude

$$\bigwedge_{1 \le k \le n} w\ R_\Sigma(a_{i_k})\ u$$

Therefore by the definition of $R_\Sigma(a_{i_k})$ (see Definition 9.6) we conclude

$$\bigwedge_{1 \le k \le n} \delta \le a_{i_k}$$

Hence we infer that $\delta \le \tau$. We conclude $\vdash_{\Sigma_2} [\tau]p \rightarrow [\delta]p$ directly by Lemma 8.4. On the other hand, since $\delta \le \tau$ then $I_\Sigma \models [\tau]p \rightarrow [\delta]p$, whence we infer $\vdash_{\Sigma_1} [\tau]p \rightarrow [\delta]p$ by Lemma 6.3. Therefore $\vdash_\Sigma [\tau]p \rightarrow [\delta]p$, whence by rule US_G we obtain $\vdash_\Sigma [\tau]\psi \rightarrow [\delta]\psi$. Thus $[\tau]\psi \rightarrow [\delta]\psi \in w[1]$. Since $[\tau]\psi \in w[1]$, we infer $[\delta]\psi \in w[1]$. As $w\ R_\Sigma(a_{i_1})\ u$, we conclude from the definition of relation $R_\Sigma(a_{i_1})$ (see Definition 9.6) that $\{C \mid [\delta]C \in w[1]\} \subseteq u[1]$. Therefore, as $[\delta]\psi \in w[1]$, we conclude that $\psi \in u[1]$. By the induction hypothesis of the main induction, we now obtain $M_\Sigma, u \models \psi$. We have therefore shown for an arbitrary $u \in W_\Sigma$ that $w\ R_\Sigma(\tau)\ u \Rightarrow M_\Sigma, u \models \psi$. Thus $M_\Sigma, w \models [\tau]\psi$. This concludes the first half of the case $A = [\tau]\psi$.

Let us then assume that $[\tau]\psi \notin w[1]$. Since $w[1]$ is a negation complete set of formulae, we conclude $\neg[\tau]\psi \in w[1]$. By Lemma 4.15 and Lindenbaum's lemma we know that there exists a negation complete and Σ-consistent set of formulae Δ such that $\{C \mid [\tau]C \in w[1]\} \cup \{\neg\psi\} \subseteq \Delta$. Therefore there exists a point $v = \langle \Delta, \tau \rangle$ in W_Σ. Now let a_i be an arbitrary atom in term τ. Clearly $\tau \le a_i$. Now, since we also know that $\{C \mid [\tau]C \in w[1]\} \subseteq v[1]$, we conclude $w\ R_\Sigma(a_i)\ v$ from the definition of relation $R_\Sigma(a_i)$ in Definition 9.6. Since a_i was assumed to be an arbitrary atom in τ, we conclude $w\ R_\Sigma(\tau)\ v$. Since $\neg\psi \in v[1]$, then $\psi \notin v[1]$, whence by the induction hypothesis $M_\Sigma, v \not\models \psi$. Therefore, as $w\ R_\Sigma(\tau)\ v$ and $M_\Sigma, v \not\models \psi$, we obtain $M_\Sigma, w \not\models [\tau]\psi$. This concludes the latter half of case $A = [\tau]\psi$. Therefore the basis of the sub-induction has now been established.

We then consider the induction step of the sub-induction. The hypothesis of the sub-induction is that

$$[\eta]\psi \in w[1] \text{ iff } M_\Sigma, w \models [\eta]\psi$$

and

$$[\epsilon]\psi \in w[1] \text{ iff } M_\Sigma, w \models [\epsilon]\psi$$

where $\eta, \epsilon \in \Lambda$ are in disjunctive normal form. The base clause covers all the cases where $A = [\alpha]\psi$ such that α does not include symbol $+$. Therefore we only need to discuss cases where $A = [\alpha]\psi$ and α includes at least one instance of symbol $+$. Since α is in disjunctive normal form and contains at least one instance of symbol $+$, it is clear that the main connective of α must be $+$. Therefore we assume that $A = [\eta + \epsilon]\psi$.

As $[\eta+\epsilon]\psi \leftrightarrow [\eta]\psi \wedge [\epsilon]\psi$ is an axiom, we know that $[\eta+\epsilon]\psi \leftrightarrow [\eta]\psi \wedge [\epsilon]\psi \in w[1]$. Since $I_\Sigma(+) = \cup$, by Lemma 3.10 we also know that $M_\Sigma \models [\eta+\epsilon]\psi \leftrightarrow [\eta]\psi \wedge [\epsilon]\psi$. Therefore we now have the following chain of equivalences:

$$\begin{aligned}
& [\eta + \epsilon]\psi \in w[1] \\
\Leftrightarrow\ & [\eta]\psi \wedge [\epsilon]\psi \in w[1] \\
\Leftrightarrow\ & [\eta]\psi \in w[1] \text{ and } [\epsilon]\psi \in w[1] \\
\Leftrightarrow\ & M_\Sigma, w \models [\eta]\psi \text{ and } M_\Sigma, w \models [\epsilon]\psi \\
\Leftrightarrow\ & M_\Sigma, w \models [\eta]\psi \wedge [\epsilon]\psi \\
\Leftrightarrow\ & M_\Sigma, w \models [\eta + \epsilon]\psi
\end{aligned}$$

The first equivalence is due to the fact that $[\eta+\epsilon]\psi \leftrightarrow [\eta]\psi \wedge [\epsilon]\psi \in w[1]$, as already mentioned above. The second equivalence follows from $w[1]$ being a negation complete and Σ-consistent set of formulae. The third equivalence is due to the induction hypothesis of the sub-induction. The fourth equivalence is trivial. The fifth equivalence follows from the fact that $M_\Sigma \models [\eta+\epsilon]\psi \leftrightarrow [\eta]\psi \wedge [\epsilon]\psi$, also mentioned above. We have now proved the lemma. □

For the next result, we need the following definitions:

Definition 9.8. Consider a modal term $\alpha \in \Lambda$. With $DNF(\alpha)$ we refer to a term that is a disjunctive normal form version of α. For the proof of completeness of Σ_2 it is also assumed that $DNF(\alpha)$ has been obtained from α by applying the distributive law $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ (cf. proof of Lemma 8.3).

Definition 9.9. Consider a formula A of the language under discourse. With $DNF(A)$ we refer to a formula obtained from A by replacing each subformula of type $[\alpha]\varphi$ by $[DNF(\alpha)]\varphi$.

With the help of Lemma 9.7 it is now easy to prove a general truth lemma:

Lemma 9.10 (Truth Lemma)**.** *Let M_Σ be the canonical model defined above and w an arbitrary member of the domain W_Σ of M_Σ. Now $A \in w[1]$ iff $M_\Sigma, w \models A$ for all formulae A.*

Proof. We shall first show that $\vdash_\Sigma A \leftrightarrow DNF(A)$. Consider arbitrary formulae B and $[\gamma]\varphi$. By Theorem 5.10 we conclude that $\vdash_{\Sigma_1} [\gamma]p \leftrightarrow$

$[DNF(\gamma)]p$, where p is a proposition symbol. On the other hand, we infer $\vdash_{\Sigma_2} [\gamma]p \leftrightarrow [DNF(\gamma)]p$ by Lemma 8.3. Thus $\vdash_\Sigma [\gamma]p \leftrightarrow [DNF(\gamma)]p$, whence $\vdash_\Sigma [\gamma]\varphi \leftrightarrow [DNF(\gamma)]\varphi$ by rule US_G. By Lemma 4.10 we obtain $\vdash_\Sigma B([\gamma]\varphi/[\gamma]\varphi) \leftrightarrow B([DNF(\gamma)]\varphi/[\gamma]\varphi)$. We have thus shown that for all formulae B and $[\gamma]\varphi$ it is the case that $\vdash_\Sigma B \leftrightarrow B([DNF(\gamma)]\varphi/[\gamma]\varphi)$. Therefore, for some n we obtain

$$\begin{array}{l} \vdash_\Sigma A_1 \leftrightarrow A_2 \\ \vdash_\Sigma A_2 \leftrightarrow A_3 \\ \vdots \\ \vdash_\Sigma A_{n-1} \leftrightarrow A_n \end{array}$$

where for all $i \in \{1, 2, ..., n-1\}$ there exists some formula ψ and some $\tau \in \Lambda$ such that $A_{i+1} = A_i([DNF(\tau)]\psi/[\tau]\psi)$, and where $A_1 = A$ and $A_n = DNF(A)$. Therefore we conclude $\vdash_\Sigma A \leftrightarrow DNF(A)$ by propositional logic.

Let us then assume that $A \in w[1]$. Since $A \leftrightarrow DNF(A) \in w[1]$, then $DNF(A) \in w[1]$. Therefore by Lemma 9.7 we obtain $M_\Sigma, w \models DNF(A)$. Since $M_\Sigma, w \models DNF(A)$, then also $M_\Sigma, w \models A$. For the converse, assume that $M_\Sigma, w \models A$. Thus $M_\Sigma, w \models DNF(A)$ and hence $DNF(A) \in w[1]$ by Lemma 9.7. Now since $A \leftrightarrow DNF(A) \in w[1]$, we conclude $A \in w[1]$. We have therefore shown that $A \in w[1]$ iff $M_\Sigma, w \models A$. □

We are now ready for the proof of completeness:

Theorem 9.11 (Completeness)**.** *Consider a language with $\mathcal{F} = \{+, \cdot\}$. Let $I = \{\langle +, \cup \rangle\ , \langle \cdot, \cap \rangle\}$. Then for all formulae A it the case that $I \models A$ iff $\vdash_\Sigma A$.*

Proof. Assume $I \models A$. Thus $M_\Sigma \models A$, whence by Lemma 9.10 we conclude that $A \in \Gamma$ for all $\Gamma \in \mathcal{M}_\Sigma$. Therefore, by Lindenbaum's Lemma (Lemma 4.16), we conclude that $\{\neg A\}$ is not a Σ-consistent set of formulae. Thus $\vdash_\Sigma \neg\neg A$ by Definition 4.14. Therefore $\vdash_\Sigma A$ by propositional logic.

The converse follows from the fact that the axioms included in deduction system Σ are I-valid and all rules of inference of Σ preserve I-validity of formulae. □

We finish the section by noting that interestingly it is not possible to apply a trivially generalized version of the presented method of proof of completeness to the case with $I = \{\langle +, \cup \rangle, \langle \cdot, \cap \rangle, \langle ^-, \sim \rangle, \langle 0, Z \rangle, \langle 1, U \rangle\}$ (cf. Example 5.8). Indeed it is natural to presume that if all literals (atoms and negated atoms) were treated alike, the proof should generalize to this case. However, defining $R_\Sigma(\bar{a})$ for negated atoms $\bar{a}$ in the same way as for atoms (see Definition 9.6) leads to a contradictory situation: If $v[2] = b \in \mathcal{A}$, then the fact that either $\langle u, v \rangle \in R_\Sigma(a)$ or $\langle u, v \rangle \in R_\Sigma(\bar{a})$ implies that either $b \leq a$ or $b \leq \bar{a}$, which is clearly not the case.

10 Concluding Remarks

In this paper we have discussed multimodal logics with operations on modalities from a general point of view. The emphasis has been on the theory of formal deductions.

The purpose of the first part of the article (sections 2-6) was to develop generally applicable methods that assist in the process of constructing (complete) axiomatizations of $MLOM$s. We defined a rule of inference RS that enables the algebraic manipulation of modal terms. The rule is applicable to any particular $MLOM$ (see section 5). We continued by proving partial completeness results that apply to $MLOM$s with an effectively axiomatizable algebra of modal terms (see section 6). In addition to this we considered how deduction systems can be extended in order to deal with more general algebras of modal terms (see section 5). The overall aim was to provide a generally applicable framework for the process of axiomatizing different $MLOM$s. We note however that questions related to axiomatizability of different algebras of binary relations are not at all trivial (see [14] for a thorough discussion of a number of algebras of binary relations), whence the axiomatizations of different $MLOM$s remain an interesting area of inquiry (see for example [4] and [5] for more general complete axiomatizations than those discussed in this article).

In the second part of the paper (from section 7 onwards) we applied the results of the first part and gave two alternative axiomatizations of multimodal logic with union and intersection of modalities and also provided a relatively straightforward proof of completeness for the axiomatizations.

We consider both of the two axiomatizations relatively simple. It is clear that a sufficiently simple axiomatization as opposed to a more complicated one may prove advantageous especially in mathematically oriented work. Indeed, when questions of appearance do not matter, a large number of $MLOM$s can be directly associated with a complete proof calculus; we obtain the following theorem with ease:

Theorem 10.1. *Consider an interpretation I such that all the related operations are first-order definable. Assume also that we have an effective process that associates each syntactic operator in $Dom(I)$ with a corresponding first-order definition. Then we can directly construct a proof system that is complete with respect to I-validity.*

Proof. Let A be some formula containing no other syntactic operators than those in $Dom(I)$. Let $\{p_1, ..., p_m\}$ and $\{a_1, ..., a_n\}$ be the sets of proposition symbols and atoms occurring in A. Let St_x^I denote the *standard translation* (see [7] for example) of formula A with the additional effect of interpreting the syntactic operators of A in first-order logic: For example if $I(\cdot) = \cap$, then we have $St_x^I([a \cdot b]p) = \forall y(R_a(x,y) \wedge R_b(x,y) \rightarrow P(y))$. Here term $xR(a \cdot b)y$ has been turned into its first-order form $R_a(x,y) \wedge R_b(x,y)$. We have the following chain of equivalences:

$$
\begin{aligned}
& I \models A \\
\Leftrightarrow\ & \models_{SO} \forall R_{a_1} ... \forall R_{a_n} \forall P_1 ... \forall P_m \forall x\ St_x^I(A) \\
\Leftrightarrow\ & \models_{FO} \forall x\ St_x^I(A) \\
\Leftrightarrow\ & \vdash_{FO} \forall x\ St_x^I(A)
\end{aligned}
$$

where $\models_{SO}$ and $\models_{FO}$ denote the second-order and first-order satisfaction relations respectively, and $\vdash_{FO}$ is the first-order syntactic turnstile. Note that we are considering a first-order language with symbols $R_{a_1}, ..., R_{a_n}$ and $P_1, ..., P_m$.

If A is I-valid, we can confirm it by a simple procedure: We first obtain $\forall x\ St_x^I(A)$. This can be done effectively, as by assumption the first-order definitions of the syntactic operators in A are effectively obtainable. With the help of a suitable complete first-order proof calculus, we can construct an enumeration procedure for the set of theorems of $FO(R_{a_1}, ..., R_{a_n}, P_1, ..., P_m)$. We begin listing the theorems, comparing them to $\forall x\ St_x^I(A)$. If A is I-valid, then $\vdash_{FO} \forall x\ St_x^I(A)$, and we will ultimately find our match. Therefore any I-valid formula has a proof by this proof system. $\square$

While certainly various different kinds of $MLOM$s can be applied to real-life problems, the construction of axiomatizations of these logics is also interesting in its own right. One natural question at this stage is whether the trivial generalization (following Stones Representation Theorem for Boolean Algebras) of system Σ_1 is sufficiently strong to completely axiomatize the case with a Boolean algebra of modalities. Finally, it would be interesting to find a simple characterization of the class of all $MLOM$s that can be given a complete axiomatization with respect to I-validity.

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