Tampere University

JANI JOKELA

Mixed Lattice Structures

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PREFACE

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Finally, thanks to my family for putting up with this nonsense for so long.

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Jani Jokela

ABSTRACT

In this thesis we develop the theory of mixed lattice structures, which are partially ordered algebraic systems (semigroups, groups or vector spaces) with two partial orderings. These two orderings are linked by mixed lattice operations which resemble the join and meet operations of a lattice but, unlike join and meet, the mixed lattice operations are neither commutative nor associative. A mixed lattice structure is a generalization of a lattice ordered structure in the sense that if the two partial orderings coincide then the resulting structure is a lattice.

We study several aspects of the mixed lattice theory. A generalization of the theory of vector lattices is developed by investigating how some of the fundamental concepts and the structure theory of vector lattices can be extended to mixed lattice structures. We introduce a fundamental classification of mixed lattice groups based on the properties of their orderings. We define the upper and lower parts of elements and derive various related identities and inequalities. Next we introduce the notions of ideals and bands in a mixed lattice space and prove several fundamental results related to these concepts. We then begin a study of the structure theory by investigating the notion of disjoint elements and the related decomposition of mixed lattice spaces into a direct sum of complemented bands.

In addition to the algebraic theory, we also study topologies on mixed lattice spaces. As our main result we present a characterization of compatible topologies on mixed lattice spaces and study the fundamental properties of such topologies. Applications to functional analysis and convex analysis are also presented. Finally, we introduce a generalization of a mixed lattice structure and show how such structure arises in convex analysis in the context of the cone projection problem.

CONTENTS

LIST OF ORIGINAL PUBLICATIONS

[P1] S.-L. Eriksson, J. Jokela and L. Paunonen, *Generalized absolute values, ideals and homomorphisms in mixed lattice groups*. Positivity **25** (3), (2021), 939–972. https://doi.org/10.1007/s11117-020-00794-2

[P2] J. Jokela, *Ideals, bands and direct sum decompositions in mixed lattice vector spaces*. Positivity **27** (2023). https://doi.org/10.1007/s11117-023-00985-7

[P3] J. Jokela, *Compatible topologies on mixed lattice vector spaces*. Topol. Appl. **320** (2022). https://doi.org/10.1016/j.topol.2022.108241

[P4] J. Jokela, *Mixed lattice structures and cone projections*. Optim. Lett. (2023). https://doi.org/10.1007/s11590-023-02075-9

Author contribution

The author is the sole author of the articles [P2]–[P4].
In [P1], the author of this thesis carried out the mathematical research and writing. S.–L. Eriksson suggested the topic and commented on the finished manuscript. L. Paunonen gave comments and suggestions during the preparation of the manuscript, and commented on the finished manuscript.

1 INTRODUCTION

Various kinds of partially ordered structures appear in many branches of pure and applied mathematics, and *partially ordered vector spaces* are of particular importance in the field of mathematical analysis. The most important special case of a partially ordered vector space is a *vector lattice*, also known as a *Riesz space*. In this thesis, we study vector spaces with *two* partial orderings. These spaces have certain lattice-type properties, and they generalize the notion of a Riesz space. In this chapter, we first give a brief historical background of the subject, after which we outline the main research questions and objectives of this thesis.

1.1 Historical background

The development of the theory of vector lattices is generally recognized as having initiated by F. Riesz at the Bologna International Mathematical Congress in 1928. Because of this, vector lattices are commonly called Riesz spaces. Riesz studied linear functionals on partially ordered spaces of continuous functions, and his investigations were a part of functional analysis, which was then new and rapidly developing branch of mathematics. The theory of Riesz spaces was further developed during the 1930s, most notably by Riesz, H. Freudenthal, and the Soviet school led by L. Kantorovich. Significant contributions to the theory were also made by a group of Japanese mathematicians led by H. Nakano during the 1940s, and the first monographs on the subject were written in the 1950s [24, 27, 12] (the last cited book is a later English translation of the original book written in Romanian). Other important research groups that focused on partially ordered spaces were those led by M. H. Stone (United States), H. Schaefer (Germany) and A. Zaanen (Netherlands). At the same time, other more general partially ordered algebraic structures (such as groups and semigroups) were studied by many different authors. Some of these developments can be found in the monographs [9] and [19].

Riesz spaces were originally studied as part of functional analysis but according to Schaefer [32], the rapid development of functional analysis caused the theory of Riesz spaces to gradually become more of a specialized theory that existed in its own right, rather than an integral part of functional analysis. Nevertheless, the theory and methods of partially ordered spaces were frequently applied in different areas of pure and applied mathematics, such as operator theory [2], theory of measure and integration [18], approximation theory [32], optimization [25] and economics [1, 3]. In addition, several monographs on different aspects of the theory of ordered spaces have been written [21, 26, 30, 33, 34, 35]. Today, the theory and applications of Riesz spaces and some of their generalizations remain an active area of research. For an example of these more recent developments, we refer to [23].

Potential theory, which studies the properties of harmonic functions and their generalizations, was one of the mathematical disciplines that began to make use of the Riesz space methods. Using two distinct partial orderings in a vector space turned out to be a fruitful idea in the context of potential theory [14]. Initially the two orderings were considered separately but these ideas eventually led to the development of the theory of *mixed lattice structures*, in which the two orderings are intimately related through asymmetric operations that resemble the join and meet operations of a lattice. This theory is a generalization of the Riesz space theory, and also the topic of this thesis.

The theory of *mixed lattice semigroups* was developed by Maynard Arsove and Heinz Leutwiler, mainly during the 1970s, in connection with an axiomatic treatment of potential theory. Their goal was to find a suitable mathematical framework in which the essential concepts and ideas of classical and axiomatic potential theory could be formulated in a purely algebraic way. They published their theory in a series of research articles [4, 5, 6, 7], and a monograph [8], in which they presented the algebraic theory of mixed lattice semigroups. Their work resulted in a unified theory which generalized some of the earlier axiomatizations of potential theory, such as the theory of cones of potentials, developed by Mokobodzki and Sibony (see [8] and references therein), and the theory of H-cones developed by the Romanian group of mathematicians, Boboc, Bucur and Cornea [10]. These theories can be regarded as special cases of the Arsove–Leutwiler theory.

The main difference between these earlier developments and the Arsove–Leutwiler theory is that the former are formulated in Riesz spaces, or vector lattices, while the latter theory is built around a different type of algebraic structure, called a mixed lattice semigroup, which is a positive additive semigroup with two distinct partial orderings. Although the idea of equipping a vector space with two partial orderings had appeared in the literature much earlier, the distinguishing feature of mixed lattice semigroups is that the two partial orderings are mixed together and their interaction gives rise to a rather intricate mathematical structure. The usual symmetrical upper and lower envelopes (that is, the supremum and the infimum) of Riesz space theory are replaced by unsymmetrical "mixed" envelopes which are formed with respect to the two partial orderings. If the two partial orderings are identical, then the mixed envelopes become the ordinary supremum and infimum, and the structure is thus reduced to a lattice ordered semigroup. In this sense, the mixed lattice semigroup can be viewed as a generalization of a lattice ordered semigroup.

As the basic theory of mixed lattice semigroups had been extensively developed, it would seem natural to consider the mixed lattice structure also in groups and in vector spaces. Interestingly, Arsove and Leutwiler did not extend their study in this direction. In fact, they only briefly mentioned the possibility of extending the theory of mixed lattice semigroups to a group setting in [8], while adding that "there is no assurance that the mixed lattice structure can be carried over to this group [of formal differences]". It wasn't until 1991, when Eriksson–Bique studied group extensions of mixed lattice semigroups in [16], and showed that most of the basic properties of a mixed lattice semigroup are indeed preserved in such extensions. Later, in 1999, Eriksson–Bique gave a more general definition of a mixed lattice group in [17], and discussed their fundamental structure and algebraic properties. However, after these initial studies the research on the subject has been virtually nonexistent, and consequently, the theory of mixed lattice groups and vector spaces has remained relatively unexplored.

1.2 Research objectives

The purpose of this thesis is to initiate a systematic study of mixed lattice spaces and other related structures. Below we outline the most important research objectives of this thesis.

1. Introducing the most natural generalizations of the various fundamental no-

tions of vector lattice theory in the mixed lattice setting

The main theme of this dissertation is to develop the basic theory of mixed lattice groups and vector spaces, and to examine to what extent the theory of lattice ordered groups and vector lattices can be generalized to the mixed lattice setting. In order to obtain such generalization, the first step is to introduce appropriate definitions that provide the most natural generalizations of certain basic concepts in the theory of vector lattices. These include the absolute value of an element and the decomposition of elements into positive and negative parts. Some other closely related important concepts in the study of vector lattices are ideals, homomorphisms and quotient groups. Finding the most natural definitions for all these concepts in mixed lattice structures is not really an obvious or a trivial task. The main difficulty in generalizing certain aspects of Riesz space theory to mixed lattice structures is the fact that the mixed lattice operations are not commutative, associative or distributive, unlike the operations of supremum and infimum in Riesz spaces and lattice ordered groups. Because of this, many aspects of the vector lattice theory cannot be generalized to the mixed lattice setting in a very straightforward way. These fundamental questions are studied in [P1], and they are further developed in [P2].

2. Development of the basic structure theory

Ideals are the main building blocks of vector lattices, and consequently ideal theory plays a central part in the study of the structure of vector lattices. One of the fundamental problems here is the decomposition of vector lattices into disjoint bands. Such decomposition can be given in terms of band projections. This problem is quite similar to the decomposition of Hilbert spaces into a direct sum of a closed subspace and its orthogonal complement, and the associated projection operators that give the corresponding decomposition for each element. The main difference is that in vector lattices, the notion of ”orthogonality” is defined in terms of the partial order, and the closed subspaces are replaced by order-closed ideals (which are called *bands*), but the basic properties of such decompositions are similar in both settings. In a mixed lattice space, the question of existence of this kind of decompositions and the associated projections becomes a bit more complicated issue, again due to the non-commutative nature of the mixed lattice operations. Before tackling these problems, some additional properties of ideals must be established, a suitable definition of a band must be given, and the disjoint complements must be properly defined. These questions are investigated in [P2].

3. Developing the basic topological theory

The preceding discussion is focused on algebraic theory, but there are other aspects of the theory of mixed lattice structures that can be studied. From the analysis perspective, it is important to investigate the topological properties of mixed lattice structures. The theory of topological Riesz spaces, and more generally, topological ordered spaces, is well-developed and there is a vast literature devoted to these topics [1, 13, 18, 21, 30]. As mixed lattice spaces are simply ordered vector spaces with two partial orderings, we may ask if it is possible to apply the ideas from these known theories (with suitable modifications) to develop a topological theory of mixed lattice spaces. For ordered topological vector spaces, there are well-known conditions under which the topological structure of a vector space is compatible with the order structure. In particular, the well-known theorem due to G. T. Roberts and I. Namioka gives a characterization of compatible topologies on vector lattices. Can we find similar characterization of compatible topologies on a mixed lattice space? The most natural first step in developing a topological theory of mixed lattice structures is to introduce these compatibility conditions and study their immediate implications. This is the main topic of [P3].

4. Exploring potential applications

Whenever a new type of mathematical object is defined and studied, it is natural to ask whether it has any potential applications to motivate its study. Although the mixed lattice theory has its origins in potential theory, different mixed lattice structures appear not only in the context of potential theory, but also in analysis, algebra, probability theory, and convex analysis and its applications. Various examples of these are given in articles [P1], [P2], [P3] and [P4]. In this thesis, we focus on a couple of selected applications from convex analysis and optimization theory. Moreover, in functional analysis, mixed lattice spaces provide a natural setting for the study of asymmetric norms and their vector-valued generalizations. These topics have been an active area of research during the last couple of decades, and they are also discussed in [P3] and [P4].

1.3 The structure of the thesis

This thesis consists of four research articles, each of which explores different aspects of the theory of mixed lattice structures, as described in the preceding section. Article [P1] can be viewed as a general introduction to the theory of mixed lattice structures, as several fundamental definitions and basic properties are given there. Article [P2] focuses on the structure theory of mixed lattice spaces, in particular, the ideal theory. Article [P3] is dealing with topological aspects of mixed lattice theory, while [P4] presents some applications of mixed lattice structures in convex analysis and optimization theory.

The rest of the thesis is structured as follows. In Chapter 2 we give a short review of the fundamentals of the theory of vector lattices and mixed lattice structures. Chapter 3 presents the main results of the thesis. In the final chapter we make some concluding remarks and suggestions for future research.

2 MATHEMATICAL BACKGROUND

In this chapter, we collect the necessary definitions and basic results from the theory of vector lattices (Section 2.1) and mixed lattice structures (Section 2.2).

2.1 Riesz spaces and lattice ordered groups

The purpose of this thesis is to generalize the theory of Riesz spaces and lattice ordered groups to the more general mixed lattice structures. To compare the corresponding theories and results, we will first give a short overview of the basic definitions and theorems of Riesz spaces and lattice ordered groups. For proofs of the theorems in this section we refer to [21] and [26].

2.1.1 Basic concepts

Definition 2.1.1. Let G be a commutative additive group and $\leq$ a partial order on G. Then G is called a *partially ordered group* if the following condition holds for all $x, y \in G$:

(i) If $x \leq y$, then $x + z \leq y + z$ for every $z \in G$.

A *lattice ordered group* is a partially ordered group G, which is a lattice, that is, the elements $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist for all $x, y \in G$.
A real vector space E together with a partial order $\leq$ is called an *ordered vector space*, if the following conditions hold for all $x, y \in E$:

(i) If $x \leq y$, then $x + z \leq y + z$ for every $z \in E$,

(ii) If $x \leq y$, then $ax \leq ay$ for every real number $a \geq 0$.

A *Riesz space* (or a *vector lattice*) is an ordered vector space E, which is a lattice, that is, the elements $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist for all $x, y \in E$.

We also recall that a subset K of a vector space is called a *cone* if (i) $tK \subseteq K$ for all $t \geq 0$, (ii) $K + K \subseteq K$ and (iii) $K \cap (-K) = \{0\}$. For any cone K in a vector space there is an associated partial ordering defined by $x \leq y$ iff $y - x \in K$. In this case, K is called the *positive cone* for the ordering $\leq$.

We will now introduce some additional basic concepts together with various results that hold in all Riesz spaces and lattice ordered groups. The results will be formulated for Riesz spaces, but those that make no use of the scalar multiplication hold in lattice ordered groups as well. In most cases, the proofs are also the same.

Definition 2.1.2. Let E be a Riesz space. Define $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$. The elements x^+, x^- and $|x|$ are called the *positive part*, the *negative part* and the *absolute value* of x, respectively. The set $E_+ = \{x \in E \ : \ x \geq 0\}$ is called the *positive cone* of E. If A is a subspace of E then we define $A_+ = A \cap E_+$.

A partially ordered space E is a Riesz space if and only if x^+ (or equivalently, x^-) exists for all $x \in E$.

The next theorem lists some basic facts about positive and negative parts and the absolute value.

Theorem 2.1.3. *The following hold for all elements x and y of a Riesz space E:*

(a) $x^+, x^- \in E_+$, $x^+ = (-x)^-$, $x^- = (-x)^+$ *and* $|x| = |-x|$

(b) $||x|| = |x|$, $|x| = x$ *if and only if* $x \geq 0$, *and* $|x| = 0$ *if and only if* $x = 0$

(c) $|x| = x^+ + x^- = x^+ \vee x^-$

(d) $x = x^+ - x^-$

(e) $x^+ \wedge x^- = 0$

(f) $x \vee y + x \wedge y = x + y$

(g) $x \vee y - x \wedge y = |x - y|$

(h) $2(x \vee y) = x + y + |x - y|$

(i) $(x + y)^+ \leq x^+ + y^+$, $(x + y)^- \leq x^- + y^-$ *and* $|x + y| \leq |x| + |y|$.

(j) *If $x = u - v$ and $u \wedge v = 0$, then $u = x^+$ and $v = x^-$.*

2.1.2 Riesz subspaces, ideals and bands

Subspaces with special order-related properties play an important role in the theory of Riesz spaces.

Definition 2.1.4. Let E be a Riesz space. A subspace $F \subseteq E$ is called a *Riesz subspace* if $x \vee y \in F$ and $x \wedge y \in F$ whenever $x, y \in F$. A subset $A \subseteq E$ is called *full*, or *order-convex*, if $x, y \in A$ and $x \leq z \leq y$ imply $z \in A$. A subset $A \subseteq E$ is called *solid*, if $x \in A$ and $|y| \leq |x|$ imply $y \in A$. A solid subspace is called an *ideal*. An ideal $I \subseteq E$ is called a *band* if $\sup A \in I$ whenever $A \subseteq I$ and $\sup A$ exists in E.

Every ideal is a Riesz subspace. A subspace I is an ideal if and only if I is an order-convex Riesz subspace. The sum of two ideals is an ideal, and an intersection of Riesz subspaces, ideals, or bands is again a Riesz subspace, an ideal, or a band, respectively. As a consequence, the set of all ideals of a Riesz space is itself a distributive lattice, where the lattice operations are taken as the intersection and the sum of ideals.

A subset A of a Riesz space E is called *upwards-directed* if for every $x, y \in A$ there exists an element $z \in A$ such that $x \leq z$ and $y \leq z$. Bands have the following characterization.

Theorem 2.1.5. *Let U be an ideal of a Riesz space E. The following conditions are equivalent.*

(a) *U is a band.*

(b) $\sup A \in U$ *whenever A is a subset of U_+ such that* $\sup A$ *exist in E.*

(c) $\sup A \in U$ *whenever A is an upwards-directed subset of U_+ such that* $\sup A$ *exist in E.*

A vector lattice E is called *Archimedean* if $nx \leq y$ for all $n \in \mathbb{N}$ implies that $x \leq 0$. Two elements x and y of a vector lattice E are called *disjoint* if $|x| \wedge |y| = 0$. If A is any non-empty subset of E then the set $A^{\perp} = \{y \in E : |x| \wedge |y| = 0 \text{ for all } x \in A\}$ is called the *disjoint complement* of A. The disjoint complement of any non-empty set is a band in E. The inclusion $A \subseteq (A^{\perp})^{\perp}$ always holds. If B is a band in an Archimedean vector lattice then $B = (B^{\perp})^{\perp}$. A band B is called a *projection band* if E has the direct sum decomposition $E = B \oplus B^{\perp}$. In this case, every $x \in E$ has a unique decomposition $x = u + v$ where $u \in B$ and $v \in B^{\perp}$. The elements u and v are called the *components* of x and they are given by the *projection* $P : E \to B$, which is an idempotent positive linear operator such that $u = Px$ and $v = (I - P)x$.

Definition 2.1.6. Let E and F be Riesz spaces. A linear mapping $T : E \to F$ is a *Riesz homomorphism* if $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$. A bijective Riesz homomorphism is called a *Riesz isomorphism*.

If T is a Riesz homomorphism then T is increasing, that is, $x \leq y$ in E implies $Tx \leq Ty$ in F. The kernel of a Riesz homomorphism is an ideal in E, and the set $T(E)$ is a Riesz subspace of F.

Homomorphisms and quotient spaces are closely related, and in case of a Riesz space E, the quotient space E/I will also be a Riesz space if I is an ideal in E.

Theorem 2.1.7. *If I is an ideal in a Riesz space E, then E/I is a Riesz space, and the canonical projection $x \mapsto [x]$ is a Riesz homomorphism whose kernel is I.*

2.2 Algebraic mixed lattice structures

As stated in Chapter 1, mixed lattice structures are partially ordered structures (semigroups, groups or vector spaces) with two partial orderings. We now introduce these partial orderings before giving the definition of a mixed lattice semigroup, which is due to Arsove and Leutwiler [8].

Let $(S, +, \leq)$ be a positive partially ordered Abelian semigroup with zero element. That is, we require that $u \geq 0$ for all $u \in S$, and $u \leq v$ implies $u + w \leq v + w$ for all $w \in S$. In what follows, we will also assume that the cancellation law

$$u + w \leq v + w \quad \Longrightarrow \quad u \leq v \tag{2.2.1}$$

holds in S. This partial order $\leq$ is called the *initial order*. Furthermore, the semigroup S itself induces another partial ordering, called the *specific order*, and it is defined by

$$u \preccurlyeq v \quad \Longleftrightarrow \quad v = u + w \quad \text{for some } w \in S. \tag{2.2.2}$$

Now $(S, +, \leq, \preccurlyeq)$ is a positive partially ordered semigroup with respect to two partial orderings $\leq$ and $\preccurlyeq$, both satisfying (2.2.1).

With these two partial orders $\leq$ and $\preccurlyeq$ we define the *mixed lower and upper en-*

velopes

$$u \curlywedge v \;=\; \max\{\, w \in S : \; w \preccurlyeq u \;\text{ and }\; w \le v \,\}$$

and

$$u \curlyvee v \;=\; \min\{\, w \in S : \; w \succcurlyeq u \;\text{ and }\; w \ge v \,\},$$

respectively, where the minimum and maximum are taken with respect to the initial order $\le$, whenever they exist.

Definition 2.2.1. Let $(S, +, \le, \preccurlyeq)$ be a partially ordered Abelian semigroup with zero element and two partial orders $\le$ and $\preccurlyeq$ such that $u \ge 0$ for all $u \in S$ and (2.2.1) holds in S, and $\preccurlyeq$ is given by (2.2.2). If the mixed upper and lower envelopes $u \curlyvee v$ and $u \curlywedge v$ exist in S for all $u, v \in S$, and they satisfy the identity

$$u \curlyvee v \,+\, v \curlywedge u \;=\; u + v \tag{2.2.3}$$

then $(S, +, \le, \preccurlyeq)$ is called a *mixed lattice semigroup*.

The following basic properties hold in a mixed lattice semigroup (see [8], Section 2).

$$x \curlywedge y \preccurlyeq x \preccurlyeq x \curlyvee y \;\text{ and }\; x \curlywedge y \le y \le x \curlyvee y \tag{2.2.4}$$

$$x \preccurlyeq y \;\Longrightarrow\; x \le y \tag{2.2.5}$$

$$x \curlyvee y \,+\, y \curlywedge x \;=\; x + y \tag{2.2.6}$$

$$z + x \curlyvee y \;=\; (x+z) \curlyvee (y+z) \;\text{ and }\; z + x \curlywedge y \;=\; (x+z) \curlywedge (y+z) \tag{2.2.7}$$

$$x \preccurlyeq u \;\text{ and }\; y \le v \;\Longrightarrow\; x \curlyvee y \le u \curlyvee v \;\text{ and }\; x \curlywedge y \le u \curlywedge v \tag{2.2.8}$$

$$x \le y \iff y \curlyvee x = y \iff x \curlywedge y = x \tag{2.2.9}$$

$$x \preccurlyeq y \iff x \curlyvee y = y \iff y \curlywedge x = x \tag{2.2.10}$$

$$x \preccurlyeq y \implies z \curlyvee x \preccurlyeq z \curlyvee y \text{ and } z \curlywedge y \preccurlyeq z \curlywedge y \tag{2.2.11}$$

$$u \preccurlyeq x \preccurlyeq z \text{ and } u \preccurlyeq y \preccurlyeq z \implies x \curlyvee y \preccurlyeq z \text{ and } u \preccurlyeq x \curlywedge y \tag{2.2.12}$$

The following definition of a mixed lattice group was given by Eriksson–Bique in [17].

Definition 2.2.2. Let $(G, +, \leq, \preccurlyeq)$ be a partially ordered Abelian group with respect to two partial orders $\leq$ and $\preccurlyeq$. If the mixed lower envelope $x \curlywedge y$ exists for all $x, y \in G$, then $(G, +, \leq, \preccurlyeq)$ is called a *mixed lattice group*.

It should be emphasized that in Definition 2.2.2, $\leq$ and $\preccurlyeq$ are any two partial orderings compatible with the group structure. This definition clearly shows that a lattice ordered group is a special case of a mixed lattice group. Indeed, if the two partial orderings $\leq$ and $\preccurlyeq$ are chosen to be identical, then the mixed lattice group reduces to an ordinary lattice ordered group.

Eriksson–Bique has also shown that every mixed lattice semigroup S can be extended to a group of formal differences of elements of S and the mixed lattice structure is preserved in this extension [16, Theorem 3.2].

The essential property of a mixed lattice group G is that the group operation is distributive over the mixed envelopes, that is, the following identities hold for all $x, y, z \in G$.

$$(x \curlyvee y) + z = (x + z) \curlyvee (y + z) \tag{2.2.13}$$

$$(x \curlywedge y) + z = (x + z) \curlywedge (y + z) \tag{2.2.14}$$

These imply that the identity (2.2.3) also holds in every mixed lattice group. The above identities were proved in [16, Lemma 3.2].

3 RESULTS

In this chapter, we present the most important results of the articles [P1]–[P4]. The chapter is divided into three sections, each of which covers different aspects of the theory of mixed lattice structures.

3.1 Structure theory of mixed lattice spaces

In paper [P1] we establish the foundations for a systematic study of algebraic mixed lattice structures by introducing fundamental definitions and concepts that generalize various notions from the theory of vector lattices and lattice ordered groups. These include concepts such as the positive and negative parts of an element, absolute values and certain substructures that play an important role in the theory. The article [P2] is a continuation of the work that started in [P1]. In [P2], the ideal theory is studied in much greater detail, and the important notion of a band is also introduced. These concepts are then used to study the direct sum decompositions of mixed lattice spaces, where the components are bands (i.e. ideals with certain order-completeness properties).

3.1.1 Fundamental definitions and basic properties

The definition of a mixed lattice space is introduced by the author in [P1] as follows.

Definition 3.1.1. Let $(V, \leq, \preccurlyeq)$ be a partially ordered real vector space with respect to any two partial orders $\leq$ and $\preccurlyeq$. If the mixed upper and lower envelopes $x \curlyvee\!\!\!\!\curlywedge y$ and $x \curlyvee y$ exist for all $x, y \in V$, then $(V, \leq, \preccurlyeq)$ is called a *mixed lattice vector space*. The partial orderings $\leq$ and $\preccurlyeq$ are called *initial order* and *specific order*, respectively.

We will use the notation $V_{sp} = \{x \in V : x \succcurlyeq 0\}$ for the set of $(\preccurlyeq)$-positive elements of V. If A is a subspace of V then we define $A_{sp} = A \cap V_{sp}$. We will also use the following terminology. If G is a mixed lattice group then a sub-semigroup S

of G_{sp} is called a *mixed lattice sub-semigroup* of G if $0 \in S$ and the elements $x \curlywedge y$ and $x \curlyvee y$ belong to S for all $x, y \in S$. This definition is equivalent to the statement that $M = (S, \leq, \preccurlyeq)$ is a mixed lattice semigroup (with the partial orderings $\leq$ and $\preccurlyeq$ inherited from G) and the mixed envelopes $x \curlyvee y$ and $x \curlywedge y$ in M are the same as in G for all $x, y \in S$. If V is a mixed lattice space and C is a cone in V_{sp} then C is called a *mixed lattice cone* if $x \curlywedge y$ and $x \curlyvee y$ belong to C for all $x, y \in C$.

In mathematical theories, whenever different kind of mathematical structures or objects are combined, it is usually required that the combined structures are somehow connected because otherwise the resulting theory would be too general to yield any useful results. In the case of mixed lattice theory, it is reasonable to require the two partial orderings to be connected in some way. In [P1] we introduce a classification of mixed lattice groups into *regular*, *quasi-regular* and *pre-regular* cases. Roughly speaking, this classification represents how strong the relationship between the two partial orderings is, and it enables us to isolate those properties that are essential assumptions for various results presented in this thesis.

Definition 3.1.2. Let G be a mixed lattice group.

(i) G is called *pre-regular* if $x \preccurlyeq y$ implies that $x \leq y$ in G.

(ii) G is called *quasi-regular* if the set $S = \{w \in G : w \succcurlyeq 0\}$ is a mixed lattice subsemigroup in G.

(iii) G is called *regular* if it is quasi-regular and the set S defined above is generating, that is, $G = S - S$.

In the above classification, it is obvious that regularity implies quasi-regularity, and in this thesis we show that quasi-regularity implies pre-regularity [P1, Theorem 2.10]. Moreover, G is quasi-regular if and only if the property (2.2.12) of mixed lattice semigroups holds in G ([17, Theorem 2.7]).

In article [P2] we prove some important additional basic properties of quasi-regular mixed lattice groups. Perhaps the most fundamental of these properties is the following (here G is a quasi-regular mixed lattice group):

$$x \preccurlyeq y \quad \Longrightarrow \quad z \curlywedge x \preccurlyeq z \curlywedge y \quad \text{for all } z \in G. \tag{3.1.1}$$

The property (3.1.1) is crucial for many results that follow, and consequently, only quasi-regular mixed lattice spaces are considered in [P2]. Incidentally, it can be shown that (3.1.1) is equivalent to the property (2.2.12) that characterizes quasi-regular mixed lattice groups. This is shown in [22].

Another fundamental result discussed in [P2] is the following mixed lattice version of the Riesz decomposition property [P2, Theorem 3.8], which is crucial for the subsequent development of the ideal theory.

Theorem 3.1.3. *Let G be a quasi-regular mixed lattice group.*

(a) Let $u \succcurlyeq 0$, $v_1 \geq 0$ and $v_2 \succcurlyeq 0$ be elements of G satisfying $u \leq v_1 + v_2$. Then there exist elements u_1 and u_2 such that $0 \leq u_1 \leq v_1$, $0 \preccurlyeq u_2 \leq v_2$ and $u = u_1 + u_2$. Moreover, if $v_1 \succcurlyeq 0$ then $u_1 \succcurlyeq 0$, and if $u \preccurlyeq v_1 + v_2$ then $u_2 \preccurlyeq v_2$.

(b) Let $u \geq 0$, $v_1 \succcurlyeq 0$ and $v_2 \geq 0$ be elements of G satisfying $u \preccurlyeq v_1 + v_2$. Then there exist elements u_1 and u_2 such that $0 \leq u_1 \preccurlyeq v_1$, $0 \leq u_2 \leq v_2$ and $u = u_1 + u_2$. Moreover, if $u \succcurlyeq 0$ then $u_1 \succcurlyeq 0$.

Next we introduce the generalized absolute values and the representation of elements as differences of the upper and lower parts. These are generalizations of the vector lattice concepts given in Definition 2.1.2.

Definition 3.1.4. Let G be a mixed lattice group and $x \in G$. The elements ${}^u x = x \curlyvee 0$ and ${}^l x = (-x) \curlyvee 0$ are called the *upper part* and *lower part* of x, respectively. Similarly, the elements $x^u = 0 \curlyvee x$ and $x^l = 0 \curlyvee (-x)$ are called the *specific upper part* and *specific lower part* of x, respectively. The elements ${}^u x^l = {}^u x + x^l$ and ${}^l x^u = {}^l x + x^u$ are called the *asymmetric generalized absolute values* of x. The element $s(x) = \frac{1}{2}({}^u x^l + {}^l x^u)$ is called the *symmetric generalized absolute value* of x.

The elements ${}^u x^l$ and ${}^l x^u$ are distinct, in general, and they are "asymmetric" in the sense that ${}^u x^l = {}^l(-x)^u$ for all x. The upper and lower parts and the generalized absolute values have the following basic properties, which show that they are indeed proper generalizations of the corresponding notions in vector lattice theory (see Theorem 2.1.3). The properties given next are proved in [P1, Theorems 3.2, 3.5, 3.7 and 3.10] and [P2, Theorem 5.17]. Here they are stated for mixed lattice spaces but, apart from properties related to scalar multiplication, they hold in mixed lattice groups as well.

Theorem 3.1.5. *Let V be a pre-regular mixed lattice space and $x \in V$, $\alpha \in \mathbb{R}$. Then the following hold.*

(a) $\quad {}^{u}x = {}^{l}(-x), \quad x^{u} = (-x)^{l} \quad$ *and* $\quad {}^{u}x^{l} = {}^{l}(-x)^{u}.$

(b) $\quad x = x^{u} - {}^{l}x = {}^{u}x - x^{l}.$

(c) $\quad {}^{u}x^{l} = {}^{u}x \curlyvee x^{l} = {}^{u}x + x^{l} \quad$ *and* $\quad {}^{l}x^{u} = {}^{l}x \curlyvee x^{u} = {}^{l}x + x^{u}.$

(d) $\quad x^{u} \curlywedge {}^{l}x = 0 = x^{l} \curlywedge {}^{u}x.$

(e) $\quad s(x) = x^{u} \curlyvee {}^{l}x = {}^{u}x + {}^{l}x = x^{l} + x^{u} = x^{l} \curlyvee {}^{u}x$

(f) $\quad x \succcurlyeq 0$ *if and only if* $x = {}^{l}x^{u} = {}^{u}x^{l} = {}^{u}x = x^{u}$ *and* ${}^{l}x = x^{l} = 0.$

(g) $\quad x \geq 0$ *if and only if* $x = {}^{u}x^{l} = {}^{u}x$ *and* $x^{l} = 0.$

(h) $\quad s(x) \succcurlyeq 0, \ {}^{u}x^{l} \geq 0$ *and* ${}^{l}x^{u} \geq 0.$ *Moreover,* $s(x) = {}^{u}x^{l} = {}^{l}x^{u} = 0$ *if and only if* $x = 0.$

(i) $\quad {}^{u}(\alpha x)^{l} = \alpha\, {}^{u}x^{l}$ *and* ${}^{l}(\alpha x)^{u} = \alpha\, {}^{l}x^{u}$ *for all* $\alpha \geq 0.$

(j) $\quad {}^{u}(\alpha x)^{l} = |\alpha|\, {}^{l}x^{u}$ *and* ${}^{l}(\alpha x)^{u} = |\alpha|\, {}^{u}x^{l}$ *for all* $\alpha < 0.$

(k) $\quad s(\alpha x) = |\alpha|\, s(x)$ *for all* $\alpha \in \mathbb{R}.$

(l) $\quad {}^{u}(x+y) \leq {}^{u}x + {}^{u}y, \quad (x+y)^{l} \leq x^{l} + y^{l} \quad$ *and* $\quad {}^{u}(x+y)^{l} \leq {}^{u}x^{l} + {}^{u}y^{l}.$

(m) $\quad (x+y)^{u} \leq x^{u} + y^{u}, \quad {}^{l}(x+y) \leq {}^{l}x + {}^{l}y \quad$ *and* $\quad {}^{l}(x+y)^{u} \leq {}^{l}x^{u} + {}^{l}y^{u}.$

(n) $\quad s(x+y) \leq s(x) + s(y).$

(o) $\quad 2(x \curlywedge y) = x + y - {}^{l}(x-y)^{u} \quad$ *and* $\quad 2(y \curlywedge x) = x + y - {}^{u}(x-y)^{l}.$

3.1.2 Ideals in mixed lattice spaces

The most fundamental structure theoretic notions are given in the next definition. We will again formulate these concepts in mixed lattice spaces since they are mostly used in this context, but they can be defined similarly in the more general group setting.

Definition 3.1.6. Let V be a mixed lattice space.

(i) A subspace S of V is called a *mixed lattice subspace* of V if $x \curlyvee y$ and $x \curlywedge y$ belong to S whenever x and y are in S. Here $x \curlyvee y$ and $x \curlywedge y$ are the upper and lower mixed envelopes in V.

(ii) A subset $F \in V$ is called $(\leq)$-*order convex* (or $(\leq)$-*full*), if $x \leq z \leq y$ and $x, y \in F$ imply that $z \in F$. Similarly, F is called $(\preccurlyeq)$-*order convex* (or $(\preccurlyeq)$-*full*), if $x \preccurlyeq z \preccurlyeq y$ and $x, y \in F$ imply that $z \in F$. A subset $F \in V$ is called *mixed-order convex* (or *mixed-full*), if $x \preccurlyeq z \leq y$ and $x, y \in F$ imply that $z \in F$.

(iii) A $(\leq)$-order convex mixed lattice subspace of V is called an *ideal* of V, a $(\preccurlyeq)$-order convex mixed lattice subspace of V is called a *specific ideal* of V, and a mixed-order convex mixed lattice subspace of V is called a *quasi-ideal* of V.

If there is any danger of confusion, we will use the term *mixed lattice ideal* to distinguish mixed lattice ideals from lattice ideals.

Various properties and characterizations of different types of ideals are studied in [P1] and [P2]. In particular, mixed lattice ideals can be characterized using the generalized absolute value, in a similar way as the usual definition of a lattice ideal (see Definition 3.1.6). For example, a subspace A is a mixed lattice ideal if and only if $s(x) \leq s(y)$ with $y \in A$ implies that $x \in A$ [P2, Theorem 5.18]. A similar characterization based on the unsymmetrical absolute values is given in [P1, Theorem 4.2]. It is also shown that if A is an ideal then the subspace generated by the set A_{sp} is the largest regular quasi-ideal contained in A, and every ideal is uniquely determined by the set A_{sp}, which is a mixed lattice cone [P2, Theorems 4.12 and 4.15]. It should be noted that in mixed lattice semigroups there is no distinction between the notions of ideal and quasi-ideal, because all the elements of a mixed lattice semigroup are specifically positive. This distinction becomes meaningful only in mixed lattice groups and vector spaces.

The set of all ideals of a vector lattice, partially ordered by inclusion, is a distributive lattice where the supremum and infimum of two ideals are given by their sum and intersection, respectively [32, Proposition 2.3]. This brings up the question whether the same holds for ideals in mixed lattice spaces. It is shown in [P2, Theorem 4.18] that the set of all regular quasi-ideals of a mixed lattice space V is indeed a distributive lattice. Moreover, if V is a lattice with respect to the partial order $\preccurlyeq$ then the set of all specific ideals is also a distributive lattice [P2, Corollary 4.20]. It is unclear whether the latter result holds more generally, or if the set of all mixed lattice ideals is always a lattice.

3.1.3 Bands in mixed lattice spaces

In vector lattices, the definition of a band and its characterizations are given in Theorem 2.1.5. Some authors use the condition (c) of Theorem 2.1.5 as a definition of a band in a vector lattice. When considering the notion of a band in the mixed lattice setting, there are a couple of issues that must be taken into account. Firstly, the presence of different types of ideals, as described above, gives rise to different notions of bands. Secondly, we do not know if the equivalent conditions of Theorem 2.1.5 can be generalized to mixed lattice bands. These conditions are quite important because various proofs concerning bands become much easier if we only need to consider positive elements. The main difficulty is that if E is a subset of a vector lattice then there is a rather simple method of constructing an upwards-directed set with the same suprema as the set E (see [26, Theorem 15.11]). However, this method depends on the associativity laws and therefore it does not work in a mixed lattice space as such. The author has shown that a similar method can be devised in mixed lattice spaces, but it is not included in this thesis.

We define bands in a mixed lattice space using basically the same definition as in vector lattices, but the characterization analogous to Theorem 2.1.5 becomes slightly more complicated, and we need to consider a weaker notion of a band by imposing certain additional assumptions. For this we need the concepts of a *strong supremum* and a *strong infimum*.

Let E be a subset of V such that $\sup E$ and $\mathrm{sp}\,\sup E$ exist in V, and $u_0 = \sup E = \mathrm{sp}\,\sup E$. Then the element u_0 is called the *strong supremum* of E and it is denoted by $u_0 = \mathrm{str}\,\sup E$. The *strong infimum* v_0 of E is defined similarly, and it is denoted by $v_0 = \mathrm{str}\,\inf E$.

Definition 3.1.7. Let V be a mixed lattice space.

(i) A specific ideal A is called a *specific band* if $\mathrm{sp}\,\sup E \in A$ whenever E is a non-empty subset of A such that $\mathrm{sp}\,\sup E$ exists in V. If A is a quasi-ideal with the above property then A is called a *quasi-band*. An ideal B is called a *band* if $\sup E \in A$ whenever E is a non-empty subset of A such that $\sup E$ exists in V.

(ii) A specific ideal A is called a *weak specific band* if $\mathrm{str}\,\sup E \in A$ whenever E is a non-empty subset of A such that $\mathrm{str}\,\sup E$ exists in V. A quasi-ideal with the above property is called a *weak quasi-band*, and an ideal with the above property is called a *weak band*.

As noted in Subsection 2.1.2, the set of elements disjoint to some subset of a Riesz space is a band. This is where things get more complicated once we begin to study this concept in a mixed lattice space. First of all, because of the asymmetry, we have two one-sided notions of disjointness. We shall say that the element x is *left-disjoint to the element* y if $x \curlywedge y = 0$, or that y is *right-disjoint to* x. Now, let A be a mixed lattice subspace of a mixed lattice space V. Since the condition $x \curlywedge y = 0$ implies that $x \succcurlyeq 0$ and $y \geq 0$, we define the *left disjoint complement* of A as the specific ideal generated by the set $({}^{\perp}A)_{sp} = \{x \succcurlyeq 0 : x \curlywedge z = 0 \text{ for all } z \in A_p\}$. The *right disjoint complement* of A is then defined as the ideal generated by the set $(A^{\perp})_{sp} = \{x \succcurlyeq 0 : z \curlywedge x = 0 \text{ for all } z \in A_{sp}\}$, and it is denoted by $A^{\perp}$. The sets $({}^{\perp}A)_{sp}$ and $(A^{\perp})_{sp}$ are shown to be mixed lattice cones [P2, Theorems 5.2 and 5.8]. The right disjoint complement $A^{\perp}$ is a weak band [P2, Theorem 5.11], and the left disjoint complement ${}^{\perp}A$ is a regular weak specific band [P2, Theorem 5.5].

We do not know if ${}^{\perp}A$ is a specific band in general, but we show in [P2, Theorem 6.1] that if V can be written as a direct sum $V = A \oplus B$, where A is a regular specific ideal and B is an ideal, then B is a band and A is a regular specific band such that $A = {}^{\perp}B = {}^{\perp}(A^{\perp})$ and $B = A^{\perp} = ({}^{\perp}B)^{\perp}$. Hence, this kind of direct sum decomposition of V implies a stronger band property for the left disjoint complement of B.

In conclusion, the most important structure-theoretic results in [P2] are Theorem 4.15, describing the relationship between ideals and regular quasi-ideals; Theorems 4.18 and 4.19, which reveal the lattice structure of regular quasi-ideals and specific ideals; the definitions of disjoint complemets and theorems showing their band properties; Theorem 6.1 that gives the decompositions of V as a direct sum of disjoint bands, and the related existence of order projection operators associated with the components (Theorem 6.3). Moreover, Theorem 5.23 and its Corollary 5.24 discuss the relationship between quasi-ideals and their second disjoint complements in Archimedean mixed lattice spaces. This result is related to the problem of finding bands generated by ideals, a topic that is planned to be studied in future research by the author.

3.1.4 On homomorphisms and quotient groups

In [P1, Sections 5 and 6] we study homomorphisms and quotient group constructions in mixed lattice groups. These topics are closely connected to the ideal theory, as can be seen from Theorem 2.1.7. Although these results are not used elsewhere in this thesis, they will most likely be important for future research. Indeed, it is well known that homomorphisms, or structure preserving maps, and the related quotient constructions are important in many areas of pure mathematics, especially in connection with different kinds of representation theorems. In Riesz space theory, quotient spaces and lattice homomorphisms are important tools for theorems concerning the structural properties of Riesz spaces. As some examples of application, we could mention the Dedekind completion of certain Archimedean Riesz spaces ([1, Theorem 2.18]), and also the proof of the fundamental Dodds–Fremlin domination theorem for compact operators in Banach lattices incorporates the idea of the quotient space construction ([15]). Moreover, the representation theorems (such as those by Kakutani, Krein and Nakano, see [2, Theorems 4.27 and 4.28]) allow us to represent certain types of Riesz spaces as concrete function spaces. Currently, such representation theorems for mixed lattice groups are not known, but it is reasonable to expect that homomorphisms will play a crucial role if such theorems are to be found.

3.2 Topological mixed lattice vector spaces

In [P3], we study the topological theory of mixed lattice spaces, a different aspect of the theory that leans more towards analysis. As in the earlier discussion, we take the well-known theory of topological Riesz spaces as a starting point and try to generalize this theory to the mixed lattice setting. Again, the main differences and difficulties are related to the asymmetry that comes from the definition of the mixed envelopes.

The main results of [P3] are the generalizations of the Roberts–Namioka Theorem that gives a characterization of locally solid topologies on Riesz spaces.

Theorem 3.2.1 (Roberts–Namioka)**.** *Let (V, τ) be a Riesz space equipped with a vector topology τ. The following conditions are equivalent.*

(a) τ is a locally solid topology.

(b) τ is locally full and the lattice operations are continuous at zero.

(c) The lattice operations are uniformly continuous.

We introduce two different versions of this theorem for mixed lattice spaces. First, under quite general assumptions, we obtain a partial result. A complete characterization is then obtained in the locally convex case, assuming the mixed lattice space is regular.

Before proving, or even stating these theorems, it is necessary to introduce new definitions and develop some preliminary results.

As in the case of topological Riesz spaces, we introduce the following terminology. A mixed lattice space V with a vector topology τ is called a *topological mixed lattice space* if the mixed lattice operations are continuous. A subset S of a mixed lattice space V is called *($\leq$)-full* if $x, y \in S$ and $y \leq z \leq x$ together imply that $z \in S$. Similarly, a subset $S \subseteq V$ is called *($\preccurlyeq$)-full* if $x, y \in S$ and $y \preccurlyeq z \preccurlyeq x$ together imply that $z \in S$. Moreover, S is called *symmetric-solid* if $x \in S$ and $s(y) \leq s(x)$ together imply that $y \in S$.

A vector topology τ on a mixed lattice vector space is called *locally symmetric-solid* if τ has a base at zero consisting of symmetric-solid sets. A vector topology τ is called *locally ($\leq$)-full* if τ has a base at zero consisting of ($\leq$)-full sets. Similarly, τ is called *locally ($\preccurlyeq$)-full* if τ has a base at zero consisting of ($\preccurlyeq$)-full sets.

In [P3] we also introduce the following type of sets that turn out to be important for the topological theory of mixed lattice spaces. A neighborhood U of zero is called *mixed-full* if $x \in U$ whenever $0 \preccurlyeq x \leq y$ or $0 \leq x \preccurlyeq y$ holds with $y \in U$. A vector topology τ on a mixed lattice vector space is called *locally mixed-full* if τ has a base at zero consisting of mixed-full sets. The fundamental properties of mixed-full topologies are given in [P3, Section 3].

In order to study locally convex topologies, it is necessary to investigate the properties of seminorms. In [P3, Section 4] the notions of *mixed-monotone seminorm* and *mixed lattice seminorm* are introduced and their basic properties are established. A seminorm p in a mixed lattice space V is called a *mixed-monotone seminorm* if $0 \preccurlyeq x \leq y$ implies $p(x) \leq p(y)$, and p is called a *mixed lattice seminorm* if $s(x) \leq s(y)$ implies $p(x) \leq p(y)$. It is shown that the locally convex convex topology generated by a family of mixed-monotone seminorms is locally mixed-full [P3, Theorem 4.2], and the locally convex convex topology generated by a family of mixed lattice seminorms is locally symmetric-solid [P3, Proposition 4.3].

The main result of [P3] is now the following ([P3, Corollary 4.7]).

Theorem 3.2.2. *Let V be a regular mixed lattice space. If τ is a locally convex topology on V then the following conditions are equivalent.*

(a) The mixed lattice operations are uniformly continuous.

(b) τ *is locally symmetric-solid.*

(c) τ *is locally mixed-full and the mixed lattice operations are continuous at zero.*

(d) τ *is locally* $(\leq)$*-full and the mixed lattice operations are continuous at zero.*

In a more general case, without assuming the regularity and local convexity, we have established the equivalence of (b) and (d) in the above theorem, and shown that they imply (a), which in turn implies (c). This is the content of [P3, Theorem 3.12].

Another important fundamental result is [P3, Theorem 3.13] which gives several additional facts concerning topological mixed lattice spaces. Some of these properties are stated in the next theorem.

Theorem 3.2.3. *Let* (V, τ) *be a topological mixed lattice vector space. Then the following statements hold.*

(a) If τ *is a Hausdorff topology then* V *is Archimedean and the positive cones* V_p *and* V_{sp} *are closed.*

(b) If τ *is Hausdorff and* $\{x_n\}$ *is a* $(\leq)$*-increasing (or* $(\preccurlyeq)$*-increasing) sequence such that* $x_n \xrightarrow{\tau} x$ *then* $\sup\{x_n\} = x$ *(or* $\operatorname{sp}\sup\{x_n\} = x$*, respectively).*

(c) If τ *is locally mixed-full then the* mixed-order intervals $\{x \in V : z \preccurlyeq x \leq y\}$ *and* $\{x \in V : z \leq x \preccurlyeq y\}$ *are bounded for all* $z, y \in V$*. In particular, every* $(\preccurlyeq)$*-order interval is bounded.*

(d) The closure of a quasi-ideal is a quasi-ideal, the closure of a specific ideal is a specific ideal, and the closure of a mixed lattice subspace is a mixed lattice subspace.

As an application of these results, we show that any regular finite-dimensional normed mixed lattice space is a lattice with respect to specific order $\preccurlyeq$ [P3, Theorem 4.11]. In fact, this holds more generally for any regular topological mixed lattice space with the Bolzano–Weierstrass property.

We also show that if V is a mixed lattice vector space which is a lattice with respect to specific order $\preccurlyeq$ and τ is a locally $(\preccurlyeq)$-solid Riesz space topology on V then the mixed lattice operations are continuous [P3, Theorem 4.8], and if V is a lattice with respect to initial order $\leq$ and τ is a locally symmetric-solid topology on V then

τ is a locally $(\leq)$-solid Riesz space topology [P3, Theorem 4.9].

The results of [P3] indicate that mixed lattice spaces provide a natural setting for the study of asymmetric norms and their vector-valued generalizations. This topic is briefly discussed in [P3, Section 5], and the observations made there could be used as a starting point for further study of functional analysis in mixed lattice spaces. A specific application is given in [P3, Section 5], where it is shown that if a cone C in $\mathbb{R}^n$ is given, then with an appropriate choice of the second partial ordering $\mathbb{R}^n$ can be turned into a topological mixed lattice space such that $C = V_p$. The upper part mapping can then be viewed as a continuous asymmetric cone norm. This application is closely connected to the cone projection problem discussed in [P4], since continuous asymmetric cone norms are generalizations of cone projections. A more in-depth study of such generalizations can be found in [29]. The theory of asymmetric normed spaces is relatively recent, and it has seen a rapid progress during the last couple of decades. An account of the basic theory of asymmetric normed spaces is given in [11].

3.3 Some applications and generalizations

Although this thesis is mainly concerned with the structure theory of mixed lattice spaces, we also investigate the connection of the mixed lattice theory to other areas of study, and present some non-trivial applications. In particular, we introduce an application from convex analysis and optimization theory. These ideas are presented in articles [P4] and [P3].

An important concept in convex analysis is the cone projection problem, which is encountered in connection to optimization problems. Cone projections have been studied extensively in the past, but in [P4] we present a new theoretical approach to these problems using the mixed lattice structures.

3.3.1 Generalized mixed lattice structure

The basic mixed lattice vector space structure, as described in this thesis, is too limited for certain applications involving cone projections. We therefore present the following generalization of a mixed lattice vector space.

Let E be a subset of a vector space V. We recall that an element $x \in E$ is called a *minimal element* of E if $y \in E$ and $y \leq x$ implies $y = x$. A dual notion of a *maximal* *element* is defined similarly.

Next we introduce the following set notation. If x and y are elements of V then we write

$$[x \curlyvee y] = \{w \in V : w \succcurlyeq x \text{ and } w \geq y\} \quad \text{and} \quad [x \curlywedge y] = \{w \in V : w \preccurlyeq x \text{ and } w \leq y\}.$$

The set of minimal elements of the set $[x \curlyvee y]$ will be denoted by $\mathrm{Min}[x \curlyvee y]$, and the set of maximal elements of the set $[x \curlywedge y]$ will be denoted by $\mathrm{Max}[x \curlywedge y]$.

We now introduce a generalization of the mixed lattice structure in which the elements $x \curlyvee y$ and $x \curlywedge y$ are replaced by set-valued mappings $(x, y) \mapsto \mathrm{Min}[x \curlyvee y]$ and $(x, y) \mapsto \mathrm{Max}[x \curlywedge y]$.

Definition 3.3.1. Let V be a partially ordered vector space with respect to two partial orders $\leq$ and $\preccurlyeq$, and let V_p and V_{sp} be the corresponding positive cones, respectively. Then $V = (V, \leq, \preccurlyeq)$ is called a *generalized mixed lattice structure* if the sets $\mathrm{Min}[x \curlyvee y]$ and $\mathrm{Max}[x \curlywedge y]$ are non-empty for all $x, y \in V$.

We observe that if the sets $\mathrm{Min}[x \curlyvee y]$ and $\mathrm{Max}[x \curlywedge y]$ contain only one element for every $x, y \in V$ then these elements are equal to $x \curlyvee y$ and $x \curlywedge y$, respectively, and the generalized mixed lattice structure then reduces to an ordinary mixed lattice space. Hence we have defined a proper generalization of a mixed lattice vector space.

This generalized structure has certain basic properties that are similar to the properties of the ordinary mixed lattice space, which were outlined in Sections 3.1 and 2.2. Of particular importance is the fact that in a generalized mixed lattice structure every element can be written as a difference of a "positive part" and a "negative part", but the representation is not generally unique. The following is thus a generalized version of [P1, Theorem 3.6] and parts (b) and (e) of Theorem 3.1.5.

Theorem 3.3.2. *Let V be a generalized mixed lattice structure and $x \in V$.*

(a) For any $u \in \mathrm{Min}[x \curlyvee 0]$ there exist an element $w \in \mathrm{Min}[0 \curlyvee (-x)]$ such that $x = u - w$. Moreover, if $u \in \mathrm{Min}[x \curlyvee 0]$ and $w \in \mathrm{Min}[0 \curlyvee (-x)]$ are any such elements that $x = u - w$ then $0 \in \mathrm{Max}[w \curlywedge u]$ and $u + w \in \mathrm{Min}[u \curlyvee w]$. On the other hand, if $x = u - w$ and $0 \in \mathrm{Max}[w \curlywedge u]$ then $u \in \mathrm{Min}[x \curlyvee 0]$ and $w \in \mathrm{Min}[0 \curlyvee (-x)]$.

(b) For any $u \in \mathrm{Min}[0 \curlyvee x]$ there exist an element $w \in \mathrm{Min}[(-x) \curlyvee 0]$ such that $x = u - w$. Moreover, if $u \in \mathrm{Min}[0 \curlyvee x]$ and $w \in \mathrm{Min}[(-x) \curlyvee 0]$ are any such

elements that $x = u - w$ *then* $0 \in \mathrm{Max}[\, u \curlywedge w \,]$ *and* $u + w \in \mathrm{Min}[\, w \curlyvee u \,]$. *On the other hand, if* $x = u - w$ *and* $0 \in \mathrm{Max}[\, u \curlywedge w \,]$ *then* $u \in \mathrm{Min}[\, 0 \curlyvee x \,]$ *and* $w \in \mathrm{Min}[\, (-x) \curlyvee 0 \,]$.

By the preceding result, for any $x \in V$ we can choose an element $u \in \mathrm{Min}[\, 0 \curlyvee x \,]$, and there always exists a corresponding element $v \in \mathrm{Min}[\, (-x) \curlyvee 0 \,]$ such that $x = u - v$. Similar remarks apply to the sets $\mathrm{Min}[\, x \curlyvee 0 \,]$ and $\mathrm{Min}[\, 0 \curlyvee (-x) \,]$.

Finally, with regards to the generalized mixed lattice structures discussed above, we point out that another type of generalization of mixed lattice structures is given in [22], where an example from the theory of divisibility is also discussed. However, the above cited paper is not included in this thesis.

3.3.2 Cone projections and the mixed lattice structure

The concept of the cone projection in $\mathbb{R}^n$ can be outlined as follows. Let K be a closed and convex pointed cone in $\mathbb{R}^n$ with the dual cone $K^* = \{y : \langle x, y \rangle \geq 0 \text{ for all } x \in K\}$. Let $\preccurlyeq_K$ be the partial ordering induced by the cone K and let $\leq_*$ be the partial ordering given by the dual cone K^*.

Let $P_K : \mathbb{R}^n \to K$ be the projection mapping that gives the unique point $P_K x$ on K nearest to x. That is,

$$P_K x \in K \quad \text{and} \quad ||x - P_K x|| = \inf\{||x - y|| : y \in K\}.$$

This nearest point $P_K x$ has the following characterization: $z = P_K x$ if and only if

$$z \in K \quad \text{and} \quad \langle z - x, P_K x - y \rangle \leq 0 \text{ for all } y \in K. \tag{3.3.1}$$

A fundamental tool in the study of cone projections is the following classical theorem of J. Moreau.

Theorem 3.3.3 (Moreau)**.** *Let* K *be a closed convex cone in* $\mathbb{R}^n$ *and* K^* *its dual cone. Every* $x \in \mathbb{R}^n$ *can be written as* $x = P_K x - P_{K^*}(-x)$ *where* $\langle P_K x, P_{K^*}(-x) \rangle = 0$. *Moreover,* $P_K x = 0$ *holds if and only if* $x \in -K^*$.

With the notation introduced above we can now state the main result of [P4], which shows that the cone projection problem can be given an order theoretic interpretation in terms of a mixed lattice structure, and the projection element is a minimal element of the set $\mathrm{Min}[\,0 \curlyvee x\,]$ introduced above.

Theorem 3.3.4. *Let K be a closed and convex cone in $\mathbb{R}^n$ and K^* its dual cone, and let $\preccurlyeq_K$ and $\leq_*$ be the partial orderings determined by the cones K and K^*, respectively. Then $V = (\mathbb{R}^n, \leq_*, \preccurlyeq_K)$ is a generalized mixed lattice structure and for every $x \in \mathbb{R}^n$ the projection element $P_K x$ satisfies $P_K x \in \mathrm{Min}[\,0 \curlyvee x\,]$.*

Furthermore, Moreau's Theorem 3.3.3 can now be viewed as a special case of Theorem 3.3.2.

The generalizations of asymmetric norms mentioned in the preceding section have applications in convex analysis, and this is where the topic of paper [P4] is closely connected to article [P3], in which a proof is given of a result that in $\mathbb{R}^n$ every closed convex cone has an associated asymmetric cone norm which arises from the mixed lattice structure [P3, Theorem 5.3]. This result is not new, as it has been proved recently in [29, Theorem 2] using different methods, but it is a fine example of an application that clearly shows the potential usefulness of the mixed lattice theory.

4 CONCLUSIONS, OPEN QUESTIONS AND FUTURE RESEARCH

In this thesis, the theory of vector lattices and lattice ordered groups was generalized by developing the basic theory of mixed lattice structures. The work started with the introduction of the most fundamental definitions needed for further development of the theory. Most importantly, basic properties of mixed envelopes were studied, and the representation of elements as differences of upper and lower parts was introduced. Many fundamental identities and inequalities were proved. The important subspaces related to the order structure were introduced. In particular, the notions of ideals and bands were studied in detail. These concepts and results provided the necessary tools for the development of the structure theory, where the main research problem was the decomposition of a mixed lattice space into a direct sum of order-disjoint bands. Such decomposition is a fundamental structure theoretic result in the theory of vector lattices, and in this thesis this idea was generalized to the mixed lattice setting. All this algebraic theory was developed in [P1] and [P2].

In [P3], we turned our attention to the study of topologies on mixed lattice spaces. As the main result, a characterization of compatible topologies on mixed lattice spaces was obtained. Other properties of such topologies were also studied, an application to convex analysis was presented and some interesting connections between the mixed lattice theory and asymmetric functional analysis were indicated. Finally, in [P4] a generalization of the mixed lattice structure on a vector space was introduced, and its basic properties were established. We then showed how the cone projection problem of convex analysis is related to generalized mixed lattice structures.

However, the subject matter of this thesis is vast, and there are many topics that we haven't even touched upon. The parts of the theory developed in this thesis is also far from complete, and there are some "gaps" in the details that remain unclear. We will now briefly go through some of the most interesting and important open questions that we haven't been able to answer, or we simply haven't had enough time to study them properly. Some of the following questions may turn out to be relatively easy, while some others could be highly non-trivial.

1. Does there exist a mixed lattice space which is pre-regular but not quasi-regular?

2. Does there exist a mixed lattice group that is regular but not a lattice with respect to specific order?

3. Does there exist non-regular proper specific ideals or quasi-ideals?

4. Is every ideal of V regular if V itself is regular?

5. Is there an example of a mixed lattice subspace A such that ${}^{\perp}(A^{\perp})$ is not a specific band? (We know that ${}^{\perp}(A^{\perp})$ is always a weak specific band [P2, Theorem 5.5].)

6. If A is a specific ideal in an Archimedean mixed lattice space, then is ${}^{\perp}(A^{\perp})$ the weak specific band generated by A? Similarly, is $({}^{\perp}A)^{\perp}$ the weak band generated by A?

7. Is the convex hull of a symmetric-solid set always symmetric-solid?

8. Does the characterization of locally convex mixed lattice topologies given in [P3, Corollary 4.7] hold in a more general setting (i.e. without the assumptions of local convexity and/or regularity)?

In addition to these particular open questions, there are also entire topics to be explored in the future research. We shall now discuss some of these topics.

It was already mentioned at the end of Section 3.1 that currently there are no known representation theorems for mixed lattice spaces. Finding such results, even under certain special assumptions, would be significant as it would give us a better understanding of the fundamental nature of mixed lattice spaces. A well known representation theorem for Riesz spaces states that any Archimedean Riesz space can be represented as the space of continuous real valued functions defined on some compact Hausdorff space. The proof of this theorem is based on the properties of the set of all maximal ideals of the Riesz space. It is not unreasonable to expect that something similar could be done with mixed lattice spaces, and therefore it might be worthwile to begin by studying maximal ideals of an Archimedean mixed lattice space. The author of this thesis has already began preliminary work on this topic, including some further results on bands, some of which were briefly mentioned in Section 3.1.

Another unexplored topic is operator theory in mixed lattice spaces. In particular, if the operators are linear functionals then this would lead to a duality thoery of (topological) mixed lattice spaces.

Apart from the applications discussed in Section 3.3, we would like to mention a few particular examples that suggest connections between the mixed lattice theory and other branches of mathematics. In [P1, Example 2.14], a connection between mixed lattice structures and stochastic processes is presented, and [22, Example 2.11] gives an example related to the theory of divisibility. One example that is frequently studied in papers [P1], [P2] and [P3] is the set of functions of bounded variation, an important class of functions that appears in many different contexts in mathematics. A more thorough study of these connections would be of great interest, as it could potentially reveal some interesting applications of the mixed lattice theory.

Another possible line of investigation is the study of *general mixed lattices*, that is, the mixed lattice order structure on some set without any other algebraic structure. This can be viewed as a generalization of lattice theory, which is a well-developed and important area of mathematics. Initial studies in this direction have already been made by the author, and some basic results can be found in [22].

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Generalized absolute values, ideals and homomorphisms in mixed lattice groups

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Abstract
A mixed lattice group is a generalization of a lattice ordered group. The theory of mixed lattice semigroups dates back to the 1970s, but the corresponding theory for groups and vector spaces has been relatively unexplored. In this paper we investigate the basic structure of mixed lattice groups, and study how some of the fundamental concepts in Riesz spaces and lattice ordered groups, such as the absolute value and other related ideas, can be extended to mixed lattice groups and mixed lattice vector spaces. We also investigate ideals and study the properties of mixed lattice group homomorphisms and quotient groups. Most of the results in this paper have their analogues in the theory of Riesz spaces.

Keywords Mixed lattice · Mixed lattice group · Mixed lattice semigroup · Riesz space · Lattice ordered group · Absolute value · Ideal

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1 Introduction

The theory of mixed lattice semigroups was developed by Arsove and Leutwiler in [4–7]. Their aim was to develop an axiomatic approach to classical potential theory from a purely algebraic point of view. Their theory unified and generalized some of the earlier axiomatizations of potential theory, such as the theory of cones of potentials, developed by Mokobodzki and Sibony (see [7] and references therein), and the theory of H-cones developed by the Romanian group of mathematicians, Boboc, Bucur and Cornea [9]. These theories can be regarded as special cases of the Arsove–Leutwiler theory. Since the properties of mixed lattice semigroups had been extensively studied, it seemed natural to investigate the mixed lattice structure in the group setting. This was done by Eriksson–Bique in [10,11]. Group extensions of mixed lattice semigroups were considered in [10], where it was shown that the mixed lattice structure is indeed preserved in such extensions. A more general definition of a mixed lattice group was given in [11], and some of the basic properties of mixed lattice groups were derived there. However, after the publication of the aforementioned papers the research on the subject has been virtually nonexistent. The purpose of the present paper is to initiate a systematic study of mixed lattice groups.

A mixed lattice group is a partially ordered Abelian group with two partial orderings. The usual supremum and infimum of two elements is replaced by unsymmetrical mixed envelopes which are formed with respect to the two partial orderings. It is then required that these mixed upper and lower envelopes exist for every pair of elements. If the two partial orderings are identical, then the mixed upper and lower envelopes become the usual supremum and infimum, and the mixed lattice group reduces to an ordinary lattice ordered group. In this sense, a mixed lattice group is a generalization of a lattice ordered group. However, due to the unsymmetrical nature of the mixed envelopes, some familiar properties of lattice ordered groups (such as the distributive laws) no longer hold in a mixed lattice group.

Since we are considering a generalization of lattice ordered groups and Riesz spaces, it is natural to begin our study by considering some of the most fundamental concepts in these structures, such as the absolute value and ideals, and examine how these concepts could be carried over to the theory of mixed lattice structures. In Sect. 2 of the present paper we give the essential definitions and results that will be needed in the subsequent sections. These include the concept of a *mixed lattice vector space* which has not been studied before. We also introduce a classification of mixed lattice groups based on their algebraic structure, and present some examples to illustrate the relationships between different classes of mixed lattice groups. Roughly speaking, these so-called regularity classes describe how closely the set of positive elements resembles a mixed lattice semigroup. The importance of this classification becomes apparent as we develop our theory further.

In Sect. 3 we define the generalized absolute values and derive their basic properties. The generalized absolute values are defined in terms of upper and lower parts, which are analogous to the positive and negative parts in the theory of Riesz spaces, but now we have two partial orderings which leads to two distinct upper and lower parts for each element. The fundamental identities as well as the triangle inequalities for the generalized absolute values are given in Theorems 3.2, 3.4 and 3.5. In addition to these we give some other interesting results, such as Theorem 3.6, which is the mixed lattice version of the representation of an element as a difference of two disjoint elements, a well known result in Riesz space theory. To finish Sect. 3, we study the behavior of mixed envelopes under scalar multiplication in vector spaces and the relationship between the ordinary absolute value and the generalized absolute values.

The related concept of *mixed lattice ideal* is introduced in Sect. 4, and some basic facts concerning ideals will be presented there. The definitions of ideals are based on the notion of order convex sets. We give characterizations of mixed lattice ideals and examine their relationship with lattice ideals.

In Sect. 5 we study homomorphisms between mixed lattice groups. First we give the fundamental characterization of mixed lattice homomorphisms (Theorem 5.2) and show that a mixed lattice homomorphism preserves the regularity properties of its domain (Theorem 5.8). We study the kernel of a mixed lattice homomorphism and show that the kernel has similar ideal properties as the kernels of Riesz homomorphisms between two Riesz spaces (Theorem 5.9). Theorem 5.10 characterizes mixed lattice isomorphisms, and in Theorem 5.13 we show that a mixed lattice semigroup homomorphism can be extended to a mixed lattice group homomorphism. The closely related idea of a quotient group is discussed in Sect. 6, where we give necessary conditions for the quotient group to be a mixed lattice group (Theorem 6.2).

Most of the concepts and results in the present paper have their counterparts in the theory of Riesz spaces and lattice ordered groups. For a comprehensive treatment of these topics, we refer to [8,12,15,16]. We should point out that although the proofs of some results in this paper may appear to be similar to their counterparts in Riesz spaces, the algebraic structure of mixed lattice groups is different than the structure of Riesz spaces. In fact, many well known fundamental properties of Riesz spaces such as the commutativity of the lattice operations no longer hold in mixed lattice groups. To emphasize these differences, we will give references on Riesz spaces and lattice ordered groups at many places. Moreover, since this is a rather unexplored topic, we try to give fairly complete proofs for most results. It is our hope that this paper will serve as a good starting point for future research on mixed lattice groups and mixed lattice vector spaces.

2 Mixed lattice semigroups and groups

We begin by stating the definitions of the basic structures. Let $(\mathcal{S}, +, \leq)$ be a positive partially ordered Abelian semigroup with zero element. That is, we require that $u \geq 0$ for all $u \in \mathcal{S}$, and $u \leq v$ implies $u + w \leq v + w$. In what follows, we will also assume that the cancellation law

$$u + w \leq v + w \quad \Longrightarrow \quad u \leq v \tag{2.1}$$

holds in $\mathcal{S}$. This partial order $\leq$ is called the *initial order*. It follows from the assumptions that the initial order has the following property.

$$a \leq u, \quad b \leq v \quad \text{and} \quad a + b = u + v \quad \Longrightarrow \quad a = u \quad \text{and} \quad b = v \tag{2.2}$$

Indeed, if the above assumptions hold then $u + v = a + b \le a + v$, and so by (2.1) we have $u \le a$. But by assumption $a \le u$, so $u = a$. A similar argument shows that $v = b$ and (2.2) follows.

The semigroup $\mathcal{S}$ itself gives rise to another partial ordering. This intrinsic order on $\mathcal{S}$ is called the *specific order*, and it is defined by

$$u \preccurlyeq v \quad \Longleftrightarrow \quad v = u + w \quad \text{for some } w \in \mathcal{S}. \tag{2.3}$$

Clearly, (2.3) means that $u \succcurlyeq 0$ for all $u \in \mathcal{S}$ (since $u = u + 0$). Hence, if we formally use the difference notation $v - u$ to denote the unique element that satisfies $u + (v - u) = v$ (whenever such an element exists), then the above definition is equivalent to

$$u \preccurlyeq v \quad \Longleftrightarrow \quad \text{The difference element } v - u \text{ exists in } \mathcal{S},$$

but traditionally definition (2.3) is used for the specific order since there are no negative elements in $\mathcal{S}$. Note that (2.3) does not hold for the initial order, in general.

We should still verify that $\preccurlyeq$ is a partial ordering which is compatible with the semigroup structure and satisfies (2.1). It is easy to see that $\preccurlyeq$ is reflexive and transitive. To prove antisymmetry, assume $u \preccurlyeq v$ and $v \preccurlyeq u$. Then $u = v + v'$ and $v = u + u'$ for some u', $v' \in \mathcal{S}$. Adding v' to the last equality gives $u = v + v' = u + u' + v'$. As $u' + v' \ge 0$, it follows by (2.2) that $u' + v' = 0$. But this implies that $u' = v' = 0$, again by (2.2). Hence, $u = v$. This proves that $\preccurlyeq$ is a partial order. Next, if $u \preccurlyeq v$ then $v = u + u'$ for some $u' \in \mathcal{S}$. It follows that $v + w = u + w + u'$ for any element $w \in \mathcal{S}$, and so $u + w \preccurlyeq v + w$. Conversely, if $u + w \preccurlyeq v + w$, then $v + w = u + w + u'$ for some $u' \in \mathcal{S}$. Since equality is just a special case of the relation $\le$, it follows by (2.1) that $v = u + u'$, or equivalently $u \preccurlyeq v$. Hence, the specific order $\preccurlyeq$ is compatible with the semigroup structure and has the cancellation property (2.1).

If we equip the semigroup $\mathcal{S}$ with the partial order $\preccurlyeq$, then $(\mathcal{S}, +, \le, \preccurlyeq)$ is a positive partially ordered semigroup with respect to two partial orderings $\le$ and $\preccurlyeq$, both satisfying (2.1). Notice that $u \preccurlyeq v$ implies $u \le v$ in $\mathcal{S}$, but not conversely. Moreover, it follows from positivity and (2.2) that no nonzero element of $\mathcal{S}$ has an inverse in $\mathcal{S}$. In particular, $\mathcal{S}$ cannot be a group, unless $\mathcal{S} = \{0\}$.

With these two partial orders $\le$ and $\preccurlyeq$ we define the *mixed lower and upper envelopes*

$$u \curlywedge v = \max\{w \in \mathcal{S} : w \preccurlyeq u \text{ and } w \le v\}$$

and

$$u \curlyvee v = \min\{w \in \mathcal{S} : w \succcurlyeq u \text{ and } w \ge v\},$$

respectively, where the minimum and maximum are taken with respect to the initial order $\le$, whenever they exist. The following definition was given by Arsove and Leutwiler in [7].

Definition 2.1 Let $(\mathcal{S}, +, \leq, \preccurlyeq)$ be a partially ordered Abelian semigroup with zero element and two partial orders $\leq$ and $\preccurlyeq$ such that $u \geq 0$ for all $u \in \mathcal{S}$ and (2.1) holds in $\mathcal{S}$, and $\preccurlyeq$ is given by (2.3). If the mixed upper and lower envelopes $u \curlyvee v$ and $u \curlywedge v$ exist in $\mathcal{S}$ for all $u, v \in \mathcal{S}$, and they satisfy the identity

$$u \curlyvee v + v \curlywedge u = u + v$$

then $(\mathcal{S}, +, \leq, \preccurlyeq)$ is called a *mixed lattice semigroup*.

Remark 2.2 There are other equivalent formulations for the definition of a mixed lattice semigroup given in [7]. We have chosen to use the above definition in this paper, as it is the most convenient for our purposes. We do not discuss the other possible formulations here, but we would like to remark that the additional identity in Definition 2.1 does not follow from the other assumptions and it is needed in order to make these different formulations equivalent. The existence of the mixed envelopes is not sufficient for that, and it is possible to construct a semigroup where all the other assumptions hold except the last identity in Definition 2.1. The next example clarifies this.

Example 2.3 Consider the semigroup $\mathcal{S} = \{0\} \cup [2, \infty)$ with the usual addition of real numbers. Let $\leq$ be the usual order inherited from $\mathbb{R}$ and define $\preccurlyeq$ as the partial order induced by $\mathcal{S}$ itself, that is, $y \succcurlyeq x$ if and only if $y = x$ or $y \geq x + 2$. Note that this is equivalent to $y = x + w$ for some $w \in \mathcal{S}$. Clearly, $\leq$ and $\preccurlyeq$ are partial orderings that are compatible with the semigroup structure of $\mathcal{S}$ and satisfy condition (2.1). Moreover, the elements $x \curlyvee y$ and $x \curlywedge y$ exist in $\mathcal{S}$ for all $x, y \in \mathcal{S}$. For example, the element $x \curlywedge y$ is given by

$$x \curlywedge y = \begin{cases} x, & \text{if } x \leq y \\ y, & \text{if } x \geq y + 2 \\ x - 2, & \text{if } y < x < y + 2 \text{ and } x \notin (2, 4) \\ 0, & \text{if } y < x < y + 2 \text{ and } x \in (2, 4) \end{cases}$$

Thus $(\mathcal{S}, +, \leq, \preccurlyeq)$ satisfies all the assumptions of Definition 2.1, except that the identity $x \curlyvee y + y \curlywedge x = x + y$ does not hold in $\mathcal{S}$. For instance, $2 \curlyvee 3 + 3 \curlywedge 2 = 4 + 0 \neq 2 + 3$. Hence $\mathcal{S}$ is not a mixed lattice semigroup.

Remark 2.4 The ideas introduced so far can be given a more informal description. Loosely speaking, the construction of a mixed lattice semigroup is as follows. We start with some "initial" partially ordered Abelian semigroup $(\mathcal{S}_0, +, \leq)$ with zero element (hence the term *initial order*), assuming that $(\mathcal{S}_0, +, \leq)$ is positive with respect to $\leq$, and the cancellation property (2.1) holds. Then we take some specific subsemigroup $\mathcal{S}$ of $\mathcal{S}_0$ and define the specific order as the partial order determined by $\mathcal{S}$. Thus we end up with a semigroup $(\mathcal{S}, +, \leq, \preccurlyeq)$ with two partial orders, which is then a mixed lattice semigroup, provided that all the assumptions regarding the mixed envelopes are satisfied. In this process, some elements of the initial semigroup $\mathcal{S}_0$ are "left out" and the semigroup $\mathcal{S}$ does not contain all the original elements that are positive with respect to the initial order $\leq$, but it consists precisely of those elements that are positive with respect to the specific order $\preccurlyeq$. Many of the examples given in this paper are also constructed in this manner.

Now we come to the topic of mixed lattice groups rather than semigroups. The following definition of a mixed lattice group was given by Eriksson–Bique in [11].

Definition 2.5 Let $(\mathcal{G}, +, \leq, \preccurlyeq)$ be a partially ordered Abelian group with respect to two partial orders $\leq$ and $\preccurlyeq$. If the mixed lower envelope $x \curlywedge y$ exists for all $x, y \in \mathcal{G}$, then $(\mathcal{G}, +, \leq, \preccurlyeq)$ is called a *mixed lattice group*.

We note that the above definition is not as restrictive as the definition of a mixed lattice semigroup. Most importantly, the two partial orders are not required to be related in any way. Definition 2.5 also clearly shows that a lattice ordered group is a special case of a mixed lattice group. Indeed, if the two partial orderings $\leq$ and $\preccurlyeq$ are chosen to be identical, then the mixed lattice group reduces to an ordinary lattice ordered group.

Eriksson–Bique [10] has also shown that every mixed lattice semigroup $\mathcal{S}$ can be extended to a group of formal differences of elements of $\mathcal{S}$ and the mixed lattice structure is preserved in this extension. Many important examples of mixed lattice groups arise in this way. We will now discuss this construction in more detail.

Let $\mathcal{S}$ be a mixed lattice semigroup and define an equivalence relation on $\mathcal{S} \times \mathcal{S}$ by putting

$$(u, v) \sim (x, y) \qquad \Longleftrightarrow \qquad u + y = v + x.$$

The equivalence class generated by (u, v) is denoted by $[(u, v)]$ and the set of all equivalence classes by $[(\mathcal{S}, \mathcal{S})]$. We define addition in $[(\mathcal{S}, \mathcal{S})]$ by

$$[(u, v)] + [(x, y)] = [(u + x, v + y)].$$

Now the zero element of $[(\mathcal{S}, \mathcal{S})]$ is $[(0, 0)]$ and $[(u, 0)] + [(0, u)] = [(0, 0)]$, and so $[(u, 0)] = -[(0, u)]$ for all $u \in \mathcal{S}$. Thus, $[(\mathcal{S}, \mathcal{S})]$ is a group. Next we define partial ordering by

$$[(u, v)] \leq [(x, y)] \qquad \Longleftrightarrow \qquad u + y \leq x + v.$$

It is not difficult to show that the addition and partial ordering are well-defined. The above definitions imply that the property

$$[(u, v)] \leq [(x, y)] \qquad \Longrightarrow \qquad [(u, v)] + [(a, b)] \leq [(x, y)] + [(a, b)]$$

holds in $[(\mathcal{S}, \mathcal{S})]$, and hence $[(\mathcal{S}, \mathcal{S})]$ is a partially ordered group. Moreover, We observe that the sets $\mathcal{S}$ and $\{[(u, 0)] : u \in \mathcal{S}\}$ are isomorphic, so we identify $[(u, 0)]$ with u, and similarly we identify $[(0, u)]$ with $-u$. We now denote this group by $[(\mathcal{S}, \mathcal{S})] = \mathcal{S} - \mathcal{S}$ and we call $\mathcal{S} - \mathcal{S}$ the *group of formal differences* of elements of $\mathcal{S}$.

The specific order in $\mathcal{S} - \mathcal{S}$ is defined as

$$u - v \preccurlyeq x - y \quad \text{in } \mathcal{S} - \mathcal{S} \qquad \Longleftrightarrow \qquad u + y \preccurlyeq x + v \quad \text{in } \mathcal{S}.$$

The following results are due to Eriksson–Bique.

Lemma 2.6 ([10, Lemma 3.1]) *The specific order in $\mathcal{S}-\mathcal{S}$ defined above is a partial order that is well-defined and compatible with the group structure of $\mathcal{S}-\mathcal{S}$. Moreover, it has the property*

$$u-v \preccurlyeq x-y \qquad \Longleftrightarrow \qquad x-y=u-v+w \quad \textit{for some } w \in \mathcal{S},$$

for $u, v, x, y \in \mathcal{S}$.

Theorem 2.7 ([10, Theorem 3.2]) *Let $\mathcal{S}$ be a mixed lattice semigroup and $u_1, u_2, v_1, v_2 \in \mathcal{S}$. Then the mixed envelopes of elements u_1-u_2 and v_1-v_2 in $\mathcal{S}-\mathcal{S}$ exist and they are given by*

$$(u_1-u_2) \curlywedge (v_1-v_2)=(u_1+v_2) \curlywedge (v_1+u_2)-(u_2+v_2)$$

and

$$(u_1-u_2) \curlyvee (v_1-v_2)=(u_1+v_2) \curlyvee (v_1+u_2)-(u_2+v_2).$$

In particular, $\mathcal{S}-\mathcal{S}$ is a mixed lattice group.

Let $\mathcal{G}$ be a mixed lattice group. The definition of the mixed envelopes implies that the inequalities

$$x \curlywedge y \preccurlyeq x \preccurlyeq x \curlyvee y \quad \text{and} \quad x \curlywedge y \leq y \leq x \curlyvee y$$

hold for all $x, y \in \mathcal{G}$. In the sequel, we will use these inequalities without further reference. There are several other identities and calculation rules that we will apply frequently. For easy reference, we list all these fundamental properties of mixed lattice groups below. The following properties hold for all elements x, y and z in an arbitrary mixed lattice group.

$(P1)$ $\quad x \curlyvee y+y \curlywedge x=x+y$

$(P2a)$ $\; z+x \curlyvee y=(x+z) \curlyvee (y+z)$

$(P2b)$ $\; z+x \curlywedge y=(x+z) \curlywedge (y+z)$

$(P3)$ $\quad x \curlyvee y=-(-x \curlywedge -y)$

$(P4)$ $\quad x \preccurlyeq y \iff x \curlyvee y=y \iff y \curlywedge x=x$

$(P5a)$ $\; x \preccurlyeq u \;$ and $\; y \leq v \implies x \curlyvee y \leq u \curlyvee v \;$ and $\; x \curlywedge y \leq u \curlywedge v$

$(P5b)$ $\; x \preccurlyeq z \;$ and $\; y \leq z \implies x \curlyvee y \leq z \;$ and $\; x \curlywedge y \leq z$

Identities (P1), (P2a), (P2b) and (P3) were proved in [11, Lemma 3.1]. Note, in particular, that in a mixed lattice group (P1) is a consequence of the definitions of the mixed envelopes and the existence of negative elements, while in the definition of mixed lattice semigroup (P1) must be included as an axiom. Properties (P4), (P5a) and (P5b) are easy consequences of the definition of the mixed envelopes.

A subset $\mathcal{S}$ of $\mathcal{G}$ is said to be a *mixed lattice subsemigroup* in $\mathcal{G}$ if $\mathcal{S}$ is a mixed lattice semigroup and the mixed envelopes $x \curlyvee y$ and $x \curlywedge y$ in $\mathcal{S}$ are the same as in $\mathcal{G}$ for all $x, y \in \mathcal{S}$. It follows from the definition of the mixed lattice semigroup that if the semigroup $\mathcal{U} = \{w \in \mathcal{G} : w \succcurlyeq 0\}$ is a mixed lattice subsemigroup in $\mathcal{G}$ then it is the largest (with respect to set inclusion) mixed lattice subsemigroup in $\mathcal{G}$.

In addition to properties (P1)–(P5), there are the following ”regularity properties” which require certain additional assumptions that will be discussed below.

$$
\begin{array}{ll}
(R0) & x \leq y \iff y \curlyvee x = y \iff x \curlywedge y = x \\
(R1) & x \preccurlyeq y \implies x \leq y \\
(R2a) & x \preccurlyeq z \;\text{ and }\; y \preccurlyeq z \implies x \curlyvee y \preccurlyeq z \\
(R2b) & z \preccurlyeq x \;\text{ and }\; z \preccurlyeq y \implies z \preccurlyeq x \curlywedge y
\end{array}
$$

We begin with properties (R2a) and (R2b), which have the following characterization.

Theorem 2.8 ([11, Theorem 3.5]) *Let $\mathcal{G}$ be a mixed lattice group and let $\mathcal{U} = \{w \in \mathcal{G} : w \succcurlyeq 0\}$. Then the following conditions are equivalent.*

(a) $\mathcal{U}$ is a mixed lattice subsemigroup of $\mathcal{G}$.
(b) $x \curlywedge y \succcurlyeq 0$ for all $x, y \in \mathcal{U}$, where $\curlywedge$ is the mixed lower envelope in $\mathcal{G}$.
(c) $\mathcal{G}$ has the properties (R2a) and (R2b).

Property (R1) holds in every mixed lattice semigroup, where it is a simple consequence of the definition of specific order. As we develop our theory further in the following sections, it turns out that this property is quite important in many situations. We will show below that (R0) is implied by (R1). Moreover, (R2a) and (R2b) are also essential for certain results and, in many cases, a mixed lattice group can be viewed as a group extension of some mixed lattice semigroup. Due to the importance of these properties, we introduce the following classification of mixed lattice groups.

Definition 2.9 Let $\mathcal{G}$ be a mixed lattice group.

(i) $\mathcal{G}$ is called *pre-regular* if $x \preccurlyeq y$ implies that $x \leq y$ in $\mathcal{G}$.
(ii) $\mathcal{G}$ is called *quasi-regular* if the set $\mathcal{S} = \{w \in \mathcal{G} : w \succcurlyeq 0\}$ is a mixed lattice subsemigroup in $\mathcal{G}$.
(iii) $\mathcal{G}$ is called *regular* if it is quasi-regular and the set $\mathcal{S}$ defined above is generating, that is, $\mathcal{G} = \mathcal{S} - \mathcal{S}$.
(iv) A mixed lattice group that is not pre-regular is called *irregular*.

By Theorem 2.8, $\mathcal{G}$ is quasi-regular if and only if the properties (R2a) and (R2b) hold in $\mathcal{G}$. It is now a simple matter to show that every regular mixed lattice group is quasi-regular, and every quasi-regular mixed lattice group is pre-regular. Moreover, pre-regularity implies property (R0). The converse implications do not hold in general (see Examples 2.17 through 2.23 below).

Theorem 2.10 *The following hold.*

(a) Every regular mixed lattice group is quasi-regular.
(b) Every quasi-regular mixed lattice group is pre-regular.
(c) Every pre-regular mixed lattice group has the property (R0).

Proof *(a)* Obvious.

(b) Assume that $\mathcal{G}$ is quasi-regular and $x \succcurlyeq y$. Then $x - y \succcurlyeq 0$, and the set $\{w \in \mathcal{G} : w \succcurlyeq 0\}$ is a mixed lattice semigroup, where (R1) holds. So we have $x - y \geq 0$, or $x \geq y$. Hence, $\mathcal{G}$ is pre-regular.

(c) Assume first that $x \leq y$. Since $y \preccurlyeq y$, by (P5) we have $y \curlyvee x \leq y \curlyvee y = y$, and so by (P1) we get $x + y = x \curlywedge y + y \curlyvee x \leq x \curlywedge y + y$. This implies that $x \leq x \curlywedge y$. On the other hand, we have $x \succcurlyeq x \curlywedge y$, and so by (R1) we have $x \geq x \curlywedge y$. Hence $x = x \curlywedge y$. Conversely, if $x = x \curlywedge y$ then $x \leq y$. For the other equivalence, note that if $x \curlywedge y = x$ then the identity $x + y = x \curlywedge y + y \curlyvee x$ implies that $y \curlyvee x = y$. The reverse implication is proved similarly. □

The foregoing theory can also be formulated in vector spaces. For this, recall that $(\mathcal{V}, \leq)$ is a *partially ordered vector space* if

$$u \leq v \implies u + w \leq v + w \quad \text{and} \quad u \leq v \implies \alpha u \leq \alpha v$$

hold for all $u, v, w \in \mathcal{V}$ and $\alpha \geq 0$. We now define a mixed lattice vector space by introducing another partial ordering and requiring that the mixed envelopes exist for every pair of elements. In this paper, we only consider real vector spaces.

Definition 2.11 Let $(\mathcal{V}, \leq, \preccurlyeq)$ be a partially ordered real vector space with respect to two partial orders $\leq$ and $\preccurlyeq$. If the mixed upper and lower envelopes $x \curlywedge y$ and $x \curlyvee y$ exist for all $x, y \in \mathcal{V}$, then $(\mathcal{V}, \leq, \preccurlyeq)$ is called a *mixed lattice vector space*. The partial orderings $\leq$ and $\preccurlyeq$ are called initial order and specific order, respectively.

Definition 2.11 clearly implies that a Riesz space is a special case of a mixed lattice vector space. Indeed, if the two partial orderings $\leq$ and $\preccurlyeq$ coincide, then the mixed upper and lower envelopes become the usual supremum and infimum, and the mixed lattice vector space reduces to an ordinary Riesz space. Thus every Riesz space is a mixed lattice vector space, and every mixed lattice vector space is an ordered vector space, but the converse inclusions do not hold. In this sense, the concept of a mixed lattice vector space is intermediate between a Riesz space and a general ordered vector space.

Most results in this paper are given in the more general group setting and the proofs can be translated to vector spaces with no difficulty. In fact, many of our results do not depend on the scalar multiplication at all. Therefore, we shall formulate our results in mixed lattice vector spaces only if they are not applicable to the group setting.

Before discussing the examples, we give a useful result that helps to determine whether a given structure is actually a mixed lattice group. The next theorem is an extension of [11, Theorem 3.3], in which the equivalence *(a)* $\iff$ *(e)* was proved.

Theorem 2.12 *Let $\mathcal{G}$ be a partially ordered group with two partial orderings $\leq$ and $\preccurlyeq$. Then the following conditions are equivalent:*

(a) $\mathcal{G}$ *is a mixed lattice group.*
(b) The mixed lower envelope $0 \curlywedge x$ *exists for all* $x \in \mathcal{G}$.
(c) The mixed upper envelope $0 \curlyvee x$ *exists for all* $x \in \mathcal{G}$.
(d) The mixed lower envelope $x \curlywedge 0$ *exists for all* $x \in \mathcal{G}$.
(e) The mixed upper envelope $x \curlyvee 0$ *exists for all* $x \in \mathcal{G}$.

Proof The equivalence *(a)* $\Longleftrightarrow$ *(e)* was proved in [11, Theorem 3.3]. The other equivalences *(a)* $\Longleftrightarrow$ *(b)*, *(a)* $\Longleftrightarrow$ *(c)*, and *(a)* $\Longleftrightarrow$ *(d)* are proved similarly. □

We now discuss several examples of mixed lattice structures. The first one is due to Arsove and Leutwiler [7].

Example 2.13 The historical background of mixed lattice semigroups is in potential theory, and an archetypical example of a mixed lattice semigroup is the set $\mathcal{U}$ of positive superharmonic functions on some region D in $\mathbb{R}^n$. Here the initial order is defined pointwise, that is, $f \le g$ if $f(x) \le g(x)$ for almost every $x \in D$ and the specific order is defined by $f \preccurlyeq g$ if $f \le g$ and $g - f$ is superharmonic on D. Then $(\mathcal{U}, \le, \preccurlyeq)$ is a mixed lattice semigroup [7, Theorem 21.2].

A special case of positive superharmonic functions is the semigroup of positive concave real functions on some interval $[a, b]$. Hence, the set of those functions that can be written as a difference of two positive concave functions is a mixed lattice vector space. Differences of concave functions typically arise in certain optimization problems.

The next example is from probability theory, and it was also given by Arsove and Leutwiler [7, pp. 126–127].

Example 2.14 A *sub-stochastic matrix* is a (possibly infinite) non-negative matrix $P = (p_{ij})$ such that the elements of P satisfy the condition $\sum_j p_{ij} \le 1$ for all i. Vectors are ordered in the usual way, that is, $x \le y$ if $x_i \le y_i$ for all i. A vector $x \ge 0$ is called *excessive* with respect to the sub-stochastic matrix P if $Px \le x$. Define the specific order by $x \preccurlyeq y$ if $x \le y$ and $P(y - x) \le y - x$, or $(I - P)(y - x) \ge 0$. For a given sub-stochastic matrix P, let $\mathcal{U}$ be the set of all bounded vectors that are excessive with respect to P. Then $(\mathcal{U}, \le, \preccurlyeq)$ is a mixed lattice semigroup.

The dual concept of *excessive measure* is defined as a non-negative row vector x^T such that $x^T P \le x^T$. The set of finite measures that are excessive with respect to P is similarly a mixed lattice semigroup.

More generally, it has been shown that the cones of excessive functions and excessive measures associated with Markov processes with continuous state-space have a mixed lattice structure. For proofs and more details, see [13, Theorems 5.9 and 5.11]

The following example of a regular mixed lattice group (indeed, a mixed lattice vector space) was given in [11, Example 1].

Example 2.15 Let $\mathcal{G} = BV([a, b])$ be the set of all functions of bounded variation on an interval $[a, b]$. Functions of bounded variation have the following well known characterization (see for example [3, Theorem 6.13]): $f \in BV([a, b])$ if and only if

$f = f_1 - f_2$, where f_1 and f_2 are positive increasing functions. It was shown in [7, Theorem 21.2], that the set $\mathcal{S}$ of all positive increasing functions is a mixed lattice semigroup. Hence, we can write $\mathcal{G} = \mathcal{S} - \mathcal{S}$ and so $\mathcal{G}$ is a mixed lattice group where the initial and specific orders are defined by

$$f \leq g \qquad \Longleftrightarrow \qquad f(x) \leq g(x) \quad \text{for all } x \in [a, b]$$

and

$$f \preccurlyeq g \qquad \Longleftrightarrow \qquad g - f \text{ is increasing and } g - f \geq 0,$$

respectively. It is well known that the space $BV([a, b])$ is a Riesz space with respect to both partial orders defined above (see [1, Section 9.8]). So, in a sense, this mixed lattice vector space is a "mixture" of the two Riesz spaces $(BV([a, b]), \leq)$ and $(BV([a, b]), \preccurlyeq)$.

Next we consider different positive cones in $\mathbb{R}^n$ to obtain more examples of mixed lattice vector spaces.

Example 2.16 Let $\mathcal{V} = (\mathbb{R}^n, \leq, \preccurlyeq)$ where $\leq$ is given by the standard positive cone, that is, $x \geq 0$ if $x_k \geq 0$ for all $k = 1, 2, \ldots, n$ and define $\preccurlyeq$ by the positive cone $C = \{x : x_{k+1} \geq x_k \geq 0, k = 1, 2, \ldots, n - 1\}$. Then the cone C is generating and the element $u = 0 \curlyvee x$ exists for every $x \in \mathcal{V}$ and hence $\mathcal{V}$ is a regular mixed lattice vector space. In fact, u is a vector such that

$$u_1 = \begin{cases} x_1 & \text{if } x_1 \geq 0 \\ 0 & \text{if } x_1 < 0 \end{cases} \qquad \text{and} \quad u_{k+1} = \begin{cases} x_{k+1} & \text{if } x_{k+1} \geq u_k \\ u_k & \text{if } x_{k+1} < u_k \end{cases}$$

for $k = 1, 2, \ldots, n - 1$. This example applies also for the infinite dimensional sequence space consisting of those sequences (x_k), where x_k is defined as above for $k = 1, 2, 3, \ldots$. In this case, the definition of the specific order can be written as $x \succcurlyeq 0$ if $x \geq 0$ and x is increasing.

In the sequel, we denote the supremum and infimum formed with respect to initial order by sup and inf, respectively (whenever they exist), and the supremum and infimum formed with respect to specific order by sp sup and sp inf, respectively (whenever they exist). Our next examples are mixed lattice vector spaces that are quasi-regular but not regular.

Example 2.17 Let $\mathcal{V} = \mathbb{R}^3$ and define $\leq$ as the partial order induced by the positive cone $C_1 = \{(x, y, z) : z \geq \sqrt{x^2 + y^2}\}$. Define specific order $\preccurlyeq$ as the partial order with the positive cone $C_2 = \{(x, y, z) : x = y = 0, z \geq 0\}$. Geometrically it is obvious that the element $x \curlyvee 0$ exists for all $x \in \mathbb{R}^3$, since the cone C_1 and any line parallel to z-axis always intersect. Moreover, the two partial orders $\leq$ and $\preccurlyeq$ coincide on the z-axis and hence the set $\{x \in \mathcal{G} : x \succcurlyeq 0\}$ can be identified with the set of positive real numbers with the usual ordering. This set is obviously a very special case of a mixed lattice semigroup. Thus $(\mathcal{V}, \leq, \preccurlyeq)$ is a mixed lattice group (in fact, a mixed lattice vector space) which is quasi-regular, but not regular since the specific cone C_2 is not generating. This example also shows that a mixed lattice group need not be a lattice with respect to either partial ordering. Again, this is easy to see geometrically for partial order $\leq$, since the intersection of two circular cones of type C_1 is not necessarily a circular cone and thus the element $\sup(x, y)$ does not exists if the elements x and y are not comparable. $\mathcal{G}$ is also not a lattice with respect to $\preccurlyeq$, since the element sp $\sup\{x, y\}$ does not exist, unless the points x and y both lie on the same line parallel to the z-axis.

Example 2.18 Let $\mathcal{V}$ be the vector space of all bounded differentiable real functions on an interval (a, b) and define the initial order $\leq$ as the usual pointwise ordering, and specific order by $f \preccurlyeq g$ if $g - f \geq 0$ and $g - f$ is a constant function. Let $f \in \mathcal{V}$ and put $m = \inf\{f(x) : x \in (a, b)\}$ and $m_f = \min\{0, m\}$. Then $g = f \curlyvee 0$ exists and $g(x) = f(x) - m_f$ for all $x \in (a, b)$. Clearly, $g \in \mathcal{V}$ and so $(\mathcal{V}, \leq, \preccurlyeq)$ is a mixed lattice vector space that is quasi-regular but not regular since the specific positive cone $C_s = \{f \in \mathcal{V} : f \geq 0 \text{ and } f \text{ is constant}\}$ is not generating.

Example 2.19 Let $\mathcal{G}$ be the set of real $n \times n$ matrices and let $\mathcal{B}$ be the class of all symmetric $n \times n$ matrices. We define order relations $A \leq B$ if $a_{ij} \leq b_{ij}$ for all $i, j = 1, \ldots, n$ and $A \preccurlyeq B$ if $A \leq B$ and $B - A \in \mathcal{B}$. Then $C = 0 \curlyvee A$ exists for every $A \in \mathcal{G}$ and the elements of C are given by $c_{ij} = \max\{0, a_{ij}, a_{ji}\}$. Hence, $\mathcal{G}$ is a mixed lattice group. $\mathcal{G}$ is easily seen to be quasi-regular (by the condition (b) of Theorem 2.8) but not regular since $\mathcal{B}$ does not generate $\mathcal{G}$.

Note that $\mathcal{B}$ could be replaced by some other suitable class of matrices, such as integer matrices. This choice for $\mathcal{B}$ yields a different mixed lattice group.

Example 2.20 Let $\mathcal{V}$ be the space of continuous real functions on the interval $[-1, 1]$. Let $f \leq g$ be the usual pointwise ordering and $f \preccurlyeq g$ if $f \leq g$ and $g - f$ is an even function. Then $h = 0 \curlyvee f$ exists in $\mathcal{V}$ for every $f \in \mathcal{V}$, and $h(x) = \max\{0, f(x), f(-x)\}$. Hence, $(\mathcal{V}, \leq, \preccurlyeq)$ is a mixed lattice vector space.

The following two examples are due to Eriksson–Bique [11].

Example 2.21 Consider the partially ordered group $(\mathbb{Z}, \leq, \preccurlyeq)$, where $\leq$ is the usual order. Let p be a strictly positive integer and define specific order $\preccurlyeq$ by setting

$$n \preccurlyeq m \quad \text{if } m - n \geq 0 \text{ and } m - n \text{ is divisible by } p.$$

Then $(\mathbb{Z}, \leq, \preccurlyeq)$ is a quasi-regular mixed lattice group [11, Example 2].

Example 2.22 The set $\mathbb{R}$ of real numbers is a mixed lattice group with $\leq$ as the usual order and specific order defined by $x \preccurlyeq y$ if $y - x \geq 0$ and $y - x \in \mathbb{Z}$. Now, for example, if $n \in \mathbb{Z}$ then

$$n \curlywedge x = \begin{cases} n & \text{if } n \leq x \\ \lfloor x \rfloor & \text{if } n > x \end{cases}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. [11, Example 3]

Note that the above construction could also be applied to $\mathbb{Q}$ or any subfield of $\mathbb{R}$ to yield a mixed lattice group.

Next we give an example of a pre-regular mixed lattice group that is not quasi-regular.

Example 2.23 Let $\mathcal{G} = (\mathbb{Z} \times \mathbb{Z}, \leq, \preccurlyeq)$ and define partial orders $\leq$ and $\preccurlyeq$ as follows. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ then $x \leq y$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$. In addition, $x \preccurlyeq y$ iff $x = y$ or $y_1 \geq x_1 + 1$ and $y_2 \geq x_2 + 1$. Using Theorem 2.12 it is easy to check that $\mathcal{G}$ is a mixed lattice group which is clearly pre-regular. Now let $x = (1, 1)$ and $y = (1, 2)$, and let $W = \{w \in \mathcal{G} : w \succcurlyeq 0\}$. Then $x \in W$ and $y \in W$ but $y \curlywedge x = (0, 1) \notin W$. By Theorem 2.8 the set W is not a mixed lattice subsemigroup of $\mathcal{G}$ and so $\mathcal{G}$ is not quasi-regular.

Remark 2.24 There is an interesting problem related to the previous example and the regularity properties of mixed lattice vector spaces. The above counterexample was based on the fact that the set of positive elements in a mixed lattice group is not necessarily convex. However, in an ordered vector space the positive cone is always convex, so this raises the following question: Does there exist a pre-regular mixed lattice vector space that is not quasi-regular? A negative answer to this question would result in a somewhat simplified theory for mixed lattice vector spaces.

The next example presents a mixed lattice group which is irregular, and does not have properties (R0), (R1) and (R2).

Example 2.25 Let $\mathcal{G} = (\mathbb{Z} \times \mathbb{Z}, \leq, \preccurlyeq)$, where $\leq$ and $\preccurlyeq$ are lexicographic orders defined as

$$x \leq y \quad \Longleftrightarrow \quad (x_1 \leq y_1 \quad \text{or} \quad x_1 = y_1 \quad \text{and} \quad x_2 \leq y_2\,)$$

and

$$x \preccurlyeq y \quad \Longleftrightarrow \quad (\,x_2 \leq y_2 \quad \text{or} \quad x_2 = y_2 \quad \text{and} \quad x_1 \leq y_1\,).$$

Then $\mathcal{G}$ is a partially ordered group with the usual coordinatewise addition, and since both partial orders are total orders they obviously have the properties

$$x \leq y \iff x + z \leq y + z \quad \text{and} \quad x \preccurlyeq y \iff x + z \preccurlyeq y + z.$$

Clearly, the element $x \curlyvee 0$ exists for every $x \in \mathcal{G}$, and so $\mathcal{G}$ is a mixed lattice group. However, none of the regularity properties (R1)–(R2) hold in $\mathcal{G}$. To see this, let $x = (0, 1)$ and $y = (1, 0)$. Then $x \leq y$ but $x \curlywedge y = y \neq x$. Hence, (R0) does not hold and this implies that (R1) and (R2) do not hold either.

Irregular mixed lattice structures are perhaps a bit too general to be very useful. It turns out that more interesting results can be obtained under the pre-regularity assumption, and in later sections we will mostly focus on the pre-regular case.

3 Generalized absolute values

One of the fundamental concepts in the theory of Riesz spaces and lattice ordered groups is the absolute value of an element. However, in mixed lattice groups this idea is completely unexplored. In this section we introduce the generalization of the absolute value for elements of mixed lattice groups and mixed lattice vector spaces.

We start by introducing the upper and lower parts of an element which play the roles of the positive and negative parts of an element in a lattice ordered group.

Definition 3.1 Let $\mathcal{G}$ be a mixed lattice group and $x \in \mathcal{G}$. The elements ${}^u x = x \curlyvee 0$ and ${}^l x = (-x) \curlyvee 0$ are called the *upper part* and *lower part* of x, respectively. Similarly, the elements $x^u = 0 \curlyvee x$ and $x^l = 0 \curlyvee (-x)$ are called the *specific upper part* and *specific lower part* of x, respectively.

From the above definitions we observe that for the specific upper and lower parts we have $x^u \succcurlyeq 0$ and $x^l \succcurlyeq 0$, and for the upper and lower parts ${}^u x \geq 0$ and ${}^l x \geq 0$. By Theorem 2.12, a partially ordered group $\mathcal{G}$ is a mixed lattice group if and only if one of the upper or lower parts exists for all $x \in \mathcal{G}$.

The upper and lower parts have the following basic properties.

Theorem 3.2 *Let $\mathcal{G}$ be a mixed lattice group and $x \in \mathcal{G}$. Then we have*

(a) ${}^u x = {}^l(-x)$ *and* $x^u = (-x)^l$.
(b) ${}^u x \curlyvee x^l = {}^u x + x^l$ *and* ${}^l x \curlyvee x^u = {}^l x + x^u$
(c) $x = x^u - {}^l x = {}^u x - x^l$
(d) $x^u \curlywedge {}^l x = 0 = x^l \curlywedge {}^u x$
(e) $x^u \curlyvee {}^l x = {}^u x + {}^l x = x^l + x^u = x^l \curlyvee {}^u x$
(f) ${}^u x - {}^l x = x + {}^u x \curlywedge x^l$ *and* $x^u - x^l = x + {}^l x \curlywedge x^u$.

Proof *(a)* These follow immediately from the definitions.

(b) Using (P2a) repeatedly we get

$$\begin{aligned} {}^u x + x^l &= x \curlyvee 0 + 0 \curlyvee (-x) \\ &= (x + 0 \curlyvee (-x)) \curlyvee (0 + 0 \curlyvee (-x)) \\ &= (x \curlyvee 0) \curlyvee (0 \curlyvee (-x)) \\ &= {}^u x \curlyvee x^l. \end{aligned}$$

The proof for the second equality is similar.

(c) Using (P1) and (P3) we get

$${}^u x - x^l = x \curlyvee 0 - 0 \curlyvee (-x) = x \curlyvee 0 + 0 \curlywedge x = x + 0 = x.$$

Similarly, $x^u - {}^l x = x$.

(d) Combining (P1) and (b) gives

$${}^u x \curlyvee x^l + x^l \curlywedge {}^u x = {}^u x + x^l = {}^u x \curlyvee x^l.$$

From this it follows that $x^l \curlywedge {}^u x = 0$. The other equality is similar.

(e) For the first equality, we use (P2a) twice to get

$$\begin{aligned} {}^u x + {}^l x &= x \curlyvee 0 + (-x) \curlyvee 0 \\ &= (x + (-x) \curlyvee 0) \curlyvee (0 + (-x) \curlyvee 0) \\ &= (0 \curlyvee x) \curlyvee ((-x) \curlyvee 0) \\ &= x^u \curlyvee {}^l x. \end{aligned}$$

The last equality is proved similarly. The middle equality follows from (c).

(f) First we note that by (P2a) and (P3)

$$(x^u \curlyvee {}^l x) - x^u - {}^l x = (-{}^l x) \curlyvee (-x^u) = -({}^l x \curlywedge x^u)$$

and

$$(x^l \curlyvee {}^u x) - {}^u x - x^l = (-{}^u x) \curlyvee (-x^l) = -({}^u x \curlywedge x^l).$$

So by adding $-x^u - {}^l x$ and $-{}^u x - x^l$, respectively, to the two equations in (e) we get

$${}^u x \curlywedge x^l = {}^u x - x^u \qquad \text{and} \qquad -({}^l x \curlywedge x^u) = x^l - {}^l x.$$

Now we use (c) and substitute $x^u = x + {}^l x$ into the first equality and $x^l = {}^u x - x$ into the second equality to get ${}^u x - {}^l x = x + {}^u x \curlywedge x^l$ and $x^u - x^l = x + {}^l x \curlywedge x^u$. □

From the preceding theorem we observe that both ”unsymmetrical” expressions ${}^u x \curlyvee x^l$ and ${}^l x \curlyvee x^u$ have properties that are similar to the absolute value of an element in lattice ordered groups, and they can be expressed as the sum of upper and lower parts, just as the absolute value can be expressed as the sum of positive and negative parts. These observations motivate the following definition.

Definition 3.3 Let $\mathcal{G}$ be a mixed lattice group and $x \in \mathcal{G}$. The elements ${}^u x^l = {}^u x + x^l$ and ${}^l x^u = {}^l x + x^u$ are called the *unsymmetrical generalized absolute values* of x.

We will now examine the basic properties of the generalized absolute values. As we might expect, they turn out to have many similarities with the ordinary absolute value, as the next few theorems show.

Theorem 3.4 *Let $\mathcal{G}$ be a mixed lattice group and $x \in \mathcal{G}$. Then we have*

(a) ${}^u x^l = {}^u x \curlyvee x^l = {}^u x + x^l$ *and* ${}^l x^u = {}^l x \curlyvee x^u = {}^l x + x^u$.
(b) ${}^u x^l = {}^l(-x)^u$.
(c) $x \succcurlyeq 0$ *if and only if* $x = {}^l x^u = x^u$ *and* ${}^l x = 0$.
(d) $(x^u)^u = {}^l(x^u)^u = x^u$ *and* $(x^l)^u = {}^l(x^l)^u = x^l$.
(e) ${}^u x^l + {}^l x^u \succcurlyeq 0$ *and* ${}^u x^l + {}^l x^u \geq 0$.

Proof (a) is just a restatement of Theorem 3.2.

(b) By Theorem 3.2 (a) we have ${}^{l}(-x)^{u} = {}^{l}(-x) \curlyvee (-x)^{u} = {}^{u}x \curlyvee x^{l} = {}^{u}x^{l}$.

(c) If $x = x^{u}$ then $x \succcurlyeq 0$. Conversely, let $x \succcurlyeq 0$. Then ${}^{l}x = (-x) \curlyvee 0 \geq 0$ and since $-x \preccurlyeq 0$ and $0 \leq 0$, it follows by (P5) that ${}^{l}x = (-x) \curlyvee 0 \leq 0 \curlyvee 0 = 0$. Hence, ${}^{l}x = 0$. This implies that $x = x^{u} - {}^{l}x = x^{u}$ and ${}^{l}x^{u} = {}^{l}x + x^{u} = x^{u} = x$.

(d) These identities follow immediately from (f), since $x^{u} \succcurlyeq 0$ and $x^{l} \succcurlyeq 0$.

(e) These follow from (c) and Theorem 3.2 (e), since
${}^{u}x^{l} + {}^{l}x^{u} = {}^{u}x + {}^{l}x + x^{l} + x^{u} = 2({}^{u}x + {}^{l}x) = 2(x^{l} + x^{u})$. □

The next theorem gives some useful additional properties of the generalized absolute values. The first two are the triangle inequalities.

Theorem 3.5 *Let $\mathcal{G}$ be a mixed lattice group. For all $x, y \in \mathcal{G}$ the following hold.*

(a) ${}^{u}x + {}^{u}y \geq {}^{u}(x+y)$, $x^{l} + y^{l} \geq (x+y)^{l}$ *and* ${}^{u}x^{l} + {}^{u}y^{l} \geq {}^{u}(x+y)^{l}$
(b) $x^{u} + y^{u} \geq (x+y)^{u}$, ${}^{l}x + {}^{l}y \geq {}^{l}(x+y)$ *and* ${}^{l}x^{u} + {}^{l}y^{u} \geq {}^{l}(x+y)^{u}$
(c) $x \curlyvee y - y \curlywedge x = {}^{u}(x-y)^{l}$ *and* $y \curlyvee x - x \curlywedge y = {}^{l}(x-y)^{u}$
(d) $2(x \curlyvee y) = x + y + {}^{u}(x-y)^{l}$ *and* $2(y \curlyvee x) = x + y + {}^{l}(x-y)^{u}$.

Proof *(a)* Since ${}^{u}x = x \curlyvee 0 \succcurlyeq x$ and ${}^{u}x = x \curlyvee 0 \geq 0$ and similarly ${}^{u}y = y \curlyvee 0 \succcurlyeq y$ and ${}^{u}y = y \curlyvee 0 \geq 0$, it follows by (P5) that ${}^{u}x + {}^{u}y \geq (x+y) \curlyvee 0 = {}^{u}(x+y)$. The proof for the second inequality is similar. It then follows that

$$ {}^{u}(x+y)^{l} = {}^{u}(x+y) + (x+y)^{l} \leq {}^{u}x + {}^{u}y + x^{l} + y^{l} = {}^{u}x^{l} + {}^{u}y^{l}. $$

(b) Just repeat the arguments in (a).

(c) Applying Theorem 3.2 we get

$$ \begin{aligned} y \curlyvee x - x \curlywedge y &= y \curlyvee x + (-x \curlyvee -y) \\ &= [y + (-x \curlyvee -y)] \curlyvee [x + (-x \curlyvee -y)] \\ &= [(y-x) \curlyvee 0] \curlyvee [0 \curlyvee (x-y)] \\ &= {}^{u}(y-x) \curlyvee (y-x)^{l} \\ &= {}^{u}(y-x)^{l}. \end{aligned} $$

The second equality follows from the first one by exchanging x and y and applying Theorem 3.2 (a).

(d) From (P1) we get $y \curlywedge x = x + y - (x \curlyvee y)$. The desired result follows by substituting this into (c). The second identity is similar. □

In Theorem 3.2 we showed that any $x \in \mathcal{G}$ can be written as a difference of upper and lower parts. Just as in Riesz space theory, these decompositions $x = x^{u} - {}^{l}x$ and $x = {}^{u}x - x^{l}$ with $x^{u} \curlywedge {}^{l}x = 0 = x^{l} \curlywedge {}^{u}x$ are minimal in the sense of the following theorem.

Theorem 3.6 *Let $\mathcal{G}$ be a mixed lattice group. If $x \in \mathcal{G}$ and $x = u - v$ then the following hold.*

(a) *If $u \succcurlyeq 0$ and $v \geq 0$ then $u \geq x^{u}$ and $v \geq {}^{l}x$,*

(b) if $u \geq 0$ *and* $v \succcurlyeq 0$ *then* $u \geq {}^u x$ *and* $v \geq x^l$,

and conversely,

(c) if $x = u - v$ *with* $u \curlywedge v = 0$ *then* $u = x^u$ *and* $v = {}^l x$,

(d) if $x = u - v$ *with* $v \curlywedge u = 0$ *then* $u = {}^u x$ *and* $v = x^l$.

Proof *(a)* Let $x = u - v$ with $u \succcurlyeq 0$ and $v \geq 0$. Then $u = x + v \geq x$ and it follows by (P5a) and (P4) that $x^u = 0 \curlyvee x \leq 0 \curlyvee u = u$. Then, using (P5) again we get

$$ {}^l x = (-x) \curlyvee 0 = (v - u) \curlyvee 0 = v + (-u \curlyvee -v) \leq v + 0 = v. $$

(b) Similar.

(c) Assume that $x = u - v$ with $u \curlywedge v = 0$. Then we have

$$ 0 = u \curlywedge v = u + 0 \curlywedge (v - u) = u - 0 \curlyvee (u - v) = u - (u - v)^u = u - x^u, $$

and so $u = x^u$. It now follows that $v = u - x = x^u - x = {}^l x$.

(d) Similar. □

The preceding theorem corresponds to the representation of elements in a Riesz space as a difference of two disjoint elements. The theory of disjointness in mixed lattice semigroups has been studied by Arsove and Leutwiler [7].

In an arbitrary mixed lattice group, the generalized absolute values can behave somewhat unexpectedly. However, in pre-regular mixed lattice groups we can say more about the upper and lower parts as well as the generalized absolute values.

Theorem 3.7 *Let $\mathcal{G}$ be a pre-regular mixed lattice group and $x \in \mathcal{G}$. Then the following hold.*

(a) $x \succcurlyeq 0$ *if and only if* $x = {}^l x^u = {}^u x^l = {}^u x = x^u$ *and* ${}^l x = x^l = 0$.

(b) $x \geq 0$ *if and only if* $x = {}^u x^l = {}^u x$ *and* $x^l = 0$.

(c) ${}^u x^l \geq 0$ *and* ${}^l x^u \geq 0$. *Moreover,* ${}^u x^l = {}^l x^u = 0$ *if and only if* $x = 0$.

(d) ${}^u({}^u x) = {}^u({}^u x)^l = {}^u x$ *and* ${}^u({}^l x) = {}^u({}^l x)^l = {}^l x$.

(e) ${}^u({}^u x^l)^l = {}^u x^l$ *and* ${}^u({}^l x^u)^l = {}^l x^u$.

Proof *(a)* Let $x \succcurlyeq 0$. By (P4) it follows that $x = 0 \curlyvee x = x^u$. Then we have $x = x^u = x^u - {}^l x$ and it follows that ${}^l x = 0$ and ${}^l x^u = {}^l x + x^u = x^u = x$. Furthermore, we have $x^l = 0 \curlyvee (-x) \succcurlyeq 0$, which implies that $x^l \geq 0$ by (R1). But if $x \succcurlyeq 0$ then $-x \preccurlyeq 0$, which implies that $-x \leq 0$. Hence, since $0 \preccurlyeq 0$, by (P5) we have $x^l = 0 \curlyvee (-x) \leq 0 \curlyvee 0 = 0$. Thus $x^l = 0$ and so $x = {}^u x - x^l = {}^u x$ and ${}^u x^l = {}^u x + x^l = {}^u x = x$. Conversely, if $x = {}^l x^u = {}^l x \curlyvee x^u$ then $x \geq x^u = 0 \curlyvee x \geq x$ and so $x = x^u = 0 \curlyvee x$ and it follows from (P4) that $x \succcurlyeq 0$.

(b) Similar.

(c) Clearly, ${}^u x^l \geq 0$ and ${}^l x^u \geq 0$. If $x = 0$ then evidently ${}^u x^l = {}^l x^u = 0$. Conversely, assume that ${}^l x^u = {}^l x + x^u = 0$. Then $x^u = -{}^l x$, which implies that $2x^u = x^u - {}^l x = x$. Now $x = 2x^u \succcurlyeq 0$, and so by (a) we have $x = {}^l x^u = 0$. If ${}^u x^l = 0$ then similar arguments show that $x = 0$.

(d) These follow immediately from (b), since ${}^u x \geq 0$ and ${}^l x \geq 0$.

(e) This follows immediately from (a) and (b). □

Remark 3.8 The last theorem is not valid without the pre-regularity condition, and the generalized absolute value may even be negative. The next example clarifies these points.

Example 3.9 Consider the lexicographically ordered mixed lattice group of Example 2.25. If $x = (1, -1)$ then $x \geq 0$ but ${}^u x = (0, 0) \leq x$. Moreover, $x^l = -x \leq 0$ and so ${}^u x^l = {}^u x + x^l = -x \leq 0$. This shows that in general, one of the generalized absolute values as well as the upper or lower part can actually be negative with respect to one of the partial orders. However, Theorem 3.4 (e) implies that both unsymmetrical absolute values cannot be negative at the same time.

Next we will study the behavior of the mixed envelopes and generalized absolute values under scalar multiplication. In a mixed lattice vector space the generalized absolute values have alternative expressions, which bear close resemblance to the definition of the absolute value in Riesz spaces, that is $|x| = x \vee (-x)$.

Proposition 3.10 *Let $\mathcal{V}$ be a mixed lattice vector space. Then for all $x, y \in \mathcal{V}$ the following hold.*

(a) $(ax) \curlywedge (ay) = a(x \curlywedge y)$ *and* $(ax) \curlyvee (ay) = a(x \curlyvee y)$ *for all* $a \geq 0$,
(b) $(ax) \curlywedge (ay) = a(x \curlyvee y)$ *and* $(ax) \curlyvee (ay) = a(x \curlywedge y)$ *for all* $a < 0$,
(c) ${}^u(ax)^l = a\,{}^u x^l$ *and* ${}^l(ax)^u = a\,{}^l x^u$ *for all* $a \geq 0$,
(d) ${}^u(ax)^l = |a|\,{}^l x^u$ *and* ${}^l(ax)^u = |a|\,{}^u x^l$ *for all* $a < 0$.
(e) ${}^u x^l = x \curlyvee (-x)$ *and* ${}^l x^u = (-x) \curlyvee x$.

Proof *(a)* Assume that $a > 0$ (the case $a = 0$ is trivial). Then by the definition of the lower envelope we have

$$\begin{aligned}(ax) \curlywedge (ay) &= \max\{w : w \preccurlyeq ax, w \leq ay\} \\ &= \max\{w : \tfrac{w}{a} \preccurlyeq x, \tfrac{w}{a} \leq y\} \\ &= \max\{au : u \preccurlyeq x, u \leq y\} \\ &= a \max\{u : u \preccurlyeq x, u \leq y\} \\ &= a(x \curlywedge y).\end{aligned}$$

The proof for the upper envelope is similar.
(b) These identities follow from (a) and property (P3).
(c) These follow from (a) and (b) and Theorem 3.2.
(d) Note that if $a < 0$ then $a = -|a|$, and apply part (c) and Theorem 3.2(b).
(e) By Theorem 3.2 and part (b) we have

$$\begin{aligned}x \curlyvee (-x) &= ({}^u x - x^l) \curlyvee (x^l - {}^u x) \\ &= (-2x^l) \curlyvee (-2\,{}^u x) + {}^u x + x^l \\ &= -2(x^l \curlywedge {}^u x) + {}^u x^l \\ &= {}^u x^l.\end{aligned}$$

This proves the first identity in (e). The second identity is proved in a similar manner. □

To conclude this section, we show that under certain conditions the results of the preceding proposition hold also in mixed lattice groups. For the discussion that follows, we will need the next definition.

Definition 3.11 A mixed lattice group $\mathcal{G}$ is called *Archimedean* if the condition $nx \leq y$ for all $n \in \mathbb{N}$ implies $x \leq 0$.

Erkisson–Bique has proved the following result concerning Archimedean mixed lattice semigroups.

Theorem 3.12 ([10, Theorem 4.1]) *If $\mathcal{S}$ is an Archimedean mixed lattice semigroup, then*

$$nx \curlywedge ny = n(x \curlywedge y) \text{ for all } n \in \mathbb{N} \text{ and } x, y \in \mathcal{S}.$$

Combining the result of Theorem 3.12 with property (P3) and the identities in Theorem 2.7 yields the following corollary.

Corollary 3.13 *If $\mathcal{G}$ is a regular Archimedean mixed lattice group, then the following hold for all $x, y \in \mathcal{G}$.*

(a) $nx \curlywedge ny = n(x \curlywedge y)$ *and* $nx \curlyvee ny = n(x \curlyvee y)$ *for* $n = 0, 1, 2, \ldots$

(b) $nx \curlywedge ny = n(x \curlyvee y)$ *and* $nx \curlyvee ny = n(x \curlywedge y)$ *for* $n = -1, -2, \ldots$.

(c) ${}^{u}x^{l} = x \curlyvee (-x)$ *and* ${}^{l}x^{u} = (-x) \curlyvee x$.

Remark 3.14 It should be noted that the Archimedean property in the preceding theorem is not a necessary condition, as the above result holds in every non-Archimedean Riesz space. In general, the identities in Corollary 3.13 do not hold as the following example shows.

Example 3.15 Consider the partially ordered group $(\mathbb{Z}, \leq, \preccurlyeq)$ of Example 2.21, where $\leq$ is the usual order and

$$n \preccurlyeq m \quad \text{if } m - n \geq 0 \text{ and } m - n \text{ is divisible by } p.$$

Choose $p = 3$. Then we find that $-1 \curlyvee 1 = 2$ but $-3 \curlyvee 3 = 3 \neq 3(-1 \curlyvee 1) = 6$ and so the identity (a) of Corollary 3.13 fails to hold. Next, let $x = -1$ and compute ${}^{u}x = -1 \curlyvee 0 = 2$ and $x^{l} = 0 \curlyvee 1 = 3$. Then by Theorem 3.2 we have ${}^{u}x^{l} = 2 + 3 = 5$. However, if we try to use the formula of Corollary 3.13 (c) we get $x \curlyvee (-x) = -1 \curlyvee 1 = 2 \neq {}^{u}x^{l}$.

If $\mathcal{G}$ is a lattice with respect to specific order $\preccurlyeq$ then the absolute value with respect to $\preccurlyeq$ exists for all $x \in \mathcal{G}$ and we denote it by $\mathrm{sp}\,|x| = \mathrm{sp}\sup\{x, -x\}$. Similarly, we denote the positive and negative parts with respect to $\preccurlyeq$ by $\mathrm{sp}(x^{+})$ and $\mathrm{sp}(x^{-})$, respectively.

Proposition 3.16 *Let $\mathcal{G}$ be a quasi-regular mixed lattice group that is a lattice with respect to its specific order. Then*

(a) $\mathrm{sp}\,|x| = \mathrm{sp}\,\sup\{{}^u x^l, {}^l x^u\}$ *for all* $x \in \mathcal{G}$.
(b) $\mathrm{sp}(x^+) = \mathrm{sp}\,\sup\{{}^u x, x^u\}$ *for all* $x \in \mathcal{G}$.
(c) $\mathrm{sp}(x^-) = \mathrm{sp}\,\sup\{{}^l x, x^l\}$ *for all* $x \in \mathcal{G}$.

Proof (a) We have ${}^u x^l = {}^u x \curlyvee x^l \succcurlyeq {}^u x \succcurlyeq x$ and ${}^l x^u = {}^l x \curlyvee x^u \succcurlyeq {}^l x \succcurlyeq -x$. From this it follows that $\mathrm{sp}\,\sup\{{}^u x^l, {}^l x^u\} \succcurlyeq \mathrm{sp}\,\sup\{x, -x\} = \mathrm{sp}\,|x|$. On the other hand, we have $\mathrm{sp}\,|x| \succcurlyeq x$, $\mathrm{sp}\,|x| \succcurlyeq -x$ and $\mathrm{sp}\,|x| \succcurlyeq 0$. Then by (R2) $\mathrm{sp}\,|x| \succcurlyeq 0 \curlyvee x = x^u$ and $\mathrm{sp}\,|x| \succcurlyeq (-x) \curlyvee 0 = {}^l x$. So $\mathrm{sp}\,|x| \succcurlyeq {}^l x \curlyvee x^u = {}^l x^u$. Similarly we show that $\mathrm{sp}\,|x| \succcurlyeq {}^u x^l$. Hence, $\mathrm{sp}\,\sup\{{}^u x^l, {}^l x^u\} \preccurlyeq \mathrm{sp}\,|x|$ and the proof is complete.

(b) Since ${}^u x \succcurlyeq x$ and $x^u \succcurlyeq 0$ we get $\mathrm{sp}\,\sup\{{}^u x, x^u\} \succcurlyeq \mathrm{sp}\,\sup\{x, 0\} = \mathrm{sp}(x^+)$. On the other hand, $\mathrm{sp}(x^+) \succcurlyeq 0$ and $\mathrm{sp}(x^+) \succcurlyeq x$, and so $\mathrm{sp}(x^+) \succcurlyeq x \curlyvee 0 = {}^u x$ and $\mathrm{sp}(x^+) \succcurlyeq 0 \curlyvee x = x^u$. It follows that $\mathrm{sp}(x^+) \succcurlyeq \mathrm{sp}\,\sup\{{}^u x, x^u\}$. Hence, $\mathrm{sp}(x^+) = \mathrm{sp}\,\sup\{{}^u x, x^u\}$. The proof for (c) is similar. □

4 Mixed lattice ideals

Ideals play a fundamental role in the theory of Riesz spaces. In this section we extend the study of ideals to mixed lattice groups. A lattice ideal in a Riesz space E is defined as a subspace A of E such that if $x \in A$ and $|y| \le |x|$ then $y \in A$. We could define ideals in a mixed lattice group similarly with respect to initial and specific orders. However, for the specific order such a definition is problematic, since the absolute values ${}^u x^l$ and ${}^l x^u$ are not necessarily positive with respect to specific order. The notion of an order convex subspace can be used as a generalization of the lattice ideal in an ordered vector space, and lattice ideals in Riesz spaces are precisely those sublattices that are order convex. We use this approach to define ideals in mixed lattice structures.

Definition 4.1 (i) A subgroup $\mathcal{S}$ of a mixed lattice group $\mathcal{G}$ is called a *mixed lattice subgroup* of $\mathcal{G}$ if $x \curlyvee y$ and $x \curlywedge y$ belong to $\mathcal{S}$ whenever x and y are in $\mathcal{S}$. Here $x \curlyvee y$ and $x \curlywedge y$ are the upper and lower mixed envelopes in $\mathcal{G}$.

(ii) A subset $U \in \mathcal{G}$ is called *order convex*, if $x \le z \le y$ and $x, y \in U$ imply that $z \in U$. Similarly, a subset $U \in \mathcal{G}$ is called *specifically order convex*, if $x \preccurlyeq z \preccurlyeq y$ and $x, y \in U$ imply that $z \in U$.

(iii) If $\mathcal{A}$ is an order convex mixed lattice subgroup of $\mathcal{G}$ then $\mathcal{A}$ is called a *mixed lattice ideal* of $\mathcal{G}$. Similarly, a specifically order convex mixed lattice subgroup of $\mathcal{G}$ is called a *specific mixed lattice ideal* of $\mathcal{G}$.

Next we state a useful characterization for mixed lattice ideals in a pre-regular mixed lattice group. The conditions (c)–(f) correspond to the usual definition of a lattice ideal.

Theorem 4.2 *Let $\mathcal{G}$ be a pre-regular mixed lattice group and $\mathcal{A}$ a subgroup of $\mathcal{G}$. The following conditions are equivalent.*

(a) $\mathcal{A}$ is a mixed lattice ideal in $\mathcal{G}$
(b) The following two conditions hold.
(i) $x \in \mathcal{A}$ *implies* ${}^u x^l \in \mathcal{A}$. (ii) $0 \le x \le y$ *with* $y \in \mathcal{A}$ *implies* $x \in \mathcal{A}$.

(c) *If* ${}^u y^l \leq {}^u x^l$ *and* $x \in \mathcal{A}$ *then* $y \in \mathcal{A}$.
(d) *If* ${}^l y^u \leq {}^u x^l$ *and* $x \in \mathcal{A}$ *then* $y \in \mathcal{A}$.
(e) *If* ${}^l y^u \leq {}^l x^u$ *and* $x \in \mathcal{A}$ *then* $y \in \mathcal{A}$.
(f) *If* ${}^u y^l \leq {}^l x^u$ *and* $x \in \mathcal{A}$ *then* $y \in \mathcal{A}$.

Proof The implication $(a) \implies (b)$ is evident.

To show that $(b) \implies (c)$, assume that conditions (i) and (ii) in (b) hold. If $0 \leq {}^u y^l \leq {}^u x^l$ and $x \in \mathcal{A}$, then ${}^u x^l \in \mathcal{A}$ by (i). Then (ii) implies that ${}^u y^l \in \mathcal{A}$ and the inequality $0 \leq {}^u y \leq {}^u y^l$ together with (ii) imply that ${}^u y \in \mathcal{A}$. So ${}^u y^l - {}^u y = y^l \in \mathcal{A}$, and hence ${}^u y - y^l = y \in \mathcal{A}$.

Next we show that $(c) \implies (a)$. Assuming that (c) holds, let $x \in \mathcal{A}$. Then $0 \leq {}^u x = {}^u({}^u x)^l \leq {}^u x^l$ and so ${}^u x \in \mathcal{A}$. This proves that $\mathcal{A}$ is a mixed lattice subgroup (by Theorem 2.12). Next, assume that $u \leq z \leq v$ with $u, v \in \mathcal{A}$. We want to show that $z \in \mathcal{A}$. Now $0 \leq z - u \leq v - u$, and it follows that $0 \leq {}^u(z-u)^l \leq {}^u(v-u)^l$. Since $\mathcal{A}$ is a subgroup, we have $v - u \in \mathcal{A}$ and so by assumption $z - u \in \mathcal{A}$. Thus, $z = u - (z - u) \in \mathcal{A}$ and $\mathcal{A}$ is order convex, and hence a mixed lattice ideal.

For $(c) \implies (d)$, let $x \in \mathcal{A}$ and observe that the inequality ${}^l y^u \leq {}^u x^l$ is equivalent to ${}^u(-y)^l \leq {}^u x^l$. If the condition (c) holds, then $-y \in \mathcal{A}$ and since $\mathcal{A}$ is a subgroup, we have $y \in \mathcal{A}$. This establishes (d). The implications $(d) \implies (e)$, $(e) \implies (f)$ and $(f) \implies (c)$ are proved similarly. □

As it turns out, the equivalent conditions for specific mixed lattice ideals are slightly different. More details are given in the following results.

Proposition 4.3 *Let $\mathcal{G}$ be a pre-regular mixed lattice group and $\mathcal{A}$ a subgroup of $\mathcal{G}$. The following conditions are equivalent.*

(a) *$\mathcal{A}$ is a specific mixed lattice ideal in $\mathcal{G}$*
(b) *The following two conditions hold.*
(i) $x \in \mathcal{A}$ *implies* ${}^u x^l \in \mathcal{A}$. (ii) $0 \preccurlyeq x \preccurlyeq y$ *with* $y \in \mathcal{A}$ *implies* $x \in \mathcal{A}$.

Proof The implication $(a) \implies (b)$ is clear.

To prove that $(b) \implies (a)$, suppose that conditions (i) and (ii) in (b) hold. If $x \in \mathcal{A}$ then ${}^u x^l \in \mathcal{A}$ and this implies that $\mathcal{A}$ is a mixed lattice subgroup. Moreover, if $x, y \in \mathcal{A}$ and $x \preccurlyeq z \preccurlyeq y$ holds, then $0 \preccurlyeq z - x \preccurlyeq y - x \in \mathcal{A}$ and by condition (ii) we have $z - x \in \mathcal{A}$, and consequently, $x + (z - x) = z \in \mathcal{A}$. Hence $\mathcal{A}$ is a specific mixed lattice ideal. □

The related conditions (c)–(f) of Theorem 4.2 are not equivalent for specific mixed lattice ideals, but they imply the conditions given in Proposition 4.3.

Proposition 4.4 *Let $\mathcal{G}$ be a pre-regular mixed lattice group and $\mathcal{A}$ a subgroup of $\mathcal{G}$. If one of the following equivalent conditions holds then $\mathcal{A}$ is a specific mixed lattice ideal.*

(a) *If* ${}^u y^l \preccurlyeq {}^u x^l$ *and* $x \in \mathcal{A}$ *then* $y \in \mathcal{A}$.
(b) *If* ${}^l y^u \preccurlyeq {}^u x^l$ *and* $x \in \mathcal{A}$ *then* $y \in \mathcal{A}$.
(c) *If* ${}^l y^u \preccurlyeq {}^l x^u$ *and* $x \in \mathcal{A}$ *then* $y \in \mathcal{A}$.
(d) *If* ${}^u y^l \preccurlyeq {}^l x^u$ *and* $x \in \mathcal{A}$ *then* $y \in \mathcal{A}$.

Proof The equivalence of the conditions is proved in the same way as in Theorem 4.2. Assume that (a) holds. If $x \in \mathcal{A}$ then it follows from the identity ${}^{u}({}^{u}x^{l})^{l} = {}^{u}x^{l}$ that ${}^{u}x^{l} \in \mathcal{A}$. Moreover, if $0 \preccurlyeq y \preccurlyeq x$ then $0 \preccurlyeq {}^{u}y^{l} \preccurlyeq {}^{u}x^{l}$ and so $y \in \mathcal{A}$. Hence by Proposition 4.3 $\mathcal{A}$ is a specific mixed lattice ideal. □

The next counterexample shows that the conditions of Propositions 4.3 and 4.4 are not equivalent.

Example 4.5 Consider $\mathbb{Z} \times \mathbb{Z}$ with the lexicographic order $\leq$ given in Example 2.25 and define $\preccurlyeq$ by $(x, y) \preccurlyeq (u, v)$ if $x \leq u$ and $y \leq v$. Then $\mathcal{G} = (\mathbb{Z} \times \mathbb{Z}, \leq, \preccurlyeq)$ is a regular mixed lattice group. Consider the subgroup $\mathcal{A} = \{(x, y) : y = 0\}$. It is easy to see that the condition (b) of Theorem 4.3 holds in $\mathcal{A}$ and so $\mathcal{A}$ is a specific mixed lattice ideal in $\mathcal{G}$. However, the conditions of Proposition 4.4 do not hold in $\mathcal{A}$. Choose for example $x = (2, 0)$ and $y = (1, -1)$. Then $x \in \mathcal{A}$ and $y = {}^{u}y^{l} \preccurlyeq {}^{u}x^{l} = x$, but $y \notin \mathcal{A}$.

As an easy consequence of the preceding results we have the following relationship between lattice ideals and mixed lattice ideals.

Proposition 4.6 *(a) If $\mathcal{G}$ is a mixed lattice group that is a lattice with respect to initial order $\leq$ then every mixed lattice ideal is a lattice ideal (with respect to $\leq$).*
(b) If $\mathcal{G}$ is a mixed lattice group that is a lattice with respect to specific order $\preccurlyeq$ then every lattice ideal (with respect to $\preccurlyeq$) is a specific mixed lattice ideal.

Proof (a) Let $\mathcal{A}$ be a mixed lattice ideal in $\mathcal{G}$ and $x \in \mathcal{A}$. Assume that $|y| \leq |x|$. By Theorem 4.2 we have ${}^{u}x^{l}, {}^{l}x^{u} \in \mathcal{A}$ and inequalities $x \leq {}^{l}x^{u}$ and $-x \leq {}^{u}x^{l}$ imply that $0 \leq |x| \leq {}^{u}x^{l} + {}^{l}x^{u}$. Then Theorem 4.2 implies that $|x| \in \mathcal{A}$. Since $0 \leq y^{+} \leq |y| \leq |x|$, it follows that $y^{+} \in \mathcal{A}$. Similarly, $y^{-} \in \mathcal{A}$ and so $y = y^{+} - y^{-} \in \mathcal{A}$. This proves that $\mathcal{A}$ is a lattice ideal in $\mathcal{G}$.
(b) Let $\mathcal{A}$ be a lattice ideal with respect to $\preccurlyeq$ and $x \in \mathcal{A}$. Then $0 \preccurlyeq x^{u} = \operatorname{sp}|x^{u}| \preccurlyeq \operatorname{sp}|x|$. This implies that $x^{u} \in \mathcal{A}$ and so ${}^{l}x = x^{u} - x \in \mathcal{A}$. Hence ${}^{u}x^{l} = x^{u} + {}^{l}x \in \mathcal{A}$. Moreover, if $0 \preccurlyeq y \preccurlyeq x$ with $x \in \mathcal{A}$ then $0 \preccurlyeq \operatorname{sp}|y| \preccurlyeq \operatorname{sp}|x|$ and since $\mathcal{A}$ is a lattice ideal, this implies that $y \in \mathcal{A}$. Hence the conditions of Proposition 4.3 hold and $\mathcal{A}$ is a specific mixed lattice ideal. □

The converse of Proposition 4.6 does not hold (see Example 4.10 below).

The following facts are now evident.

Proposition 4.7 *In a mixed lattice group $\mathcal{G}$ the following hold.*

(a) If $\mathcal{G}$ is pre-regular and $\mathcal{A}$ is a mixed lattice ideal in $\mathcal{G}$, then $\mathcal{A}$ is a specific mixed lattice ideal in $\mathcal{G}$.
(b) Intersections of (specific) mixed lattice ideals are (specific) mixed lattice ideals.

Before discussing examples we give the following simple but useful result.

Lemma 4.8 *If $\mathcal{G}$ is a quasi-regular mixed lattice group and $\mathcal{U} = \{w : w \succcurlyeq 0\}$ then the subgroup $\mathcal{S}$ generated by $\mathcal{U}$ (i.e. $\mathcal{S} = \mathcal{U} - \mathcal{U}$) is a specific mixed lattice ideal in $\mathcal{G}$.*

Proof If $x \in \mathcal{S}$ then $x = u - v$ for some $u, v \succcurlyeq 0$ and $0 \curlyvee x = 0 \curlyvee (u - v) = v \curlyvee u - v \succcurlyeq 0$ and so $0 \curlyvee x \in \mathcal{U} \subset \mathcal{S}$. This proves that $\mathcal{S}$ is a mixed lattice subgroup of $\mathcal{G}$. Moreover, $0 \preccurlyeq y \preccurlyeq x$ clearly implies that $y \in \mathcal{S}$ and hence $\mathcal{S}$ is a specific mixed lattice ideal. □

In the remainder of this paper, we shall use simply the terms *ideal* and *specific ideal* for mixed lattice ideals and specific mixed lattice ideals, respectively, if there is no danger of confusion.

Example 4.9 Consider the mixed lattice group $(\mathcal{G}, \leq, \preccurlyeq)$ of Example 2.21. The only ideals in this mixed lattice group are $\{0\}$ and $\mathcal{G}$. Indeed, if $\mathcal{S}$ is an ideal in $\mathcal{G}$ and $x \in \mathcal{S}$ with $x \geq 1 \geq 0$, then since $\mathcal{G}$ is pre-regular, we have ${}^u x^l = x \geq 1$, and so ${}^u 1^l = 1 \leq {}^u x^l$. Hence $1 \in \mathcal{S}$, and consequently, $n \cdot 1 = n \in \mathcal{S}$ for every $n \in \mathbb{Z}$. Thus $\mathcal{S} = \mathcal{G}$.

The subgroup $\mathcal{A} = \{pn : n \in \mathbb{Z}\}$ is a specific ideal in $\mathcal{G}$, by Lemma 4.8.

Example 4.10 Let $\mathcal{G} = \mathbb{R}^2$ and define $\leq$ as the partial order induced by the positive cone $C_1 = \{(x, y) : x \geq 0 \text{and} y \geq 0\}$. Define specific order $\preccurlyeq$ as the partial order with the positive cone $C_2 = \{(x, x) : x \geq 0\}$. Both of these are compatible partial orders in $\mathbb{R}^2$. An easy verification shows that the element $0 \curlyvee x$ exists for all $x \in \mathcal{G}$. Thus, by Theorem 2.12, $(\mathcal{G}, \leq, \preccurlyeq)$ is a mixed lattice vector space which is clearly pre-regular but not regular, since the cone C_2 does not generate the whole space $\mathbb{R}^2$. It should be clear, however, that the set $\mathcal{S} = \{x \in \mathcal{G} : x \succcurlyeq 0\}$ is a mixed lattice semigroup and thus $\mathcal{G}$ is quasi-regular.

In $(\mathcal{G}, \leq, \preccurlyeq)$, the set $\mathcal{A} = \{(x, y) : x = y\}$ is a specific mixed lattice ideal by Lemma 4.8, but not a mixed lattice ideal. To show that $\mathcal{A}$ is not a mixed lattice ideal, choose $x = (1, 1)$ and $y = (1, 0)$. Then $x \in \mathcal{A}$ and $y = {}^u y^l \leq {}^u x^l = x$, but $y \notin \mathcal{A}$. Finally, note that $\mathcal{G}$ is a lattice with respect to $\leq$ and the x-axis is a lattice ideal in $\mathcal{G}$ but not a mixed lattice ideal. Hence, the converse of Proposition 4.6 does not hold.

Example 4.11 As another example, consider $\mathcal{G} = BV([0, 1])$, the functions of bounded variation on $[0, 1]$ (see Example 2.15). Let $\mathcal{S}$ be the set of all constant functions. Then $\mathcal{S}$ is clearly a subgroup of $\mathcal{G}$, and it is also a mixed lattice subgroup, where the upper and lower envelopes are given by $f \curlyvee g = \max\{f, g\}$ and $f \curlywedge g = \min\{f, g\}$. However, $\mathcal{S}$ is not a (specific) ideal in $\mathcal{G}$. This can be seen by choosing $f(x) = 1$ and $g(x) = 1 - x$. Then $g \geq 0$ and $f \succcurlyeq 0$, and since $\mathcal{G}$ is regular, we have $g = {}^u g^l$ and $f = {}^u f^l$. Now ${}^u g^l \leq {}^u f^l$ and ${}^u g^l \preccurlyeq {}^u f^l$, and $f \in \mathcal{S}$ but $g \notin \mathcal{S}$. Hence $\mathcal{S}$ is not an ideal or a specific ideal in $\mathcal{G}$.

Next, consider the subset $\mathcal{A} = \{f \in \mathcal{G} : f(0) = 0\}$. This is both an ideal and a specific ideal in $\mathcal{G}$. To see this, let $f \in \mathcal{A}$ and ${}^u g^l \leq {}^u f^l$. By the definitions of initial and specific orders (see Example 2.15), it is clear that ${}^u f(0) = f^l(0) = 0$, and so ${}^u f^l \in \mathcal{A}$. Moreover, since $\mathcal{G}$ is regular, we have $0 \leq {}^u g^l$ and hence $0 \leq {}^u g^l(0) \leq {}^u f^l(0) = 0$. This implies that ${}^u g^l(0) = 0$ and so $g \in \mathcal{A}$. Hence, $\mathcal{A}$ is an ideal in $\mathcal{G}$, and it then follows from Proposition 4.7 that $\mathcal{A}$ is also a specific ideal in $\mathcal{G}$.

Example 4.12 Let $\mathcal{V}$ be the mixed lattice vector space of Example 2.18 with the specific positive cone $C_s = \{f \in \mathcal{V} : f \geq 0 \text{ and } f \text{ is constant}\}$. The subspace $S = C_s - C_s$ consisting of all constant functions is a specific ideal in $\mathcal{V}$ but not an ideal.

Example 4.13 Let $\mathcal{G}$ be the mixed lattice group of Example 2.19. It is easy to verify that the set of diagonal matrices is a mixed lattice ideal in $\mathcal{G}$ and the set of symmetric matrices is a specific mixed lattice ideal in $\mathcal{G}$ (by Lemma 4.8) but not an ideal. Finally, the set of integer matrices is a mixed lattice subgroup in $\mathcal{G}$ but not a specific mixed lattice ideal.

5 Mixed lattice homomorphisms

Next we use the results of the preceding sections to study mixed lattice group homomorphisms.

Definition 5.1 Let $\mathcal{G}$ and $\mathcal{H}$ be mixed lattice (semi)groups. An additive mapping $T : \mathcal{G} \to \mathcal{H}$ is a *mixed lattice (semi)group homomorphism* if $T(x \curlyvee y) = Tx \curlyvee Ty$ and $T(x \curlywedge y) = Tx \curlywedge Ty$ for all $x, y \in \mathcal{G}$.

If T is a mixed lattice group homomorphism, then by additivity $T(0) = 0$ and $T(-x) = -Tx$ for all $x \in \mathcal{G}$.

Our next result gives a characterization of mixed lattice group homomorphisms. Similar characterizations are well known in Riesz spaces ([2, Theorem 2.14]) and lattice ordered groups ([14, Theorem 4.5]).

Theorem 5.2 *Let $T : \mathcal{G} \to \mathcal{H}$ be an additive mapping between two mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$. Then the following statements are equivalent.*

(a) T *is a mixed lattice group homomorphism.*
(b) $T(x \curlyvee y) = Tx \curlyvee Ty$ *for all* $x, y \in \mathcal{G}$.
(c) $T(x^l) = (Tx)^l$ *for all* $x \in \mathcal{G}$.
(d) $T(x^u) = (Tx)^u$ *for all* $x \in \mathcal{G}$.
(e) $T({}^l x) = {}^l(Tx)$ *for all* $x \in \mathcal{G}$.
(f) $T({}^u x) = {}^u(Tx)$ *for all* $x \in \mathcal{G}$.
(g) $T(x \curlywedge y) = Tx \curlywedge Ty$ *for all* $x, y \in \mathcal{G}$.
(h) If $x \curlywedge y = 0$ *in* $\mathcal{G}$, *then* $Tx \curlywedge Ty = 0$ *holds in* $\mathcal{H}$.

Proof *(a)* $\Longrightarrow$ *(b)* Obvious.
(b) $\Longrightarrow$ *(c)* Assuming that (b) holds, we have

$$T(x^l) = T(0 \curlyvee (-x)) = T(0) \curlyvee T(-x) = 0 \curlyvee (-Tx) = (Tx)^l.$$

(c) $\Longrightarrow$ *(d)* If (c) holds, then by Theorem 3.2 (a) we have

$$T(x^u) = T((-x)^l) = (T(-x))^l = (-Tx)^l = Tx^u.$$

(d) $\Longrightarrow$ *(e)* Assume (d) holds. From the identity ${}^l x = x^u - x$ it follows that

$$T({}^l x) = T(x^u - x) = T(x^u) - Tx = (Tx)^u - Tx = {}^l(Tx).$$

(e) $\Longrightarrow$ *(f)* If (e) holds, then

$$T({}^u x) = T({}^l(-x)) = {}^l(T(-x)) = {}^l(-Tx) = {}^u Tx.$$

(f) $\Longrightarrow$ *(g)* First we note that for any elements u and v in a mixed lattice group we have

$$u - {}^u(u - v) = u - (u - v) \curlyvee 0 = u + (v - u) \curlywedge 0 = v \curlywedge u,$$

so by additivity of T and part (f) we have

$$T(y \curlywedge x) = T(x - {}^u(x - y)) = Tx - {}^u(Tx - Ty) = Ty \curlywedge Tx.$$

(g) $\Longrightarrow$ *(h)* If $x \curlywedge y = 0$ then (g) implies $Tx \curlywedge Ty = T(x \curlywedge y) = T(0) = 0$.
(h) $\Longrightarrow$ *(g)* First we note that $0 = (x \curlywedge y) - (x \curlywedge y) = (x - x \curlywedge y) \curlywedge (y - x \curlywedge y)$, so by (h) we have

$$\begin{aligned} 0 &= T(x - x \curlywedge y) \curlywedge T(y - x \curlywedge y) \\ &= (Tx - T(x \curlywedge y)) \curlywedge (Ty - T(x \curlywedge y)) \\ &= Tx \curlywedge Ty - T(x \curlywedge y). \end{aligned}$$

Hence, $T(x \curlywedge y) = Tx \curlywedge Ty$.
(g) $\Longrightarrow$ *(a)* If *(g)* holds, then

$$\begin{aligned} T(x \curlyvee y) &= T(-(-x \curlywedge -y)) \\ &= -T(-x \curlywedge -y) \\ &= -(T(-x) \curlywedge T(-y)) \\ &= -(-Tx \curlywedge -Ty) \\ &= Tx \curlyvee Ty. \end{aligned}$$

□

Under the additional assumption that $\mathcal{H}$ contains no elements of order 2 we can add the following conditions to the list of the preceding theorem.

Proposition 5.3 *Let $T : \mathcal{G} \to \mathcal{H}$ be an additive mapping between two mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$. If $\mathcal{H}$ contains no elements of order 2 then the following statements are equivalent.*

(a) T is a mixed lattice homomorphism.
(b) $T({}^l x^u) = {}^l(Tx)^u$ for all $x \in \mathcal{G}$.
(c) $T({}^u x^l) = {}^u(Tx)^l$ for all $x \in \mathcal{G}$.

Proof *(a)* $\Longrightarrow$ *(b)* Applying Theorem 5.2 we get

$$T({}^l x^u) = T({}^l x + x^u) = T({}^l x) + T(x^u) = {}^l(Tx) + (Tx)^u = {}^l(Tx)^u.$$

(b) $\Longrightarrow$ *(c)* Follows by the identity ${}^u x^l = {}^l(-x)^u$ in Theorem 3.4.

(c) $\implies$ *(a)* By Theorem 3.5 (d) we have $2(x \curlyvee y) = x + y + {}^u(x-y)^l$, so by additivity of T we get

$$\begin{aligned} 2T(x \curlyvee y) = T(2(x \curlyvee y)) &= Tx + Ty + T({}^u(x-y)^l) \\ &= Tx + Ty + {}^u(Tx - Ty)^l \\ &= 2(Tx \curlyvee Ty). \end{aligned}$$

It follows that $2(T(x \curlyvee y) - Tx \curlyvee Ty) = 0$, and since $\mathcal{H}$ contains no elements of order 2, this implies that $T(x \curlyvee y) = Tx \curlyvee Ty$. Hence T is a mixed lattice homomorphism. □

In particular, the additional assumption in the preceding proposition holds in mixed lattice vector spaces. Moreover, it holds if the range of T is a mixed lattice group that is a lattice with respect to one of the partial orderings. In fact, we have the following well-known result.

Lemma 5.4 ([8, Corollary 1, pp. 294]) *If G is a lattice ordered group with $x \in G$ and $n \in \mathbb{N}$, then $nx = 0$ implies that $x = 0$.*

The conclusion of the above lemma holds also in a regular Archimedean mixed lattice group. For this, we need the following lemma (which is due to the referee, to whom we are grateful).

Lemma 5.5 *If $\mathcal{G}$ is a regular Archimedean mixed lattice group with $x \in G$ and $n \in \mathbb{N}$, then $nx = 0$ implies that $x = 0$.*

Proof First assume that $x \geq 0$. Then $0 \leq x \leq 2x \leq \cdots \leq nx = 0$, so $x = 0$. Now let x be arbitrary. It follows from Corollary 3.13 that

$$n({}^u x^l) = {}^u(nx)^l = {}^u 0^l = 0 \quad \text{and} \quad n({}^l x^u) = {}^l(nx)^u = {}^l 0^u = 0.$$

Moreover, Theorem 3.7(c) shows that ${}^u x^l \geq 0$ and ${}^l x^u \geq 0$, so the first line of our proof implies that ${}^u x^l = 0$ and ${}^l x^u = 0$. Consequently, $x = 0$, again by Theorem 3.7(c). □

Combining the results of Proposition 5.3, Lemma 5.4 and Lemma 5.5 we get the following special cases of Proposition 5.3.

Corollary 5.6 *Let $T : \mathcal{G} \to \mathcal{H}$ be an additive mapping between two mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$. Consider the following statements.*

(a) T is a mixed lattice group homomorphism.
(b) $T({}^l x^u) = {}^l(Tx)^u$ for all $x \in G$.
(c) $T({}^u x^l) = {}^u(Tx)^l$ for all $x \in G$.

The statements (a), (b) and (c) are equivalent if at least one of the following conditions hold.

(i) $\mathcal{G}$ and $\mathcal{H}$ are mixed lattice vector spaces.
(ii) $\mathcal{H}$ is a lattice with respect to one of the partial orderings.

(iii) $\mathcal{H}$ *is a regular Archimedean mixed lattice group.*

Remark 5.7 Note that in Example 2.17 we showed that there does exist mixed lattice groups that are not lattices with respect to either partial ordering. We should also point out that the conditions (b) and (c) in the above corollary always hold for any mixed lattice homomorphism but they are not necessarily equivalent to T being a homomorphism. Furthermore, the assumption in Lemma 5.4 that $\mathcal{G}$ is a lattice is not a necessary condition.

We shall say that a mapping $T : \mathcal{G} \to \mathcal{H}$ between two mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$ is *increasing* if $x \le y$ in $\mathcal{G}$ implies that $Tx \le Ty$ in $\mathcal{H}$. Similarly, T is *specifically increasing*, if $x \preccurlyeq y$ in $\mathcal{G}$ implies that $Tx \preccurlyeq Ty$ in $\mathcal{H}$. We now prove the following result which gives some additional basic facts about mixed lattice homomorphisms.

Theorem 5.8 *Let* $T : \mathcal{G} \to \mathcal{H}$ *be a mixed lattice group homomorphism between two mixed lattice groups* $\mathcal{G}$ *and* $\mathcal{H}$*. Then the following hold.*

(a) T *is specifically increasing. In addition, if* $\mathcal{G}$ *is pre-regular then* T *is increasing.*
(b) If $\mathcal{S} \subset \mathcal{G}$ *is a mixed lattice subgroup (or subsemigroup) of* $\mathcal{G}$*, then* $T(\mathcal{S})$ *is a mixed lattice subgroup (or subsemigroup) of* $\mathcal{H}$*.*
(c) If $\mathcal{G}$ *is pre-regular then* $T(\mathcal{G})$ *is pre-regular.*
(d) If $\mathcal{G}$ *is (quasi-)regular then* $T(\mathcal{G})$ *is (quasi-)regular.*
(e) If $\mathcal{G}$ *is pre-regular and* $\mathcal{A}$ *is a (specific) ideal in* $\mathcal{G}$*, then* $T(\mathcal{A})$ *is a (specific) ideal in* $T(\mathcal{G})$*.*
(f) If T *is bijective then* T^{-1} *is also a mixed lattice homomorphism.*

Proof *(a)* If $x \succcurlyeq y$ then $x - y \succcurlyeq 0$, and by Theorem 3.4 we have $(x-y)^u = x - y$. Since T is a homomorphism, it follows that

$$Tx - Ty = T(x-y) = T((x-y)^u) = [T(x-y)]^u \succcurlyeq 0,$$

and so $Tx \succcurlyeq Ty$.

If $\mathcal{G}$ is pre-regular, then $x \ge 0$ if and only if $x = {}^u x$, by 3.7. Similar arguments as above now establish that $x \ge y$ implies that $Tx \ge Ty$.

(b) These statements are evident by the definition of a mixed lattice homomorphism.

(c) It is sufficient to show that if $y \in T(\mathcal{G})$, then $y \succcurlyeq 0$ implies that $y \ge 0$. If $y \succcurlyeq 0$ then by Theorem 3.4 (f) $y = y^u$. There exists $x \in \mathcal{G}$ such that $y = Tx$, and we have $y = y^u = (Tx)^u = T(x^u)$. Now $x^u \succcurlyeq 0$ and $\mathcal{G}$ is pre-regular, so it follows that $x^u \ge 0$. By (a) T is increasing, and so $y = T(x^u) \ge 0$. Hence $T(\mathcal{G})$ is pre-regular.

(d) Assume that $\mathcal{G}$ is quasi-regular. Let $\mathcal{U} = \{w \in T(\mathcal{G}) : w \succcurlyeq 0\}$. We need to show that $\mathcal{U}$ is a mixed lattice semigroup. If $u, v \in \mathcal{U}$, then $u = u^u$ and $v = v^u$ by Theorem 3.4, and there exist elements $x, y \in \mathcal{G}$ such that $u = Tx$ and $v = Ty$. Since $x^u \succcurlyeq 0$, $y^u \succcurlyeq 0$ and $\mathcal{G}$ is quasi-regular, it follows by Theorem 2.8 that $x^u \curlywedge y^u \succcurlyeq 0$. By (a) T is specifically increasing, so we have

$$u \curlywedge v = u^u \curlywedge v^u = (Tx)^u \curlywedge (Ty)^u = T(x^u \curlywedge y^u) \succcurlyeq 0.$$

Hence, by Theorem 2.8 the set $\mathcal{U}$ is a mixed lattice semigroup and so $T(\mathcal{G})$ is quasi-regular. Moreover, if $\mathcal{G}$ is regular, then $\mathcal{G} = \mathcal{S} - \mathcal{S}$ and $T(\mathcal{G}) = T(\mathcal{S}) - T(\mathcal{S})$, where $T(\mathcal{S})$ is a mixed lattice semigroup, as we have just proved. Hence $T(\mathcal{G})$ is regular.

(e) By Theorem 4.2 we need to show that if $0 \le Tx \le Ty$ with $Ty \in T(\mathcal{A})$, then $Tx \in T(\mathcal{A})$. Since $Tx \ge 0$ it follows that $Tx = {}^u(Tx)$. Hence, $(Tx)^l = T(x^l) = 0$, which implies that $x^l = 0$, and so $x \ge 0$. Moreover, $0 \le Tx \le Ty$ implies that $T(y - x) \ge 0$, and again similar reasoning shows that $y - x \ge 0$. Hence we have $0 \le x \le y$ with $y \in \mathcal{A}$, and since $\mathcal{A}$ is an ideal, it follows that $x \in \mathcal{A}$ and so $Tx \in T(\mathcal{A})$. Moreover, since ${}^u(Tx)^l = T({}^u x^l)$ and $\mathcal{A}$ is an ideal it follows that ${}^u(Tx)^l \in T(\mathcal{A})$ if and only if $Tx \in T(\mathcal{A})$. Hence, $T(\mathcal{A})$ is an ideal, by Theorem 4.2. The proof is essentially the same for specific ideals.

(f) Let $y \in \mathcal{H}$ and set $x = T^{-1}y$. Then $T(x^u) = y^u$, and we have $T^{-1}(y^u) = x^u = (T^{-1}y)^u$. Thus T^{-1} is a mixed lattice homomorphism, by Theorem 5.2. □

Let $T : \mathcal{G} \to \mathcal{H}$ be a group homomorphism, and recall that the *kernel* of T is the set $N(T) = \{x \in \mathcal{G} : Tx = 0\}$. We give the following result concerning the kernel of a mixed lattice homomorphism.

Theorem 5.9 *Let $T : \mathcal{G} \to \mathcal{H}$ be a mixed lattice group homomorphism between two mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$. If $\mathcal{G}$ is pre-regular then the kernel $N(T)$ is an ideal, and hence a specific ideal.*

Proof The kernel $N(T)$ is certainly a subgroup of $\mathcal{G}$, and it is also a mixed lattice subgroup of $\mathcal{G}$, for if $x, y \in N(T)$ then $T(x \curlyvee y) = Tx \curlyvee Ty = 0 \curlyvee 0 = 0$, hence $x \curlyvee y \in N(T)$. Assume then that $\mathcal{G}$ is pre-regular. Then by 3.7 we have ${}^u x^l \ge 0$ for all $x \in \mathcal{G}$. Suppose now that $0 \le {}^u y^l \le {}^u x^l$ with $x \in N(T)$. By Theorem 5.8 T is increasing, and so we have

$$T(0) = 0 \le T({}^u y^l) \le T({}^u x^l) = {}^u(Tx)^l = {}^u0^l = 0.$$

Hence, $T({}^u y^l) = {}^u(Ty)^l = 0$, and it follows by Theorem 3.7 (c) that $Ty = 0$, or $y \in N(T)$. This shows that $N(T)$ is an ideal in $\mathcal{G}$, and hence a specific ideal by Proposition 4.7. □

The converse of the preceding theorem holds too, in the sense that if $\mathcal{A}$ is an ideal in a pre-regular mixed lattice group $\mathcal{G}$, then there exists a mixed lattice homomorphism whose kernel is $\mathcal{A}$. We will prove this later in Theorem 6.3.

If $T : \mathcal{G} \to \mathcal{H}$ is a bijective mixed lattice homomorphism, then so is T^{-1} (by Theorem 5.8), and T is called a *mixed lattice isomorphism*, and the mixed lattice groups $\mathcal{G}$ and $\mathcal{H}$ are said to be *isomorphic*. In pre-regular mixed lattice groups, isomorphisms have the following characterization, which is analogous to the characterization of Riesz isomorphisms between two Riesz spaces (see [16, Theorem 18.5]). Additionally, Eriksson–Bique has proved a similar result for isomorphisms between two mixed lattice semigroups ([10, Theorem 4.3]).

Theorem 5.10 *Let $\mathcal{G}$ and $\mathcal{H}$ be pre-regular mixed lattice groups and $T : \mathcal{G} \to \mathcal{H}$ a bijective additive mapping. Then T is a mixed lattice isomorphism if and only if both T and T^{-1} are increasing and specifically increasing.*

Proof If T is an isomorphism, then T^{-1} is also an isomorphism and they are both increasing and specifically increasing by Theorem 5.8. Conversely, assume that T and T^{-1} are increasing and specifically increasing. If $x, y \in \mathcal{G}$, then $x \preccurlyeq x \curlyvee y$ and $y \leq x \curlyvee y$, which implies that $Tx \preccurlyeq T(x \curlyvee y)$ and $Ty \leq T(x \curlyvee y)$. Hence, $Tx \curlyvee Ty \leq T(x \curlyvee y)$. Similar reasoning shows that $T^{-1}(u) \curlyvee T^{-1}(v) \leq T^{-1}(u \curlyvee v)$ for all $u, v \in \mathcal{H}$. Since T is bijective, there exist unique u and v in $\mathcal{H}$ such that $u = Tx$ and $v = Ty$. So we have $x \curlyvee y \leq T^{-1}(Tx \curlyvee Ty)$. Now since T is increasing, we can apply T and obtain $T(x \curlyvee y) \leq Tx \curlyvee Ty$. Hence, $T(x \curlyvee y) = Tx \curlyvee Ty$, and so T is a mixed lattice homomorphism. Since T was assumed to be bijective, T is an isomorphism. □

Mixed lattice homomorphisms also have a domination property described in the next theorem. For a similar result in Riesz spaces, see ([15, 2.6.9]).

Proposition 5.11 *Let $\mathcal{G}$ and $\mathcal{H}$ be mixed lattice groups and $T : \mathcal{G} \to \mathcal{H}$ a mixed lattice homomorphism. If $S : \mathcal{G} \to \mathcal{H}$ is an additive mapping such that S and $T - S$ are both increasing and specifically increasing, then S is a mixed lattice homomorphism.*

Proof Let x and y be elements of $\mathcal{G}$ satisfying $x \curlywedge y = 0$. This implies that $x \succcurlyeq 0$ and $y \geq 0$. To see this, we write $x \curlywedge y = x + 0 \curlywedge (y - x) = x - 0 \curlyvee (x - y) = x - (x - y)^u = 0$, or $x = (x - y)^u \succcurlyeq 0$. Similarly we get $y = {}^u(y - x) \geq 0$. By our hypothesis it follows that $0 \preccurlyeq Sx \preccurlyeq Tx$ and $0 \leq Sy \leq Ty$. Since T is a homomorphism, we have $0 \leq Sx \curlywedge Sy \leq Tx \curlywedge Ty = T(x \curlywedge y) = T(0) = 0$. Hence $Sx \curlywedge Sy = 0$ and so by Theorem 5.2 S is a mixed lattice homomorphism. □

We will now examine the extensions of mixed lattice semigroup homomorphisms to mixed lattice group homomorphisms. The following lemma is a slight modification of Kantorovich's classical extension lemma in Riesz spaces (see [2, Theorem 1.10]).

Lemma 5.12 *Let $\mathcal{S}$ and $\mathcal{T}$ be Abelian semigroups and $T : \mathcal{S} \to \mathcal{T}$ an additive mapping. If $\mathcal{G}$ and $\mathcal{H}$ are the groups generated by $\mathcal{S}$ and $\mathcal{T}$, respectively (i.e. $\mathcal{G} = \mathcal{S} - \mathcal{S}$ and $\mathcal{H} = \mathcal{T} - \mathcal{T}$), then there exists a unique additive mapping $S : \mathcal{G} \to \mathcal{H}$ such that S extends T, that is, $Sx = Tx$ for all $x \in \mathcal{S}$.*

The next result shows that if a mixed lattice group is extended to a group of formal differences, then a mixed lattice semigroup homomorphism can be extended to a mixed lattice group homomorphism.

Theorem 5.13 *Let $T : \mathcal{S} \to \mathcal{T}$ be a mixed lattice semigroup homomorphism between two mixed lattice semigroups $\mathcal{S}$ and $\mathcal{T}$. If $\mathcal{G} = \mathcal{S} - \mathcal{S}$ and $\mathcal{H} = \mathcal{T} - \mathcal{T}$ are the mixed lattice group extensions of $\mathcal{S}$ and $\mathcal{T}$, then T can be extended to a mixed lattice group homomorphism $S : \mathcal{G} \to \mathcal{H}$.*

Proof Let $T : \mathcal{S} \to \mathcal{T}$ be a mixed lattice semigroup homomorphism. Since T is an additive map, by Lemma 5.12 it has an additive extension $S : \mathcal{G} \to \mathcal{H}$ where $\mathcal{G} = \mathcal{S} - \mathcal{S}$ and $\mathcal{H} = \mathcal{T} - \mathcal{T}$ are the mixed lattice group extensions of $\mathcal{S}$ and $\mathcal{T}$, respectively. We need to show that the extension S preserves the mixed lattice operations. Let $w, z \in \mathcal{G}$ and write $w = u - v$ and $z = x - y$ where $u, v, x, y \in \mathcal{S}$. Then by Theorem 2.7

$$w \curlyvee z = (u - v) \curlyvee (x - y) = (u + y) \curlyvee (x + v) - (v + y).$$

Since $S = T$ on $\mathcal{S}$, we get

$$\begin{aligned} S(w \curlyvee z) &= S((u+y) \curlyvee (x+v)) - S((v+y)) \\ &= T((u+y) \curlyvee (x+v)) - T(v+y) \\ &= T(u+y) \curlyvee T(x+v) - T(v+y) \\ &= (Tu+Ty) \curlyvee (Tx+Tv) - Tv - Ty \\ &= (Tu-Tv) \curlyvee (Tx-Ty) \\ &= Sw \curlyvee Sz \end{aligned}$$

for all $w, z \in \mathcal{G}$. It now follows from Theorem 5.2 that S is a mixed lattice group homomorphism. □

6 Quotient mixed lattice groups

In this section we examine how quotient group constructions work in mixed lattice groups, and we obtain results that have analogues in Riesz space theory. For these we refer to [16, Chapter 3, §18].

Let $\mathcal{G}$ be an additive group and $\mathcal{S}$ a subgroup of $\mathcal{G}$. An equivalence relation $\sim$ in $\mathcal{G}$ is defined as $x \sim y$ iff $x - y \in \mathcal{S}$. The equivalence class (or coset) of $x \in \mathcal{G}$ is denoted by $[x]$, and we identify $\mathcal{S} = [0]$. The quotient group of $\mathcal{G}$ modulo $\mathcal{S}$ is denoted by $\mathcal{G}/\mathcal{S}$. The set $\mathcal{G}/\mathcal{S}$ with addition defined by

$$[x] + [y] = [x + y] \quad \text{for all } [x] \text{ and } [y],$$

is a group, where $-[x] = [-x]$.

Assume now that $\mathcal{G}$ is a pre-regular mixed lattice group. We will prove that if $\mathcal{A}$ is an ideal in $\mathcal{G}$ then we can define initial and specific orders in $\mathcal{G}/\mathcal{A}$ in such manner that $\mathcal{G}/\mathcal{A}$ becomes a mixed lattice group. First we define the partial orderings.

Definition 6.1 Let $\mathcal{A}$ be an ideal in a pre-regular mixed lattice group $\mathcal{G}$. If $[x], [y] \in \mathcal{G}/\mathcal{A}$ then we define $[x] \leq [y]$ whenever there exists elements $x \in [x]$ and $y \in [y]$ such that $x \leq y$ in $\mathcal{G}$. Similarly, $[x] \preccurlyeq [y]$ whenever there exists elements $x \in [x]$ and $y \in [y]$ such that $x \preccurlyeq y$ in $\mathcal{G}$.

Before we proceed we need to verify that the relations given in the above definition are indeed partial orderings in $\mathcal{G}/\mathcal{A}$. We will begin with the initial order. Reflexivity is obvious. For transitivity, assume that $[x] \leq [y]$ and $[y] \leq [z]$. Let $x \in [x]$, $y_1, y_2 \in [y]$ and $z \in [z]$ be such that $x \leq y_1$ and $y_2 \leq z$. Then

$$x \leq y_1 = y_2 + (y_1 - y_2) \leq z + (y_1 - y_2),$$

but by definition $y_1 - y_2 \in \mathcal{A}$, and so $z + (y_1 - y_2) \in [z]$. This shows that $[x] \leq [z]$.

Finally, to prove antisymmetry, let $[x] \leq [y]$ and $[y] \leq [x]$. Then there exist $x_1, x_2 \in [x]$ and $y_1, y_2 \in [y]$ such that $x_1 \leq y_1$ and $y_2 \leq x_2$. Since $\mathcal{A}$ is a subgroup, it follows that

$$0 \leq y_1 - x_1 \leq (y_1 - x_1) + (x_2 - y_2) = (x_2 - x_1) + (y_1 - y_2) \in \mathcal{A}.$$

Since $\mathcal{A}$ is an ideal, it follows that $y_1 - x_1 \in \mathcal{A}$, that is, $[x] = [y]$. Noting that $\mathcal{A}$ is also a specific ideal (by Proposition 4.7), we can show similarly that the specific order given in Definition 6.1 is likewise a partial ordering in $\mathcal{G}/\mathcal{A}$.

We will now show that with these orderings the set $\mathcal{G}/\mathcal{A}$ becomes a pre-regular mixed lattice group.

Theorem 6.2 *If $\mathcal{A}$ is an ideal in a pre-regular mixed lattice group $\mathcal{G}$, then $\mathcal{G}/\mathcal{A}$ is a pre-regular mixed lattice group with respect to initial and specific orders given in Definition 6.1.*

Proof First we must show that these partial orderings are compatible with the group structure, that is, $\mathcal{G}/\mathcal{A}$ is a partially ordered group with respect to orderings $\leq$ and $\preccurlyeq$. To this end, assume that $[x] \leq [y]$ and choose $x \in [x]$ and $y \in [y]$ such that $x \leq y$, and let $z \in [z]$. Then $x+z \leq y+z$, and so $[x+z] \leq [y+z]$. By definition of addition in $\mathcal{G}/\mathcal{A}$, this is equivalent to $[x] + [z] \leq [y] + [z]$. The specific order $\preccurlyeq$ is treated similarly.

It remains to show that $\mathcal{G}/\mathcal{A}$ is a mixed lattice group. We will do this by showing that $[x]^u$ exists for all $x \in \mathcal{G}$ and it is equal to $[x^u]$. Let $x \in \mathcal{G}$ and notice that $x^u \geq x$ and $x^u \succcurlyeq 0$, so we have $[x] \leq [x^u]$ and $[0] \preccurlyeq [x^u]$. Assume then that $[x] \leq [y]$ and $[0] \preccurlyeq [y]$. Then there exist $x_1 \in [x]$ and $y_1, y_2 \in [y]$ such that $x_1 \leq y_1$ and $0 \preccurlyeq y_2$. It follows that $x_1 \leq y_1 \leq y_2 \curlyvee y_1$ and $x - x_1 \leq 0 \curlyvee (x - x_1) = (x - x_1)^u$. This implies that

$$x = x_1 + (x - x_1) \leq y_2 \curlyvee y_1 + (x - x_1)^u = y_2 + (y_1 - y_2)^u + (x - x_1)^u.$$

Now we also have

$$y_2 + (y_1 - y_2)^u + (x - x_1)^u \succcurlyeq 0,$$

and therefore

$$x^u = 0 \curlyvee x \leq y_2 + (y_1 - y_2)^u + (x - x_1)^u.$$

Now, since $y_1, y_2 \in [y]$ and $x, x_1 \in [x]$, we have $y_1 - y_2 \in \mathcal{A}$ and $x - x_1 \in \mathcal{A}$, and since $\mathcal{A}$ is a mixed lattice subgroup, it follows that $(y_1 - y_2)^u \in \mathcal{A}$ and $(x - x_1)^u \in \mathcal{A}$. Hence we have

$$x^u \leq y_2 + (y_1 - y_2)^u + (x - x_1)^u \in y_2 + \mathcal{A} = [y_2] = [y].$$

Thus, $[x^u] \leq [y]$ and so we have proved that

$$[x^u] = \min\{[w] \in \mathcal{G}/\mathcal{A} : [0] \preccurlyeq [w] \text{ and } [x] \leq [w]\} = [0] \curlyvee [x] = [x]^u,$$

and so $\mathcal{G}/\mathcal{A}$ is a mixed lattice group, by Theorem 2.12.

Finally, assume that $\mathcal{G}$ is pre-regular and let $[x] \preccurlyeq [y]$. Then there exist $x \in [x]$ and $y \in [y]$ satisfying $x \preccurlyeq y$ in $\mathcal{G}$. Since $\mathcal{G}$ is pre-regular, this implies that $x \leq y$. Consequently, $[x] \leq [y]$ and so $\mathcal{G}/\mathcal{A}$ is pre-regular. □

The mixed lattice group $\mathcal{G}/\mathcal{A}$ is called the *quotient mixed lattice group of $\mathcal{G}$ modulo $\mathcal{A}$*.

Now we observe that there is a close connection between mixed lattice homomorphisms and quotient mixed lattice groups.

Theorem 6.3 *If $\mathcal{A}$ is an ideal in a pre-regular mixed lattice group $\mathcal{G}$, then $\mathcal{G}/\mathcal{A}$ is a mixed lattice homomorphic image of $\mathcal{G}$ under the canonical mapping $x \mapsto [x]$. Conversely, if $\mathcal{G}$ is pre-regular and $T : \mathcal{G} \to \mathcal{H}$ is a surjective mixed lattice homomorphism with kernel $N(T)$, then $\mathcal{G}/N(T)$ and $\mathcal{H}$ are isomorphic.*

Proof Let $\mathcal{G}$ be a pre-regular mixed lattice group and $\mathcal{A}$ an ideal in $\mathcal{G}$. Inspecting the proof of Theorem 6.2 reveals that $[x^u] = [x]^u$ in $\mathcal{G}/\mathcal{A}$, and so by Theorem 5.2 the mapping $x \mapsto [x]$ is a mixed lattice homomorphism of $\mathcal{G}$ onto $\mathcal{G}/\mathcal{A}$, and $\mathcal{A}$ is the kernel of this homomorphism.

Conversely, let $T : \mathcal{G} \to \mathcal{H}$ be a surjective mixed lattice homomorphism with kernel $N(T)$. Then by Theorem 5.9, $N(T)$ is an ideal in $\mathcal{G}$, and $\mathcal{G}/N(T)$ is a mixed lattice group by Theorem 6.2. Now the mapping $f : \mathcal{G}/N(T) \to \mathcal{H}$, where $f([x]) = Tx$, is a mixed lattice isomorphism of $\mathcal{G}/N(T)$ onto $\mathcal{H}$. To prove this, we need to show that f is a bijective mixed lattice homomorphism. First, if $Tx = Ty$ then $T(x-y) = 0$, and so $x - y \in N(T)$. But this means that x and y belong to the same equivalence class in $\mathcal{G}/N(T)$, that is, $[x] = [y]$. Thus f is injective. Next, let $z \in \mathcal{H}$. Since T is surjective, there exists an element $x \in \mathcal{G}$ such that $z = Tx$. Now we have $f([x]) = Tx = z$, and so f is surjective. Finally, since $[x^u] = [x]^u$ in $\mathcal{G}/N(T)$, and T is a homomorphism, it follows that

$$f([x]^u) = f([x^u]) = T(x^u) = (Tx)^u = (f([x]))^u.$$

Hence, f is a bijective homomorphism, and therefore an isomorphism. □

Combining the two preceding theorems with Theorem 5.8(d) we get the following result.

Corollary 6.4 *If $\mathcal{A}$ is an ideal in a quasi-regular (regular) mixed lattice group $\mathcal{G}$, then $\mathcal{G}/\mathcal{A}$ is a quasi-regular (regular, respectively) mixed lattice group with respect to initial and specific orders given in Definition 6.1.*

We conclude this section with an example.

Example 6.5 Let $\mathcal{G} = BV([0, 1])$ be the set of all functions of bounded variation on the interval $[0, 1]$ (see Example 2.15). In Example 4.11 we saw that the set $\mathcal{S}$ of all constant functions is a mixed lattice subgroup of $\mathcal{G}$, but not an ideal. In Example 4.11 we also considered the set $\mathcal{A} = \{f \in \mathcal{G} : f(0) = 0\}$, which was seen to be an ideal in $\mathcal{G}$. As a group, the quotient group $\mathcal{G}/\mathcal{S}$ can be identified with the subgroup $\mathcal{A}$ of $\mathcal{G}$. If we define a mapping $T : \mathcal{G}/\mathcal{S} \to \mathcal{A}$ by $T([f]) = f - f(0)$, then T is clearly an additive map (that is, a group homomorphism), which is bijective. Indeed, if $T([f]) = T([g])$ and choose any $f \in [f]$ and $g \in [g]$, then $f - f(0) = g - g(0)$, or $f - g = f(0) - g(0)$. Now $f(0) - g(0)$ is a constant and so $f - g \in \mathcal{S}$, or $[f] = [g]$. Thus T is injective. Let $f \in \mathcal{A}$. Then $f \in \mathcal{G}$ and so $f \in [f] \in \mathcal{G}/\mathcal{S}$, and we have

$T([f]) = f - f(0) = f$, since $f(0) = 0$ for all $f \in \mathcal{A}$. Hence T is surjective. This shows that the groups $\mathcal{G}/\mathcal{S}$ and $\mathcal{A}$ are isomorphic.

Now, since $\mathcal{S}$ is not an ideal, Theorem 6.1 does not apply here. In particular, the canonical mapping $x \mapsto [x]$ is not a mixed lattice homomorphism, because the kernel of this mapping is $\mathcal{S}$, and the kernel of a mixed lattice homomorphism is necessarily an ideal (by Theorem 5.9).

Since $\mathcal{A}$ is an ideal in $\mathcal{G}$ (see Example 4.11), the quotient group $\mathcal{G}/\mathcal{A}$ is a mixed lattice group and the canonical map $f \mapsto [f]$ is a mixed lattice homomorphism. We observe that the elements of $\mathcal{G}/\mathcal{A}$ are the equivalence classes determined by constant functions, that is, $\mathcal{G}/\mathcal{A} = \{[f] : f \text{ is constant}\}$. Indeed, if f and g are in $[f]$, then $f - g \in \mathcal{A}$, or $(f - g)(0) = 0$. This implies that $f(0) = g(0)$ and hence $[f] = [g]$ if and only if $f(0) = g(0)$. Since $f(0)$ is a constant, we have $[f] = [f(0)]$.

Now the mapping $T : \mathcal{G} \to \mathcal{S}$ defined by $Tf = f(0)$ is a surjective mixed lattice homomorphism, since

$$T(f \curlyvee g) = (f \curlyvee g)(0) = \max\{f(0), g(0)\} = Tf \curlyvee Tg,$$

and the kernel of T is $\mathcal{A}$. Hence, by Theorem 6.3, $\mathcal{G}/\mathcal{A}$ and $\mathcal{S}$ are isomorphic mixed lattice groups.

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Ideals, bands and direct sum decompositions in mixed lattice vector spaces

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Positivity

Ideals, bands and direct sum decompositions in mixed lattice vector spaces

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Abstract
A mixed lattice vector space is a partially ordered vector space with two partial orderings and certain lattice-type properties. In this paper we first give some fundamental results in mixed lattice groups, and then we investigate the structure theory of mixed lattice vector spaces, which can be viewed as a generalization of the theory of Riesz spaces. More specifically, we study the properties of ideals and bands in mixed lattice spaces, and the related idea of representing a mixed lattice space as a direct sum of disjoint bands. Under certain conditions, these decompositions can also be given in terms of order projections.

Keywords Mixed lattice · Mixed lattice semigroup · Riesz space · Ideal · Band · Disjointness · Direct sum

Mathematics Subject Classification Primary 06F20 · 46A40

1 Introduction

The idea of equipping a vector space with two partial orderings plays an important role in classical potential theory [4]. During the 1970 s and early 1980 s, M. Arsove and H. Leutwiler introduced the notion of a mixed lattice semigroup which provides a rather general setting for axiomatic potential theory [2]. The novelty of their theory was that it mixed two partial orderings in a semigroup in such way that the resulting structure is not a lattice, in general, but it has many lattice-type properties, and the interplay between the two partial orderings plays a fundamental role in the theory. Although Arsove and Leutwiler formulated their theory in the semigroup setting, a similar mixed lattice order structure can also be imposed on groups and vector spaces

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where non-positive elements are present. This gives rise to the notions of mixed lattice groups and mixed lattice vector spaces. In this paper, we are mostly concerned with the latter. A mixed lattice space is a real partially ordered vector space with two partial orderings, where the usual lattice operations (i.e. the supremum and infimum of two elements) are replaced with asymmetric mixed envelopes that are formed with respect to the two partial orderings. As a consequence, the theory of mixed lattice structures is asymmetric in nature. For example, the mixed envelopes do not have commutative or distributive properties, in contrast to the theory of Riesz spaces, where the lattice operations are commutative and distributive. On the other hand, mixed lattice spaces have also many similarities to Riesz spaces. In fact, a Riesz space is just a special case of a mixed lattice vector space, in which the two partial orderings coincide. Early studies of mixed lattice groups were done by Eriksson-Bique [6, 7], and more recently by the authors in [5]. The present paper is a continuation of the research that was commenced in [5–7].

Ideals and bands are the main structural components of a Riesz space, and Riesz spaces can be decomposed into order direct sums of disjoint bands. Apart from Riesz spaces, these concepts have recently been studied in more general ordered vector spaces [8, 9], as well as in mixed lattice semigroups in [2]. It is therefore natural to explore these ideas also in the mixed lattice space setting, and this is the main topic of the present paper.

First we give a brief survey of terminology, definitions and basic results that will be needed in the subsequent sections. Section 3 contains some important results on the properties of the mixed envelopes. Many of these results are known in mixed lattice semigroups through the work of Arsove and Leutwiler in [2], but they haven't been studied in the group setting. The main difficulty here is that many of the results that hold in a mixed lattice semigroup depend on the fact that all the elements are positive. Therefore, some of these properties do not hold in the group setting without some limitations. We also give the mixed lattice version of the fundamental dominated decomposition theorem.

The concept of an ideal in mixed lattice groups and vector spaces was introduced in [5]. Many properties of the most relevant subspaces, including ideals, are determined by their sets of positive elements. In this context, positive cones play a fundamental role and we discuss some properties of mixed lattice cones, which generate mixed lattice subspaces. In Sect. 4 we study the structure of ideals, and we introduce the notion of a band in mixed lattice spaces and give some basic results concerning bands. The two partial orderings give rise to two different types of ideals, called ideals and specific ideals, depending on which partial ordering is considered. Consequently, we also have different notions of a band, depending on the type of the underlying ideal. In addition to these, we introduce an intermediate notion of a quasi-ideal, which plays an important role in the structure theory.

Disjoint complements are introduced in Sect. 5. The disjointness of two positive elements is defined in a similar manner as in Riesz spaces, by requiring that the lower envelope of the two elements is zero. However, the asymmetric nature of the mixed envelopes leads to two distinct one-sided notions of disjointness, the left and right disjointness. The disjoint complements and direct sum decompositions in mixed lattice semigroups were studied by Arsove and Leutwiler in [2], where they showed that the left and right disjoint complements of any non-empty set in a mixed lattice semigroup have certain band-type properties. In mixed lattice vector spaces the situation is a bit more complicated, again due to the existence of non-positive elements. In order to obtain a satisfactory theory, we consider disjointness first for positive elements, and we then define the left and right disjoint complements as the specific ideals and ideals, respectively, generated by certain cones of positive elements that satisfy the disjointness conditions. This approach turns out to be quite natural, and we also find it to be compatible with the existing theory of mixed lattice semigroups. We then investigate the structure and properties of the disjoint complements, such as the band properties and the structure of the sub-cones that generate the disjoint complements. We also introduce the notion of symmetric absolute value in mixed lattice spaces and show that the right disjoint complement can be described in terms of the absolute value. This approach corresponds to the usual definition of the disjoint complement in a Riesz space. Finally, we give some results concerning bands in Archimedean mixed lattice spaces.

In the final section we study decompositions of mixed lattice spaces, where the space can be written as a direct sum of an ideal and a specific ideal that are disjoint complements of each other. The direct sum decompositions give rise to band projections which can be expressed in terms of order projection operators. Of course, these projection properties have their well-known analogues in Riesz space theory. For an account of the theory and terminology of Riesz spaces, we refer to [10] and [11].

2 Preliminaries

The fundamental structure on which most of the theory of mixed lattice groups and vector spaces is based on, is called a *mixed lattice semigroup*. Let $S(+, \leq)$ be a positive partially ordered abelian semigroup with zero element. The semigroup $S(+, \leq)$ is also assumed to have the cancellation property: $x + z \leq y + z$ implies $x \leq y$ for all $x, y, z \in S$. The partial order $\leq$ is called the *initial order*. A second partial ordering $\preccurlyeq$ (called the *specific order*) is then defined on S by $x \preccurlyeq y$ if $y = x + m$ for some $m \in S$. Note that this implies that $x \succcurlyeq 0$ for all $x \in S$. The semigroup $(S, +, \leq, \preccurlyeq)$ is called a *mixed lattice semigroup* if, in addition, the *mixed lower envelope*

$$x \curlywedge y \;=\; \max\{\, w \in S : \; w \preccurlyeq x \;\text{ and }\; w \leq y \,\}$$

and the *mixed upper envelope*

$$x \curlyvee y \;=\; \min\{\, w \in S : \; w \succcurlyeq x \;\text{ and }\; w \geq y \,\},$$

exist for all $x, y \in S$, and they satisfy the identity

$$x \curlyvee y + y \curlywedge x = x + y. \tag{2.1}$$

In the above expressions the minimum and maximum are taken with respect to the initial order $\leq$.

Typical examples of mixed lattice semigroups are constructed by starting with some vector space V and a cone C in V which generates a partial ordering $\leq$. Then a sub-cone S of C is taken as the semigroup in which the specific order $\preccurlyeq$ is defined as the partial ordering induced by the semigroup S itself, that is, S is taken as the positive cone for the partial ordering $\preccurlyeq$. This procedure turns S into a mixed lattice semigroup, provided that the conditions in the above definitions are satisfied. Many examples of mixed lattice structures are given in [5].

A *mixed lattice group* is a partially ordered commutative group $(G, +, \leq, \preccurlyeq)$ with two partial orderings $\leq$ and $\preccurlyeq$ (called again the initial order and the specific order, respectively) such that the mixed upper and lower envelopes $x \curlyvee y$ and $x \curlywedge y$, as defined above, exist in G for all $x, y \in G$. A sub-semigroup S of a mixed lattice group G is called a *mixed lattice sub-semigroup* of G if $x \curlyvee y$ and $x \curlywedge y$ belong to S whenever $x, y \in S$. It should be remarked here that the additional identity (2.1) in the definition of a mixed lattice semigroup is then automatically satisfied because the identity holds in G (see further properties of the mixed envelopes below). A mixed lattice group G is called *quasi-regular* if the set $G_{sp} = \{x \in G : x \succcurlyeq 0\}$ is a mixed lattice sub-semigroup of G, and G is called *regular* if G is quasi-regular and every $x \in G$ can be written as $x = u - v$ where $u, v \in G_{sp}$.

There are several fundamental rules for the mixed envelopes that hold in quasi-regular mixed lattice groups. These are listed below, and will be used in this paper frequently. For more details concerning these rules and their proofs we refer to [5] and [7].

The following hold for all elements x, y and z in a quasi-regular mixed lattice group G.

$$(M1) \quad x \curlyvee y + y \curlywedge x = x + y$$

$$(M2a) \quad z + x \curlyvee y = (x + z) \curlyvee (y + z)$$

$$(M2b) \quad z + x \curlywedge y = (x + z) \curlywedge (y + z)$$

$$(M3) \quad x \curlyvee y = -(-x \curlywedge - y)$$

$$(M4) \quad x \curlywedge y \preccurlyeq x \preccurlyeq x \curlyvee y \quad \text{and} \quad x \curlywedge y \leq y \leq x \curlyvee y$$

$$(M5) \quad x \preccurlyeq u \quad \text{and} \quad y \leq v \implies x \curlyvee y \leq u \curlyvee v \quad \text{and} \quad x \curlywedge y \leq u \curlywedge v$$

$$(M6a) \quad x \preccurlyeq y \iff x \curlyvee y = y \iff y \curlywedge x = x$$

$$(M6b) \quad x \leq y \iff y \curlyvee x = y \iff x \curlywedge y = x$$

$$(M7) \quad x \preccurlyeq y \implies x \leq y$$

$$(M8a) \quad x \preccurlyeq z \quad \text{and} \quad y \preccurlyeq z \implies x \curlyvee y \preccurlyeq z$$

$$(M8b) \quad z \preccurlyeq x \quad \text{and} \quad z \preccurlyeq y \implies z \preccurlyeq x \curlywedge y$$

We should mention here that (M1)–(M6a) hold more generally in every mixed lattice group. Moreover, quasi-regularity is equivalent to the properties (M8a) and (M8b), and these in turn imply (M7) (but not conversely). A mixed lattice group in which (M7) holds is called *pre-regular*.

3 Additional properties of mixed lattice groups

In this section we give several further properties of mixed lattice groups. Many of these have been studied in mixed lattice semigroups in [2], but some of them do not carry over to the group setting without some restrictions.

We begin by introducing a useful tool (which is due to M. Arsove and H. Leutwiler, [2]) for studying the properties of the mixed envelopes. For $x, u \in G$ we define the mapping $S_x(u) = \min\{w \succcurlyeq 0 : u \leq w + x\}$, where the minimum is taken with respect to $\leq$. We will first show that the element $S_x(u)$ exists for all $x, u \in G$ and it has certain basic properties.

Proposition 3.1 *Let G be a quasi-regular mixed lattice group. The mapping S_x defined above has the following properties.*

(a) The element $S_x(u) = \min\{w \succcurlyeq 0 : u \leq w + x\}$ exists for all $x, u \in G$ and it is given by $S_x(u) = x \curlyvee u - x = u - u \curlywedge x$.
(b) If $x \succcurlyeq 0$ then $S_x S_y(u) = S_{x+y}(u)$ holds for all $u, y \in G$. Moreover, if $x, y \succcurlyeq 0$ then $S_x S_y(u) = S_y S_x(u) = S_{x+y}(u)$ for all $u \in G$.
(c) $S_{x+a}(u + a) = S_x(u)$ for all $x, u, a \in G$.
(d) If $u \preccurlyeq v$ then $S_x(u) = S_{v-u} S_x(v)$ for all $x \in G$.

Proof (a) Let $A = \{w \succcurlyeq 0 : u \leq w + x\}$. Let $m = x \curlyvee u - x$. The equality $x \curlyvee u - x = u - u \curlywedge x$ follows immediately from (M1). First we note that $m \succcurlyeq 0$ and

$$m + x = x \curlyvee u - x + x = x \curlyvee u \geq u,$$

and so $m \in A$. Assume then that $z \in A$. Then $u \leq z + x$ and since $z \succcurlyeq 0$, we also have $u \preccurlyeq z + u$. By (M5) and (M2) it now follows that $u \leq (z + u) \curlywedge (z + x) = z + u \curlywedge x$ and this implies that $m = u - u \curlywedge x \leq z$. This shows that $m = S_x(u) = \min\{w \succcurlyeq 0 : u \leq w + x\}$.

(b) By part (a) we have $u \leq y \curlyvee u = (y \curlyvee u - y) + y = S_y(u) + y$ for all $y, u \in G$, and similarly $S_y(u) \leq S_x S_y(u) + x$. It follows that $u \leq S_x S_y(u) + x + y$ and by the definition of S_x we obtain the inequality $S_{x+y}(u) \leq S_x S_y(u)$. Similarly, exchanging the roles of x and y gives $S_{x+y}(u) \leq S_y S_x(u)$. On the other hand, we have $u \leq S_{x+y}(u) + x + y$, and if $x \succcurlyeq 0$ then $S_{x+y}(u) + x \succcurlyeq 0$, and it follows again from the definition that $S_y(u) \leq S_{x+y}(u) + x$. Hence, $S_x S_y(u) \leq S_{x+y}(u)$ and so the equality $S_x S_y(u) = S_{x+y}(u)$ holds for all $u, y \in G$. If also $y \succcurlyeq 0$ then we can exchange the roles of x and y above to get $S_x S_y(u) = S_y S_x(u) = S_{x+y}(u)$ for all $u \in G$.

(c) The translation invariance property follows immediately from the definition of S_x.

(d) Let $u \preccurlyeq v$. Then $a = v - u \succcurlyeq 0$ and using (b) and (c) gives

$$S_x(u) = S_{x+a}(u + a) = S_a S_x(u + a) = S_{v-u} S_x(v).$$

□

Using the above results for S_x we can prove several additional properties of the mixed envelopes. We begin with the following theorem, which gives two important inequalities for the mixed envelopes. They were first studied in mixed lattice semigroups by Arsove and Leutwiler [2, Theorem 3.2]. Later, Eriksson–Bique showed that they hold also in a group extension of a mixed lattice semigroup, and therefore in every regular mixed lattice group [6, Theorem 3.5]. We will now prove that the inequalities hold more generally in every quasi-regular mixed lattice group. Later on, these inequalities will play an important role in the development of the theory.

Theorem 3.2 *Let G be a quasi-regular mixed lattice group. If $x \preccurlyeq y$ then the inequalities*

$$u \curlywedge x \preccurlyeq u \curlywedge y \quad \text{and} \quad u \curlyvee x \preccurlyeq u \curlyvee y$$

hold for all $u \in G$.

Proof First we observe that the first inequality is equivalent to

$$0 \preccurlyeq u \curlywedge y - u \curlywedge x = (u - u \curlywedge x) - (u - u \curlywedge y) = S_x(u) - S_y(u).$$

Now if $x \preccurlyeq y$ then $y = a + x$, where $a = y - x \succcurlyeq 0$. Using the properties of the mapping S_x we get

$$S_y(u) = S_{x+a}(u) = S_a(S_x(u)) = S_{y-x}(S_x(u)) = S_x(u) - S_x(u) \curlywedge (y - x),$$

and this yields

$$S_x(u) - S_y(u) = S_x(u) \curlywedge (y - x) = (u - u \curlywedge x) \curlywedge (y - x).$$

Hence we have $u \curlywedge y - u \curlywedge x = (u - u \curlywedge x) \curlywedge (y - x)$. Now $u - u \curlywedge x \succcurlyeq 0$ and $y - x \succcurlyeq 0$, and since G is quasi-regular, it follows by (M8b) that $(u - u \curlywedge x) \curlywedge (y - x) \succcurlyeq 0$. Thus we have shown that $u \curlywedge y - u \curlywedge x \succcurlyeq 0$, or $u \curlywedge x \preccurlyeq u \curlywedge y$.

The proof of the other inequality is similar. □

The next example shows that the assumptions in the last theorem cannot be weakened, that is, the inequalities do not hold if G is not quasi-regular.

Example 3.3 Let $G = (\mathbb{Z} \times \mathbb{Z}, \leq, \preccurlyeq)$ and define partial orders $\leq$ and $\preccurlyeq$ as follows. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ then $x \leq y$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$. In addition, $x \preccurlyeq y$ iff $x = y$ or $y_1 \geq x_1 + 1$ and $y_2 \geq x_2 + 1$. Then G is a mixed lattice group which is pre-regular but not quasi-regular (see [5, Example 2.23]). If $x = (1, 0)$, $y = (2, 1)$ and $u = (0, 0)$ then $u \curlyvee x = (1, 1)$ and $u \curlyvee y = (2, 1)$. Now $x \preccurlyeq y$ but $u \curlyvee x \preccurlyeq u \curlyvee y$ does not hold.

The inequalities in Theorem 3.2 have several implications. For example, the following one-sided associative and distributive laws hold for the mixed envelopes. The proof is exactly the same as in the mixed lattice semigroup case, see [2, Theorems 3.3 and 3.5].

Theorem 3.4 *If G is a quasi-regular mixed lattice group then the following one-sided associative laws*

$$(x \curlywedge y) \curlywedge z \geq x \curlywedge (y \curlywedge z) \quad \textit{and} \quad (x \curlyvee y) \curlyvee z \leq x \curlyvee (y \curlyvee z)$$

and one-sided distributive laws

$$x \curlywedge (y \curlyvee z) \geq (x \curlywedge y) \curlyvee (x \curlywedge z) \quad \textit{and} \quad x \curlyvee (y \curlywedge z) \leq (x \curlyvee y) \curlywedge (x \curlyvee z)$$

hold for all $x, y, z \in G$.

In the sequel, $\sup E$ and $\inf E$ stand for the supremum and infimum of a subset E with respect to the initial order $\leq$. The supremum and infimum of E with respect to the specific order $\preccurlyeq$ are denoted by $\operatorname{sp}\sup E$ and $\operatorname{sp}\inf E$, respectively.

The following result is originally due to Boboc and Cornea (see [3, Proposition 2.1.4]). The more general mixed lattice version is given in [2, Theorem 4.2].

Lemma 3.5 *Let E be a subset of a quasi-regular mixed lattice group G such that $u_0 = \sup E$ exists in G. If $w \in G$ is an element such that $x \preccurlyeq w$ for all $x \in E$ then $u_0 \preccurlyeq w$. Similarly, if $v_0 = \inf E$ exists and $w \in G$ is such that $w \preccurlyeq x$ for all $x \in E$ then $w \preccurlyeq v_0$.*

Proof If $u_0 = \sup E$ then $x \preccurlyeq u_0$ for all $x \in E$, and so if $x \preccurlyeq w$ for all $x \in E$ then by (M5), $x \leq w \curlywedge u_0 \leq u_0$ for all $x \in E$. This implies that $u_0 = w \curlywedge u_0 \preccurlyeq w$. The result concerning the infimum can be proved by a similar argument. □

We obtain stronger results if the supremum and specific supremum of a subset are equal.

Definition 3.6 Let E be a subset of a mixed lattice group G such that $\sup E$ and $\operatorname{sp}\sup E$ exist in G, and $u_0 = \sup E = \operatorname{sp}\sup E$. The element u_0 is called the *strong supremum* of E and it is denoted by $u_0 = \operatorname{str}\sup E$. The *strong infimum* v_0 of E is defined similarly, and it is denoted by $v_0 = \operatorname{str}\inf E$.

The following result can now be proved exactly the same way as in the theory of mixed lattice semigroups (see [2], p. 23). We include the proof here for completeness.

Proposition 3.7 *Let E be a subset of a quasi-regular mixed lattice group G such that $u_0 = \operatorname{str}\sup E$ exists in G. Then for all $x \in G$*

$$\operatorname{str}\sup_{u \in E}(x \curlywedge u) = x \curlywedge u_0 \quad \textit{and} \quad \operatorname{str}\sup_{u \in E}(x \curlyvee u) = x \curlyvee u_0.$$

Similarly, if $v_0 = \operatorname{str}\inf E$ exists then for all $x \in G$

$$\operatorname{str}\inf_{u \in E}(x \curlywedge u) = x \curlywedge v_0 \quad \textit{and} \quad \operatorname{str}\inf_{u \in E}(x \curlyvee u) = x \curlyvee v_0.$$

Proof If $x \in G$ and $u_0 = \operatorname{str}\sup E$ then By (M5), $x \curlywedge u \leq x \curlywedge u_0$ and $u \curlyvee x \leq u_0 \curlyvee x$ for all $u \in E$. Let w be any element such that $x \curlywedge u \leq w$ for all $u \in E$. Then by (M1) we have

$$x + u = x \curlywedge u + u \curlyvee x \leq x \curlywedge u + u_0 \curlyvee x \leq w + u_0 \curlyvee x$$

for all $u \in E$. Thus $w + u_0 \curlyvee x - x$ is a ($\leq$)-upper bound of E, and since $u_0 = \sup E$, by (M1) this implies that

$$x \curlywedge u_0 + u_0 \curlyvee x = u_0 + x \leq (w + u_0 \curlyvee x - x) + x = w + u_0 \curlyvee x.$$

Hence $x \curlywedge u_0 \leq w$, and this shows that $x \curlywedge u_0 = \sup\{x \curlywedge u : u \in E\}$. On the other hand, by Theorem 3.2 we have $x \curlywedge u \preccurlyeq x \curlywedge u_0$ for all $u \in E$. If $x \curlywedge u \preccurlyeq w$ for all $u \in E$ then it follows by Lemma 3.5 that $x \curlywedge u_0 \preccurlyeq w$, and so $x \curlywedge u_0 = \operatorname{sp}\sup\{x \curlywedge u : u \in E\}$. This shows that $x \curlywedge u_0 = \operatorname{str}\sup\{x \curlywedge u : u \in E\}$. The other identities can be proved in a similar manner. □

In the sequel, we will use the following notation. The inequalities $x \preccurlyeq y$ and $y \leq z$ are written more concisely as $x \preccurlyeq y \leq z$. Similarly, the notation $x \leq y \preccurlyeq z$ means that $x \leq y$ and $y \preccurlyeq z$.

We conclude this section by showing that mixed lattice groups have the dominated decomposition property. We actually have different variants of this property that will be useful.

Theorem 3.8 *Let G be a quasi-regular mixed lattice group.*

(a) Let $u \succcurlyeq 0$, $v_1 \geq 0$ and $v_2 \succcurlyeq 0$ be elements of G satisfying $u \leq v_1 + v_2$. Then there exist elements u_1 and u_2 such that $0 \leq u_1 \leq v_1$, $0 \preccurlyeq u_2 \leq v_2$ and $u = u_1 + u_2$. Moreover, if $v_1 \succcurlyeq 0$ then $u_1 \succcurlyeq 0$, and if $u \preccurlyeq v_1 + v_2$ then $u_2 \preccurlyeq v_2$.

(B) Let $u \geq 0$, $v_1 \succcurlyeq 0$ and $v_2 \geq 0$ be elements of G satisfying $u \preccurlyeq v_1 + v_2$. Then there exist elements u_1 and u_2 such that $0 \leq u_1 \preccurlyeq v_1$, $0 \leq u_2 \leq v_2$ and $u = u_1 + u_2$. Moreover, if $u \succcurlyeq 0$ then $u_1 \succcurlyeq 0$.

Proof *(a)* The element $u_1 = u \curlywedge v_1$ satisfies $u_1 \leq v_1$ and $u_1 \geq 0$ (and $u_1 \succcurlyeq 0$ if $v_1 \succcurlyeq 0$). Let $u_2 = u - u_1$. Then $u = u_1 + u_2$ and $u_2 \succcurlyeq 0$, since $u_1 \preccurlyeq u$. It remains to show that $u_2 \leq v_2$. For this, we note that $0 \preccurlyeq v_2$ and $u - v_1 \leq v_2$, and so we have

$$u_2 = u - u \curlywedge v_1 = u + (-u) \curlyvee (-v_1) = 0 \curlyvee (u - v_1) \leq v_2 \curlyvee v_2 = v_2.$$

If $u \preccurlyeq v_1 + v_2$ then $u - v_1 \preccurlyeq v_2$ and we can replace $\leq$ by $\preccurlyeq$ in the above inequality, by (M8a).

(b) The element $u_1 = v_1 \curlywedge u$ satisfies $u_1 \leq u$, $u_1 \preccurlyeq v_1$ and $u_1 \geq 0$ (and $u_1 \succcurlyeq 0$ if $u \succcurlyeq 0$). It follows from $u \preccurlyeq v_1 + v_2$ and $v_1 \leq v_1 + v_2$ that $u \curlyvee v_1 \leq v_1 + v_2$, and so if we set $u_2 = u - u_1$ we have $u_2 \geq 0$ and

$$u_2 = u - v_1 \curlywedge u = u \curlyvee v_1 - v_1 \leq v_1 + v_2 - v_1 = v_2,$$

and the proof is complete. □

4 Ideals and bands in a mixed lattice vector space

A *mixed lattice vector space* (or briefly, a *mixed lattice space*) was defined in [5] as a partially ordered real vector space V with two partial orderings $\leq$ and $\preccurlyeq$ (called the initial order and the specific order, respectively) such that the mixed upper and lower envelopes $x \curlyvee y$ and $x \curlywedge y$, as defined in Sect. 2, exist in V for all $x, y \in V$.

We recall that a subset C of a vector space is called a *cone* if (i) $\alpha C \subseteq C$ for all $\alpha \geq 0$, (ii) $C + C \subseteq C$ and (iii) $C \cap (-C) = \{0\}$. Cones play an important role in the theory of ordered vector spaces because a partial ordering in a vector space can be given in terms of the corresponding positive cone. For more information about cones and their properties we refer to [1].

Before we proceed, let us introduce some notation. If V is a mixed lattice space then $V_p = \{x \in V : x \geq 0\}$ and $V_{sp} = \{x \in V : x \succcurlyeq 0\}$ are the $(\leq)$-*positive cone* and the $(\preccurlyeq)$-*positive cone* of V, respectively. Accordingly, an element $x \in V_p$ is called *positive*, and if $x \in V_{sp}$ then x is said to be *specifically positive*, or $(\preccurlyeq)$-*positive*. In mixed lattice spaces we are particularly interested in cones that are also mixed lattice semigroups.

Definition 4.1 A cone $C \subseteq V_{sp}$ in a mixed lattice space V is called a *mixed lattice cone* if $x \curlyvee y$ and $x \curlywedge y$ belong to C whenever $x, y \in C$.

If E is any subset of V then we define $E_p = E \cap V_p$ and $E_{sp} = E \cap V_{sp}$. The notions of regular and quasi-regular mixed lattice space are defined similarly as in mixed lattice groups. Hence, using the notation just introduced, V is *quasi-regular* if V_{sp}, the positive cone associated with the specific order, is a mixed lattice cone. If C is a cone in a vector space V then $S = C - C$ is called the *subspace generated by* C. A mixed lattice space V is *regular* if V_{sp} is a generating mixed lattice cone, that is, $V = V_{sp} - V_{sp}$.

The rules (M1)–(M8) for the mixed envelopes remain valid in quasi-regular mixed lattice spaces. In a mixed lattice space we can add to the list the following rules concerning the scalar multiplication. If V is a mixed lattice space and $0 \leq a \in \mathbb{R}$ then the following hold for all $x, y \in V$

$$(M9)\ (ax) \curlywedge (ay) = a(x \curlywedge y) \ \text{ and } \ (ax) \curlyvee (ay) = a(x \curlyvee y) \quad (a \geq 0).$$

The notions of upper and lower parts of an element and the generalized asymmetrical absolute values were introduced in [5]. If $x \in V$ then the elements ${}^u x = x \curlyvee 0$ and ${}^l x = (-x) \curlyvee 0$ are called the *upper part* and *lower part* of x, respectively. Similarly, the elements $x^u = 0 \curlyvee x$ and $x^l = 0 \curlyvee (-x)$ are called the *specific upper part* and *specific lower part* of x, respectively. These play a similar role as the positive and negative parts of an element in a Riesz space. The *generalized absolute values* of x are then defined as ${}^u x^l = {}^u x + x^l$ and ${}^l x^u = {}^l x + x^u$. The elements ${}^u x^l$ and ${}^l x^u$ are distinct, in general, and they are "asymmetrical" in the sense that ${}^u x^l = {}^l(-x)^u$ for all x.

The upper and lower parts and the generalized absolute values have several important basic properties, which were proved in [5]. These properties are given in the next theorem.

Theorem 4.2 *Let V be a quasi-regular mixed lattice space and $x \in V$. Then the following hold.*

(a) $^u x = {}^l(-x)$ *and* $x^u = (-x)^l$.
(b) $x = x^u - {}^l x = {}^u x - x^l$.
(c) $^u x^l = {}^u x \curlyvee x^l = {}^u x + x^l$ *and* $^l x^u = {}^l x \curlyvee x^u = {}^l x + x^u$.
(d) $^u x^l = {}^l(-x)^u$.
(e) $^u x + {}^u y \geq {}^u(x+y)$, $x^l + y^l \geq (x+y)^l$ *and* $^u x^l + {}^u y^l \geq {}^u(x+y)^l$.
(f) $x^u + y^u \geq (x+y)^u$, $^l x + {}^l y \geq {}^l(x+y)$ *and* $^l x^u + {}^l y^u \geq {}^l(x+y)^u$.
(g) $x^u \curlywedge {}^l x = 0 = x^l \curlywedge {}^u x$.
(h) $x^u \curlyvee {}^l x = {}^u x + {}^l x = x^l + x^u = x^l \curlyvee {}^u x$
(i) $x \succcurlyeq 0$ *if and only if* $x = {}^l x^u = {}^u x^l = {}^u x = x^u$ *and* $^l x = x^l = 0$.
(j) $x \geq 0$ *if and only if* $x = {}^u x^l = {}^u x$ *and* $x^l = 0$.
(k) $^u x^l \geq 0$ *and* $^l x^u \geq 0$. *Moreover,* $^u x^l = {}^l x^u = 0$ *if and only if* $x = 0$.
(l) $^u(ax)^l = a\,{}^u x^l$ *and* $^l(ax)^u = a\,{}^l x^u$ *for all* $a \geq 0$.
(m) $^u(ax)^l = |a|\,{}^l x^u$ *and* $^l(ax)^u = |a|\,{}^u x^l$ *for all* $a < 0$.
(n) $2(x \curlywedge y) = x + y - {}^l(x-y)^u$ *and* $2(y \curlywedge x) = x + y - {}^u(x-y)^l$.

The next theorem was also proved in [5], but under more general assumptions. If V is assumed to be quasi-regular then we have a sharper version of the same theorem, as given below. The proof is almost identical to the more general case, see [5, Theorem 3.6].

Theorem 4.3 *Let V be a quasi-regular mixed lattice space and $x \in V$.*

(a) If $x = u - v$ with $u \succcurlyeq 0$ and $v \geq 0$, then $0 \leq x^u \leq u$ and $0 \leq v \leq {}^l x$. If also $v \succcurlyeq 0$ then $0 \preccurlyeq x^u \preccurlyeq u$ and $0 \leq {}^l x \preccurlyeq v$.
(b) Conversely, if $x = u - v$ with $u \curlywedge v = 0$ then $u = x^u$ and $v = {}^l x$.

For the remainder of this paper, we shall always assume that V is a quasi-regular mixed lattice vector space unless otherwise stated. Due to this convention, we shall drop the term "quasi-regular", and henceforth, by a mixed lattice space we mean a quasi-regular mixed lattice space.

The notions of mixed lattice subspaces and ideals were introduced in [5]. A subspace S of a mixed lattice vector space V is called a *mixed lattice subspace* of V if $x \curlyvee y$ and $x \curlywedge y$ belong to S whenever x and y are in S. A subset $U \subseteq V$ is called ($\leq$)-*order convex*, if $x \leq z \leq y$ and $x, y \in U$ imply that $z \in U$. Similarly, a subset $U \subseteq V$ is called ($\preccurlyeq$)-*order convex*, if $x \preccurlyeq z \preccurlyeq y$ and $x, y \in U$ imply that $z \in U$. A subspace A is ($\leq$)-order convex if and only if $0 \leq y \leq x$ and $x \in A$ imply that $y \in A$. Similarly, a subspace A is ($\preccurlyeq$)-order convex if and only if $0 \preccurlyeq y \preccurlyeq x$ and $x \in A$ imply that $y \in A$. If A is a ($\leq$)-order convex mixed lattice subspace of V then A is called a *mixed lattice ideal* of V. Similarly, a ($\preccurlyeq$)-order convex mixed lattice subspace of V is called a *specific mixed lattice ideal* of V. For brevity, these will be called simply *ideals* and *specific ideals*, respectively. This should not cause any confusion as we are mostly dealing with mixed lattice ideals in this paper. If we refer to other type of ideals (such as lattice ideals) then we will emphasize it accordingly.

To study the relationships between ideals and specific ideals, as well as the structure of mixed lattice spaces in more detail, we introduce the following definitions.

Definition 4.4

(i) A subspace S is called *regular*, if $S = S_{sp} - S_{sp}$.
(ii) A subspace S is called *positively generated*, if $S = S_p - S_p$.
(iii) A subspace A is called *mixed–order convex* if $y \in A$ and $0 \preccurlyeq x \leq y$ together imply that $x \in A$.
(iv) A mixed–order convex mixed lattice subspace is called a *quasi-ideal*.
(v) A specific ideal A is called a *proper specific ideal* if A is not a quasi-ideal. Similarly, a quasi-ideal A is called a *proper quasi-ideal* if A is not an ideal.

Every mixed lattice subspace is positively generated. It is also easy to see that a mixed lattice subspace S is regular if and only if for every $x \in S$ there exists some $z \in S_{sp}$ such that $x \preccurlyeq z$. There are ideals that are not regular (Example 5.15). In many cases, proper specific ideals and proper quasi-ideals are regular, but it is not known whether or not this is true in general.

Before we proceed, we should remark that if A is a subspace of V then, in order to show that A is a mixed lattice subspace it is sufficient to show that $x^u \in A$ for every $x \in A$. Indeed, if this holds and $x, y \in A$ then $y \curlyvee x = y + (x - y)^u \in A$. Consequently, $x \curlywedge y = x + y - y \curlyvee x \in A$. This observation simplifies many of the proofs that follow.

Now the following result holds.

Proposition 4.5 *Every regular* $(\preccurlyeq)$*-order convex subspace is a mixed lattice subspace, and hence a specific ideal.*

Proof If $x \in A$ then $x = u - v$ with $u, v \in A_{sp}$. By Theorem 4.3 we have $0 \preccurlyeq x^u \preccurlyeq u$ and so it follows that $x^u \in A$. Hence, A is a regular $(\preccurlyeq)$-order convex mixed lattice subspace, by the remark made before the proposition. □

The notion of a mixed-order convex subspace is intermediate between $(\preccurlyeq)$-order convex and $(\leq)$-order convex subspaces. Every $(\leq)$-order convex subspace is mixed-order convex, and every mixed-order convex subspace is $(\preccurlyeq)$-order convex. In particular, every ideal is a quasi-ideal, and every quasi-ideal is a specific ideal.

The following simple lemma is useful in the study of mixed-order convex subspaces.

Lemma 4.6 *A subspace A is mixed-order convex if and only if* $0 \leq x \preccurlyeq y$ *with* $y \in A$ *imply that* $x \in A$.

Proof Assume that $0 \preccurlyeq x \leq y$ with $y \in A$ implies that $x \in A$. We observe that if $0 \leq u \preccurlyeq v$ with $v \in A$ then $0 \preccurlyeq v - u \leq v$, and this implies that $v - u \in A$ by assumption. Then $u = v - (v - u) \in A$ since A is a subspace. The converse implication is similar. □

Next we will study some basic properties of mixed lattice cones.

Theorem 4.7 *Let V be a mixed lattice space and C a mixed lattice cone in V. Then the subspace generated by C is a mixed lattice subspace. Conversely, if S is a mixed lattice subspace of V then S_{sp} is a mixed lattice cone in V.*

Proof Let C be a mixed lattice cone. It is clear that $S = C - C$ is a subspace. If $x, y \in S$ then $x = u - v$ and $y = a - b$ for some $u, v, a, b \in C$. Then

$$x \curlywedge y = (u - v) \curlywedge (a - b) = (u + b) \curlywedge (a + v) - (v + b),$$

where $(u + b) \curlywedge (a + v) \in C$. Also $v + b \in C$, so $x \curlywedge y \in S$. Similarly $x \curlyvee y \in S$ and so S is a mixed lattice subspace.

Conversely, if S is a mixed lattice subspace then S_{sp} is clearly a cone. If $x, y \in S_{sp}$ then $x \curlywedge y \in S$ and $x \curlywedge y \succcurlyeq 0$, by (M8b), and so $x \curlywedge y \in S_{sp}$. Similarly, $x \curlyvee y \in S_{sp}$, and the identity (2.1) automatically holds, by (M1). Hence S_{sp} is a mixed lattice cone. □

To gain more information about the structure of specific ideals, we introduce some additional terminology. A mixed lattice cone C_1 is called a *mixed lattice sub-cone* of another mixed lattice cone C_2 if $C_1 \subseteq C_2$. A sub-cone F of a cone C is called a *face* of C if $x + y \in F$ with $x, y \in C$ imply that $x, y \in F$.

First we observe that every face of V_{sp} is a mixed lattice sub-cone.

Proposition 4.8 *Every face of V_{sp} is a mixed lattice sub-cone of V_{sp}.*

Proof Let F be a face of V_{sp} and $x, y \in F$. Then $x \curlyvee y + y \curlywedge x = x + y \in F$. It follows that $x \curlyvee y \in F$ and $y \curlywedge x \in F$, and so F is a mixed lattice sub-cone of V_{sp}. □

The next two results are well-known but we give proofs for completeness.

Lemma 4.9 *A sub-cone C of V_p is $(\leq)$-order convex if and only if $0 \leq x \leq y$ with $y \in C$ implies that $x \in C$. Similarly, a sub-cone C of V_{sp} is $(\preccurlyeq)$-order convex if and only if $0 \preccurlyeq x \preccurlyeq y$ with $y \in C$ implies that $x \in C$.*

Proof Clearly the given condition is necessary. Assume the condition holds and let $z \leq x \leq y$ with $z, y \in C$. Then $0 \leq x - z \leq y - z \leq y$ which implies that $x - z \in C$, and since $z \in C$ and C is a sub-cone, it follows that $x = (x - z) + z \in C$. The proof of the second statement is essentially the same. □

Proposition 4.10 *A sub-cone F of V_p is a face of V_p if and only if F is $(\leq)$-order convex. Similarly, F is a face of V_{sp} if and only if F is a $(\preccurlyeq)$-order convex sub-cone of V_{sp}.*

Proof Suppose that F is a face of V_p and let $0 \leq x \leq y$ with $y \in F$. Then $x \in V_p$, $0 \leq y - x \in V_p$ and $(y - x) + x = y \in F$. It follows that $x \in F$ and this shows that F is $(\leq)$-order convex, by the preceding lemma. Conversely, if F is $(\leq)$-order convex and $x + y \in F$ then $0 \leq x \leq x + y$ and $0 \leq y \leq x + y$ imply that $x, y \in F$. Hence F is a face. The proof of the second assertion is identical, just replace $\leq$ by $\preccurlyeq$. □

The following result is now an immediate consequence of the preceding results and the definition of a specific ideal.

Corollary 4.11 *A mixed lattice subspace A is a specific ideal if and only if A_{sp} is a face of V_{sp}.*

Our next result is important for the theory of ideals. It shows that every ideal contains a quasi-ideal.

Theorem 4.12 *If A is an ideal and $W = A_{sp} - A_{sp}$ then W is a quasi-ideal, and there is no ideal B such that $W \subseteq B \subseteq A$ and $B \neq A$. In particular, if W is an ideal then $W = A$.*

Proof By Theorem 4.7, W is a mixed lattice subspace. Moreover, if $0 \preccurlyeq x \leq y$ with $y \in W$ then $y \in A$, and A is an ideal, so $x \in A_{sp} \subseteq W$. This shows that W is mixed-order convex. Let B be any ideal contained in A such that $W \subseteq B$. Then for every $x \in A_p$ we have $0 \leq x \leq x^u \in B$. This implies that $x \in B$, so $B = A$. In particular, if W is an ideal then $W = A$. □

We recall that if E is a subset of V then the smallest ideal (with respect to set inclusion) that contains E is called the *ideal generated by* E. The next result gives a description of ideals generated by a mixed lattice cone.

Theorem 4.13 *The ideal A generated by a mixed lattice cone C equals the subspace generated by the cone $S = \{x \in V : 0 \leq x \leq u \text{ for some } u \in C\}$.*

Proof Since C is a cone, it follows immediately that S is also a cone. Evidently, every ideal that contains C must contain S, so it contains also the subspace $S - S$. Therefore, it is sufficient to show that the subspace $S - S$ is an ideal. $S - S$ is obviously ($\leq$)-order convex, so it remains to show that it is a mixed lattice subspace. Let $x \in S - S$. Then $x = x_1 - x_2$ with $x_1, x_2 \in S$, and so $x \leq x_1 \leq u_1$ for some $u_1 \in C$. Since $C \subseteq V_{sp}$, it follows by Theorem 4.2(i) that $(u_1)^u = u_1$, and so by (M5) we have

$$0 \leq x^u = 0 \curlyvee x \leq 0 \curlyvee x_1 \leq 0 \curlyvee u_1 = (u_1)^u = u_1 \in C.$$

Thus, $x^u \in S$, and so $S - S$ is a mixed lattice subspace. Hence we have proved that $S - S$ is a smallest ideal containing C. □

If A is an ideal then by Theorem 4.7 the set A_{sp} is a mixed lattice cone, and by Theorem 4.12 A is the smallest ideal that contains A_{sp}. Now the two preceding theorems yield the following corollary which tells that ideals are uniquely determined by their sets of ($\preccurlyeq$)-positive elements.

Corollary 4.14 *Every ideal A equals the ideal generated by A_{sp}, that is, the subspace generated by the set $S = \{x \in V : 0 \leq x \leq u \text{ for some } u \in A_{sp}\}$. In fact, $S = A_p$. Consequently, if A and B are two ideals such that $A_{sp} = B_{sp}$ then $A = B$.*

Proof The only thing that still needs proof is the claim that $S = A_p$. The inclusion $S \subseteq A_p$ is clear, and the reverse inclusion holds too, for if $x \in A_p$ then $0 \leq x \leq x^u \in A_{sp}$. □

The preceding results have yet another consequence. By Theorem 4.12, if A is an ideal then the subspace $W = A_{sp} - A_{sp}$ is a regular quasi-ideal. The following gives the converse.

Theorem 4.15 *If W is a regular quasi-ideal then there exists an ideal A such that $W = A_{sp} - A_{sp}$. Hence, W is a regular quasi-ideal if and only if $W = A_{sp} - A_{sp}$ for some ideal A.*

Proof Let $W = W_{sp} - W_{sp}$ be a regular quasi-ideal and let A be the ideal generated by W_{sp}. By Theorem 4.7 the set W_{sp} is a mixed lattice cone, so by Theorem 4.13 $A = S - S$, where $S = \{x \in V : 0 \leq x \leq u \text{ for some } u \in W_{sp}\}$. Now we only need to show that $A_{sp} = W_{sp}$. It is clear that $W_{sp} \subseteq A_{sp}$, so let $x \in A_{sp}$. Then $x = y - z$ where $y, z \in S$, so we have $0 \preccurlyeq x \leq y \leq u$ for some $u \in W_{sp}$. But this implies that $x \in W_{sp}$ since W is a quasi-ideal, so we conclude that $A_{sp} = W_{sp}$. □

Next we will study the properties of algebraic sums of different types of subspaces. For instance, if A and B are quasi-ideals, then what can be said about the sum $A + B$? Another important question is if the sets of positive elements are preserved in these sums, that is, does $(A + B)_{sp} = A_{sp} + B_{sp}$ or $(A + B)_p = A_p + B_p$ hold.

Theorem 4.16 *If A is a quasi-ideal and B is a positively generated ($\leq$)-order convex subspace then $A+B$ is a mixed-order convex subspace. Moreover, if B is an ideal then $A + B$ is a quasi-ideal, and if, in addition, A is regular then $(A + B)_p = A_p + B_p$.*

Proof It is clear that $A + B$ is a subspace. Assume first that $0 \preccurlyeq x \leq y$ with $y = y_1 + y_2 \in A + B$. By assumption, $y_2 = v_1 - v_2$ with $0 \leq v_1, v_2 \in B$, and A is a mixed lattice subspace, so $(y_1)^u \in A$. Then $0 \preccurlyeq x \leq y_1 + y_2 \leq (y_1)^u + v_1$, so we can now apply the dominated decomposition property (Theorem 3.8(a)) and write $x = x_1 + x_2$, where $0 \preccurlyeq x_1 \leq (y_1)^u$ and $0 \leq x_2 \leq v_1$. It follows that $x_1 \in A$ and $x_2 \in B$, so $x \in A + B$. This shows that $A + B$ is mixed-order convex.

Assume next that B is an ideal and let $y = y_1 + y_2 \in A + B$. Then $0 \preccurlyeq y^u \leq (y_1)^u + (y_2)^u$ and, applying Theorem 3.8(a) again we can find elements u_1 and u_2 such that $y^u = u_1 + u_2$, $0 \preccurlyeq u_1 \leq (y_1)^u$ and $0 \preccurlyeq u_2 \leq (y_2)^u$. This implies that $u_1 \in A$ and $u_2 \in B$, and hence $y^u \in A + B$. This shows that $A + B$ is a mixed lattice subspace, and hence a quasi-ideal. It is clear that $A_p + B_p \subseteq (A + B)_p$. Conversely, if $0 \leq x \in A + B$ then $x = x_1 + x_2$ with $x_1 \in A$ and $x_2 \in B$. If A is regular, we can choose an element $0 \preccurlyeq v \in A$ such that $x_1 \preccurlyeq v$. Also, $x_2 \preccurlyeq {}^u x_2 \in B$, and so we have $0 \leq x \preccurlyeq v + {}^u x_2$. We can now apply Theorem 3.8(b) to find positive elements $u_1 \in A$ and $u_2 \in B$ such that $x = u_1 + u_2$, completing the proof. □

Along the same lines we have the following result.

Theorem 4.17 *In a mixed lattice space the following hold.*

(a) If A and B are quasi-ideals then $(A + B)_{sp} = A_{sp} + B_{sp}$. Moreover, if A and B are regular then $A + B$ is also a regular quasi-ideal.

(b) If A is a regular specific ideal and B is a regular quasi-ideal then $A + B$ is a regular specific ideal and $(A + B)_{sp} = A_{sp} + B_{sp}$. The last equality holds, in particular, if B is an ideal.

Proof *(a)* If $x \in (A+B)_{sp}$ then $0 \preccurlyeq x = x_1 + x_2 \leq (x_1)^u + (x_2)^u$ with $(x_1)^u \in A_{sp}$ and $(x_2)^u \in B_{sp}$. By Theorem 3.8(a) $x = a + b$ for some elements a and b such that $0 \preccurlyeq a \leq (x_1)^u$ and $0 \preccurlyeq b \leq (x_2)^u$. It follows that $a \in A_{sp}$ and $b \in B_{sp}$, hence $(A + B)_{sp} \subseteq A_{sp} + B_{sp}$. The reverse inclusion is obvious. Assume then that $0 \preccurlyeq y \leq x$ with $x \in A + B$. If A and B are regular it follows that $A + B$ is also regular, and we may thus assume that $x \succcurlyeq 0$. Hence, $0 \preccurlyeq y \leq x = a + b$ where $a \in A_{sp}$ and $b \in B_{sp}$. Now the same argument as above (using Theorem 3.8(a)) shows that $y \in A + B$, and so $A + B$ is mixed-order convex. Then $A + B$ is a mixed lattice subspace, by Proposition 4.5.

(b) If $x \in (A + B)_{sp}$ then $x = x_1 + x_2$ with $x_1 \in A$ and $x_2 \in B$. Since A and B are regular, there exist elements $u \in A_{sp}$ and $v \in B_{sp}$ such that $x_1 \preccurlyeq u$ and $x_2 \preccurlyeq v$. Hence $0 \preccurlyeq x = x_1 + x_2 \preccurlyeq u + v$, and by Theorem 3.8(a) we have $x = a + b$ for some elements a and b such that $0 \preccurlyeq a \preccurlyeq u$ and $0 \preccurlyeq b \leq v$. It follows that $a \in A_{sp}$ and $b \in B_{sp}$, hence $(A + B)_{sp} \subseteq A_{sp} + B_{sp}$. The reverse inclusion is clear. In particular, if B is an ideal then $W = B_{sp} - B_{sp}$ is a regular quasi-ideal, and so $V_{sp} = A_{sp} + W_{sp} = A_{sp} + B_{sp}$. The proof that $A + B$ is a regular specific ideal is similar to part (a). □

Regarding the above theorem we note that by Theorem 4.15 regular quasi-ideals are precisely those subspaces W that $W = A_{sp} - A_{sp}$ for some ideal A. As with ideals in Riesz spaces, these regular quasi-ideals form a distributive lattice.

Theorem 4.18 *Let V be a mixed lattice space and denote by $\mathcal{R}(V)$ the set of all regular quasi-ideals of V, ordered by inclusion. Then $\mathcal{R}(V)$ is a distributive lattice where $A \vee B = A + B$ and $A \wedge B = A \cap B$. Moreover, $\mathcal{R}(V)$ has the smallest element $\{0\}$ and the largest element $V_{sp} - V_{sp}$.*

Proof If $A, B \in \mathcal{R}(V)$ then by Theorem 4.17 we have $A + B \in \mathcal{R}(V)$, and $A + B$ is clearly the smallest regular quasi-ideal that contains both A and B, hence $A + B = A \vee B$. It is also clear that $A \cap B$ is the largest quasi-ideal that is contained in both A and B. To see that $A \cap B$ is regular, let $x \in A \cap B$. Since A and B are regular, there exist elements $u \in A_{sp}$ and $v \in B_{sp}$ such that $x \preccurlyeq u$ and $x \preccurlyeq v$ hold. Then $x \preccurlyeq u \curlywedge v$ and the inequalities $0 \preccurlyeq u \curlywedge v \preccurlyeq u$ and $0 \preccurlyeq u \curlywedge v \leq v$ imply that $u \curlywedge v \in A \cap B$, proving that $A \cap B$ is regular, and hence $A \cap B = A \wedge B$. To prove the distributivity, it is sufficient to show that $[(A \cap B) + (A \cap C)]_{sp} = [A \cap (B + C)]_{sp}$, since $(A \cap B) + (A \cap C)$ and $A \cap (B + C)$ are both regular quasi-ideals. The inclusion $(A \cap B) + (A \cap C) \subseteq A \cap (B + C)$ is rather trivial, so let $x \in [A \cap (B + C)]_{sp}$. Then $x \in (B + C)_{sp}$, so $x = x_1 + x_2$ where $x_1 \in B_{sp}$ and $x_2 \in C_{sp}$, by Theorem 4.17. Moreover, $0 \preccurlyeq x_1 \preccurlyeq x \in A$ implies that $x_1 \in A_{sp}$, and similarly, $x_2 \in A_{sp}$. Hence, $x_1 \in (A \cap B)_{sp}$ and $x_2 \in (A \cap C)_{sp}$, and therefore $x \in [(A \cap B) + (A \cap C)]_{sp}$. This shows that $\mathcal{R}(V)$ is distributive. Finally, it is clear that $\{0\}$ is the smallest element in $\mathcal{R}(V)$, and by Theorem 4.12 the largest element is $W = V_{sp} - V_{sp}$. □

We do not know if the preceding theorem can be stated for regular specific ideals in general. However, in many cases, a mixed lattice space is also a lattice with respect to one (or both) partial orderings. In this case, we obtain stronger results as all the lattice-theoretic tools are available to us. We will now briefly consider this situation.

To avoid any confusion, we need to fix some terminology. If $(V, \preccurlyeq)$ is a vector lattice then we will say that V is a $(\preccurlyeq)$-*lattice*. If $(V, \preccurlyeq)$ is a vector lattice and A is a lattice ideal in $(V, \preccurlyeq)$, then A is called a $(\preccurlyeq)$-*lattice ideal*. We also denote the absolute value and the positive and negative parts of an element x with respect to $\preccurlyeq$ by $\mathrm{sp}\,|x|$, $\mathrm{sp}(x^+)$ and $\mathrm{sp}(x^-)$, respectively.

Theorem 4.19 *If V is a mixed lattice space such that $(V, \preccurlyeq)$ is a vector lattice then A is a regular specific ideal in V if and only if A is a $(\preccurlyeq)$-lattice ideal of $(V, \preccurlyeq)$.*

Proof It was proved in [5, Proposition 4.6] that every $(\preccurlyeq)$-lattice ideal is a specific ideal. Moreover, a $(\preccurlyeq)$-lattice ideal A is regular (since every $x \in A$ can be written as $x = \mathrm{sp}(x^+) - \mathrm{sp}(x^-)$ where $\mathrm{sp}(x^+), \mathrm{sp}(x^+) \in A_{sp}$), so we only need to prove the converse. Let A be a regular specific ideal. Then A is $(\preccurlyeq)$-order convex and if $x \in A$ then also ${}^u x^l \in A$ and ${}^l x^u \in A$. Since A is regular, there is an element $u \in A_{sp}$ such that ${}^u x^l \preccurlyeq u$ and ${}^l x^u \preccurlyeq u$. By [5, Proposition 3.16] the absolute value of x formed with respect to $\preccurlyeq$ is given by $\mathrm{sp}\,|x| = \mathrm{sp}\sup\{{}^u x^l, {}^l x^u\}$. Thus we have $0 \preccurlyeq \mathrm{sp}\,|x| = \mathrm{sp}\sup\{{}^u x^l, {}^l x^u\} \preccurlyeq u \in A$. Since A is $(\preccurlyeq)$-order convex, it follows that $\mathrm{sp}\,|x| \in A$. This shows that A is a $(\preccurlyeq)$-lattice ideal. $\square$

It is well known that the set of lattice ideals in a Riesz space is a distributive lattice. Also, every regular quasi-ideal is a regular specific ideal, so putting all this together with Theorems 4.19 and 4.18 we obtain the following:

Corollary 4.20 *Let V be a mixed lattice space that is a lattice with respect to $\preccurlyeq$, and denote by $\mathcal{L}(V)$ the set of all regular specific ideals of V, ordered by inclusion. Then $\mathcal{L}(V)$ is a distributive lattice where $A \vee B = A + B$ and $A \wedge B = A \cap B$. Moreover, $\mathcal{L}(V)$ has the smallest element $\{0\}$ and the largest element V, and the set $\mathcal{R}(V)$ of all regular quasi-ideals of V is a sub-lattice of $\mathcal{L}(V)$.*

Now we turn to the discussion of bands in mixed lattice spaces.

Definition 4.21 Let V be a mixed lattice space. A specific ideal A is called a *specific band* if $\mathrm{sp}\sup E \in A$ whenever E is a non-empty subset of A such that $\mathrm{sp}\sup E$ exists in V. If A is a quasi-ideal with the above property then A is called a *quasi-band*. An ideal B is called a *band* if $\sup E \in A$ whenever E is a non-empty subset of A such that $\sup E$ exists in V.

It follows from the identity $\inf E = -\sup(-E)$ that if A is a band and E is a non-empty subset of A such that $\inf E$ exists in V then $\inf E \in A$. Similarly, if A is a specific band and $\mathrm{sp}\inf E$ exists in V then $\mathrm{sp}\inf E \in A$.

It is also clear that every quasi-band is a specific band. For the sequel, we need to introduce the following notions.

Definition 4.22 A specific ideal A is called a *weak specific band* if $\mathrm{str}\sup E \in A$ whenever E is a non-empty subset of A such that $\mathrm{str}\sup E$ exists in V. A quasi-ideal with the above property is called a *weak quasi-band*, and an ideal with the above property is called a *weak band*.

Weak bands and weak specific bands in mixed lattice semigroups were introduced by Arsove and Leutwiler [2].

Clearly, every specific band is a weak specific band, every quasi-band is a weak quasi-band and every band is a weak band. Now we can state the following characterization for weak bands and weak specific bands.

Lemma 4.23 *If A is a (specific) ideal in V then the following are equivalent.*

(a) *A is a weak (specific) band.*
(b) $\operatorname{str}\sup E \in A$ *whenever E is a non-empty subset of A_p such that* $\operatorname{str}\sup E$ *exists in V.*
(c) $\operatorname{str}\sup E \in A$ *whenever E is a non-empty subset of A_{sp} such that* $\operatorname{str}\sup E$ *exists in V.*

Proof The implication $(a) \implies (b)$ is clear. Condition (b) obviously implies (c), since $A_{sp} \subseteq A_p$. Assume that (c) holds and let E be a non-empty subset of A such that $u_0 = \operatorname{str}\sup E$ exists in V. Fix an element $x \in E$ and define $D = \{x \curlyvee u - x : u \in E\}$. Then D is a non-empty subset of A_{sp} and since $u_0 \succcurlyeq x$, by property (M6a) we have $x \curlyvee u_0 = u_0$. Using Proposition 3.7 we then have $\operatorname{str}\sup D = \operatorname{str}\sup\{x \curlyvee u - x : u \in E\} = x \curlyvee u_0 - x = u_0 - x \in A$, and hence $u_0 = x + (u_0 - x) \in A$. □

5 Disjoint complements

If $x, y \in V$ and $x \curlywedge y = 0$ then x is said to be *left-disjoint* with y and y is *right-disjoint* with x. The reason for this terminology is, of course, the fact that in general $x \curlywedge y \neq y \curlywedge x$. Next we will investigate the sets of those elements that are left or right disjoint with each element of a given subspace A. It follows immediately from the inequalities $x \succcurlyeq x \curlywedge y$ and $y \geq x \curlywedge y$ that if $x \curlywedge y = 0$ then we must have $x \succcurlyeq 0$ and $y \geq 0$. Because of this, we will first consider disjointness for positive elements only. As it turns out, these sets of positive disjoint elements are in fact mixed lattice cones. We then define the left and right disjoint complements as the specific ideal and the ideal generated by these cones.

We begin with the left disjoint complement.

Lemma 5.1 *If $x, y \succcurlyeq 0$ and $z \geq 0$ with $y \curlywedge z = 0$ then $(x + y) \curlywedge z = x \curlywedge z$. In particular, if also $x \curlywedge z = 0$ then $(x + y) \curlywedge z = 0$.*

Proof If $y \curlywedge z = 0$ and $x \succcurlyeq 0$ then, using Theorem 3.2 we obtain

$$(x + y) \curlywedge z \preccurlyeq (x + y) \curlywedge (x + z) = x + y \curlywedge z = x.$$

On the other hand, $(x + y) \curlywedge z \leq z$, so we have $(x + y) \curlywedge z \leq x \curlywedge z$. The reverse inequality $0 \leq x \curlywedge z \leq (x + y) \curlywedge z$ holds by (M5), and the lemma is proved. □

We now introduce the set $({}^{\perp}A)_{sp} = \{x \in V : x \curlywedge z = 0 \text{ for all } z \in A_p\}$ and study its properties.

Theorem 5.2 *Let V be a mixed lattice space and A a mixed lattice subspace of V. Then the set $(^{\perp}A)_{sp}$ defined above is a mixed lattice cone in V.*

Proof Let $x, y \in (^{\perp}A)_{sp}$ and $z \in A_p$. Then $(x + y) \curlywedge z = 0$ by the preceding lemma, and hence $x + y \in (^{\perp}A)_{sp}$. If $0 \le \alpha \in \mathbb{R}$ and we put $c = \max\{\alpha, 1\}$, then $0 \le (\alpha x) \curlywedge z \le c(x \curlywedge z) = 0$. Thus, $\alpha x \in (^{\perp}A)_{sp}$. This shows that $(^{\perp}A)_{sp}$ is a cone in V. Next we note that if $v = x \curlywedge y$ then $0 \preccurlyeq v \preccurlyeq x$ and it follows that for every $z \in A_p$ we have $0 \le v \curlywedge z \le x \curlywedge z = 0$, and thus $v \curlywedge z = 0$. This shows that $v \in (^{\perp}A)_{sp}$. If we set $w = x - x \curlywedge y$ then $0 \preccurlyeq w \preccurlyeq x$, which implies that $0 \le w \curlywedge z \le x \curlywedge z = 0$ for all $z \in A_p$. Thus $w \in (^{\perp}A)_{sp}$, and since $(^{\perp}A)_{sp}$ is a cone, we have $y + w = y + x - x \curlywedge y = y \curlyvee x \in (^{\perp}A)_{sp}$. Hence, $(^{\perp}A)_{sp}$ is a mixed lattice cone in V. □

The preceding result motivates the following definition.

Definition 5.3 Let A a mixed lattice subspace of a mixed lattice space V. The *left disjoint complement* of A is the specific ideal $^{\perp}A$ generated by the cone $(^{\perp}A)_{sp} = \{x \succcurlyeq 0 : x \curlywedge z = 0 \text{ for all } z \in A_p\}$.

Remark 5.4 We should point out that, more generally, if E is any subset of V such that E_p is non-empty, then $^{\perp}E_p$ is a mixed lattice cone in V. However, for the purposes of the present paper it is sufficient to restrict ourselves to mixed lattice subspaces. By doing so we can avoid some unnecessary complications that arise if one considers non-trivial subspaces S such that $S_p = \{0\}$. For instance, in such cases the algebraic sum of a subspace and its disjoint complement would not be a direct sum, in general. We will discuss these matters further at the end of this paper.

Theorem 5.5 *If E is a subset of $(^{\perp}A)_{sp}$ such that* $\operatorname{sp}\sup E$ *exists in V then* $\operatorname{sp}\sup E \in (^{\perp}A)_{sp}$. *In particular, $^{\perp}A$ is a regular weak specific band in V. Moreover, if $^{\perp}A$ is an ideal, then it is a weak regular band.*

Proof We will first show that $^{\perp}A$ is regular. Let $W = (^{\perp}A)_{sp} - (^{\perp}A)_{sp}$. It is clear that W is a subspace and $W \subseteq {}^{\perp}A$, so we only need to show that W is a specific ideal. It follows from Theorem 5.2 and Theorem 4.7 that W is a mixed lattice subspace in V. Moreover, if $0 \preccurlyeq y \preccurlyeq x$ with $x \in W$ then for every $z \in A_p$ we have $0 \le y \curlywedge z \le x \curlywedge z = 0$, so $y \curlywedge z = 0$ and thus $y \in W$. Hence, W is a regular specific ideal, and so $W = {}^{\perp}A$.

Let E be a non-empty subset of $(^{\perp}A)_{sp}$ such that $u_0 = \operatorname{sp}\sup E$ exists in V. Then, using (M1) and Theorem 3.2 we have

$$u - u \curlywedge z = z \curlyvee u - z \preccurlyeq z \curlyvee u_0 - z = u_0 - u_0 \curlywedge z$$

for all $u \in E$ and $z \in A_p$. But $u \curlywedge z = 0$, so the above inequality reduces to $u \preccurlyeq u_0 - u_0 \curlywedge z$. Thus the element $u_0 - u_0 \curlywedge z$ is a $(\preccurlyeq)$-upper bound of the set E, so we have $u_0 \preccurlyeq u_0 - u_0 \curlywedge z$. This implies that $0 \le u_0 \curlywedge z \preccurlyeq 0$ (where the inequality $0 \le u_0 \curlywedge z$ follows by (M5), since $u_0 \succcurlyeq 0$ and $z \ge 0$), and so $u_0 \curlywedge z = 0$. Hence, $u_0 \in (^{\perp}A)_{sp}$. In particular, if u_0 is the strong supremum then $^{\perp}A$ is a weak specific band, and if $^{\perp}A$ is an ideal then $^{\perp}A$ is a weak band, by Lemma 4.23. □

Next we turn to the right disjoint complement which we define in a similar manner as the left disjoint complement, but the situation is slightly more complicated. Let A be a mixed lattice subspace and consider the set $\mathcal{S}(A) = \{x \geq 0 : z \curlywedge x = 0 \text{ for all } z \in A_{sp}\}$. It is easy to see that $\mathcal{S}(A)$ is closed under multiplication by positive scalars, by using a similar argument as in the proof of Theorem 5.2. However, in general, $\mathcal{S}(A)$ is not closed under addition (see Example 5.15).

To get a better understanding of the situation, let us briefly examine the set $\mathcal{S}(A)$ more closely. Let $\mathcal{B}$ be the family of all those subsets of $\mathcal{S}(A)$ that are closed under addition. Notice that $\mathcal{B}$ is non-empty since $\{0\} \in \mathcal{B}$. Let $\mathcal{B}$ be ordered by inclusion. If $\mathcal{C}$ is a totally ordered subset of $\mathcal{B}$ then $\bigcup\{C : C \in \mathcal{C}\}$ is an upper bound of $\mathcal{C}$, and so by Zorn's lemma $\mathcal{B}$ has maximal elements. Let us denote by $\mathcal{M}(A)$ the set of maximal elements of $\mathcal{B}$, that is, the set of those subsets of $\mathcal{S}(A)$ that are maximal with respect to the property of being closed under addition. If $\mathcal{S}(A)$ itself is closed under addition then $\mathcal{S}(A)$ is the only element of $\mathcal{M}(A)$. Clearly, $\mathcal{S}(A) = \bigcup\{C : C \in \mathcal{M}(A)\}$, and each set in $\mathcal{M}(A)$ is a $(\leq)$-order convex cone.

Proposition 5.6 *The set $\mathcal{S}(A)$ is $(\leq)$-order convex, and each $C \in \mathcal{M}(A)$ is a $(\leq)$-order convex cone.*

Proof We divide the proof into 5 steps.

(1) If $0 \leq x \leq y$ with $y \in \mathcal{S}(A)$ then $0 \leq z \curlywedge x \leq z \curlywedge y = 0$ for all $z \in A_{sp}$, and so $z \curlywedge x = 0$ for all $z \in A_{sp}$, proving that $\mathcal{S}(A)$ is $(\leq)$-order convex.

(2) Let $C \in \mathcal{M}(A)$. Then C is closed under addition by definition.

(3) If $0 \leq x \leq y$ with $y \in C$, then $x \in \mathcal{S}(A)$ by step (1). Now, if $x \notin C$ and $x + z \in \mathcal{S}(A)$ for all $z \in C$ then $0 \leq ax + bz \leq (a + b)(x + z) \in \mathcal{S}(A)$ for all $z \in C$ and $a, b \in \mathbb{R}_+$. By step (1) this implies that the set $D = \{ax + bz : z \in C,\ a, b \in \mathbb{R}_+\}$ is contained in $\mathcal{S}(A)$. Clearly, D is closed under addition, and C is contained in D, contradicting the maximality of C. Hence, there exists some $z \in C$ such that $x + z \notin \mathcal{S}(A)$. But then $0 \leq x + z \leq y + z \in C$, and by step (1) this implies that $x + z \in \mathcal{S}(A)$, a contradiction. Thus $x \in C$, and C is $(\leq)$-order convex.

(4) If $x \in C$ then an inductive argument applied to step (2) shows that $nx \in C$ for all $n \in \mathbb{N}$. If $a \in \mathbb{R}_+$ then we can find some $m \in \mathbb{N}$ such that $a \leq m$, and so $0 \leq ax \leq mx \in C$. By step (3) this implies that $ax \in C$, showing that C is closed with respect to multiplication by positive scalars.

(5) Finally, $C \cap (-C) = \{0\}$, because the elements of C are positive and $0 \in C$. By steps (2)–(5), C is a $(\leq)$-order convex cone. □

Due to the above result, we will call $\mathcal{M}(A)$ the *set of maximal right disjoint cones of A*. It was noted above that, in general, the set $\mathcal{S}(A)$ is not closed under addition. In this sense, the set $\mathcal{S}(A)$ is "too large". However, we have the following result which is the analogue of Lemma 5.1.

Lemma 5.7 *If $z, x \succcurlyeq 0$ and $y \geq 0$ with $z \curlywedge y = 0$ then $z \curlywedge (x + y) = z \curlywedge x$. In particular, if also $z \curlywedge x = 0$ then $z \curlywedge (x + y) = 0$.*

Proof Let $v = z \curlywedge (x + y)$. Then $v \leq x + y$, so $v - x \leq y$. On the other hand, since $x \succcurlyeq 0$ we have $v - x \preccurlyeq v \preccurlyeq z$. Hence, $v - x \leq z \curlywedge y = 0$, and so $v \leq x$.

Since we also have $v \preccurlyeq z$, it follows that $0 \leq v \leq z \curlywedge x$. The reverse inequality $0 \leq z \curlywedge x \leq z \curlywedge (x + y)$ holds by (M5), and the desired result follows. $\square$

It is clear that the set $(A^{\perp})_{sp} = \{x \succcurlyeq 0 : z \curlywedge x = 0 \text{ for all } z \in A_{sp}\}$ is contained in $\mathcal{S}(A)$. The set $(A^{\perp})_{sp}$ is in fact a mixed lattice cone.

Theorem 5.8 *If A is a mixed lattice subspace of V then the set $(A^{\perp})_{sp}$ defined above is a mixed lattice cone in V.*

Proof The proof of the fact that $(A^{\perp})_{sp}$ is a mixed lattice cone is similar to the proof of Theorem 5.2. The only real difference is in showing that the set $(A^{\perp})_{sp}$ is closed under addition, and this follows immediately from the preceding lemma. $\square$

Definition 5.9 Let V be a mixed lattice space and A a mixed lattice subspace of V. The *right disjoint complement* of A is the ideal $A^{\perp}$ generated by the cone $(A^{\perp})_{sp} = \{x \succcurlyeq 0 : z \curlywedge x = 0 \text{ for all } z \in A_{sp}\}$.

Remark 5.10 In the above definition we require A to be a mixed lattice subspace for similar reasons that were explained in Remark 5.4 considering the left disjoint complement. The assumption that A is a mixed lattice subspace guarantees that A contains non-zero specifically positive elements (except, of course, in the trivial case $A = \{0\}$).

Theorem 5.11 *$A^{\perp}$ is a weak band and $z \curlywedge x = 0$ for all $0 \leq x \in A^{\perp}$ and $z \in A_{sp}$.*

Proof We will first show that $(A^{\perp})_{sp} = \{w \in A^{\perp} : w \succcurlyeq 0\}$, that is, the ideal $A^{\perp}$ does not contain any $(\preccurlyeq)$-positive elements that are not in $(A^{\perp})_{sp}$. This will also justify the notation used for the set $(A^{\perp})_{sp}$ in Definition 5.9. For this, let $W = (A^{\perp})_{sp} - (A^{\perp})_{sp}$ and $0 \preccurlyeq x \leq y \in W$. Then $y = u - v$ for some $u, v \in (A^{\perp})_{sp}$, and so $0 \preccurlyeq x \leq u \in (A^{\perp})_{sp}$. This implies that $0 \preccurlyeq z \curlywedge x \leq z \curlywedge u = 0$ for all $z \in A_{sp}$, and therefore $z \curlywedge x = 0$, so $x \in (A^{\perp})_{sp} \subseteq W$. This shows that W is a regular quasi-ideal, and so by Corollary 4.14 and Theorem 4.15, $W_{sp} = (A^{\perp})_{sp} = \{w \in A^{\perp} : w \succcurlyeq 0\}$.

Now, if $z \in A_{sp}$ and $0 \leq x \in A^{\perp}$ then $0 \leq x \leq x^{u} \in (A^{\perp})_{sp}$, so $0 \leq z \curlywedge x \leq z \curlywedge x^{u} = 0$, which implies that $z \curlywedge x = 0$. Next, let $E \subseteq (A^{\perp})_{sp}$ be a non-empty set and assume that $x_0 = \text{str} \sup E$ exists in V. Then $x_0 \succcurlyeq 0$, and since $z \curlywedge x = 0$ for all $x \in E$ and $z \in A_{sp}$, it follows by Proposition 3.7 that $z \curlywedge x_0 = \text{str} \sup_{x \in E}(z \curlywedge x) = 0$, and hence $x_0 \in (A^{\perp})_{sp}$. By Lemma 4.23 this shows that $A^{\perp}$ is a weak band. $\square$

The next result provides more information about the relationship between $A^{\perp}$ and $\mathcal{S}(A)$.

Theorem 5.12 *If A is a specific ideal then $(A^{\perp})_{p} \subseteq \bigcap\{C : C \in \mathcal{M}(A)\}$.*

Proof We will first show that that $(A^{\perp})_{sp} \subseteq \bigcap\{C : C \in \mathcal{M}(A)\}$. Let A be a specific ideal and $C \in \mathcal{M}(A)$ with $x \in C$. Then it follows by Lemma 5.7 that for every $w \in A_{sp}$ and $z \in (A^{\perp})_{sp}$ we have $w \curlywedge (x + z) = 0$, and so $x + z \in \mathcal{S}(A)$. This shows that $(A^{\perp})_{sp} + C \subseteq \mathcal{S}(A)$. But C is, by definition, a maximal cone in $\mathcal{S}(A)$, so we must have $(A^{\perp})_{sp} \subseteq C$. This proves that $(A^{\perp})_{sp} \subseteq \bigcap\{C : C \in \mathcal{M}(A)\}$.

Next, if $0 \le y \in A^\perp$ then $0 \preccurlyeq y^u \in (A^\perp)_{sp}$. Thus, by what was just proved above, we have $y^u \in \bigcap\{C : C \in \mathcal{M}(A)\}$, and the inequality $0 \le y \le y^u$ then implies that $y \in \bigcap\{C : C \in \mathcal{M}(A)\}$, by Proposition 5.6. Hence, $(A^\perp)_p \subseteq \bigcap\{C : C \in \mathcal{M}(A)\}$. □

In the next theorem we collect some basic properties of the disjoint complements. Some of them are straightforward consequences of the definitions.

Theorem 5.13 *If A is a mixed lattice subspace then the following hold.*

(a) $A \cap {}^\perp A = \{0\}$ *and* $A \cap A^\perp = \{0\}$.
(b) $A \subseteq ({}^\perp A)^\perp$.
(c) If A is regular then $A \subseteq {}^\perp(A^\perp)$.
(d) If A is a quasi-ideal then $(A^\perp)_{sp} \subseteq ({}^\perp A)_{sp}$.
(e) If A is a quasi-ideal and ${}^\perp A$ is quasi-ideal then $(A^\perp)_{sp} = ({}^\perp A)_{sp}$. Moreover, if A is a quasi-ideal and ${}^\perp A$ is an ideal then ${}^\perp A = A^\perp$.

Proof (a) If $x \in A \cap {}^\perp A$ then also $x^u, x^l \in A \cap {}^\perp A$, since A and ${}^\perp A$ are mixed lattice subspaces. It follows that $x^u = x^u \curlywedge x^u = 0$ and so $x \le 0$. Similarly $x^l = 0$, which implies that $x \ge 0$. Hence $x = 0$. By a similar argument we have $A \cap A^\perp = \{0\}$.

(b) This follows from the observation that $A_p \subseteq (({}^\perp A)^\perp)_p$. Indeed, it follows from the definitions that $A_{sp} \subseteq (({}^\perp A)^\perp)_{sp}$. Hence, if $x \in A_p$, we have $x^u \in A_{sp} \subseteq (({}^\perp A)^\perp)_{sp}$. Since $({}^\perp A)^\perp$ is an ideal, it follows from $0 \le x \le x^u$ that $x \in (({}^\perp A)^\perp)_p$, and so $A_p \subseteq (({}^\perp A)^\perp)_p$.

(c) It is evident that $A_{sp} \subseteq {}^\perp(A^\perp)_{sp}$ and if A is regular then $A = A_{sp} - A_{sp} \subseteq {}^\perp(A^\perp)_{sp} - {}^\perp(A^\perp)_{sp} = {}^\perp(A^\perp)$.

(d) Let $x \in (A^\perp)_{sp}$ and $z \in A_{sp}$. Then the inequalities $0 \preccurlyeq x \curlywedge z \preccurlyeq x$ and $0 \preccurlyeq x \curlywedge z \le z$ imply that $x \curlywedge z \in (A^\perp)_{sp} \cap A_{sp} = \{0\}$, and so $x \in ({}^\perp A)_{sp}$.

(e) If $x \in ({}^\perp A)_{sp}$ and $z \in A_{sp}$ then it follows from $0 \preccurlyeq z \curlywedge x \preccurlyeq z$ and $0 \preccurlyeq z \curlywedge x \le x$ that $z \curlywedge x \in A \cap {}^\perp A = \{0\}$. Hence, $x \in (A^\perp)_{sp}$ and so $({}^\perp A)_{sp} \subseteq (A^\perp)_{sp}$. The reverse inclusion follows from (e). Then ${}^\perp A = ({}^\perp A)_{sp} - ({}^\perp A)_{sp} = (A^\perp)_{sp} - (A^\perp)_{sp}$, by Theorem 5.5, so if ${}^\perp A$ is an ideal then the equality ${}^\perp A = A^\perp$ follows from Theorem 4.12.

□

Remark 5.14 We note that, in particular, $\{0\}^\perp = V$ and ${}^\perp V = \{0\}$. Moreover, $V^\perp = \{0\}$ holds always, and ${}^\perp\{0\} = V$ holds if and only if V is regular. Indeed, if V is not regular then ${}^\perp\{0\} = V_{sp} - V_{sp} \ne V$.

We now give an example to illustrate some of the ideas presented above.

Example 5.15 Let $V = \mathbb{R}^3$ where $\le$ is the standard partial ordering where $(x, y, z) \ge (0, 0, 0)$ if $x, y, z \ge 0$. Define $\preccurlyeq$ to be the partial ordering with the positive cone $V_{sp} = \{(x, x, z) : x, z \ge 0\}$. Then V is a quasi-regular mixed lattice space. Consider the following subspaces: $A = \{(0, 0, z) : z \in \mathbb{R}\}$, $B = \{(x, y, 0) : x, y \in \mathbb{R}\}$, $C = \{(x, x, z) : x, z \in \mathbb{R}\}$, $D = \{(x, x, 0) : x \in \mathbb{R}\}$, $E = \{(x, x, x) : x \in \mathbb{R}\}$. Then A is a regular ideal and B is an ideal (but not regular) such that $B = A^\perp$ and

$A = {}^{\perp}B$. Moreover, $C = V_{sp} - V_{sp}$ is a quasi-ideal, and it is the largest quasi-ideal in V. Also $D = B_{sp} - B_{sp}$ is a quasi-ideal, and it is the largest quasi-ideal contained in B. Finally, E is a regular mixed lattice subspace but not a specific ideal.

Next, $K_1 = \{(0, y, z) : y, z \geq 0\}$ and $K_2 = \{(x, 0, z) : x, z \geq 0\}$ are the maximal right disjoint cones of D and $\mathcal{S}(D) = K_1 \cup K_2$, which is clearly not closed under addition. Moreover, $(D^{\perp})_p = A_{sp} = K_1 \cap K_2$ (Theorem 5.12). The inclusion in Theorem 5.12 may be proper. To see this, let us modify the mixed lattice space, and consider $U = \mathbb{R}^3$ where $\leq$ is the same as above, and define $\preccurlyeq$ as the partial ordering with the positive cone $U_{sp} = \{\alpha(0, 0, 1) + \beta(0, 1, 0) + \gamma(1, 1, 0) : \alpha, \beta, \gamma \geq 0\}$. Let all the subspaces be the same as above. Then U is a regular mixed lattice space, and $F = \{(0, y, 0) : y \in \mathbb{R}\}$ is a regular ideal in U. Now $\mathcal{S}(F) = K_2$ which is closed under addition, but $(F^{\perp})_p = A_{sp} \subseteq K_2$. Note also that A and B are still ideals in U, this time they are both regular, and $B = A^{\perp}$ and $A = {}^{\perp}B$ holds.

Our definitions of the left and right disjoint complements differ from the corresponding definition in the theory of Riesz spaces. We recall that if E is a subset of a Riesz space L then the disjoint complement of E is defined as $E^{\perp} = \{x \in L : |x| \wedge |y| = 0 \text{ for all } y \in E\}$.

In mixed lattice spaces the generalized absolute values exist, and this naturally raises the question whether it is possible to give the definitions of the disjoint complements in terms of the absolute values, like in Riesz spaces. The main difficulty here is that the asymmetric generalized absolute values are not necessarily positive with respect to the specific order. To deal with this issue we introduce the notion of a symmetric absolute value, which is defined in terms of the asymmetric absolute values. It has the advantage of being positive with respect to both partial orderings while retaining most of the other important properties of the absolute value.

Definition 5.16 Let V be a mixed lattice vector space and $x \in V$. The element $s(x) = \frac{1}{2}({}^{u}x^{l} + {}^{l}x^{u})$ is called the *symmetric generalized absolute value* of x.

Next we derive some basic properties of the symmetric absolute value. The first item gives useful alternative expressions and the rest of the properties show that, in many ways, the symmetric generalized absolute value behaves like the ordinary absolute value in Riesz spaces.

Theorem 5.17 *Let V be a mixed lattice vector space and $x \in V$. Then the following statements hold.*

(a) $s(x) = x^{u} \curlyvee {}^{l}x = {}^{u}x + {}^{l}x = x^{l} + x^{u} = x^{l} \curlyvee {}^{u}x$.
(b) $s(\alpha x) = |\alpha| s(x)$ *for all* $\alpha \in \mathbb{R}$.
(c) $s(x) \succcurlyeq 0$ *and* $s(x) \geq 0$. *Moreover,* $s(x) = 0$ *if and only if* $x = 0$.
(d) $x \succcurlyeq 0$ *if and only if* $x = s(x)$. *In particular,* $s(s(x)) = s(x)$.
(e) $s(x + y) \leq s(x) + s(y)$.
(f) $x \curlywedge y + y \curlywedge x = x + y - s(x - y)$.

Proof (a) All the equalities apart from the first one were given in Theorem 4.2(h). The first equality follows from

$$2s(x) = {}^{u}x^{l} + {}^{l}x^{u} = {}^{u}x + {}^{l}x + x^{l} + x^{u} = 2(x^{l} + x^{u}) = 2(x^{u} \curlyvee {}^{l}x),$$

where we used Theorem 4.2 (c) and (h).

(b) This follows from Theorem 4.2 (l) and (m).

(c) Since $x^u \succcurlyeq 0$ and $x^l \succcurlyeq 0$ we have $s(x) = x^u + x^l \succcurlyeq 0$. Also, ${}^u x \geq 0$ and ${}^l x \geq 0$ imply $s(x) = {}^u x + {}^l x \geq 0$. It is clear that $x = 0$ implies $s(x) = 0$. Assume then that $s(x) = 0$. Now $s(x) = {}^u x + {}^l x = 0$, or ${}^u x = -{}^l x$. Hence $0 \leq {}^u x = -{}^l x \leq 0$, which implies ${}^u x = 0$. On the other hand, $s(x) = x^u + x^l = 0$, or $x^l = -x^u$. So, $0 \preccurlyeq x^l = -x^u \preccurlyeq 0$, which in turn implies $x^l = 0$. Consequently, $x = {}^u x - x^l = 0$.

(d) This follows immediately from (a), (c) and Theorem 4.2 (i).

(e) Using (a) together with Theorem 4.2 (e) and (f) we get

$$s(x+y) = {}^u(x+y) + {}^l(x+y) \leq {}^u x + {}^u y + {}^l x + {}^l y = s(x) + s(y).$$

(f) This follows by adding the two identities given in Theorem 4.2 (n).

□

We can now characterize ideals in terms of the symmetric absolute value.

Theorem 5.18 *Let V be a mixed lattice space and A a subspace of V. The following conditions are equivalent.*

(a) A is an ideal.

(b) If $s(x) \leq s(y)$ and $y \in A$ then $x \in A$.

Proof Assume that (a) holds and let $s(x) \leq s(y)$ with $y \in A$. Then $s(y) \in A$, and the inequalities $0 \leq x^l \leq s(x) \leq s(y)$ and $0 \leq {}^u x \leq s(x) \leq s(y)$ imply that ${}^u x, x^l \in A$ and so ${}^u x - x^l = x \in A$.

Conversely, assume that (b) holds and let $y \in A$. Then by Theorem 5.17 $0 \preccurlyeq y^u = s(y^u) \leq s(y)$. It follows that $y^u \in A$ and so A is a mixed lattice subspace. Next, assume that $0 \leq x \leq y$ with $y \in A$. Then $y^l = x^l = 0$ and so $s(y) = y^u$ and $s(x) = x^u$. Now the above assumption implies that $x^u = s(x^u) \leq y^u = s(y^u) \leq s(y)$. It follows that $x^u = s(x) \in A$ and since $s(x) \leq s(s(x))$ by Theorem 5.17(d), we infer that $x \in A$ and so A is $(\leq)$-order convex, and hence an ideal. □

Next we will show that the right disjoint complement can also be given in terms of the symmetric absolute value, and for Riesz subspaces, the usual Riesz space definition of the disjoint complement can thus be viewed as a special case of the next result.

Theorem 5.19 *If A is a mixed lattice subspace then the right disjoint complement is given by $A^\perp = \{x \in V : s(z) \curlywedge s(x) = 0$ for all $z \in A\}$.*

Proof Let $X = \{x \in V : s(z) \curlywedge s(x) = 0 \text{ for all } z \in A\}$. We first note that $A_{sp} = \{s(z) : z \in A\}$ and so $X = \{x \in V : z \curlywedge s(x) = 0 \text{ for all } z \in A_{sp}\}$. Let $x, y \in X$ and $z \in A_{sp}$. Since $s(x) \succcurlyeq 0$ and $s(y) \succcurlyeq 0$ we have

$$0 \leq z \curlywedge s(x+y) \leq z \curlywedge (s(x) + s(y)) \leq z \curlywedge s(x) + z \curlywedge s(y) = 0,$$

and so $z \curlywedge s(x+y) = 0$ (here we used Theorem 5.17 and Lemma 5.7). This shows that X is a subspace. To show that X is an ideal, we note that if $s(x) \le s(y)$ and $y \in X$ then $0 \le z \curlywedge s(x) \le z \curlywedge s(y) = 0$, and so $x \in X$ and by Theorem 5.18 X is an ideal.

Clearly, $A^\perp \subseteq X$. If $x \in X$ then $s(x) \in A^\perp_{sp}$, and it follows from $0 \le x^u \le s(x)$ and $0 \le {}^l x \le s(x)$ that $x = x^u - {}^l x \in A^\perp$. Hence $A^\perp = X$. □

Theorem 5.19 thus gives an alternative (but equivalent) definition of the right disjoint complement by means of the symmetric absolute value. The reason this works is implied by Theorem 5.18, for if A is an ideal then $x \in A$ if and only if $s(x) \in A$. However, if A is a specific ideal, then $x \in A$ implies $s(x) \in A$, but not conversely in general. This is the fundamental reason why we cannot similarly characterize the left disjoint complement ${}^\perp A$ in terms of the symmetric absolute value.

In the next section we consider the situation where A is a band such that $A = ({}^\perp A)^\perp$. We will now give sufficient conditions for this to hold. The next couple of results have their counterparts in the theory of Riesz spaces, and the methods used in their proofs are also similar. First we recall some terminology. A mixed lattice space V is called $(\le)$-*Archimedean* if the condition $nx \le y$ for all $n \in \mathbb{N}$ implies $x \le 0$. Similarly, V is called $(\preccurlyeq)$-*Archimedean* if the condition $nx \preccurlyeq y$ for all $n \in \mathbb{N}$ implies $x \preccurlyeq 0$. It is easy to see that if V is $(\le)$-Archimedean then it is also $(\preccurlyeq)$-Archimedean, but the converse is not true as the next example shows.

Example 5.20 Let $V = (\mathbb{R}^2, \le, \preccurlyeq)$, where $\le$ is the lexicographic ordering defined as

$$(x_1, x_2) \le (y_1, y_2) \quad \Longleftrightarrow \quad (\, x_1 < y_1 \quad \text{or} \quad x_1 = y_1 \quad \text{and} \quad x_2 \le y_2\,),$$

and $\preccurlyeq$ is the usual coordinatewise ordering $(x_1, x_2) \preccurlyeq (y_1, y_2)$ if $x_1 \le y_1$ and $x_2 \le y_2$. Then V is a mixed lattice space which is $(\preccurlyeq)$-Archimedean but not $(\le)$-Archimedean.

Next we will show that in $(\preccurlyeq)$-Archimedean mixed lattice spaces every band A has the property that $A = ({}^\perp A)^\perp$. First we consider the following order-denseness property of quasi-ideals with respect to their second right disjoint complements.

Proposition 5.21 *If A is a quasi-ideal and $x \in ({}^\perp A)^\perp$ is a non-zero element such that $x \succcurlyeq 0$, then there exists a non-zero element $y \in A$ such that $0 \preccurlyeq y \preccurlyeq x$.*

Proof Let $0 \preccurlyeq x \in ({}^\perp A)^\perp$ be a non-zero element. Then $x \notin {}^\perp A$, and there exists an element $0 \preccurlyeq w \in A$ such that $x \curlywedge w \neq 0$. Since A is a quasi-ideal, the inequality $0 \preccurlyeq x \curlywedge w \le w$ implies that $x \curlywedge w \in A$. The element $y = x \curlywedge w$ then satisfies $0 \preccurlyeq y \preccurlyeq x$, as required. □

Remark 5.22 The above proposition holds also if $A = \{0\}$, because then ${}^\perp A = V_{sp} - V_{sp}$ and $({}^\perp A)^\perp = \{0\}$, so there are no non-zero elements in $({}^\perp A)^\perp$ which makes the proposition vacuously true.

In $(\preccurlyeq)$-Archimedean mixed lattice spaces the order-denseness property of the preceding proposition has the following consequence.

Theorem 5.23 *Let V be a $(\preccurlyeq)$-Archimedean mixed lattice space and A a quasi-ideal in V. If $M_u = \{v \in A : 0 \preccurlyeq v \preccurlyeq u\}$ then $u = \operatorname{sp}\sup M_u$ for every $u \in (({}^\perp A)^\perp)_{sp}$.*

Proof Let $0 \preccurlyeq u \in ({}^{\perp}A)^{\perp}$ and $M_u = \{v \in A : 0 \preccurlyeq v \preccurlyeq u\}$. Clearly, u is a specific upper bound of M_u. Assume that $u \neq \operatorname{sp}\sup M_u$. Then there exists some specific upper bound w_0 of M_u such that $u \preccurlyeq w_0$ does not hold. If we put $w = u \curlywedge w_0$ then $0 \preccurlyeq w \preccurlyeq u$ holds, and $w \in ({}^{\perp}A)^{\perp}$, since $({}^{\perp}A)^{\perp}$ is an ideal. Moreover, $0 \preccurlyeq u - w \in ({}^{\perp}A)^{\perp}$ and so we can apply Proposition 5.21 to find a non-zero element $z \in A$ such that $0 \preccurlyeq z \preccurlyeq u - w \preccurlyeq u$. In particular, we have $z \preccurlyeq u$ and $z \preccurlyeq w_0$, and consequently, $z \preccurlyeq w$. Hence

$$0 \preccurlyeq 2z = z + z \preccurlyeq (u - w) + w = u,$$

and clearly $2z \in A$, so in fact we have $2z \in M_u$. Hence, we can repeat the same argument for the element $2z$. That is,

$$0 \preccurlyeq 3z = z + 2z \preccurlyeq (u - w) + w = u,$$

and so $3z \in M_u$. Continuing this way we find that $nz \in M_u$ for all n, and so $0 \preccurlyeq nz \preccurlyeq u$ for all n. But $0 \preccurlyeq nz \neq 0$, which is impossible since V is $(\preccurlyeq)$-Archimedean. Hence, $u = \operatorname{sp}\sup M_u$, and the proof is complete. □

The following result is now a rather immediate consequence of the preceding theorem.

Corollary 5.24 *If V is a $(\preccurlyeq)$-Archimedean mixed lattice space and A is a quasi-band in V, then $A_{sp} = (({}^{\perp}A)^{\perp})_{sp}$.*

6 Direct sum decompositions and projections

The sum of subspaces A and B of a mixed lattice space V is called a *direct sum* if $A \cap B = \{0\}$, and this is denoted by $A \oplus B$. If V has a direct sum decomposition $V = A \oplus B$ then each element $x \in V$ has a unique representation $x = x_1 + x_2$, where $x_1 \in A$ and $x_2 \in B$. The elements x_1 and x_2 are called *components* of x in A and B, respectively. A vector space can usually be written as a direct sum of subspaces in many ways, but we are mainly concerned with direct sum decompositions that are compatible with the mixed lattice structure in the sense that the order structure of V is determined by its direct sum components.

In this section we give some results concerning direct sum decompositions of a mixed lattice space in terms of ideals and specific ideals. First we show that if a mixed lattice space V can be written as a direct sum of a regular specific ideal A and an ideal B then A and B are necessarily the disjoint complements of each other. Moreover, A has stronger band properties than those implied by Theorem 5.5.

Theorem 6.1 *If A is a regular specific ideal and B is an ideal such that $V = A \oplus B$ then A is a regular specific band and B is a weak band such that $A = {}^{\perp}B = {}^{\perp}(A^{\perp})$ and $B = A^{\perp} = ({}^{\perp}B)^{\perp}$. Moreover, $V_{sp} = A_{sp} + B_{sp}$.*

Proof Assume that $V = A \oplus B$, that is, $A \cap B = \{0\}$. Let $u \in A$ and $v \in B$ such that $u \succcurlyeq 0$ and $v \geq 0$. Then, since B is an ideal, it follows from $0 \leq u \curlywedge v \leq v$ that $u \curlywedge v \in B$, and consequently $(u \curlywedge v)^u \in B$. On the other hand, $u \curlywedge v \preccurlyeq u$ and by Theorem 3.2 we have

$$0 \preccurlyeq (u \curlywedge v)^u = 0 \curlyvee (u \curlywedge v) \preccurlyeq 0 \curlyvee u = u^u = u,$$

and so $(u \curlywedge v)^u \in A$, since A is a specific ideal. Hence, we have $(u \curlywedge v)^u \in A \cap B$, and so $(u \curlywedge v)^u = 0$. Since $u \curlywedge v \geq 0$, this implies that $u \curlywedge v = 0$. Hence, $A \subseteq {}^{\perp}B$ and $B \subseteq A^{\perp}$.

To prove the reverse inclusions, let $x \in {}^{\perp}B$. Since $V = A \oplus B$, the element x has a unique decomposition $x = x_1 + x_2$ with $x_1 \in A$ and $x_2 \in B$. But we showed above that $A \subseteq {}^{\perp}B$, and so $x_1 \in {}^{\perp}B$. Since ${}^{\perp}B$ is a subspace, we have $x - x_1 = x_2 \in {}^{\perp}B$. Thus $x_2 = 0$ (by Theorem 5.13) and $x = x_1 \in A$. Hence, ${}^{\perp}B \subseteq A$ and so $A = {}^{\perp}B$. From this it follows that $A^{\perp} = ({}^{\perp}B)^{\perp}$ and $V = A \oplus B = {}^{\perp}B \oplus B$. Thus, every $x \in A^{\perp} = ({}^{\perp}B)^{\perp}$ has a unique decomposition $x = x_1 + x_2$ with $x_1 \in {}^{\perp}B$ and $x_2 \in B \subseteq ({}^{\perp}B)^{\perp}$, and so by Theorem 5.13 we have $x_1 = x - x_2 \in ({}^{\perp}B)^{\perp}$. But then $x_1 \in {}^{\perp}B \cap ({}^{\perp}B)^{\perp} = \{0\}$, so we deduce that $x = x_2 \in B$, and therefore $A^{\perp} \subseteq B$, and so $A^{\perp} = B$. Hence, we have $A = {}^{\perp}B = {}^{\perp}(A^{\perp})$ and $B = A^{\perp} = ({}^{\perp}B)^{\perp}$.

The equality $V_{sp} = A_{sp} + B_{sp}$ holds by Theorem 4.17(b). By Theorem 5.11, B is a weak band, and by Theorem 5.5, A is a weak specific band, but we must show that A is a specific band. For this, let E be a subset of A such that $w = \operatorname{sp}\sup E$ exists in V. Then for any $x \in E$ we have $z = w - x \succcurlyeq 0$, and so $z = z_1 + z_2$ with $z_1 \in A_{sp}$ and $z_2 \in B_{sp}$. On the other hand, we have $w = w_1 + w_2$ with $w_1 \in A$ and $w_2 \in B$, and therefore $z = (w_1 - x) + w_2$ where $w_1 - x \in A$. Since the representation of z is unique, we have $z_1 = w_1 - x \in A_{sp}$ and $z_2 = w_2 \in B_{sp}$. Hence, $x \preccurlyeq w_1 \preccurlyeq w_1 + w_2 = w$. Since this holds for any $x \in E$, it follows that $w_2 = 0$ and $w = w_1 \in A$, and so A is a specific band. □

For a regular mixed lattice space we have the following result.

Theorem 6.2 *Let A be a regular specific ideal and B an ideal such that $V = A \oplus B$. Then V is regular if and only if A and B are both regular.*

Proof If A and B are both regular then clearly V is also regular. If V is regular and $x \in B$ then $x \preccurlyeq u$ for some $u \in V_{sp}$. Now by the preceding theorem $u = u_1 + u_2$ with $u_1 \in A_{sp}$ and $u_2 \in B_{sp}$. The inequality $x \preccurlyeq u_1 + u_2$ is equivalent to $0 \preccurlyeq u_1 + (u_2 - x)$, and, again by Theorem 6.1, we have $u_2 - x \in B_{sp}$. This implies that B is regular. □

If there is a weak band A such that $V = {}^{\perp}A \oplus A$ then the components of specifically positive elements in A and ${}^{\perp}A$ are given by the next theorem.

Theorem 6.3 *Let A be a weak band such that $V = {}^{\perp}A \oplus A$. Then for every $x \in V_{sp}$ the elements*

$$x_1 = \operatorname{sp}\sup\{v \in A : 0 \preccurlyeq v \preccurlyeq x\} \quad \textit{and} \quad x_2 = \operatorname{sp}\sup\{v \in {}^{\perp}A : 0 \preccurlyeq v \preccurlyeq x\}$$

exist, and x_1 and x_2 are the components of x in A and ${}^{\perp}A$, respectively.

Proof Let $x \in V_{sp}$ have the decomposition $x = x_1 + x_2$ with $x_1 \in A$ and $x_2 \in {}^{\perp}A$. Let $M = \{v \in A : 0 \preccurlyeq v \preccurlyeq x\}$. If $v \in M$ then the element $x - v \succcurlyeq 0$ has the unique decomposition $x - v = (x_1 - v) + x_2$ where $x_1 - v \in A_{sp}$ and $x_2 \in ({}^{\perp}A)_{sp}$, by Theorem 6.1. Thus, $v \preccurlyeq x_1$ and so x_1 is a $(\preccurlyeq)$-upper bound of M. But we also have $0 \preccurlyeq x_1 = x - x_2 \preccurlyeq x$, and so $x_1 \in M$. This shows that $x_1 = \operatorname{sp}\sup M$. The formula for x_2 is proved similarly. □

In accordance with the Riesz space terminology, the components given by the above theorem are called *specific projections* of x on A and ${}^{\perp}A$. We will return to the discussion of projection elements later.

In the above theorems we considered a direct sum $V = A \oplus B$ such that $V_{sp} = A_{sp} + B_{sp}$. If also $V_p = A_p + B_p$ holds then we obtain stronger results, and this motivates the following definition.

Definition 6.4 Let A and B be subspaces of a mixed lattice space V such that $V = A \oplus B$. This direct sum is called a *mixed-order direct sum* if $V_{sp} = A_{sp} + B_{sp}$ and $V_p = A_p + B_p$.

Mixed-order direct sums can be characterized as follows.

Theorem 6.5 *Let A and B be subspaces of V such that $V = A \oplus B$. Then the following are equivalent.*

(a) $V = A \oplus B$ is a mixed-order direct sum
(b) A is a regular band and B is a band such that $A = {}^{\perp}B$ and $B = A^{\perp}$.

Proof Assuming that $V = A \oplus B$ is a mixed-order direct sum, we first show that A is $(\leq)$-order convex. Let $0 \leq v \leq u$ with $u \in A$. If we put $w = u - v$ then $u = v + w$ and $v, w \geq 0$. Now by assumption v and w have representations $v = v_1 + v_2$ and $w = w_1 + w_2$ where $v_1, w_1 \in A_p$ and $v_2, w_2 \in B_p$. Hence, $u = (v_1 + v_2) + (w_1 + w_2)$, or $v_2 + w_2 = u - (v_1 + w_1)$. A and B are subspaces, so $v_2 + w_2 \in B$ and $u - (v_1 + w_1) \in A$. But then $v_2 + w_2 \in A \cap B = \{0\}$, and it follows that $v_2 = w_2 = 0$. Thus, $v = v_1 \in A_p$ and this shows that A is $(\leq)$-order convex.

Next we need to prove that A is a mixed lattice subspace. By assumption, the elements x^u and ${}^l x$ can be written as $x^u = a_1 + b_1$ and ${}^l x = a_2 + b_2$ where $a_1 \in A_{sp}$, $b_1 \in B_{sp}$ and $a_2 \in A_p$, $b_2 \in B_p$. Now $x = x^u - {}^l x = (a_1 - a_2) - (b_2 - b_1)$, so $b_2 - b_1 = x - (a_1 - a_2)$. Since A and B are subspaces, we have $b_2 - b_1 \in B$ and $x - (a_1 - a_2) \in A$. But then $b_2 - b_1 \in A \cap B = \{0\}$, and it follows that $x = a_1 - a_2$, and so we have two representations $x = x^u - {}^l x$ and $x = a_1 - a_2$ with $a_1 \in A_{sp}$ and $a_2 \in A_p$. By Theorem 4.3 we have $0 \leq x^u \leq a_1$ and $0 \leq {}^l x \leq a_2$. It follows that ${}^l x, x^u \in A$, since A is order-convex. This shows that A is a mixed lattice subspace, and hence an ideal. Similar arguments show that B is also an ideal.

To show that A is regular, we note that $0 \leq x \curlywedge y \leq x$ and $0 \leq x \curlywedge y \leq y$ for all $x \in A_{sp}$ and $y \in B_p$, and since A and B are ideals such that $A \cap B = \{0\}$, it follows that $x \curlywedge y = 0$ for all $x \in A_{sp}$ and $y \in B_p$. Hence, $A \subseteq {}^{\perp}B$. Next, let $0 \leq z \in {}^{\perp}B$. By assumption we can write $z = a + b$ where $a \in A_p$ and $b \in B_p$. Then $b = z - a \in {}^{\perp}B$, because $A \subseteq {}^{\perp}B$ and ${}^{\perp}B$ is a subspace. Thus, $b \in B \cap {}^{\perp}B = \{0\}$, by Theorem 5.13. Hence, $z = a \in A$ and so $A = {}^{\perp}B$, which is a regular weak band by Theorem 5.5. It now follows from Theorem 6.1 that B is a weak band such that $B = A^{\perp}$. To see that B is actually a band, let E be a subset of B such that $w = \sup E$ exists in V. Then for any $x \in E$ we have $z = w - x \geq 0$, and so $z = z_1 + z_2$ with $z_1 \in A_p$ and $z_2 \in B_p$. On the other hand, $w = w_1 + w_2$ with $w_1 \in A$ and $w_2 \in B$, and so $z = w_1 + (w_2 - x)$ where $w_2 - x \in B$. Since this representation is unique, we have $z_1 = w_1 \in A_p$ and $z_2 = w_2 - x \in B_p$. Hence, $x \leq w_2 \leq w_1 + w_2 = w$. Since this holds for any $x \in E$, it follows that $w_1 = 0$ and $w = w_2 \in B$, and so B is a band. A similar argument shows that A is also a band.

To prove the implication $(b) \Rightarrow (a)$, assume that A is a regular band such that $V = A \oplus A^{\perp}$. If $W = (A^{\perp})_{sp} - (A^{\perp})_{sp}$ then by Theorem 4.12 W is a quasi-ideal, so by Theorem 4.17 we have $V_{sp} = A_{sp} + (A^{\perp})_{sp} = A_{sp} + W_{sp}$. Then $V_p = A_p + (A^{\perp})_p$, by Theorem 4.16, and hence $V = A \oplus A^{\perp}$ is a mixed-order direct sum. □

Now, if $V = A \oplus A^{\perp}$ is a mixed-order direct sum then the left and right disjoint complements of A are very closely related, and they are equal if V is regular.

Theorem 6.6 *If $V = A \oplus A^{\perp}$ is a mixed-order direct sum then $A = {}^{\perp}(A^{\perp}) = (A^{\perp})^{\perp}$. Moreover, ${}^{\perp}A \subseteq A^{\perp}$, and if $A^{\perp}$ is regular then ${}^{\perp}A = A^{\perp}$. In this case, $A = {}^{\perp}(A^{\perp}) = (A^{\perp})^{\perp} = {}^{\perp}({}^{\perp}A) = ({}^{\perp}A)^{\perp}$. In particular, all these equalities hold if V is regular.*

Proof The equality $A = {}^{\perp}(A^{\perp})$ follows from Theorem 6.5, and since A and $A^{\perp}$ are both ideals, it then follows from Theorem 5.13(e) that ${}^{\perp}(A^{\perp}) = A = (A^{\perp})^{\perp}$. By Theorem 5.13(d) $(A^{\perp})_{sp} \subseteq ({}^{\perp}A)_{sp}$. To prove the reverse inclusion, let $x \in ({}^{\perp}A)_{sp}$. Then by assumption we can write $x = x_1 + x_2$ where $x_1 \in A_{sp}$ and $x_2 \in (A^{\perp})_{sp} \subseteq ({}^{\perp}A)_{sp}$. Then for every $z \in A_{sp}$ we have $x_2 \curlywedge z = 0$, and it follows by by Lemma 5.1 that $0 = x \curlywedge z = (x_1 + x_2) \curlywedge z = x_1 \curlywedge z$. Hence $x_1 \in ({}^{\perp}A)_{sp}$, and so $x_1 \in {}^{\perp}A \cap A = \{0\}$. Thus $x = x_2 \in (A^{\perp})_{sp}$, and so $({}^{\perp}A)_{sp} = (A^{\perp})_{sp}$. Since ${}^{\perp}A = ({}^{\perp}A)_{sp} - ({}^{\perp}A)_{sp}$, it follows that ${}^{\perp}A \subseteq A^{\perp}$, and if $A^{\perp}$ is regular then the equality ${}^{\perp}A = A^{\perp}$ holds. In this case we therefore have $A = {}^{\perp}(A^{\perp}) = (A^{\perp})^{\perp} = {}^{\perp}({}^{\perp}A) = ({}^{\perp}A)^{\perp}$. In particular, this holds if V is regular, by Theorem 6.2. □

The next theorem shows that disjoint components of elements in a mixed lattice space have similar properties as in Riesz spaces. In particular, the symmetric absolute value behaves as one would expect.

Theorem 6.7 *Let $V = A \oplus B$ be a mixed-order direct sum. If $x \in A$ and $y \in B$ then the following hold.*

(a) $(x + y)^u = x^u + y^u$ *and* ${}^l(x + y) = {}^l x + {}^l y$.
(b) ${}^u(x + y) = {}^u x + {}^u y$ *and* $(x + y)^l = x^l + y^l$.
(c) ${}^u(x + y)^l = {}^u x^l + {}^u y^l$ *and* ${}^l(x + y)^u = {}^l x^u + {}^l y^u$.
(d) $s(s(x) - s(y)) = s(x + y) = s(x - y) = s(x) + s(y) = s(x) \curlyvee s(y) = s(y) \curlyvee s(x) = \mathrm{sp}\sup\{s(x), s(y)\}$.

Proof To prove (a), we start by noting that $0 \leq x^u \curlywedge ({}^l x + {}^l y) \leq x^u \curlywedge ({}^l x + ({}^l y)^u)$. Now $x^u \curlywedge {}^l x = 0$ (by Theorem 4.2) and $x^u \curlywedge ({}^l y)^u = 0$ (since $({}^l y)^u \in B_{sp}$). Hence, by Lemma 5.7 we have $x^u \curlywedge ({}^l x + ({}^l y)^u) = 0$, and consequently, $x^u \curlywedge ({}^l x + {}^l y) = 0$.

Now let $w = (x^u + y^u) \curlywedge ({}^l x + {}^l y)$. Then $w \leq {}^l x + {}^l y$ and, since $y^u \succcurlyeq 0$, by Theorem 3.2 we also have

$$w \preccurlyeq (x^u + y^u) \curlywedge (y^u + {}^l x + {}^l y) = y^u + x^u \curlywedge ({}^l x + {}^l y) = y^u.$$

This implies that $0 \leq w \leq y^u \curlywedge ({}^l x + {}^l y)$. Next we note that $y^u \in B_{sp} = (A^{\perp})_{sp} \subseteq ({}^{\perp}A)_{sp}$ (by Theorem 5.13(d)), and $({}^l x)^u \in A_{sp}$, so $0 \leq y^u \curlywedge {}^l x \leq y^u \curlywedge ({}^l x)^u = 0$. Moreover, $y^u \curlywedge {}^l y = 0$, so we can use Lemma 5.7 again to obtain $0 \leq y^u \curlywedge ({}^l x + {}^l y) \leq y^u \curlywedge (({}^l x)^u + {}^l y) = 0$. Hence, we have shown that $y^u \curlywedge ({}^l x + {}^l y) = 0$, and consequently, $w = 0$.

Since $x + y = (x^u + y^u) - ({}^l x + {}^l y)$ and $(x^u + y^u) \curlywedge ({}^l x + {}^l y) = 0$, we have $(x + y)^u = x^u + y^u$ and ${}^l(x + y) = {}^l x + {}^l y$, by Theorem 4.3. Similar reasoning proves (b), and adding the equalities in (a) and (b) gives the equalities in (c). Adding the equalities in (c) then gives $s(x + y) = s(x) + s(y)$. Since this holds for all x and y, we can replace y by $-y$ to get $s(x - y) = s(x) + s(-y) = s(x) + s(y)$. The equality $s(s(x) - s(y)) = s(x) + s(y)$ then follows immediately from Theorem 5.17(f) by replacing x with $s(x)$ and y with $s(y)$, and using the fact that $s(x) \curlywedge s(y) = s(y) \curlywedge s(x) = 0$. Indeed, since $s(x) \in A_{sp}$ and $s(y) \in B_{sp} = (A^{\perp})_{sp}$, then by Theorem 5.13(d) we have $s(y) \in ({}^{\perp}A)_{sp}$, and so $s(x) \curlywedge s(y) = s(y) \curlywedge s(x) = 0$. Consequently, $s(x) \curlyvee s(y) = s(y) \curlyvee s(x) = s(x) + s(y) = \mathrm{sp}\sup\{s(x), s(y)\}$. Here the last equality follows by noting that $s(x) + s(y)$ is clearly a $(\preccurlyeq)$-upper bound of $\{s(x), s(y)\}$, and if $s(x) \preccurlyeq c$ and $s(y) \preccurlyeq c$ then $s(x) \curlyvee s(y) \preccurlyeq c$, completing the proof. □

If C is a mixed lattice cone then the following inequalities hold for all $x, y, z \in C$. (For the proofs we refer to [2, pp.13] and [2, Theorem 3.6.])

$$z \curlywedge (x + y) \leq z \curlywedge x + z \curlywedge y \tag{6.1}$$

$$(x + y) \curlywedge z \preccurlyeq x \curlywedge z + y \curlywedge z \tag{6.2}$$

With these inequalities we obtain the following formulae for the mixed envelopes of specifically positive elements.

Proposition 6.8 *Let $V = A \oplus B$ be a mixed-order direct sum. If $x, y \in V_{sp}$ are elements with components $x = x_1 + x_2$ and $y = y_1 + y_2$ where $x_1, y_1 \in A_{sp}$ and $x_2, y_2 \in B_{sp}$, then*

$$x \curlywedge y = x_1 \curlywedge y_1 + x_2 \curlywedge y_2 \quad \textit{and} \quad x \curlyvee y = x_1 \curlyvee y_1 + x_2 \curlyvee y_2.$$

Proof A_{sp} and B_{sp} are mixed lattice cones, so using 6.1 and 6.2 together with the fact that $x_1 \curlywedge y_2 = 0 = x_2 \curlywedge y_1$ (where the last equality follows from Theorem 6.6, since $y_1 \in A = {}^{\perp}B \subseteq B^{\perp}$) we get

$$
\begin{aligned}
x \curlywedge y &= (x_1 + x_2) \curlywedge (y_1 + y_2) \\
&\preccurlyeq x_1 \curlywedge (y_1 + y_2) + x_2 \curlywedge (y_1 + y_2) \\
&\leq x_1 \curlywedge y_1 + x_1 \curlywedge y_2 + x_2 \curlywedge y_1 + x_2 \curlywedge y_2 \\
&= x_1 \curlywedge y_1 + x_2 \curlywedge y_2.
\end{aligned}
$$

On the other hand, we have $x_1 \curlywedge y_1 + x_2 \curlywedge y_2 \preccurlyeq x_1 + x_2 = x$ and $x_1 \curlywedge y_1 + x_2 \curlywedge y_2 \leq y_1 + y_2 = y$, so $x_1 \curlywedge y_1 + x_2 \curlywedge y_2 \leq x \curlywedge y$, and the first identity follows. Similar reasoning shows that $y \curlywedge x = y_1 \curlywedge x_1 + y_2 \curlywedge x_2$.

Substituting these into $x \curlyvee y = x + y - y \curlywedge x$ gives

$$
\begin{aligned}
x \curlyvee y &= x_1 + x_2 + y_1 + y_2 - (y_1 \curlywedge x_1 + y_2 \curlywedge x_2) \\
&= (x_1 + y_1 - y_1 \curlywedge x_1) + (x_2 + y_2 - y_2 \curlywedge x_2) \\
&= x_1 \curlyvee y_1 + x_2 \curlyvee y_2.
\end{aligned}
$$

□

If $V = A \oplus A^{\perp}$ is a mixed order direct sum then the components of a positive element in A and $A^{\perp}$ are given by the next theorem, which is proved exactly as Theorem 6.3.

Theorem 6.9 *Let A be a band such that $V = A \oplus A^{\perp}$ is a mixed order direct sum. Then for every $x \in V_p$ the elements*

$$
x_1 = \sup\{v \in A : 0 \leq v \leq x\} \quad \textit{and} \quad x_2 = \sup\{v \in A^{\perp} : 0 \leq v \leq x\}
$$

exist, and x_1 and x_2 are the components of x in A and $A^{\perp}$, respectively.

As in the theory of Riesz spaces, it turns out that the existence of a mixed order direct sum is equivalent to the existence of the associated order projection. Let $V = A \oplus A^{\perp}$ be a mixed order direct sum. If $x \in V$ has the components $x_1 \in A$ and $x_2 \in A^{\perp}$ then we define the mapping $P_A : V \to V$ by $P_A(x) = x_1$. We can immediately see that P_A has the following properties:

$$
\begin{aligned}
&(P1)\ \ P_A \text{ is a linear operator and } P_A^2 = P_A. \\
&(P2)\ \ 0 \leq P_A(x) \leq x \text{ for every } x \geq 0. \\
&(P3)\ \ 0 \preccurlyeq P_A(x) \preccurlyeq x \text{ for every } x \succcurlyeq 0.
\end{aligned}
$$

A mapping with these properties is called a *mixed order projection* and the associated band A is called a *projection band*. (Note that the specific projection elements in Theorem 6.3 can be given similarly in terms of a specific projection operator that has the properties (P1) and (P3)).

Conversely, if P is a mixed order projection on V then there exists a corresponding mixed order direct sum decomposition of V. Apart from a few minor modifications, the proof is essentially the same as in the case of Riesz spaces (see [11, Theorem 11.4]).

Theorem 6.10 *If $P : V \to V$ is a mapping with properties (P1)–(P3) then there exists a regular band A such that $V = A \oplus A^{\perp}$ is a mixed order direct sum and P is the associated projection on A and $I - P$ is the projection on $A^{\perp}$.*

Proof Let $A = \{Px : x \in V\}$ and $B = \{(I - P)x : x \in V\}$. Then A and B are clearly subspaces of V. If $x \in A \cap B$ then $x = Py$ and $x = (I - P)z$ for some $y, z \in V$. Using the property (P1) we get

$$x = Py = P(Py) = P(I - P)z = Pz - P^2 z = 0.$$

This shows that $A \cap B = \{0\}$, so $A \oplus B$ is a direct sum. Moreover, for any $x \in V$ we have $x = Px + (x - Px) = Px + (I - P)x$, so $V = A \oplus B$. It now follows from (P2) that if $x \geq 0$ then $Px \geq 0$ and $0 \leq x - Px = (I - P)x$. Hence, $V_p = A_p + B_p$. Similarly, (P3) implies that $V_{sp} = A_{sp} + B_{sp}$, and so $V = A \oplus B$ is a mixed order direct sum. It then follows from Theorem 6.5 that A is a regular band and $B = A^{\perp}$. Clearly, P and $I - P$ are the projections on A and $A^{\perp}$, respectively. □

If $x \geq 0$ then by Theorem 6.9 the element $P_A(x)$ is given by $P_A(x) = \sup\{v \in A : 0 \leq v \leq x\}$. Since every element can be written as a difference of positive elements, we obtain the projection of an arbitrary element.

Theorem 6.11 *If $V = A \oplus A^{\perp}$ is a mixed order direct sum and P_A and $P_{A^{\perp}} = I - P_A$ are the associated projections on A and $A^{\perp}$, respectively, then for any $x \in V$ we have*

$$x = P_A(x) + P_{A^{\perp}}(x) = P_A(x^u) - P_A({}^l x) + P_{A^{\perp}}(x^u) - P_{A^{\perp}}({}^l x),$$

or alternatively,

$$x = P_A(x) + P_{A^{\perp}}(x) = P_A({}^u x) - P_A(x^l) + P_{A^{\perp}}({}^u x) - P_{A^{\perp}}(x^l).$$

Proof Every $x \in V$ can be written as $x = x^u - {}^l x$ (or alternatively, $x = {}^u x - x^l$, but this case is treated similarly). The elements x^u and ${}^l x$ have the components $x^u = a + b$ and ${}^l x = u + v$, where $a, u \in A$ and $b, v \in A^{\perp}$. On the other hand, $x = x_1 + x_2$ with the components $x_1 \in A$ and $x_2 \in A^{\perp}$. By Theorem 6.7 we have $a + b = x^u = {x_1}^u + {x_2}^u$ and $u + v = {}^l x = {}^l x_1 + {}^l x_2$, where ${x_1}^u, {}^l x_1 \in A$ and ${x_2}^u, {}^l x_2 \in A^{\perp}$. Since $A \cap A^{\perp} = \{0\}$, this implies that ${x_1}^u = a$, ${x_2}^u = b$, ${}^l x_1 = u$ and ${}^l x_2 = v$. These components are the projection elements, and the proof is thus complete. □

Next we consider some examples. First it should be noted that there are non-trivial mixed lattice spaces in which non-trivial ideals do not exist, and hence non-trivial mixed lattice decompositions do not always exist either.

Example 6.12 Let $V = \mathbb{R}^2$ and define $\leq$ as the partial ordering with the usual positive cone $V_p = \{(x, y) : x \geq 0, y \geq 0\}$. Let $\preccurlyeq$ to be the partial ordering induced by the positive cone $V_{sp} = \{\alpha(2, 1) + \beta(1, 2) : \alpha, \beta \geq 0\}$. Then V is a regular mixed lattice space. If $C_1 = \{(x, y) : y = \frac{1}{2}x\}$ and $C_2 = \{(x, y) : y = 2x\}$ then C_1 and C_2 are both specific ideals, but there are no other ideals in V than $\{0\}$ and V itself.

If we change $\preccurlyeq$ to be the partial ordering with the positive cone $V_{sp} = \{\alpha(1,0) + \beta(1,1) : \alpha, \beta \geq 0\}$ then V is again a regular mixed lattice space. Let $A = \{(x,y) : y = x\}$ and $B = \{(x,y) : y = 0\}$. It is easy to see that A is a regular specific ideal and B is a regular ideal such that $V = A \oplus B$. The conditions of Theorem 6.1 are thus satisfied and $A = {}^{\perp}B$ and $B = A^{\perp}$. Moreover, $(A+B)_{sp} = A_{sp} + B_{sp}$ holds. However, this is not a mixed-order direct sum. For instance, the element $x = (1,2)$ cannot be written as $x = x_1 + x_2$ where $x_1 \in A_p$ and $x_2 \in B_p$.

The next one is related to the setting of Theorem 6.1.

Example 6.13 This example is adapted from [2, pp. 34]. See also [5, Example 2.15]. Let $V = BV([0,1])$ be the set of all functions of bounded variation on the interval $[0,1]$. Define the initial order as $f \leq g$ if $f(x) \leq g(x)$ for all $x \in [0,1]$, and the specific order as $f \preccurlyeq g$ if $f \leq g$ and $g - f$ is non-decreasing on $[0,1]$. Then V is a regular mixed lattice space.

Now fix some $c \in (0,1)$ and let A be the subspace consisting of those functions that are constant on the closed interval $[c,1]$, and let B be the subspace consisting of those functions that vanish on the closed interval $[0,c]$. Then A is a regular specific ideal and B is a regular ideal such that $A \cap B = \{0\}$. It is well known that every $g \in V$ can be written as a difference of two non-decreasing non-negative functions on $[0,1]$. Hence, to see that $V = A \oplus B$ it is sufficient to note that every non-decreasing non-negative function f on $[0,1]$ can be written as $f = f_1 + f_2$ where $f_1 \in A$ and $f_2 \in B$. Indeed, define f_1 by $f_1(x) = f(x)$ for all $x \in [0,c]$, and $f_1(x) = f(c)$ for all $x \in [c,1]$. Then define f_2 by $f_2(x) = 0$ for all $x \in [0,c]$ and $f_2(x) = f(x) - f(c)$ for all $x \in [c,1]$. Then $f_1 \in A$, $f_2 \in B$ and $f = f_1 + f_2$, and by Theorem 6.1 $A = {}^{\perp}B$ is a specific band and $B = A^{\perp}$ is a weak band.

If we consider the space W of all continuous functions of bounded variation on $[0,1]$, and we put $c = 0$ with A and B defined as above, then A is just the set of all constant functions on $[0,1]$ and $B = \{g \in W : g(0) = 0\}$. Then $W = A \oplus B$ as above, and B is a weak band by Theorem 6.1, but not a band. This can be seen by choosing $f(x) = 1$ for all $x \in [0,1]$ and defining f_n by $f_n(x) = nx$ for $x \in [0, 1/n]$, and $f_n(x) = 1$ for $x \in (1/n, 1]$. Then $\{f_n\} \subseteq B$ for all $n \in \mathbb{N}$ and $\sup\{f_n\} = f$, but $f \notin B$. Note also, that now f is not the specific supremum (and hence not the strong supremum) of $\{f_n\}$.

Special cases of mixed-order direct sums are provided by Dedekind complete Riesz spaces, where all bands are projection bands. We give some other examples below.

Example 6.14 Let U be the same mixed lattice space as in Example 5.15, with the subspaces $C = \{(0,y,z) : y,z \in \mathbb{R}\}$ and $D = \{(x,x,0) : x \in \mathbb{R}\}$. Now C is a regular ideal and D is a regular specific ideal such that $U = C \oplus D$. Hence, the conditions of Theorem 6.1 are satisfied and we have $C = D^{\perp}$ and $D = {}^{\perp}C$. Moreover, $(C+D)_{sp} = C_{sp} + D_{sp}$ but this is not a mixed-order direct sum. If $A = \{(0,0,z) : z \in \mathbb{R}\}$ and $B = \{(x,y,0) : x,y \in \mathbb{R}\}$ then A and B are both regular ideals such that $U = A \oplus B$. This is a mixed-order direct sum where $B = A^{\perp}$ and $A = {}^{\perp}B$.

Similarly, if V is the same as in Example 5.15, then in V, A is a regular band and B is a band such that $B = A^{\perp}$, $A = {}^{\perp}B$ and $V = A \oplus B$ is a mixed order direct sum.

Example 6.15 Let V be the set of all n-by-n matrices and define $\leq$ as the usual element-wise ordering and define specific order by $M \succcurlyeq 0$ if $M \geq 0$ and M is a symmetric matrix. Then V is a quasi-regular mixed lattice space (see [5, Example 2.19]), where the set A consisting of all diagonal matrices is a band and the set B of those matrices with zero diagonal elements is a band such that $A = {}^{\perp}B$, $B = A^{\perp}$ and $V = A \oplus B$ is a mixed order direct sum.

As we have seen in Example 6.13, the mixed lattice space of functions of bounded variation has direct sum decompositions of the type described in Theorem 6.1, but it does not possess non-trivial mixed-order direct sum decompositions. The following example is somewhat analogous to the situation in the Riesz space $C([0, 1])$ of continuous real functions on the interval $[0, 1]$, where non-trivial projection bands do not exist. Before discussing the example we need the following lemma.

Lemma 6.16 *The ideal $I(u)$ generated by a single element $u \in V$ is given by $I(u) = \{x \in V : s(x) \leq ns(u)$ for some $n \in \mathbb{N}\}$.*

Proof We first show that $I(u)$ is a subspace. If $x, y \in I(u)$ then $s(x) \leq ns(u)$ and $s(y) \leq ms(u)$ for some $n, m \in \mathbb{N}$. Then for all $a, b \in \mathbb{R}$,

$$s(ax + by) \leq |a|s(x) + |b|s(y) \leq (|a|n + |b|m)s(u) \leq ps(u),$$

where $p \in \mathbb{N}$ is a number such that $|a|n + |b|m \leq p$. Thus, $ax + ny \in I(u)$. Next, let $x \in I(u)$ and $s(y) \leq s(x)$. Then $s(y) \leq s(x) \leq ns(u)$ for some $n \in \mathbb{N}$ and so $y \in I(u)$. It follows that $I(u)$ is an ideal, by the condition of Theorem 5.18. To show that $I(u)$ is the smallest ideal that contains u, let J be another ideal such that $u \in J$. Then also $ns(u) \in J$ for all $n \in \mathbb{N}$, and if $x \in I(u)$ the inequality $s(x) \leq ms(u) = s(mu)$ holds for some $m \in \mathbb{N}$. It follows again by Theorem 5.18 that $x \in J$. Hence, $I(u) \subseteq J$.

Example 6.17 Consider the regular mixed lattice space $V = BV([0, 1])$, as in Example 6.13. If $f, g \in V$ then the mixed lower and upper envelopes are given by ([2, Theorem 21.1.])

$$(f \curlywedge g)(u) = \inf \{f(u) - (f(x) - g(x))^+ : x \in [0, u]\}$$

and

$$(f \curlyvee g)(u) = \sup \{f(u) + (g(x) - f(x))^+ : x \in [0, u]\},$$

where $r^+ = \max\{0, r\}$ is the positive part of the real number r.

Now $V = V \oplus \{0\}$ is the only mixed order direct sum decomposition of V. This can be seen by considering the constant function $h(x) = 1$ for all $x \in [0, 1]$. First we note that the ideal generated by h is V. Indeed, since every $f \in V$ is bounded, then for any $f \in V$ there exists some $n \in \mathbb{N}$ such that $s(f) \leq nh = ns(h)$. Hence $I(h) = V$, by the preceding lemma.

Now if $V = A \oplus B$ is a mixed order direct sum then h has the components $f \in A$ and $g \in B$ such that $f(x) + g(x) = 1$ for all $x \in [0, 1]$ and $f \curlywedge g = 0$. This implies that f and g are positive, so $0 \leq f(x) \leq 1$ and $0 \leq g(x) \leq 1$ for all $x \in [0, 1]$. Then the above expression for $f \curlywedge g$ gives

$$(f \curlywedge g)(u) = f(u) - \sup\{(f(x) - g(x))^+ : x \in [0, u]\}.$$

If $f \curlywedge g = 0$ then we have $f(u) = \sup\{(f(x) - g(x))^+ : x \in [0, u]\}$. In particular, $f(0) = (f(0) - g(0))^+ = (2f(0) - 1)^+$. Now, if $0 \leq f(0) \leq \frac{1}{2}$ then $f(0) = 0$. If $\frac{1}{2} < f(0) \leq 1$ then $f(0) = 2f(0) - 1$, or $f(0) = 1$. Hence, we must have either $f(0) = 0$ and $g(0) = 1$, or $f(0) = 1$ and $g(0) = 0$. Since $h \succcurlyeq 0$ and $V = A \oplus B$ is a mixed order direct sum then also the components of h satisfy $f \succcurlyeq 0$ and $g \succcurlyeq 0$. In other words, f and g are non-decreasing. But this and $f(x) + g(x) = 1$ imply that either $f(x) = 0$ and $g(x) = 1$, or $f(x) = 1$ and $g(x) = 0$ for all $x \in [0, 1]$. This shows that either A or B contains the constant function $h(x) = 1$. Since the ideal generated by h equals V, we must therefore have either $A = V$ and $B = \{0\}$, or $B = V$ and $A = \{0\}$.

A few concluding remarks are in order to further explain and justify our choice of certain definitions. Since we have defined $A^\perp$ to be the ideal generated by the cone $(A^\perp)_{sp}$, one might be inclined to ask why did we define ${}^\perp A$ as the specific ideal generated by the cone $({}^\perp A)_{sp}$, and not the ideal generated by $({}^\perp A)_{sp}$. The main reason for this stems from the fact that if B is the ideal generated by $({}^\perp A)_{sp}$ then, in general, B_{sp} is a larger set than $({}^\perp A)_{sp}$. As a consequence, properties such as $A \cap {}^\perp A = \{0\}$ would no longer hold. Moreover (and perhaps most importantly), there is a rather well developed theory of direct sum decompositions in mixed lattice semigroups, as presented in [2]. Indeed, our Theorems 6.1 and 6.5 have their counterparts in the theory of mixed lattice semigroups ([2, Theorems 7.1 and 7.2]). (Note however, that the authors in [2] use different terminology, as they use the potential-theoretic notions of *pre-harmonic band* and *potential band*. The corresponding objects in this paper are called specific bands and weak bands, respectively.) Our present definitions of $A^\perp$ and ${}^\perp A$ are in agreement with the existing theory of mixed lattice semigroups. In fact, if A is an ideal such that $V = {}^\perp A \oplus A$, then the set V_{sp} is a mixed lattice semigroup in its own right (with the orderings inherited from V), and $V_{sp} = ({}^\perp A)_{sp} \oplus A_{sp}$ is the corresponding mixed lattice semigroup decomposition of V_{sp}, by Theorem 6.1.

With all these considerations, our definitions of the disjoint complements indeed seem to be the most natural, and also compatible with the theory of mixed lattice semigroups, at least if we restrict ourselves to mixed lattice subspaces, as we have done in this paper. However, the corresponding definitions for more general sets would be more problematic, as pointed out in Remark 5.4.

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Compatible topologies on mixed lattice vector spaces

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ABSTRACT

A mixed lattice vector space is a partially ordered vector space with two partial orderings, generalizing the notion of a Riesz space. The purpose of this paper is to develop the basic topological theory of mixed lattice spaces. A vector topology is said to be compatible with the mixed lattice structure if the mixed lattice operations are continuous. We give a characterization of compatible mixed lattice topologies, similar to the well known Roberts-Namioka characterization of locally solid Riesz spaces. We then study locally convex topologies and the associated seminorms, as well as connections between mixed lattice topologies and locally solid topologies on Riesz spaces. In the locally convex case, we obtain a more complete characterization of compatible mixed lattice topologies. We also briefly discuss asymmetric norms and cone norms on mixed lattice spaces with a particular application to finite dimensional spaces.

1. Introduction

We recall that a real vector space $\mathcal{V}$ together with a partial ordering $\leq$ is called a *partially ordered vector space*, if

$$u \leq v \implies u + w \leq v + w \quad \text{and} \quad u \leq v \implies au \leq av$$

holds for all $u, v, w \in \mathcal{V}$ and $a \in \mathbb{R}_+$. A partially ordered space $\mathcal{V}$ is called a *Riesz space*, or *vector lattice*, if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for all $x, y \in \mathcal{V}$.

A mixed lattice vector space is a partially ordered vector space with two partial orderings. The precise definition will be given in Section 2, but the idea is that the usual notions of supremum and infimum of two elements in a Riesz space are replaced by asymmetric mixed envelopes which are formed with respect to the two partial orderings. It is then required that these mixed upper and lower envelopes exist for every pair of elements. If the two partial orderings are identical, then the mixed upper and lower envelopes become

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the usual supremum and infimum, and the mixed lattice vector space is reduced to a Riesz space. In this sense, the concept of a mixed lattice vector space is a generalization of a Riesz space. However, due to the asymmetric behavior of the mixed envelopes, some familiar properties of Riesz spaces (such as the distributive laws and commutativity of the lattice operations) no longer hold in a mixed lattice vector space. In particular, a mixed lattice space is not necessarily a lattice with respect to either of the partial orderings.

The notion of a mixed lattice semigroup was introduced by Arsove and Leutwiler in connection to their work on axiomatization of potential theory [2,3]. The mixed lattice theory in a group setting was later studied by Eriksson–Bique [7,8]. More recently, in [6] and [11] the theory of mixed lattice vector spaces has been developed in a direction that is more parallel with the theory of Riesz spaces, as presented, for example, in [12].

All the previous research on mixed lattice structures has focused on the algebraic and order structure, and the main purpose of this paper is to develop the topological theory of mixed lattice spaces. The natural starting point is that the topology should be compatible with the mixed lattice structure in the sense that the mixed lattice operations are continuous. This requirement leads to different conditions the topology should satisfy in order to be compatible. A similar topological theory for Riesz spaces and more general ordered vector spaces is well-established (cf. [1,5,16]).

In a general partially ordered vector space the notion of locally full topology provides the most natural setting for topological considerations, and with some modifications, these ideas can be applied also in mixed lattice spaces. We recall here that a subset S of an ordered vector space is called *full* if $x, y \in S$ and $x \leq z \leq y$ imply that $z \in S$. A vector topology is called *locally full* if it has a neighborhood base at zero consisting of full sets. On the other hand, in Riesz spaces the fundamental idea is that there is a base of neighborhoods of zero consisting of solid sets. A subset S of a Riesz space is called *solid* if $y \in S$ and $|x| \leq |y|$ imply that $x \in S$. Solid sets are defined in terms of the absolute value of an element, which does not necessarily exist in a mixed lattice vector space. However, there are ways to generalize the notion of absolute value in mixed lattice spaces, as introduced in [6] and [11]. Such generalization can be used to define an analogous notion of a solid set in mixed lattice spaces, resulting in a topological theory that is similar to the theory of locally solid Riesz spaces.

The abovementioned ideas are discussed in Section 3, where we study adaptations of locally full and locally solid topologies in the mixed lattice setting. In Section 4 it is shown that in the locally convex case the topology is determined by a family of seminorms that have additional properties related to the partial orderings. Our main results in Sections 3 and 4 give fundamental characterizations of mixed lattice topologies, which can be viewed as mixed lattice versions of the well-known Roberts–Namioka theorem in locally solid Riesz spaces. In the general case we obtain a partial result, but in the locally convex case we can give a more complete characterization. We also study the connections between mixed lattice topologies and Riesz space topologies in the case that the mixed lattice space is a lattice with respect to one of the partial orderings. It turns out that a compatible topological structure on a mixed lattice space places some restrictions on the order structure. We show in Section 4 that a finite dimensional normed mixed lattice space is necessarily a lattice with respect to one of the partial orderings.

In Section 5 we briefly discuss asymmetric norms on mixed lattice spaces. Vector spaces with asymmetric norms are a relatively recent area of research (see [4] and the references therein), and in mixed lattice spaces asymmetric norms appear naturally due to the asymmetric nature of the mixed envelopes. We close the paper by presenting an application of our results to asymmetric cone norms on $\mathbb{R}^n$. More specifically, we show that given any closed cone C in $\mathbb{R}^n$ there is an associated mixed lattice structure on $\mathbb{R}^n$ that gives rise to an asymmetric cone norm corresponding to the cone C. These types of applications are of interest in other branches of mathematics, particularly in convex analysis and related topics [15].

2. Mixed lattice vector spaces

We begin by recalling some definitions and terminology related to mixed lattice spaces. A subset K of a vector space $\mathcal{V}$ is called a *cone* if (i) $tK \subseteq K$ for all $t \geq 0$, (ii) $K + K \subseteq K$ and (iii) $K \cap (-K) = \{0\}$. For any cone K in a vector space there is an associated partial ordering defined by $x \leq y$ iff $y - x \in K$. Then K is called the *positive cone* for the ordering $\leq$.

Suppose next that we have two partial orderings $\leq$ and $\preccurlyeq$ on $\mathcal{V}$. Here $\leq$ is called the *initial order* and $\preccurlyeq$ is called the *specific order*. For these two partial orderings $\leq$ and $\preccurlyeq$ we define the *mixed upper and lower envelopes*

$$u \curlyvee v \;=\; \min\{\, w \in \mathcal{V} :\; w \succcurlyeq u \;\text{ and }\; w \geq v \,\} \tag{2.1}$$

and

$$u \curlywedge v \;=\; \max\{\, w \in \mathcal{V} :\; w \preccurlyeq u \;\text{ and }\; w \leq v \,\}, \tag{2.2}$$

respectively, where the minimum and maximum (whenever they exist) are taken with respect to the initial order $\leq$. These definitions were introduced by Arsove and Leutwiler in [2]. We observe that these operations are not commutative, i.e. $x \curlyvee y$ and $y \curlyvee x$ are not equal, in general.

A mixed lattice vector space is defined by requiring that the mixed envelopes exist for every pair of elements.

Definition 2.1. Let $\mathcal{V}$ be a partially ordered real vector space with respect to two partial orderings $\leq$ and $\preccurlyeq$, and let $\mathcal{V}_p = \{x \in \mathcal{V} : x \geq 0\}$ and $\mathcal{V}_{sp} = \{x \in \mathcal{V} : x \succcurlyeq 0\}$ be the corresponding positive cones. Then $(\mathcal{V}, \leq, \preccurlyeq)$ is called a *mixed lattice vector space* if the following conditions hold:

(1) The elements $x \curlywedge y$ and $x \curlyvee y$ exist in $\mathcal{V}$ for all $x, y \in \mathcal{V}$,
(2) $\mathcal{V}_{sp}$ is a *mixed lattice cone*, that is, the elements $x \curlyvee y$ and $x \curlywedge y$ are in $\mathcal{V}_{sp}$ whenever $x, y \in \mathcal{V}_{sp}$.

Remark 2.2. A more general definition of a mixed lattice space does not assume the condition (2) in the above definition. This condition is included here because it provides a sufficiently rich structure for developing an interesting theory for mixed lattice spaces. In fact, many important properties of mixed lattice structures depend on this assumption. For more details on these technicalities, as well as many examples of mixed lattice spaces, we refer to [6] and [11].

Below we have listed several basic properties of the mixed envelopes that hold in every mixed lattice space. For proofs and further discussion on these properties we refer to [8], [6], and [11]. In the following identities and inequalities, x, y, z, u, v are elements of a mixed lattice space $\mathcal{V}$ and $a \in \mathbb{R}$.

$$x \curlywedge y \preccurlyeq x \preccurlyeq x \curlyvee y \;\text{ and }\; x \curlywedge y \leq y \leq x \curlyvee y \tag{2.3}$$

$$x \curlyvee y \,+\, y \curlywedge x \;=\; x + y \tag{2.4}$$

$$z \,+\, x \curlyvee y \;=\; (x+z) \curlyvee (y+z) \;\text{ and }\; z \,+\, x \curlywedge y \;=\; (x+z) \curlywedge (y+z) \tag{2.5}$$

$$x \curlyvee y \;=\; -(-x \curlywedge -y) \tag{2.6}$$

$$x \preccurlyeq y \;\Longrightarrow\; x \leq y \tag{2.7}$$

$$x \preccurlyeq u \;\text{ and }\; y \leq v \;\Longrightarrow\; x \curlyvee y \leq u \curlyvee v \;\text{ and }\; x \curlywedge y \leq u \curlywedge v \tag{2.8}$$

$$x \leq y \;\Longleftrightarrow\; y \curlyvee x = y \;\Longleftrightarrow\; x \curlywedge y = x \tag{2.9}$$

$$x \preccurlyeq y \iff x \curlyvee y = y \iff y \curlywedge x = x \tag{2.10}$$

$$x \preccurlyeq y \implies z \curlyvee x \preccurlyeq z \curlyvee y \text{ and } z \curlywedge y \preccurlyeq z \curlywedge y \tag{2.11}$$

$$u \preccurlyeq x \preccurlyeq z \text{ and } u \preccurlyeq y \preccurlyeq z \implies x \curlyvee y \preccurlyeq z \text{ and } u \preccurlyeq x \curlywedge y \tag{2.12}$$

$$(ax) \curlywedge (ay) = a(x \curlywedge y) \text{ and } (ax) \curlyvee (ay) = a(x \curlyvee y) \quad (a \geq 0) \tag{2.13}$$

$$(ax) \curlywedge (ay) = a(x \curlyvee y) \text{ and } (ax) \curlyvee (ay) = a(x \curlywedge y) \quad (a < 0) \tag{2.14}$$

The following notions of upper and lower parts of an element were introduced in [6]. They generalize the concepts of the positive and negative parts of an element in a Riesz space.

Definition 2.3. Let $\mathcal{V}$ be a mixed lattice vector space and $x \in \mathcal{V}$. The elements ${}^u x = x \curlyvee 0$ and ${}^l x = (-x) \curlyvee 0$ are called the *upper part* and *lower part* of x, respectively. Similarly, the elements $x^u = 0 \curlyvee x$ and $x^l = 0 \curlyvee (-x)$ are called *specific upper part* and *specific lower part* of x, respectively. The elements ${}^u x^l = x \curlyvee (-x)$ and ${}^l x^u = (-x) \curlyvee x$ are called the *(asymmetric) generalized absolute values* of x.

From the above definitions we observe that for the specific upper and lower parts we have $x^u \succcurlyeq 0$ and $x^l \succcurlyeq 0$, and for the upper and lower parts ${}^u x \geq 0$ and ${}^l x \geq 0$.

The following useful result is from [6, Theorem 2.12].

Theorem 2.4 *([6, Theorem 2.12]). Let $(\mathcal{V}, \leq, \preccurlyeq)$ be a partially ordered vector space with two partial orderings. Then $\mathcal{V}$ is a mixed lattice vector space if and only if one of the elements ${}^u x$, x^u, ${}^l x$ or x^l exists for all $x \in \mathcal{V}$.*

The upper and lower parts and the generalized absolute values have several important basic properties, which were proved in [6]. These properties are given in the next theorem.

Theorem 2.5. *Let $\mathcal{V}$ be a mixed lattice vector space and $x \in \mathcal{V}$. Then the following hold.*

(a) ${}^u x = {}^l(-x)$ *and* $x^u = (-x)^l$.
(b) $x = x^u - {}^l x = {}^u x - x^l$.
(c) ${}^u x^l = {}^u x \curlyvee x^l = {}^u x + x^l$ *and* ${}^l x^u = {}^l x \curlyvee x^u = {}^l x + x^u$.
(d) ${}^u x^l = {}^l(-x)^u$.
(e) ${}^u(x+y) \leq {}^u x + {}^u y$, $(x+y)^l \leq x^l + y^l$ *and* ${}^u(x+y)^l \leq {}^u x^l + {}^u y^l$.
(f) $(x+y)^u \leq x^u + y^u$, ${}^l(x+y) \leq {}^l x + {}^l y$ *and* ${}^l(x+y)^u \leq {}^l x^u + {}^l y^u$.
(g) $x^u \curlywedge {}^l x = 0 = x^l \curlywedge {}^u x$.
(h) $x^u \curlyvee {}^l x = {}^u x + {}^l x = x^l + x^u = x^l \curlyvee {}^u x$
(i) $x \succcurlyeq 0$ *if and only if* $x = {}^l x^u = {}^u x^l = {}^u x = x^u$ *and* ${}^l x = x^l = 0$.
(j) $x \geq 0$ *if and only if* $x = {}^u x^l = {}^u x$ *and* $x^l = 0$.
(k) ${}^u x^l \geq 0$ *and* ${}^l x^u \geq 0$. *Moreover,* ${}^u x^l = {}^l x^u = 0$ *if and only if* $x = 0$.
(l) ${}^u(ax)^l = a\, {}^u x^l$ *and* ${}^l(ax)^u = a\, {}^l x^u$ *for all* $a \geq 0$.
(m) ${}^u(ax)^l = |a|\, {}^l x^u$ *and* ${}^l(ax)^u = |a|\, {}^u x^l$ *for all* $a < 0$.

We should note here that in a mixed lattice vector space $\mathcal{V}$ the positive cone $\mathcal{V}_p = \{x \in \mathcal{V} : x \geq 0\}$ is always generating. This is a consequence of Theorem 2.5(b).

It is possible to combine the two asymmetric generalized absolute values to define a symmetric version of the generalized absolute value that retains most of the important properties of the absolute value. This notion was introduced in [11], and it will be essential for describing topologies on mixed lattice vector spaces.

Definition 2.6. Let $\mathcal{V}$ be a mixed lattice vector space and $x \in \mathcal{V}$. The element $s(x) = \frac{1}{2}({}^u x^l + {}^l x^u)$ is called the *symmetric generalized absolute value* of x.

The next theorem lists the most important properties of the symmetric absolute value. They are rather straightforward consequences of Theorem 2.5 (for proofs, see [11, Theorem 5.17]).

Theorem 2.7 *([11, Theorem 5.17]). Let $\mathcal{V}$ be a mixed lattice vector space and $x \in \mathcal{V}$. Then the following hold.*

(a) $s(x) = x^u \curlyvee {}^l x = {}^u x + {}^l x = x^l + x^u = x^l \curlyvee {}^u x$.
(b) $s(\alpha x) = |\alpha| s(x)$ *for all* $\alpha \in \mathbb{R}$.
(c) $s(x) \succcurlyeq 0$ *and* $s(x) \geq 0$. *Moreover,* $s(x) = 0$ *if and only if* $x = 0$.
(d) $x \succcurlyeq 0$ *if and only if* $x = s(x) = x^u$.
(e) $s(s(x)) = s(x)$.
(f) $s(x + y) \leq s(x) + s(y)$.

The following result will also be useful in the next section.

Lemma 2.8. *If $\mathcal{V}$ is a mixed lattice vector space then $s(x^u - y^u) \leq s(x - y)$ for all $x, y \in \mathcal{V}$.*

Proof. Since $x = y + (x - y)$, Theorem 2.5(f) implies that $x^u \leq y^u + (x - y)^u$ and thus $x^u - y^u \leq (x - y)^u$. Since $0 \preccurlyeq 0$, we have from (2.8) and Theorem 2.5(i) that

$$(x^u - y^u)^u = 0 \curlyvee (x^u - y^u) \leq 0 \curlyvee (x - y)^u = ((x - y)^u)^u = (x - y)^u.$$

Exchanging x and y gives similarly $(y^u - x^u)^u \leq (y - x)^u$. But by Theorem 2.5(a) we have $(y^u - x^u)^u = (x^u - y^u)^l$ and $(y - x)^u = (x - y)^l$, and hence $(x^u - y^u)^l \leq (x - y)^l$. Adding the two inequalities gives

$$(x^u - y^u)^u + (x^u - y^u)^l \leq (x - y)^u + (x - y)^l,$$

or equivalently, $s(x^u - y^u) \leq s(x - y)$, by Theorem 2.7(a). $\square$

3. Topological mixed lattice vector spaces

We collect some basic facts and terminology from the theory of topological vector spaces. A more detailed account is given in [5] and [14]. Let S be a subset of a vector space $\mathcal{V}$. Then S is called *symmetric*, if $S = -S$, and S is called *balanced* if $x \in S$ implies $ax \in S$ whenever $|a| \leq 1$. Moreover, S is called *absorbing* if for every $x \in \mathcal{V}$ there exists a number $r > 0$ such that $x \in aS$ whenever $|a| \geq r$, and S is called *convex* if $ax + (1 - a)y \in S$ for all $x, y \in S$ and $a \in [0, 1]$.

If τ is a topology on $\mathcal{V}$ such that vector addition and scalar multiplication are continuous mappings then τ is called a *vector topology* and $(\mathcal{V}, \tau)$ is a *topological vector space*. A collection $\mathcal{B}$ of neighborhoods of zero of $\mathcal{V}$ is called a *neighborhood base at zero* if for every neighborhood U of zero there exists $V \in \mathcal{B}$ such that $V \subseteq U$. Every vector topology is translation invariant, that is, for any $y \in \mathcal{V}$ the map $x \mapsto x + y$ is a homeomorphism. This implies that if $x \in \mathcal{V}$ and $\mathcal{B}$ is a neighborhood base at zero, then $x + \mathcal{B} = \{x + B : b \in \mathcal{B}\}$ is a neighborhood base at x. This means that the topology is completely determined by the neighborhoods of zero, and for this reason we usually only need to work with neighborhoods of zero.

Every topological vector space has a base at zero consisting of balanced absorbing sets. Furthermore, if U is any neighborhood of zero, then there is a balanced neighborhood V of zero such that $V + V \subseteq U$. A vector topology τ is called *locally convex*, if τ has a neighborhood base at zero consisting of convex sets. In this case, for any neighborhood U of zero there exists a convex, balanced and absorbing neighborhood V of zero such that $V \subseteq U$.

A mapping $p : \mathcal{V} \to \mathbb{R}$ is called a *seminorm* if (*i*) $p(x) \geq 0$ for all $x \in \mathcal{V}$, (*ii*) $p(ax) = |a|p(x)$ for all $x \in \mathcal{V}$, $a \in \mathbb{R}$, and (*iii*) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in \mathcal{V}$. A locally convex vector topology is determined by a family of seminorms. We may always assume that the generating family $\mathcal{P}$ of seminorms is *directed*, i.e. for every $p_1, p_2 \in \mathcal{P}$ there exists $p \in \mathcal{P}$ such that $p(x) \geq \max\{p_1(x), p_2(x)\}$ for all $x \in \mathcal{V}$. In this case, a base of neighborhoods of zero is given by the sets $V(p, \varepsilon) = \{x : p(x) < \varepsilon\}$ where $p \in \mathcal{P}$ and $\varepsilon > 0$. For an absorbing subset U the mapping defined as $p_U(x) = \inf\{t > 0 : x \in tU\}$ for each $x \in \mathcal{V}$ is called the *Minkowski functional* of U. If U is an absorbing, convex and balanced set then p_U is a seminorm.

Let $\mathcal{V}$ be a topological vector space. A map $f : \mathcal{V} \to \mathcal{V}$ is called *uniformly continuous*, if for every neighborhood V of zero there exists a neighborhood U of zero such that $x - y \in U$ implies $f(x) - f(y) \in V$. For example, the inversion map $x \mapsto -x$, addition and translation are all uniformly continuous. Uniform continuity implies continuity, and the composition of uniformly continuous maps is uniformly continuous.

If $(\mathcal{V}, \leq)$ is an ordered vector space then a subset U is called *full* if $x, y \in U$ and $x \leq z \leq y$ imply that $z \in U$. An ordered vector space equipped with a vector topology τ is called an *ordered topological vector space* if τ is a *locally full* topology, i.e. it has a neighborhood base at zero consisting of full sets. A vector topology is locally full if and only if every neighborhood U of zero contains a neighborhood V of zero such that $y \in V$ and $0 \leq x \leq y$ imply that $x \in V$. For more information on ordered topological vector spaces, see [5] and [16].

We will now turn our attention to topologies on a mixed lattice space. In order to obtain useful results the topology should be compatible with the mixed lattice structure, and this motivates the following definition.

Definition 3.1. Let $\mathcal{V}$ be a mixed lattice vector space equipped with a vector space topology τ. Then τ is called a *mixed lattice topology* and $(\mathcal{V}, \tau)$ is called a *topological mixed lattice space* if the mixed upper and lower envelopes are continuous mappings from $\mathcal{V} \times \mathcal{V}$ to $\mathcal{V}$ (here it is understood that $\mathcal{V} \times \mathcal{V}$ is equipped with the product topology).

In Riesz spaces the compatibility is achieved by requiring that the topology is *locally solid*, i.e. it has a local base consisting of solid sets. In fact, the following well-known Roberts–Namioka theorem holds in topological Riesz spaces. The theorem is due to G. T. Roberts, who introduced the notion of a locally solid topology in [17], and I. Namioka, who later extended the theorem in [13].

Theorem 3.2 *(Roberts–Namioka). Let $(\mathcal{V}, \tau)$ be a Riesz space equipped with a vector topology τ. The following conditions are equivalent.*

(a) τ is a locally solid topology.
(b) τ is locally full and the lattice operations are continuous at zero.
(c) The lattice operations are uniformly continuous.

As one of the main results of this paper we will show that a similar characterization holds in topological mixed lattice spaces. Let us begin by considering the uniform continuity of the mixed lattice operations.

Proposition 3.3. *Let $(\mathcal{V}, \tau)$ be a mixed lattice vector space equipped with a vector topology τ. Then the following conditions are mutually equivalent.*

(a) The map $(x, y) \mapsto x \curlyvee y$ from $\mathcal{V} \times \mathcal{V}$ to $\mathcal{V}$ is uniformly continuous.
(b) The map $(x, y) \mapsto x \curlywedge y$ from $\mathcal{V} \times \mathcal{V}$ to $\mathcal{V}$ is uniformly continuous.
(c) The map $x \mapsto {}^u x$ from $\mathcal{V}$ to $\mathcal{V}$ is uniformly continuous.

(d) The map $x \mapsto {}^l x$ from $\mathcal{V}$ to $\mathcal{V}$ is uniformly continuous.
(e) The map $x \mapsto {}^l x^u$ from $\mathcal{V}$ to $\mathcal{V}$ is uniformly continuous.
(f) The map $x \mapsto {}^u x^l$ from $\mathcal{V}$ to $\mathcal{V}$ is uniformly continuous.
(g) The map $x \mapsto x^l$ from $\mathcal{V}$ to $\mathcal{V}$ is uniformly continuous.
(h) The map $x \mapsto x^u$ from $\mathcal{V}$ to $\mathcal{V}$ is uniformly continuous.

Proof. *(a)* $\Longrightarrow$ *(b)* This follows from the identity $x \curlywedge y = -(-x \curlyvee -y)$ and uniform continuity of the map $x \mapsto -x$.
(b) $\Longrightarrow$ *(c)* Follows from the identity ${}^u x = -((-x) \curlywedge 0)$.
(c) $\Longrightarrow$ *(d)* Follows from the identity ${}^l x = {}^u(-x)$.
(d) $\Longrightarrow$ *(e)* This follows since ${}^l x^u = 2\,{}^l x + x$ and addition and scalar multiplication are uniformly continuous.
(e) $\Longrightarrow$ *(f)* Follows from the identity ${}^u x^l = {}^l(-x)^u$.
(f) $\Longrightarrow$ *(g)* Follows from the identity $2x^l = {}^u x^l - x$.
(g) $\Longrightarrow$ *(h)* Follows from the identity $x^u = (-x)^l$.
(h) $\Longrightarrow$ *(a)* If $x \mapsto x^u$ is uniformly continuous then, since addition is uniformly continuous, it follows that

$$x \curlyvee y = x + 0 \curlyvee (y - x) = x + (y - x)^u$$

is also uniformly continuous as a composition of uniformly continuous mappings. $\square$

Remark 3.4. In any topological mixed lattice space the mapping $x \mapsto s(x)$ is also continuous. This follows from the identities in Theorem 2.7(a).

We will next define different types of sets that will be used in the study of mixed lattice topologies.

Definition 3.5. Let A be a subset of $\mathcal{V}$ and define the sets

$$MF_1(A) = \{y \in \mathcal{V} : x \preccurlyeq y \leq z \text{ for some } x, z \in A\}$$

and

$$MF_2(A) = \{y \in \mathcal{V} : x \leq y \preccurlyeq z \text{ for some } x, z \in A\}.$$

The set $MF_1(A)$ is called the *type 1 mixed-full hull* of A, and $MF_2(A)$ is called the *type 2 mixed-full hull* of A. If $A = MF_1(A)$ then A is called a *type 1 mixed-full set*, and if $A = MF_2(A)$ then A is called a *type 2 mixed-full set*.

The mixed-full hulls of a subset A are the smallest mixed-full sets containing A. Alternatively, these sets are given by $MF_1(A) = (A + \mathcal{V}_{sp}) \cap (A - \mathcal{V}_p)$ and $MF_2(A) = (A + \mathcal{V}_p) \cap (A - \mathcal{V}_{sp})$.

Next we show that with regards to neighborhood bases at zero the two different notions of mixed-full sets are equivalent, and so for all practical purposes, we only need to consider one type of mixed-full neighborhoods.

Proposition 3.6. *Let $(\mathcal{V}, \tau)$ be a mixed lattice vector space equipped with a vector topology τ. The following conditions are equivalent.*

(a) Each neighborhood of zero contains a type 2 mixed-full neighborhood of zero.
(b) Each neighborhood of zero contains a neighborhood V of zero with the following property: if $y \in V$ and $0 \leq x \preccurlyeq y$ then $x \in V$.

(c) *Each neighborhood of zero contains a type 1 mixed-full neighborhood of zero.*
(d) *Each neighborhood of zero contains a neighborhood V of zero with the following property: if $y \in V$ and $0 \preccurlyeq x \leq y$ then $x \in V$.*

Proof. The implication $(a) \implies (b)$ is obvious. Suppose that (b) holds. Let U be a neighborhood of zero and let W_1 be a neighborhood of zero such that $W_1 + W_1 \subseteq U$. Let $W_2 \subseteq W_1$ be a neighborhood of zero with the property given in (b). Choose a balanced neighborhood V of zero such that $V + V \subseteq W_2$. If $x \in MF_1(V)$ then there exist $y, z \in V$ such that $y \preccurlyeq x \leq z$. Then $0 \leq z - x \preccurlyeq z - y$ where $z - y \in W_2$, and it follows that $z - x \in W_2 \subseteq W_1$. Consequently, since $z \in V \subseteq W_1$, we have $x = z - (z - x) \in W_1 + W_1 \subseteq U$. Hence, $MF_1(V) \subseteq U$ and so (c) holds. The implication $(c) \implies (d)$ is obvious and the implication $(d) \implies (a)$ is proved in a similar way as $(b) \implies (c)$, and so the proof is complete. □

The above results motivate the following definition.

Definition 3.7. Let $\mathcal{V}$ be a mixed lattice vector space with a vector topology τ. Then τ is called *locally mixed-full* if τ satisfies the equivalent conditions of Proposition 3.6.

The equivalence of the conditions given in Proposition 3.6 also motivates the following convention for simplifying the terminology. We shall say that a neighborhood V of zero is *mixed-full* if $x \in V$ whenever $0 \preccurlyeq x \leq y$ or $0 \leq x \preccurlyeq y$ holds with $y \in V$.

The following basic result provides more information about mixed-full sets and locally mixed-full topologies.

Proposition 3.8. *Let $(\mathcal{V}, \tau)$ be a topological mixed lattice space.*

(a) *If A is a balanced and absorbing set then $MF_1(A) = -MF_2(A)$ and the set $B = MF_1(A) + MF_2(A)$ is balanced, absorbing and mixed-full (i.e. it has the properties (b) and (d) of Proposition 3.6). If, in addition, A is convex then $MF_1(A)$, $MF_2(A)$ and B are convex too.*
(b) *If τ is a locally mixed-full topology then every neighborhood of zero contains a balanced absorbing neighborhood of zero with the properties (b) and (d) of Proposition 3.6. If τ is also locally convex then every neighborhood of zero contains a convex, balanced and absorbing neighborhood of zero with the aforementioned properties.*

Proof. (a) First we note that if A is absorbing, then so are $MF_1(A)$ and $MF_2(A)$ since A is contained in these sets. Let A be a balanced set. If $x \in MF_1(A)$ then $y \preccurlyeq x \leq z$ for some $y, z \in A$. This implies that $-z \leq -x \preccurlyeq -y$ where $-y, -z \in A$ since A is symmetric. Hence $-x \in MF_2(A)$. The reverse inclusion is similar, so $MF_1(A) = -MF_2(A)$. Consequently, $B = MF_1(A) + MF_2(A)$ is a symmetric set. Let $x \in B$ and $0 \leq t \leq 1$. Then $x = x_1 + x_2$ where $x_1 \in MF_1(A)$ and $x_2 \in MF_2(A)$. If $x_1 \in MF_1(A)$ then $y \preccurlyeq x_1 \leq z$ for some $y, z \in A$. This implies that $ty \preccurlyeq tx_1 \leq tz$, and since A is balanced, we have $ty, tz \in A$, and the last inequality shows that $tx_1 \in MF_1(A)$. A similar argument shows that $tx_2 \in MF_2(A)$, and so $tx \in B$. This shows that B is balanced. In addition, if A is convex then $MF_1(A) = (A + \mathcal{V}_{sp}) \cap (A - \mathcal{V}_p)$ and $MF_2(A) = (A + \mathcal{V}_p) \cap (A - \mathcal{V}_{sp})$ are also convex, as the sum and intersection of convex sets are again convex. It follows that B is also convex. It remains to show that B has the stated properties. For this, let $0 \preccurlyeq x \leq y$ where $y \in B$. Then $y = y_1 + y_2$ where $y_1 \in MF_1(A)$ and $y_2 \in MF_2(A)$. From the above inequality we get $-y_2 \preccurlyeq x - y_2 \leq y_1$, where $y_1 \in MF_1(A)$ and $-y_2 \in -MF_2(A) = MF_1(A)$. Hence, $x - y_2 \in MF_1(A)$ and so $x = (x - y_2) + y_2 \in MF_1(A) + MF_2(A) = B$. A similar argument shows that B has the property (d) of Proposition 3.6.

(b) Let τ be a locally mixed-full topology. Let U be a neighborhood of zero and choose a balanced neighborhood W_0 of zero and a type 1 mixed-full neighborhood W_1 of zero such that $W_0 + W_0 \subseteq U$

and $W_1 \subseteq W_0$. Next, let V be a balanced absorbing neighborhood of zero such that $V \subseteq W_1$. Then $MS_1(V) \subseteq W_1 \subseteq W_0$, and since W_0 is balanced, we also have $MF_2(V) = -MF_1(V) \subseteq W_0$. It follows that $B = MF_1(V) + MF_2(V) \subseteq W_0 + W_0 \subseteq U$, where B is balanced and absorbing, and it has the stated properties, by part (a). If τ is also locally convex then the set V can be chosen to be convex and balanced. The desired result then follows by part (a). □

For the characterization of mixed lattice topologies we need a few more definitions.

Definition 3.9. Let $\mathcal{V}$ be a mixed lattice vector space.

(i) A subset $A \subseteq \mathcal{V}$ is called *symmetric-solid* if $x \in A$ and $s(y) \leq s(x)$ together imply that $y \in A$. The set $SH(A) = \{y \in \mathcal{V} : s(y) \leq s(x) \text{ for some } x \in A\}$ is called the *symmetric-solid hull* of A.
(ii) A subset $S \subseteq \mathcal{V}$ is called *($\leq$)-full* if $x, y \in S$ and $y \leq z \leq x$ together imply that $z \in S$. Similarly, a subset $S \subseteq \mathcal{V}$ is called *($\preccurlyeq$)-full* if $x, y \in S$ and $y \preccurlyeq z \preccurlyeq x$ together imply that $z \in S$.

Remark 3.10. We observe that $SH(A)$ is the smallest symmetric-solid set containing A (with respect to set inclusion). Moreover, it follows easily from the properties given in Theorem 2.7 that every symmetric-solid set is symmetric, and if A is balanced and absorbing, then so is $SH(A)$.

Definition 3.11. A vector topology τ on a mixed lattice vector space is called *locally symmetric-solid* if τ has a base at zero consisting of symmetric-solid sets. A vector topology τ is called *locally ($\leq$)-full* if τ has a base at zero consisting of ($\leq$)-full sets. Similarly, τ is called *locally ($\preccurlyeq$)-full* if τ has a base at zero consisting of ($\preccurlyeq$)-full sets.

The next theorem is the first version of our main result on mixed lattice topologies.

Theorem 3.12. *Let $(\mathcal{V}, \tau)$ be a mixed lattice vector space equipped with a vector topology τ. Consider the following statements:*

(a) τ is locally symmetric-solid.
(b) τ is locally ($\leq$)-full and the mixed lattice operations are continuous at zero.
(c) The mixed lattice operations are uniformly continuous.
(d) τ is locally mixed-full and the mixed lattice operations are continuous at zero.

Then the statements (a) and (b) are equivalent, and they both imply (c), and (c) implies (d).

Proof. *(a) $\Longrightarrow$ (b)* We will first show that the condition *(a)* implies uniform continuity of the map $x \mapsto x^u$. Let U be a neighborhood of zero. Then there exists a symmetric-solid neighborhood V of zero such that $V \subseteq U$. If $x - y \in V$, then $s(x - y) \in V$ and by Lemma 2.8 we have $s(x^u - y^u) \leq s(x - y)$ and since V is symmetric-solid we have $x^u - y^u \in V \subseteq U$. This shows that the map $x \mapsto x^u$ is uniformly continuous, and so by Proposition 3.3 the mixed lattice operations are uniformly continuous. In particular, they are continuous at zero.

To show that τ is locally ($\leq$)-full we only need to show that every symmetric-solid set is ($\leq$)-full. For this, let V be a symmetric-solid set with $x \in V$ and $0 \leq y \leq x$. Then $y^l = x^l = 0$ and $y^u = 0 \curlyvee y \leq 0 \curlyvee x = x^u$, by (2.8). Thus,

$$s(y) = y^u + y^l = y^u \leq x^u = x^u + x^l = s(x).$$

Since V is symmetric-solid, we deduce that $y \in V$. Hence, V is ($\leq$)-full.

(b) $\Longrightarrow$ *(a)* Let U be a neighborhood of zero and choose a $(\leq)$-full and balanced neighborhood V of zero such that $V \subseteq U$. Choose another neighborhood W of zero such that $W + W \subseteq V$. Since the map $x \mapsto {}^u x$ is continuous we can choose a balanced neighborhood V_1 of zero such that $x \in V_1$ implies ${}^u x \in W$. Since V_1 is balanced we have $-x \in V_1$ and so ${}^u(-x) = {}^l x \in W$. Then $s(x) = {}^u x + {}^l x \in W + W \subseteq V$. Now, if $s(y) \leq s(x)$ then we have the following inequalities

$$-s(x) \leq -s(y) \leq -y^l \leq y \leq {}^u y \leq s(y) \leq s(x).$$

Since V is balanced we have $\pm s(x) \in V$, and so $y \in V$ since V is $(\leq)$-full. Hence U contains the symmetric-solid hull of V_1 and so *(a)* holds.

The implication *(a)* $\Longrightarrow$ *(c)* was already proved. To finish the proof, we will show that uniform continuity of the mixed lattice operations implies the statement (d). If the mixed lattice operations are uniformly continuous, then they are certainly continuous at zero. Let U be a neighborhood of zero and choose a balanced neighborhood W of zero such that $W + W \subseteq U$. Since the maps $x \mapsto {}^l x$ and $x \mapsto x^u$ are uniformly continuous we can choose a neighborhood V of zero such that $z - y \in V$ implies $z^u - y^u \in W$ and ${}^l z - {}^l y \in W$. Assume that $u \preccurlyeq x \leq v$ with $u, v \in V$. We write $v = x - (x - v)$ and $u = x - (x - u)$ and note that $x - v \leq 0$ and $x - u \succcurlyeq 0$ imply that $(x - v)^u = 0$ and ${}^l(x - u) = 0$. It follows that $x^u = x^u - (x - v)^u \in W$ and ${}^l x = {}^l x - {}^l(x - u) \in W$. Hence $x = x^u - {}^l x \in W + W \subseteq U$. This shows that $MF_1(V)$ is contained in U. Hence, the condition (d) holds and the proof is complete. $\square$

Since every mixed-full set is $(\preccurlyeq)$-full, the preceding theorem shows in particular that locally symmetric-solid mixed lattice spaces are always ordered topological vector spaces with respect to both partial orderings, and therefore all the well-known results for ordered topological vector spaces hold in locally symmetric-solid mixed lattice spaces. For more on these results, see [5] and [16].

We do not know if all the conditions in Theorem 3.12 are equivalent in general. However, under certain stronger assumptions we obtain a more complete characterization, which will be given later (cf. Theorem 4.7).

The notions of Riesz subspaces and ideals are in a central role in the theory of Riesz spaces. In the theory of mixed lattice spaces, the two partial orderings give rise to different types of ideals (see [6,11]). A subspace $\mathcal{S}$ of a mixed lattice space $\mathcal{V}$ is a *mixed lattice subspace* of $\mathcal{V}$ if the elements $x \curlyvee y$ and $x \curlywedge y$ are in $\mathcal{S}$ whenever $x, y \in \mathcal{S}$. A mixed-full mixed lattice subspace of $\mathcal{V}$ is called a *quasi-ideal* of $\mathcal{V}$, and a $(\preccurlyeq)$-full mixed lattice subspace of $\mathcal{V}$ is called a *specific ideal* of $\mathcal{V}$. If $y, z \in \mathcal{V}$ then the sets $\{x \in \mathcal{V} : z \preccurlyeq x \leq y\}$ and $\{x \in \mathcal{V} : z \leq x \preccurlyeq y\}$ are called *mixed-order intervals.*

We also recall that an ordered vector space $\mathcal{V}$ is called $(\leq)$-*Archimedean* if $nx \leq y$ for all $n \in \mathbb{N}$ implies that $x \leq 0$. We shall also say that a sequence $\{x_n\}$ is $(\leq)$-*increasing* if $x_n \leq x_m$ whenever $n \leq m$. A $(\preccurlyeq)$-*increasing* sequence is defined similarly. The following theorem gives some additional properties of topological mixed lattice spaces. Similar results hold in topological Riesz spaces (see [1, Theorems 2.19 and 2.21]).

Theorem 3.13. *Let $(\mathcal{V}, \tau)$ be a topological mixed lattice vector space. Then the following statements hold.*

(a) The positive cones $\mathcal{V}_p = \{x \in \mathcal{V} : x \geq 0\}$ and $\mathcal{V}_{sp} = \{x \in \mathcal{V} : x \succcurlyeq 0\}$ are closed if and only if τ is a Hausdorff topology.
(b) If τ is Hausdorff then $\mathcal{V}$ is $(\leq)$-Archimedean.
(c) If τ is Hausdorff and $\{x_n\}$ is a $(\leq)$-increasing (or $(\preccurlyeq)$-increasing) sequence such that $x_n \xrightarrow{\tau} x$ then $\sup\{x_n\} = x$ (or $\mathrm{sp}\sup\{x_n\} = x$, respectively).
(d) If τ is locally mixed-full and A is a bounded set then $MF_1(A)$ and $MF_2(A)$ are bounded.
(e) If τ is locally mixed-full then every mixed-order interval is bounded. In particular, every $(\preccurlyeq)$-order interval is bounded.

(f) If $\mathcal{S}$ is a mixed lattice subspace of $\mathcal{V}$ then the closure $\overline{\mathcal{S}}$ is also a mixed lattice subspace.
(g) The closure of a quasi-ideal is a quasi-ideal, and the closure of a specific ideal is a specific ideal.

Proof. (a) If τ is Hausdorff then the set $\{0\}$ is closed, and so the cones $\mathcal{V}_p = \{x \in \mathcal{V} : x^l = 0\}$ and $\mathcal{V}_{sp} = \{x \in \mathcal{V} : {}^l x = 0\}$ are inverse images of a closed set under continuous mappings $x \mapsto x^l$ and $x \mapsto {}^l x$, respectively, and are therefore closed. Conversely, if $\mathcal{V}_p$ and $\mathcal{V}_{sp}$ are closed, then $\mathcal{V}_p \cap (-\mathcal{V}_{sp}) = \{0\}$ is closed and this implies that τ is Hausdorff.

(b) If $nx \leq y$ for all $n \in \mathbb{N}$ then $n^{-1}y - x \geq 0$ for all $n \in \mathbb{N}$, and so $-x = \lim_{n\to\infty} n^{-1}y - x \geq 0$ since $\mathcal{V}_p$ is closed by (a). Hence $x \leq 0$ and $\mathcal{V}$ is $(\leq)$-Archimedean.

(c) Let $\{x_n\}$ be a $(\leq)$-increasing sequence such that $x_n \xrightarrow{\tau} x$. Then $x_n \leq x_{n+m}$ for all $m \in \mathbb{N}$, so $0 \leq x_{n+m} - x_n$ and since by (a) $\mathcal{V}_p$ is closed, we have $\lim_{m\to\infty}(x_{n+m} - x_n) = x - x_n \geq 0$. Hence $x_n \leq x$ for all n. Suppose that y is another upper bound of $\{x_n\}$. Then $y - x_n \geq 0$ for all n, and $\lim_{n\to\infty}(y - x_n) = y - x \geq 0$, or $x \leq y$. This shows that $x = \sup\{x_n\}$. The case of a $(\preccurlyeq)$-increasing sequence is treated similarly.

(d) Let A be a bounded set. If U is any neighborhood of zero then there is an absorbing mixed-full neighborhood V of zero such that $V \subseteq U$. Since A is bounded, there is some $\alpha \geq 0$ such that $A \subseteq \alpha V$. Then, since αV is mixed-full, we have $MF_1(A) \subseteq \alpha V \subseteq \alpha U$. This shows that $MF_1(A)$ is bounded. Similarly, $MF_2(A)$ is bounded.

(e) Let $S = \{z : x \preccurlyeq z \leq y\}$ and let V be a mixed-full and absorbing neighborhood of zero. Then there exist $\alpha > 0$ such that $x \in \alpha V$ and $y \in \alpha V$. Then $S \subseteq \alpha V$, and so S is bounded. A $(\preccurlyeq)$-order interval $\{z : x \preccurlyeq z \preccurlyeq y\}$ is contained in S, and is therefore bounded.

(f) Let $\mathcal{S}$ be a mixed lattice subspace of $\mathcal{V}$. Evidently, $\overline{\mathcal{S}}$ is a subspace. If $x \in \overline{\mathcal{S}}$ then there exists a net $\{x_\alpha\} \subset \mathcal{S}$ such that $x_\alpha \xrightarrow{\tau} x$. Since $\mathcal{S}$ is a mixed lattice subspace, we have $(x_\alpha)^u \in \mathcal{S}$ for all α, and $(x_\alpha)^u \xrightarrow{\tau} x^u$ by the continuity of the map $x \mapsto x^u$. Hence $x^u \in \overline{\mathcal{S}}$ and so $\overline{\mathcal{S}}$ is a mixed lattice subspace.

(g) Let $\mathcal{A}$ be a quasi-ideal. If $x \in \overline{\mathcal{A}}$ then there exists a net $\{x_\alpha\} \subset \mathcal{A}$ such that $x_\alpha \xrightarrow{\tau} x$. If we define another net $\{v_\alpha\}$ by $v_\alpha = x - x_\alpha$, then $v_\alpha \xrightarrow{\tau} 0$. Now let $0 \preccurlyeq y \leq x$. Since $x - v_\alpha \in \mathcal{A}$ it follows that $(x - v_\alpha)^u \in \mathcal{A}$, and so $y - v_\alpha \leq x - v_\alpha$ for all α. By (2.8) this implies that $0 \preccurlyeq (y - v_\alpha)^u \leq (x - v_\alpha)^u$, and consequently, $(y - v_\alpha)^u \in \mathcal{A}$ for all α. But $0 \preccurlyeq y$, and by the continuity of the map $x \mapsto x^u$ we have $(y - v_\alpha)^u \xrightarrow{\tau} y^u = y$. Hence, $y \in \overline{\mathcal{A}}$ proving that $\overline{\mathcal{A}}$ is mixed-full. It now follows from part (f) that $\mathcal{A}$ is a quasi-ideal. The statement concerning specific ideals is proved in a similar way (just replace the assumption $0 \preccurlyeq y \leq x$ by $0 \preccurlyeq y \preccurlyeq x$ and apply (2.11) in place of (2.8)). □

In the next section, we discuss the special case where the $(\preccurlyeq)$-positive cone $\mathcal{V}_{sp}$ is generating. Under this assumption we can obtain stronger results than in the general case. For this, we need to introduce some additional concepts. Let A be a subset of $\mathcal{V}$ and define the sets

$$MS_1(A) = \{y \in \mathcal{V} : -s(x) \preccurlyeq y \leq s(x) \text{ for some } x \in A\}$$

and

$$MS_2(A) = \{y \in \mathcal{V} : -s(x) \leq y \preccurlyeq s(x) \text{ for some } x \in A\}.$$

These sets are useful technical devices in the study of topological properties of mixed lattice spaces. We should point out that the set A is not usually contained in $MS_1(A)$ or $MS_2(A)$. However, it is clear that if A is any non-empty set then $0 \in MS_1(A)$ and $0 \in MS_2(A)$. Other basic properties of these sets are given in the next proposition. In particular, it will be shown why these concepts are mainly useful in the case that the cone $\mathcal{V}_{sp}$ is generating.

Proposition 3.14. *Let A be a non-empty subset of a mixed lattice space $\mathcal{V}$.*

(a) If $x \in A$ then $\pm s(x) \in MS_1(A)$ and $\pm s(x) \in MS_2(A)$. Moreover, $MS_1(A) = -MS_2(A)$.
(b) If A is an absorbing set then $MS_1(A)$ and $MS_2(A)$ are absorbing if and only if the cone $\mathcal{V}_{sp}$ is generating.
(c) If $0 \leq t \leq 1$ and $x \in MS_1(A)$ then $tx \in MS_1(A)$, and similarly, if $x \in MS_2(A)$ then $tx \in MS_2(A)$. Consequently, the set $MS_1(A) \cup MS_2(A)$ is balanced.

Proof. (a) If $x \in A$ then, in particular, $-s(x) \preccurlyeq \pm s(x) \leq s(x)$ holds. This implies that $\pm s(x) \in MS_1(A)$. Similarly, $\pm s(x) \in MS_2(A)$. If $x \in MS_1(A)$ then $-s(y) \preccurlyeq x \leq s(y)$ holds for some $y \in A$. This is equivalent to $-s(y) \leq -x \preccurlyeq s(y)$ for some $y \in A$, and so $-x \in MS_2(A)$. Hence $MS_1(A) = -MS_2(A)$.

(b) If $x \in \mathcal{V}$ and the cone $\mathcal{V}_{sp}$ is generating then $x = u - v$ for some $u, v \in \mathcal{V}_{sp}$. If A is an absorbing set, there exists some $r > 0$ such that $t(u + v) \in A$ whenever $|t| \leq r$. Since $u + v \succcurlyeq 0$, we have $s(u + v) = u + v$ and $-|t|(u + v) \preccurlyeq tx \preccurlyeq |t|(u + v)$. Thus $-s(t(u + v)) \preccurlyeq tx \preccurlyeq s(t(u + v))$, and by (2.7) this implies that $tx \in MS_1(A)$ and $tx \in MS_2(A)$. Hence, $MS_1(A)$ and $MS_2(A)$ are absorbing.

For the converse, let A be an absorbing set and assume that the cone $\mathcal{V}_{sp}$ is not generating. Then the subspace $W = \mathcal{V}_{sp} - \mathcal{V}_{sp}$ is a proper subspace, and it has been shown in [11, Theorem 4.12] that W is a quasi-ideal in $\mathcal{V}$. For any $x \in \mathcal{V}$ we have $s(x) \succcurlyeq 0$, and so $\pm s(x) \in W$ for all $x \in \mathcal{V}$. Since W is a quasi-ideal, it follows from $-s(x) \preccurlyeq y \leq s(x)$ that $y \in W$. In particular, $MS_1(A) \subseteq W$ and $MS_2(A) = -MS_1(A) \subseteq W$. Since W is a proper subspace, there exists an $x \in \mathcal{V}$ such that $x \notin W$. This implies that $tx \notin W$ for all $t > 0$. In particular, $tx \notin MS_1(A)$ and $tx \notin MS_2(A)$ for all $t > 0$, and hence $MS_1(A)$ and $MS_2(A)$ are not absorbing.

(c) Assume first that $0 \leq t \leq 1$ and $x \in MS_1(A)$ for some set A. Then there exist some $y \in A$ such that $-s(y) \preccurlyeq x \leq s(y)$. Since $s(y) \succcurlyeq 0$, this implies that $-s(y) \preccurlyeq -ts(y) \preccurlyeq tx \leq ts(y) \leq s(y)$, and consequently, $tx \in MS_1(A)$. The case of $MS_2(A)$ is proved similarly.

Now if $B = MS_1(A) \cup MS_2(A)$ and $x \in B$ then $x \in MS_1(A)$ or $x \in MS_2(A) = -MS_1(A)$. If $x \in MS_1(A)$ and $0 \leq t \leq 1$ then $tx \in MS_1(A)$ by what was proved above. If $-1 \leq t \leq 0$ then $-tx \in MS_1(A)$, so $tx \in MS_2(A)$. In either case, $tx \in B$. The case $x \in MS_2(A)$ is similar. Therefore, $tx \in B$ whenever $x \in B$ and $|t| \leq 1$, and this shows that B is balanced. $\square$

The next result provides additional conditions for our main characterization of mixed lattice topologies. As with the two types of mixed-full sets discussed earlier (see Proposition 3.6), we usually only need to work with one of the sets $MS_1(A)$ or $MS_2(A)$.

Theorem 3.15. *Let $(\mathcal{V}, \tau)$ be a mixed lattice space with a generating cone $\mathcal{V}_{sp}$ and a vector topology τ. Then the following conditions are equivalent.*

(a) τ is locally mixed-full and the mixed lattice operations are continuous at zero.
(b) For every neighborhood U of zero there exists a neighborhood V of zero such that $V \subseteq U$ and $MS_1(V) \subseteq U$.
(c) For every neighborhood U of zero there exists a neighborhood V of zero such that $V \subseteq U$ and $MS_2(V) \subseteq U$.

Proof. *(a) $\Longrightarrow$ (b)* Let U be a neighborhood of zero and let W be a balanced mixed-full neighborhood of zero such that $W + W \subseteq U$. By the continuity of $x \mapsto {}^l x$ and $x \mapsto s(x)$ at zero (see Remark 3.4) we can find a neighborhood V of zero such that $x \in V$ implies ${}^l x \in W$ and $\pm s(x) \in W$. Now if $-s(x) \preccurlyeq y \leq s(x)$ then $y \in W$ since W is mixed-full. Hence, $MS_1(V) \subseteq W \subseteq U$. In particular, $-s(x) \preccurlyeq x^u \leq s(x)$ implies $x^u \in W$, and so $x = x^u - {}^l x \in W + W \subseteq U$. Hence $V \subseteq U$, as required.

(b) $\Longrightarrow$ (c) If U is a neighborhood of zero and W is a balanced neighborhood of zero such that $W \subseteq U$, then we can choose a neighborhood V of zero such that $MS_1(V) \subseteq W$. Since W is balanced we have $-MS_1(V) = MS_2(V) \subseteq W \subseteq U$. The converse implication *(c) $\Longrightarrow$ (b)* is similar.

(b) $\Longrightarrow$ *(a)* Let U be a neighborhood of zero and let V be a neighborhood of zero such that $MS_1(V) \subseteq U$. If $0 \preccurlyeq y \leq x$ with $x \in MS_1(V)$ then $-s(z) \preccurlyeq x \leq s(z)$ for some $z \in V$. Hence $-s(z) \preccurlyeq y \leq s(z)$, so $y \in MS_1(V)$. This shows that $MS_1(V)$ is a mixed-full neighborhood of zero, so τ is locally mixed-full by Proposition 3.6. Now, if $x \in V$ then $-s(x) \preccurlyeq x^u \leq s(x)$, and so $x^u \in MS_1(V) \subseteq U$. This shows that the map $x \mapsto x^u$ is continuous at zero, and hence the equivalence of (a) and (b) is proved. $\square$

4. Locally convex topologies on mixed lattice spaces

In a locally convex space the topology is determined by a family of seminorms, and in the case of a locally convex mixed lattice space the topology is compatible if the seminorms satisfy an additional condition.

Definition 4.1. Let p be a (semi)norm in a mixed lattice vector space $\mathcal{V}$.

(i) If $0 \preccurlyeq x \leq y$ implies $p(x) \leq p(y)$ then p is called a *mixed-monotone (semi)norm.*
(ii) If $s(x) \leq s(y)$ implies $p(x) \leq p(y)$ then p is called a *mixed lattice (semi)norm.*

If the topology of $\mathcal{V}$ is given by a mixed lattice norm then $\mathcal{V}$ is called a *normed mixed lattice space.*

Locally convex mixed-full topologies have the following characterization.

Theorem 4.2. *A locally convex topology τ on a topological mixed lattice vector space is locally mixed-full if and only if τ is generated by a family of mixed-monotone seminorms.*

Proof. If τ is locally convex and mixed-full then by Proposition 3.8 there exists a neighborhood base at zero $\{V_\alpha : \alpha \in I\}$ consisting of convex, absorbing, balanced, mixed-full sets. If V_α is any such neighborhood then the Minkowski functional p_α associated to V_α is a mixed-monotone seminorm. Indeed, let $p_\alpha(x) = \inf\{t > 0 : x \in tV_\alpha\}$ for each $x \in \mathcal{V}$. If $0 \preccurlyeq y \leq x$ then for every $\alpha \in I$ we have $x \in p_\alpha(x)V_\alpha$. Since V_α is mixed-full, so is $p_\alpha(x)V_\alpha$ and it follows that $y \in p_\alpha(x)V_\alpha$. By the definition of p_α, this implies that $p_\alpha(y) \leq p_\alpha(x)$. Hence, $\{p_\alpha : \alpha \in I\}$ is a family of mixed-full seminorms generating τ.

Conversely, let $\{p_\alpha : \alpha \in I\}$ be a family of mixed-full seminorms generating τ. For every finite subset S of I and $\varepsilon > 0$ we put $V(S,\varepsilon) = \{x : p_\alpha(x) < \varepsilon \text{ for all } \alpha \in S\}$. Then $0 \preccurlyeq x \leq y$ and $y \in V(S,\varepsilon)$ implies $x \in V(S,\varepsilon)$, and so the sets $V(S,\varepsilon)$ form a base of mixed-full neighborhoods of zero. $\square$

It follows from Definition 4.1 that the topology generated by a family of mixed lattice seminorms is locally symmetric-solid, and so the mixed lattice operations are continuous, by Theorem 3.12. The proof of the next result is similar to the latter part of the proof of Theorem 4.2.

Proposition 4.3. *If a vector topology τ on a mixed lattice vector space is generated by a family of mixed lattice seminorms then τ is locally symmetric-solid.*

We also have the following relationship between mixed lattice seminorms and mixed-monotone seminorms.

Proposition 4.4. *Let p be a seminorm on $\mathcal{V}$. The following conditions are equivalent.*

(a) p is a mixed lattice seminorm.
(b) p is a mixed-monotone seminorm and $p(s(x)) = p(x)$ for all $x \in \mathcal{V}$.

Proof. If (a) holds then, since $s(x) = s(s(x))$ for all $x \in \mathcal{V}$, it follows that $p(s(x)) = p(x)$. If $0 \preccurlyeq x \leq y$ then $0 \preccurlyeq s(x) = x^u \leq y^u = s(y)$, and by assumption this implies that $p(x) \leq p(y)$. Conversely, if (b) holds and $0 \preccurlyeq s(x) \leq s(y)$ then $p(s(x)) \leq p(s(y))$. By assumption, this is equivalent to $p(x) \leq p(y)$, and so (a) follows. $\square$

As an immediate consequence of the preceding result, we notice that every mixed-monotone seminorm p gives rise to a mixed-lattice seminorm q, defined by $q(x) = p(s(x))$ for all x, which we call a *mixed lattice seminorm associated with* p.

Proposition 4.5. *If p is any mixed-monotone seminorm on $\mathcal{V}$ then $q(x) = p(s(x))$ is a mixed lattice seminorm. Moreover, if p is a mixed-monotone norm, then q is a mixed lattice norm.*

Proof. If $0 \preccurlyeq x \leq y$ then $0 \preccurlyeq s(x) = x^u \leq y^u = s(y)$. Since p is a mixed-monotone seminorm, it follows that $q(x) = p(s(x)) \leq p(s(y)) = q(y)$, so q is also mixed-monotone. Moreover, $q(s(x)) = p(s(s(x))) = p(s(x)) = q(x)$, and so q is a mixed lattice seminorm, by Proposition 4.4. If p is a norm then so is q, because then $q(x) = p(s(x)) = 0$ iff $s(x) = 0$ iff $x = 0$. $\square$

With the concepts and results introduced above we can now complete our characterization of mixed lattice topologies. It was shown in Theorem 3.12 that a locally symmetric-solid topology is locally mixed-full. If $\mathcal{V}$ is locally convex then we have the following converse.

Theorem 4.6. *Let $(\mathcal{V}, \tau)$ be a locally convex topological mixed lattice space such that the cone $\mathcal{V}_{sp}$ is generating. If τ is locally mixed-full then τ is locally symmetric-solid.*

Proof. If τ is a locally convex and locally mixed-full then τ is determined by a directed family $\{p_\alpha : \alpha \in I\}$ of mixed-monotone seminorms, by Theorem 4.2. For each p_α, let q_α be the associated mixed lattice seminorm, that is, $q_\alpha(x) = p_\alpha(s(x))$ for all $x \in \mathcal{V}$, and let us denote the topology determined by the family $\{q_\alpha : \alpha \in I\}$ by $\widetilde{\tau}$. By Proposition 4.3 $\widetilde{\tau}$ is locally symmetric solid. We will show that $\tau = \widetilde{\tau}$.

For every $\varepsilon > 0$ and $\alpha \in I$ the set $W = \{x : p_\alpha(x) < \frac{\varepsilon}{2}\}$ is a mixed-full τ-neighborhood of zero, and $\widetilde{W} = \{x : q_\alpha(x) < \varepsilon\}$ is a symmetric-solid $\widetilde{\tau}$-neighborhood of zero. By Theorem 3.15 there exists a τ-neighborhood $W_0 \subseteq W$ of zero such that $MS_1(W_0) \subseteq W$. Thus, for every $x \in MS_1(W_0)$ there exists $y \in W_0$ such that $-s(y) \preccurlyeq x \leq s(y)$ and $s(y) \in MS_1(W_0)$. This inequality implies that $0 \preccurlyeq x^u \leq s(y)$ and $0 \preccurlyeq x^l \leq s(y)$, and summing these inequalities gives $s(x) \leq 2s(y)$. This implies that $q_\alpha(x) \leq 2q_\alpha(y) = 2p_\alpha(s(y)) < \varepsilon$. Hence $x \in \widetilde{W}$, and so $MS_1(W_0) \subseteq \widetilde{W}$, proving that $\widetilde{\tau} \subseteq \tau$.

For the converse, choose the neighborhoods $V = \{x : p_\alpha(x) < \varepsilon\}$ and $\widetilde{V} = \{x : q_\alpha(x) < \frac{\varepsilon}{3}\}$. Then, in particular, $\widetilde{V}$ is mixed-full since q_α is a mixed-monotone seminorm, by Proposition 4.4. Thus, there exists a $\widetilde{\tau}$-neighborhood $\widetilde{W_0} \subseteq \widetilde{W}$ of zero such that $MS_1(\widetilde{W_0}) \subseteq \widetilde{W}$. Thus, for every $x \in MS_1(\widetilde{W_0})$ there exists $y \in \widetilde{W_0}$ such that $-s(y) \preccurlyeq x \leq s(y)$ and $s(y) \in MS_1(\widetilde{W_0})$. Then $0 \preccurlyeq x + s(y) \leq 2s(y)$, and since p_α is a mixed-monotone seminorm, we get $p_\alpha(x+s(y)) \leq 2p_\alpha(s(y))$. The triangle inequality then gives $p_\alpha(x) - p_\alpha(s(y)) \leq 2p_\alpha(s(y))$, or $p_\alpha(x) \leq 3p_\alpha(s(y)) = 3q_\alpha(x) < \varepsilon$. Therefore, $x \in W$, and so $MS_1(\widetilde{W_0}) \subseteq W$. This shows that $\tau \subseteq \widetilde{\tau}$, and hence $\tau = \widetilde{\tau}$. $\square$

Combining the results of the preceding theorem with Theorem 3.12 gives a complete characterization of mixed lattice topologies in the locally convex case, assuming the cone $\mathcal{V}_{sp}$ is generating.

Corollary 4.7. *Let $\mathcal{V}$ be a mixed lattice space such that the cone $\mathcal{V}_{sp}$ is generating. If τ is a locally convex topology on $\mathcal{V}$ then the following conditions are equivalent.*

(a) The mixed lattice operations are uniformly continuous.
(b) τ is locally symmetric-solid.
(c) τ is locally mixed-full and the mixed lattice operations are continuous at zero.

(d) τ is locally $(\leq)$-full and the mixed lattice operations are continuous at zero.

Next we investigate the relationships between mixed lattice topologies and locally solid Riesz space topologies. If $\mathcal{V}$ is a mixed lattice vector space which is a lattice with respect to the specific order $\preccurlyeq$ then $\mathcal{V}$ can be equipped with a locally solid (with respect to $\preccurlyeq$) Riesz space topology, and such topology will be called a *locally* $(\preccurlyeq)$*-solid* Riesz space topology. Similarly, if $\mathcal{V}$ is a lattice with respect to $\leq$ then we use the term *locally* $(\leq)$*-solid* Riesz space topology. As usual, if $(\mathcal{V}, \leq)$ is a Riesz space then the lattice-theoretic positive part of an element x is denoted by x^+. Similarly, if $(\mathcal{V}, \preccurlyeq)$ is a Riesz space then the positive part of x with respect to $\preccurlyeq$ is denoted by $\mathrm{sp}(x^+)$, and the least upper bound of elements x and y with respect to $\preccurlyeq$ is denoted by $\mathrm{sp}\sup\{x, y\}$.

Theorem 4.8. *Let $(\mathcal{V}, \leq, \preccurlyeq)$ be a mixed lattice vector space which is a lattice with respect to specific order $\preccurlyeq$. If τ is a locally $(\preccurlyeq)$-solid Riesz space topology on $\mathcal{V}$ then the mixed lattice operations are continuous.*

Proof. In [6, Proposition 3.16] it was shown that $\mathrm{sp}(x^+) = \mathrm{sp}\sup\{{}^u x, x^u\}$, and so we have $0 \preccurlyeq x^u \preccurlyeq \mathrm{sp}(x^+)$. Now if U is a neighborhood of zero and τ is locally $(\preccurlyeq)$-solid, we can choose a $(\preccurlyeq)$-solid neighborhood W of zero such that $W \subset U$, and by the continuity of the mapping $x \mapsto \mathrm{sp}(x^+)$ there exists a neighborhood V of zero such that $x \in V$ implies $\mathrm{sp}(x^+) \in W$. Hence, by the above inequality $x \in V$ implies $x^u \in W \subset U$, and hence the map $x \mapsto x^u$ is continuous at zero. By Proposition 3.3, all the mixed lattice operations are then continuous. □

Theorem 4.9. *Let $(\mathcal{V}, \leq, \preccurlyeq)$ be a mixed lattice vector space which is a lattice with respect to initial order $\leq$. If τ is a locally symmetric-solid topology on $\mathcal{V}$ then τ is a locally $(\leq)$-solid Riesz space topology.*

Proof. By Theorem 3.12 τ is locally $(\leq)$-full and the map $x \mapsto {}^u x$ continuous at zero. By the Roberts–Namioka Theorem 3.2 it is sufficient to show that the map $x \mapsto x^+$ is continuous at zero. To this end, we just need to note that ${}^u x \geq x$ and ${}^u x \geq 0$, and hence ${}^u x \geq x^+$. Now if U is a neighborhood of zero, we can choose a $(\leq)$-full neighborhood V of zero such that $V \subset U$, and by the continuity of $x \mapsto {}^u x$ there is a neighborhood W of zero such that $x \in W$ implies ${}^u x \in V$. But $0 \leq x^+ \leq {}^u x$ and V is full, so $x^+ \in V \subset U$ and this completes the proof. □

As for the lattice properties of topological mixed lattice spaces, our results have the consequence that finite-dimensional normed mixed lattice spaces with a generating cone $\mathcal{V}_{sp}$ are necessarily lattices with respect to specific order $\preccurlyeq$. To prove this, we need the following theorem which was proved by Arsove and Leutwiler for mixed lattice semigroups ([3, Theorem 9.7]), and essentially the same proof carries over to mixed lattice vector spaces (cf. [8, Theorem 3.9]).

Theorem 4.10 *([8, Theorem 3.9]). Let $\mathcal{V}$ be a mixed lattice space such that the cone $\mathcal{V}_{sp}$ is generating and* $\mathrm{sp}\sup\{x_n\}$ *exists for every $(\preccurlyeq)$-increasing and $(\preccurlyeq)$-bounded sequence $\{x_n\}$. Then $\mathcal{V}$ is a lattice with respect to $\preccurlyeq$.*

The following theorem is now rather immediate.

Theorem 4.11. *Let $\mathcal{V}$ be a finite-dimensional normed mixed lattice space. If the cone $\mathcal{V}_{sp}$ is generating then $\mathcal{V}$ is a lattice with respect to $\preccurlyeq$.*

Proof. Let $\{x_n\}$ be a $(\preccurlyeq)$-increasing sequence such that $x_n \preccurlyeq u$ for all n and some $u \in \mathcal{V}$. Since $\{x_n\}$ is contained in the $(\preccurlyeq)$-order interval $\{z : x_1 \preccurlyeq z \preccurlyeq u\}$, it follows by Theorem 3.13(e) that $\{x_n\}$ is norm bounded. As $\mathcal{V}$ is finite-dimensional and $\{x_n\}$ is $(\preccurlyeq)$-increasing, it follows from the Bolzano–Weierstrass theorem that

$\{x_n\}$ converges to a limit x, and by Theorem 3.13(c) we have $x = \mathrm{sp}\sup\{x_n\}$. Now Theorem 4.10 implies that $\mathcal{V}$ is a lattice with respect to $\preccurlyeq$. $\square$

Locally solid Riesz spaces are of course a special case of topological mixed lattice spaces. Let us consider the following example to illustrate our results.

Example. Let $BV([a,b])$ be the space of functions of bounded variation on an interval $[a,b]$. We define initial order in $BV([a,b])$ by

$$f \leq g \qquad \Longleftrightarrow \qquad f(x) \leq g(x) \quad \text{for all } x \in [a,b]$$

and specific order by

$$f \preccurlyeq g \iff f(x) \leq g(x) \text{ for all } x \in [a,b] \text{ and } g-f \text{ is increasing on } [a,b].$$

It has been shown in [8] that $\mathcal{V} = (BV([a,b]), \leq, \preccurlyeq)$ is a mixed lattice vector space where the mixed lower and upper envelopes are given by

$$(f \curlywedge g)(u) = \inf\{f(u) - (f(x)-g(x))^+ \,:\, x \in [a,u]\}$$

and

$$(f \curlyvee g)(u) = \sup\{f(u) + (g(x)-f(x))^+ \,:\, x \in [a,u]\},$$

where $c^+ = \max\{0,c\}$ is the positive part of the real number c. It is well-known that $\mathcal{V}$ is a lattice with respect to both partial orderings $\leq$ and $\preccurlyeq$. In particular, the positive cone $\mathcal{V}_{sp}$ is generating.

First we will show that if $\mathcal{V}$ is equipped with the sup-norm $\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\}$ then $\mathcal{V}$ is a topological mixed lattice space. By Proposition 4.3 and Corollary 4.7 it is sufficient to show that $s(f) \leq s(g)$ implies $\|f\|_\infty \leq \|g\|_\infty$. Let $g \in V$. Let us first find $s(g)$ using the above formulae for the mixed envelopes. We have

$$g^u(u) = (0 \curlyvee g)(u) = \sup_{x\in[a,u]} \{(g(x))^+\},$$

and

$$g^l(u) = (0 \curlyvee (-g))(u) = \sup_{x\in[a,u]} \{(-g(x))^+\} = \sup_{x\in[a,u]} \{(g(x))^-\}.$$

Since $s(g) = g^u + g^l$, we obtain

$$s(g)(u) = \sup_{x\in[a,u]} \{(g(x))^+ + (g(x))^-\} = \sup_{x\in[a,u]} \{|g(x)|\}.$$

Hence we have

$$\|g\|_\infty = \sup\{|g(x)| \,:\, x \in [a,b]\} = s(g)(b).$$

This shows that if $s(f) \leq s(g)$ then $\|f\|_\infty = s(f)(b) \leq s(g)(b) = \|g\|_\infty$, and so $\|\cdot\|_\infty$ is a mixed lattice norm on $\mathcal{V}$ which generates a mixed lattice topology (by Proposition 4.3). In this case, this topology is the same as the locally solid Riesz space topology on $(\mathcal{V}, \leq)$ induced by the sup-norm.

On the other hand, if $\mathcal{V}$ is equipped with the total variation norm $\|f\|_{BV} = |f(a)| + V_a^b(f)$, (where $V_a^b(f)$ is the total variation of f on $[a,b]$) then the mixed lattice operations are continuous. This follows from Theorem 4.10, since $\mathcal{V}$ is a vector lattice with respect to $\preccurlyeq$ and $\|\cdot\|_{BV}$ is a ($\preccurlyeq$)-lattice norm. Moreover, if $0 \preccurlyeq f \leq g$ then f is positive and non-decreasing, and $f(x) \leq g(x)$ for all $x \in [a,b]$, so we have $\|f\|_{BV} = f(b) \leq g(b) \leq \|g\|_{BV}$. This shows that the topology generated by the total variation norm is locally mixed-full, and so $(\mathcal{V}, \|\cdot\|_{BV})$ is a locally symmetric-solid mixed lattice space also with respect to the BV-norm topology, by Corollary 4.7. Consequently, $(\mathcal{V}, \|\cdot\|_{BV})$ is also a locally ($\leq$)-solid Riesz space, by Theorem 4.9. Note, however, that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{BV}$ are not equivalent.

5. On asymmetric norms and cone norms

The theory of asymmetric normed spaces is relatively recent. A detailed account of the theory is given in the monograph [4]. Although we shall not explore these ideas in detail in the present paper, the discussion in the preceding sections suggests that topologies determined by asymmetric seminorms arise rather naturally in mixed lattice spaces. For example, we have seen that there are two types of mixed-full sets that behave in an asymmetric fashion, that is, if $A \subseteq \mathcal{V}$ is a balanced set then $MF_1(A) = -MF_2(A)$. On the other hand, if a mixed lattice norm on $\mathcal{V}$ is given, then the properties of the upper and lower parts give rise to asymmetric norms on $\mathcal{V}$, as the following discussion shows. The definitions and notations that follow are adopted from [4].

Let X be a real vector space. A real-valued mapping $p : X \to [0,\infty)$ is called an *asymmetric norm* on X if for all $x, y \in X$ the following conditions hold.

$$\begin{array}{ll} \text{(A1)} & p(x) = 0 \quad \text{and} \quad p(-x) = 0 \quad \text{implies} \quad x = 0 \\ \text{(A2)} & p(\alpha x) = \alpha p(x) \quad \text{for all} \quad \alpha \geq 0 \\ \text{(A3)} & p(x+y) \leq p(x) + p(y) \end{array}$$

If p satisfies conditions (A2) and (A3) then p is called an *asymmetric seminorm*.

Given an asymmetric norm p, its *conjugate asymmetric norm* $\overline{p}$ is defined as $\overline{p}(x) = p(-x)$. If we put $p^s(x) = p(x) + \overline{p}(x)$ then p^s is a norm associated with the asymmetric norm p.

To see how these concepts are related to the results of the preceding sections, we note that if $\mathcal{V}$ is a mixed lattice space and U is a convex, absorbing and balanced neighborhood of zero then by Proposition 3.8 the sets $MF_1(U)$ and $MF_2(U)$ are also convex and absorbing neighborhoods of zero (but not balanced, since $x \in MF_1(U)$ implies $tx \in MF_1(U)$ only for $0 \leq t \leq 1$, and similarly for $MF_2(U)$). Hence, the Minkowski functionals p_1 of the set $MF_1(U)$ and p_2 of the set $MF_2(U)$ are asymmetric seminorms. In fact, they are conjugate asymmetric seminorms since $MF_1(U) = -MF_2(U)$. Furthermore, by Proposition 3.8, the sum $MF_1(U) + MF_2(U)$ is a balanced, convex, absorbing and mixed-full neighborhood of zero, whose Minkowski functional is a mixed-monotone seminorm by Theorem 4.2.

In normed Riesz spaces, asymmetric norms are related to lattice norms in a quite natural way through the positive and negative parts of an element. Similarly, in mixed lattice spaces the upper and lower parts give rise to asymmetric norms. If ρ is any mixed lattice norm on $\mathcal{V}$ then we can define $p_1(x) = \rho(x^u)$ and $p_2(x) = \rho(x^l)$. Then p_1 and p_2 are conjugate asymmetric seminorms on $\mathcal{V}$. It follows immediately from the properties of x^u and x^l (Theorem 2.5) that p_1 and p_2 satisfy the conditions in the definition of an asymmetric norm.

Asymmetric cone norms are further generalizations of asymmetric norms, and they have found applications in different areas of mathematics. In recent years, they have been used for studying generalizations of asymmetric normed spaces and some related results in analysis [10,9]. The basic idea in such generalizations is that the usual asymmetric norm is replaced by a vector-valued mapping that has properties similar to those of an asymmetric norm, but its range is a positive cone of a partially ordered vector space. Asymmetric cone norms are also closely related to certain problems in optimization theory [15].

Definition 5.1. Let $C \subset X$ be a cone in a topological vector space X and let $\leq$ be the associated order relation. A continuous mapping $Q : X \to C$ is an *asymmetric cone norm* if

(1) $Q(x) = x$ for all $x \in C$ and $Q(X) = C$
(2) $Q(tx) = tQ(x)$ for all $t \in \mathbb{R}_+$ and $x \in X$
(3) $Q(x + y) \leq Q(x) + Q(y)$ for all $x, y \in X$
(4) If $Q(x) = 0$ and $Q(-x) = 0$ then $x = 0$.

Moreover, we say that Q is a *proper asymmetric cone norm* if $Q(I - Q) = 0$. Here I denotes the identity operator on X.

In a general topological vector space, the existence of a mapping satisfying the conditions in the above definition is not at all clear. However, the properties of the mixed lattice operations given in Theorem 2.5 yield the following existence result.

Theorem 5.2. *Let $\mathcal{V} = (\mathcal{V}, \leq, \preccurlyeq)$ be a topological mixed lattice space. Then the following hold.*

(a) The mapping $Q : \mathcal{V} \to \mathcal{V}_p$ given by $Q(x) = {}^u x$ is a proper asymmetric cone norm on $\mathcal{V}$.
(b) The mapping $Q : \mathcal{V} \to \mathcal{V}_{sp}$ given by $Q(x) = x^u$ is a proper asymmetric cone norm on $\mathcal{V}$ which is increasing with respect to both partial orderings (that is, $x \preccurlyeq y$ implies $Q(x) \preccurlyeq Q(y)$ and $x \leq y$ implies $Q(x) \leq Q(y)$).

Proof. *(a)* The properties of ${}^u x$ imply that the mapping $Q(x) = {}^u x$ has the properties listed in Definition 5.1. Property (1) follows from Theorem 2.5(g), and (2) follows from (2.13). Property (3) is an immediate consequence of Theorem 2.5(c), while (4) follows from Theorem 2.5(a) and (g). To check that Q is proper, we note that

$$Q(I - Q)(x) = Q(x - {}^u x) = Q(-x^l) = {}^u(-x^l) = {}^l(x^l) = 0,$$

by Theorem 2.5 (since $x^l \succcurlyeq 0$).

The proof of *(b)* is similar. The property that Q is increasing with respect to both partial orderings follows by (2.8) and (2.11). □

Suppose that C is a cone in a topological vector space X, and we ask if there exists an asymmetric cone norm Q on X satisfying the conditions in Definition 5.1. Theorem 5.2 shows that a sufficient condition for the existence of such mapping is that there exists a mixed lattice order structure on X such that C is a positive cone for the partial order $\leq$.

As an application of our results, we show that given any closed convex cone C in $\mathbb{R}^n$ we can always turn $\mathbb{R}^n$ into a topological mixed lattice space $(\mathbb{R}^n, \leq, \preccurlyeq)$ in such way that C will be the positive cone associated with the partial order $\leq$, and so by Theorem 5.2 there always exists a continuous asymmetric cone norm associated with the given cone in $\mathbb{R}^n$. The latter part (that is, the existence of a proper asymmetric cone norm) has been proved recently in [15, Theorem 2] using different methods. This result is of importance in problems related to optimization theory (see [15] and references therein).

Theorem 5.3. *Let C be a generating closed convex pointed cone in $\mathbb{R}^n$ and let $x \in \mathrm{int}(C)$. If $\leq$ is the partial ordering induced by C and $\preccurlyeq$ is another partial ordering induced by the cone $R_x = \{tx : t \geq 0\}$ then $(\mathbb{R}^n, \leq, \preccurlyeq)$ is a normed mixed lattice vector space with respect to the usual norm of $\mathbb{R}^n$. In particular, the mapping $Q : \mathbb{R}^n \to C$ given by $Q(x) = {}^u x$ is a continuous proper asymmetric cone norm.*

In the proof of Theorem 5.3 we use the well known fact that the interior $\operatorname{int}(C)$ of a generating cone C in $\mathbb{R}^n$ is non-empty. We also recall that a subset B of a cone C is called a *base of* C if for every $x \in C \setminus \{0\}$ there exists a unique number $t > 0$ such that $tx \in B$. We will also need the following geometric result, which is a consequence of the existence of a hyperplane supporting C at 0 (cf. [15, Lemma 2]).

Lemma 5.4. *If $C \subset \mathbb{R}^n$ is a closed convex cone such that $\operatorname{int}(C) \neq \emptyset$ then there exists a closed, bounded and convex base B of C and an interior point $x \in \operatorname{int}(C)$ such that the line spanned by x is orthogonal to B.*

Proof of Theorem 5.3. Let $\leq$ be the partial ordering induced by the cone C. By the preceding lemma, there is a closed, bounded and convex base B of C and an interior point x of C such that B is orthogonal to ray R_x generated by x (i.e. $R_x = \{tx : t \geq 0\}$). Clearly, R_x is a cone and it induces a partial ordering $\preccurlyeq$ on $\mathbb{R}^n$ by $u \preccurlyeq v$ if $v - u \in R_x$, or equivalently, $v = u + tx$ for some $t \geq 0$. Evidently, R_x is also a mixed lattice cone, as it is "one dimensional" and both partial orderings coincide on R_x. Now $\mathcal{V} = (\mathbb{R}^n, \leq, \preccurlyeq)$ is a partially ordered vector space with two partial orderings such that $u \preccurlyeq v$ implies $u \leq v$. To show that $\mathcal{V}$ is a mixed lattice space it is sufficient to show that the element

$$^{u}y = y \curlyvee 0 = \min\{\, w \in \mathcal{V} : \ w \succcurlyeq y \ \text{ and } \ w \geq 0\,\}$$

exists for all $y \in \mathcal{V}$ (by Theorem 2.4). We note that in the present situation this is equivalent to showing that

$$\min\left(\{y + tx : t \geq 0\} \cap C\right) = \min\{t \geq 0 : y + tx \in C\}$$

exists for all $y \in \mathcal{V}$. If we denote the boundary of C by ∂C then the distance of x to ∂C is defined as $d(x, \partial C) = \inf\{\|x - y\| : y \in \partial C\}$. Then the set D bounded by B and ∂C and containing 0 is convex, closed and bounded. Moreover, $tx \in B$ for some $t > 0$, so we may thus assume that $x \in D$. Now $d(x, \partial C) > 0$ since x is an interior point of C, and since D is closed, convex and bounded there is a point $v \in D \cap \partial C$ (not unique, in general) such that $d(x, v) = d(x, D \cap \partial C) = d(x, \partial C)$. The last equality follows from the orthogonality of B and R_x. Hence, we have shown that for any $x \in \operatorname{int}(C)$ there is some point $v \in \partial C$ such that $d(x, \partial C) = d(x, v)$. Now for any $t > 0$ we have $d(tx, \partial C) = d(tx, tv) = td(x, v)$. Indeed, if there was some $u \in \partial C$ such that $\|tx - u\| < \|tx - tv\| = t\|x - v\|$, then this would imply that $\|x - t^{-1}u\| < \|x - v\|$, a contradiction. Therefore, if $0 < s < t$ then $d(sx, \partial C) < d(tx, \partial C)$, and so the distance of tx to the boundary of C increases as t increases. But $\{tx : t \geq 0\}$ and $\{y + tx : t \geq 0\}$ are parallel half-lines, so $d(tx, y + tx) = \|y\|$ is a constant. Thus, there is some t_1 such that $d(tx, \partial C) \geq d(tx, y + tx) = \|y\|$ for all $t \geq t_1$, and so $y + tx$ lies inside C for all $t \geq t_1$. Since C is closed, the number $t_0 = \min\{t \geq 0 : y + tx \in C\}$ exists, and so the element $^{u}y = y + t_0 x$ exists for all $y \in \mathbb{R}^n$. This shows that $(\mathbb{R}^n, \leq, \preccurlyeq)$ is a mixed lattice vector space, by Theorem 2.4. Then by Proposition 5.2 the mapping $Q(x) = {}^{u}x$ is a proper asymmetric cone norm.

We still need to prove the continuity of Q. Let $\|\cdot\|_2$ be the usual Euclidean norm on $\mathbb{R}^n$. Define a new norm by $\|z\|_0 = \|s(z)\|_2$ for all $z \in \mathbb{R}^n$. We will show that $\|\cdot\|_0$ is a mixed lattice norm. Assume that $s(x) \leq s(y)$. Then, since $s(x), s(y) \succcurlyeq 0$, it follows that $s(x)$ and $s(y)$ are both contained in the ray R_x. Hence, $s(x) \leq s(y)$ implies that $s(y) = ts(x)$ for some $t \geq 1$, and so $\|s(x)\|_2 \leq t\|s(x)\|_2 = \|s(y)\|_2$, or $\|x\|_0 \leq \|y\|_0$. This shows that $\|\cdot\|_0$ is a mixed lattice norm, and hence the mapping Q is continuous in the topology generated by the norm $\|\cdot\|_0$. Since all norms on $\mathbb{R}^n$ are equivalent, it follows that the mapping Q is continuous in the usual topology of $\mathbb{R}^n$. $\square$

We remark that the proof of Theorem 5.3 is based on rather simple geometric arguments which do not necessarily work in infinite-dimensional spaces. For instance, a generating cone in an infinite dimensional space does not always possess interior points.

Data availability

No data was used for the research described in the article.

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4

Mixed lattice structures and cone projections

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ORIGINAL PAPER

Mixed lattice structures and cone projections

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Abstract
Problems related to projections on closed convex cones are frequently encountered in optimization theory and related fields. To study these problems, various unifying ideas have been introduced, including asymmetric vector-valued norms and certain generalized lattice-like operations. We propose a new perspective on these studies by describing how the problem of cone projection can be formulated using an order-theoretic formalism developed in this paper. The underlying mathematical structure is a partially ordered vector space that generalizes the notion of a vector lattice by using two partial orderings and having certain lattice-type properties with respect to these orderings. In this note we introduce a generalization of these so-called mixed lattice spaces, and we show how such structures arise quite naturally in some of the applications mentioned above.

Keywords Mixed lattice · Cone projection · Isotone retraction cone · Asymmetric cone norm

1 Introduction

The problem of cone projection and the related decomposition method with respect to dual cones is of significant importance in convex analysis, and it has been studied extensively since the pioneering work of Moreau [11]. In this paper we present an order-theoretic setting in which the basic projection problem can be stated and studied. The theoretical framework on which this note is based is called a *mixed lattice space* which is a partially ordered vector space (or more generally, a group) with two partial orderings and certain "mixed" lattice-type properties with respect to these two orders. In a mixed lattice space, the vector lattice operations of supremum and infimum are replaced by asymmetric mixed upper and lower envelopes, which are formed with respect to the two partial orders. A mixed lattice space is a generalization of a vector

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lattice in the sense that if the two partial orders are identical, then the mixed lattice space reduces to a vector lattice. However, the two partial orders of a mixed lattice space do not need to be lattice orderings. A systematic development of this theory is quite recent [4, 9, 10], although some earlier studies on mixed lattice groups date back to 1990s [5, 6]. The theory of mixed lattice spaces and groups is heavily based on an earlier theory of mixed lattice semigroups, which was developed by M. Arsove and H. Leutwiler in the 1970s for the purposes of axiomatic potential theory (see [2] and references therein).

In Sect. 2 we give a brief overview of the terminology and the elementary properties of mixed lattice spaces. In Sect. 3 we present a generalization of mixed lattice structures by relaxing some of the assumptions in the definitions, resulting in a significant gain in generality. This modification gives the structure more flexibility in terms of applications to cone projections, and this is what our main results are concerned with. In Sect. 4 we formulate certain problems in convex optimization using this generalized order-theoretic framework. Here we observe a close connection between the mixed lattice theory and the theory of cone projections.

In metric cone projections there are two cones involved (a cone and its dual), and we show that this setting has a generalized mixed lattice structure. We then look at some other fundamental facts and results concerning cone projections, and formulate them in the language of mixed lattice theory. We also observe that the so-called *lattice like operations* introduced in [12] can be viewed as the generalized mixed lattice operations.

We should also mention the more general concept of *isotone cone retraction*, introduced by Németh [14], and the related notion of an *asymmetric cone norm*, recently studied in [13]. Some of the connections between these and the mixed lattice theory is investigated in [10].

2 Mixed lattice spaces

In this section we collect the essential terminology, definitions and basic results. For a more detailed presentation with proofs, we refer to [4, 6, 9].

Let V be a partially ordered real vector space with two partial orderings $\leq$ and $\preccurlyeq$ (i.e. we assume that both orderings are compatible with the linear structure of V). The partial order $\leq$ is called the *initial order* and $\preccurlyeq$ is called the *specific order*. With these two partial orders $\leq$ and $\preccurlyeq$ we define the *mixed upper and lower envelopes*

$$u \curlyvee v \;=\; \min\{\, w \in V \,:\, w \succcurlyeq u \;\text{ and }\; w \geq v \,\} \tag{2.1}$$

and

$$u \curlywedge v \;=\; \max\{\, w \in V \,:\, w \preccurlyeq u \;\text{ and }\; w \leq v \,\}, \tag{2.2}$$

respectively, where the minimum and maximum (whenever they exist) are taken with respect to the initial order $\leq$. These definitions were introduced by Arsove and Leutwiler in [1]. We observe that these operations are not commutative, i.e. $x \curlyvee y$ and $y \curlyvee x$ are not necessarily equal. We recall that a subset K of a vector space is called a *cone* if (i) $tK \subseteq K$ for all $t \geq 0$, (ii) $K + K \subseteq K$ and (iii) $K \cap (-K) = \{0\}$. Although the above definition of a cone is rather standard in the theory of partially ordered spaces, we should point out that there are also different definitions in use, and sometimes a subset satisfying the above conditions is called a *full-dimensional convex pointed cone*.

Definition 2.1 Let V be a partially ordered real vector space with respect to two partial orders $\leq$ and $\preccurlyeq$, and let V_p and V_{sp} be the corresponding positive cones, respectively. Then $V = (V, \leq, \preccurlyeq)$ is called a *mixed lattice vector space* if the following conditions hold:

(1) The elements $x \curlywedge y$ and $x \curlyvee y$ exist in V for all $x, y \in V$,
(2) The elements $x \curlyvee y$ and $x \curlywedge y$ are in V_{sp} whenever $x, y \in V_{sp}$.

Remark 2.2 A more general definition of a mixed lattice structure does not require the condition (2) in the above definition. It is included here for convenience, as many important properties depend on it. For more details on these technicalities, as well as many examples of mixed lattice spaces, we refer to [4]. If the condition (2) holds, then V_{sp} is a called a *mixed lattice cone*. A mixed lattice space satisfying (2) is called *quasi-regular*, and if in addition the cone V_{sp} is generating then V is called *regular*. It follows from the condition (2) that $V_{sp} \subseteq V_p$ ([4, Theorem 2.10]). We also note that the cone V_p is always generating, that is $V = V_p - V_p$. This is a consequence of Theorem 2.5(b) given below.

The definition of the mixed envelopes implies that in a mixed lattice space V the inequalities $x \curlywedge y \preccurlyeq x \preccurlyeq x \curlyvee y$ and $x \curlywedge y \leq y \leq x \curlyvee y$ hold for all $x, y \in V$. For proofs and further discussion on the properties given in the next theorem, see [4, 6, 9].

Theorem 2.3 *Let V be a mixed lattice space. The mixed envelopes have the following properties for all $x, y, z, u, v \in V$ and $a \in \mathbb{R}$.*

(a) $x \curlyvee y + y \curlywedge x = x + y$
(b) $z + x \curlyvee y = (x + z) \curlyvee (y + z)$ *and* $z + x \curlywedge y = (x + z) \curlywedge (y + z)$
(c) $x \curlyvee y = -(-x \curlywedge -y)$
(d) $x \preccurlyeq u$ *and* $y \leq v \implies x \curlyvee y \leq u \curlyvee v$ *and* $x \curlywedge y \leq u \curlywedge v$
(e) $x \leq y \iff y \curlyvee x = y \iff x \curlywedge y = x$
(f) $x \preccurlyeq y \iff x \curlyvee y = y \iff y \curlywedge x = x$
(g) $(ax) \curlywedge (ay) = a(x \curlywedge y)$ *and* $(ax) \curlyvee (ay) = a(x \curlyvee y)$ *for all* $a \geq 0$
(h) $(ax) \curlywedge (ay) = a(x \curlyvee y)$ *and* $(ax) \curlyvee (ay) = a(x \curlywedge y)$ *for all* $a < 0$
(i) $x \preccurlyeq y \implies z \curlyvee x \preccurlyeq z \curlyvee y$ *and* $z \curlywedge y \preccurlyeq z \curlywedge y$
(j) $u \preccurlyeq x \preccurlyeq z$ *and* $u \preccurlyeq y \preccurlyeq z \implies x \curlyvee y \preccurlyeq z$ *and* $u \preccurlyeq x \curlywedge y$

The upper and lower parts of an element were introduced in [4, Definition 3.1]. Their roles are similar to those of the positive and negative parts of an element in a vector lattice.

Definition 2.4 Let V be a mixed lattice space and $x \in V$. The elements ${}^u x = x \curlyvee 0$ and ${}^l x = (-x) \curlyvee 0$ are called the *upper part* and *lower part* of x, respectively.

Similarly, the elements $x^u = 0 \curlyvee x$ and $x^l = 0 \curlyvee (-x)$ are called *specific upper part* and *specific lower part* of x, respectively.

From the above definitions we observe that for the specific upper and lower parts we have $x^u \succcurlyeq 0$ and $x^l \succcurlyeq 0$, and for the upper and lower parts ${}^u x \geq 0$ and ${}^l x \geq 0$. The upper and lower parts have several important basic properties, which were proved in [4, Section 3]. Some of these properties are given in the next theorem.

Theorem 2.5 *Let V be a mixed lattice space and $x \in V$. Then we have*

(a) ${}^u x = {}^l(-x)$ *and* $x^u = (-x)^l$.
(b) $x = x^u - {}^l x = {}^u x - x^l$.
(c) ${}^u(x+y) \leq {}^u x + {}^u y$ *and* $(x+y)^l \leq x^l + y^l$.
(d) $(x+y)^u \leq x^u + y^u$ *and* ${}^l(x+y) \leq {}^l x + {}^l y$.
(e) $x^u \curlywedge {}^l x = 0 = x^l \curlywedge {}^u x$.
(f) $x \succcurlyeq 0$ *if and only if* $x = {}^u x = x^u$ *and* ${}^l x = x^l = 0$.
(g) $x \geq 0$ *if and only if* $x = {}^u x$ *and* $x^l = 0$.

3 The generalized mixed lattice structure

In a mixed lattice space V the existence of the mixed envelopes places rather strict restrictions on the cones V_p and V_{sp} which limits the range of possible applications. In this section we present a generalization of a mixed lattice space to overcome this limitation. Our main motivation for the generalization is that when studying cone projections in optimization problems, the upper and lower parts of elements do not typically exist.

Let $(V, \leq, \preccurlyeq)$ be a partially ordered vector space with two partial orderings, as in the preceding section. We introduce the following set notation.

$$[x \curlyvee y] = \{w : w \succcurlyeq x \text{ and } w \geq y\} \quad \text{and} \quad [x \curlywedge y] = \{w : w \preccurlyeq x \text{ and } w \leq y\}.$$

Let $E \subset V$. An element $x \in E$ is called a *minimal element* of E if $y \in E$ and $y \leq x$ implies $y = x$. A dual notion of a *maximal element* is defined similarly. The set of minimal elements of the set $[x \curlyvee y]$ will be denoted by $\mathrm{Min}[x \curlyvee y]$, and the set of maximal elements of the set $[x \curlywedge y]$ will be denoted by $\mathrm{Max}[x \curlywedge y]$.

We now define a generalization of the mixed lattice structure in which the elements $x \curlyvee y$ and $x \curlywedge y$ are replaced by set-valued mappings $(x, y) \mapsto \mathrm{Min}[x \curlyvee y]$ and $(x, y) \mapsto .\mathrm{Max}[x \curlywedge y]$. This provides a considerable increase in generality, at the expense of losing some good properties of mixed lattice spaces and becoming somewhat more difficult to work with.

Definition 3.1 Let V be a partially ordered vector space with respect to two partial orders $\leq$ and $\preccurlyeq$, and let V_p and V_{sp} be the corresponding positive cones, respectively. Then $V = (V, \leq, \preccurlyeq)$ is called a *generalized mixed lattice structure* if the sets $\mathrm{Min}[x \curlyvee y]$ and $\mathrm{Max}[x \curlywedge y]$ are non-empty for all $x, y \in V$.

Clearly, if the sets $\mathrm{Min}[x \curlyvee y]$ and $\mathrm{Max}[x \curlywedge y]$ contain only one element for every $x, y \in V$ then these elements are equal to $x \curlyvee y$ and $x \curlywedge y$, respectively, and the generalized mixed lattice structure then reduces to an ordinary mixed lattice space.

In the following we introduce selected fundamental properties of generalized mixed lattice structures to facilitate the study of cone projections in Sect. 4. The next theorem gives some of the basic properties of generalized mixed lattice structures corresponding to the properties of the mixed envelopes listed in Theorem 2.3.

Theorem 3.2 *Let V be a generalized mixed lattice structure. The following hold for all $x, y \in V$.*

(a) $\mathrm{Min}[x \curlyvee y] = -\mathrm{Max}[(-x) \curlywedge (-y)]$
(b) *If $u \in \mathrm{Min}[x \curlyvee y]$ then there exists an element $w \in \mathrm{Max}[y \curlywedge x]$ such that $x+y = u+w$. Similarly, for any $w \in \mathrm{Max}[y \curlywedge x]$ there exists an element $u \in \mathrm{Min}[x \curlyvee y]$ such that $x + y = u + w$. Hence, we have $x + y \in \mathrm{Min}[x \curlyvee y] + \mathrm{Max}[y \curlywedge x]$.*
(c) $\mathrm{Min}[(z + x) \curlyvee (z + y)] = z + \mathrm{Min}[x \curlyvee y]$ *for all $z \in V$.*
(d) $\mathrm{Max}[(z + x) \curlywedge (z + y)] = z + \mathrm{Max}[x \curlywedge y]$ *for all $z \in V$.*
(e) $x \succcurlyeq y \iff x \in \mathrm{Min}[y \curlyvee x] \iff \mathrm{Min}[y \curlyvee x] = \{x\} \iff y \in \mathrm{Max}[x \curlywedge y] \iff \mathrm{Max}[x \curlywedge y] = \{y\}$.
(f) *For all $a \in \mathbb{R}$, $a \geq 0$ we have $\mathrm{Max}[(ax) \curlywedge (ay)] = a\,\mathrm{Max}[x \curlywedge y]$ and $\mathrm{Min}[(ax) \curlyvee (ay)] = a\,\mathrm{Min}[x \curlyvee y]$.*
(g) *For all $a \in \mathbb{R}$, $a < 0$ we have $\mathrm{Max}[(ax) \curlywedge (ay)] = a\,\mathrm{Min}[x \curlyvee y]$ and $\mathrm{Min}[(ax) \curlyvee (ay)] = a\,\mathrm{Max}[x \curlywedge y]$.*

Proof *(a)* If $u \in \mathrm{Min}[x \curlyvee y]$ then $u \succcurlyeq x$ and $u \geq y$, so $-u \preccurlyeq -x$ and $-u \leq -y$, and thus $-u \in [(-x) \curlywedge (-y)]$. If $w \in [(-x) \curlywedge (-y)]$ and $w \geq -u$ then by a similar argument $-w \in [x \curlyvee y]$. But $-w \leq u$ and u is a minimal element of $[x \curlyvee y]$, so we must have $-w = u$. Hence, $u \in -\mathrm{Max}[(-x) \curlywedge (-y)]$. The reverse inclusion is proved similarly, so $\mathrm{Min}[x \curlyvee y] = -\mathrm{Max}[(-x) \curlywedge (-y)]$.

(b) If $u \in \mathrm{Min}[x \curlyvee y]$ then $u \succcurlyeq x$ and $u \geq y$. This implies that $x + y \preccurlyeq u + y$ and $x + y \leq x + u$, and so $x + y - u \in [y \curlywedge x]$. We want to show that $x + y - u$ is a maximal element of the set $[y \curlywedge x]$. For this, suppose that $w \in .[y \curlywedge x]$ and $w \geq x + y - u$. Then $w \preccurlyeq y$ and $w \leq x$, and it follows that $u \geq x + y - w \succcurlyeq x$ and $u \geq x+y-w \geq y$. Hence $u \geq x+y-w \in [x \curlyvee y]$. But u is a minimal element of the set $[x \curlyvee y]$, so we must have $u = x + y - w$ and thus $w = x + y - u \in \mathrm{Max}[y \curlywedge x]$ and $u + w = x + y$. The dual statement is proved similarly.

(c) If $v \in \mathrm{Min}[x \curlyvee y]$ then $v \succcurlyeq x$ and $v \geq y$. Consequently, $v + z \succcurlyeq x + z$ and $v+z \geq y+z$ for all $z \in V$, and so $v+z \in [(x+z) \curlyvee (y+z)]$. If $w \in [(x+z) \curlyvee (y+z)]$ and $w \leq v + z$ then $v \geq w - z \succcurlyeq x + z - z = x$ and $v \geq w - z \geq y + z - z = y$. But then $v \geq w - z \in [x \curlyvee y]$, and since v is a minimal element, it follows that $v = w - z$, or $w = v + z$. Hence, $z + v \in \mathrm{Min}[(z + x) \curlyvee (z + y)]$ for all $z \in V$.

For the converse, if $w \in \mathrm{Min}[(z + x) \curlyvee (z + y)]$ then $w \succcurlyeq z + x$ and $w \geq z + y$. Thus $w - z \succcurlyeq x$ and $w - z \geq y$, so $w - z \in [x \curlyvee y]$. Again, if $v \in [x \curlyvee y]$ and $v \leq w - z$ then $w \geq v + z \succcurlyeq x + z$ and $w \geq v + z \geq y + z$, and it follows that $w \geq v+z \in [(z+x) \curlyvee (z+y)]$. Since w is minimal, we have $w = v+z$, or $v = w-z$.

Hence, $w - z \in \operatorname{Min}[x \curlyvee y]$. This shows that $w = z + (w - z) \in z + \operatorname{Min}[x \curlyvee y]$, proving the equality of the two sets.

(d) is similar to *(c)*.

(e) Let $x \succcurlyeq y$. Since $x \geq x$ we have $x \in [y \curlyvee x]$. If $u \in [y \curlyvee x]$ is an element such that $u \leq x$ then $x \leq u \leq x$, and hence $x = u$ because $V_p \cap (-V_p) = \{0\}$ holds by assumption. This shows that $x \in \operatorname{Min}[y \curlyvee x]$. Next, if $x \in \operatorname{Min}[y \curlyvee x]$ and $w \in \operatorname{Min}[y \curlyvee x]$ then $w \geq x$. But then w and x are both minimal elements, so they are comparable only if $w = x$. Hence $\operatorname{Min}[y \curlyvee x] = \{x\}$.

If $\operatorname{Min}[y \curlyvee x] = \{x\}$ then it follows by part (b) that $y \in \operatorname{Max}[x \curlywedge y]$. If there is some other element $w \in \operatorname{Max}[x \curlywedge y]$ then $y \leq w$. But y and w are both maximal elements of the set $[x \curlywedge y]$, and so $w = y$. This shows that $\operatorname{Max}[x \curlywedge y] = \{y\}$. Finally, if $\operatorname{Max}[x \curlywedge y] = \{y\}$ then of course $y \preccurlyeq x$, finishing the proof.

(f) The case $a = 0$ is trivial. If $a > 0$ and $z \in \operatorname{Min}[(ax) \curlyvee (ay)]$ then $z \preccurlyeq ax$ and $z \leq ay$. Hence $\frac{z}{a} \preccurlyeq x$ and $\frac{z}{a} \leq y$, and so $z = a\frac{z}{a} \in a \operatorname{Min}[x \curlyvee y]$. The reverse inclusion is straightforward, and the other equality is similar.

(g) follows from *(a)* and *(f)*. □

The preceding theorem suggests a useful characterization for general mixed lattice structures.

Theorem 3.3 *Let V be a partially ordered vector space with two partial orders $\leq$ and $\preccurlyeq$, and let V_p and V_{sp} be the corresponding positive cones, respectively. Then $V = (V, \leq, \preccurlyeq)$ is a generalized mixed lattice structure if and only if the set* $\operatorname{Min}[0 \curlyvee x]$ *is non-empty for all $x \in V$.*

Proof The given condition is obviously necessary. To prove that it is sufficient, let $x, y \in V$. By assumption, the set $\operatorname{Min}[0 \curlyvee (y - x)]$ is non-empty, and the same arguments as in the proof of Theorem 3.2(c) show that $\operatorname{Min}[x \curlyvee y] = x + \operatorname{Min}[0 \curlyvee (y - x)]$, and so the set $\operatorname{Min}[x \curlyvee y]$ is also non-empty. Since x and y were arbitrary, we can apply the arguments from the proof of Theorem 3.2(a) to see that the set $\operatorname{Max}[x \curlywedge y] = -\operatorname{Min}[(-x) \curlyvee (-y)]$ is non-empty too. Hence V is a mixed lattice structure. □

The next result shows that in a generalized mixed lattice structure every element can be written as a difference of a positive part and a negative part, but the representation is not unique. The following is thus a generalized version of [4, Theorem 3.6] and parts (b) and (e) of Theorem 2.5.

Theorem 3.4 *Let V be a generalized mixed lattice structure and $x \in V$.*

(a) *For any $u \in \operatorname{Min}[x \curlyvee 0]$ there exist an element $w \in \operatorname{Min}[0 \curlyvee (-x)]$ such that $x = u - w$. Moreover, if $u \in \operatorname{Min}[x \curlyvee 0]$ and $w \in \operatorname{Min}[0 \curlyvee (-x)]$ are any such elements that $x = u - w$ then $0 \in \operatorname{Max}[w \curlywedge u]$ and $u + w \in \operatorname{Min}[u \curlyvee w]$. On the other hand, if $x = u - w$ and $0 \in \operatorname{Max}[w \curlywedge u]$ then $u \in \operatorname{Min}[x \curlyvee 0]$ and $w \in \operatorname{Min}[0 \curlyvee (-x)]$.*
(b) *For any $u \in \operatorname{Min}[0 \curlyvee x]$ there exist an element $w \in \operatorname{Min}[(-x) \curlyvee 0]$ such that $x = u - w$. Moreover, if $u \in \operatorname{Min}[0 \curlyvee x]$ and $w \in \operatorname{Min}[(-x) \curlyvee 0]$ are any such elements that $x = u - w$ then $0 \in \operatorname{Max}[u \curlywedge w]$ and $u + w \in \operatorname{Min}[w \curlyvee u]$. On*

the other hand, if $x = u - w$ *and* $0 \in \operatorname{Max}[u \curlywedge w]$ *then* $u \in \operatorname{Min}[0 \curlyvee x]$ *and* $w \in \operatorname{Min}[(-x) \curlyvee 0]$.

Proof (a) The first part follows immediately from Theorem 3.2(b). Indeed, if $x \in V$ and $u \in \operatorname{Min}[x \curlyvee 0]$ then there exists $v \in \operatorname{Max}[0 \curlywedge x]$ such that $x = u + v$. If we put $w = -v$ then $w \in \operatorname{Min}[0 \curlyvee (-x)]$ and $x = u - w$.

Next, let $u \in \operatorname{Min}[x \curlyvee 0]$ and $w \in \operatorname{Min}[0 \curlyvee (-x)]$ be such that $x = u - w$. Since $0 \preccurlyeq w$ and $0 \leq u$, it follows that $0 \in [w \curlywedge u]$. Suppose there exists some $y \in [w \curlywedge u]$ such that $y \geq 0$. Now $y \in [w \curlywedge u] = [w \curlywedge (x+w)] = w + [0 \curlywedge x]$ (by Theorem 3.2(d)), so there exists $z \in [0 \curlywedge x]$ such that $y = w + z$. But then $z = y - w \geq -w$, and since $-w$ is a maximal element of the set $[0 \curlywedge x]$, it follows that $z = -w$, and so $y = 0$. This shows that $0 \in \operatorname{Max}[w \curlywedge u]$, and by Theorem 3.2(b) there exists an element $r \in \operatorname{Min}[u \curlyvee w]$ such that $u + w = 0 + r = r$. Therefore, $u + w \in \operatorname{Min}[u \curlyvee w]$.

On the other hand, if $x = u - w$ and $0 \in \operatorname{Max}[w \curlywedge u]$ then by Theorem 3.2(c) and (b) we have $u + w \in \operatorname{Min}[u \curlyvee w] = \operatorname{Min}[(x + w) \curlyvee w] = w + \operatorname{Min}[x \curlyvee 0]$. Hence, $u \in \operatorname{Min}[x \curlyvee 0]$. But then $u = x + w \in \operatorname{Min}[x \curlyvee 0] = x + \operatorname{Min}[0 \curlyvee (-x)]$ and this shows that $w \in \operatorname{Min}[0 \curlyvee (-x)]$.

(b) This is similar to the proof of part of (a), but we will prove the second statement just to indicate how to prove dual statements such as this. Let $u \in \operatorname{Min}[0 \curlyvee x]$ and $w \in \operatorname{Min}[(-x) \curlyvee 0]$ be such that $x = u - w$. Since $0 \leq w$ and $0 \preccurlyeq u$, it follows that $0 \in [u \curlywedge w]$. Assume that $y \in .[u \curlywedge w]$ and $y \geq 0$. Then $y \in [u \curlywedge w] = [(x + w) \curlywedge w] = w + [x \curlywedge 0]$, so there exists $z \in [x \curlywedge 0]$ such that $y = w + z$. But then $z = y - w \geq -w$, and since $-w$ is a maximal element of the set $[x \curlywedge 0]$ (by Theorem 3.2(a)), it follows that $z = -w$, and so $y = 0$. This shows that $0 \in \operatorname{Max}[u \curlywedge w]$, and by Theorem 3.2(b) there exists an element $r \in \operatorname{Min}[w \curlyvee u]$ such that $u + w = 0 + r = r$. Therefore, $u + w \in \operatorname{Min}[w \curlyvee u]$. The last statement is again proved as in part (a). ◻

By the preceding theorem, the set $\operatorname{Min}[0 \curlyvee x]$ can be called the *set of specific upper parts* of x, and $\operatorname{Min}[(-x) \curlyvee 0]$ the *set of lower parts* of x. For any $x \in V$ we can choose an upper part $u \in \operatorname{Min}[0 \curlyvee x]$, and there always exists a corresponding lower part $v \in \operatorname{Min}[(-x) \curlyvee 0]$ such that $x = u - v$. Similar remarks apply to the sets $\operatorname{Min}[x \curlyvee 0]$ and $\operatorname{Min}[0 \curlyvee (-x)]$, called the *set of upper parts* of x and the *set of specific lower parts* of x, respectively.

Under an additional assumption we can add the following properties to the list of Theorem 3.2.

Proposition 3.5 *Let* V *be a generalized mixed lattice structure such that* $V_{sp} \cap -V_p = \{0\}$. *Then the following equivalences hold:*

$$x \geq y \iff x \in \operatorname{Min}[x \curlyvee y] \iff y \in \operatorname{Max}[y \curlywedge x].$$

Proof Let $x \geq y$. Since $x \succcurlyeq x$ we have $x \in [x \curlyvee y]$. Let $u \in [x \curlyvee y]$ and $u \leq x$. Then $x \preccurlyeq u$ and $u \leq x$, or $0 \preccurlyeq u - x$ and $u - x \leq 0$. By assumption this implies that $u - x = 0$, or $u = x$. This shows that $x \in \operatorname{Min}[x \curlyvee y]$. Conversely, if $x \in \operatorname{Min}[x \curlyvee y]$

then $x \geq y$. The equivalence $x \geq y \iff y \in \mathrm{Max}[\, y \curlywedge x\,]$ can be proved by a similar argument. $\square$

We introduce some additional terminology for the next section. The set

$$V_{sp}^{*} = \{y \in V : V_p \cap \mathrm{Max}[x \curlywedge y] \neq \emptyset \text{ for all } x \in V_{sp}\}$$

is called the *right dual of* V_{sp}, and the set

$$^{*}V_p = \{x \in V : V_p \cap \mathrm{Max}[x \curlywedge y] \neq \emptyset \text{ for all } y \in V_p\}$$

is called the *left dual of* V_p.

Proposition 3.6 *$V_{sp}^{*} \subseteq V_p$ and $^{*}V_p \subseteq V_{sp}$. Moreover, $V_{sp} \subseteq {}^{*}(V_{sp}^{*})$ and $V_p \subseteq ({}^{*}V_p)^{*}$.*

Proof Let $y \in V_{sp}^{*}$. Then for any $x \in V_{sp}$ there exist $w \in \mathrm{Max}[x \curlywedge y]$ and $0 \leq w \leq y$. Thus $y \in V_p$, proving that $V_{sp}^{*} \subseteq V_p$. Next, if $x \in^{*} V_p$ then for any $y \in V_p$ there exist $w \in \mathrm{Max}[x \curlywedge y]$ such that $w \geq 0$. In particular, if $y = 0$ then there exist $v \in \mathrm{Max}[x \curlywedge 0]$ such that $v \geq 0$. But then $0 \leq v \leq 0$, so $v = 0$ and $v \preccurlyeq x$ implies that $x \in V_{sp}$. This shows that $^{*}V_p \subseteq V_{sp}$.

Since $V_{sp}^{*} \subseteq V_p$ and $^{*}V_p \subseteq V_{sp}$, it makes sense to consider the sets

$$^{*}(V_{sp}^{*}) = \{x \in V : V_p \cap \mathrm{Max}[x \curlywedge y] \neq \emptyset \text{ for all } y \in V_{sp}^{*}\}$$

and

$$(^{*}V_p)^{*} = \{y \in V : V_p \cap \mathrm{Max}[x \curlywedge y] \neq \emptyset \text{ for all } x \in^{*} V_p\}.$$

If $x \in V_{sp}$ then by the definition of V_{sp}^{*} we have $V_p \cap \mathrm{Max}[x \curlywedge y] \neq \emptyset$ for all $y \in V_{sp}^{*}$, and so $x \in^{*}(V_{sp}^{*})$. Hence $V_{sp} \subseteq^{*}(V_{sp}^{*})$, and by a similar argument, $V_p \subseteq (^{*}V_p)^{*}$. $\square$

4 Mixed lattice structure in the problem of cone projection

The main application of the results of this paper are given in this section. The aim is to show how the problem of cone projection can be stated in a purely order-theoretic form in the framework of generalized mixed lattice structure, thus providing a new perspective on such problems. We assume the knowledge of basic notions and terminology of convex optimization. For these we refer to [3, 8].

Let K be a closed and convex pointed cone in $\mathbb{R}^n$ with the dual cone $K^{*} = \{y : \langle x, y\rangle \geq 0 \text{ for all } x \in K\}$. Let $\preccurlyeq_K$ be the partial ordering induced by the cone K and let $\leq_{*}$ be the partial ordering given by the dual cone K^{*}.

Let $P_K : \mathbb{R}^n \to K$ be the projection mapping that gives the unique point $P_K x$ on K nearest to x. That is,

$$P_K x \in K \quad \text{and} \quad ||x - P_K x|| = \inf\{||x - y|| : y \in K\}.$$

This nearest point $P_K x$ has the characterization ([8, Theorem 3.1.1])

$$P_K x \in K \quad \text{and} \quad \langle P_K x - x, P_K x - y \rangle \le 0 \text{ for all } y \in K. \tag{4.1}$$

The projection mapping also has the translation property

$$P_{x+K} y = x + P_K(y - x) \quad \text{for all } x, y \in \mathbb{R}^n. \tag{4.2}$$

The mapping P_K is called K*-isotone* if $x \preccurlyeq_K y$ implies $P_K x \preccurlyeq_K P_K y$.

A fundamental tool in the study of cone projections is the following classical theorem of Moreau [11].

Theorem 4.1 (Moreau) *Let K be a closed convex cone in $\mathbb{R}^n$ and K^* its dual cone. Every $x \in \mathbb{R}^n$ can be written as $x = P_K x - P_{K^*}(-x)$ where $\langle P_K x, P_{K^*}(-x) \rangle = 0$. Moreover, $P_K x = 0$ holds if and only if $x \in -K^*$.*

With the notation introduced above, the projection $P_K x$ clearly satifies $P_K x \succcurlyeq_K 0$ and $P_K x \ge_* x$. Now we can show that $V = (\mathbb{R}^n, \le_*, \preccurlyeq_K)$ is a generalized mixed lattice structure in the sense of Definition 3.1, and the orthogonal projection $P_K x$ is in fact a minimal element satisfying the inequalities $P_K x \succcurlyeq_K 0$ and $P_K x \ge_* x$.

Theorem 4.2 *Let K be a closed and convex cone in $\mathbb{R}^n$ and K^* its dual cone, and let $\preccurlyeq_K$ and $\le_*$ be the partial orderings determined by the cones K and K^*, respectively. Then $V = (\mathbb{R}^n, \le_*, \preccurlyeq_K)$ is a generalized mixed lattice structure and for every $x \in \mathbb{R}^n$ the projection element $P_K x$ satisfies $P_K x \in \mathrm{Min}[0 \curlyvee x]$.*

Proof Let $x \in \mathbb{R}^n$. By Theorem 4.1 we have $\langle P_K x - x, P_K x \rangle = 0$, and from this we get $\langle P_K x, P_K x \rangle = \langle x, P_K x \rangle$. As noted above, the element $P_K x$ satifies $P_K x \succcurlyeq_K 0$ and $P_K x \ge_* x$. Suppose there is some other element w such that $w \succcurlyeq_K 0$, $w \ge_* x$ and $w \le_* P_K x$. Then $P_K x \in K$ and $P_K x - w \ge_* 0$, or $P_K x - w \in K^*$, so by the definition of K^* we have $\langle P_K x - w, P_K x \rangle \ge 0$, so $\langle P_K x, P_K x \rangle \ge \langle w, P_K x \rangle$. On the other hand, $w - x \in K^*$, and so $\langle w - x, P_K x \rangle \ge 0$, which gives $\langle w, P_K x \rangle \ge \langle x, P_K x \rangle$. Hence, we have

$$\langle P_K x, P_K x \rangle = \langle x, P_K x \rangle \le \langle w, P_K x \rangle \le \langle P_K x, P_K x \rangle.$$

Thus, $\langle P_K x, P_K x \rangle = \langle x, P_K x \rangle = \langle w, P_K x \rangle$, and it follows that

$$\langle P_K x, P_K x \rangle - \langle w, P_K x \rangle = \langle P_K x - w, P_K x \rangle = 0.$$

Now, for every $y \in K$ we have

$$\langle P_K x - w, P_K x - y \rangle = \langle P_K x - w, P_K x \rangle - \langle P_K x - w, y \rangle,$$

where the first term on the right hand side is zero, as we have shown, and for the second term we have $\langle P_K x - w, y \rangle \ge 0$ by the definition of the dual cone. Hence, we have $\langle P_K x - w, P_K x - y \rangle \le 0$ for all $y \in K$. By characterization (4.1) this means that $P_K x$

is the unique point on K nearest to w. But $w \in K$, so we must have $P_K x = P_K w = w$. This shows that $P_K x \in \mathrm{Min}[0 \curlyvee x]$, and since this holds for any $x \in V$ it follows by Theorem 3.3 that V is a generalized mixed lattice structure. $\square$

Now Theorem 4.2 allows us to translate the projection problem to the mixed lattice setting. Let us choose an element $x^u \in \mathrm{Min}[0 \curlyvee x]$ by the criterion of shortest distance, that is, $x^u = P_K x$. Then, if we denote by ${}^l x$ the corresponding element in $\mathrm{Min}[(-x) \curlyvee 0]$ (see Theorem 3.4), we have the unique representation $x = x^u - {}^l x$ for every $x \in V$, and this is the most natural representation in the present setting.

After fixing the "representatives" x^u and ${}^l x$ of each x in this way, we can now simplify (or rather abuse) the notation and write $x^u = 0 \curlyvee x$ and ${}^l x = (-x) \curlyvee 0 = -(x \curlywedge 0)$ (we observe here that the element ${}^l x$ gives the projection of $-x$ on K^*). It now follows from the results of Sect. 3 that, in essence, our generalized mixed lattice structure behaves much like an ordinary mixed lattice space, and (with some care) we can apply the rules $(a) - (h)$ of Theorem 2.3. Hence, by Theorem 3.2(c) equation (4.2) becomes

$$P_{x+K} y = x + P_K(y-x) = x + (y-x)^u = x + 0 \curlyvee (y-x) = x \curlyvee y. \quad (4.3)$$

In other words, $x \curlyvee y$ is the point on the cone $x + K$ that is nearest to the point y, or equivalently, the point on the cone $y + K^*$ that is nearest to the point x. In a similar manner, the lower envelope $x \curlywedge y$ is associated with the projections on the cones $x - K$ and $y - K^*$, i.e. $x \curlywedge y = P_{x-K} y = P_{y-K^*} x$.

Moreover, we observe that the duality between the cones K and K^* is a special case of the order-theoretic notion of duality, which was discussed in Proposition 3.6. We next show that orthogonality in the usual sense implies the order-theoretic version of the orthogonality condition.

Proposition 4.3 *If $x \in K$ and $y \in K^*$ are elements such that $\langle x, y\rangle = 0$ then $0 \in \mathrm{Max}[x \curlywedge y]$.*

Proof Let $z \in K$. Then by the definition of K^* we have $\langle z, y\rangle \geq 0$, and since $\langle x, y\rangle = 0$ we obtain $\langle x - z, y\rangle = \langle x, y\rangle - \langle z, y\rangle = -\langle z, y\rangle \leq 0$. This holds for all $z \in K$, so if we define $u = x - y$ then $y = x - u$ and the above inequality becomes $\langle x - u, x - z\rangle \leq 0$ for all $z \in K$. By the characterization (4.1) this means that $x = P_K u$. Then by Theorem 4.2 we have $x \in \mathrm{Min}[0 \curlyvee u]$, so by Theorem 3.2(c) we get $y = x - u \in \mathrm{Min}[0 \curlyvee u] - u = \mathrm{Min}[(-u) \curlyvee 0]$. We have now shown that $u = x - y$ where $x \in \mathrm{Min}[0 \curlyvee u]$ and $y \in \mathrm{Min}[(-u) \curlyvee 0]$. Hence, $0 \in \mathrm{Max}[x \curlywedge y]$, by Theorem 3.4. $\square$

Using the notation introduced above, we get the following special case of Theorem 3.4, which can be viewed as the order-theoretic version of Moreau's theorem.

Theorem 4.4 *Let $V = (\mathbb{R}^n, \leq_*, \preccurlyeq_K)$ be the generalized mixed lattice structure on $\mathbb{R}^n$, where $\preccurlyeq_K$ is the partial order defined by a closed convex cone K and $\leq_*$ is the partial order defined by the dual cone K^*. Then every $x \in V$ can be written as $x = x^u - {}^l x$ where $x^u \succcurlyeq_K 0$, ${}^l x \geq_* 0$ and $x^u \curlywedge {}^l x = 0$. Moreover, $x^u = 0$ if and only if $x \leq_* 0$.*

Proof By Theorem 4.2, for any $x \in V$ we have $P_K x \in \mathrm{Min}[0 \curlyvee x]$, and as noted above, if we put $x^u = P_K x$ then $x^u \succcurlyeq_K 0$ and by Theorem 3.4, if ${}^l x$ is the corresponding element in $\mathrm{Min}[(-x) \curlyvee 0]$, then ${}^l x = P_{K^*}(-x) \geq_* 0$ and we have the unique representation $x = x^u - {}^l x$. Since $\langle x^u, {}^l x \rangle = 0$, it follows by Proposition 4.3 that $0 \in \mathrm{Max}[x^u \curlywedge {}^l x]$. Also, by Theorem 3.2 we have $\mathrm{Max}[x^u \curlywedge {}^l x] = -\mathrm{Min}[(-x^u) \curlyvee (-{}^l x)]$, so from Eq. (4.3) we get $P_{-x^u+K}(-{}^l x) = -(x^u \curlywedge {}^l x)$. On the other hand, by (4.2) we obtain

$$P_{-x^u+K}(-{}^l x) = -x^u + P_K(-{}^l x + x^u) = -x^u + P_K x = -x^u + x^u = 0,$$

and this gives justification for writing $x^u \curlywedge {}^l x = 0$. (Again, this just amounts to the fact that we have chosen the ”representative” from the set $\mathrm{Max}[x^u \curlywedge {}^l x]$ by the criterion of shortest distance, which is consistent with our earlier choice of x^u, that is, if x^u and ${}^l x$ are chosen as above then the element in the set $\mathrm{Max}[x^u \curlywedge {}^l x]$ corresponding to this choice is 0.)

For the last statement, we first note that if $z \in (-K) \cap K^*$ then by the definition of K^* we have $\langle z, z \rangle = \|z\|^2 \leq 0$, which implies that $z = 0$, or $(-K) \cap K^* = \{0\}$. Thus, using Theorem 4.2 and Proposition 3.5, we have $x \leq_* 0$ if and only if $0 = x^u = P_K x \in \mathrm{Min}[0 \curlyvee x]$. □

It should be stressed again that the notation we use here is not entirely correct because our structure is not a mixed lattice space in the sense of Definition 2.1, but rather a generalized structure as described in the preceding section. Although the element $P_K x$ is the minimum in terms of distance to the cone K, it is not necessarily the order-theoretic minimum in the sense of (2.1). Because of this, some properties of mixed lattice space do not hold in the present situation, and some care should be taken when manipulating expressions that contain the mixed envelopes. For instance, the inequalities in Theorem 2.3(i) do not necessarily hold, which means that the projection mapping is not isotone, in general. The conditions for the isotonicity of the cone projection have been extensively studied (see [12] and the references therein). However, the following discussion gives further justification for the use of this notation.

The authors in [12] introduced what they called the *lattice-like operations* for studying questions related to cone projections. These operations are a generalization of similar operations that were introduced in [7] for the special case of self-dual cones (i.e. $K = K^*$). The lattice-like operations are defined by

$$x \sqcup y = P_{x+K} y, \quad x \sqcap y = P_{x-K} y, \quad x \sqcup_* y = P_{x+K^*} y, \quad x \sqcap_* y = P_{x-K^*} y.$$

There is an interesting connection between the lattice-like operations and the generalized mixed lattice operations. As we have shown, the projection element $P_{x+K} y$ is a minimal element of the set $[x \curlyvee y]$, and in the notation of (4.3) we denote this element by $P_{x+K} y = x \curlyvee y$. We can now observe that in the case of the projection elements, the generalized mixed lattice operations reduce to the lattice-like operations as follows:

$$x \sqcup y = x \curlyvee y, \quad x \sqcup_* y = y \curlyvee x, \quad x \sqcap y = x \curlywedge y \quad \text{and} \quad x \sqcap_* y = y \curlywedge x.$$

In fact, most of the properties of the lattice-like operations ([12, Lemma 2 and Lemma 3]) are identical to the properties of the mixed envelopes listed in Theorem 2.3 $(a)-(h)$. The above discussion places the lattice-like operations in the present order-theoretic context.

5 Conclusions

In this paper we presented a new approach to the problem of cone projection based on an ordered algebraic structure called a mixed lattice space. We first introduced a generalization of the notion of mixed lattice space, and we showed that many of the basic properties of mixed lattice spaces can be extended to the generalized mixed lattice structure. The motivation for this generalization is that it can be applied in a broader range of situations. As our main application, we showed how the mixed lattice structure arises quite naturally in the study of cone projections. We demonstrated how the problem of cone projection can be formulated in the mixed lattice setting, and we also observed that the related notion of lattice-like operations can be interpreted as the generalized mixed lattice operations.

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