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Saturated Output Regulation of Distributed Parameter Systems with Collocated Actuators and Sensors ⋆

Thavamani Govindaraj * **Lassi Paunonen** *
Jukka-Pekka Humaloja **

* *Faculty of Information Technology and Communication Sciences, Tampere University, P.O.Box 692, 33101, Tampere, Finland (e-mails: thavamani.govindaraj@tuni.fi, lassi.paunonen@tuni.fi).*
** *University of Alberta, 9211-116 St, Edmonton, AB T6G 1H9, Canada (e-mail: jphumaloja@ualberta.ca)*

Abstract: This paper addresses the problem of output regulation of infinite-dimensional linear systems subject to input saturation. We focus on strongly stabilizable linear dissipative systems with collocated actuators and sensors. We generalize the output regulation theory for finite-dimensional linear systems subject to input saturation to the class of considered infinite-dimensional linear systems. The theoretic results are illustrated with an example where we consider the output regulation of a flexible satellite model that is composed of two identical flexible solar panels and a center rigid body.

Keywords: Distributed parameter systems, output regulation, input saturation, strongly stabilizable systems, collocated input and output.

1. INTRODUCTION

For the past few decades, there has been interest in studying linear systems subject to input saturation due to limitations on the control input. Stabilization and output regulation of such systems have been studied, for example, in Fuller (1969), Sontag and Sussmann (1990), Teel (1992), Saberi et al. (2003), Logemann et al. (1998), Slemrod (1989), Oostveen (2000), Lasiecka and Seidman (2003), Prieur et al. (2015), Marx et al. (2015), Mironchenko et al. (2021) and the references therein. However, there are only few results in the literature dealing with output regulation of infinite-dimensional linear systems subject to input saturation Logemann et al. (1998), Oostveen (2000), Fliegner et al. (2001).

In this paper, we study output regulation of infinite-dimensional abstract linear systems subject to input saturation. We focus on the class of abstract systems given by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\phi(u(t)) + B_d w_d(t), \quad x(0) = x_0,\\ y(t) &= B^* x(t)\end{aligned} \tag{1}$$

on a real Hilbert space X. Here $x(t) \in X$ is the state variable, $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}$ is the output, $w_d(t) \in \mathbb{R}^{n_d}$ is an external disturbance and ϕ is a saturation function. The saturation function ϕ is defined as

$$\phi(u) = \begin{cases} u, & |u| \le 1\\ 1, & u > 1\\ -1, & u < -1.\end{cases} \tag{2}$$

Our goal is to find a linear feedback control law such that the output $y(t)$ of the system (1) tracks the given reference signal $y_{ref}(t)$ asymptotically despite disturbances $w_d(t)$ in the system. The reference $y_{ref}(\cdot)$ and the disturbance $w_d(\cdot)$ signals are assumed to be generated by an exosystem

$$\begin{aligned}\dot{v}(t) &= Sv(t), \quad v(0) = v_0,\\ w_d(t) &= Ev(t),\\ y_{ref}(t) &= -Fv(t)\end{aligned} \tag{3}$$

on a finite-dimensional space $W = \mathbb{R}^q$. Here $S \in \mathbb{R}^{q\times q}$, $F \in \mathbb{R}^{1\times q}$ and $E \in \mathbb{R}^{n_d\times q}$. Furthermore, we make the following assumptions on the system (1) and the exosystem (3).

Assumption 1.1. (1) The operator A generates a C_0-semigroup $T(t)$ of contractions on X, $B \in \mathcal{L}(\mathbb{R}, X)$ and the operator $A - \kappa BB^*$ generates a strongly stable contraction semigroup $T_{-\kappa BB^*}(t)$ for any $\kappa > 0$.
(2) The spectrum $\sigma(S)$ of S lies on the imaginary axis.

As the main contribution, we extend the output regulation theory in (Saberi et al., 2003, Ch. 3) for finite-dimensional linear systems subject to input saturation to the class of systems in (1)-(2) under Assumption 1.1. The considered class of systems (1) arise in the study of systems with collocated actuators and sensors Oostveen (2000). We present a linear output feedback control law that solves the output regulation problem. In addition, we demonstrate the results on a flexible satellite model subject to input saturation.

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Stabilization is an important part of control design for the output regulation. Stabilization problem for infinite-dimensional linear systems subject to input saturation has been studied, for example in Slemrod (1989), Lasiecka and Seidman (2003), Prieur et al. (2015), Curtain and Zwart (2016), Marx et al. (2015) and Mironchenko et al. (2021). The output regulation of infinite-dimensional linear systems subject to input saturation has been studied, for example in Logemann et al. (1998), Logemann et al. (1999), Logemann and Adam (2001) and Fliegner et al. (2003) for exponentially stable single-input single-output regular linear systems and in Oostveen (2000) for strongly stable single-input single-output linear systems. The results in these references use integral control to achieve output tracking of constant reference signals. The key novelty in our work is that we allow the reference and disturbance signals to be combination of sinusoids. The output tracking is achieved by using a linear output feedback control law which is a generalization of the control law presented in (Saberi et al., 2003, Thm. 3.3.3).

The paper is organized as follows. In Section 2, we present preliminaries on semilinear systems and the output regulation problem. Section 3 is devoted to our main results where we present a linear feedback control law and the solvability conditions for the output regulation of the system (1). In Section 4, we present a numerical example where we consider output regulation of a flexible satellite model subject to input saturation. Concluding remarks and further research directions are presented in Section 5.

1.1 Notation

For normed linear spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from X to Y. For a linear operator A, $D(A)$, $\mathcal{R}(A)$ and $\sigma(A)$ denote the domain, the range and the spectrum of A, respectively.

2. PRELIMINARIES

In this section, we present definitions and lemmas that are used in proving the main results. Consider the system (1) on a real Hilbert space X with $A : D(A) \subset X \to X$, $B \in \mathcal{L}(\mathbb{R}, X)$ and $B_d \in \mathcal{L}(\mathbb{R}^{n_d}, X)$.

Definition 2.1. Let $G(\cdot) = B^*(\cdot I - A)^{-1}B$ be the transfer function of the system (A, B, B^*). Then $s \in \mathbb{C}$ is called a transmission zero if $G(s) = 0$.

Lemma 2.2. (Curtain and Zwart, 2020, Thm. 11.1.5). Consider the semilinear differential equation

$$\dot{x}(t) = Ax(t) + f(x(t)), \quad t \geq 0, \quad x(0) = x_0, \tag{4}$$

where A is the infinitesimal generator of the C_0-semigroup on the Hilbert space X. If $f : X \to X$ is uniformly Lipschitz continuous, then the system (4) has a unique mild solution on $[0, \infty)$ with the following properties:

(i) For $0 \leq t < \infty$ the solution depends continuously on the initial condition, uniformly on any bounded interval $[0, \tau] \subset [0, \infty)$.
(ii) If $x_0 \in D(A)$, then the mild solution is a classical solution on $[0, \infty)$.

Definition 2.3. (Curtain and Zwart, 2020, Def. 11.2.2). Consider the semilinear differential equation (4) on the Hilbert space X. Assume that $f : X \to X$ is locally Lipschitz continuous.

Then the origin of (4) is *stable* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\|x_0\| < \delta$ there exists a solution $x(t)$ of (4) on $[0, \infty)$ satisfying $\|x(t)\| < \epsilon$ for all $t \geq 0$. If, in addition, there exists $\gamma > 0$ such that $\|x_0\| < \gamma$ implies that $\|x(t)\| \to 0$ as $t \to \infty$, then the origin is said to be *asymptotically stable*. The origin is said to be *globally asymptotically stable* if for every $x_0 \in X$ we have $\|x(t)\| \to 0$ as $t \to \infty$.

From the theory of output regulation of finite-dimensional linear systems subject to input saturation we know that the output regulation problem, in general, is not solvable for all initial conditions $v_0 \in \mathbb{R}^q$ of the exosystem (Saberi et al., 2003, Rem. 3.2.2). However, if we restrict to the initial conditions v_0 of the exosystem lying inside a given compact set, then the output regulation problem is solvable. In this work, we focus on this semi-global output regulation problem of (1).

Semi-Global Output Regulation Problem. Consider the systems (1)-(3) and a compact set $\mathcal{W}_0 \subset \mathbb{R}^q$. Find a linear output feedback control law in the form

$$u(t) = -\kappa y(t) + Lv(t) \tag{5}$$

such that $\kappa > 0,\ L \in \mathbb{R}^{1\times q}$ and

(1) The origin of the system $\dot{x}(t) = Ax(t) + B\phi(-\kappa y(t))$, $x(0) = x_0$ is globally asymptotically stable.
(2) For all $x_0 \in X$ and $v_0 \in \mathcal{W}_0$, the error between the output $y(t)$ and the reference signal $y_{ref}(t)$ satisfies

$$\lim_{t\to\infty} y(t) - y_{ref}(t) = 0.$$

3. MAIN RESULTS

In this section, we present our main theorem which provides the solvability conditions and the control law for the semi-global output regulation of the system (1). The theorem is an infinite-dimensional generalization of (Saberi et al., 2003, Thm. 3.3.3) where a low-and-high-gain state feedback control design is used to achieve semi-global output regulation of finite-dimensional linear systems subject to input saturation. In our case, since the considered class of systems can be stabilized strongly using negative output feedback, it is not necessary to find a stabilizing state feedback law separately. Consequently, there is no low-gain requirement on the stabilizing feedback law and there is only one gain parameter that corresponds to negative output feedback. So, the strong stabilizability property of the system (1) by output feedback enables simplifying the control design compared to the original one in (Saberi et al., 2003, Thm. 3.3.3). Our approach for showing the asymptotic convergence of the regulation error is motivated by the techniques in (Curtain and Zwart, 2020, Thm. 11.2.11).

Theorem 3.1. Consider the systems (1), (3) and the given compact set $\mathcal{W}_0 \subset \mathbb{R}^q$. Under the Assumption 1.1, the semi-global output regulation problem is solvable if there exist $\Pi \in \mathcal{L}(\mathbb{R}^q, X)$ with $\mathcal{R}(\Pi) \subset D(A)$ and $\Gamma \in \mathbb{R}^{1\times q}$ such that they solve the regulator equations

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + B_d E \\ 0 &= B^*\Pi + F \end{aligned} \tag{6}$$

and there exists a $\delta > 0$ such that $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1-\delta$ for all $v(t) = e^{St}v_0$ with $v_0 \in \mathcal{W}_0$. In this case, for any $\kappa > 0$ the feedback law

$$u(t) = -\kappa y(t) + (\kappa B^*\Pi + \Gamma)v(t) \tag{7}$$

solves the semi-global output regulation problem.

Proof. By Assumption 1.1, we have that $A - \kappa BB^*$ generates a strongly stable contraction semigroup for any $\kappa > 0$. In addition, the saturation function ϕ is uniformly Lipschitz continuous on $\mathbb{R}$, $\phi(0) = 0$ and

$$\langle u, \phi(u)\rangle_{\mathbb{R}} = u^2, \quad \text{if } |u| \leq 1,$$
$$\langle u, \phi(u)\rangle_{\mathbb{R}} > 1, \quad \text{if } u > 1,$$
$$\langle u, \phi(u)\rangle_{\mathbb{R}} > 1, \quad \text{if } u < -1.$$

Therefore, by (Curtain and Zwart, 2020, Thm. 11.2.11), we have that the origin of

$$\dot{x}(t) = Ax(t) + B\phi(-\kappa y(t)),$$
$$x(0) = x_0$$

is globally asymptotically stable.

Next, using the feedback law (7), we will show that $y(t) - y_{ref}(t) \to 0$ as $t \to \infty$. Assume that $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1-\delta$. Let us introduce a new variable $\xi(t) = x(t) - \Pi v(t)$ which is the mild solution of

$$\begin{aligned}\dot{\xi}(t) &= A\xi(t) + B[\phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t)]\\ \xi(0) &= \xi_0.\end{aligned} \tag{8}$$

on X, where we have used $u(t) = -\kappa y(t) + (\kappa B^*\Pi + \Gamma)v(t) = -\kappa B^*\xi(t) + \Gamma v(t)$. We will begin by showing that the mild solution $\xi(t)$ of (8) exists for $t \in [0, \infty)$. Let us consider the composite system

$$\begin{aligned}\dot{\xi}_e(t) &= A_e\xi_e(t) + f_e(\xi_e(t))\\ \xi_e(0) &= \xi_{e0}\end{aligned} \tag{9}$$

on $X \times \mathbb{R}^q$ where

$$\xi_e(t) = \begin{bmatrix}\xi(t)\\ v(t)\end{bmatrix},\ \xi_{e0} = \begin{bmatrix}\xi_0\\ v_0\end{bmatrix},\ A_e = \begin{bmatrix}A & 0\\ 0 & S\end{bmatrix},$$
$$f_e(\xi_e(t)) = \begin{bmatrix}B[\phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t)]\\ 0\end{bmatrix}.$$

Here the operator A_e generates a C_0-semigroup (since it is block-diagonal and A generates a C_0-semigroup) and since ϕ is uniformly Lipschitz continuous and B and Γ are bounded linear operators, we have that f_e is uniformly Lispschitz continuous. In fact, using $\|\phi(u_1) - \phi(u_2)\| \leq \|u_1 - u_2\|$ for $u_1, u_2 \in \mathbb{R}$, for $\xi_{e1} = (\xi_1, v_1)^T, \xi_{e2} = (\xi_2, v_2)^T \in X \times \mathbb{R}^q$, we have

$$\begin{aligned}&\|f_e(\xi_{e1}) - f_e(\xi_{e2})\|\\ &\leq \|B\|\|\phi(-\kappa B^*\xi_1 + \Gamma v_1) - \phi(-\kappa B^*\xi_2 + \Gamma v_2)\|\\ &\qquad + \|B\|\|\Gamma\|\|v_1 - v_2\|\\ &\leq \|B\|\kappa\|B^*\|\|\xi_1 - \xi_2\| + 2\|B\|\|\Gamma\|\|v_1 - v_2\|\\ &\leq C\|\xi_{e1} - \xi_{e2}\|,\end{aligned}$$

where $C = \max\{\kappa\|B\|^2, 2\|B\|\|\Gamma\|\}$. Thus by Lemma 2.2, the system (9) has a unique mild solution $\xi_e(t)$ for $t \in [0, \infty)$. The mild solution $\xi_e(t)$ satisfies

$$\xi_e(t) = \begin{bmatrix}T(t) & 0\\ 0 & e^{St}\end{bmatrix}\xi_{e0} + \int_0^t \begin{bmatrix}T(t-s) & 0\\ 0 & e^{S(t-s)}\end{bmatrix} f_e(\xi_e(s))ds.$$

Furthermore, if $\xi_{e0} \in D(A_e)$, then $\xi_e(t)$ is a classical solution for $t \in [0, \infty)$. In particular, we have

$$\begin{aligned}\xi(t) &= T(t)\xi_0\\ &\quad + \int_0^t T(t-s)B[\phi(-\kappa B^*\xi(s) + \Gamma v(s)) - \Gamma v(s)]ds\end{aligned}$$

which is the mild solution for the system (8) and if $\xi_0 \in D(A)$, then $\xi(t)$ is a classical solution for $t \in [0, \infty)$.

Next, we show that the solution $\xi(t)$ is uniformly bounded. For $\xi_0 \in D(A)$, we have

$$\begin{aligned}\frac{d}{dt}\|\xi(t)\|^2 &= 2\left\langle \dot{\xi}(t), \xi(t)\right\rangle\\ &= 2\left\langle A\xi(t) + B[\phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t)], \xi(t)\right\rangle_X\\ &\leq 2\left\langle B[\phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t)], \xi(t)\right\rangle_X\\ &= 2\left\langle \phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t), B^*\xi(t)\right\rangle_{\mathbb{R}}\end{aligned} \tag{10}$$

where we have used the contractivity of A. Now by using the definition (2) of the saturation function ϕ and the assumption $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1-\delta$, we show that the right hand side of (10) is always non-positive. If we consider those $t \geq 0$ such that $|-\kappa B^*\xi(t) + \Gamma v(t)| \leq 1$, then

$$\begin{aligned}&\langle \phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}}\\ &\quad = \langle -\kappa B^*\xi(t) + \Gamma v(t) - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}}\\ &\quad = -\kappa\|B^*\xi(t)\|^2 \leq 0.\end{aligned}$$

If we consider those $t \geq 0$ such that $-\kappa B^*\xi(t) + \Gamma v(t) > 1$, then $-\kappa B^*\xi(t) > 1 - \Gamma v(t) > 0$. This implies that $B^*\xi(t) < 0$. Therefore

$$\begin{aligned}&\langle \phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}}\\ &\quad = \langle 1 - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}} \leq 0.\end{aligned}$$

Finally, if we consider those $t \geq 0$ such that $-\kappa B^*\xi(t) + \Gamma v(t) < -1$, then $-\kappa B^*\xi(t) < -1 - \Gamma v(t) < 0$. This implies that $B^*\xi(t) > 0$. Therefore

$$\begin{aligned}&\langle \phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}}\\ &\quad = \langle -1 - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}} \leq 0.\end{aligned}$$

Therefore $\frac{d}{dt}\|\xi(t)\|^2 \leq 0$. Integrating (10), we obtain for all $t \geq 0$

$$\begin{aligned}\|\xi(t)\|^2 &\leq \|\xi_0\|^2\\ &\quad + 2\int_0^t \langle \phi(-\kappa B^*\xi(s) + \Gamma v(s)) - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds\\ &\leq \|\xi_0\|^2.\end{aligned} \tag{11}$$

By the continuity of $\xi(t)$ with respect to the initial conditions, the above inequality holds for all $\xi_0 \in X$. This implies that for all $\xi_0 \in X$, $\xi(t)$ is bounded uniformly in t on $[0, \infty)$. Next, we show that the mild solution $\xi(t)$ converges to zero as $t \to \infty$. Let us reformulate the system (8) as

$$\begin{aligned}\dot{\xi}(t) &= (A - \kappa BB^*)\xi(t)\\ &\qquad - B[-\kappa B^*\xi(t) + \Gamma v(t) - \phi(-\kappa B^*\xi(t) + \Gamma v(t))]\\ \xi(0) &= \xi_0.\end{aligned}$$

Denote $\hat{u}(t) := -\kappa B^*\xi(t) + \Gamma v(t) - \phi(-\kappa B^*\xi(t) + \Gamma v(t))$. Since $B\hat{u} \in L^1_{loc}(0, \infty; X)$, the solution of the above system is given by

$$\xi(t) = T_{-\kappa BB^*}(t)\xi_0 - \int_0^t T_{-\kappa BB^*}(t-s)B\hat{u}(s)ds. \tag{12}$$

We will first show that $\hat{u} \in L^2(0, \infty; \mathbb{R})$. We will begin by splitting the interval $[0, \infty)$ into three parts. Let $\Omega_1 := \{t \in [0, \infty) \mid -\kappa B^*\xi(t) + \Gamma v(t) > 1\}$, $\Omega_2 := \{t \in [0, \infty) \mid -$

$\kappa B^*\xi(t) + \Gamma v(t) < -1\}$ and $\Omega_3 := \{t \in [0,\infty) \mid |-\kappa B^*\xi(t) + \Gamma v(t)| \le 1\}$. Then using the definition of ϕ, the assumption $\sup_{t\ge0}\|\Gamma v(t)\| \le 1-\delta$ and $\kappa B^*\xi(t) < \Gamma v(t) - 1$ on Ω_1, we obtain

$$\begin{aligned}&\int_{\Omega_1} \langle \phi(-\kappa B^*\xi(s) + \Gamma v(s)) - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds\\ &= \int_{\Omega_1} \langle 1 - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds\\ &\le \int_{\Omega_1} \left\langle 1 - \Gamma v(s), \frac{\Gamma v(s) - 1}{\kappa}\right\rangle_{\mathbb{R}} ds\\ &= -\frac{1}{\kappa}\int_{\Omega_1} \|1 - \Gamma v(s)\|^2 ds\\ &\le -\frac{\delta^2}{\kappa}\nu(\Omega_1),\end{aligned}$$

where ν is a Lebesgue measure. Moreover, from (11), we have

$$\int_0^\infty \langle \phi(-\kappa B^*\xi(s) + \Gamma v(s)) - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds < \infty \tag{13}$$

which implies that Ω_1 has finite measure. Similarly, using the definition of ϕ and $-\kappa B^*\xi(t) < -1 - \Gamma v(t)$ on Ω_2, we obtain

$$\begin{aligned}&\int_{\Omega_2} \langle \phi(-\kappa B^*\xi(s) + \Gamma v(s)) - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds\\ &= \int_{\Omega_2} \langle -1 - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds\\ &\le \int_{\Omega_2} \left\langle 1 + \Gamma v(s), \frac{-\Gamma v(s) - 1}{\kappa}\right\rangle_{\mathbb{R}} ds\\ &= -\frac{1}{\kappa}\int_{\Omega_2} \|1 + \Gamma v(s)\|^2 ds\\ &\le -\frac{\delta^2}{\kappa}\nu(\Omega_2)\end{aligned}$$

which from (13) implies that Ω_2 has finite measure. Furthermore, by using the definition of ϕ, we obtain

$$\begin{aligned}&\int_{\Omega_3} |\hat{u}(s)|^2 ds\\ &= \int_{\Omega_3} |-\kappa B^*\xi(s) + \Gamma v(s) - \phi(-\kappa B^*\xi(s) + \Gamma v(s))|^2 ds\\ &= 0.\end{aligned}$$

Since $B^* \in \mathcal{L}(X,\mathbb{R})$, $\sup_{t\ge0}\|\Gamma v(t)\| \le 1-\delta$ and $\xi(t)$ is uniformly bounded, we have that $\hat{u}(t)$ is uniformly bounded and therefore by using the above arguments, we obtain

$$\int_0^\infty |\hat{u}(s)|^2 ds = \int_{\Omega_1} |\hat{u}(s)|^2 ds + \int_{\Omega_2} |\hat{u}(s)|^2 ds < \infty.$$

Thus $\hat{u} \in L^2(0,\infty;\mathbb{R})$. By Assumption 1.1, A generates a contraction semigroup $T(t)$ and $B \in \mathcal{L}(\mathbb{R}, X)$. Therefore, by (Curtain and Zwart, 2020, Thm. 6.4.4), we have that the system $(A - \kappa BB^*, B, B^*, 0)$ is input stable, i.e., there exists a constant $\beta > 0$ such that for all $t > 0$ and $\tilde{u} \in L^2(0,\infty;\mathbb{R})$, we have

$$\left\|\int_0^t T_{-\kappa BB^*}(t-s)B\tilde{u}(s)ds\right\|^2 \le \beta^2 \int_0^t \|\tilde{u}(s)\|^2 ds.$$

Moreover, by Assumption 1.1, $T_{-\kappa BB^*}(t)$ is strongly stable. Since $\hat{u} \in L^2(0,\infty;\mathbb{R})$, (Curtain and Zwart, 2020, Thm. 5.2.3) implies that $\xi(t) \to 0$ as $t \to \infty$.

Finally, using $\xi(t) = x(t) - \Pi v(t)$ and $B^*\Pi + F = 0$ from (6), we obtain

$$\begin{aligned}y(t) - y_{ref}(t) &= B^*x(t) - y_{ref}(t)\\ &= B^*(\xi(t) + \Pi v(t)) + Fv(t)\\ &= B^*\xi(t) + (B^*\Pi + F)v(t)\\ &= B^*\xi(t) \to 0\end{aligned}$$

as $t \to \infty$ which completes the proof.

Remark 3.2. From the proof of Theorem 3.1, we note that the control law (7) can be written as $u(t) = -\kappa B^*\xi(t) + \Gamma v(t)$. Now we can see that the system (1) asymptotically operates in the linear region of saturation since $\xi(t) \to 0$ as $t \to \infty$ and $\sup_{t\ge0}\|\Gamma v(t)\| \le 1-\delta$.

Based on (Byrnes et al., 2000, Sec. V), the solvability conditions for the regulator equations are given in the following lemma.

Lemma 3.3. (Byrnes et al., 2000, Sec. V). Let $A - \kappa BB^*$ generate a strongly stable contraction semigroup for any $\kappa > 0$. Assume that $\sigma(S) \subset i\mathbb{R}$ and S has no nontrivial Jordan blocks. Then the regulator equations (6) are solvable if and only if no eigenvalue of S coincides with a transmission zero of the system (1). In this case, Π and Γ are given by

$$\begin{aligned}\Pi\Phi_k &= (i\omega_k - A)^{-1}(B\Gamma + B_dE)\Phi_k\\ \Gamma\Phi_k &= -G(i\omega_k)^{-1}(B^*(i\omega_k - A)^{-1}B_dE + F)\Phi_k,\end{aligned}$$

$k = 1, 2, \cdots, q$, where $i\omega_k$ and Φ_k are the eigenvalues and the corresponding orthonormal eigenvectors of S, respectively and $G(\cdot) = B^*(\cdot I - A)^{-1}B$ is the transfer function of the system (1).

Corollary 3.4. Let the assumptions of Lemma 3.3 hold and no eigenvalue of S coincides with a transmission zero of the system (1). Let $i\omega_k, k = 1, 2, \cdots q$ be the eigenvalues and $\{\Phi_k\}_{k=1}^q$ be the corresponding orthonormal basis of S. Then for any $v_0 \in \mathcal{W}_0$, the control input (7) has the representation

$$\begin{aligned}u(t) = -\kappa y(t) &- \sum_{k=1}^q e^{i\omega_k t}\langle v_0, \Phi_k\rangle (\kappa + G(i\omega_k)^{-1})F\Phi_k\\ &- \sum_{k=1}^q e^{i\omega_k t}\langle v_0, \Phi_k\rangle G(i\omega_k)^{-1}B^*(i\omega_k - A)^{-1}B_dE\Phi_k.\end{aligned}$$

Remark 3.5. Since the expressions for Γ and Π use information from the exosystem and the exosystem is determined by the reference and disturbance signals, we can derive expressions for $\Gamma v(t)$ and $B^*\Pi v(t)$ in terms of frequency, phase and amplitude of the reference and disturbance signals. This is illustrated in the following.

For simplicity, we assume that $y_{ref}(t) = a\cos(\omega t + \varphi)$ and $w_d(t) \equiv 0$. Then the exosystem can be chosen as

$$\begin{aligned}&\dot{v}(t) = Sv(t),\ v(0) = v_0 = \begin{bmatrix}1\\0\end{bmatrix},\ S = \begin{bmatrix}0 & \omega\\ -\omega & 0\end{bmatrix},\\ &F = -a\left[\cos(\varphi)\ \sin(\varphi)\right],\ E = 0.\end{aligned}$$

Moreover, the eigenvalues of S are $\pm i\omega$ and the corresponding orthonormal eigenvectors of S are given by $\frac{1}{\sqrt{2}}\left\{\begin{bmatrix}1\\i\end{bmatrix}, \begin{bmatrix}1\\-i\end{bmatrix}\right\}$.

Now, substituting the above information and the expression for $\Gamma\Phi_k$ from Lemma 3.3 in

$$\Gamma v(t) = \sum_{k=1}^{2} e^{i\omega_k t} \langle v_0, \Phi_k \rangle \Gamma \Phi_k$$

we obtain $\Gamma v(t) = a|G(i\omega)^{-1}|\cos(\omega t + \varphi + \theta)$, where $\theta = \tan^{-1}(\beta/\alpha)$, $\alpha = \mathrm{Re}(G(i\omega)), \beta = -\mathrm{Im}(G(i\omega))$. Similarly, we obtain $B^*\Pi v(t) = a\cos(\omega t + \varphi)$.

Furthermore, the condition $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1 - \delta$ in Theorem 3.1 can be reformulated as $|aG(i\omega)^{-1}| \leq 1 - \delta$ and the control input (7) can be written as

$$\begin{aligned} u(t) = -\kappa y(t) + \kappa a \cos(\omega t + \varphi) \\ + a|G(i\omega)^{-1}|\cos(\omega t + \varphi + \theta). \end{aligned} \tag{14}$$

This implies that the above feedback law (14) solves the semi-global output regulation problem provided that the frequency ω from the reference signal satisfies $G(i\omega) \neq 0$ and $|aG(i\omega)^{-1}| \leq 1-\delta$. This shows that it is not necessary to formulate the exosystem in order to solve the semi-global output regulation problem.

4. NUMERICAL EXAMPLE

In this section, we illustrate our main results in Section 3 on a flexible satellite model that is composed of two symmetrical flexible solar panels and a center rigid body (Bontsema et al. (1988), He and Ge (2015)). Modeling the satellite panels as viscously damped Euler-Bernoulli beams of length 1, the satellite model that we consider is described by (Govindaraj et al. (2020))

$$\begin{gathered} \ddot{w}_l(\xi,t) + w_l''''(\xi,t) + 5\dot{w}_l(\xi,t) = 0,\ -1 < \xi < 0, t > 0, \\ \ddot{w}_r(\xi,t) + w_r''''(\xi,t) + 5\dot{w}_r(\xi,t) = 0,\ 0 < \xi < 1, t > 0, \\ \ddot{w}_c(t) = w_l'''(0,t) - w_r'''(0,t) + \phi(u(t)) + w_d(t), \\ \ddot{\theta}_c(t) = -w_l''(0,t) + w_r''(0,t), \\ w_l''(-1,t) = 0, \quad w_l'''(-1,t) = 0, \\ w_r''(1,t) = 0, \quad w_r'''(1,t) = 0, \\ \dot{w}_l(0,t) = \dot{w}_r(0,t) = \dot{w}_c(t), \\ \dot{w}_l'(0,t) = \dot{w}_r'(0,t) = \dot{\theta}_c(t), \\ y(t) = \dot{w}_c(t), \end{gathered} \tag{15}$$

where $w_l(\xi,t)$ and $w_r(\xi,t)$ are the transverse displacements of the left and the right beam, respectively, $\dot{w}_l(\xi,t)$ and $w_l'(\xi,t)$ denote time and spatial derivatives of $w_l(\xi,t)$, respectively, $w_c(t)$ and $\theta_c(t)$ are the linear and angular displacements of the rigid body, respectively, the function $\phi(u(t))$ is the saturated external control input defined in (2) and $w_d(t)$ is an external disturbance. Here $\dot{w}_c(t) = \dot{w}_l(\xi,t)|_{\xi=0} = \dot{w}_r(\xi,t)|_{\xi=0}$ and $\dot{\theta}_c(t) = \dot{w}_l'(\xi,t)|_{\xi=0} = \dot{w}_r'(\xi,t)|_{\xi=0}$ are linear and angular velocities of the rigid body, respectively.

As shown in Govindaraj et al. (2023), the satellite model (15) can be written in the form (1) and the operator A generates an exponentially stable contraction semigroup on the state space $X = L^2(-1,0;\mathbb{R}^2) \times L^2(0,1;\mathbb{R}^2) \times \mathbb{R}^2$. It can be also verified that $A - \kappa BB^*$ generates an exponentially stable contraction semigroup on X for any $\kappa > 0$ (Govindaraj et al., 2020, Sec. 3). This implies that Assumption 1.1(1) is satisfied.

Our goal is to track the reference signal $y_{ref}(t) = 0.09\sin(1.5t)$ and reject the disturbance $w_d(t) \equiv 0.08$. Motivated by this, we choose the exosystem

$$\dot{v}(t) = Sv(t), \quad v(0) = \begin{bmatrix} 0 \\ 0.09 \\ 0.08 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1.5 & 0 \\ -1.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $F = [1\ 0\ 0]$, $E = [0\ 0\ 1]$. The eigenvalues of S are given by $\{0, \pm 1.5i\}$ and therefore, Assumption 1.1(2) is satisfied. Moreover, it can be verified that the system (1) does not have any transmission zeros at 0, $1.5i$ and $-1.5i$ (Govindaraj et al., 2023, Lem. 4.1) implying that the regulator equations are solvable.

The control input from Section 3 is given by $u(t) = -\kappa y(t) + (\kappa B^*\Pi + \Gamma)v(t)$. The control parameters Γ and Π can be obtained by using Lemma 3.3 as in Remark 3.5 and they are given by $\Gamma v(t) = 0.09|G(1.5i)^{-1}|\sin(1.5t + \theta) + 0.08$, $\theta = \tan^{-1}(\beta/\alpha)$, $\alpha = \mathrm{Re}(G(i\omega)), \beta = -\mathrm{Im}(G(i\omega))$ and $B^*\Pi v(t) = 0.09\sin(1.5t)$ where $G(\cdot)$ is the transfer function of the satellite system (A, B, B^*). Simulations are carried out in Matlab with $\kappa = 100$ on the time interval $[0, 15]$. The solutions of the satellite system (15) are approximated by using Legendre Spectral Galerkin method with number of basis functions $N = 10$ Asti (2020). Figure 1 shows that after the transient period the controller operates in the linear region of the saturation function and $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1 - \delta$. The output tracking and the tracking error are depicted in Figures 2 and 3 respectively and the velocity profile of the right solar panel is depicted in Figure 4.

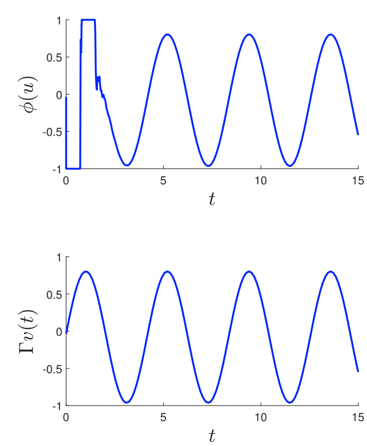

Fig. 1. Behaviour of saturated control input $\phi(u)$ (above) and $\Gamma v(t)$ (below) over the time interval [0,15]

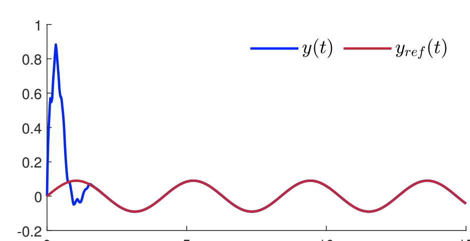

Fig. 2. Output tracking

5. CONCLUSION

We considered output regulation problem for the class of strongly stabilizable infinite-dimensional linear systems

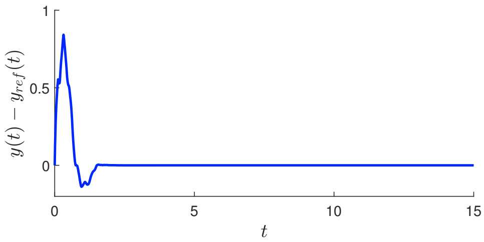

Fig. 3. Tracking error

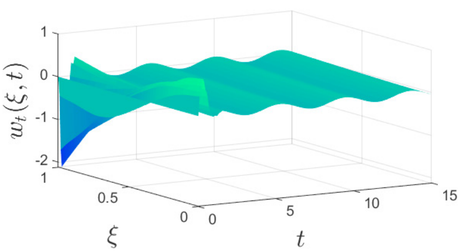

Fig. 4. Velocity profile of the right solar panel

with collocated actuators and sensors subject to input saturation. Strong stabilization of the system enabled us to construct a linear feedback control law that solves the semi-global output regulation problem. The results were illustrated on a flexible satellite model subject to input saturation where output tracking of a given sinusoidal reference signal and rejection of a constant disturbance signal were achieved by using the proposed control law.

Many future research directions are possible. In this work, we considered a particular class of infinite-dimensional systems with bounded input and output operators. So, the theory can be developed for wider class of systems, for example, for the systems with unbounded input and output operators and multi-input multi-output systems.

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