Logics of Imperfect Information without Identity

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Abstract

We investigate the expressive power of sentences of the family of independence-friendly (IF) logics in the equality-free setting. Various natural equality-free fragments of logics in this family translate into the version of existential second-order logic with prenex quantification of function symbols only and with the first-order parts of formulae equality-free. We study this version of existential second-order logic. Our principal result is that over finite models with a vocabulary consisting of unary relation symbols only, this fragment of second-order logic is weaker in expressive power than first-order logic. Such results could turn out useful in the study of independence-friendly modal logics. In addition to proving results of a technical nature, we consider issues related to a perspective where IF logic is regarded as a specification framework for games, and also discuss the significance of understanding fragments of second-order logic in investigations related to non-classical logics.

1 Introduction

We investigate the family of *independence-friendly* (IF) *logics* introduced by Hintikka and Sandu in [8]. See also [7] for an early exposition of the main ingredients leading to the idea of IF logic, and of course [5] for an

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even earlier discussion of ideas closely related to IF logic. Variants of IF logic have received a lot of attention recently; see [1, 4, 9, 10, 11, 13, 16] for example. Therefore we believe that the time is beginning to be mature for investigations not directly related to technical aspects concerning semantical issues. The focus of our work is the expressive power of the equality-free fragment of IF logic without slashed *connectives*. To be exact, we study the fragment of the system IF* (see [1]) without equality and without slashed connectives. We denote this fragment by IF$_{wo=}$.

Even though motivated by questions related to the expressive power of IF$_{wo=}$, our study concerns a wider range of logics. In fact, our study focuses on the system fESO$_{wo=}$ which is the version of existential second-order logic where the second-order quantifiers quantify function symbols only and where equality is not used. Here we allow for the function symbols to be nullary, i.e., to be interpreted as constants. With a careful inside-out Skolemization procedure preceeded by some preprocessing, any sentence of IF$_{wo=}$ can be turned into a sentence of fESO$_{wo=}$ that defines exactly the same class of models as the original IF$_{wo=}$ sentence. However, results about fESO$_{wo=}$ automatically apply to a wider range of logics. For example, the delightfully exotic looking expressions of the form

$$\begin{pmatrix} \forall x_1 & \exists x_2 \\ \forall x_3 & \exists x_4 \end{pmatrix} \varphi(x_1, x_2, x_3, x_4),$$

where a finite partially ordered quantifier preceeds an equality-free FO formula $\varphi(x_1, x_2, x_3, x_4)$ (with the free variables x_1, x_2, x_3, x_4), are equivalent to sentences of fESO$_{wo=}$ by the definition of Henkin [5]. Hence, whatever is inexpressible in fESO$_{wo=}$, is automatically inexpressible with expressions of the above type. Thus by studying fESO$_{wo=}$ we can kill multiple birds with one stone. This is part of a more general phenomenon. Results about fragments of second-order logic are very useful in the study of non-classical logics with devices giving them the capacity to express genuinely second-order properties. A typical such non-classical logic often immediately translates into a fragment of second-order logic. Then, armed with theorems about fragments of second-order logic, one may immediately obtain a range of metatheoretic results concerning the non-classical logic in question. Such results could be, for example, related to decidability issues. By directing attention to *fragments* of second-order logic rather than the full system of second-order logic, one can often easily identify, for example, truth preserving model transformations etc. The very high expressive power of second-order logic seems to often make it very difficult to obtain results like truth preserving model transformation theorems applying to *all* sentences of the system. These considerations provide part of the motivation for our study of the system fESO$_{wo=}$.

In addition to contributing to the general program of studying fragments of second-order logic, we believe that insights about sentences of the equality-free systems $\mathrm{IF}_{wo=}$ and $\mathrm{fESO}_{wo=}$ can be more or less directly useful in the study of the independence-friendly modal logics of Tulenheimo [15] and Tulenheimo and Sevenster [14] and others. This is due to the fact that formulae of such systems tend to translate to formulae of $\mathrm{IF}_{wo=}$. This realization provides an example that demonstrates the significance of the claim made about the study of fragments of second-order logic above.

In this paper we study the expressivity of *sentences* of $\mathrm{IF}_{wo=}$ only. A sentence of $\mathrm{IF}_{wo=}$ *defines* the class of models on which *Eloise* has a winning strategy in the related semantic game. We begin the paper by observing that $\mathrm{fESO}_{wo=}$ can define properties not definable in first-order logic FO (with equality), when the vocabulary under consideration contains at least one binary relation symbol. We then define a simple model-transformation that preserves the truth of $\mathrm{fESO}_{wo=}$ sentences, but not FO sentences. Therefore we observe that $\mathrm{fESO}_{wo=}$ and FO are incomparable with regard to expressive power. The same transformation of course also preserves the truth of $\mathrm{IF}_{wo=}$ sentences. We discuss the significance of this observation in relation to the use of IF logic as a specification language for games.

Finally, we ask whether $\mathrm{fESO}_{wo=}$ and FO are also incomparable with regard to expressive power when attention is limited to vocabularies containing only unary relation symbols. Our principal result is that over finite models with such a vocabulary,

$$\mathrm{FO}_{wo=} < \mathrm{fESO}_{wo=} < \mathrm{FO},$$

where $\mathrm{FO}_{wo=}$ denotes first-order logic without equality. So far we have not succeeded in establishing these results without the use of somewhat involved combinatorial arguments.

2 Preliminary Considerations

We assume the reader is familiar with first-order logic and independence-friendly logic. For a tour of properties of IF logic, see [1]. The version of IF logic studied in this paper is the version where *slashed quantifiers* $\exists x/\{y_0, ...y_i\}$, $\forall x/\{y_0, ...y_i\}$ are allowed, but disjunctions and conjunctions do not have slash sets associated with them. To be exact, we study the fragment of the system IF^* (see [1]) without equality and without slashed connectives. We call this logic $\mathrm{IF}_{wo=}$. For the semantics of $\mathrm{IF}_{wo=}$, see Definition 4.2 in [1].

Our main tool in investigating $\mathrm{IF}_{wo=}$ is the logic $\mathrm{fESO}_{wo=}$, whose formulae are exactly the formulae of the type $\exists \overline{f}\varphi$, where $\overline{f}$ is a finite vector of function symbols and φ is an FO formula without equality. The function symbols are allowed to be nullary, i.e., to be interpreted as constants. The formulae of $\mathrm{fESO}_{wo=}$ are interpreted according to the natural semantics.

A sentence φ of IF* is called equivalent to a sentence ψ of fESO$_{wo=}$ (or, equivalently, a sentence of FO) if and only if *Eloise* has a winning strategy in the semantic game defined by φ exactly on those models where ψ is true. Any equality-free sentence of IF* without slashed *connectives*, i.e., a sentence of IF$_{wo=}$, can be transformed to an equivalent sentence of fESO$_{wo=}$. We base this claim on Theorem 10.2 of [1] which implies that any sentence of IF$_{wo=}$ can be put to an equivalent prenex normal form with exactly the original propositional skeleton, and the transformation can be done so that connectives and quantifiers without slash sets associated with them remain unslashed. As the propositional skeleton of the new prenex sentence is the same as that of the original sentence, the transformation process does not introduce equality symbols. Furthermore, we obtain a sentence that is *regular*, implying that no quantifier for a variable occurs within the scope of another quantifier for the same varible. See [1] for details. A sentence in this normal form can then be Skolemized in a careful inside-out fashion. The procedure eliminates existential quantifiers and introduces fresh function symbols. The related functions encode the way *Eloise* can play the semantic game. The procedure does not introduce equality or slashed connectives. The slash sets associated with universal quantifiers get eliminated. Finally, the fresh function symbols are prenex quantified existentially, resulting in a sentence of fESO$_{wo=}$ equivalent to the original IF$_{wo=}$ sentence.

The reader uneasy about this translation should note that the results in the current paper are mainly about for fESO$_{wo=}$, and the statements about IF$_{wo=}$ are mostly nothing more than direct corollaries to results concerning fESO$_{wo=}$.

3 Expressivity of IF$_{wo=}$ and fESO$_{wo=}$ over Models with a Relational Vocabulary

We begin the section by making the simple observation that IF$_{wo=}$ and FO are incomparable with regard to the expressive power of sentences over vocabularies containing at least one binary relation symbol.[1] Here we do not limit our attention to finite models only.

Proposition 3.1. *Let V be a vocabulary containing at least one binary relation symbol R. Then there is a class of V-models definable by a sentence of* IF$_{wo=}$ *and also a sentence of* fESO$_{wo=}$ *that is not definable by a sentence of* FO.

[1] A trick similar to the one used in the proof below can be easily used to show that fESO$_{wo=}$ is not closed under negation. The argument is based on the fact that existential second-order logic ESO can easily define *infinity*, but cannot define *finiteness* due to compactness.

Proof. It is well known that there is an IF sentence φ (*with* equality and without slashed connectives) that is true in exactly those models whose domain has an even or an infinite cardinality. Let φ' be the sentence obtained from φ by replacing each atom of the type $x = y$ by the atom $R(x, y)$. Let C be the class of finite V-models $\mathfrak{A}$ such that

$$R^{\mathfrak{A}} = \{\ (a, a) \mid a \in Dom(\mathfrak{A})\ \}.$$

It is clear that with respect to C, the sentence φ' defines the class C_{even} of models whose domain is even. A straightforward Ehrenfeucht-Fraïssé argument shows that the class C_{even} is not definable with respect to C by any FO sentence. Since there is no FO sentence that is equivalent over C to φ', there is no FO sentence equivalent to φ'.

Since φ' can be transformed to an equivalent fESO$_{wo=}$ sentence, it follows that fESO$_{wo=} \not\leq$ FO with regard to expressive power of sentences over the class of V-models. □

3.1 Bloating Models

We now define a model-transformation under which the truth of fESO$_{wo=}$ sentences is preserved.

Definition 3.2. Let V be a relational vocabulary containing only unary and binary relation symbols. (We restrict our attention to at most binary relation symbols for the sake of simplicity.) Let $\mathfrak{A}$ be a V-model with the domain A, and let $a \in A$. Let S be some set such that $S \cap A = \emptyset$. Define the V-model $\mathfrak{B}$ as follows.

1. The domain of $\mathfrak{B}$ is the set $A \cup S$.
2. Let $P \in V$ be a unary relation symbol. We define $P^{\mathfrak{B}}$ as follows.
   (a) For all $v \in A$, $v \in P^{\mathfrak{B}}$ iff $v \in P^{\mathfrak{A}}$.
   (b) For all $s \in S$, $s \in P^{\mathfrak{B}}$ iff $a \in P^{\mathfrak{A}}$.
3. Let $R \in V$ be a binary relation symbol. We define $R^{\mathfrak{B}}$ as follows.
   (a) For all $\overline{v} \in A \times A$, $\overline{v} \in R^{\mathfrak{B}}$ iff $\overline{v} \in R^{\mathfrak{A}}$.
   (b) For all $s \in S$ and all $v \in A$, $(v, s) \in R^{\mathfrak{B}}$ iff $(v, a) \in R^{\mathfrak{A}}$.
   (c) For all $s \in S$ and all $v \in A$, $(s, v) \in R^{\mathfrak{B}}$ iff $(a, v) \in R^{\mathfrak{A}}$.
   (d) For all $s, s' \in S$, $(s, s') \in R^{\mathfrak{B}}$ iff $(a, a) \in R^{\mathfrak{A}}$.

We call the model $\mathfrak{B}$ a *bloating* of $\mathfrak{A}$. Figure 1 illustrates how this model transformation affects models.

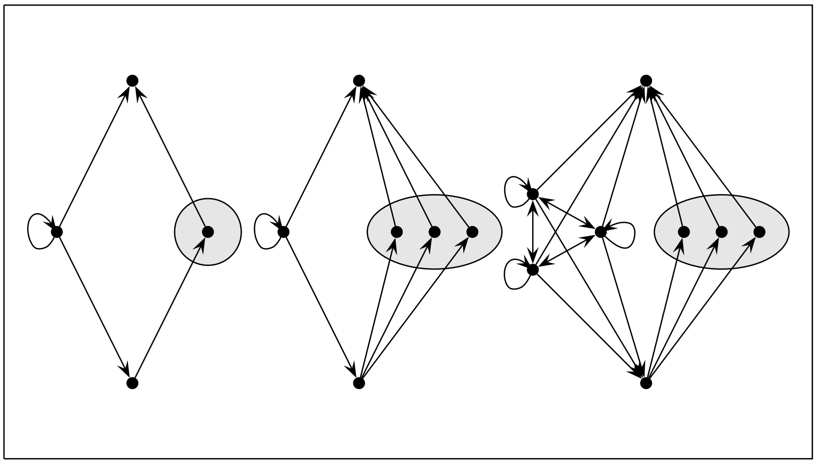

Figure 1: The figure shows three structures of a vocabulary consisting of one binary and one unary relation symbol. The shaded areas correspond to the extensions of the unary relation symbol. The structure in the middle is a bloating of the structure on the left. The structure in the middle is obtained from the one on the left by adding two new copies of the middle right element. The structure on the right is a bloating of the structure in the middle obtained by adding two copies of the middle left element.

Theorem 3.3. *Let V be a vocabulary containing unary and binary relation symbols only. The truth of* $\mathrm{fESO}_{wo=}$ *sentences is preserved from V-models to their bloatings.*

Proof. Let $\mathfrak{A}$ be a V-model and φ a sentence of $\mathrm{fESO}_{wo=}$. We assume that φ is of the form $\exists\overline{f}\psi$, where the symbols f are function symbols (some of them perhaps nullary) and ψ is a first-order sentence without existential quantifiers and with negations pushed to the atomic level. This normal form is obtained by first transferring the first-order part of φ into negation normal form and then Skolemizing the resulting sentence.[2] The freshly introduced Skolem functions are prenex quantified existentially, so the vocabulary of $\exists\overline{f}\psi$ is the same as that of φ. The process of transferring φ into the described normal form does not introduce equality.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be as in Definition 3.2. The models there had the domains A and $A \cup S$, respectively, and the element $a \in A$ was used in order to define $\mathfrak{B}$. We assume that $\mathfrak{A} \models \exists\overline{f}\psi$ and expand $\mathfrak{A}$ to a model $\mathfrak{A}' = (\mathfrak{A}, \overline{f^{\mathfrak{A}'}})$ such that $\mathfrak{A}' \models \psi$. We then expand $\mathfrak{B}$ to a model $\mathfrak{B}' = (\mathfrak{B}, \overline{f^{\mathfrak{B}'}})$ as follows.

1. For each k-ary symbol f, we let $f^{\mathfrak{B}'} \upharpoonright A^k = f^{\mathfrak{A}'} \upharpoonright A^k$. Note that when $k = 0$, i.e., when f is a constant symbol, then $f^{\mathfrak{B}'} = f^{\mathfrak{A}'}$.

[2] Again, before Skolemizing, we preprocess the sentence by putting it into prenex normal form without nested quantification of the same variable.

2. For each k-tuple $\overline{w} \in (A \cup S)^k$ containing points from the set S, we define the k-tuple $\overline{w}'$, where each co-ordinate value $s \in S$ of $\overline{w}$ is replaced by the element a. We then set $f^{\mathfrak{B}'}(\overline{w}) = f^{\mathfrak{A}'}(\overline{w}')$.

We then establish that $\mathfrak{B}' \models \psi$. The proof is a simple induction on the structure of ψ. For each variable assignment h with codomain A, let $g(h)$ denote the set of all variable assignments with codomain $A \cup S$ that can be obtained from h by allowing some subset of the variables mapping to the element a to map to elements in S. We prove that for every variable assignment h with codomain A and every subformula χ of ψ,

$$\mathfrak{A}', h \models \chi \ \Rightarrow\ \forall h' \in g(h)(\mathfrak{B}', h' \models \chi).$$

The cases for atomic and negated atomic formulae form the basis of the induction. The claim for these formulae follows immediately with the help of the observation that $h(t) = h'(t)$ for all h and $h' \in g(h)$ and terms t that *contain function symbols*, i.e., terms that are not variable symbols. We will next establish this claim by induction on the nesting depth of function symbols.

The basis of the induction deals with the terms of nesting depth one, i.e., terms of the type $f(x_1, ..., x_k)$ and c, where the symbols $x_1, ..., x_k$ are variable symbols and the symbol c is a constant symbol. It is immediate that $h(t) = h'(t)$ for all h and $h' \in g(h)$ and all such terms t of nesting depth one.

Now let $f(t_1, ..., t_k)$ be a term of nesting depth $n + 1$. By the induction hypothesis, for each one of the terms t_i that is not a variable symbol, we have $h(t_i) = h'(t_i)$. For the terms t_i that are variable symbols and for which $h(t_i) \neq a$, we have $h(t_i) = h'(t_i)$. For the terms t_i that are variable symbols and for which $h(t_i) = a$, we have either $h'(t_i) = a$ or $h'(t_i) \in S$. We therefore notice that we obtain the tuple $(h(t_1), ..., h(t_k))$ from the tuple $(h'(t_1), ..., h'(t_k))$ by replacing the elements $u \in S$ of the tuple $((h'(t_1), ..., h'(t_k))$ by the element a. Therefore we conclude, by the definition of the function $f^{\mathfrak{B}'}$, that

$$f^{\mathfrak{B}'}(h'(t_1), ..., h'(t_k)) = f^{\mathfrak{A}'}(h(t_1), ..., h(t_k)).$$

This concludes the induction on terms and therefore the basis of the original induction on the structure of ψ has now been established. We return to the original induction.

The connective cases are trivial and the quantifier case relatively straightforward. We discuss the details of the quantifier case here.

Assume $\mathfrak{A}', h \models \forall x \alpha(x)$. We need to show that for all $h' \in g(h)$, $\mathfrak{B}', h' \models \forall x \alpha(x)$. Assume, for contradiction, that for some $h'' \in g(h)$ we have $\mathfrak{B}', h'' \not\models \forall x \alpha(x)$. Therefore, for some $u \in A \cup S$, we have $\mathfrak{B}', h'' \frac{u}{x} \not\models \alpha(x)$. It suffices to show that $h'' \frac{u}{x} \in g(h \frac{v}{x})$ for some $v \in A$. This suffices, as the assumption $\mathfrak{A}', h \models \forall x\alpha(x)$ first implies that $\mathfrak{A}', h\frac{v}{x} \models \alpha(x)$, which in turn then implies, by the induction hypothesis, that $\mathfrak{B}', h''\frac{u}{x} \models \alpha(x)$.

If $u \in A$, let $v = u$. Then, as $h'' \in g(h)$, we have $h''\frac{u}{x} = h''\frac{v}{x} \in g(h\frac{v}{x})$. If $u \in S$, we let $v = a$. Then, as $h'' \in g(h)$, we have $h''\frac{u}{x} \in g(h\frac{a}{x}) = g(h\frac{v}{x})$. □

An immediate consequence of Theorem 3.3 is that FO $\not\leq$ fESO$_{wo=}$ because there exist first-order sentences whose truth is not preserved under bloating.

Theorem 3.3 is interesting when regarding IF logic as a kind of a specification language for games. Let V be a vocabulary of the type defined in Theorem 3.3. Let the equality-free and slash connective-free IF sentence φ of the vocabulary V specify some class of games and assume we know some board (i.e., a V-model) on which *Eloise* wins the game (i.e., φ is true on that model). The theorem then gives us a whole range of new, larger boards where she wins the game specified by φ. On the other hand, *non-winning* and in fact even *indeterminacy* are clearly preserved in reverse bloatings. This follows directly by a dualization argument.

4 Expressivity of fESO$_{wo=}$ and IF$_{wo=}$ over Finite Models with a Unary Relational Vocabulary

We now turn our attention to finite models whose vocabulary contains only unary relation symbols. Over such finite models, the picture is quite different from the case where there is a binary relation symbol in the vocabulary. We will show that over the class of finite models whose vocabulary contains only unary relation symbols,

$$\mathrm{FO}_{wo=} < \mathrm{fESO}_{wo=} < \mathrm{FO}.$$

We first discuss the latter inequality and then the former one.

4.1 fESO$_{wo=}$ < FO over the Class of Finite Models with a Unary Vocabulary

In this subsection, we establish that fESO$_{wo=}$ < FO over the class of finite models with a unary relational vocabulary. Therefore also IF$_{wo=}$ < FO over that class. We begin by making a number of auxiliary definitions.

Let U be a finite vocabulary containing unary relation symbols only. A *unary U-type* (with the free variable x) is a conjunction τ with $|U|$ conjuncts such that for each $P \in U$, exactly one of the formulae $P(x)$ and $\neg P(x)$ is a conjunct of τ. Let $T = \{\tau_1, ..., \tau_{|T|}\}$ be the set of unary U-types.[3] The

[3]We assume some standard ordering of conjuncts and bracketing, so that there are exactly $2^{|U|}$ different unary U-types, and different unary U-types are non-equivalent. Here and everywhere below, we consider explicitly only the cases where $U \neq \emptyset$.

domain of each (finite) U-model $\mathfrak{A}$ is partitioned into some number $n \leq |T|$ of sets S_i such that the elements of S_i *realize*, i.e., satisfy, the type $\tau_i \in T$. (Here an element $a \in Dom(\mathfrak{A})$ realizes (satisfies) the type τ_i if and only if $\mathfrak{A} \models \tau_i(a)$ in the usual sense of first-order logic.)

Let $n \in \mathbb{N}_{\geq 1}$, and let $k = 2^n$. Any relation

$$R \subseteq \mathbb{N}^k \setminus \{0\}^k$$

is called a *spectrum*. We associate sentences of FO and fESO$_{wo=}$ with spectra in a way specified in the following definition.

Definition 4.1. Let V be a vocabulary containing unary relation symbols only. Let φ be a sentence of FO or fESO$_{wo=}$ of the vocabulary V. Let $U \subseteq V$ be the finite set of relation symbols occurring in φ. Let $T = \{\tau_1, ..., \tau_{|T|}\}$ be the finite set of unary U-types, and let $\leq^T$ denote a linear ordering of the types in T defined such that $\tau_i \leq^T \tau_j$ iff $i \leq j$. Define the relation $R_\varphi \subseteq \mathbb{N}^{|T|}$ such that $(n_1, ..., n_{|T|}) \in R_\varphi$ iff there exists a finite U-model $\mathfrak{A}$ of φ such that for all $i \in \{1, ..., |T|\}$, the number of points in the domain of $\mathfrak{A}$ that satisfy τ_i is n_i. We call such a relation R_φ the *spectrum of* φ (with respect to the ordering $\leq^T$).

Notice that the class of finite V-models defined by φ is completely characterized by the spectrum $R_\varphi \subseteq \mathbb{N}^{|T|}$. We next define a special family of spectra and then establish that this family exactly characterizes the expressive power of FO over the class of (finite) models with a vocabulary containing unary relation symbols only. See Figure 2 for an illustration of a spectrum of a sentence of FO with a unary relational vocabulary.

Definition 4.2. Let $l = 2^{l'}$ for some $l' \in \mathbb{N}_{\geq 1}$. Let $R \subseteq \mathbb{N}^l$ be a spectrum for which there exists a number $n \in \mathbb{N}_{\geq 1}$ such that for all co-ordinate positions $i \in \{1, ..., l\}$, all integers $k, k' > n$ and all $m_1, ..., m_{i-1}, m_{i+1}, ..., m_l \in \mathbb{N}$, we have

$$(m_1, ..., m_{i-1}, k, m_{i+1}, ..., m_l) \in R$$
$$\Leftrightarrow$$
$$(m_1, ..., m_{i-1}, k', m_{i+1}, ..., m_l) \in R.$$

We call such a number n a *stabilizer* of the spectrum R. A spectrum with a stabilizer is called a *stabilizing spectrum.*

Proposition 4.3. *A spectrum R is a stabilizing spectrum if and only if R is a spectrum of some* FO *sentence.*

Proof. Given a stabilizing spectrum, it is easy to write a corresponding FO sentence by applying the quantifiers $\exists^{=j}$ and $\exists^{\geq j}$ expressible with the use of the equality symbol. (Here $\exists^{=j}x\ \varphi(x)$ states that there exists exactly j elements a such that $\varphi(a)$ holds, and $\exists^{\geq j}$ is defined analogously.)

The fact that each spectrum of an FO sentence is stabilizing follows by a straightforward Ehrenfeucht-Fraïssé argument. □

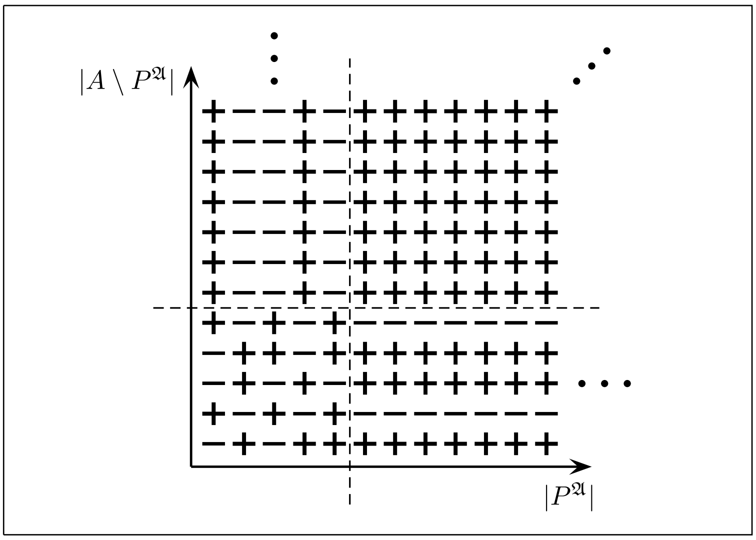

Figure 2: The figure illustrates a stabilizing spectrum that corresponds to some FO sentence φ of the vocabulary $\{P\}$, where P is a unary relation symbol. A plus symbol occurs at the position (i, j) iff there exists a $\{P\}$-model $\mathfrak{A}$ satisfying φ such that $|P^{\mathfrak{A}}| = i$ and $|A \setminus P^{\mathfrak{A}}| = j$, where $A = Dom(\mathfrak{A})$. In other words, the number of points in the domain of $\mathfrak{A}$ satisfying the type $P(x)$ is i and the number of points satisfying the type $\neg P(x)$ is j. The spectra for FO sentences divide the xy-plane into four distinct regions. The upper right region always contains either only plus symbols or only minus symbols. In the bottom left region, any distribution is possible. (The point $(0, 0)$ always contains a minus symbol though since we do not allow for models to have an empty domain.)

Next we define some order theoretic concepts and then prove a number of related results that are needed for the proof of the main theorem (Theorem 4.7) of the current section.

A structure $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$ is a *partial order* if $\leq^{\mathfrak{A}} \subseteq A \times A$ is a reflexive, transitive and antisymmetric binary relation. Given a partial order $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$, we let $<^{\mathfrak{A}}$ denote the irreflexive version of the order $\leq^{\mathfrak{A}}$. A partial order is *well-founded* if no strictly decreasing infinite sequence occurs in it. That is, a partial order $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$ is well-founded if for each each sequence $s : \mathbb{N} \longrightarrow A$ there exist numbers $i, j \in \mathbb{N}$ such that $i < j$ and $s(j) \not<^{\mathfrak{A}} s(i)$. An *antichain* $S \subseteq A$ of a partial order $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$ is a set such that for all distinct elements $s, s' \in S$, we have $s \not\leq^{\mathfrak{A}} s'$ and $s' \not\leq^{\mathfrak{A}} s$. In other words, the distinct elements s and s' are incomparable. A well-founded partial order that does not contain an infinite antichain is called a *partial well order*, or a *pwo*.

Let $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$ and $\mathfrak{B} = (B, \leq^{\mathfrak{B}})$ be partial orders. The *Cartesian* *product* $\mathfrak{A} \times \mathfrak{B}$ of the structures is the partial order defined in the following way.

1. The domain of $\mathfrak{A} \times \mathfrak{B}$ is the Cartesian product $A \times B$.
2. The binary relation $\leq^{\mathfrak{A}\times\mathfrak{B}} \subseteq (A\times B)\times(A\times B)$ is defined in a pointwise fashion as follows.
$$(a, b) \leq^{\mathfrak{A}\times\mathfrak{B}} (a', b') \Leftrightarrow \Big(a \leq^{\mathfrak{A}} a' \text{ and } b \leq^{\mathfrak{B}} b'\Big)$$

For each $k \in \mathbb{N}_{\geq 1}$ and each partial order $\mathfrak{A} = (A, \leq^{\mathfrak{A}})$, we let $\mathfrak{A}^k = (A^k, \leq^{\mathfrak{A}^k})$ denote the partial order where the relation $\leq^{\mathfrak{A}^k} \subseteq A^k \times A^k$ is again defined in the pointwise fashion as follows.

$$(a_1, ..., a_k) \leq^{\mathfrak{A}^k} (a'_1, ..., a'_k) \quad \Leftrightarrow \quad \forall i \in \{1, ..., k\} : a_i \leq^{\mathfrak{A}} a'_i$$

We call the structure $\mathfrak{A}^k$ the k^{th} *Cartesian power* of $\mathfrak{A}$. We let $(\mathbb{N}^k, \leq)$ denote the k^{th} Cartesian power of the linear order $(\mathbb{N}, \leq)$. When $S \subseteq \mathbb{N}^k$, we let $(S, \leq)$ denote the partial order with the domain S and with the ordering relation inherited from the structure $(\mathbb{N}^k, \leq)$. In other words, for all $\overline{s}, \overline{s}' \in S$, we have $\overline{s} \leq^{(S,\leq)} \overline{s}'$ if and only if $\overline{s} \leq^{(\mathbb{N}^k,\leq)} \overline{s}'$. We simply write $\overline{u} \leq \overline{v}$ in order to assert that $\overline{u} \leq^{(\mathbb{N}^k,\leq)} \overline{v}$, when $\overline{u}, \overline{v} \in \mathbb{N}^k$.

The following lemma is a paraphrase of Lemma 5 of [12], where the lemma is credited to Higman [6].

Lemma 4.4. *The Cartesian product of any two partial well orders is a partial well order.*

Variations of the following lemma are often attributed to Dickson [3]. The lemma follows immediately from Lemma 4.4.

Lemma 4.5. *Let* $k \in \mathbb{N}_{\geq 1}$. *The structure* $(\mathbb{N}^k, \leq)$ *does not contain an infinite antichain.*

Proof. The structure $(\mathbb{N}, \leq)$ is a pwo, and the property of being a pwo is preserved under taking finite Cartesian products by Lemma 4.4. Therefore the structure $(\mathbb{N}^k, \leq)$ is a pwo. By definition, a pwo does not contain an infinite antichain. □

Let $l \in \mathbb{N}_{\geq 1}$ and let $R \subseteq \mathbb{N}^l$ be a relation such that for all $\overline{u}, \overline{v} \in \mathbb{N}^l$, if $\overline{u} \in R$ and $\overline{u} \leq \overline{v}$, then $\overline{v} \in R$. We call the relation R *upwards closed* with respect to $(\mathbb{N}^l, \leq)$. When the exponent l is irrelevant or known from the context, we simply say that the relation R is upwards closed.

Theorem 4.6. *Let* $l' \in \mathbb{N}_{\geq 1}$ *and* $l = 2^{l'}$. *Let* $R \subseteq \mathbb{N}^l$ *be a relation*[4] *that is upwards closed with respect to* $(\mathbb{N}^l, \leq)$. *Then* R *is a stabilizing spectrum.*

[4] There is a typo inherited from the original workshop version of the article here. Instead of "relation" it should read "spectrum".

Proof. We begin[5] the proof by defining a function f that maps each non-empty subset of the set $\{1, ..., l\}$ to a natural number. Let $C \subseteq \{1, ..., l\}$ be a non-empty set. Let $R(C)$ denote the set consisting of exactly those tuples $\overline{w} \in R$ that have a non-zero co-ordinate value at each co-ordinate position $i \in C$ and a zero co-ordinate value at each co-ordinate position $j \in \{1, ..., l\} \setminus C$. Define the value $f(C) \in \mathbb{N}$ as follows.

1. If $R(C) = \emptyset$, let $f(C) = 0$.
2. If $R(C) \neq \emptyset$, choose some $\overline{w} \in R(C)$. Let $W \subseteq R(C)$ be a maximal antichain of $(R(C) \leq)$ with $\overline{w} \in W$, i.e., let W be an antichain of $(R(C), \leq)$ such that for all $\overline{u} \in R(C) \setminus W$, there exists some $\overline{v} \in W$ such that $\overline{u} < \overline{v}$ or[6] $\overline{v} > \overline{u}$. By Lemma 4.5, we see that the set W is finite. Thus there exists a maximum co-ordinate value occurring in the tuples in W. Let $f(C)$ to be equal to this value.

(Notice that we have some freedom of *choice* when defining the function f, so there need not be a unique way of defining the function.)

With the function f defined, call

$$n = max(\{\ f(C) \mid C \subseteq \{1, ..., l\},\ C \neq \emptyset\ \}).$$

We establish that n is a stabilizer for the relation R. We assume, for the sake of contradiction, that there there exist integers $k, k' > n$ and $m_1, ..., m_{i-1}, m_{i+1}, ..., m_l \in \mathbb{N}$ such that the equivalence

$$\begin{array}{ll} & (m_1, ..., m_{i-1}, k, m_{i+1}, ..., m_l) \in R \\ \Leftrightarrow & \\ & (m_1, ..., m_{i-1}, k', m_{i+1}, ..., m_l) \in R. \end{array}$$

does not hold. Let $k < k'$. As by assumption the relation R is upwards closed, it must be the case that

$$\begin{array}{ll} & (m_1, ..., m_{i-1}, k, m_{i+1}, ..., m_l) \notin R \\ \text{and} & \\ & (m_1, ..., m_{i-1}, k', m_{i+1}, ..., m_l) \in R. \end{array}$$

Otherwise we would immediately reach a contradiction. Call

$$\begin{array}{ll} & \overline{w}_k = (m_1, ..., m_{i-1}, k, m_{i+1}, ..., m_l) \\ \text{and} & \\ & \overline{w}_{k'} = (m_1, ..., m_{i-1}, k', m_{i+1}, ..., m_l). \end{array}$$

Let $C^* \subseteq \{1, ..., l\}$ be the set of co-ordinate positions where the tuple $\overline{w}_{k'}$ (and therefore also the tuple $\overline{w}_k$) has a non-zero co-ordinate value. Let

[5] As $\emptyset$ is a stabilizing spectrum, we assume without loss of generality that $R \neq \emptyset$.

[6] There is a typo inherited from the original workshop version of the article here. Instead of "$\overline{u} < \overline{v}$ or $\overline{v} > \overline{u}$" it should read "$\overline{u} < \overline{v}$ or $\overline{v} < \overline{u}$".

$W(C^*)$ denote the domain of the maximal antichain of $(R(C^*), \leq)$ chosen when defining the value of the function f on the input C^*. The tuple $\overline{w}_{k'}$ cannot belong to the set $W(C^*)$, since the co-ordinate value k' is greater than n, and therefore greater than any of the co-ordinate values of the tuples in $W(C^*)$. Hence, as $W(C^*)$ is a maximal antichain of $(R(C^*), \leq)$ and $\overline{w}_{k'} \in R(C^*)$, we conclude that there exists a tuple $\overline{u} \in W(C^*)$ such that $\overline{w}_{k'} < \overline{u}$ or $\overline{u} < \overline{w}_{k'}$. Since $k' > f(C^*)$, we must have $\overline{u} < \overline{w}_{k'}$. Therefore, as also $k > f(C^*)$, we conclude that $\overline{u} < \overline{w}_k$. Since R is upwards closed and $\overline{u} \in R$, we have $\overline{w}_k \in R$. This is a contradiction, as desired. □

The following theorem is the main result of the current section.

Theorem 4.7. *Over the class of finite models of a vocabulary V containing only unary relation symbols,* $\mathrm{fESO}_{wo=} < \mathrm{FO}$.

Proof. It is immediate by Theorem 3.3 that $\mathrm{fESO}_{wo=} \neq \mathrm{FO}$ (over finite V-models). It thus suffices to show that $\mathrm{fESO}_{wo=} \leq \mathrm{FO}$ over finite V-models.

To show that $\mathrm{fESO}_{wo=} \leq \mathrm{FO}$, by Proposition 4.3 it suffices to establish that the spectrum R_φ of an arbitrarily chosen $\mathrm{fESO}_{\mathrm{wo=}}$ sentence φ is stabilizing. By Theorem 3.3, the spectrum R_φ is upwards closed. Therefore, by Theorem 4.6, R_φ is a stabilizing spectrum. □

Corollary 4.8. *Over finite models of a vocabulary containing only unary relation symbols,* $\mathrm{IF}_{wo=} < \mathrm{FO}$.

Note that Theorem 4.7 applies not only to $\mathrm{fESO}_{wo=}$ but to any system such that the definable classes of models with a unary vocabulary are closed under bloating.[7] Note also that the method of proof seems nonconstructive in the sense that it seems to leave open the question whether there is an *effective translation* from the system considered into FO.

4.2 $\mathrm{FO}_{wo=} < \mathrm{IF}_{wo=}$ over the Class of Finite Models with a Unary Vocabulary

In this subsection we establish that over the class of finite $\{P\}$-models, where P is a unary relation symbol, we have $\mathrm{FO}_{wo=} < \mathrm{IF}_{wo=}$. The $\mathrm{IF}_{wo=}$ sentence

$$\forall x \exists y \exists z/\{x\}\big(P(y) \wedge (P(x) \leftrightarrow P(z))\big)$$

is true on a model $\mathfrak{M}$ with three points, two of which satisfy P. The sentence is not true on a model $\mathfrak{N}$ with two points, one satisfying P and one not. However, $\mathrm{FO}_{wo=}$ cannot separate the models $\mathfrak{M}$ and $\mathfrak{N}$. This is seen by a straightforward Ehrenfeucht-Fraïssé argument involving a version of the Ehrenfeucht-Fraïssé game that characterizes $\mathrm{FO}_{wo=}$. Instead of the usual

[7] Here we assume the standard convention that studies definability w.r.t. a class of finite V-models, V being finite.

partial isomorphism condition, this game involves the following end condition between the pebbles $a_1, ..., a_k \in A = Dom(\mathfrak{A})$ and $b_1, ..., b_k \in B = Dom(\mathfrak{B})$ picked during a play of the k-round game involving models $\mathfrak{A}$ and $\mathfrak{B}$ with a relational vocabulary. A play of the game defines a binary relation $Z = \{(a_1, b_1), ..., (a_k, b_k)\}$. The relation Z is called a *partial relativeness correspondence* between the models $\mathfrak{A}$ and $\mathfrak{B}$ if for all relation symbols R in the vocabulary of the models, the condition $Z(a'_1, b'_1), ..., Z(a'_n, b'_n)$ implies $R^{\mathfrak{A}}(a'_1, ..., a'_n) \Leftrightarrow R^{\mathfrak{B}}(b'_1, ..., b'_n)$. Here n is the arity of the symbol R. The duplicator wins the play of the game if the relation Z defined by the play is a partial relativeness correspondence. A discussion concerning the related Ehrenfeucht-Fraïssé characterization theorem can be found in [2].

Theorem 4.9. *Over finite models of a vocabulary V containing only unary relation symbols,* $\mathrm{FO}_{wo=} < \mathrm{IF}_{wo=}$.

Proof. It suffices to establish that the duplicator has a winning strategy in the Ehrenfeucht-Fraïssé game for $\mathrm{FO}_{wo=}$ for any number k of rounds played on the models $\mathfrak{M}$ and $\mathfrak{N}$ defined above. The duplicator employs a strategy where the reply to each one of the spoiler's moves is simply a pick of any element in the correct model that satisfies exactly the same unary $\{P\}$-type as the element chosen by the spoiler. □

Corollary 4.10. *Over finite models of a vocabulary V containing only unary relation symbols,* $\mathrm{FO}_{wo=} < \mathrm{fESO}_{wo=}$.

5 Concluding Remarks

We have investigated the expressive power of the equality-free version of IF logic without slashed connectives. The results obtained have been established through a study of the logic $\mathrm{fESO}_{wo=}$. Our principal result is that over finite models with a vocabulary containing only unary relation symbols, the logics $\mathrm{IF}_{wo=}$ and $\mathrm{fESO}_{wo=}$ are weaker than FO. We have also identified a model-transformation that preserves the truth of $\mathrm{IF}_{wo=}$ sentences.

In the future we expect to tie up some loose ends that were left undiscussed here. This includes considering infinite models. Furthermore, we wish to identify differences (rather than similarities) in the roles that different logical constructors – such as negation and identity – play in versions of IF logic and other logics of the same family such as dependence logic [16]. The full systems of dependence logic and IF* coincide in expressive power on the level of sentences, both being able to exactly capture existential second-order logic. However, the systems might perhaps differ in expressive power when a suitable subset of the available logical constructors is uniformly removed from both systems. Another possibility is to restrict the number of available variable symbols to some finite number. The possibilities are endless indeed. Investigations along such lines should lead to a deeper understanding of the strengths and weaknesses different systems have in relation to different applications.[8]

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[8] A different kind of an intriguing possibility is to consider systems that extend FO in a similar but not the same way as IF^* logic and dependence logic. One could, for example, consider *generalized atoms* that could be defined atop team semantics. Such atoms would make assertions about teams in the spirit of the dependence atoms of [16]; one could assert for example that the variable x obtains only finitely many values in a team – to list one possibility.

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