EXOTIC CLOSED SUBIDEALS OF ALGEBRAS OF BOUNDED OPERATORS

HANS-OLAV TYLLI AND HENRIK WIRZENIUS

Abstract. We exhibit a Banach space Z failing the approximation property, for which there is an uncountable family $\mathscr{F}$ of closed subideals contained in the Banach algebra $\mathcal{K}(Z)$ of the compact operators on Z, such that the subideals in $\mathscr{F}$ are mutually isomorphic as Banach algebras. This contrasts with the behaviour of closed ideals of the algebras $\mathcal{L}(X)$ of bounded operators on X, where closed ideals $\mathcal{I} \neq \mathcal{J}$ are never isomorphic as Banach algebras. We also construct families of non-trivial closed subideals contained in the strictly singular operators $\mathcal{S}(X)$ for classical spaces such as $X = L^p$ with $p \neq 2$, where pairwise isomorphic as well as pairwise non-isomorphic subideals occur.

1. Introduction

Let X be a Banach space and $\mathcal{L}(X)$ be the Banach algebra of bounded linear operators $X \to X$. It was pointed out in [11, Added in Proof] that *if $\mathcal{I}$ and $\mathcal{J}$ are closed ideals of $\mathcal{L}(X)$ for which there is a Banach algebra isomorphism $\theta : \mathcal{I} \to \mathcal{J}$, then $\mathcal{I} = \mathcal{J}$*. In other words, distinct closed ideals of $\mathcal{L}(X)$ are never isomorphic as Banach algebras.

We will show that, surprisingly enough, the above property fails in general for closed subideals of $\mathcal{L}(X)$. We will adhere to the terminology suggested by Patnaik and Weiss [16], [17], and say that $\mathcal{J}$ is an $\mathcal{I}$-subideal of $\mathcal{L}(X)$, if $\mathcal{J} \subset \mathcal{I}$, where $\mathcal{I}$ is an ideal of $\mathcal{L}(X)$ and $\mathcal{J}$ is an ideal of $\mathcal{I}$. We are only concerned with *closed linear* subideals, that is, $\mathcal{J} \subset \mathcal{I}$ are closed linear subspaces of $\mathcal{L}(X)$, such that $US \in \mathcal{J}$ and $SU \in \mathcal{J}$ whenever $S \in \mathcal{J}$ and $U \in \mathcal{I}$ (and similarly for $\mathcal{I} \subset \mathcal{L}(X)$). It will be convenient to say here that $\mathcal{J}$ is a *non-trivial* subideal of $\mathcal{L}(X)$ if $\mathcal{J}$ is not an ideal of $\mathcal{L}(X)$. (Note that subideals $\mathcal{J}$ depend on the intermediary ideal $\mathcal{I}$, but we will occasionally suppress its role.) Subideals of $\mathcal{L}(H)$ for Hilbert spaces H were first considered by Fong and Radjavi in [8]. In particular, they obtained examples of non-trivial singly generated (but non-closed) $\mathcal{K}(H)$-subideals $\mathcal{J}$ of $\mathcal{L}(H)$, see e.g. [8, Theorem 1] or [16, Example 1.3].

Our main result is based on an example constructed in [24, Theorem 4.5] for different purposes. This produces a family $\{\mathcal{I}_A : \emptyset \neq A \subsetneq \mathbb{N}\}$ having the size of the continuum of non-trivial closed $\mathcal{K}(Z)$-subideals, for which the subideals $\mathcal{I}_A$ are mutually isomorphic as Banach algebras. Here $\mathcal{K}(Z)$ denotes the closed ideal of $\mathcal{L}(Z)$ of the compact operators $Z \to Z$, where the above Banach space Z fails to have the approximation property (abbreviated A.P.). In Section 3 we obtain, by different methods, families of pairwise non-isomorphic as well as isomorphic non-trivial closed $\mathcal{S}(X)$-subideals of $\mathcal{L}(X)$ for classical Banach spaces including $X = L^p(0, 1)$, where $1 \leq p < \infty$ and $p \neq 2$. Above $\mathcal{S}(X)$ is the closed ideal of the strictly singular

2020 *Mathematics Subject Classification.* Primary 46H10, 46B28, 47L10.

operators $X \to X$. Our results demonstrate that the closed subideals of $\mathcal{L}(X)$ behave quite differently compared with closed ideals.

References [1], [12] and [13] will be our standard sources for undefined concepts related to Banach spaces, and [4] for notions related to Banach algebras. We use $X \approx Y$ to indicate linearly isomorphic Banach spaces, and $\mathcal{A} \cong \mathcal{B}$ for isomorphic Banach algebras (that is, there is a Banach algebra isomorphism $\theta : \mathcal{A} \to \mathcal{B}$). Recall that these notions can differ for spaces of operators, as for instance $\mathcal{L}(L^p)$ and $\mathcal{L}(\ell^p)$ are linearly isomorphic as Banach spaces for $1 < p < \infty$ and $p \neq 2$ by [2], but they are not isomorphic as Banach algebras by Eidelheit's theorem (see below).

2. Closed subideals of $\mathcal{L}(X)$ which are isomorphic as Banach algebras

Let X and Y be Banach spaces. It is a classical result of Eidelheit [7] (see also [4, Theorem 2.5.7]) that if $\theta : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a Banach algebra isomorphism, then there is a linear isomorphism $U \in \mathcal{L}(X, Y)$ such that $\theta(S) = USU^{-1}$ for all $S \in \mathcal{L}(X)$. Chernoff [3, Corollary 3.2] (see also [15, Section 1.7.15]) established the following extension: *Suppose that $\mathcal{A} \subset \mathcal{L}(X)$ and $\mathcal{B} \subset \mathcal{L}(Y)$ are subalgebras such that the bounded finite rank operators $\mathcal{F}(X) \subset \mathcal{A}$ and $\mathcal{F}(Y) \subset \mathcal{B}$. If $\theta : \mathcal{A} \to \mathcal{B}$ is a bijective algebra homomorphism, then there is a linear isomorphism $U \in \mathcal{L}(X, Y)$ such that $\theta(S) = USU^{-1}$ for all $S \in \mathcal{A}$.* As a consequence, if $\mathcal{I}, \mathcal{J} \subset \mathcal{L}(X)$ are closed ideals for which there is a Banach algebra isomorphism $\theta : \mathcal{I} \to \mathcal{J}$, then $\mathcal{I} = \mathcal{J}$ (cf. also Remarks 2.5.(ii)). The purpose of this section is to exhibit Banach spaces Z, where the above consequence fails very dramatically within a large class of closed $\mathcal{K}(Z)$-subideals.

Let $\mathcal{A}(X, Y) = \overline{\mathcal{F}(X, Y)}$ denote the class of the approximable operators $X \to Y$, where the closure is taken in the uniform operator norm. We note for reference that

$$\mathcal{A}(X) \subset \mathcal{J} \tag{2.1}$$

for any non-zero closed $\mathcal{I}$-subideal $\mathcal{J}$ of $\mathcal{L}(X)$, see e.g. [4, Theorem 2.5.8.(ii)] or [16, Remark 6.1].

We proceed to describe the Banach spaces and the closed subideals from [24]. Let (X, Y) be a pair of Banach spaces such that

$$X \text{ has the A.P., and } \mathcal{A}(X, Y) \subsetneq \mathcal{K}(X, Y). \tag{2.2}$$

We recall that $\mathcal{A}(X, Y) \subsetneq \mathcal{K}(X, Y)$ for some Banach space Y if and only if the dual space X^* fails the A.P., see e.g. [12, Theorem 1.e.5]. Moreover, there are spaces X such that X has the A.P., but X^* fails to have the A.P., see e.g. [12, Theorem 1.e.7].

Fix $1 < p < \infty$. For any pair (X, Y) that satisfies condition (2.2) we consider the direct sum

$$Z_p := \big(\oplus_{j=0}^{\infty} X_j \big)_{\ell^p}, \tag{2.3}$$

where we put $X_0 = Y$ and $X_j = X$ for $j \geq 1$ for unity of notation. Bounded operators $S \in \mathcal{L}(Z_p)$ can be represented as operator matrices $S = (S_{m,n})$ with $S_{m,n} = P_m S J_n$, where $P_m : Z_p \to X_m$ and $J_n : X_n \to Z_p$ are the natural projections and inclusions associated to the component spaces of Z_p for $m, n \in \mathbb{N} \cup \{0\}$. For any subset $\emptyset \neq A \subsetneq \mathbb{N}$ define

$$\mathcal{I}_A := \{S = (S_{m,n}) \in \mathcal{K}(Z_p) : S_{0,0} \in \mathcal{A}(Y), S_{0,k} \in \mathcal{A}(X, Y) \text{ for all } k \in A\}. \tag{2.4}$$

It is shown in [24, Theorem 4.5] that the family

$$\mathscr{F} := \{\mathcal{I}_A : \emptyset \neq A \subsetneq \mathbb{N}\} \tag{2.5}$$

has the following properties:

(i) $\mathcal{I}_A$ is a closed ideal of $\mathcal{K}(Z_p)$, and $\mathcal{A}(Z_p) \subsetneq \mathcal{I}_A \subsetneq \mathcal{K}(Z_p)$ for $\emptyset \neq A \subsetneq \mathbb{N}$.

(ii) $\mathcal{I}_A$ is a left ideal of $\mathcal{L}(Z_p)$ but not a right ideal of $\mathcal{L}(Z_p)$ for $\emptyset \neq A \subsetneq \mathbb{N}$, see [24, Remark 4.8] and [26, Remarks 6.2]. In particular, $\mathcal{I}_A$ is a non-trivial closed $\mathcal{K}(Z_p)$-subideal of $\mathcal{L}(Z_p)$ for $\emptyset \neq A \subsetneq \mathbb{N}$.

(iii) if $A \subset B$, then $\mathcal{I}_B \subset \mathcal{I}_A$, and $\mathcal{I}_A \neq \mathcal{I}_B$ whenever $A \neq B$.

We stress that above (i)-(iii) hold for the spaces Z_p in (2.3) which are obtained from any pair (X, Y) that satisfies (2.2). We will later impose further conditions on X or Y, and in our main result it is assumed that X also satisfies

$$\big(\oplus_{n=1}^{\infty} X\big)_{\ell^p} \approx X, \tag{2.6}$$

whence also $X \oplus X \approx X$. A typical way to achieve this is as follows: if X_0 is any space such that X_0 has the A.P., but X_0^* fails the A.P., then $\mathcal{A}(X_0, Y) \subsetneq \mathcal{K}(X_0, Y)$ holds for some space Y by [12, Theorem 1.e.5]. Let $X = (\oplus_{n=1}^{\infty} X_0)_{\ell^p}$. It is not difficult to check that X has the A.P. and $\mathcal{A}(X, Y) \subsetneq \mathcal{K}(X, Y)$, and in addition that $X \approx (\oplus_{n=1}^{\infty} X)_{\ell^p}$ holds.

Our main result highlights surprising features of the non-trivial closed $\mathcal{K}(Z_p)$-subideals from the above family $\mathscr{F}$. This answers a query of Gideon Schechtman (private communication).

Theorem 2.1. *Fix $1 < p < \infty$, and let Z_p be as in* (2.3)*, where the pair (X, Y) satisfies* (2.2) *and X satisfies* (2.6)*. Then all the non-trivial closed subideals from the family $\mathscr{F}$ defined by* (2.5) *are mutually isomorphic as Banach algebras, that is,*

$$\mathcal{I}_A \cong \mathcal{I}_B \quad \text{for all } \emptyset \neq A, B \subsetneq \mathbb{N}.$$

Before the proof we comment on the form of Banach algebra isomorphisms between closed subideals. Let X be any Banach space and suppose that $\mathcal{I}, \mathcal{J} \subset \mathcal{L}(X)$ are non-zero closed subideals. Recall from (2.1) that non-trivial closed subideals of $\mathcal{L}(X)$ are closed subalgebras that contain the approximable operators $\mathcal{A}(X)$. Hence, if $\theta : \mathcal{I} \to \mathcal{J}$ is a Banach algebra isomorphism, then by [3, Corollary 3.2] there is a linear isomorphism $U \in \mathcal{L}(X)$, such that the restriction to $\mathcal{I}$ of the inner automorphism

$$\psi(S) = USU^{-1}, \quad S \in \mathcal{L}(X),$$

equals $\theta : \mathcal{I} \to \mathcal{J}$. In the proof of Theorem 2.1 we will construct inner automorphisms ψ of $\mathcal{L}(Z_p)$ for which $\psi(\mathcal{I}_A) = \mathcal{I}_B$. The novel feature is that such a phenomenon is possible among non-trivial closed subideals of $\mathcal{L}(Z_p)$, whereas it is impossible for the smaller class of the closed ideals of $\mathcal{L}(X)$ for any X.

The argument will be split into auxiliary steps, where we first construct Banach algebra isomorphisms $\mathcal{I}_A \cong \mathcal{I}_B$ for various basic combinations of the cardinalities of A and $A^c = \mathbb{N} \setminus A$, respectively of B and B^c. In the final step we deduce Theorem 2.1 from these lemmas. Let $|A| \in \mathbb{N} \cup \{\infty\}$ denote the cardinality of the non-empty set $A \subset \mathbb{N}$. The spaces X and Y are as in the definition of Z_p.

Lemma 2.2. *Suppose that $\emptyset \neq A, B \subsetneq \mathbb{N}$ are subsets for which there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(A) = B$. Then*

$$\mathcal{I}_A \cong \mathcal{I}_B.$$

The assumption is satisfied if (and only if) $|A| = |B| \in \mathbb{N} \cup \{\infty\}$ *and* $|A^c| = |B^c| \in \mathbb{N} \cup \{\infty\}$.

Proof. Define the linear isometry $U \in \mathcal{L}(Z_p)$ by

$$U(y, x_1, x_2, \ldots) = (y, x_{\sigma(1)}, x_{\sigma(2)}, \ldots), \quad (y, x_1, x_2, \ldots) \in Z_p.$$

Clearly U is a linear isomorphism $Z_p \to Z_p$, whose inverse U^{-1} satisfies

$$U^{-1}(y, x_1, x_2, \ldots) = (y, x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots), \quad (y, x_1, x_2, \ldots) \in Z_p.$$

Let $\theta(S) = USU^{-1}$ for $S \in \mathcal{L}(Z_p)$. It follows that θ is a Banach algebra isomorphism $\mathcal{L}(Z_p) \to \mathcal{L}(Z_p)$, as well as $\mathcal{K}(Z_p) \to \mathcal{K}(Z_p)$. Its inverse θ^{-1} on $\mathcal{L}(Z_p)$ has the form $\theta^{-1}(T) = U^{-1}TU$ for $T \in \mathcal{L}(Z_p)$. It will be enough to verify the following

Claim. $\theta(\mathcal{I}_A) \subset \mathcal{I}_B$ and $\theta^{-1}(\mathcal{I}_B) \subset \mathcal{I}_A$.

Namely, in this event the restriction of θ to $\mathcal{I}_A$ will be a Banach algebra isomorphism $\mathcal{I}_A \to \mathcal{I}_B$: for any $T \in \mathcal{I}_B$ one has $T = \theta(\theta^{-1}(T))$, where $\theta^{-1}(T) \in \mathcal{I}_A$, so that $\theta(\mathcal{I}_A) = \mathcal{I}_B$.

Towards $\theta(\mathcal{I}_A) \subset \mathcal{I}_B$ we will verify that for any $S \in \mathcal{I}_A$ we have $P_0(USU^{-1})J_0 \in \mathcal{A}(Y)$ and $P_0(USU^{-1})J_r \in \mathcal{A}(X, Y)$ for any $r \in B$. Suppose that $y \in Y$ is arbitrary. In this case

$$SU^{-1}J_0y = SU^{-1}(y, 0, 0, \ldots) = S(y, 0, 0, \ldots) = (S_{0,0}y, S_{1,0}y, \ldots),$$

so that $P_0(USU^{-1})J_0 = S_{0,0} \in \mathcal{A}(Y)$ since $P_0U = P_0$.

Next, let $r = \sigma(k) \in B = \sigma(A)$, where $k = \sigma^{-1}(r) \in A$. If $x_r \in X_r$, then $SU^{-1}J_rx_r = SJ_kx_r$, so that

$$SU^{-1}J_rx_r = (S_{0,k}x_r, S_{1,k}x_r, \ldots).$$

It follows that $P_0(USU^{-1})J_r = S_{0,k} \in \mathcal{A}(X, Y)$, because $S \in \mathcal{I}_A$ and $k \in A$.

The second inclusion $\theta^{-1}(\mathcal{I}_B) \subset \mathcal{I}_A$ can be deduced from the symmetry. Namely, the inverse permutation σ^{-1}, for which $\sigma^{-1}(B) = A$, corresponds to the Banach algebra isomorphism $\psi(S) = U^{-1}SU$ for $S \in \mathcal{L}(Z_p)$. The first part of the Claim implies that $\psi(\mathcal{I}_B) \subset \mathcal{I}_A$, where $\psi = \theta^{-1}$.

Finally, if $|A| = |B| \in \mathbb{N} \cup \{\infty\}$ and $|A^c| = |B^c| \in \mathbb{N} \cup \{\infty\}$, then there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(A) = B$ (and $\sigma(A^c) = B^c$). ☐

Put $[r] = \{1, \ldots, r\}$ for $r \in \mathbb{N}$.

Lemma 2.3. *Suppose that* $X \oplus X \approx X$. *Then for all* $r, s \in \mathbb{N}$ *the following hold:*

(a) $\mathcal{I}_{[r]} \cong \mathcal{I}_{[s]}$
(b) $\mathcal{I}_{[r]^c} \cong \mathcal{I}_{[s]^c}$.

Proof. Let $V : X \to X \oplus X$ be a linear isomorphism.

(a) It will be enough to show that $\mathcal{I}_{[r]} \cong \mathcal{I}_{[r+1]}$ for all $r \in \mathbb{N}$. Namely, if $r < s$, then $\mathcal{I}_{[r]} \cong \mathcal{I}_{[s]}$ follows by transitivity.

Let $r \in \mathbb{N}$ and define the bounded linear isomorphism $U \in \mathcal{L}(Z_p)$ by

$$U(y, x_1, x_2, \ldots) = (y, x_1, \ldots, x_{r-1}, V(x_r), x_{r+1}, \ldots), \quad (y, x_1, x_2, \ldots) \in Z_p,$$

whose inverse map is

$$U^{-1}(y, x_1, x_2, \ldots) = (y, x_1, \ldots, x_{r-1}, V^{-1}(x_r, x_{r+1}), x_{r+2}, \ldots), \quad (y, x_1, \ldots) \in Z_p.$$

Let $\widetilde{J}_k : X \to X \oplus X$ denote the inclusion maps and $\widetilde{P}_k$ the corresponding projections for $k = 1, 2$ (relative to $X \oplus X$). Observe that

$$UJ_k = \begin{cases} J_k & \text{if } k \le r-1, \\ J_r\widetilde{P}_1V + J_{r+1}\widetilde{P}_2V & \text{if } k = r, \\ J_{k+1} & \text{if } k > r, \end{cases} \tag{2.7}$$

$$U^{-1}J_k = \begin{cases} J_k & \text{if } k \le r-1, \\ J_rV^{-1}\widetilde{J}_1 & \text{if } k = r, \\ J_rV^{-1}\widetilde{J}_2 & \text{if } k = r+1, \\ J_{k-1} & \text{if } k > r+1. \end{cases} \tag{2.8}$$

Moreover, $P_0U = P_0U^{-1} = P_0$, since the 0:th component of Z_p is not affected by U or U^{-1}.

Let $\theta(S) = USU^{-1}$ for $S \in \mathcal{L}(Z_p)$, so that $\theta^{-1}(S) = U^{-1}SU$ for $S \in \mathcal{L}(Z_p)$. It will suffice to verify, as explained in the proof of Lemma 2.2, the

Claim. $\theta(\mathcal{I}_{[r]}) \subset \mathcal{I}_{[r+1]}$ and $\theta^{-1}(\mathcal{I}_{[r+1]}) \subset \mathcal{I}_{[r]}$.

(i) We verify that $\theta(T) = UTU^{-1} \in \mathcal{I}_{[r+1]}$ for any $T \in \mathcal{I}_{[r]}$. Note first that $P_0UTU^{-1}J_0 = P_0TJ_0 \in \mathcal{A}(Y)$. Assume that $k \in [r+1]$. If $k \le r-1$, then

$$P_0UTU^{-1}J_k = P_0TJ_k \in \mathcal{A}(X, Y)$$

since $U^{-1}J_k = J_k$ by (2.8). If $k = r$, then again by (2.8) we have

$$P_0UTU^{-1}J_r = P_0TJ_rV^{-1}\widetilde{J}_1 \in \mathcal{A}(X, Y),$$

since $P_0TJ_r \in \mathcal{A}(X, Y)$ by assumption. Finally, if $k = r+1$, then similarly

$$P_0UTU^{-1}J_{r+1} = P_0TJ_rV^{-1}\widetilde{J}_2 \in \mathcal{A}(X, Y).$$

(ii) We next verify that $\theta^{-1}(T) = U^{-1}TU \in \mathcal{I}_{[r]}$ for any $T \in \mathcal{I}_{[r+1]}$. As above $P_0U^{-1}TUJ_0 = P_0TJ_0 \in \mathcal{A}(Y)$. Let $k \in [r]$. If $k \le r-1$, then since $UJ_k = J_k$ by (2.7) we get that

$$P_0U^{-1}TUJ_k = P_0TJ_k \in \mathcal{A}(X, Y)$$

by assumption. If $k = r$, then from (2.7) we get that

$$P_0U^{-1}TUJ_r = P_0T(J_r\widetilde{P}_1V + J_{r+1}\widetilde{P}_2V) \in \mathcal{A}(X, Y),$$

since $T \in \mathcal{I}_{[r+1]}$ implies that P_0TJ_r and P_0TJ_{r+1} belong to $\mathcal{A}(X, Y)$.

(b) Let $U \in \mathcal{L}(Z_p)$ be the linear isomorphism from part (a), and let $\theta(S) = USU^{-1}$ be the corresponding inner automorphism $\mathcal{L}(Z_p) \to \mathcal{L}(Z_p)$. We claim that also here

$$\theta(\mathcal{I}_{[r]^c}) \subset \mathcal{I}_{[r+1]^c} \text{ and } \theta^{-1}(\mathcal{I}_{[r+1]^c}) \subset \mathcal{I}_{[r]^c}.$$

As in part (a) we get that $P_0(\theta(S))J_0 = P_0SJ_0 \in \mathcal{A}(Y)$ and $P_0(\theta^{-1}(T))J_0 = P_0TJ_0 \in \mathcal{A}(Y)$ for any $S \in \mathcal{I}_{[r]^c}$ and $T \in \mathcal{I}_{[r+1]^c}$.

(iii) Suppose that $S \in \mathcal{I}_{[r]^c}$ and $s \ge r+2$. From (2.8) we have

$$P_0(USU^{-1})J_s = P_0USJ_{s-1} = P_0SJ_{s-1} \in \mathcal{A}(Y)$$

as $s - 1 \ge r+1$ and $S \in \mathcal{I}_{[r]^c}$.

(iv) Suppose next that $T \in \mathcal{I}_{[r+1]^c}$ and $s \geq r+1$. From (2.7) we get that

$$P_0(U^{-1}TU)J_s = P_0U^{-1}TJ_{s+1} = P_0TJ_{s+1} \in \mathcal{A}(Y)$$

as $s+1 \geq r+2$ and $T \in \mathcal{I}_{[r+1]^c}$.

This completes the proof of part (b). □

Condition (2.6) on X enables us to find Banach algebra isomorphisms $\mathcal{I}_A \to \mathcal{I}_B$ for sets A and B of very unequal size. We first isolate two particular cases.

Lemma 2.4. *Suppose that X satisfies condition* (2.6)*. Then the following hold:*

(a) $\mathcal{I}_{\{1\}} \cong \mathcal{I}_{\{2,3,4,\ldots\}}$.

(b) $\mathcal{I}_{\{2,3,4,\ldots\}} \cong \mathcal{I}_{\{2,4,6,\ldots\}}$.

Proof. Let $V : X \to \big(\oplus_{n=1}^{\infty} X\big)_{\ell^p}$ be a linear isomorphism.

(a) We define $U : Z_p \to Z_p$ by

$$U(y, x_1, x_2, \ldots) = (y, V^{-1}(x_2, x_3, \ldots), V(x_1)), \quad (y, x_1, x_2, \ldots) \in Z_p,$$

where $V^{-1}(x_2, x_3, \ldots)$ sits in the first component of Z_p. Clearly $U \in \mathcal{L}(Z_p)$ is a linear isomorphism for which $U^{-1} = U$. Let $\psi : \mathcal{L}(Z_p) \to \mathcal{L}(Z_p)$ be the Banach algebra isomorphism $\psi(S) = USU$, for which $\psi^{-1} = \psi$. Put $B = \{1\}^c$.

<u>Claim</u>. $\psi(\mathcal{I}_{\{1\}}) \subset \mathcal{I}_B$ and $\psi(\mathcal{I}_B) \subset \mathcal{I}_{\{1\}}$.

(i) Suppose first that $S \in \mathcal{I}_{\{1\}}$, so that $S_{0,0}$ and $S_{0,1}$ are approximable operators. Clearly $P_0(USU)J_0 = S_{0,0} \in \mathcal{A}(Y)$. Next, let $r \geq 2$ and $x_r \in X_r$ be arbitrary. Then

$$SUJ_rx_r = SU(0, 0, \ldots, 0, x_r, 0, \ldots) = S(0, z, 0, \ldots) = (S_{0,1}z, S_{1,1}z, \ldots),$$

where $z = V^{-1}\widetilde{J}_{r-1}x_r$ and $\widetilde{J}_k$ is the inclusion $X \to \big(\oplus_{n=1}^{\infty} X\big)_{\ell^p}$ into the k:th position of the right-hand direct sum. Deduce that

$$P_0(USU)J_r = S_{0,1}V^{-1}\widetilde{J}_{r-1} \in \mathcal{A}(X, Y),$$

since $S \in \mathcal{I}_{\{1\}}$ and $P_0U = P_0$.

(ii) We next claim that $P_0(UTU)J_1 \in \mathcal{A}(X, Y)$ for any $T \in \mathcal{I}_B$. For this purpose observe that $\sum_{k=0}^{n} TJ_kP_k \to T$ as $n \to \infty$ in the operator norm by the proof of [24, Lemma 4.6], since $T \in \mathcal{K}(Z_p)$ and $1 < p < \infty$. It follows that

$$\|\sum_{k=0}^{n} P_0U(TJ_kP_k)UJ_1 - P_0UTUJ_1\| \to 0 \text{ as } n \to \infty.$$

By approximation it will suffice to verify that $P_0U(TJ_kP_k)UJ_1 \in \mathcal{A}(X, Y)$ for all $k \geq 0$, that is, $P_0TJ_kP_kUJ_1 \in \mathcal{A}(X, Y)$ for all $k \geq 0$ (since $P_0U = P_0$). Towards this observe that $P_0TJ_k \in \mathcal{A}(X, Y)$ for $k = 0$ and for $k > 1$ since $T \in \mathcal{I}_B$. Moreover, $P_1UJ_1 = 0$ for $k = 1$. Thus $\psi(\mathcal{I}_B) \subset \mathcal{I}_{\{1\}}$, which completes the proof of part (a).

(b) Define $U : Z_p \to Z_p$ by

$$U(y, x_1, x_2, \ldots) = (y, (Vx_1)_1, x_2, (Vx_1)_2, x_3, \ldots), \quad (y, x_1, x_2, \ldots) \in Z_p,$$

where $(Vx_1)_k$ denotes the k:th component of Vx_1 in the direct sum $\big(\oplus_{n=1}^{\infty} X\big)_{\ell^p}$. Then $U \in \mathcal{L}(Z_p)$ is a linear isomorphism, whose inverse $U^{-1} : Z_p \to Z_p$ is defined by

$$U^{-1}(y, x_1, x_2, \ldots) = (y, V^{-1}(x_1, x_3, \ldots), x_2, x_4, \ldots), \quad (y, x_1, x_2, \ldots) \in Z_p.$$

(iii) We first claim that $U^{-1}SU \in \mathcal{I}_{\{2,3,4,\dots\}}$ for any $S \in \mathcal{I}_{\{2,4,6,\dots\}}$. Towards this, note that $P_0U^{-1} = P_0$ and $UJ_0 = J_0$. Thus

$$P_0U^{-1}SUJ_0 = P_0SJ_0 \in \mathcal{A}(Y).$$

Suppose next that $r \geq 2$. Observe that $UJ_r = J_{2r-2}$, and thus

$$P_0U^{-1}SUJ_r = P_0SJ_{2r-2} \in \mathcal{A}(X,Y).$$

(iv) We next claim that $USU^{-1} \in \mathcal{I}_{\{2,4,6,\dots\}}$ for any $S \in \mathcal{I}_{\{2,3,4,\dots\}}$. For this, note again that $P_0U = P_0$ and $U^{-1}J_0 = J_0$, so that $P_0USU^{-1}J_0 \in \mathcal{A}(Y)$. Let $2n \in \{2,4,6,\dots\}$. In this event $U^{-1}J_{2n} = J_{n+1}$, so that

$$P_0USU^{-1}J_{2n} = P_0SJ_{n+1} \in \mathcal{A}(X,Y).$$

Put $\chi(S) := USU^{-1}$ for $S \in \mathcal{L}(Z_p)$. By combining parts (iii) and (iv) we deduce that $\chi(\mathcal{I}_{\{2,3,4,\dots\}}) = \mathcal{I}_{\{2,4,6,\dots\}}$, so χ yields a Banach algebra isomorphism $\mathcal{I}_{\{2,3,4,\dots\}} \to \mathcal{I}_{\{2,4,6,\dots\}}$. □

We are now in position to complete the argument of the main result.

Proof of Theorem 2.1. By transitivity and symmetry it suffices to show that $\mathcal{I}_A \cong \mathcal{I}_{\{1\}}$ for any subset $\emptyset \neq A \subsetneq \mathbb{N}$. We consider the cases $|A| < \infty$, $|A^c| < \infty$ and $|A| = |A^c| = \infty$ separately:

Case 1. *Suppose that* $|A| = s < \infty$. From Lemmas 2.2 and 2.3.(a) we get that

$$\mathcal{I}_A \cong \mathcal{I}_{[s]} \cong \mathcal{I}_{\{1\}}.$$

Case 2. *Suppose that* $|A^c| = r < \infty$. From Lemmas 2.2, 2.3.(b) and 2.4.(a) we get that

$$\mathcal{I}_A \cong \mathcal{I}_{[r]^c} \cong \mathcal{I}_{\{1\}^c} \cong \mathcal{I}_{\{1\}}.$$

Case 3. *Suppose that* $|A| = |A^c| = \infty$. According to the assumption there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(A) = \{2,4,6,\dots\}$. Hence we find that

$$\mathcal{I}_A \cong \mathcal{I}_{\{2,4,6,\dots\}} \cong \mathcal{I}_{\{2,3,4,\dots\}} \cong \mathcal{I}_{\{1\}}$$

from Lemmas 2.2, 2.4.(b) and 2.4.(a). □

Remarks 2.5. (i) If $\mathcal{K}(Z_p)$ is separable in Theorem 2.1, then it can be verified that $\mathcal{K}(Z_p)$ has at most continuum many closed subspaces (as well as non-trivial $\mathcal{K}(Z_p)$-subideals). Hence the size of the family $\mathscr{F}$ from (2.5) is as large as possible.

The pair (X,Y) satisfying (2.2) and (2.6) can be chosen so that $\mathcal{K}(Z_p)$ is separable. Recall first that $\mathcal{K}(Z_p)$ is separable if and only if Z_p^* is separable, see e.g. [23, page 272]. Secondly, if X^* and Y^* are separable, then in (2.3) the dual $Z_p^* = \big(\oplus_{j=0}^{\infty} X_j^*\big)_{\ell^{p'}}$ is separable. Here $X_0^* = Y^*$ as well as $X_j^* = X^*$ for $j \geq 1$, and p' is the dual exponent of $p \in (1,\infty)$. Next, to choose X we follow the argument of [12, Theorem 1.e.7.(b)]. For this purpose let U be a separable reflexive space such that U^* fails the A.P. By [12, Theorem 1.d.3] there is a Banach space W such that W^{**} has a Schauder basis and $W^{**}/W \approx U$. It follows that $X = W^{**}$ has the A.P., but $X^* \approx W^* \oplus U^*$ is separable and fails the A.P. Apply [12, Theorem 1.e.5] to pick a Banach space Y_0 and $S_0 \in \mathcal{K}(X,Y_0) \setminus \mathcal{A}(X,Y_0)$. By Terzioğlu's compact factorization theorem [22] there is a closed subspace $Y \subset c_0$ and a factorization $S_0 = BS$ with $S \in \mathcal{K}(X,Y)$. Here $S \notin \mathcal{A}(X,Y)$ and Y^* is separable.

(ii) A variant of the fact in [11, Added in Proof] implies that the non-trivial subideals $\mathcal{I}_A \in \mathscr{F}$ in Theorem 2.1 are not isomorphic as Banach algebras to either $\mathcal{A}(Z_p)$ or $\mathcal{K}(Z_p)$: *Suppose that X is a Banach space, let $\mathcal{I}$ be a closed ideal of $\mathcal{L}(X)$ and $\mathcal{J}$ be a closed subalgebra of $\mathcal{L}(X)$ such that $\mathcal{A}(X) \subset \mathcal{J}$. If $\theta : \mathcal{I} \to \mathcal{J}$ is a Banach algebra isomorphism, then $\mathcal{I} = \mathcal{J}$. In particular, if $\mathcal{J}$ is a non-trivial closed subideal of $\mathcal{L}(X)$, then $\mathcal{I}$ and $\mathcal{J}$ are not isomorphic as Banach algebras.*

To see this fact, by [3, Corollary 3.2] there is a linear isomorphism $U \in \mathcal{L}(X)$ so that $\theta(S) = USU^{-1}$ for $S \in \mathcal{I}$. If $T \in \mathcal{J}$, then there is $S \in \mathcal{I}$ such that $T = USU^{-1}$, where $USU^{-1} \in \mathcal{I}$ as $\mathcal{I}$ is an ideal of $\mathcal{L}(X)$. Thus $\mathcal{J} \subset \mathcal{I}$. Conversely, if $S \in \mathcal{I}$ then $U^{-1}SU \in \mathcal{I}$, so that $S = \theta(U^{-1}SU) \in \mathcal{J}$. This yields $\mathcal{I} = \mathcal{J}$.

Thomas Schlumprecht asked whether it is possible to identify the closed ideal $[\mathcal{I}_A]$ of $\mathcal{L}(Z_p)$ generated by the subideal $\mathcal{I}_A \in \mathscr{F}$ for $\emptyset \neq A \subsetneq \mathbb{N}$. It turns out that $\mathcal{I}_A$ generate the same closed ideal of $\mathcal{L}(Z_p)$. We first record another general consequence of Chernoff's result.

Lemma 2.6. *Let X be a Banach space and suppose that $\mathcal{A} \subset \mathcal{L}(X)$, $\mathcal{B} \subset \mathcal{L}(X)$ are closed subalgebras that contain $\mathcal{F}(X)$, for which $\mathcal{A} \cong \mathcal{B}$. Then the subalgebras $\mathcal{A}$ and $\mathcal{B}$ generate the same closed ideal of $\mathcal{L}(X)$, that is,*

$$[\mathcal{A}] = [\mathcal{B}].$$

Proof. Let $\theta : \mathcal{A} \to \mathcal{B}$ be a Banach algebra isomorphism. By [3, Corollary 3.2] there is a linear isomorphism $U \in \mathcal{L}(X)$ such that $\theta(S) = USU^{-1}$ for $S \in \mathcal{A}$. If $S \in \mathcal{A}$ is arbitrary, then $\theta(S) = USU^{-1} \in \mathcal{B}$, so that $S = U^{-1}\theta(S)U \in [\mathcal{B}]$. Deduce that $[\mathcal{A}] \subset [\mathcal{B}]$, and by symmetry that $[\mathcal{B}] \subset [\mathcal{A}]$. □

Lemma 2.6 together with Theorem 2.1 imply that $[\mathcal{I}_A] = [\mathcal{I}_B]$ holds for all non-trivial closed subideals $\mathcal{I}_A, \mathcal{I}_B \in \mathscr{F}$, where $\mathscr{F}$ is given by (2.5). For this application one requires that the pair (X, Y) of component spaces of Z_p satisfies (2.2) and that X satisfies (2.6). Actually the resulting closed ideal of $\mathcal{L}(Z_p)$ can be identified explicitly, and condition (2.6) on X can even be removed.

Proposition 2.7. Suppose that $1 < p < \infty$, and let Z_p be defined by (2.3), where (X, Y) satisfies (2.2). Then

$$[\mathcal{I}_A] = [\mathcal{I}] \tag{2.9}$$

for all $\emptyset \neq A \subsetneq \mathbb{N}$, where $\mathcal{I} := \{T \in \mathcal{K}(Z_p) \mid P_0 T J_0 \in \mathcal{A}(Y)\}$.

Proof. Let $\emptyset \neq A \subsetneq \mathbb{N}$ be arbitrary. Since $\mathcal{I}_A \subset \mathcal{I}$ it will suffice to verify that $\mathcal{I} \subset [\mathcal{I}_A]$. Let $T \in \mathcal{I}$. Since $T \in \mathcal{K}(Z_p)$ and $1 < p < \infty$ we know that

$$||\sum_{k=0}^{r} TJ_kP_k - T|| \to 0 \text{ as } r \to \infty \tag{2.10}$$

(see e.g. the proof of [24, Lemma 4.6]). Thus, in order to show that $T \in [\mathcal{I}_A]$, it will be enough by (2.10) to verify that $TJ_kP_k \in [\mathcal{I}_A]$ for all $k \geq 0$. We need to consider the following mutually exclusive cases.

<u>Case $k = 0$</u>. We know that $P_0(TJ_0P_0)J_0 = P_0TJ_0 \in \mathcal{A}(Y)$ by assumption. Moreover, for any $r \in A$ we get that $P_0(TJ_0P_0)J_r = 0$ since $P_0J_r = 0$. Thus $TJ_0P_0 \in \mathcal{I}_A$.

<u>Case $k \in A^c$</u>. Here $TJ_kP_k \in \mathcal{I}_A$ since $P_0(TJ_kP_k)J_s = 0$ for any $s \in A \cup \{0\}$.

Case $k \in A$. Pick $r \in A^c$ and let $J_{r,k} : X_r \to X_k$ and $J_{k,r} : X_k \to X_r$ denote the identity operator on $X = X_r = X_k$. Clearly $J_{r,k}P_rJ_rJ_{k,r}$ is the identity operator $X_k \to X_k$, so that

$$TJ_kP_k = (TJ_kJ_{r,k}P_r)(J_rJ_{k,r}P_k).$$

We claim that $TJ_kJ_{r,k}P_r \in \mathcal{I}_A$, so that $TJ_kP_k \in [\mathcal{I}_A]$. In fact, for any $s \in A \cup \{0\}$ we have $P_rJ_s = 0$, and thus $P_0(TJ_kJ_{r,k}P_r)J_s = 0$. □

Remark 2.8. In Proposition 2.7 there are pairs (X, Y) satisfying (2.2), for which $\mathcal{I}$ is a non-trivial closed $\mathcal{K}(Z_p)$-subideal of $\mathcal{L}(Z_p)$. Hence the closed ideal $[\mathcal{I}]$ of $\mathcal{L}(Z_p)$ is required on the right-hand side of (2.9) instead of $\mathcal{I}$.

In fact, if X has the A.P. and X^* fails this property, then first apply [12, Theorem 1.e.5] to pick Y_0 and $S_0 \in \mathcal{K}(X, Y_0) \setminus \mathcal{A}(X, Y_0)$. Let $Y = Y_0 \oplus X$. Then (X, Y) satisfies (2.2), since $Sx = (S_0x, 0)$ for $x \in X$ defines a compact, non-approximable operator $X \to Y$. Moreover, $U(y, x) = x$ for $(y, x) \in Y$ is a non-compact operator $U : Y \to X$ for which $SU : Y \to Y$ is compact and non-approximable. Fix $r \in \mathbb{N}$ and define $V \in \mathcal{L}(Z_p)$ and $T \in \mathcal{I}$ by $V = J_rUP_0$, respectively $T = J_0SP_r$. Then

$$P_0(TV)J_0 = P_0(J_0SP_rJ_rUP_0)J_0 = SU \notin \mathcal{A}(Y),$$

that is, $TV \notin \mathcal{I}$.

3. Non-trivial closed $\mathcal{S}(X)$-subideals

It is a natural question whether $\mathcal{L}(X)$ contains non-trivial closed subideals for classical Banach spaces X. Recall that $\mathcal{S}(X)$, the class of the strictly singular operators $X \to X$, is a closed ideal of $\mathcal{L}(X)$ that satisfies $\mathcal{K}(X) \subset \mathcal{S}(X)$ for any X. Here we describe large families of non-trivial closed $\mathcal{S}(X)$-subideals of $\mathcal{L}(X)$ for many classical Banach spaces X, including $L^p := L^p(0, 1)$ with $p \neq 2$. We first briefly discuss closed subideals of Banach algebras.

Let $\mathcal{A}$ be a Banach algebra, and suppose that $\mathcal{J} \subset \mathcal{I} \subset \mathcal{A}$. We say that $\mathcal{J}$ is a closed $\mathcal{I}$-subideal of $\mathcal{A}$ if $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ and $\mathcal{J}$ is a closed ideal of $\mathcal{I}$. This setting reveals that the existence of non-trivial closed $\mathcal{I}$-subideals is related to the absence of approximate identities for $\mathcal{I}$. Recall that the net $(e_\alpha) \subset \mathcal{I}$ is a left approximate identity (LAI) of $\mathcal{I}$ if $y = \lim_\alpha e_\alpha y$ for all $y \in \mathcal{I}$. Right approximate identities (RAI) of $\mathcal{I}$ are defined analogously. The following fact is a variant and reformulation of [4, Proposition 2.9.4].

Lemma 3.1. *Suppose that $\mathcal{J} \subset \mathcal{I} \subset \mathcal{A}$, where $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ and $\mathcal{J}$ is a closed $\mathcal{I}$-subideal of $\mathcal{A}$.*

(i) If $\mathcal{I}$ or $\mathcal{J}$ has a RAI, then $\mathcal{J}$ is a right ideal of $\mathcal{A}$.

(ii) If $\mathcal{I}$ or $\mathcal{J}$ has a LAI, then $\mathcal{J}$ is a left ideal of $\mathcal{A}$.

Proof. (i) Suppose that (e_α) is a RAI for $\mathcal{I}$, and let $x \in \mathcal{J}$ and $z \in \mathcal{A}$ be arbitrary. It follows that

$$xz = \lim_\alpha (xe_\alpha)z = \lim_\alpha x(e_\alpha z) \in \mathcal{J},$$

since $e_\alpha z \in \mathcal{I}$ for all α and $\mathcal{J}$ is a closed ideal of $\mathcal{I}$. The other cases are similar. □

Remarks 3.2. (i) By Lemma 3.1 there are Banach algebras without non-trivial closed subideals. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{I} \subset \mathcal{A}$ a closed ideal. It is known that there is a bounded net $(e_\alpha) \subset \mathcal{I}$ which is a LAI as well as a RAI for $\mathcal{I}$, see e.g. [5, Proposition 1.8.5] or [4, Theorem 3.2.21]. Moreover, $\mathcal{L}(X)$ fails to have non-trivial closed subideals by (2.1) for $X = \ell^p$ with $1 \le p < \infty$ or $X = c_0$, since here $\mathcal{K}(X)$ is the unique proper closed ideal of $\mathcal{L}(X)$, see e.g. [19, sections 5.1–5.2].

(ii) If X has the A.P., then $\mathcal{K}(X) = \mathcal{A}(X)$, so by (2.1) there are no non-trivial closed $\mathcal{K}(X)$-subideals. The existence of a LAI or a RAI in $\mathcal{K}(X)$ is related to the compact approximation property, see [6, Theorem 2.7] and [27, Proposition 7].

(iii) Let Z_p be the Banach space in (2.3), where $1 < p < \infty$. Lemma 3.1 implies that $\mathcal{K}(Z_p)$ cannot have a RAI. Namely, for any $\emptyset \neq A \subsetneq \mathbb{N}$ the closed $\mathcal{K}(Z_p)$-subideal $\mathcal{I}_A$ is not a right ideal of $\mathcal{L}(Z_p)$ by property (ii) following (2.5).

For classical spaces X that have the A.P., including L^p for $1 \le p < \infty$ or $C(0,1)$, the algebra $\mathcal{L}(X)$ does not contain any non-trivial closed $\mathcal{K}(X)$-subideals, see Remarks 3.2.(ii). The following elementary observation will lead to large families of non-trivial closed $\mathcal{S}(X)$-subideals of $\mathcal{L}(X)$, where the conditions apply to many classical Banach spaces (see Proposition 3.6).

Proposition 3.3. Suppose that X is a Banach space such that

$$dim\big(\mathcal{S}(X)/\mathcal{K}(X)\big) \ge 2, \tag{3.1}$$

$$UV \in \mathcal{K}(X) \text{ for any } U, V \in \mathcal{S}(X). \tag{3.2}$$

(i) If $\mathcal{K}(X) \subsetneq M \subsetneq \mathcal{S}(X)$ is any closed linear subspace, then M is a closed $\mathcal{S}(X)$-subideal of $\mathcal{L}(X)$.

(ii) Let $\mathcal{K}(X) \subsetneq M_1, M_2 \subsetneq \mathcal{S}(X)$ be closed linear subspaces. Then the subideals $M_1 \cong M_2$ if and only if $M_2 = UM_1U^{-1}$ for some linear isomorphism $U \in \mathcal{L}(X)$.

Proof. (i) If $S \in M$ and $U \in \mathcal{S}(X)$, then $US \in M$ and $SU \in M$ by (3.2).

(ii) Let $\theta : M_1 \to M_2$ be a Banach algebra isomorphism. Use [3, Corollary 3.2] to find a linear isomorphism $U \in \mathcal{L}(X)$ for which $\theta(S) = USU^{-1} \in M_2$ for $S \in \mathcal{M}_1$, so that $UM_1U^{-1} \subset M_2$. The inner automorphism $\theta : \mathcal{L}(X) \to \mathcal{L}(X)$ has the unique inverse θ^{-1} given by $\theta^{-1}(T) = U^{-1}TU$ for $T \in \mathcal{L}(X)$. If $T \in M_2$ is arbitrary, then

$$T = U(U^{-1}TU)U^{-1} = U(\theta^{-1}(T))U^{-1} \in UM_1U^{-1}$$

as $\theta^{-1}(T) \in M_1$. Conclude that $M_2 = UM_1U^{-1}$.

Conversely, suppose that $M_2 = UM_1U^{-1}$ for some linear isomorphism $U \in \mathcal{L}(X)$. Then the restriction of the inner automorphism $S \mapsto \theta(S) = USU^{-1}$ on $\mathcal{L}(X)$ is a Banach algebra isomorphism $M_1 \to M_2$. □

Recall that Tarbard [21] constructed a Banach space X_2, for which (3.2) holds and $dim\big(\mathcal{S}(X_2)/\mathcal{K}(X_2)\big) = 1$. We next provide examples in the above setting of closed linear subspaces M, such that M is not an ideal of $\mathcal{L}(X)$ and $UMU^{-1} \neq M$ for some linear isomorphism U. It will be convenient to work on $X \oplus X$, but the spaces X listed below in Proposition 3.6 satisfy $X \oplus X \approx X$. Note from Eidelheit's theorem that if $V : X \oplus X \to X$ is a linear isomorphism, then $S \mapsto \psi(S) = VSV^{-1}$ is a Banach algebra isomorphism $\mathcal{L}(X \oplus X) \to \mathcal{L}(X)$. Moreover, $\psi(\mathcal{K}(X \oplus X)) = \mathcal{K}(X)$ and $\psi(\mathcal{S}(X \oplus X)) = \mathcal{S}(X)$, so that a non-trivial closed $\mathcal{S}(X \oplus X)$-subideal $\mathcal{I}$ transfers to a non-trivial closed $\mathcal{S}(X)$-subideal $\psi(\mathcal{I})$ of $\mathcal{L}(X)$. We will often write operators $U \in \mathcal{L}(X \oplus X)$ as $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$, where $U_{kl} = P_k U J_l \in \mathcal{L}(X)$ for $k, l = 1, 2$. Here

P_k and J_l are the canonical projections and inclusions associated to $X \oplus X$. Given closed linear subspaces $M_{ij} \subset \mathcal{L}(X)$ for $i, j = 1, 2$ we denote

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \Big\{ U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in \mathcal{L}(X \oplus X) : U_{ij} \in M_{ij} \text{ for } i, j = 1, 2 \Big\}.$$

We write $\mathcal{A} \ncong \mathcal{B}$ to indicate non-isomorphic Banach algebras $\mathcal{A}$ and $\mathcal{B}$.

Theorem 3.4. *Suppose that X is a Banach space that satisfies* (3.2) *and*

$$\mathcal{S}(X)/\mathcal{K}(X) \text{ is infinite-dimensional.} \tag{3.3}$$

(i) For any closed linear subspace $\mathcal{K}(X) \subsetneq M \subsetneq \mathcal{S}(X)$ put

$$\mathcal{I}(M) = \begin{bmatrix} M & \mathcal{K}(X) \\ \mathcal{K}(X) & \mathcal{K}(X) \end{bmatrix}. \tag{3.4}$$

Let $U \in \mathcal{L}(X \oplus X)$ be the isomorphism $U(x, y) = (y, x)$ for $(x, y) \in X \oplus X$, and $\theta(S) = USU^{-1}$ for $S \in \mathcal{L}(X \oplus X)$. Then $\mathcal{I}(M)$ is a non-trivial closed $\mathcal{S}(X \oplus X)$-subideal of $\mathcal{L}(X \oplus X)$, and $\mathcal{J}(M) := \theta(\mathcal{I}(M))$ is also a non-trivial closed $\mathcal{S}(X \oplus X)$-subideal, for which $\mathcal{J}(M) \cong \mathcal{I}(M)$ and $\mathcal{J}(M) \neq \mathcal{I}(M)$.

(ii) Let $(T_k) \subset \mathcal{S}(X)$ be a linearly independent sequence modulo $\mathcal{K}(X)$, and let $M_n \subset \mathcal{S}(X)$ be the closed linear subspace spanned by $\{T_j : 1 \le j \le n\} \cup \mathcal{K}(X)$ for $n \in \mathbb{N}$. Then $\{\mathcal{I}(M_n) : n \in \mathbb{N}\}$ is an increasing sequence of non-trivial closed $\mathcal{S}(X \oplus X)$-subideals of $\mathcal{L}(X \oplus X)$, such that $\mathcal{I}(M_n) \ncong \mathcal{I}(M_k)$ for any $n \neq k$.

(iii) Suppose that $\mathcal{K}(X) \subsetneq \mathcal{J}_1, \mathcal{J}_2 \subsetneq \mathcal{S}(X)$ are closed ideals of $\mathcal{L}(X)$ such that $\mathcal{J}_1 \neq \mathcal{J}_2$. Then $\mathcal{I}(\mathcal{J}_1)$ and $\mathcal{I}(\mathcal{J}_2)$ are non-trivial closed $\mathcal{S}(X \oplus X)$-subideals for which $\mathcal{I}(\mathcal{J}_1) \ncong \mathcal{I}(\mathcal{J}_2)$.

Proof. (i) $\mathcal{I}(M)$ is a closed $\mathcal{S}(X \oplus X)$-subideal of $\mathcal{L}(X \oplus X)$ by part (i) of Proposition 3.3. Fix $T \in M \setminus \mathcal{K}(X)$, let $S = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{I}(M)$ and $V = \begin{bmatrix} 0 & 0 \\ I_X & 0 \end{bmatrix}$. It follows that $VS = \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix} \notin \mathcal{I}(M)$, so that $\mathcal{I}(M)$ is a non-trivial subideal.

The above θ defines an inner automorphism θ on $\mathcal{L}(X \oplus X)$, so that $\mathcal{J}(M) = \theta(\mathcal{I}(M)) \cong \mathcal{I}(M)$. It is not difficult to check that

$$\mathcal{J}(M) = \begin{bmatrix} \mathcal{K}(X) & \mathcal{K}(X) \\ \mathcal{K}(X) & M \end{bmatrix} \neq \mathcal{I}(M),$$

since $U = \begin{bmatrix} 0 & I_X \\ I_X & 0 \end{bmatrix}$. As above $\mathcal{J}(M)$ is not an ideal of $\mathcal{L}(X \oplus X)$.

(ii) The linear span $M_n = span(\{T_j : 1 \le j \le n\} \cup \mathcal{K}(X))$ is a closed linear subspace that satisfies $\mathcal{K}(X) \subsetneq M_n \subsetneq \mathcal{S}(X)$ for all $n \in \mathbb{N}$. Hence $\mathcal{I}(M_n) \subsetneq \mathcal{I}(M_{n+1})$ are non-trivial closed $\mathcal{S}(X \oplus X)$-subideals of $\mathcal{L}(X \oplus X)$ for $n \in \mathbb{N}$ by part (i).

Suppose to the contrary that $k < n$ and $\theta : \mathcal{I}(M_n) \to \mathcal{I}(M_k)$ is a Banach algebra isomorphism. By [3, Corollary 3.2] pick a linear isomorphism $U \in \mathcal{L}(X \oplus X)$, such that $\theta(T) = UTU^{-1}$ for all $T \in \mathcal{I}(M_n)$. Since θ is a Banach algebra isomorphism of $\mathcal{L}(X \oplus X)$ for which $\theta(\mathcal{K}(X \oplus X)) = \mathcal{K}(X \oplus X)$, there is an induced linear isomorphism $\mathcal{I}(M_n)/\mathcal{K}(X \oplus X) \to \mathcal{I}(M_k)/\mathcal{K}(X \oplus X)$. This cannot happen since $\mathcal{I}(M_r)/\mathcal{K}(X \oplus X)$ is r-dimensional for $r \in \mathbb{N}$.

(iii) $\mathcal{I}(\mathcal{J}_1)$ and $\mathcal{I}(\mathcal{J}_2)$ are non-trivial closed subideals of $\mathcal{L}(X \oplus X)$ by part (i). Suppose that $\theta : \mathcal{I}(\mathcal{J}_2) \to \mathcal{I}(\mathcal{J}_1)$ is a Banach algebra isomorphism. We claim that $\mathcal{J}_1 \subset \mathcal{J}_2$, so that $\mathcal{J}_1 = \mathcal{J}_2$ by symmetry.

By [3, Corollary 3.2] there is a linear isomorphism $V \in \mathcal{L}(X \oplus X)$, such that $\theta(T) = VTV^{-1}$ for all $T \in \mathcal{I}(\mathcal{J}_2)$. Let $S_0 \in \mathcal{J}_1$ be arbitrary. Since $S = J_1 S_0 P_1 \in \mathcal{I}(\mathcal{J}_1)$, there is $T \in \mathcal{I}(\mathcal{J}_2)$ with $S = VTV^{-1}$. Write $T = \begin{bmatrix} T_{11} & 0 \\ 0 & 0 \end{bmatrix} + R$, where $T_{11} \in \mathcal{J}_2$ and $R \in \mathcal{K}(X \oplus X)$. We get that

$$S_0 = P_1 S J_1 = (P_1 V J_1) T_{11} (P_1 V^{-1} J_1) + P_1 (V R V^{-1}) J_1 \in \mathcal{J}_2,$$

since $\mathcal{J}_2$ is an ideal of $\mathcal{L}(X)$. □

Remarks 3.5. (i) Conditions (3.2) and (3.3) imply that $\mathcal{S}(X)$ has neither a LAI nor a RAI. There are also versions of Proposition 3.3 and Theorem 3.4 for certain other pairs of closed ideals $\mathcal{J} \subset \mathcal{I} \subset \mathcal{L}(X)$, but we do not pursue this here.

(ii) No examples of non-trivial closed $\mathcal{K}(X)$-subideals are available along the line of Theorem 3.4, since it is unknown whether there is a Banach space X such that $\mathcal{K}(X)/\mathcal{A}(X)$ is non-zero and 2-nilpotent, see e.g. [24, Remarks 5.4.(ii)].

We briefly recall some classical Banach spaces X to which Theorem 3.4 applies. The spaces listed here are known to satisfy $X \oplus X \approx X$.

Proposition 3.6. Conditions (3.2) and (3.3) are satisfied by $X = L^p$ with $1 \le p < \infty$ and $p \neq 2$, $C(0,1)$, ℓ^∞, as well as $\ell^p \oplus \ell^q$ and $\ell^p \oplus c_0$ with $1 \le p < q < \infty$.

Proof. We first make a preliminary observation towards (3.3). Fix a partition $\{A_j : j \in \mathbb{N}\}$ of $\mathbb{N}$ into infinite subsets, and suppose that $1 \le p < q < \infty$. Put $a = \chi_A \in \ell^\infty$ for $A \subset \mathbb{N}$ and let $D_a : \ell^p \to \ell^q$ be the corresponding diagonal operator. Here $D_a \in \mathcal{S}(\ell^p, \ell^q)$ by the total incomparability of ℓ^p and ℓ^q, see e.g. [12, Proposition 2.a.2].

Claim. The family $\{D_{a_j} : j \in \mathbb{N}\}$ is linearly independent modulo $\mathcal{K}(\ell^p, \ell^q)$.

Namely, let $c_1, \dots, c_m$ be scalars for some $m \in \mathbb{N}$, and let (e_k) be the unit vector basis in ℓ^p. If $c_j \neq 0$ for some $j \in \{1, \dots, m\}$, then $\sum_{r=1}^m c_r D_{a_r} \notin \mathcal{K}(\ell^p, \ell^q)$. In fact, for $k \in A_j$ one has

$$\sum_{r=1}^{m} c_r D_{a_r} e_k = c_j e_k,$$

which fails to have any norm-convergent subsequences. An analogous claim also holds for $\{D_{a_j} : j \in \mathbb{N}\} \subset \mathcal{S}(\ell^p, c_0)$, with a similar argument.

Suppose that $X = L^p$ with $1 \le p < \infty$ and $p \neq 2$. Recall that if $p \neq 1$, then ℓ^p and ℓ^2 are isomorphic to complemented subspaces of L^p. For $p > 2$ let $P : L^p \to \ell^2$ be a surjection and $J : \ell^p \to L^p$ a linear embedding. Deduce from the above Claim that $\{J D_{a_j} P : j \in \mathbb{N}\} \subset \mathcal{S}(X)$ is linearly independent modulo $\mathcal{K}(X)$. For $1 < p < 2$ reverse the roles of 2 and p, and for $p = 1$ use the facts that ℓ^1 is complemented in L^1 and ℓ^2 embeds isomorphically into L^1.

Simple modifications yield (3.3) in the other cases. For $X = \ell^p \oplus \ell^q$, where $1 \le p < q < \infty$, it suffices to consider $\{D_{a_j} : j \in \mathbb{N}\} \subset \mathcal{S}(\ell^p, \ell^q) \subset \mathcal{S}(\ell^p \oplus \ell^q)$, and analogously for $X = \ell^p \oplus c_0$. Finally, [20, Proposition 1.3] and standard duality imply that ℓ^2 is a quotient space of both $C(0,1)$ and ℓ^∞. Since ℓ^p embeds isometrically into $C(0,1)$ and ℓ^∞ for $p > 2$, we may again proceed as above.

Milman [14, Teorema 7] showed that (3.2) holds for $X = L^p$ with $1 < p < \infty$ and $p \neq 2$. For $p = 1$ it was shown by Pełczynski [18, Theorem II.1] that $\mathcal{S}(L^1)$ equals the class of weakly compact operators on L^1, so that (3.2) follows from the Dunford-Pettis property of L^1, see e.g. [1, Theorem 5.4.5.(i)]. The same argument also applies to $X = C(0,1)$ and $X = \ell^\infty$ in view of [18, Theorem I.1] and [1, Theorem 5.4.5.(ii)]. Finally, the identification of the component ideals of $\mathcal{S}(\ell^p \oplus \ell^q)$ and $\mathcal{S}(\ell^p \oplus c_0)$, see e.g. [19, Theorem 5.3.2], easily yields (3.2) for $\ell^p \oplus \ell^q$ and $\ell^p \oplus c_0$. □

Part (iii) of Theorem 3.4 can be improved e.g. for L^p, with $1 < p < \infty$, $p \neq 2$, by using the existence of specific large families of closed ideals of $\mathcal{L}(L^p)$.

Theorem 3.7. *(i) Let X be L^p with $1 < p < \infty$ and $p \neq 2$, $\ell^p \oplus \ell^q$ with $1 \leq p < q < \infty$ or $\ell^p \oplus c_0$ with $1 < p < \infty$. Then there are $2^{\mathfrak{c}}$ non-trivial closed $\mathcal{S}(X)$-subideals of $\mathcal{L}(X)$ that are pairwise non-isomorphic as Banach algebras.*

(ii) For $1 < p < \infty$ and $p \neq 2$ there is a family $\{\mathcal{I}_\alpha : \alpha \in \mathcal{C}\}$ of the size of the continuum of singly generated non-trivial closed $\mathcal{S}(L^p)$-subideals of $\mathcal{L}(L^p)$ that are pairwise non-isomorphic as Banach algebras, but linearly isomorphic as Banach spaces.

Proof. (i) Johnson and Schechtman [11, Remark 4.4] proved that there are $2^{\mathfrak{c}}$ different closed ideals $\mathcal{J}$ of $\mathcal{L}(L^p)$ which satisfy $\mathcal{K}(L^p) \subset \mathcal{J} \subset \mathcal{S}(L^p)$. The claim then follows from part (iii) of Theorem 3.4, and the facts that $L^p \approx L^p \oplus L^p$ and that $\mathcal{S}(L^p)$ has at most $2^{\mathfrak{c}}$ closed subspaces, see e.g. [11, p. 107]. Analogous results about closed ideals of $\mathcal{L}(\ell^p \oplus \ell^q)$ and of $\mathcal{L}(\ell^p \oplus c_0)$ were shown by Freeman, Schlumprecht and Zsák [9, Corollary 9] (see also its preceding Remark and the Remark on p. 17 of [9]).

(ii) Let $\mathcal{C}$ be a continuum of infinite subsets of $\mathbb{N}$ such that $|\alpha \cap \beta| < \infty$ for all $\alpha, \beta \in \mathcal{C}$, and let $1 < p < 2$. By [11, Remarks 4.3 and 4.4], there is a closed complemented subspace $X \subset L^p$ together with operators $U, P, T_\alpha \in \mathcal{L}(X)$ for $\alpha \in \mathcal{C}$, such that the following properties hold:

$$T_\alpha UP \in \mathcal{S}(X) \setminus \mathcal{K}(X) \text{ for all } \alpha \in \mathcal{C}, \tag{3.5}$$

$$T_\beta UP \notin [T_\alpha UP] \text{ for any } \alpha, \beta \in \mathcal{C}, \alpha \neq \beta. \tag{3.6}$$

Here $[T_\alpha UP]$ denotes the closed ideal of $\mathcal{L}(X)$ generated by $T_\alpha UP$. Write $L^p = X \oplus M$ and for $\alpha \in \mathcal{C}$ consider the operators

$$R_\alpha = \begin{bmatrix} T_\alpha UP & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{S}(L^p) \setminus \mathcal{K}(L^p), \quad S_\alpha = \begin{bmatrix} R_\alpha & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{S}(L^p \oplus L^p). \tag{3.7}$$

For $\alpha \in \mathcal{C}$ consider the closed subspace

$$M_\alpha = \{\lambda R_\alpha + R \mid \lambda \in \mathbb{K} \text{ and } R \in \mathcal{K}(L^p)\} \subset \mathcal{S}(L^p)$$

and let

$$\mathcal{I}_\alpha := \mathcal{I}(M_\alpha) = \{\lambda S_\alpha + R \mid \lambda \in \mathbb{K} \text{ and } R \in \mathcal{K}(L^p \oplus L^p)\}$$

be as in (3.4). By part (i) of Theorem 3.4 the family $\{\mathcal{I}_\alpha : \alpha \in \mathcal{C}\}$ consists of non-trivial closed $\mathcal{S}(L^p \oplus L^p)$-subideals of $\mathcal{L}(L^p \oplus L^p)$. Here $\mathcal{I}_\alpha$ is a singly generated ideal of $\mathcal{S}(L^p \oplus L^p)$ according to (2.1), since L^p has a Schauder basis. It is clear that $\mathcal{I}_\alpha \approx \mathcal{I}_\beta$ for all $\alpha, \beta \in \mathcal{C}$, as the subideals $\mathcal{I}_\alpha$ for $\alpha \in \mathcal{C}$ are one-dimensional extensions of $\mathcal{K}(L^p \oplus L^p)$.

It follows that $\mathcal{I}_\alpha \not\cong \mathcal{I}_\beta$ by a modification of the argument of Theorem 3.4.(iii). In fact, let $\alpha \neq \beta$ and assume that there is a Banach algebra isomorphism $\theta : \mathcal{I}_\alpha \to \mathcal{I}_\beta$.

Hence there is a linear isomorphism $V \in \mathcal{L}(L^p \oplus L^p)$ such that $\theta(T) = VTV^{-1}$ for $T \in \mathcal{I}_\alpha$. Pick $\lambda \in \mathbb{K}$ and $R \in \mathcal{K}(L^p \oplus L^p)$ such that

$$S_\beta = \theta(\lambda S_\alpha + R) = V(\lambda S_\alpha + R)V^{-1} = \lambda V J_0 T_\alpha U P P_0 V^{-1} + VRV^{-1},$$

where J_0 is the inclusion map $X \to L^p \oplus L^p$ (into the first copy of L^p), and P_0 is the corresponding projection $L^p \oplus L^p \to X$. Deduce that

$$T_\beta UP = P_0 S_\beta J_0 = \lambda(P_0 V J_0) T_\alpha UP(P_0 V^{-1} J_0) + P_0 VRV^{-1} J_0 \in [T_\alpha UP], \tag{3.8}$$

which contradicts (3.6).

Let $2 < q < \infty$ and p be the dual exponent of q. For $\alpha \in \mathcal{C}$ define

$$\mathcal{J}_\alpha = \{\lambda S_\alpha^* + R \mid \lambda \in \mathbb{K} \text{ and } R \in \mathcal{K}(L^q \oplus L^q)\},$$

where S_α is given by (3.7). Observe that $S_\alpha^* \in \mathcal{S}(L^q \oplus L^q) \setminus \mathcal{K}(L^q \oplus L^q)$ by [25, Corollary 2] or [14, p. 19], so that $\mathcal{J}_\alpha \subset \mathcal{S}(L^q \oplus L^q)$ is a non-trivial closed $\mathcal{S}(L^q \oplus L^q)$-subideal of $\mathcal{L}(L^q \oplus L^q)$ for all $\alpha \in \mathcal{C}$. Moreover, $\mathcal{J}_\alpha \approx \mathcal{J}_\beta$ for all $\alpha, \beta \in \mathcal{C}$.

Finally, suppose that $\alpha \neq \beta$ and assume that there is a Banach algebra isomorphism $\theta : \mathcal{J}_\alpha \to \mathcal{J}_\beta$, where as before $\theta(T) = VTV^{-1}$ for all $T \in \mathcal{J}_\alpha$ and some linear isomorphism $V \in \mathcal{L}(L^q \oplus L^q)$. Pick $\lambda \in \mathbb{K}$ and $R \in \mathcal{K}(L^q \oplus L^q)$ such that $S_\beta^* = V(\lambda S_\alpha^* + R)V^{-1}$. Deduce from reflexivity that $S_\beta = (V^*)^{-1}(\lambda S_\alpha + R^*)V^*$, which leads to a contradiction as in (3.8). □

Remark 3.8. There are versions of part (ii) of Theorem 3.7 for $X = L^1$, $C(0,1)$ and ℓ^∞. For L^1 use the family $\{J_p Q : p \in (2,\infty)\} \subset \mathcal{S}(L^1)$ from [10, p. 701], which has similar properties to (3.5) and (3.6). Analogous results for $C(0,1)$ are contained in [10, Corollary 3.2], and for L^∞ in [10, Theorem 4.2 and Corollary 4.4], where $\ell^\infty \approx L^\infty$ by [1, Theorem 4.3.10]. The details are left to the interested reader.

Acknowledgements. We are indebted to Gideon Schechtman for a query during the conference Functional Analysis in Lille 2022, which motivated Theorem 2.1. We are also grateful to Thomas Schlumprecht for a question during IWOTA 2020 (August, 2021) which inspired Proposition 2.7, and to him and Niels Laustsen for suggestions about the largest possible size of the family of closed $\mathcal{K}(Z_p)$-subideals (see Remarks 2.5.(i)). We thank the referee for suggestions that improved the presentation. Henrik Wirzenius gratefully acknowledges the financial support of the Magnus Ehrnrooth Foundation.

References

1. F. Albiac and N.J. Kalton, *Topics in Banach Space Theory*, Grad. Texts in Math., vol. 233, Springer-Verlag, New York, 2006.
2. A. Arias and J.D. Farmer, *On the structure of tensor products of ℓ_p-spaces*, Pacific J. Math. **175** (1996), 13–37.
3. P.R. Chernoff, *Representations, Automorphisms and Derivations of Some Operator Algebras*, J. Funct. Anal. **12** (1973), 275–289.
4. H.G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monogr. (N.S.), vol. 24, Clarendon Press, Oxford, 2000.
5. J. Dixmier, *Les C^*-algèbres et leurs représentations*, Cahiers Scientifiques, Fasc. XXIX Gauthier-Villars & Cie, Éditeur-Imprimeur, Paris, 1964.
6. P.G. Dixon, *Left approximate identities in algebras of compact operators on Banach spaces*, Proc. Roy. Soc. Edinburgh Sect. A **104** (1986), 169–175.
7. M. Eidelheit, *On isomorphisms of rings of linear operators*, Studia Math. **9** (1940), 97–105

8. C.K. Fong and H. Radjavi, *On ideals and Lie ideals of compact operators*, Math. Ann. **262** (1983), 23–28.
9. D. Freeman, T. Schlumprecht and A. Zsák, *Banach spaces for which the space of operators has* $2^{\mathfrak{c}}$ *closed ideals*, Forum Math. Sigma 9 (2021), Paper No. e27, 20 pp.
10. W.B. Johnson, G. Pisier, and G. Schechtman, *Ideals in* $L(L_1)$, Math. Ann. **376** (2020), 693–705.
11. W.B. Johnson and G. Schechtman, *The number of closed ideals in* $L(L_p)$, Acta Math. **227** (2021), 103–113.
12. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 92, Springer-Verlag, Berlin, 1977.
13. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II, Function Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 97, Springer-Verlag, Berlin, 1979.
14. V.D. Milman, *Operators of class* C_0 *and* C_0^* (Russian), Teor. Funktsiĭ Funkcional. Anal. i Priložen. **10** (1970), 15–26.
15. T.W. Palmer, *Banach Algebras and The General Theory of* **-algebras. Vol. 1. Algebras and Banach Algebras*, Encyclopedia of Mathematics and its Applications, vol. 49, Cambridge University Press, 1994.
16. S. Patnaik and G. Weiss, *Subideals of operators*, J. Operator Theory **70** (2013), 355–373.
17. S. Patnaik and G. Weiss, *Subideals of operators - a survey and introduction to subideal-traces*, Operator Theory in Harmonic and Non-commutative Analysis (Cham), Oper. Theory Adv. Appl., vol. 240, Birkhäuser, 2014, pp. 221–234.
18. A. Pełczynski, *On strictly singular and strictly cosingular operators. I. Strictly singular and strictly cosingular operators in* $C(S)$*-spaces. II. Strictly singular and strictly cosingular operators in* $L(\nu)$*-spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **13** (1965), 31–41.
19. A. Pietsch, *Operator Ideals*, North-Holland Mathematical Library vol. 20, North-Holland, Amsterdam, 1980.
20. H.P. Rosenthal, *On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from* $L^p(\mu)$ *to* $L^r(\nu)$, J. Funct. Anal. **4** (1969), 176-214.
21. M. Tarbard, *Hereditarily indecomposable, separable* $\mathcal{L}^\infty$ *Banach spaces with* ℓ^1 *dual having few but not very few operators*, J. Lond. Math. Soc. (2) **85** (2012), 737–764.
22. T. Terzioğlu, *A characterization of compact linear mappings*, Arch. Math. (Basel) **22** (1971), 76–78.
23. H-O. Tylli and H. Wirzenius, *The quotient algebra of compact-by-approximable operators on Banach spaces failing the approximation property*, J. Aust. Math. Soc. **110** (2021), 266–288.
24. H-O. Tylli and H. Wirzenius, *Closed ideals in the algebra of compact-by-approximable operators*, J. Funct. Anal. **282** (2022), Paper No. 109328, 47 pp.
25. L. Weis, *On perturbations of Fredholm operators in* $L_p(\mu)$*-spaces*, Proc. Amer. Math. Soc. **67** (1977), 287–292.
26. H. Wirzenius, *Quotient algebras of Banach operator ideals related to non-classical approximation properties*, J. Math. Anal. Appl. **517** (2023), Paper No. 126637, 33 pp.
27. Y. Zhang, *Unbounded approximate identities in algebras of compact operators on Banach spaces*, Math. Proc. Camb. Phil. Soc. **134** (2003), 187–192.

Tylli: Department of Mathematics and Statistics, Box 68, FI-00014 University of Helsinki, Finland
Email address: hans-olav.tylli@helsinki.fi

Wirzenius: Department of Mathematics and Statistics, Box 68, FI-00014 University of Helsinki, Finland
Email address: henrik.wirzenius@helsinki.fi