Saturating Integral Control for Infinite-Dimensional Linear Systems

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Abstract— **We propose a saturating integrator based controller for infinite dimensional well-posed linear systems, which prevents the controller state from leaving a desired closed interval. This set usually represents actuator constraints or safety requirements. We use Lyapunov theory to prove closed-loop stability and tracking of a constant reference for a suitable set of feasible references. The performance of the proposed controller is showcased through an application: the boundary control of a string equation with localized viscous damping.**

I. Introduction

A classical result in control theory is the *internal model principle*, originally formulated for linear finite-dimensional systems in [5], [9]. This principle states that, under reasonable assumptions on the plant, the reference tracking and disturbance rejection problem, i.e., the *output regulation problem*, can be solved by including a copy of the exosystem in the controller. Similar results have been later derived in [16] for nonlinear finite-dimensional systems, and in [12], [27], [26], [29] for linear infinite-dimensional systems. As a particular application of the internal model principle, an integrator should be included in the controller when the exogenous signals (reference and disturbances) are constant.

Sufficient conditions for solving the regulator problem for finite-dimensional linear systems with constant signals were given in [23]. In particular, it was shown that if the plant is stable and its DC-gain is positive, then a (positive) small-gain integral controller robustly solves the regulator problem [23, Thm. 10]. This result has been extended to nonlinear systems in [7], where the positivity of the plant DC-gain is replaced with the monotonicity of the plant steady-state input-output map [7, Thm. 3.1]. Recently, the result from [7] has been further generalized in [30]: instead of demanding the monotonicity of the plant steady-state input-output map, the reduced dynamics are required to be infinitesimally contracting. Related results for linear systems with input nonlinearities are in [11], [18]. The theory from [23] has been extended to infinite-dimensional linear systems in [17], [28], [35] and others, while a novel integral controller for nonlinear infinite-dimensional systems using forwarding control [22] can be found in [39].

In practical applications, integral controllers can be severely affected by the *windup* phenomenon [1], which

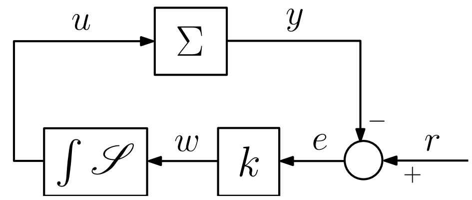

Fig. 1. The closed-loop system. Here Σ is an exponentially stable well-posed linear system, k is the constant control gain, and $\int \mathscr{S}$ is the saturating integrator from (5). The control objective is to drive the tracking error e to zero for a feasible constant reference $r \in \mathbb{R}$ (see Sect. II for details).

can drive the closed-loop system unstable or create large transients. Due to its relevance, many anti-windup techniques have been proposed in the control literature. When the plant is linear with a nonlinear actuator, LMI tools can be used to design an anti-windup compensator [15], [34], [43], which ensures that closed-loop stability is preserved. In the case of nonlinear systems, it is usually preferred to include the saturation constraints in the controller design, without the need of designing an additional compensator (this approach is sometimes referred to as *one-step* anti-windup [34]). The *bounded integral controller* (BIC) designed in [14] guarantees boundedness of the integrator state trajectories and input-to-state practical stability when the plant is input-to-state practically stable. A multi-input multi-output (MIMO) extension of the BIC was presented in [40]. A *saturating integrator* was proposed in [21] for stable nonlinear single-input single-output (SISO) systems. This controller constrains the integrator state to a desired set (thus preventing windup). The saturating integrator was generalized for MIMO systems in [20] using tools from projected dynamical systems [24]. Both [20] and [21] rely on singular perturbation tools [13, Ch. 11] to prove closed-loop stability, in the spirit of [7], [30]. A similar nonlinear integral controller, called *hybrid integrator gain*, has been proposed in [6] for SISO linear systems. This controller exploits projection techniques to keep its input-output relation constrained in a bounded sector, improving controller performance and preventing windup.

In this paper, we solve the *output tracking problem* for an exponentially stable well-posed linear system and a constant reference $r \in \mathbb{R}$ under *input constraints*, which restrict the plant input values $u(t) \in \mathbb{R}$ in a desired interval, namely, $u(t) \in U_0 = [u_{min}, u_{max}] \subset \mathbb{R}$ for all $t \geq 0$. Such constraints arise frequently as a result of actuators with limited ranges of operation, or may be imposed due to safety concerns. Our controller is based on a *saturating integrator*, see Fig. 1, which has previously been used in the control of nonlinear finite-dimensional systems in [20], [21], [25]. As our main result we show that the saturating integrator

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solves the output tracking problem with exponential rate of convergence of the tracking error, and achieves global exponential stability for the closed-loop system. Our proof is based on Lyapunov stability analysis of the infinite-dimensional nonlinear closed-loop system in Figure 1.

The paper is organized as follows. In Sec. II we recall some background on well-posed systems and state the control objective in precise terms. In Sec. III we prove the well-posedness of the closed-loop system from Fig. 1. In Sec. IV we state and we prove our main result. Finally, in Sec. V we apply our results to the boundary control of a string equation with localized viscous damping.

II. Background and Problem Statement

All the vector spaces in this paper are real. We denote by $\mathcal{L}(X,Y)$ the space of linear bounded operators from the Hilbert space X to the Hilbert space Y, and by $\rho(A)$ the resolvent set of an operator A. For any interval J and any Hilbert space U, $\mathcal{H}^1(J;U)$ denotes the Sobolev space of functions in $L^2(J;U)$ that are integrals of functions in $L^2(J;U)$. We denote by $L^2_{loc}([0,\infty);U)$ the space of functions $u:[0,\infty)\to U$ whose restriction to $[0,\tau]$ is in $L^2([0,\tau];U)$, for any $\tau\geq 0$. The space $\mathcal{H}^1_{loc}((0,\infty);U)$ consists of integrals of functions in $L^2_{loc}([0,\infty);U)$.

We recall some simple facts about well–posed linear systems, following [31], [38], [41]. Let us denote by U the *input space*, by X the *state space* and by Y the *output space* of a well-posed linear system Σ (these are Hilbert spaces). The input and the output functions are locally L^2 functions with values in U and Y, respectively. For such a function u, we denote by $\mathbf{P}_\tau u$ its truncation to the interval $[0,\tau]$.

A well-posed linear system Σ consists of four families of bounded linear operators $\Sigma=(\Sigma_\tau)_{\tau\geq 0}=(\mathbb{T},\Phi,\Psi,\mathbb{F})$ such that

$$\begin{bmatrix} x(\tau)\\ \mathbf{P}_\tau y\end{bmatrix}=\begin{bmatrix}\mathbb{T}_\tau & \Phi_\tau\\ \Psi_\tau & \mathbb{F}_\tau\end{bmatrix}\cdot\begin{bmatrix} x(0)\\ \mathbf{P}_\tau u\end{bmatrix}. \tag{1}$$

Here $x:[0,\infty)\to X$ is the *state trajectory* of Σ corresponding to the initial state $x(0)$ and the input function u, and y is the corresponding output function.

The above families of operators must satisfy functional equations expressing the causality and the time-invariance of Σ (these functional equations are parts of the definition of a well-posed system), see for instance [41]. In particular, the family $(\mathbb{T}_\tau)_{\tau\geq 0}$ is a strongly continuous operator semigroup on X and its generator A is called the *semigroup generator* of Σ. We introduce $X_1=\mathcal{D}(A)$ with the norm

$$\|x\|_1=\|(\beta I-A)x\|,\quad \beta\in\rho(A).$$

X_{-1} is the completion of X with respect to the norm $\|x\|_{-1}=\|(\beta I-A)^{-1}x\|$. These spaces are independent of the choice of β. A has a unique extension that is bounded from X to X_{-1}, and we denote this extension by the same symbol A. The semigroup $\mathbb{T}$ can be extended to an operator semigroup on X_{-1}, denoted by the same symbol, whose generator is the extension of A mentioned earlier. There exists a unique operator $B\in\mathcal{L}(U,X_{-1})$, called the *control operator* of Σ, such that for all $t\geq 0$,

$$\Phi_t u=\int_0^t \mathbb{T}_{t-\sigma}Bu(\sigma)\mathrm{d}\sigma \qquad \forall u\in L^2([0,\infty);U).$$

The above integration is done in X_{-1}. There exists a unique *observation operator* $C\in\mathcal{L}(X_1,Y)$ so that for every $\tau\geq 0$,

$$(\Psi_\tau x_0)(t)=C\mathbb{T}_t x_0 \qquad \forall x_0\in\mathcal{D}(A),\ t\in[0,\tau].$$

The well-posedness of Σ implies that B is an admissible control operator for $\mathbb{T}$, and that C is an admissible observation operator [37, Ch. 4], [41].

The operator C has an extension $\bar{C}$ to the space

$$Z=\mathcal{D}(A)+(\beta I-A)^{-1}BU. \tag{2}$$

This is a Hilbert space with the norm

$$\|z\|_Z^2=\inf\left\{\|x\|_1^2+\|v\|^2 \;\middle|\; \begin{array}{c} x\in X_1,\quad v\in U,\\ z=x+(\beta I-A)^{-1}Bv\end{array}\right\},$$

and $\bar{C}\in\mathcal{L}(Z,Y)$. The extension $\bar{C}$ may be not unique. For each such extension $\bar{C}$, there exists $D\in\mathcal{L}(U,Y)$ such that the transfer function $\mathbf{G}$ of Σ is given by

$$\mathbf{G}(s)=\bar{C}(sI-A)^{-1}B+D \quad \forall s\in\mathbb{C},\ \mathrm{Re}\,s>\omega_0(\mathbb{T}), \tag{3}$$

where $\omega_0(\mathbb{T})$ denotes the growth bound of $\mathbb{T}$. The following proposition is contained in [33, Thm. 3.1].

Proposition 2.1: Assume that $u\in\mathcal{H}^1_{loc}((0,\infty);U)$ and $x_0\in X$ are such that $Ax_0+Bu(0)\in X$. The state trajectory x and the output function y of Σ are defined as in (1). Then

$$x\in C^1([0,\infty);X),\quad Ax+Bu\in C([0,\infty);X),$$
$$y\in\mathcal{H}^1_{loc}((0,\infty);Y),$$

and for every $t\geq 0$ we have that

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=Ax(t)+Bu(t),\quad y(t)=\bar{C}x(t)+Du(t). \tag{4}$$

Throughout the paper we consider a SISO well-posed linear system $\Sigma=(\mathbb{T},\Phi,\Psi,\mathbb{F})$ described by (4), with $A:\mathcal{D}(A)\subset X\to X$ generating the strongly continuous semigroup $\mathbb{T}$ on the Hilbert space X, and with $B\in\mathcal{L}(\mathbb{R},X_{-1})$, $\bar{C}\in\mathcal{L}(Z,\mathbb{R})$ an extension of $C\in\mathcal{L}(X_1,\mathbb{R})$, and $D\in\mathbb{R}$. We assume that for practical reasons (such as actuators limitations) the input u must be constrained in $U_0=[u_{min},u_{max}]\subset\mathbb{R}$ with $u_{min}<u_{max}$.

Our goal is to design a control law so that the output y of the system Σ converges to a desired constant reference $r\in\mathbb{R}$ as $t\to\infty$. For this purpose, we introduce the *saturating integrator* from [21, Sec. II], defined by $\dot{u}=\mathscr{S}(u,w)$, where

$$\mathscr{S}(u,w)=\begin{cases}\max\{w,0\} & \text{if } u\leq u_{min},\\ w & \text{if } u\in(u_{min},u_{max}),\\ \min\{w,0\} & \text{if } u\geq u_{max}.\end{cases} \tag{5}$$

We use the controller (see Fig. 1)

$$\dot{u}=\mathscr{S}(u,k(r-y)), \tag{6}$$

with $k>0$. The *closed-loop system* consisting of (4) and (6), shown in Fig. 1, is given by

$$\dot{x}=Ax+Bu,\qquad x(0)=x_0\in X, \tag{7a}$$
$$y=\bar{C}x+Du, \tag{7b}$$
$$\dot{u}=\mathscr{S}(u,k(r-y)),\quad u(0)=u_0\in\mathbb{R}. \tag{7c}$$

Note that if (x^e, u^e) is an equilibrium state of (7), denoting $y^e = \bar{C}x^e + Du^e$, then $\mathscr{S}(u^e, k(r - y^e)) = 0$. In addition, if u^e is in the interior of U_0, then this implies $y^e = r$.

The control objective is to globally exponentially stabilize (7) and to solve the constrained tracking problem for all r in an interval $J \subset \mathbb{R}$ (to be defined) of feasible constant references. A solved constrained tracking problem means that for all $r \in J$, $\lim_{t\to\infty} y(t) = r$, while guaranteeing that if $u_0 \in U_0$, then $u(t) \in U_0$ for all $t \geq 0$.

We remark that in our main result (in Sect. IV) we allow also $u_0 \in \mathbb{R} \setminus U_0$. However, in this case, the control input $u(t)$ may or may not enter U_0 in finite time, depending on r. More details on this are given in Remark 4.5.

III. Existence of closed-loop trajectories

For a given $u_0 \in \mathbb{R}$, we define *the input to output map* $\mathbb{S}_\tau^{u_0}$ *of* (6) *corresponding to the fixed initial state* u_0 such that $u = \mathbb{S}_\tau^{u_0} w$ for a polynomial $w \in P[0,\tau]$ if u is the solution of (6) on $[0,\tau]$ with the initial condition $u(0) = u_0$. (The operator $\mathbb{S}_\tau^{u_0}$ is well-defined for a polynomial w, as explained in [21, Sec. II].) The following lemma shows that $\mathbb{S}_\tau^{u_0}$ extends to a globally Lipschitz mapping $\mathbb{S}_\tau^{u_0} : L^2[0,\tau] \to \mathcal{H}^1(0,\tau)$.

Lemma 3.1: Let $u_0 \in \mathbb{R}$ and $\tau > 0$. The mapping $\mathbb{S}_\tau^{u_0}$ has a unique continuous extension $\mathbb{S}_\tau^{u_0} : L^2[0,\tau] \to \mathcal{H}^1(0,\tau)$. Moreover, for all $w_1, w_2 \in L^2[0,\tau]$ we have

$$\|\mathbb{S}_\tau^{u_0}(w_2) - \mathbb{S}_\tau^{u_0}(w_1)\|_{L^2[0,\tau]} \leq \tau \|w_2 - w_1\|_{L^2[0,\tau]} \quad (8)$$

$$\|\mathbb{S}_\tau^{u_0}(w_2) - \mathbb{S}_\tau^{u_0}(w_1)\|_{C[0,\tau]} \leq \sqrt{\tau} \|w_2 - w_1\|_{L^2[0,\tau]}. \quad (9)$$

For any $w \in L^2[0,\tau]$, on u_0 and $\mathbb{S}_\tau^{u_0}(w) \in C[0,\tau]$ depends continuously on u_0 and $u = \mathbb{S}_\tau^{u_0}(w)$ satisfies (a.e. on $[0,\tau]$)

$$\dot{u} = \mathscr{S}(u, w), \qquad u(0) = u_0. \quad (10)$$

Proof: Let $u_0 \in \mathbb{R}$ and $\tau > 0$. Let w_1 and w_2 be polynomials and define $u_1 = \mathbb{S}_\tau^{u_0}(w_1)$ and $u_2 = \mathbb{S}_\tau^{u_0}(w_2)$. Similarly as in [42, Sec. 1], straightforward estimates show

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_2(t) - u_1(t)| \leq |w_2(t) - w_1(t)|$$

for almost every $t \in [0,\tau]$. Note in particular that u_1 and u_2 are piecewise polynomials, and thus $u_1(t) = u_2(t)$ can hold only for t in a set consisting of a finite number of points and a finite number of subintervals of $[0,\tau]$. Integrating this inequality and using $u_1(0) = u_2(0)$ leads to

$$|u_2(t) - u_1(t)| \leq |u_2(0) - u_1(0)| + \int_0^t |w_2(s) - w_1(s)| \mathrm{d}s \leq \sqrt{t} \|w_2 - w_1\|_{L^2[0,\tau]}.$$

Thus (8) and (9) hold for polynomials, and since the set of polynomials is dense in $L^2[0,\tau]$, $\mathbb{S}_\tau^{u_0}$ has a unique continuous extension $\mathbb{S}_\tau^{u_0} : L^2[0,\tau] \to L^2[0,\tau]$ satisfying (8) and (9).

To show that $\mathbb{S}_\tau^{u_0}$ maps into $\mathcal{H}^1(0,\tau)$, let $w \in L^2[0,\tau]$, define $u = \mathbb{S}_\tau^{u_0}(w)$ and let $(w_n)_n$ be a sequence of polynomials such that $w_n \to w$ in $L^2[0,\tau]$. Denoting $u_n = \mathbb{S}_\tau^{u_0}(w_n)$ we have $|\dot{u}_n(t)| = |\mathscr{S}(u_n(t), w_n(t))| \leq |w_n(t)|$ for a.e. $t \in [0,\tau]$. Thus $(u_n)_n$ is a bounded sequence in $\mathcal{H}^1(0,\tau)$ and has a subsequence which converges weakly in $\mathcal{H}^1(0,\tau)$. This limit is u due to uniqueness, and thus $u \in \mathcal{H}^1(0,\tau)$.

Finally, for a polynomial w the continuity $\mathbb{R} \ni u_0 \mapsto \mathbb{S}_\tau^{u_0}(w) \in C[0,\tau]$ and $(\mathbb{S}_\tau^{u_0}(w))(0) = u_0$ follow from the above pointwise estimate $|u_2(t) - u_1(t)| \leq |u_2(0) - u_1(0)|$, and the density of polynomials implies these properties also for $w \in L^2[0,\tau]$. We omit the nontrivial proof of (10). ■

Let us denote by $\Psi_\tau : X \mapsto L^2[0,\tau]$ the state to output map of Σ on $[0,\tau]$, and by $\mathbb{F}_\tau : L^2[0,\tau] \mapsto L^2[0,\tau]$ the input to output map of Σ on $[0,\tau]$, as in (1). Since Σ is well-posed, these operators are bounded.

Proposition 3.2: Assume that Σ is well-posed and let $k > 0$. Then for any constant reference $r \in \mathbb{R}$ and initial state $(x_0, u_0) \in X \times \mathbb{R}$ there exists a *generalized closed-loop state trajectory* of the closed-loop system (7), by which we mean a function $(x, u) \in C([0,\infty); X \times \mathbb{R})$ such that $u \in \mathcal{H}^1_{loc}(0,\infty)$ and for every $\tau \geq 0$ we have

$$x(t) = \mathbb{T}_t x_0 + \Phi_t u, \qquad t \geq 0 \quad (11a)$$

$$\mathbf{P}_\tau u = \mathbb{S}_\tau^{u_0}(k(r - \Psi_\tau x_0 - \mathbb{F}_\tau u)). \quad (11b)$$

Moreover, for any $t \geq 0$, *the state* $(x(t), u(t))$ of the closed-loop system depends continuously on (x_0, u_0) and on r.

If $(x_0, u_0) \in X \times \mathbb{R}$ are such that $Ax_0 + Bu_0 \in X$, then for any $r \in \mathbb{R}$ the closed-loop state trajectory (x, u) satisfies $x \in C^1([0,\infty); X)$ and $Ax + Bu \in C([0,\infty); X)$, and (7a)–(7b) hold pointwise on $[0,\infty)$. Lemma 3.1 implies that (7c) holds pointwise a.e. on $[0,\infty)$.

Remark 3.3: It is clear from (10) and (11a) that if (x, u) is a generalized closed-loop state trajectory of (7), corresponding to an initial state (x_0, u_0), then

$$x(0) = x_0 \qquad \text{and} \qquad u(0) = u_0.$$

Proof: Let the initial state $(x_0, u_0) \in X \times \mathbb{R}$ and the reference $r \in \mathbb{R}$ be arbitrary. We rewrite (11b) in terms of the error $e = r - y$ truncated to $[0,\tau]$, as follows:

$$\mathbf{P}_\tau e = r - \Psi_\tau x_0 - \mathbb{F}_\tau \mathbb{S}_\tau^{u_0}(k \mathbf{P}_\tau e). \quad (12)$$

This is a fixed point equation in $L^2[0,\tau]$ for the operator

$$T(\nu) = r - \Psi_\tau x_0 - \mathbb{F}_\tau \mathbb{S}_\tau^{u_0}(k\nu).$$

Lemma 3.1 implies that T is a strict contraction for $\tau > 0$ small enough. Indeed, for any $\nu_1, \nu_2 \in L^2[0,\tau]$,

$$\|T(\nu_1) - T(\nu_2)\| \leq \|\mathbb{F}_\tau\| \cdot \|\mathbb{S}_\tau^{u_0}(k\nu_1) - \mathbb{S}_\tau^{u_0}(k\nu_2)\| \leq \tau k \|\mathbb{F}_\tau\| \cdot \|\nu_1 - \nu_2\|.$$

For any $\tau \in [0,1]$ and any $u \in L^2[0,1]$, $\mathbb{F}_\tau u$ is obtained from $\mathbb{F}_1 u$ by a projection (truncation to $[0,\tau]$), so that clearly $\|\mathbb{F}_\tau\| \leq \|\mathbb{F}_1\|$. Thus, for sufficiently small $\tau > 0$, we have $\tau k \|\mathbb{F}_\tau\| < 1$. With this choice of $\tau > 0$, T is a strict contraction on $L^2[0,\tau]$, and therefore it has a unique fixed point $\mathbf{P}_\tau e \in L^2[0,\tau]$ which satisfies (12), see for instance [2, Thm. 3.1]. Moreover, $\mathbf{P}_\tau e$ depends continuously on (x_0, u_0, r), according to [2, Thm. 3.8]. We define $\mathbf{P}_\tau u = \mathbb{S}_\tau^{u_0}(k \mathbf{P}_\tau e)$, then according to Lemma 3.1 we have $\mathbf{P}_\tau u \in \mathcal{H}^1(0,\tau)$.

If we define $x(t) = \mathbb{T}_t x_0 + \Phi_t \mathbf{P}\tau u$ for $t \in [0,\tau]$, then $x \in C([0,\tau]; X)$ and (x, u) satisfies (11) on $[0,\tau]$. Repeating the analysis on $[n\tau, (n+1)\tau]$ starting with the initial conditions

$(x(n\tau), u(n\tau))$ for $n \in \mathbb{N}$ and combining the resulting state trajectories leads to a closed-loop state trajectory $(x, u) \in C([0, \infty); X \times U)$ satisfying (11) and $u \in \mathcal{H}^1_{loc}(0, \infty)$.

Assume now that $(x_0, u_0) \in X \times \mathbb{R}$ satisfy $Ax_0 + Bu_0 \in X$. Since $u \in \mathcal{H}^1_{loc}(0, \infty)$ and $u(0) = u_0$, we have from Proposition 2.1 that x has the properties claimed. ∎

IV. The main result

The following assumption states the required stability properties for the infinite-dimensional system (4).

Assumption 4.1: We assume that there exists a quadratic and coercive Lyapunov function $W_0 : X \to [0, \infty)$ associated to the abstract Cauchy problem $\dot{x} = Ax$. That is, we assume there exists $\eta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} W_0(x(t)) \leq -\eta W_0(x(t)), \qquad t > 0,$$

for all solutions x of $\dot{x} = Ax$, $t \geq 0$, with $x(0) \in \mathcal{D}(A)$.

Remark 4.2: Assumption 4.1 implies that the semigroup $\mathbb{T}$ is exponentially stable. The assumption is in particular satisfied if $\mathbb{T}$ is exponentially stable and there exists an admissible observation operator C_0 for $\mathbb{T}$ such that (A, C_0) is exactly observable (in infinite time).

Here is a short proof of the above statement: If $\mathbb{T}$ is exponentially stable and (A, C_0) is exactly observable, then by [37, Sec. 5.1] the Lyapunov equation $A^*P_0 + P_0A = -C_0^*C_0$ has a unique self-adjoint and positive solution $P_0 > 0$, which is also strictly positive (or coercive) in the sense that $P_0 \geq \varepsilon_0 I$ for some $\varepsilon_0 > 0$ (which means that $\langle P_0 x, x\rangle \geq \varepsilon_0 \|x\|^2$ for all $x \in X$). Moreover, from exponentially stability, there exists a solution $P_1 \geq 0$ of the Lyapunov equation $A^*P_1 + P_1A = -I$. The operator $P_2 = P_0 + P_1$ is strictly positive since $P_2 \geq \varepsilon_0 I$, and defining $W_0(x) = \langle x, P_2 x\rangle$ we can see that W_0 is coercive and for any $x_0 \in \mathcal{D}(A)$, the state trajectory $x(t) = \mathbb{T}_t x_0$ satisfies

$$\begin{aligned}\dot{W}_0(x) &= 2\langle P_2 Ax, x\rangle = -\|x\|^2 - \|C_0 x\|^2 \leq -\|x\|^2 \\ &\leq -\eta W_0(x), \quad \text{due to the boundedness of } P_2.\end{aligned}$$

We make the following additional assumption that the "DC-gain" of the system (the value of the transfer function at zero) is positive.

Assumption 4.3: The transfer function $\mathbf{G}$ of Σ as in (3) satisfies $\mathbf{G}(0) > 0$.

Our results can be extended easily to the case $\mathbf{G}(0) < 0$, if we redefine the input $u(t)$ of our system as $-u(t)$.

We define $u_r = \mathbf{G}(0)^{-1} r$, which is the steady-state input of the system (1) corresponding to the output $y = r$. We also define $\Xi = -A^{-1}B \in \mathcal{L}(\mathbb{R}, X)$ and $x_r = \Xi u_r = -A^{-1}Bu_r \in Z$ (from (2)). Then $Ax_r + Bu_r = 0$ and $\mathscr{S}(u_r, r - (\bar{C}x_r + Du_r)) = \mathscr{S}(u_r, r - \mathbf{G}(0)u_r) = 0$, which implies that (x_r, u_r) is an equilibrium point of the system (7). (This is true also if $u_r \notin U_0$.)

Theorem 4.4: Suppose that Assumptions 4.1 and 4.3 hold. There exists $k^* > 0$ such that for all $0 < k < k^*$ and for all $r \in \mathbb{R}$ satisfying $\mathbf{G}(0)^{-1} r \in U_0$ the point $(x_r, u_r) \in X \times U_0$ is a globally exponentially stable equilibrium of the closed-loop system (7). Moreover, for every $k \in (0, k^*)$ there exists $\alpha > 0$ such that

$$\int_0^\infty e^{\alpha t} |r - y(t)|^2 \mathrm{d}t < \infty$$

for all initial states $(x_0, u_0) \in X \times \mathbb{R}$ and $r \in \mathbb{R}$ satisfying $\mathbf{G}(0)^{-1} r \in U_0$. Finally, if $Ax_0 + Bu_0 \in X$ then, in addition, we have that $e^{\alpha t}|r - y(t)| \to 0$ as $t \to \infty$.

Remark 4.5: Note that if $u_0 \notin U_0$ and $\mathbf{G}(0)^{-1} r \in \operatorname{int} U_0$, then $u(t)$ will reach (and remain in) U_0 after some finite time. However, if $u_0 \notin U_0$ and $\mathbf{G}(0)^{-1} r$ is on the boundary of U_0, then this may not be the case. Clearly, if $u_0 \in U_0$, then $u(t) \in U_0$ for all $t \geq 0$, independently of $r \in \mathbb{R}$.

To prove Theorem 4.4 we need the following lemma.

Lemma 4.6: Suppose Assumption 4.3 holds and let $u_r \in U_0 = [u_{min}, u_{max}]$ and $u, y \in \mathbb{R}$. Then the function $\mathscr{S}$ satisfies $|\mathscr{S}(u, y)| \leq |y|$ and

$$\begin{aligned}&(u - u_r)\mathscr{S}(u, \mathbf{G}(0)(u_r - u) - y) \\ &\qquad \leq -\mathbf{G}(0)(u - u_r)^2 + |u - u_r||y|.\end{aligned}$$

Proof: The estimate $|\mathscr{S}(u, y)| \leq |y|$ follows immediately from the definition $\mathscr{S}$ in (5). If $u \in (u_{min}, u_{max})$, then $\mathscr{S}(u, \mathbf{G}(0)(u_r - u) - y) = \mathbf{G}(0)(u_r - u) - y$, and this implies the second estimate. Assume now that $u \leq u_{min}$. Then $u_r \in [u_{min}, u_{max}]$ implies that $u_r - u \geq 0$, and thus

$$\begin{aligned}&(u - u_r)\mathscr{S}(u, \mathbf{G}(0)(u_r - u) - y) \\ &\quad = (u - u_r)\max\{\mathbf{G}(0)(u_r - u) - y, 0\} \\ &\quad = \min\{-\mathbf{G}(0)(u - u_r)^2 + (u_r - u)y, 0\} \\ &\quad = \begin{cases} 0 & \text{if } y \geq \mathbf{G}(0)(u_r - u) \\ -\mathbf{G}(0)(u - u_r)^2 + (u_r - u)y & \text{otherwise} \end{cases} \\ &\quad \leq -\mathbf{G}(0)(u - u_r)^2 + |u - u_r||y|,\end{aligned}$$

since $0 \leq -\mathbf{G}(0)(u - u_r)^2 + (u_r - u)y$ and $(u_r - u)y = |u - u_r||y|$ whenever $y \geq \mathbf{G}(0)(u_r - u) \geq 0$. The case where $u \geq u_{max}$ can be analysed analogously to confirm that the desired estimate also holds in this case. ∎

Proof of Theorem 4.4. Assumption 4.1 implies that there exists a bounded and self-adjoint operator $P_0 \in \mathcal{L}(X)$ which is also strictly positive, i.e., $P_0 \geq \varepsilon_0 I$ for some $\varepsilon_0 > 0$, such that $2\langle P_0 z, Az\rangle \leq -\|z\|^2$ for all $z \in \mathcal{D}(A)$. Since the semigroup $\mathbb{T}_t$ generated by A is exponentially stable and $C \in \mathcal{L}(X_1, \mathbb{R})$ is assumed to be admissible, C is also *infinite-time admissible* [37, Sec. 4.6]. By [37, Thm. 5.1.1] there exists a non-negative operator $P_1 \in \mathcal{L}(X)$ such that $2\langle P_1 z, Az\rangle = -(Cz)^2$ for all $z \in \mathcal{D}(A)$. Thus if we define $P = P_0 + P_1 \in \mathcal{L}(X)$, then P is self-adjoint, strictly positive, and

$$2\langle Pz, Az\rangle \leq -\|z\|^2 - (Cz)^2, \qquad \forall z \in \mathcal{D}(A).$$

In order to prove that state trajectories (x, u) of the closed-loop system (7) converge to (x_r, u_r) with $u_r = \mathbf{G}(0)^{-1} r$ and $x_r = \Xi u_r$, we define the quadratic Lyapunov function candidate $\nu : X \times \mathbb{R} \to \mathbb{R}$ by

$$\nu(x, u) := \langle P(x - \Xi u), x - \Xi u\rangle + \frac{1}{2}(u - u_r)^2.$$

We consider initial conditions $x_0 \in X$ and $u_0 \in \mathbb{R}$ of the closed-loop system (7) satisfying $Ax_0 + Bu_0 \in X$.

Under this assumption, Proposition 3.2 implies that the corresponding closed-loop solution satisfies $x \in C^1([0,\infty);X)$ and $u \in \mathcal{H}^1_{loc}(0,\infty)$, and $Ax(t)+Bu(t) \in X$ for $t \geq 0$. Moreover, $x(t) \in Z$ (from (2)) and the closed-loop equations (7) are satisfied pointwise for almost every $t \geq 0$.

We denote $z(t) = x(t) - \Xi u(t)$ for $t \geq 0$ for brevity. Since $\Xi = -A^{-1}B$, for all $t \geq 0$ the values $z(t) \in Z$ satisfy $Az(t) = Ax(t) - A\Xi u(t) = Ax(t) + Bu(t) \in X$, which implies that $z(t) \in \mathcal{D}(A)$ for all $t \geq 0$. In particular, we also have $\bar{C}z(t) = Cz(t)$ for $t \geq 0$ and $y = \bar{C}x + Du = Cz + (\bar{C}\Xi + D)u = Cz + \mathbf{G}(0)u$. Recalling that $r = \mathbf{G}(0)u_r$, we have $r - y = \mathbf{G}(0)(u_r - u) - Cz$ and thus for a.e. $t \geq 0$,

$$\begin{aligned}\dot{z} &= \dot{x} - \Xi\dot{u} = Ax + Bu - \Xi\mathscr{S}(u, k(r-y)) \\ &= Az - k\Xi\mathscr{S}(u, \mathbf{G}(0)(u_r - u) - Cz).\end{aligned}$$

Because of this, we have from Lemma 4.6 that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\langle Pz, z\rangle = 2\langle Pz, \dot{z}\rangle \\ &\leq 2\langle Pz, Az\rangle - 2k\langle Pz, \Xi\mathscr{S}(u, \mathbf{G}(0)(u_r - u) - Cz)\rangle) \\ &\leq -\|z\|^2 - (Cz)^2 \\ &\quad + 2k\|z\|\|P\|\|\Xi\||\mathscr{S}(u, \mathbf{G}(0)(u_r - u) - Cz|) \\ &\leq -\|z\|^2 - (Cz)^2 + 2k\|z\|\|P\|\|\Xi\|(\mathbf{G}(0)|u - u_r| + |Cz|).\end{aligned}$$

On the other hand, Lemma 4.6 also implies that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(u - u_r)^2 &= (u - u_r)\dot{u} = (u - u_r)\mathscr{S}(u, k(r-y)) \\ &= k(u - u_r)\mathscr{S}(u, \mathbf{G}(0)(u_r - u) - Cz) \\ &\leq -k\mathbf{G}(0)(u - u_r)^2 + k|u - u_r||Cz|.\end{aligned}$$

Combining the estimates, denoting $\alpha_1 = 2\|P\|\|\Xi\| \geq 0$ and $\alpha_2 = 2\mathbf{G}(0)\|P\|\|\Xi\| \geq 0$, and using Young's inequality we get for any fixed $\varepsilon > 0$ and for a.e. $t \geq 0$ that

$$\begin{aligned}&\frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\langle Pz, z\rangle + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(u - u_r)^2 \\ &\leq -\|z\|^2 - (Cz)^2 + k\alpha_1\|z\||Cz| + k\alpha_2\|z\||u - u_r| \\ &\quad - k\mathbf{G}(0)(u - u_r)^2 + k|u - u_r||Cz| \\ &\leq -\|z\|^2 - (Cz)^2 + \frac{k\alpha_1^2}{2}\|z\|^2 + \frac{k}{2}(Cz)^2 + \frac{k\varepsilon}{2}(u - u_r)^2 \\ &\quad + \frac{k\alpha_2^2}{2\varepsilon}\|z\|^2 - k\mathbf{G}(0)(u - u_r)^2 + \frac{k\varepsilon}{2}(u - u_r)^2 + \frac{k}{2\varepsilon}(Cz)^2 \\ &= -\left[1 - \frac{k(\alpha_1^2\varepsilon + \alpha_2^2)}{2\varepsilon}\right]\|z\|^2 - \left[1 - \frac{k(1+\varepsilon)}{2\varepsilon}\right](Cz)^2 \\ &\quad - k\left(\mathbf{G}(0) - \varepsilon\right)(u - u_r)^2.\end{aligned}$$

Choosing $\varepsilon = \mathbf{G}(0)/2 \in (0, \mathbf{G}(0))$ and

$$\begin{aligned}k^* &= \min\left\{\frac{2\varepsilon}{\alpha_1^2\varepsilon + \alpha_2^2}, \frac{2\varepsilon}{1+\varepsilon}\right\} \\ &= \min\left\{\frac{1}{2\|P\|^2\|\Xi\|^2(1 + 2\mathbf{G}(0))}, \frac{2\mathbf{G}(0)}{2 + \mathbf{G}(0)}\right\},\end{aligned}$$

the above estimate shows that that for every fixed $k \in (0, k^*)$ there exists a constant $c > 0$ such that

$$\frac{\mathrm{d}\nu}{\mathrm{d}t} \leq -c\left[\|x - \Xi u\|^2 + (u - u_r)^2\right].$$

Since there exists $b_0 > 0$ such that $\nu(x,u) \geq b_0\|x - \Xi u\|^2 + b_0(u - u_r)^2$, the estimate $\|x - \Xi u_r\| \leq \|x - \Xi u\| + \|\Xi\||u - u_r|$ and a standard argument using Gronwall's Lemma shows that there exists a constant $\beta > 0$ such that for all $t \geq 0$ we have

$$\begin{aligned}&\|x(t) - \Xi u_r\|^2 + (u(t) - u_r)^2 \\ &\qquad \leq e^{-\beta t}\left(\|x_0 - \Xi u_r\|^2 + (u_0 - u_r)^2\right).\end{aligned}$$

Proposition 3.2 shows that the state trajectory $(x(t), u(t))$ depends continuously on (x_0, u_0) and, as shown in [33, Sec. 3], the space $\{\,(x_0, u_0) \in Z \times \mathbb{R} \mid Ax_0 + Bu_0 \in X\,\}$ is dense in $X \times \mathbb{R}$. These two properties imply that the above estimate also holds for all generalised closed-loop state trajectories (x, u) corresponding to initial states $x_0 \in X$ and $u_0 \in \mathbb{R}$. Because of this, (x_r, u_r) with $x_r = \Xi u_r$ is a globally exponentially stable equilibrium point of the closed-loop system (7).

It remains to prove the convergence of the outputs y to the reference r. Consider initial states $x_0 \in X$ and $u_0 \in \mathbb{R}$ of the closed-loop system. Due to global exponential stability of the closed-loop system, there exists $\beta_1 > 0$ (independent of x_0 and u_0) such that $e^{\beta_1 \cdot}(u - u_r) \in L^2[0,\infty)$ (where $e^{\beta \cdot}$ is the function $h(t) = e^{\beta t}$ with $\beta \in \mathbb{R}$ and $t \geq 0$). Since $x_r = \Xi u_r \in Z$ satisfies $Ax_r + Bu_r = A(-A^{-1}Bu_r) + Bu_r = 0$ and $\bar{C}x_r + Du_r = \bar{C}(-A^{-1}Bu_r) + Du_r = \mathbf{G}(0)u_r = r$, we have that the constant functions x_r and r are the state trajectory and output, respectively, of the well-posed system Σ corresponding to the initial state $x_r \in Z$ and input u_r. If we define $\tilde{x} = x - x_r$ and $\tilde{u} = u - u_r$, then linearity implies that $\tilde{x}$ and $y - r$ are the generalized state trajectory and output of Σ corresponding to the initial state $\tilde{x}(0) = x_0 - x_r$ and input $u - u_r$. Denote by $\omega_0(\mathbb{T}) < 0$ the growth bound of $\mathbb{T}$. Let $\alpha > 0$ be such that $0 < \alpha < \min\{\beta_1, -\omega_0(\mathbb{T})\}$. Since $e^{\alpha \cdot}(u - u_r) \in L^2[0,\infty)$, we have $e^{\alpha \cdot}(y - r) \in L^2[0,\infty)$ by [32, Thm. 2.5.4(ii–iii)], and this implies the first claim concerning the output. Finally, if $x_0 \in X$ and $u_0 \in \mathbb{R}$ are such that $Ax_0 + Bu_0 \in X$, then also $A\tilde{x}(0) + B\tilde{u}(0) \in X$. Moreover, $\dot{\tilde{u}} = \dot{u} = \mathscr{S}(u, k(r-y))$ together with the estimate $|\mathscr{S}(u, k(r-y))| \leq k|y - r|$ in Lemma 4.6 imply that $e^{\alpha \cdot}\dot{\tilde{u}} \in L^2[0,\infty)$. Thus $e^{\alpha \cdot}\tilde{u} = e^{\alpha \cdot}(u - u_r) \in \mathcal{H}^1(0,\infty)$ and we have from [32, Thm. 4.6.11(i)] that $e^{\alpha \cdot}(y - r) \in \mathcal{H}^1(0,\infty)$. Barbalat's Lemma [8, Thm. 5] then implies $e^{\alpha t}(y(t) - r) \to 0$ as $t \to \infty$. ■

V. Boundary control of a damped string

In this section we use the proposed saturating integrator for boundary control of a string equation with viscous damping.

Statement of the control problem. Consider a string equation with viscous damping on the space interval $[0,1]$. A force u is applied to the left end of the string, and the right end is fixed. Then the transverse deflection w satisfies

$$\left\{\begin{array}{l}\frac{\partial^2}{\partial t^2}w(\xi,t) = \frac{\partial^2}{\partial \xi^2}w(\xi,t) - \alpha(\xi)\frac{\partial}{\partial t}w(\xi,t), \\ \frac{\partial}{\partial \xi}w(0,t) = u(t), \qquad w(1,t) = 0, \\ w(\xi,0) = f(\xi), \qquad \frac{\partial}{\partial t}w(\xi,0) = g(\xi),\end{array}\right. \tag{13}$$

where $\xi \in [0,1]$, with $\alpha \in C[0,1]$, $\alpha(\xi) \geq 0$, $\alpha(\xi) > 0$ at some points. We take the solution space

$$Z = Z_1 \times \mathcal{H}_R^1(0,1),$$

where

$$Z_1 = \mathcal{H}^2(0,1) \cap \mathcal{H}_R^1(0,1),$$
$$\mathcal{H}_R^1(0,1) = \left\{ \phi \in \mathcal{H}^1(0,1) \mid \phi(1) = 0 \right\}.$$

The norm on $\mathcal{H}_R^1(0,1)$ is $\|f\|_{\mathcal{H}^1} = \|f'\|_{L^2}$. We choose the state $x = \left[\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right]$, so that the state space is

$$X = \mathcal{H}_R^1(0,1) \times L^2[0,1]. \tag{14}$$

The control objective is to regulate the slope $\frac{\partial}{\partial \xi} w(\xi_0, t)$ at a fixed point $\xi_0 \in [0,1]$ to a desired constant $r \in \mathbb{R}$, using the boundary control $u(t)$. Due to physical constraints, the control input u must satisfy $u(t) \in U_0 = [u_{min}, u_{max}] \subset \mathbb{R}$, for all $t \geq 0$ provided that $u_0 \in U_0$. If $u_0 \notin U_0$, then the behavior of $u(t)$ is as described in Remark 4.5.

The operators A, B, C, and the well-posedness of Σ. We introduce the operator $A : \mathcal{D}(A) \to X$ by

$$\mathcal{D}(A) = H_1 \times \mathcal{H}_R^1(0,1),$$

where $H_1 = \left\{ f \in Z_1 \;\middle|\; \frac{\mathrm{d}f}{\mathrm{d}\xi}(0) = 0 \right\}$, with

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} - \alpha(\xi) g \end{bmatrix} \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A). \tag{15}$$

The operator A generates an exponentially stable semigroup $\mathbb{T}$ on X, as shown in [3, Ex. 1, Sec. 3].

For convenience, we introduce the skew-adjoint generator $A_{\rm sk}$ of the unitary semigroup $\mathbb{T}^{\rm sk}$ associated to the "undamped" version of the string equation (13), i.e., to (13) with $\alpha = 0$. This undamped wave equation is discussed in [37, Sec. 10.2.2]. Clearly $\mathcal{D}(A_{\rm sk}) = \mathcal{D}(A)$. In many arguments, we will replace A with $A_{\rm sk}$, using the fact that A is a bounded perturbation of $A_{\rm sk}$.

We introduce $A_0 : H_1 \to L^2[0,1]$, defined by

$$A_0 f = -\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} \qquad \forall\, f \in H_1,$$

and $N : \mathbb{R} \to \mathcal{H}_R^1(0,1)$, given by

$$(N\mathrm{v})(\xi) = (\xi - 1)\mathrm{v} \qquad \forall\, \xi \in [0,1].$$

The operator A_0 has a continuous extension $A_0 : L^2[0,1] \to H_{-1}$, where H_{-1} is the dual of H_1 with respect to the pivot space $L^2[0,1]$. According to [37, Prop. 10.2.3], the operator $B : \mathbb{R} \to X_{-1}$ given by

$$B\mathrm{v} = \begin{bmatrix} 0 \\ A_0 N \mathrm{v} \end{bmatrix} \qquad \forall\, \mathrm{v} \in \mathbb{R}, \tag{16}$$

is an admissible control operator for $\mathbb{T}^{\rm sk}$. It follows from [37, Cor. 5.5.1] that B is admissible also for $\mathbb{T}$. It is shown in [37, Sec. 10.2.2] that the system of equations (13) can be equivalently represented as $\dot{x} = Ax + Bu$.

We choose the operator $C : X_1 \to \mathbb{R}$ as

$$C \begin{bmatrix} f \\ g \end{bmatrix} = \frac{\mathrm{d}f}{\mathrm{d}\xi}(\xi_0) \qquad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X_1. \tag{17}$$

Since $Z \subset \mathcal{H}^2(0,1) \times \mathcal{H}_R^1(0,1)$, we can define the extension $\bar{C} \in \mathcal{L}(Z, \mathbb{R})$ with the same formula $\bar{C}\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = \frac{\mathrm{d}f}{\mathrm{d}\xi}(\xi_0)$ on Z. Lemma 5.5 shows that this extension corresponds to the feedthrough operator $D = 0$.

Lemma 5.1: C is an admissible output operator for $\mathbb{T}$.

Proof: We prove that C is an admissible output operator for $\mathbb{T}^{\rm sk}$. Then, it follows from [37, Thm. 5.4.2] that C is admissible also for $\mathbb{T}$. The operator $A_{\rm sk}$ is diagonalizable, since it admits a complete orthonormal set of eigenvectors

$$\varphi_k(\xi) = \frac{1}{\left(k + \frac{1}{2}\right)\pi} \begin{bmatrix} \cos\left(\left(k + \frac{1}{2}\right)\pi\xi\right) \\ i\left(k + \frac{1}{2}\right)\pi \cos\left(\left(k + \frac{1}{2}\right)\pi\xi\right) \end{bmatrix},$$

for $k \in \mathbb{Z}$, corresponding to the eigenvalues $i\lambda_k = i(k + 1/2)\pi$, see [3, Ex. 1, Sec. 3] for the details. These eigenvalues have a uniform gap, i.e., $|i\lambda_{k+1} - i\lambda_k| = \pi > 0$ for all $k \in \mathbb{Z}$. Since

$$|C\varphi_k| = \left| \cos\left(\left(k + \frac{1}{2} \right) \pi \xi_0 \right) \right| \leq 1$$

for all $k \in \mathbb{Z}$, we have from [37, Thm. 5.3.2] that C is admissible with respect to $\mathbb{T}^{\rm sk}$. ∎

Lemma 5.2: The triple of operators (A, B, C) from (15), (16), (17) is well-posed on $(\mathbb{R}, X, \mathbb{R})$.

Proof: According to [38, Prop. 4.9, Prop. 4.10], to prove the well-posedness of (A, B, C), it remains to show that a transfer function $\mathbf{G}$ associated to (A, B, C) is bounded on $l(\gamma) = \{s \in \mathbb{C} \mid \operatorname{Re} s = \gamma\}$, for some $\gamma > \omega_0(\mathbb{T})$. As in the proof of Lemma 5.1, we employ a bounded perturbation argument: we will prove the well-posedness for $(A_{\rm sk}, B, C)$ by proving the boundedness of the transfer function $s \mapsto \mathbf{G}_{\rm sk}(s) = \bar{C}(sI - A_{\rm sk})^{-1}B$ on $l(\gamma)$ and then the result will follow also for (A, B, C) due to [4, Thm. 4.2]. (The cited result in [4] refers to time-varying bounded perturbations, we are using here a particular case.)

Let $s \in \rho(A_{\rm sk})$ and $\mathrm{v} \in \mathbb{R}$. Using [37, Rem. 10.1.5], we can compute $z := (sI - A_{\rm sk})^{-1}B\mathrm{v} \in Z$ by solving

$$L_{\rm sk} z = sz, \qquad Gz = \mathrm{v}, \tag{18}$$

where $L_{\rm sk} \in \mathcal{L}(Z, X)$, $G \in \mathcal{L}(Z, \mathbb{R})$ are the boundary control description of (13) with $\alpha = 0$. We have

$$L_{\rm sk} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} \end{bmatrix}, \qquad G \begin{bmatrix} f \\ g \end{bmatrix} = \frac{\mathrm{d}f}{\mathrm{d}\xi}(0).$$

The unique solution of (18) has the form $z = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$, where f is the solution of the boundary value problem

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} f(\xi) = s^2 f(\xi), & \xi \in [0,1], \\ \frac{\mathrm{d}}{\mathrm{d}\xi} f(0) = \mathrm{v}, & f(1) = 0. \end{cases} \tag{19}$$

The solution of (19) is the function

$$f(\xi) = \begin{cases} \frac{e^{s\xi} - e^{-s(\xi-2)}}{s(e^{2s}+1)} \mathrm{v} & \text{if } s \in \mathbb{C} \setminus \{0\}, \\ (\xi - 1)\mathrm{v} & \text{if } s = 0. \end{cases}$$

Applying $\bar{C}$ to $z = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with $\mathrm{v} = 1$, we get

$$\mathbf{G}_{\rm sk}(s) = \bar{C}(sI - A_{\rm sk})^{-1}B = \frac{e^{s\xi_0} + e^{-s(\xi_0 - 2)}}{e^{2s} + 1}. \tag{20}$$

Let $\gamma = 1$. From a routine computation, we have

$$|\mathbf{G}_{\rm sk}(s)| \leq \frac{e^{\xi_0} + e^{2-\xi_0}}{e^2\sqrt{1+e^{-4}-2e^{-2}}} \qquad \forall\, s \in l(1).$$

Therefore, $\mathbf{G}_{\rm sk}$ is bounded on $l(1)$. ∎

Remark 5.3: The differential equation (19) could be obtained directly from (13). Indeed, applying the Laplace transform to (13), with $\alpha(\xi) = 0$ for all $\xi \in [0,1]$, and replacing the second order time-derivative with the scalar multiplication $s^2 w$ yields the boundary value problem (19).

The closed-loop system. Lemma 5.2 guarantees that the system Σ with generating operators (A, B, C) from (15), (16), (17) is a well-posed SISO linear system. Therefore, we can solve the control problem described above using the saturating integrator from (5). The closed-loop system is

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \dot{u}(t) = k\mathscr{S}(u(t), r - \bar{C}x(t)), \tag{21}$$

where $k > 0$. In the following, we prove that Assumptions 4.1 and 4.3 hold, so that Theorem 4.4 can be used to show that (21) solves the tracking problem for all feasible $r \in \mathbb{R}$.

Lemma 5.4: There exists a Lyapunov function $W : X \to [0, \infty)$, associated to $\dot{x} = Ax$, satisfying Assumption 4.1.

Proof: Let $C_0 = [0 \quad \alpha^{\frac{1}{2}}]$. Then $A = A_{\rm sk} - C_0^* C_0$ is exponentially stable, as we have already seen when we have introduced A. The operators A, C_0^* and C_0 determine a scattering conservative well-posed system of the class "from thin air" analyzed in [36] (see also [38]). It follows from [36, Thm. 1.3] that (A, C_0) is exactly observable. (In the notation of [36] we would denote the operator of pointwise multiplication with $\alpha^{\frac{1}{2}}$ by C_0.) Thus, using the procedure shown in Remark 4.2, we can build a Lyapunov function $W_0 : X \to [0, \infty)$ satisfying Assumption 4.1. ∎

Lemma 5.5: The transfer function $\mathbf{G}$ of the string equation system Σ satisfies $\mathbf{G}(s) = \bar{C}(sI - A)^{-1}B$ for all $s \in \rho(A)$, and $\mathbf{G}(0) > 0$.

Proof: Since the semigroup $\mathbb{T}$ is exponentially stable, we have from [44, Lem. 2.10 & Sec. 4] that $\mathbf{G}(0) = \bar{C}(-A)^{-1}B + D$ can be computed by considering a constant input $u \equiv \mathrm{v}$, and finding an initial state $x_0 \in X$ of the system such that corresponding state trajectory is constant, i.e., $x \equiv x_0$, and satisfies $Ax_0 + Bu_0 \in X$. The corresponding constant output then has the form $y \equiv \mathbf{G}(0)\mathrm{v}$. Based on the choice $x = \left[\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right]$ of the state and the definitions of A and B, the desired constant state trajectory $x \equiv x_0$ has the form $x = \left[\begin{smallmatrix} f(\cdot) \\ 0 \end{smallmatrix}\right]$, where f is the solution of

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} f(\xi) = 0, & \xi \in [0,1], \\ \frac{\mathrm{d}}{\mathrm{d}\xi} f(0) = \mathrm{v}, & f(1) = 0. \end{cases} \tag{22}$$

The unique solution of this equation is $f(\xi) = (\xi - 1)\mathrm{v}$ and $f \in Z_1$. Computing the corresponding constant output $y \equiv \frac{\mathrm{d}f}{\mathrm{d}\xi}(\xi_0)$ shows that we have that $\mathbf{G}(0)\mathrm{v} = \frac{\mathrm{d}f}{\mathrm{d}\xi}(\xi_0) = \mathrm{v}$. Thus $\mathbf{G}(0) = 1 > 0$ as claimed.

Since it is easy to verify that $(-A)^{-1}B = (-A_{\rm sk})^{-1}B$, formula (20) implies $\bar{C}(-A)^{-1}B = \bar{C}(-A_{\rm sk})^{-1}B = 1 = \mathbf{G}(0)$. This finally implies that $D = 0 \in \mathbb{R}$, and thus also $\mathbf{G}(s) = \bar{C}(sI - A)^{-1}B$ for all $s \in \rho(A)$. ∎

Thanks to Lemmas 5.2, 5.4, 5.5, all the assumptions of Theorem 4.4 are satisfied. Thus, the control problem can be solved for all $r \in \mathbb{R}$ satisfying $\mathbf{G}(0)^{-1}r \in U$ using the closed-loop system (21), with $k > 0$ sufficiently small.

Remark 5.6: As we saw in the proof of Lemma 5.5, the DC-gain $\mathbf{G}(0)$ coincides with the DC-gain of the undamped system. Indeed, when computing the transfer function of (13) at $s = 0$, the effect of the damping $\alpha(\cdot)$ disappears.

Numerical results. In this numerical example we consider a controlled wave equation as in (13), with damping $\alpha(\xi) = \max\{0, 40(\xi - 1/8)(7/8 - \xi)\}$. We want the slope $\frac{\partial}{\partial\xi}w(1/2, t)$ (i.e., $\xi_0 = 1/2$) to track the reference

$$r(t) = \begin{cases} 1.5, & 0 \leq t \leq 6, \\ -1, & t > 6. \end{cases}$$

The control input is constrained in $U_0 = [-1.5, 1.5]$. We choose $k = 3$ as integrator gain, the initial conditions

$$\begin{bmatrix} w(\xi, 0) \\ \frac{\partial}{\partial t} w(\xi, 0) \end{bmatrix} = \begin{bmatrix} \cos(4.6\pi\xi) \\ 0 \end{bmatrix}$$

and $u_0 = 2.5 \notin U_0$. The simulation is carried out using Matlab, where the wave equation is approximated using the first $N = 120$ eigenmodes of the undamped wave equation. Figure 2 depicts the controlled output, the tracking error, and the control input. We can see that once $u(t)$ reaches the set U, it never leaves it. Figure 3 shows the corresponding controlled deflection profile $w(\xi, t)$ of the wave.

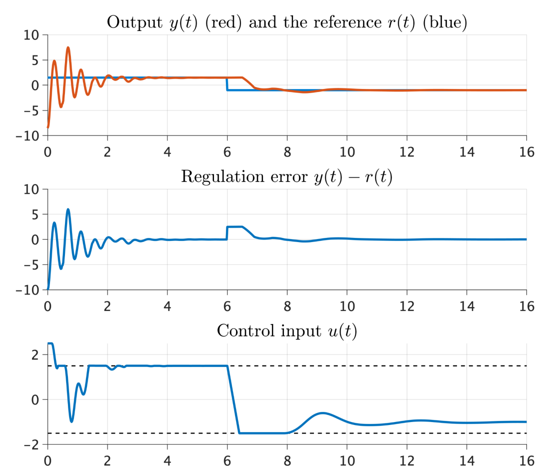

Fig. 2. The measured output, the tracking error, and the control input.

VI. Conclusion

We have extended the constrained integral control theory from [21] to plants that are well-posed linear and exponentially stable systems. We hope that the way we have overcome the obstacles encountered will pave the way for several other generalizations of constrained integral control. For instance, the next step might be to formulate a MIMO saturating integrator (as in [20]) for well-posed exponentially stable systems. Another interesting extension might be to consider infinite-dimensional nonlinear systems.

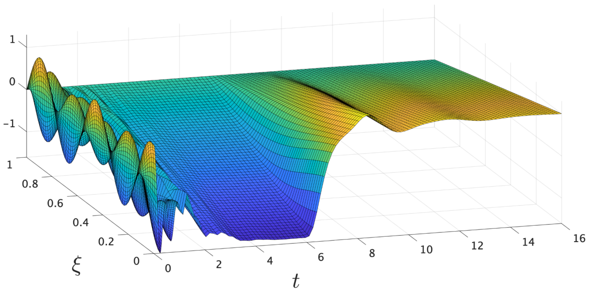

Fig. 3. The deflection $w(\xi, t)$ of the controlled wave equation.

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