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Fundamental Solutions for the Laplace–Beltrami Operator Defined by the Conformal Hyperbolic Metric and Jacobi Polynomials

Sirkka-Liisa Eriksson[1] · Heikki Orelma[2]

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Abstract
In this paper we study fundamental solutions for the Laplace–Beltrami operator

$$\Delta_\alpha f = x_n^{\frac{\alpha}{n-2}}\Big(\Delta f - \frac{\alpha}{x_n}\frac{\partial f}{\partial x_n}\Big),$$

defined on smooth enough functions in $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$. We represent explicit formulas for the fundamental solutions. Moreover, we establish fundamental solutions using Jacobi polynomials when $n = 3, 5, 7, \ldots$.

1 Introduction

In this paper, fundamental solutions of the Laplace–Beltrami operator of the hyperbolic upper half-space are considered. This is a continuation of the previous research by the authors in [6, 8–11], where we have looked at different special cases. In [12] the first author and Vuojamo found the fundamental solution in terms of associate Legendre functions of the second kind, but explicit representations in terms of elementary functions of kernels were not presented.

Dedicated to Prof. John Ryan, best wishes and new goals for retirement.

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This article is part of Topical Collection in Honor of Prof. John Ryan's Retirement.

✉ Sirkka-Liisa Eriksson
sirkka-liisa.eriksson@helsinki.fi

Heikki Orelma
heikki.orelma@tuni.fi

1 University of Helsinki, Helsinki, Finland

2 Tampere University, Tampere, Finland

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The theory is closely connected to to the axially symmetric potential theory created by Weinstein [20]. Heinz Leutwiler [14] initiated the research of Laplace–Beltrami equations connected to the differential equation of Weinstein. The general theory was also researched by Ryan et al. [4] and it has also connections to research of iterated Dirac operators of Ryan [19] (see [7]). It has interesting connections to hyperbolic Brownian motion, see e.g. [5].

The Laplace–Beltrami operator is a geometric operator, i.e. its form depends on the metric of the space. In this paper we consider the conformal metric of the hyperbolic upper half-space. It is generally interesting because the operator is with non-constant coefficients and thus considerably more difficult to handle than the constant coefficient cases. In this paper, we point out that the parity of the treated space has a fundamental effect on the shape of the fundamental solutions. We also state that in odd dimensions the basic solution can be presented using Jacobian polynomials. There is no corresponding construction in the even case. We will return to this case in the future.

The structure of the article is as follows:

- In Sect. 2, the necessary preliminaries and definitions are given.
- In Sect. 3, we consider fundamental solutions. Some examples are given.
- In Sect. 4, we simplify fundamental solutions in odd spaces using Jacobi polynomials and compute examples.

2 Conformal Hyperbolic Upper-Half Space and Laplace–Beltrami Operators

Consider the hyperbolic upper half-space

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\},$$

equipped with the metric

$$g_H = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

The geometry of the hyperbolic space $(\mathbb{R}^n_+, g_H)$ is well known and studied. The geodesics are circular arcs perpendicular to the hyperplane $x_n = 0$, that is half-circles whose origin is on $x_n = 0$, and straight vertical lines parallel to the x_n-axis.

The distance between two points $x, y \in \mathbb{R}^n_+$ with respect to the metric g_H is (see e.g. Theorem 4.6.1 in [18])

$$d_H(x, y) = \operatorname{arcosh} \lambda(x, y)$$

where

$$\lambda(x, y) = 1 + \frac{|x - y|^2}{2x_n y_n} \tag{1}$$

is a symmetric invariant, where $|x|^2 = x_1^2 + \cdots + x_n^2$ is the usual Euclidean quadratic form.

For $n > 2$ we define the conformal metric on the upper-half space by

$$g_\alpha := \frac{dx_1^2 + \cdots + dx_n^2}{x_n^{\frac{2\alpha}{n-2}}},$$

where $\alpha \in \mathbb{R}$. One reason to consider the preceding conformal metric is the simple form of the associated Laplace–Beltrami operator

$$\Delta_\alpha f = x_n^{\frac{2\alpha}{n-2}} \left(\Delta f - \frac{\alpha}{x_n} \frac{\partial f}{\partial x_n} \right), \tag{2}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Euclidean Laplacian. When $\alpha = n - 2$, we obtain the hyperbolic Laplace operator

$$\Delta_{hyp} f = x_n^2 \Delta f - (n-2)\, x_n \frac{\partial f}{\partial x_n}.$$

If $\Omega \subset \mathbb{R}^n_+$ is open, a twice continuously differentiable function $f : \Omega \to \mathbb{R}$ is called α-hyperbolic harmonic if

$$x_n^2 \Delta f - \alpha x_n \frac{\partial f}{\partial x_n} = 0.$$

If $\alpha = n - 2$, we call an α-hyperbolic harmonic function just hyperbolic harmonic. Heinz Leutwiler initiated the research of hyperbolic harmonic functions and their function theory in [15, 16]. It has been continued intensively by the first author, Leutwiler and the second author and there is a book in preparation [8].

3 Fundamental Solution for Δ_α

In this paper, we consider fundamental solutions of the operator Δ_α. A fundamental solution is a function $H_{\alpha,n}(x, y)$ that satisfies the equation

$$\Delta_\alpha H_{\alpha,n}(\,\cdot, y) = \omega_{n-1} \delta(y),$$

in the distribution sense, where δ is the usual Dirac delta distribution at $y \in \mathbb{R}^n_+$. In above the ω_{n-1} is the surface area of the unit ball $S^{n-1} \subset \mathbb{R}^n$. The necessary condition is, that $H_{\alpha,n}$ is singular at the diagonal $x = y$.

The fundamental solutions are presented in terms of associated Legendre functions of the second kind. Associated Legendre functions are defined by (see e.g. 8.703 in [13])

$$\hat{Q}_\nu^\mu(z) = \frac{\sqrt{\pi}\,\Gamma\left(\nu+\mu+1\right)\left(z^2-1\right)^{\frac{\mu}{2}}}{2^{\nu+1}z^{\nu+\mu+1}\Gamma\left(\rho+\frac{3}{2}\right)}\,{}_2F_1\left(\frac{1}{2}\nu+\frac{1}{2}\mu+1, \frac{1}{2}\nu+\frac{1}{2}\mu+\frac{1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right),$$

where Γ is the usual gamma function and ${}_2F_1$ is the hypergeometric function defined with power series representation (see e.g. [1, 2, 13])

$$ {}_2F_1\left(a, b; c; z\right) = \sum_{m=0}^{\infty} \frac{(a)_m\,(b)_m}{(c)_m\,m!} z^m. \tag{3}$$

for $|z| < 1$, and $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \ldots$. In the hypergeometric function the Pochammer symbol is defined by

$$(q)_m = \frac{\Gamma(q+m)}{\Gamma(q)} = q(q+1)\cdots(q+(m-1)). \tag{4}$$

Hence the hypergometric function terminates if a or b is a negative integer.

A reader should observe that the preceding definition for a associated Legendre function is up to the constant $e^{i\pi\nu}$ the usual one, see e.g. in [13].

In [5] the following theorem is verified.

Theorem 3.1 *Let x and y be distinct elements in $\mathbb{R}_+^n$ and $\alpha \in \mathbb{R}$. Denote $r_h = d_H\left(x, y\right)$. Define*

$$\rho_\alpha = \begin{cases} \frac{\alpha}{2}, & \text{if } \alpha \geq 0, \\ -\frac{\alpha+2}{2}, & \text{if } \alpha < 0. \end{cases}$$

(a) *If $n \in \mathbb{N}$ and $n \geq 3$, the fundamental solution is*

$$H_{\alpha,n}\left(x, y\right) = \frac{x_n^{\frac{\alpha+2-n}{2}}\, y_n^{\frac{\alpha+2-n}{2}}}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}\left(\lambda^2(x,y)-1\right)^{\frac{2-n}{4}}\hat{Q}_{\rho_\alpha}^{\frac{n-2}{2}}\left(\lambda(x,y)\right)$$
$$= x_n^{\frac{\alpha+2-n}{2}}\, y_n^{\frac{\alpha+2-n}{2}} \sinh^{2-n} r_h\, g_{\rho_\alpha,n}\left(r_h\right),$$

where

$$g_{\rho_\alpha,n}\left(r_h\right) = C\left(\alpha,n\right)\lambda^{\frac{n-4-2\rho_\alpha}{2}}\,{}_2F_1\left(\frac{2\rho_\alpha-n+4}{4}, \frac{2\rho_\alpha-n+6}{4}; \frac{2\rho_\alpha+3}{2}; \frac{1}{\cosh^2 r_h}\right)$$

and

$$C\left(\alpha, n\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{2\rho_\alpha+n}{2}\right)}{2^{\frac{2\rho_\alpha+n}{2}}\Gamma\left(\frac{2\rho_\alpha+3}{2}\right)\Gamma\left(\frac{n}{2}\right)}.$$

(b) *If $n = 2$, the fundamental solution is*

$$H_{\alpha,2}(x, y) = x_n^{\frac{\alpha}{2}} y_n^{\frac{\alpha}{2}} \hat{Q}_{\rho_\alpha}(\lambda(x, y)) \\ = x_n^{\frac{\alpha}{2}} y_n^{\frac{\alpha}{2}} \operatorname{arcoth}(\lambda(x, y))\, g_{\rho_\alpha}(\lambda(x, y)),$$

where

$$g_{\rho_\alpha}(\lambda) = \frac{\hat{Q}_{\rho_\alpha}(\lambda)}{\operatorname{arcoth}(\lambda)} = \frac{2\hat{Q}_{\rho_\alpha}(\lambda)}{\ln\left(\frac{\lambda+1}{\lambda-1}\right)}.$$

We can compute the first explicit example.

Example 3.2 Consider the case $\alpha = 0$. Using the integral representation 8.712 in [13], we have

$$\hat{Q}_\rho^\mu(\lambda) = \frac{\Gamma(\rho + \mu + 1)(\lambda^2 - 1)^{\frac{\mu}{2}}}{2^{\rho+1}\Gamma(\rho + 1)} \int_{-1}^{1} (\lambda - t)^{-\mu-\rho-1}\left(1 - t^2\right)^{\rho} dt,$$

that is,

$$\hat{Q}_0^{\frac{n-2}{2}}(\lambda) = \frac{\Gamma\left(\frac{n}{2}\right)(\lambda^2 - 1)^{\frac{n-2}{4}}}{2} \int_{-1}^{1} ((\lambda - t)^{-\frac{n-2}{2}-1} dt \\ = \frac{\Gamma\left(\frac{n}{2}\right)(\lambda^2 - 1)^{\frac{n-2}{4}}}{n-2}\left((\lambda - 1)^{-\frac{n-2}{2}} - (\lambda + 1)^{-\frac{n-2}{2}}\right).$$

Applying (1), we conclude

$$H_{0,n}(x, y) = \frac{2^{\frac{2-n}{2}}}{\Gamma\left(\frac{n}{2}\right)} x_n^{\frac{2-n}{2}} y_n^{\frac{2-n}{2}} \left(\lambda^2 - 1\right)^{\frac{2-n}{4}} \widehat{Q}_0^{\frac{n-2}{2}}(\lambda(x, y)) \\ = \frac{1}{(n-2)\, 2^{\frac{n-2}{2}} x_n^{\frac{n-2}{2}} y_n^{\frac{n-2}{2}} (\lambda - 1)^{\frac{n-2}{2}}} \\ - \frac{1}{(n-2)\, 2^{\frac{n-2}{2}} x_n^{\frac{n-2}{2}} y_n^{\frac{n-2}{2}} (\lambda + 1)^{\frac{n-2}{2}}} \\ = \frac{1}{n-2}\left(\frac{1}{|x - y|^{n-2}} - \frac{1}{|x - \hat{y}|^{n-2}}\right),$$

where $\widehat{y} = (y_1, \ldots, y_{n-1}, -y_n)$.

The previous example tells us that fundamental solutions should always be thought of as unique in the sense that some function of the operator's kernel can be added to them. This feature can be utilized when searching for Green's functions, in which case the added function takes care of the needed boundary values.

The second example shows that the fundamental solution may be also product of known harmonic fundamental solutions.

Example 3.3 If $\alpha = 2 - n$ and $n \geq 3$ we use the formula (see e.g. [13, 3.666])

$$\hat{Q}^{\mu}_{\rho}(\lambda) = \frac{\Gamma(\rho + \mu + 1)(z^2 - 1)^{-\frac{\mu}{2}}}{2^{\rho+1}\Gamma(\rho + 1)} \int_0^{\pi} (\lambda + \cos t)^{\mu - \rho - 1} (\sin t)^{2\rho+1} dt \tag{5}$$

and the Legendre dublication formula, and obtain

$$\begin{aligned} H_{2-n,n}(x, y) &= \frac{2^{\frac{2-n}{2}}}{\Gamma\left(\frac{n}{2}\right)} x_n^{2-n} y_n^{2-n} \left(\lambda^2 - 1\right)^{\frac{2-n}{4}} \widehat{Q}^{\frac{n-2}{2}}_{\frac{n-4}{2}}(\lambda(x, y)) \\ &= \frac{\Gamma(n-2)\, x_n^{2-n} y_n^{2-n} \left(\lambda^2 - 1\right)^{\frac{2-n}{2}} \int_0^{\pi} \sin^{n-3} t dt}{2^{n-2}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)} \\ &= \frac{\Gamma(n-2)\,\Gamma\left(\frac{1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)|x - y|^{n-2}\left|x - \hat{y}\right|^{n-2}} \\ &= \frac{1}{(n-2)|x - y|^{n-2}\left|x - \hat{y}\right|^{n-2}}. \end{aligned}$$

Example 3.4 If $n = 2$ and $y \in \mathbb{R}^2_+$, then

$$H_{0,2}(x, y) = \ln|x - \widehat{y}| - \ln|x - y|.$$

Indeed, if $n = 2$ and $\alpha = 0$, we compute by virtue of 8.821 (3) in [13]

$$H_{0,2}(x, y) = \widehat{Q}^0_0(\lambda(x, y)) = \frac{\ln(\lambda + 1) - \ln(\lambda - 1)}{2}.$$

3.1 Fundamental α-Hyperbolic Harmonic Functions Inductively

In the theory of harmonic functions, if you know the fundamental harmonic functions in $\mathbb{R}^2$ and $\mathbb{R}^3$, you may obtain the formula for fundamental harmonic functions in all dimensions simply by differentiating with respect to the r which is the distance from the origin. We are aiming to give a similar result for α-hyperbolic harmonic functions, but the formula depends on the parity of the space.

We recall an important tool.

Lemma 3.5 [6] *Let Ω be an open set contained in $\mathbb{R}^n_+$. A function $f : \Omega \to \mathbb{R}$ is α-hyperbolic harmonic if and only if the function $g(x) = x_n^{\frac{n-\alpha-2}{2}} f(x)$ is the eigenfunction of the hyperbolic Laplace operator corresponding to the eigenvalue $\gamma_{n,\alpha} = \frac{1}{4}\left((\alpha + 1)^2 - (n - 1)^2\right)$, that is*

$$x_n^2 \Delta g - (n - 2) x_n \frac{\partial g}{\partial x_n} = \frac{1}{4}\left((\alpha + 1)^2 - (n - 1)^2\right) g. \tag{6}$$

We are looking for eigenfunctions of the hyperbolic Laplace operator depending only on λ. Then the hyperbolic Laplace operator has the following representation.

Proposition 3.6 [9] *Let $a \in \mathbb{R}^n_+$. If $f : \mathbb{R}^n_+ \to \mathbb{R}$ is twice continuously differentiable and depending only on $\lambda = \lambda(x, a)$ then*

$$\Delta_{hyp} f(x) = \left(\lambda^2 - 1\right) f''(\lambda) + n\lambda f'(\lambda).$$

We need to reformulate the properties of associated Legendre functions for our used notations

Lemma 3.7 *If $\rho > -\frac{3}{2}$, $\mu \geq 0$, $z \in \mathbb{C}$ and $|z| > 1$, then*

$$\left(\lambda^2 - 1\right) \frac{\partial \widehat{Q}^{\mu}_{\rho}(z)}{\partial z} = (\rho - \mu + 1)\, \widehat{Q}^{\mu}_{\rho+1}(z) - (\rho + 1)\, \lambda \widehat{Q}^{\mu}_{\rho}(z),$$
$$-(\lambda^2 - 1)^{\frac{1}{2}}\, \widehat{Q}^{\mu+1}_{\rho}(z) = (\rho - \mu + 1)\, \widehat{Q}^{\mu}_{\rho+1}(z) - (\rho + \mu + 1)\, \lambda \widehat{Q}^{\mu}_{\rho}(\lambda).$$

Proof Since

$$\widehat{Q}^{\mu}_{\rho}(z) = e^{-\mu i\pi}\, Q^{\mu}_{\rho}(z),$$
$$\widehat{Q}^{\mu+1}_{\rho}(z) = e^{-(\mu+1)i\pi}\, Q^{\mu+1}_{\rho}(z) = -e^{-\mu i\pi}\, Q^{\mu+1}_{\rho}(z),$$

we obtain the result from the corresponding formulas for the associated Legendre functions 8.732 and 8.734 given in [13]. □

Applying the previous lemma and the induction principle, we deduce the following property.

Theorem 3.8 *Let $\rho > -\frac{3}{2}$ and $\mu \geq 0$. If $m \in \mathbb{N}$, then*

$$\left(\lambda^2 - 1\right)^{-\frac{\mu+m-1}{2}}\, \widehat{Q}^{m-1+\mu}_{\rho}(\lambda) = (-1)^{m-1}\, \frac{\partial^{m-1} \left(\lambda^2 - 1\right)^{-\frac{\mu}{2}}\, \widehat{Q}^{\mu}_{\rho}(\lambda)}{\partial \lambda^{m-1}},$$

where $\lambda > 1$.

Proof We first prove the assertion for $m = 1$ and for any real $\tau \geq 0$. Comparing the left side of identities of the previous corollary, we obtain

$$\left(\lambda^2 - 1\right) \frac{\partial \widehat{Q}^{\tau}_{\rho}(\lambda)}{\partial \lambda} = -(\lambda^2 - 1)^{\frac{1}{2}}\, \widehat{Q}^{\tau+1}_{\rho}(\lambda) + \tau\lambda \widehat{Q}^{\tau}_{\rho}(\lambda).$$

Dividing by $(\lambda^2 - 1)^{\frac{1}{2}}$ both sides of the equality, we compute further

$$-\widehat{Q}^{\tau+1}_{\rho}(\lambda) = (\lambda^2 - 1)^{\frac{1}{2}} \frac{\partial \widehat{Q}^{\tau}_{\rho}(\lambda)}{\partial \lambda} - \frac{\tau\lambda \widehat{Q}^{\tau}_{\rho}(\lambda)}{(\lambda^2 - 1)^{\frac{1}{2}}}.$$

Using the preceding formula, we compute

$$\frac{\partial \left(\lambda^2-1\right)^{-\frac{\tau}{2}} \widehat{Q}_\rho^\tau (\lambda)}{\partial \lambda} = -\left(\lambda^2-1\right)^{-\frac{\tau+1}{2}} \widehat{Q}_\rho^{\tau+1}(\lambda), \tag{7}$$

which implies that the result hold for $m = 1$ and for all $\tau \geq 0$. The induction hypothesis is that

$$\left(\lambda^2-1\right)^{-\frac{\mu+s}{2}} \widehat{Q}_\rho^{\mu+s}(\lambda) = (-1)^s \frac{\partial^s \left(\lambda^2-1\right)^{-\frac{\mu}{2}} \widehat{Q}_\rho^\mu (\lambda)}{\partial \lambda^s}$$

holds for some $s \in \mathbb{N}$ and all $\mu \geq 0$. Applying (7) for $\tau = \mu + s$, we obtain

$$\left(\lambda^2-1\right)^{-\frac{\mu+s+1}{2}} \widehat{Q}_\rho^{\mu+s+1}(\lambda) = -\frac{\partial \left(\lambda^2-1\right)^{-\frac{\mu+s}{2}} \widehat{Q}_\rho^{\mu+s} (\lambda)}{\partial \lambda}$$

Applying the induction hypothesis, we conclude

$$\left(\lambda^2-1\right)^{-\frac{\mu+s+1}{2}} \widehat{Q}_\rho^{\mu+s+1}(\lambda) = (-1)^{s+1} \frac{\partial^{s+1} \left(\lambda^2-1\right)^{-\frac{\mu}{2}} \widehat{Q}_\rho^\mu (\lambda)}{\partial \lambda^{s+1}}.$$

Consequetly, by the general induction principle the result holds for all $m \in \mathbb{N}$. □

The key tool is the results connecting different eigenvalues.

Proposition 3.9 *Let β and γ be real numbers. If $f :]1, \infty[\to \mathbb{R}$ is four times differentiable solution of the equation*

$$\left(\lambda^2-1\right) f''(\lambda) + \beta \lambda f'(\lambda) = \gamma f(\lambda)$$

then $g(\lambda) = f'(\lambda)$ satisfies the equation

$$\left(\lambda^2-1\right) g''(\lambda) + (\beta+2) \lambda g'(\lambda) = (\gamma - \beta) g$$

Proof We just compute

$$\begin{aligned} \gamma g = \gamma f'(\lambda) &= \left(\lambda^2-1\right) f^{(3)}(\lambda) + 2\lambda f''(\lambda) + \beta f'(\lambda) + \beta \lambda f''(\lambda) \\ &= \left(\lambda^2-1\right) g''(\lambda) + (\beta+2) \lambda g'(\lambda) + \beta g \end{aligned}$$

completing the proof. □

Applying the previous proposition, it is relatively simple to verify the result.

Theorem 3.10 *Assume that* $\beta \in \{0,1\}$ *and* $\gamma \in \mathbb{R}$*. If* $s \in \mathbb{N}$ *and* $f :]1,\infty[\to \mathbb{R}$ *is* $(s+2)$ *times differentiable solution of the equation*

$$\left(\lambda^2-1\right) f''(\lambda) + \beta\lambda f'(\lambda) = \gamma f$$

then the function $g(\lambda) = f^{(s)}(\lambda)$ *satisfies the equation*

$$\left(\lambda^2-1\right) g''(\lambda) + (2s+\beta)\lambda g'(\lambda) = \begin{cases} (\gamma-(s-1)s)\, g & \text{if } \beta = 0 \\ \left(\gamma - s^2\right) g & \text{if } \beta = 1 \end{cases}.$$

Proof The previous lemma implies that the result holds for $s = 1$. Assume that the result holds for some $s \in \mathbb{N}$. Then the function $h(\lambda) = f^{(s)}(\lambda)$ satisfies the equation

$$\left(\lambda^2-1\right) h''(\lambda) + (2s+\beta)\lambda h'(\lambda) = \begin{cases} (\gamma-(s-1)s)\, g & \text{if } \beta = 0 \\ \left(\gamma - s^2\right) g & \text{if } \beta = 1 \end{cases}$$

Using the previous lemma we obtain that the function $g(\lambda) = h'(\lambda) = f^{(s+1)}(\lambda)$ satisfies the equation

$$\begin{aligned} \left(\lambda^2-1\right) g''(\lambda) + (2(s+1)+\beta)\lambda g'(\lambda) &= \begin{cases} (\gamma-(s-1)s-2s)\, g & \text{if } \beta = 0 \\ \left(\gamma - s^2 - 2s - 1\right) g & \text{if } \beta = 1 \end{cases} \\ &= \begin{cases} (\gamma-(s+1)s)\, g & \text{if } \beta = 0 \\ \left(\gamma - (s+1)^2\right) g & \text{if } \beta = 1 \end{cases}. \end{aligned}$$

Hence the result holds for $s+1$, completing the proof. □

Applying the previous result and Proposition 3.6, we obtain easily the general formulas.

Theorem 3.11 *Let* $\gamma_{n,\alpha} = \frac{1}{4}\left((\alpha+1)^2 - (n-1)^2\right)$ *and* $\alpha \in \mathbb{R}$*. Let* $f \in \mathcal{C}^\infty(]1,\infty[)$.

(a) *If* f *satisfies the differential equation*

$$\left(\lambda^2-1\right) f''(\lambda) + \lambda f'(\lambda) = \gamma_{1,\alpha} f(\lambda),$$

then the $\frac{n-1}{2}$*:th derivative* $g(\lambda) = f^{\left(\frac{n-1}{2}\right)}(\lambda)$ *satisfies the differential equation*

$$\left(\lambda^2-1\right) g''(\lambda) + n\lambda g'(\lambda) = \gamma_{n,\alpha} g(\lambda)$$

for all odd $n \in \mathbb{N}$.

(b) *If* f *satisfies the differential equation*

$$\left(\lambda^2-1\right) f''(\lambda) = \gamma_{2,\alpha} f(\lambda)$$

then the $\frac{n}{2}$:th derivative $g(\lambda) = f^{\left(\frac{n}{2}\right)}(\lambda)$ satisfies the equation

$$\left(\lambda^2 - 1\right) g''(\lambda) + 2n\lambda g'(\lambda) = \gamma_{2n,\alpha} g(\lambda)$$

for all even $n \in \mathbb{N}$.

Applying the previous theorem, we immediately deduce the fundamental solutions depending on the parity of the space. We will denote $\widehat{Q}_\mu := \widehat{Q}_\mu^0$ and observe, that they are usual Legendre functions.

Theorem 3.12 *Let $n \in \mathbb{N}$ and $n \geq 2$.*

(a) *If n is even, then the fundamental α-hyperbolic harmonic function is given by*

$$H_{\alpha.n}(x,a) = \frac{(-1)^{\frac{n-2}{2}} x_n^{\frac{\alpha+2-n}{2}} y_n^{\frac{\alpha+2-n}{2}}}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)} \frac{\partial^{\frac{n-2}{2}} \widehat{Q}_{\rho_\alpha}(\lambda)}{\partial \lambda^{\frac{n-2}{2}}}$$

and

$$H_{\alpha,2}(x,y) = x_n^{\frac{\alpha}{2}} y_n^{\frac{\alpha}{2}} Q_{\rho_\alpha}(\lambda).$$

(b) *If $n = 2m+1$ is odd, then the fundamental solution has the representation*

$$H_{\alpha,n}(x,y) = \frac{(-1)^{m-1} x_n^{\frac{\alpha+2-n}{2}} y_n^{\frac{\alpha+2-n}{2}}}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)} \frac{\partial^{m-1}\left(\left(\lambda^2-1\right)^{-\frac{1}{4}} \widehat{Q}_{\rho_\alpha}^{\frac{1}{2}}(\lambda)\right)}{\partial \lambda^{m-1}}$$

and

$$H_{\alpha,3}(x,y) = \frac{x_n^{\frac{\alpha-1}{2}} y_n^{\frac{\alpha-1}{2}} e^{-\frac{2\rho_\alpha+1}{2} r_h}}{\sinh r_h}.$$

Proof The first assertion follows from [13, 8.752 (4)].

$$\hat{Q}_{\rho_\alpha}^{\frac{n-2}{2}}(\lambda) = \frac{\partial^{\frac{n-2}{2}} \widehat{Q}_{\rho_\alpha}(\lambda)}{\partial \lambda^{\frac{n-2}{2}}}.$$

Let $n = 3$. Applying [13, 8.754 (4)], we obtain

$$H_{\alpha,2}(x,y) = \frac{\sqrt{2} x_n^{\frac{\alpha-1}{2}} y_n^{\frac{\alpha-1}{2}} \left(\lambda^2-1\right)^{-\frac{1}{4}} \widehat{Q}_{\rho_\alpha}^{\frac{1}{2}}(\lambda)}{\sqrt{\pi}} = \frac{x_n^{\frac{\alpha-1}{2}} y_n^{\frac{\alpha-1}{2}} e^{-\left(\rho_\alpha+\frac{1}{2}\right) r_h}}{\sinh r_h}.$$

Assume next that $n = 2m+1$. Applying Theorem 3.8 for $\mu = \frac{1}{2}$ and

$$\frac{\left(\lambda^2-1\right)^{\frac{2-n}{4}} \widehat{Q}_{\rho_\alpha}^{\frac{n-2}{2}}(\lambda)}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)} = \frac{\left(\lambda^2-1\right)^{\frac{1-2m}{4}} \widehat{Q}_{\rho_\alpha}^{m-\frac{1}{2}}(\lambda)}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}$$

$$= \frac{(-1)^{m-1}}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)} \frac{\partial^{m-1}\left(\lambda^2-1\right)^{-\frac{1}{4}}\widehat{Q}^{\frac{1}{2}}_{\rho_\alpha}(\lambda)}{\partial\lambda^{m-1}}$$

where

$$\left(\lambda^2-1\right)^{-\frac{1}{4}}\widehat{Q}^{\frac{1}{2}}_{\rho_\alpha}(\lambda) = \frac{\sqrt{\pi}e^{-\left(\rho_\alpha+\frac{1}{2}\right)r_h}}{\sqrt{2}\sinh r_h}.$$

□

For $\alpha = n-2$ the fundamental solution h_n was computed by Ahlfors [3, p.57]

$$h_n(x,y) = \frac{1}{2^{n-2}}\int_{\frac{|a-x|}{|x-\bar{a}|}}^{1} \frac{\left(1-s^2\right)^{n-2}}{s^{n-1}}ds$$

and in [9] it was proved by the authors that

$$h_n(x,y) = \int_\lambda^\infty \frac{1}{\left(t^2-1\right)^{\frac{n}{2}}}dt. \tag{8}$$

In order to verify that the equality of the fundamental solutions $H_{n-2,n}$ and h_n we need a simple observation.

Lemma 3.13 *If $\rho > -\frac{1}{2}$, $\mu \geq 0$ and $\lambda > 1$, then*

$$(2\rho+1)\lambda\widehat{Q}^\mu_\rho(\lambda) - (\rho+\mu)\widehat{Q}^\mu_{\rho-1}(\lambda) = (\rho-\mu+1)\widehat{Q}^\mu_{\rho+1}(\lambda)$$

and therefore

$$\widehat{Q}^{\frac{n}{2}}_{\frac{n-4}{2}}(\lambda) = \lambda\widehat{Q}^{\frac{n}{2}}_{\frac{n-2}{2}}(\lambda)$$

for any $n \in \mathbb{N}$ and $n \geq 2$.

Proof Applying [13, 8.732 (2)], we deduce

$$(2\rho+1)\lambda Q^\mu_\rho(\lambda) = (\rho-\mu+1)Q^\mu_{\rho+1}(\lambda) - (\rho+\mu)Q^\mu_{\rho-1}(\lambda)$$

Since

$$\widehat{Q}^\mu_\nu(z) = e^{-\mu i\pi}Q^\mu_\nu(z),$$

for all $\nu > -\frac{1}{2}$ we obtain the assertion

$$(2\rho+1)\lambda\widehat{Q}^\mu_\rho(\lambda) - (\rho+\mu)\widehat{Q}^\mu_{\rho-1}(\lambda) = (\rho-\mu+1)\widehat{Q}^\mu_{\rho+1}(\lambda).$$

Since $\rho = \frac{n-2}{2} \geq 0$, substituting $\mu = \frac{n}{2}$ and $\rho = \frac{n-2}{2}$, we obtain the final statement

$$(n-1)\,\lambda\,\widehat{Q}^{\mu}_{\frac{n-2}{2}}(\lambda) - (n-1)\,\widehat{Q}^{\frac{n}{2}}_{\frac{n-4}{2}}(\lambda) = 0.$$

□

Theorem 3.14 *If $n \in \mathbb{N}$ and $n \geq 2$, then*

$$H_{n-2,n}(\lambda(x,y)) = h_n(x,y).$$

Proof We first prove the assertion for even n. Assume first that $n = 2$ and denote $\lambda = \lambda(x,a)$. Then

$$H_{0,2}(x,y) = \hat{Q}_0(\lambda) = \int_\lambda^\infty \frac{1}{t^2-1}dt = \frac{1}{2}\ln(\lambda+1) - \frac{1}{2}\ln(\lambda-1).$$

If n is even and $n > 2$, applying [13, 8.752 (4), 8.824], we obtain

$$\begin{aligned}
H_{n-2,n}(\lambda) &= \frac{1}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}\left(\lambda^2-1\right)^{\frac{2-n}{4}}\widehat{Q}^{\frac{n-2}{2}}_{\frac{n-2}{2}}(\lambda) \\
&= \frac{(-1)^{\frac{n-2}{2}}}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}\frac{\partial^{\frac{n-2}{2}}Q_{\frac{n-2}{2}}(\lambda)}{\partial\lambda^{\frac{n-2}{2}}} \\
&= \frac{(-1)^{\frac{n-2}{2}}}{\Gamma\left(\frac{n}{2}\right)}\frac{\partial^{\frac{n-2}{2}}}{\partial\lambda^{\frac{n-2}{2}}}\int_\lambda^\infty \frac{(t-\lambda)^{\frac{n-2}{2}}}{\left(t^2-1\right)^{\frac{n}{2}}}\,dt \\
&= \int_\lambda^\infty \frac{1}{\left(t^2-1\right)^{\frac{n}{2}}}dt
\end{aligned}$$

completing the proof using (8) in even case.

In odd case we first we note that

$$H_{1,3}(x,y) = \frac{\sqrt{2}\left(\lambda^2-1\right)^{-\frac{1}{4}}\widehat{Q}^{\frac{1}{2}}_{\frac{1}{2}}(\lambda)}{\sqrt{\pi}} = \frac{e^{-r_h}}{\sinh r_h} = \coth r_h - 1.$$

The assertion holds for $n = 3$, since

$$\begin{aligned}
\int_{\lambda(x,y)}^\infty \frac{1}{\left(t^2-1\right)^{\frac{3}{2}}}dt &= \frac{1}{2}\int_{\frac{|x-y|}{|x-\widehat{y}|}}^1 \frac{1-s^2}{s^2}ds \\
&= \frac{1}{2}\left(-2 + \sqrt{\frac{\lambda+1}{\lambda-1}} + \sqrt{\frac{\lambda-1}{\lambda+1}}\right) \\
&= -1 + \frac{\lambda}{\sqrt{\lambda^2-1}} = \coth r_h - 1.
\end{aligned}$$

Assume that the result holds for odd $n \geq 3$, that is

$$H_{n-2,n}(\lambda(x,a)) = \frac{1}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}\left(\lambda^2-1\right)^{-\frac{n-2}{4}}\widehat{Q}^{\frac{n-2}{2}}_{\frac{n-2}{2}}(\lambda(x,a))$$
$$= \int_{\lambda(x,a)}^{\infty}\frac{1}{\left(t^2-1\right)^{\frac{n}{2}}}dt.$$

Applying the differential formula (3.8) for $m = 1$, $\mu = \frac{n-2}{2}$ and $\rho = \frac{n}{2}$, we obtain, we obtain

$$-H_{n,n+2}(\lambda) = -\frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n+2}{2}\right)}\left(\lambda^2-1\right)^{-\frac{n}{4}}\widehat{Q}^{\frac{n}{2}}_{\frac{n}{2}}(\lambda)$$
$$= \frac{1}{n2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\frac{\partial\left(\lambda^2-1\right)^{-\frac{n-2}{4}}\widehat{Q}^{\frac{n-2}{2}}_{\frac{n}{2}}(\lambda)}{\partial\lambda}.$$

By virtue of Lemma 3.13, we infer the identity

$$\widehat{Q}^{\frac{n-2}{2}}_{\frac{n}{2}}(\lambda) = (n-1)\lambda\widehat{Q}^{\frac{n-2}{2}}_{\frac{n-2}{2}}(z) - (n-2)\widehat{Q}^{\frac{n-2}{2}}_{\frac{n-4}{2}}(z).$$

Applying the induction hypothesis, we obtain further

$$\frac{1}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}\left(\lambda^2-1\right)^{-\frac{n-2}{4}}\widehat{Q}^{\frac{n-2}{2}}_{\frac{n}{2}}(\lambda)$$
$$= (n-2)\lambda\int_{\lambda}^{\infty}\frac{1}{\left(t^2-1\right)^{\frac{n}{2}}}dt - \frac{(n-2)\left(\lambda^2-1\right)^{-\frac{n-2}{4}}\widehat{Q}^{\frac{n-2}{2}}_{\frac{n-4}{2}}(z)}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}.$$

Hence we have

$$-H_{n,n+2}(\lambda) = \frac{n-1}{n}\left(\int_{\lambda}^{\infty}\frac{1}{\left(t^2-1\right)^{\frac{n}{2}}}dt\right)$$
$$-\frac{(n-1)\lambda}{n}\frac{1}{\left(\lambda^2-1\right)^{\frac{n}{2}}} - \frac{(n-2)}{n2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}\frac{\partial\left(\lambda^2-1\right)^{-\frac{n-2}{4}}\widehat{Q}^{\frac{n-2}{2}}_{\frac{n-4}{2}}(z)}{\partial\lambda}.$$

Applying the partial integration, we deduce

$$\int_{\lambda}^{\infty}\frac{1}{\left(t^2-1\right)^{\frac{n}{2}}}dt = -\frac{\lambda}{\left(\lambda^2-1\right)^{\frac{n}{2}}} + n\int_{\lambda}^{\infty}\frac{t^2}{\left(t^2-1\right)^{\frac{n+2}{2}}}dt$$

$$
\begin{aligned}
&= n \int_{\lambda}^{\infty} \frac{1}{\left(t^2-1\right)^{\frac{n}{2}}} dt \\
&\quad + n \int_{\lambda}^{\infty} \frac{dt}{\left(t^2-1\right)^{\frac{n+2}{2}}} - \frac{\lambda}{\left(\lambda^2-1\right)^{\frac{n}{2}}},
\end{aligned}
$$

whicn by solving the formula for $\frac{n-1}{n} \int_{\lambda}^{\infty} \frac{1}{\left(t^2-1\right)^{\frac{n+1}{2}}} dt$ implies that

$$
\frac{n-1}{n} \int_{\lambda}^{\infty} \frac{1}{\left(t^2-1\right)^{\frac{n}{2}}} dt = -\int_{\lambda}^{\infty} \frac{dt}{\left(t^2-1\right)^{\frac{n+2}{2}}} + \frac{\lambda}{n} \frac{1}{\left(\lambda^2-1\right)^{\frac{n}{2}}}.
$$

Hence

$$
\begin{aligned}
-H_{n,n+2}(\lambda) &= -\int_{\lambda}^{\infty} \frac{dt}{\left(t^2-1\right)^{\frac{n+2}{2}}} + \frac{(2-n)\lambda}{n} \frac{1}{\left(\lambda^2-1\right)^{\frac{n}{2}}} \\
&\quad - \frac{(n-2)}{n 2^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right)} \frac{\partial \left(\lambda^2-1\right)^{-\frac{n-2}{4}} \widehat{Q}_{\frac{n-4}{2}}^{\frac{n-2}{2}}(z)}{\partial \lambda}.
\end{aligned}
$$

Again applying the differential formula Theorem 3.8 and Lemma 3.13, we obtain

$$
\begin{aligned}
-\frac{\partial \left(\lambda^2-1\right)^{-\frac{n-2}{4}} \widehat{Q}_{\frac{n-4}{2}}^{\frac{n-2}{2}}(z)}{\partial \lambda} &= \left(\lambda^2-1\right)^{-\frac{n}{4}} \widehat{Q}_{\frac{n-4}{2}}^{\frac{n}{2}}(z) \\
&= \lambda \left(\lambda^2-1\right)^{-\frac{n}{4}} \widehat{Q}_{\frac{n-2}{2}}^{\frac{n}{2}}(z) \\
&= -\lambda \frac{\partial \left(\lambda^2-1\right)^{-\frac{n-2}{4}} \widehat{Q}_{\frac{n-2}{2}}^{\frac{n-2}{2}}(\lambda)}{\partial \lambda}.
\end{aligned}
$$

Finally, using the induction hypothesis, we conclude

$$
\begin{aligned}
-H_{n,n+2}(\lambda) &= -\int_{\lambda}^{\infty} \frac{dt}{\left(t^2-1\right)^{\frac{n+2}{2}}} + \frac{(2-n)\lambda}{n} \frac{1}{\left(\lambda^2-1\right)^{\frac{n}{2}}} \\
&\quad - \frac{(n-2)\lambda}{n 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right)} \frac{\partial \left(\lambda^2-1\right)^{-\frac{n-2}{4}} \widehat{Q}_{\frac{n-4}{2}}^{\frac{n-2}{2}}(z)}{\partial \lambda} \\
&= -\int_{\lambda}^{\infty} \frac{dt}{\left(t^2-1\right)^{\frac{n+2}{2}}} + \frac{(2-n)\lambda}{n} \frac{1}{\left(\lambda^2-1\right)^{\frac{n}{2}}} \\
&\quad - \frac{(n-2)\lambda}{n} \frac{\partial \int_{\lambda}^{\infty} \frac{dt}{\left(t^2-1\right)^{\frac{n}{2}}}}{\partial \lambda}
\end{aligned}
$$

$$= -\int_{\lambda}^{\infty} \frac{dt}{\left(t^2-1\right)^{\frac{n+2}{2}}},$$

completing the proof. □

4 Connections to Jacobi Polynomials

Let $\alpha \in \mathbb{R}\backslash(-\mathbb{N})$ and $\beta \in \mathbb{R}$, and $n \in \mathbb{N}$. The Jacobi polynomials of degree n is defined in terms of hypergeometric functions by

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}\right) \tag{9}$$

for $z \in \mathbb{C}$. Jacobi polynomilas satisfy the following useful formula.

Proposition 4.1 (Rodrigues formula). *Let z be real and n a non negative integer. The Jacobi polynomials have the property*

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}(1-z)^{-\alpha}(1+z)^{-\beta}\frac{d^n}{dz^n}\left((1-z)^{\alpha+n}(1+z)^{\beta+n}\right).$$

The second associated Legendre function is defined by

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} {}_2F_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right) \tag{10}$$

converging $|1-z| < 2$, where $\mu \in \mathbb{R}\backslash\mathbb{N}$. Let us first prove the following connection between the second Legendre function and Jacobi polynomials.

Proposition 4.2 *If $\alpha \in \mathbb{R}\backslash(-\mathbb{N})$ and $m \in \mathbb{N}$, then*

$$P_m^{-\alpha}(z) = \frac{m!}{\Gamma(1+\alpha+m)}\left(\frac{z-1}{z+1}\right)^{\frac{\alpha}{2}} P_m^{(\alpha,-\alpha)}(z).$$

Proof We write

$$P_\nu^{-\mu}(z) = \frac{1}{\Gamma(1+\mu)}\left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} {}_2F_1\left(-\nu, \nu+1; 1+\mu; \frac{1-z}{2}\right)$$

and

$$P_m^{(\alpha,-\alpha)}(z) = \frac{(\alpha+1)_m}{m!}\, {}_2F_1\left(-m, m+1; \alpha+1; \frac{1-z}{2}\right).$$

Hence, we obtain

$$P_m^{-\alpha}(z) = \frac{1}{\Gamma(1+\alpha)}\left(\frac{z-1}{z+1}\right)^{\frac{\alpha}{2}} {}_2F_1\left(-m, m+1; 1+\alpha; \frac{1-z}{2}\right)$$
$$= \frac{m!}{(\alpha+1)_m}\frac{1}{\Gamma(1+\alpha)}\left(\frac{z-1}{z+1}\right)^{\frac{\alpha}{2}} P_m^{(\alpha,-\alpha)}(z) .$$

Using (4), we have

$$P_m^{-\alpha}(z) = \frac{m!}{\Gamma(1+\alpha+m)}\left(\frac{z-1}{z+1}\right)^{\frac{\alpha}{2}} P_m^{(\alpha,-\alpha)}(z) .$$

□

Hence, we can represent the fundamental solution using a Jacobi polynomial (see also [17])

Theorem 4.3 *Let $x, y \in \mathbb{R}^n_+$ and denote $r_h = d_h(x, y)$. If $n \geq 3$ is an odd integer then*

$$\hat{Q}_{\rho_\alpha}^{\frac{n-2}{2}}(\cosh(r_h)) = \sqrt{\frac{\pi}{2}}\left(\frac{n-3}{2}\right)! \frac{e^{-(\rho_\alpha+\frac{1}{2})r_h}}{\sqrt{\sinh(r_h)}} P_{\frac{n-3}{2}}^{\left(\rho_\alpha+\frac{1}{2},-(\rho_\alpha+\frac{1}{2})\right)}(\coth(r_h)) .$$

and

$$H_{\alpha,n}(x, y) = \frac{\sqrt{\pi}\left(\frac{n-3}{2}\right)!}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)} \frac{x_n^{\frac{\alpha+2-n}{2}} y_n^{\frac{\alpha+2-n}{2}} e^{-(\rho_\alpha+\frac{1}{2})r_h}}{\sinh^{\frac{n-1}{2}}(r_h)} P_{\frac{n-3}{2}}^{\left(\rho_\alpha+\frac{1}{2},-(\rho_\alpha+\frac{1}{2})\right)}(\coth(r_h)) .$$

Proof Assume $n \geq 3$ is odd. The formula 8.739 in [13] is in our case

$$\hat{Q}_{\rho_\alpha}^{\frac{n-2}{2}}(\cosh(r_h)) = \frac{\sqrt{\pi}\,\Gamma\left(\rho_a+\frac{n}{2}\right)}{\sqrt{2\sinh(r_h)}} P_{-\frac{n-1}{2}}^{-\rho_\alpha-\frac{1}{2}}(\coth(r_h))$$

and using the formula 8.2.1 from [1], we have

$$\hat{Q}_{\rho_\alpha}^{\frac{n-2}{2}}(\cosh(r_h)) = \frac{\sqrt{\pi}\,\Gamma\left(\rho_a+\frac{n}{2}\right)}{\sqrt{2\sinh(r_h)}} P_{\frac{n-3}{2}}^{-\rho_\alpha-\frac{1}{2}}(\coth(r_h)) .$$

The using Proposition (4.2), we obtain

$$\hat{Q}_{\rho_\alpha}^{\frac{n-2}{2}}(\cosh(r_h)) = \frac{\sqrt{\pi}\,\Gamma\left(\rho_a+\frac{n}{2}\right)}{\sqrt{2\sinh(r_h)}} \frac{\left(\frac{n-3}{2}\right)!}{\Gamma(\rho_\alpha+\frac{n}{2})}$$
$$\left(\frac{\coth(r_h)-1}{\coth(r_h)+1}\right)^{\frac{\rho_\alpha+\frac{1}{2}}{2}} P_{\frac{n-3}{2}}^{\left(\rho_\alpha+\frac{1}{2},-(\rho_\alpha+\frac{1}{2})\right)}(\coth(r_h)) .$$

We compute

$$\frac{\coth(r_h)-1}{\coth(r_h)+1}=\frac{\cosh r_h-\sinh r_h}{\cosh r_h+\sinh r_h}=e^{-2r_h},$$

that is

$$\hat{Q}_{\rho_\alpha}^{\frac{n-2}{2}}(\cosh(r_h))=\sqrt{\frac{\pi}{2}}\left(\frac{n-3}{2}\right)!\frac{e^{-(\rho_\alpha+\frac{1}{2})r_h}}{\sqrt{\sinh(r_h)}}P_{\frac{n-3}{2}}^{\left(\rho_\alpha+\frac{1}{2},-(\rho_\alpha+\frac{1}{2})\right)}(\coth(r_h)).$$

Hence

$$H_{\alpha,n}(x,y)=\frac{\sqrt{\pi}\left(\frac{n-3}{2}\right)!}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)}\frac{x_n^{\frac{\alpha+2-n}{2}}y_n^{\frac{\alpha+2-n}{2}}e^{-(\rho_\alpha+\frac{1}{2})r_h}}{\sinh^{\frac{n-1}{2}}(r_h)}P_{\frac{n-3}{2}}^{\left(\rho_\alpha+\frac{1}{2},-(\rho_\alpha+\frac{1}{2})\right)}(\coth(r_h)).$$

□

We can write the Jacobi polynomials as follows.

Proposition 4.4 *If $\alpha\in\mathbb{R}\backslash(-\mathbb{N})$ and $m\in\mathbb{N}$, then*

$$P_m^{(\alpha,-\alpha)}(z)=\Gamma(\alpha+m+1)\sum_{j=0}^{m}\frac{\binom{m+j}{m}}{(m-j)!}\frac{1}{\Gamma(\alpha+j+1)}\left(\frac{z-1}{2}\right)^j$$

and

$$P_m^{(\alpha,-\alpha)}(\coth(r_h))=\Gamma(\alpha+m+1)\sum_{j=0}^{m}\frac{\binom{m+j}{m}}{(m-j)!}\frac{1}{2^j\Gamma(\alpha+j+1)}\frac{e^{-jr_h}}{\sinh^j(r_h)}.$$

Proof Using (9) we have

$$P_m^{(\alpha,-\alpha)}(z)=\frac{(\alpha+1)_n}{n!}\sum_{j=0}^{m}\frac{(-m)_j\,(m+1)_j}{(\alpha+1)_j\,j!}\left(\frac{1-z}{2}\right)^j$$

and using $(-m)_j=(-1)^j(m-j+1)_j$, we have

$$P_m^{(\alpha,-\alpha)}(z)=\frac{(\alpha+1)_m}{m!}\sum_{j=0}^{m}\frac{(m-j+1)_j\,(m+1)_j}{(\alpha+1)_j\,j!}\left(\frac{z-1}{2}\right)^j.$$

Then applying (4) and $\Gamma(m+1)=m!$, we obtain

$$\frac{(m-j+1)_j\,(m+1)_j}{m!j!}=\frac{\Gamma(m+1)\Gamma(m+1+j)}{\Gamma(m-j+1)\Gamma(m+1)m!j!}$$

$$= \frac{(m+j)!}{(m-j)!m!j!} = \frac{\binom{m+j}{m}}{(m-j)!}$$

and using again (4), we conclude

$$P_m^{(\alpha,-\alpha)}(z) = \Gamma(\alpha+m+1)\sum_{j=0}^{m} \frac{\binom{m+j}{m}}{(m-j)!} \frac{1}{\Gamma(\alpha+j+1)} \left(\frac{z-1}{2}\right)^j .$$

We compute

$$\coth(r_h) - 1 = \frac{e^{-r_h}}{\sinh(r_h)},$$

and obtain

$$P_m^{(\alpha,-\alpha)}(\coth(r_h)) = \Gamma(\alpha+m+1)\sum_{j=0}^{m} \frac{\binom{m+j}{m}}{(m-j)!} \frac{1}{2^j\Gamma(\alpha+j+1)} \frac{e^{-jr_h}}{\sinh^j(r_h)}.$$

□

Let us complete the paper by computing the following examples.

Example 4.5 If $n = 3$, we have $P_0^{(\alpha,-\alpha)}(\coth(r_h)) = 1$ and

$$H_{\alpha,3}(x,y) = \frac{x_3^{\frac{\alpha-1}{2}} y_3^{\frac{\alpha-1}{2}} e^{-(\rho_\alpha+\frac{1}{2})r_h}}{\sinh(r_h)}.$$

Example 4.6 If $n = 5$, we have

$$\begin{aligned} P_1^{(\alpha,-\alpha)}(\coth(r_h)) &= \Gamma(\alpha+2)\sum_{j=0}^{1} \frac{\binom{1+j}{1}}{(1-j)!} \frac{1}{2^j\Gamma(\alpha+j+1)} \frac{e^{-jr_h}}{\sinh^j(r_h)} \\ &= \Gamma(\alpha+2)\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+2)}\frac{e^{-r_h}}{\sinh(r_h)}\right) \\ &= \alpha+1+\frac{e^{-r_h}}{\sinh(r_h)}. \end{aligned}$$

and

$$\begin{aligned} H_{\alpha,5}(x,y) &= \frac{\sqrt{\pi}}{4\Gamma\left(\frac{5}{2}\right)} \frac{x_5^{\frac{\alpha-3}{2}} y_5^{\frac{\alpha-3}{2}} e^{-(\rho_\alpha+\frac{1}{2})r_h}}{\sinh^2(r_h)} P_1^{\left(\rho_\alpha+\frac{1}{2},-(\rho_\alpha+\frac{1}{2})\right)}(\coth(r_h)) \\ &= \frac{x_5^{\frac{\alpha-3}{2}} y_5^{\frac{\alpha-3}{2}} e^{-(\rho_\alpha+\frac{1}{2})r_h}}{3\sinh^2(r_h)} \left(\rho_\alpha+\frac{3}{2}+\frac{e^{-r_h}}{\sinh(r_h)}\right) \end{aligned}$$

$$= \frac{x_5^{\frac{\alpha-3}{2}} y_5^{\frac{\alpha-3}{2}} e^{-(\rho_\alpha+\frac{1}{2})r_h}}{3\sinh^3(r_h)} \left(\left(\rho_\alpha + \frac{3}{2}\right) \sinh(r_h) + e^{-r_h} \right)$$
$$= \frac{x_5^{\frac{\alpha-3}{2}} y_5^{\frac{\alpha-3}{2}} e^{-(\rho_\alpha+\frac{1}{2})r_h}}{3\sinh^3(r_h)} \left(\left(\rho_\alpha + \frac{1}{2}\right) \sinh(r_h) + \cosh(r_h) \right).$$

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