Tampere University

THAVAMANI GOVINDARAJ

Robust Output Regulation of Euler-Bernoulli Beam Models

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This thesis is dedicated to the memory of my father and to my family for their endless love and support.

PREFACE

The research in this thesis is carried out at Mathematical Systems Theory Research Group, Faculty of Information Technology and Communication Sciences, Unit of Computing Sciences at Tampere University during 2019-2023.

The completion of this work would not have been possible without several individuals. First and foremost, I would like to express my deepest gratitude to my supervisor Associate Professor Lassi Paunonen for introducing me to the research world, for his guidance and continuous support throughout my thesis work. I am grateful to him for the learning that I gained from his extensive knowledge and experience. I would like to extend my gratitude to my cosupervisor Jukka-Pekka Humaloja for sharing his knowledge and support all the time.

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Last but not least, my deepest love and thanks to my mother, my husband, my son, my family and friends for their constant support and encouragement. I could not have undertaken this journey without them.

During this research, I have been financially supported by Finnish Automation Foundation for participating in conferences.

Tampere, 31.05.2023

Thavamani Govindaraj

ABSTRACT

In this thesis, we consider control and dynamical behaviour of flexible beam models which have potential applications in robotic arms, satellite panel arrays and wind turbine blades. We study mathematical models that include flexible beams described by Euler-Bernoulli beam equations. These models consist of partial differential equations or combination of partial and ordinary differential equations depending on the loads and supports in the model. Our goal is to influence the models by control inputs such as external applied forces so that measured deflection profiles of the beams in the models behave as desired.

We propose dynamic controllers for the output regulation, where the measurements from the models track desired reference signals in the given time, of flexible beam models. The controller designs are based on the so-called internal model principle and they utilize difference between measurement and desired reference trajectory. Moreover, the controllers are robust in the sense that they can achieve output regulation despite external disturbances and model uncertainties.

We also study the output regulation problem when there are certain limitations on the control input. In particular, we generalize the theory of output regulation for dynamical systems described by ordinary differential equations subject to input constraints to a particular class of systems described by partial differential equations. We present set of solvability conditions and a linear output feedback controller for the output regulation.

CONTENTS

SYMBOLS AND ABBREVIATIONS

A	system operator, generator of a strongly continuous semigroup
B	control operator
B_d	disturbance operator
C	observation operator
$D(A)$	domain of a linear operator A
E	operator associated with the disturbance signal
F	operator associated with the reference signal
$G(s)$	transfer function of a plant at $s \in \mathbb{C}$
K	output operator of a dynamic error feedback controller
$L^2(a,b;X)$	set of square integrable functions from an interval $[a,b]$ to a Hilbert space X
S	system operator of an exosystem
$T(t)$	strongly continuous semigroup generated by the linear operator A
U	input space
U_d	disturbance input space
W	state space of the exosystem
X	state space
X_1	space $D(A)$ with the norm $\|x\|_{X_1} = \|(sI - A)x\|_X$ for a fixed $s \in \rho(A)$

X_{-1}	completion of the state space X with respect to the norm $\|(sI - A)^{-1}(\cdot)\|_X$, where $s \in \rho(A)$ is fixed
Y	output space
Z	controller state space
$\mathcal{L}(X, Y)$	space of bounded linear operators from a normed space X to a normed space Y
$\mathcal{R}(A)$	range of a linear operator A
$\mathcal{A}$	system operator of a boundary control system
$\mathcal{B}$	control operator of a boundary control system
$\mathcal{C}$	observation operator of a boundary control system
$\mathcal{G}_1$	system operator of an error feedback controller
$\mathcal{G}_2$	input operator of a dynamic error feedback controller
$\mathcal{N}$	kernel of a linear operator
$\phi(\cdot)$	saturation function
$\rho(A)$	resolvent set of a linear operator A
$u(t)$	control input
$v(t)$	state of an exosystem
$w_d(t)$	disturbance input
$x(t)$	system state
$y(t)$	system output
$y_{ref}(t)$	reference output
$z(t)$	state of a dynamic error feedback controller
ODE	ordinary differential equation
PDE	partial differential equation

ORIGINAL PUBLICATIONS

Publication I	T. Govindaraj, J.-P. Humaloja and L. Paunonen. A Finite-Dimensional Controller for Robust Output Tracking of an Euler-Bernoulli Beam. *Proceedings of the American Control Conference* (2022), 988–993. DOI: 10.23919/ACC53348.2022.9867855.
Publication II	T. Govindaraj, J.-P. Humaloja and L. Paunonen. Robust Output Regulation of a Flexible Satellite. *IFAC-PapersOnLine* 53.2 (2020). 21st IFAC World Congress, 7795–7800. DOI: https://doi.org/10.1016/j.ifacol.2020.12.1871.
Publication III	T. Govindaraj, J.-P. Humaloja and L. Paunonen. Robust Controllers for a Flexible Satellite Model. *Mathematical Control and Related Fields* (2023). Published Online. DOI: https://doi.org/10.3934/mcrf.2023007.
Publication IV	T. Govindaraj, L. Paunonen and J.-P. Humaloja. Saturated Output Regulation of Distributed Parameter Systems with Collocated Actuators and Sensors. *Proceedings of the IFAC World Congress 2023* (2023). Accepted.

Author's contribution

The author was the corresponding author for the Publications I-IV.

Publication I The research was planned together with Lassi Paunonen and Jukka-Pekka Humaloja. The research was carried out by the author. The theoretical results in section III and IV were discussed with Lassi Paunonen and Jukka-Pekka Humaloja. The author drafted the manuscript. Lassi Paunonen and Jukka-Pekka Humaloja gave feedback to improve the manuscript.

Publication II — The research was planned together with Lassi Paunonen. The research was carried out by the author. The theoretical results were discussed with Lassi Paunonen. The author drafted the manuscript. Lassi Paunonen and Jukka-Pekka Humaloja gave feedback to improve the manuscript.

Publication III — The research was planned together with Lassi Paunonen. The research was carried out by the author with additional contributions from Lassi Paunonen and Jukka-Pekka Humaloja. Numerical approximation of the satellite model was obtained from Kristian Asti, Master student. The author ran the numerical simulations and drafted the manuscript except the proof of Lemma 3.8 which was written by Lassi Paunonen. Lassi Paunonen and Jukka-Pekka Humaloja gave feedback to improve the manuscript.

Publication IV — Lassi Paunonen suggested the research topic. The research was carried out by the author with additional contributions from Lassi Paunonen. The author drafted the manuscript. Lassi Paunonen and Jukka-Pekka Humaloja gave feedback to improve the manuscript.

1 INTRODUCTION

Flexible structures are widely used in the modern technology because of their advantages, for example, light weight and low energy consumption when moving the structure. Different types of flexible structures can be found, for example in satellite panels, wind turbine tower blades, robot arms and marine risers. However, the flexibility of these structures leads to problems of structural vibrations and shape deformation. It is therefore natural to ask if one can control such harmful vibrations and deformations in order to improve performances of the structures. In this thesis, we consider mathematical models consisting of flexible beams. Our goal is to influence these models, that is, control selected properties of the models, so that deflection profiles of the beams in the model behave as desired by using mathematical control theory.

Mathematical control theory studies analysis and control design of dynamical systems. Dynamical systems arise from modeling physical phenomena that change over time and they are often described by differential equations. If the dynamical system is described by ordinary differential equations (ODEs), then its state at any instance is a vector with finite number of elements and the system is called a *finite-dimensional system*. On the other hand, if the system is described by partial differential equations (PDEs), then the state lies in an infinite-dimensional vector space and the system is called an *infinite-dimensional system* or *distributed parameter system*. The dynamical systems considered in this thesis are models consisting of flexible beams and mathematical model of each of them involve at least one PDE. These systems can be controlled, for example, by external forces or moments and we assume that we can measure deflection profiles of the beams.

The general control scenario is depicted in Figure 1.1 where $\mathcal{P}$ denotes the model we consider. The goal is to find a control input $u(t)$, such as external applied forces or moments in such a way that the measured output $y(t)$, measurement from the system, behaves as desired despite external disturbances $w_d(t)$, such as external forces.

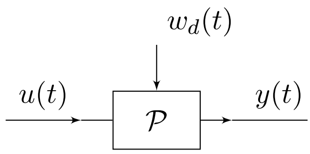


Figure 1.1 Control System

In this thesis, we consider two models consisting of one flexible beam and two flexible beams connected via a center rigid body. In both models, beam is modelled by Euler-Bernoulli beam equation of the form

$$\rho(\xi)\frac{\partial^2 w}{\partial t^2}(\xi,t) + \frac{\partial^2}{\partial \xi^2}\left(EI(\xi)\frac{\partial^2 w}{\partial \xi^2}\right)(\xi,t) = 0, \quad \xi \in \Omega \subset \mathbb{R},\ t > 0, \tag{1.1}$$

where $w(\xi,t)$ is the transverse displacement of the beam, $\rho(\xi)$ and $EI(\xi)$ are linear density and flexural rigidity of the beam, respectively. In addition, the considered models include set of initial values for the deflection profiles of the beam at time $t = 0$ and set of boundary conditions depending, for example, on the loads and supports in the models. We assume that the velocities of the beam system can be measured inside the domain Ω or at the boundaries. The measured outputs that we consider in this work include linear velocity $\frac{\partial w}{\partial t}(\xi,t)$, angular velocity $\frac{\partial^2 w}{\partial t \partial \xi}(\xi,t)$ and weighted average of velocities in the domain Ω of the beam system in the considered model. The property which we control depends on the considered model.

The main control problem is defined as follows. Our goal is to seek for a control input $u(t)$ such that the measured output $y(t)$ tracks desired reference signal $y_{ref}(t)$ asymptotically.

Output Regulation Problem. *"Find a control input $u(t)$ such that $\|y(t) - y_{ref}(t)\| \to 0$ in a suitable sense as $t \to \infty$ despite external disturbances in the system."*

In reality, it is not always possible to have accurate knowledge of the considered model. There will be for example parameter uncertainties in the model. Control designs that achieve the desired goal tolerating model uncertainties are called robust. Control designs in this work are robust in the sense that they can achieve output tracking despite a class of parameter uncertainties, for example a class of uncertainties of $\rho(\xi)$ and $EI(\xi)$ in case of (1.1), in the system.

In the output regulation problem, we consider reference and disturbance signals of the form

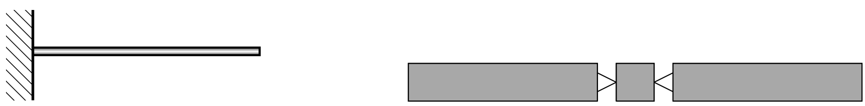
$$y_{ref}(t) = a_0 + \sum_{k=1}^{q} a_k \cos(\omega_k t) + b_k \sin(\omega_k t),$$
$$w_d(t) = c_0 + \sum_{k=1}^{q} c_k \cos(\omega_k t) + d_k \sin(\omega_k t)$$

where $(\omega_k)_{k=1}^q$ are known frequencies and $(a_k)_{k=0}^q$, $(b_k)_{k=1}^q$, $(c_k)_{k=0}^q$ and $(d_k)_{k=1}^q$ are possibly unknown constant coefficients.

Stability property of a model can affect its dynamical behaviour and therefore stability analysis is an important part of control design. Stability of a differential equation model corresponds to the behaviour of its solutions with respect to time. In general, a differential equation model is *stable* if for any initial condition, its solutions decays to zero asymptotically and the model is *stabilizable* if one can find a control input such that for any initial condition, the corresponding solutions decay to zero asymptotically. In addition to tracking of given signals, control designs in this work stabilize the considered models.

1.1 Research Objectives

In this thesis, we consider models which have potential applications in the regulation of velocities of robot arms, wind turbine blades and satellites. Particularly, we consider two models, a single beam model (Figure 1.2a) and a model of a satellite that is composed of two identical flexible solar panels and a center rigid body (Figure 1.2b).



(a) A cantilever beam

(b) A flexible satellite model

Figure 1.2 Models with flexible beam(s)

The main research objectives are:

- Solve the output regulation problem for selected mathematical models consisting of Euler-Bernoulli beams. Additionally, analyze stability properties of the considered models.
- Propose control designs that are preferably implementable in practice in the sense that the control designs are finite-dimensional and robust with respect to model uncertainties.
- Solve the output regulation problem in the presence of input constraints, i.e., when there are limitations on the control input $u(t)$.

1.2 Literature Review

In this section, we present existing results on control of Euler-Bernoulli beam models and distributed parameter systems. The control goals include stabilizing the system, influencing the system so that the outputs track given reference signals while rejecting external disturbances.

Control of Euler-Bernoulli beam models

Control problems for Euler-Bernoulli beam models have been studied widely in the literature. Stability and stabilization problems of Euler-Bernoulli beams have been studied for example in [1, 4, 32, 33, 66, 84] and [18, Sec. 8.4]. In [11, 14], stabilization problems of serially connected beams have been studied. In [8], three robust control designs have been proposed for the stabilization of three different Euler-Bernoulli beam models. Exponential stability of coupled beams with dissipative joints has been studied in [13] and [80]. Exponential stability of an Euler-Bernoulli beam with locally distributed damping has been studied in [58]. Boundary stabilization of a multiple beam system has been studied in [52]. In [22], three compensator-based robust controllers have been proposed for the stabilization of a cantilevered Euler-Bernoulli beam. In [3], stabilization problem of serially connected inhomogeneous Euler-Bernoulli beams has been studied. *Optimal control problem* for classes of hyperbolic and Euler-Bernoulli partial differential equations with boundary control has been studied in [23].

Stabilization problems of flexible beams coupled with rigid structures have been studied for example in [15, 20, 40, 59, 74, 98]. Stabilization of an Euler-Bernoulli beam with tip mass using velocity feedback controls has been studied in [59]. SCOLE model is a well-known mathematical model of a system where a flexible beam is clamped at one end and other end attached to a rigid body. The SCOLE beam system is used to model wind turbine tower [98]. Well-posedness, controllability and stability properties of SCOLE system have been studied in [96, 98, 100]. Strong stabilization of a wind turbine tower model has been studied in [99]. Vibration control of a flexible satellite model that is composed of two flexible solar panels and a center rigid body has been considered in [40] and vibrations are suppressed using a single-point control input. In comparison, the satellite model in this work consider the effect of rotation angle of the center rigid body whereas in [40] the effect of rotation angle has been ignored. Vibration control of a rigid-flexible satellite consisting of a flexible beam and a rigid body has been studied in [86] using H_∞ control design. Boundary stabilization of a flexible wing model has been studied in [56]. In [31], three approximation schemes have been compared for optimal control of a flexible beam which is attached at one end to a rotating rigid hub and at the other end to a concentrated mass. In [53], control of multiple component structures consisting of Euler-Bernoulli beams and rigid bodies has been studied.

There are some studies focusing output regulation of beam models. Stabilization and output regulation of flexible-link manipulators have been studied in [74]. In [20], stabilization and set point regulation of two-link flexible arm have been studied. Set point tracking and harmonic tracking of beam models have been studied by numerical methods in [50]. A finite-dimensional regulator has been proposed for the output tracking of an Euler-Bernoulli beam model in [19], however robustness of the control design has not been studied. In [2], a proportional derivative controller and a non-linear controller have been proposed for a rotating flexible satellite with tip masses in order to suppress vibrations and track constant signals. Output tracking of Euler-Bernoulli beams have been studied recently in [34, 35, 38, 49] and [36] using infinite-dimensional controllers which are not implementable in practice. Disturbance rejection problem of two coupled Euler-Bernoulli beams using *internal model based control design* has been studied in [81].

Control designs that have been widely used in engineering for vibration control and trajectory tracking of flexible beam models include *adaptive control*, *sliding mode control*, *iterative learning control*, *neural network control*, *fuzzy logic control* and *model-based control* [42]. A robust adaptive boundary control has been proposed in [30] to stabilize Euler-Bernoulli beam and to reject unknown external disturbances. In [29], a non-linear feedback controller based on a finite element method model has been proposed to control tip payload of a single-link flexible manipulator. In [95], a finite-dimensional model for a rotating Euler-Bernoulli beam using finite element method has been obtained and vibrations are suppressed using positive position feedback and momentum exchange feedback control laws. A sliding mode boundary controller has been proposed in [67] to reject unknown bounded disturbances and suppress vibrations of Euler-Bernoulli beam. In [37], boundary stabilization and disturbance rejection of an Euler-Bernoulli beam model have been studied by active disturbance rejection control and sliding mode control methods. In [44], a boundary control has been proposed for stabilization of a flexible marine riser. In [60], a boundary control scheme is designed to regulate vibrations of an Euler-Bernoulli beam with input and output constraints. In [27], a state feedback control and an output feedback control have been proposed for vibration control of a flexible spacecraft system with input constraints. In [43], an adaptive boundary iterative learning control has been designed for vibration control of an Euler-Bernoulli beam with input constraints. Vibration control of a flexible marine riser system with input constraints has been studied in [41].

Output Regulation of Distributed Parameter Systems

Output regulation problem has a rich history in the literature since 1970s. *Internal model principle* is the main key in constructing robust regulating controllers. The concept of internal model principle was originally introduced in [25] and [26] for finite-dimensional systems and since then it has been developed for infinite-dimensional systems, see [46, 77, 78]. Output regulation of distributed parameter systems was started by [83] without robustness analysis and then the problem has been studied actively by many authors, see for example [6, 10, 16, 19, 39, 45, 72, 75, 77, 81] and [5]. References [6, 9, 10, 75, 76, 89] and [70, Ch. 7] demonstrate how the control designs can be applied to specific PDE models. Output regulation of nonlinear systems has been studied in [47, 71, 92, 94].

Stabilization and output regulation of systems subject to input constraints have been studied by many authors, see for example [28, 54, 62, 63, 65, 68, 79, 82, 85, 88]. In [85], non-linear output feedback control design has been proposed for stabilization of linear finite-dimensional systems with input constraints. In [79], stability of one dimensional wave equation with input constraints has been studied. In [54], stability of elastic systems with input constraints has been studied. In [68], stabilization problem of *Korteweg-de Vries* equation with input constraints has been studied. In [55], output regulation of *reaction-diffusion equation* with input constraints has been studied using proportional integral control. Output tracking of constant reference signals for multi-input multi output non-linear systems with input constraints using integral control has been studied in [65]. Stabilization problem of reaction-diffusion equation with input constraints has been studied in [69]. Output regulation of finite-dimensional linear systems with input constraints has been studied in [82]. However, there are only few results focusing output regulation of infinite-dimensional linear systems subject to input constraints [24, 61, 62, 63, 64, 73] where asymptotic tracking of constant reference signals are achieved using integral controls.

1.3 Summary of the Main Results

The main contributions of this thesis are:

1. We propose robust control designs for the output regulation of the cantilever beam and the flexible satellite model.
2. We prove exponential stability of the satellite model.
3. We generalize output regulation theory for finite-dimensional linear systems subject to input constraints to a particular class of infinite-dimensional linear systems.
4. We solve the output regulation problem for the satellite model in the presence of input constraints.

The main novelty of the Result 1 compared to the existing literature is that we utilize internal model based control designs from [81], [75] and [76] designed for classes of abstract linear systems with some additional properties, such as passivity and well-posedness, see Section 3.1. The advantage of using these control designs is that they are robust by construction. Result 2 is a new contribution. In comparison, the satellite model considered in this work is different from the models in [3, 40, 96, 98, 100] where stability properties of Euler-Bernoulli beams coupled with ODE systems have been studied, therefore stability of the satellite model cannot be obtained from those results. The main novelty of the Result 3 compared to the existing results is that we allow reference signals to be linear combination of sinusoids whereas existing results focus on tracking of only constant reference signals. To our knowledge, output regulation of the satellite model with input constraints has not been considered and therefore the Result 4 is a new contribution.

In what follows, we present how the above results are addressed in this thesis. We solve the output regulation problem for the cantilever beam using a finite-dimensional robust controller in Publication I. We consider shear force control input and velocity output at the free end of the beam. In addition, we also consider a case where we have distributed control input and weighted average of velocities as the measured output.

We consider output regulation of the flexible satellite model that is composed of two identical flexible solar panels and a center rigid body in Publications II and III. Three different robust controllers are proposed for the output tracking of the satellite model. Force and moment control inputs and linear and angular velocities as outputs are considered on the center rigid body. Exponential stability of the satellite model is proved in Publication III.

Moreover, output regulation theory for finite-dimensional linear systems subject to input constraints is generalized to a particular class of infinite-dimensional linear systems in Publication IV. A linear output feedback control input is proposed for the output tracking of given reference signals. The results are illustrated with an example of the flexible satellite model subject to input constraints.

1.4 Thesis Structure

In Chapter II, we present output tracking problems for the selected two Euler-Bernoulli beam models, the cantilever beam and the model of a flexible satellite. Chapter III is devoted to the mathematical tools used to solve the considered problems in Chapter II. We introduce abstract linear systems and the controllers for the robust output regulation. In addition, we also discuss selected properties, stability and stabilization, passivity and well-posedness, of abstract linear systems. We also introduce output regulation of systems with input constraints. In Chapter IV, we discuss the main results of the thesis. Concluding remarks and future perspectives are presented in Chapter V.

2 CONTROL OF EULER-BERNOULLI BEAM MODELS

In this chapter, we describe the cantilever beam and the flexible satellite model mathematically by PDEs and coupled PDEs-ODEs. We introduce control inputs and measured outputs for both of the models. Moreover, we formulate the output regulation problem for the models.

In practice, physical beam models have some damping caused by internal or external friction forces. The types of damping on beams include *viscous, structural, Kelvin-Voigt, spatial hysteresis and time hysteresis* dampings [7, 57, 66]. If a beam is allowed to vibrate freely, the nature of damping will reduce vibrations of the beam and there will be no vibrations after some time. The decay rate of the vibrations depends on the amount of damping. Therefore, a good mathematical model for an Euler-Bernoulli beam should include some damping terms. However, undamped models have been studied widely [4, 11, 15, 97] to better understand certain properties, such as stabilizability, of the models. Moreover, undamped models can be considered to represent situations where the natural damping is very weak. In this work, our goal is to propose control designs which can introduce damping for the stabilization when the damping in the considered beam model is weak.

2.1 The Cantilever Beam

A *cantilever beam* is a beam which is clamped at one end and free at the other end (see Figure 1.2a). Assuming that the beam is of length 1, then transverse vibrations $w(\xi,t)$, $\xi \in \Omega = (0,1)$ of the beam are modelled by [32]

$$\rho(\xi)\frac{\partial^2 w}{\partial t^2}(\xi,t) + \frac{\partial^2}{\partial \xi^2}\left(EI(\xi)\frac{\partial^2 w}{\partial \xi^2}\right)(\xi,t) = 0, \quad \xi \in (0,1),\ t > 0.$$

If the end $\xi = 0$ is clamped and the end $\xi = 1$ is free, then the boundary conditions are given by

$$w(0,t) = 0, \quad \frac{\partial w}{\partial \xi}(0,t) = 0,$$
$$\left(EI(\xi)\frac{\partial^2 w}{\partial \xi^2}\right)(1,t) = 0, \quad \frac{\partial}{\partial \xi}\left(EI(\xi)\frac{\partial^2 w}{\partial \xi^2}\right)(1,t) = 0.$$

In general, control and disturbance inputs affect the beam system via boundary or inside the domain $(0,1)$. In this work, we consider the cases where the cantilever beam has boundary control, boundary measurements (see Figure 2.1) and distributed control and measurements inside the domain. Additionally, we assume that there are no external disturbances. In what follows, we introduce the two cases that we consider for the control inputs and the measured outputs.

Boundary control and observation. The beam can be controlled by an external applied force at the free end, see Figure 2.1. In this case, the control

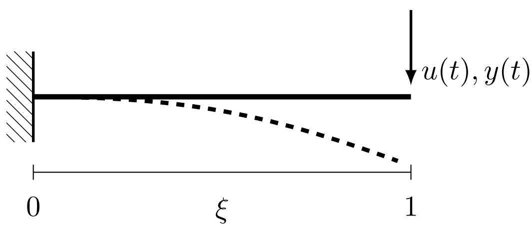


Figure 2.1 The Cantilever Beam with Boundary Control and Output

input $u(t)$ is determined by

$$\frac{\partial}{\partial \xi}\left(EI(\xi)\frac{\partial^2 w}{\partial \xi^2}\right)(1,t) = u(t).$$

We are interested in measuring velocity of the beam at the free end, i.e.,

$$y(t) = \frac{\partial w}{\partial t}(1,t).$$

In this case, our goal is to find a control design $u(t)$ such that the velocity at the free end of the beam tracks given reference signal asymptotically in the sense that

$$\int_t^{t+1} \left\| \frac{\partial w}{\partial s}(1, s) - y_{ref}(s) \right\| ds \to 0 \quad \text{as} \quad t \to \infty$$

despite external disturbances and a class of parameter uncertainties in the system. We note that we do not consider pointwise convergence of the error since boundary velocity resulting from non-smooth initial conditions may not be continuous.

Distributed control and observation. The beam can be controlled by an external applied force inside the domain $(0, 1)$, see Figure 2.2. In this case, the

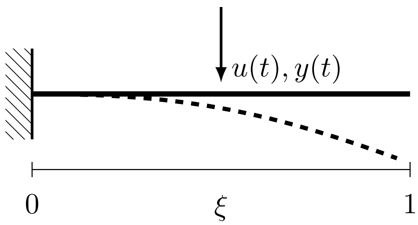


Figure 2.2 The Cantilever Beam with Distributed Control and Output

distributed force control input of the cantilever beam is determined by

$$\rho(\xi) \frac{\partial^2 w}{\partial t^2}(\xi, t) + \frac{\partial^2}{\partial \xi^2}\left(EI(\xi) \frac{\partial^2 w}{\partial \xi^2} \right)(\xi, t) = b(\xi) u(t)$$

where $b(\cdot) \in L^2(\Omega)$ is a real-valued function. We are interested in measuring weighted average of velocities in the domain $(0, 1)$ and it is given by

$$y(t) = \int_0^1 b(\xi) \frac{\partial w}{\partial t}(\xi, t) d\xi.$$

Now, the goal is to find a control design $u(t)$ such that the weighted average of velocities in the domain $(0, 1)$ tracks given reference signal asymptotically in the sense that

$$\left\| \int_0^1 b(\xi) \frac{\partial w}{\partial t}(\xi, t) d\xi - y_{ref}(t) \right\| \to 0 \quad \text{as} \quad t \to \infty$$

despite external disturbances and a class of parameter uncertainties in the system.

2.2 The Flexible Satellite Model

In this thesis, we consider the flexible satellite that is composed of two identical solar panels and a center rigid body [8, 40], see Figure 2.3. The solar panels

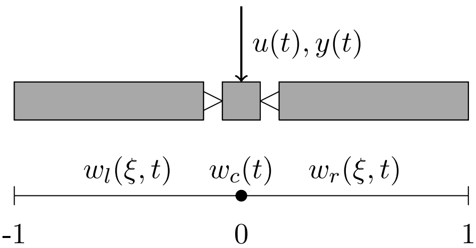


Figure 2.3 A flexible satellite model with control and observation on the center rigid body

can be modelled as Euler-Bernoulli beams. Mathematically, the considered satellite model is a coupled system of PDEs and ODEs given by

$$
\begin{aligned}
&\rho\frac{\partial^2 w_l}{\partial t^2}(\xi,t) + EI\frac{\partial^4 w_l}{\partial \xi^4}(\xi,t) + \gamma\frac{\partial w_l}{\partial t}(\xi,t) = 0, \quad -1 < \xi < 0,\\
&\rho\frac{\partial^2 w_r}{\partial t^2}(\xi,t) + EI\frac{\partial^4 w_r}{\partial \xi^4}(\xi,t) + \gamma\frac{\partial w_r}{\partial t}(\xi,t) = 0, \quad 0 < \xi < 1,\\
&m\frac{d^2 w_c}{dt^2}(t) = EI\frac{\partial^3 w_l}{\partial \xi^3}(0,t) - EI\frac{\partial^3 w_r}{\partial \xi^3}(0,t),\\
&I_m\frac{d^2 \theta_c}{dt^2}(t) = -EI\frac{\partial^2 w_l}{\partial \xi^2}(0,t) + EI\frac{\partial^2 w_r}{\partial \xi^2}(0,t),\\
&\frac{\partial^2 w_l}{\partial \xi^2}(-1,t) = 0, \quad \frac{\partial^3 w_l}{\partial \xi^3}(-1,t) = 0,\\
&\frac{\partial^2 w_r}{\partial \xi^2}(1,t) = 0, \quad \frac{\partial^3 w_r}{\partial \xi^3}(1,t) = 0,\\
&\frac{\partial w_l}{\partial t}(0,t) = \frac{\partial w_r}{\partial t}(0,t) = \frac{dw_c}{dt}(t),\\
&\frac{\partial^2 w_l}{\partial t\partial \xi}(0,t) = \frac{\partial^2 w_r}{\partial t\partial \xi}(0,t) = \frac{d\theta_c}{dt}(t).
\end{aligned}
$$

where $t > 0$, $w_l(\xi,t)$ and $w_r(\xi,t)$ are the transverse displacements of the left and the right beam, respectively, $w_c(t)$ and $\theta_c(t)$ are the linear and angular displacements of the rigid body, respectively. The parameters ρ, EI and γ are linear density, flexural rigidity and the viscous damping coefficient of the beams, respectively, and m and I_m denote the mass and the mass moment of inertia of the center rigid body. The parameters are assumed to be constants.

In general, control and disturbance inputs affect the satellite system via boundary or inside the domain $(-1, 1)$. In this work, we consider external force control input $u_1(t)$ and moment control input $u_2(t)$ on the center rigid body. Additionally, we consider external force disturbance $w_{d1}(t)$ and moment disturbance $w_{d2}(t)$ on the rigid body and possible force disturbances distributed on the solar panels. So, the control input $u(t)$ and the disturbance input $w_d(t)$ are of the form

$$\begin{aligned}
&\rho\frac{\partial^2 w_l}{\partial t^2}(\xi,t) + EI\frac{\partial^4 w_l}{\partial \xi^4}(\xi,t) + \gamma\frac{\partial w_l}{\partial t}(\xi,t) = b_{d1}(\xi)w_{d1}(t),\\
&\rho\frac{\partial^2 w_r}{\partial t^2}(\xi,t) + EI\frac{\partial^4 w_r}{\partial \xi^4}(\xi,t) + \gamma\frac{\partial w_r}{\partial t}(\xi,t) = b_{d2}(\xi)w_{d2}(t),\\
&m\frac{d^2 w_c}{dt^2}(t) = EI\frac{\partial^3 w_l}{\partial \xi^3}(0,t) - EI\frac{\partial^3 w_r}{\partial \xi^3}(0,t) + u_1(t) + w_{d3}(t),\\
&I_m\frac{d^2 \theta_c}{dt^2}(t) = -EI\frac{\partial^2 w_l}{\partial \xi^2}(0,t) + EI\frac{\partial^2 w_r}{\partial \xi^2}(0,t) + u_2(t) + w_{d4}(t)
\end{aligned}$$

where $u(t) = [u_1(t), u_2(t)]^T$, $w_d(t) = [w_{d1}(t), w_{d2}(t), w_{d3}(t), w_{d4}(t)]^T$ and $b_{d1}(\cdot) \in L^2(-1, 0)$ and $b_{d2}(\cdot) \in L^2(0, 1)$ are real-valued functions.

We are interested in the measurements of linear and angular velocities of the center rigid body. Therefore, the measured output $y(t)$ is of the form

$$y(t) = \left(\frac{dw_c}{dt}(t), \frac{d\theta_c}{dt}(t)\right)^T.$$

Our goal is to find a control design $u(t)$ such that the linear velocity and the angular velocity of the center rigid body track given reference sinusoidal signals asymptotically in the sense that

$$\left\|\frac{dw_c}{dt}(t) - y_{ref1}(t)\right\| \to 0 \quad \text{and} \quad \left\|\frac{d\theta_c}{dt}(t) - y_{ref2}(t)\right\| \to 0 \quad \text{as} \quad t \to \infty$$

despite external disturbances and a class of parameter uncertainties in the system.

2.3 Notation

In Publications I, II, III and IV, we use notations $w_\xi(\xi, t)$, $w'(\xi, t)$ for partial derivative of $w(\xi, t)$ with respect to spatial variable ξ and $w_t(\xi, t)$, $\dot{w}(\xi, t)$ for partial derivative of $w(\xi, t)$ with respect to time t.

3 THEORETICAL BACKGROUND

Mathematical models described by linear ordinary or partial differential equations can be written in the form of abstract linear systems on suitable state spaces [17, 21]. We use such a formulation since control designs we propose for the output regulation assume that the system is in the abstract form and satisfies certain required properties. In what follows, we introduce abstract linear systems and define properties such as passivity, well-posedness and stability of these systems. Then we introduce robust control designs for the output regulation. Finally, output regulation of systems subject to input constraints is introduced.

3.1 Abstract Linear Systems

For the given model, we denote the state space by X, the input space by U, the disturbance space by U_d and the output space by Y. Let $A : D(A) \subset X \to X$ be a generator of a strongly continuous semigroup $T(t)$ on X [21, Ch. I.5] with non empty resolvent set $\rho(A)$. We define X_1 to be the space $D(A)$ with norm $\|x\|_1 = \|(\beta I - A)x\|$, $\forall\ x \in D(A)$ and X_{-1} as the completion of X with norm $\|x\|_{-1} = \|(\beta I - A)^{-1}x\|$, $\forall\ x \in X$ where $\beta \in \rho(A)$ is fixed. The operator A has a unique extension $A_{-1} \in \mathcal{L}(X, X_{-1})$ [90, Sec. 2.10]. Now, the abstract linear system corresponding to the given model is of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d w_d(t), \quad x(0) = x_0, \tag{3.1a}$$

$$y(t) = Cx(t), \tag{3.1b}$$

where $x(t) \in X$ is the state variable, $u(t) \in U$ is the control input, $w_d(t) \in U_d$ is the external disturbance and $y(t) \in Y$ is the output. If the model is described by ordinary differential equations, then X is a finite-dimensional space whereas for the model described by partial differential equations X is an infinite-dimensional space. Since we consider Euler-Bernoulli beam models and their solutions are functions in suitable Hilbert spaces, X is an infinite-dimensional Hilbert space. In this work, the spaces U, U_d and Y are finite-dimensional. The operators B, B_d and C are control operator, disturbance operator and observation operator, respectively. The operators B, B_d and C are linear but not necessarily bounded. The operators B, B_d and C are called *bounded* if $B \in \mathcal{L}(U, X)$, $B_d \in \mathcal{L}(U_d, X)$ and $C \in \mathcal{L}(X, Y)$ otherwise they are called *unbounded* [90, Ch. 4.2].

Now we restrict to the situation when the operator A generates strongly continuous semigroup of contractions, i.e., the semigroup $T(t)$ satisfies $\|T(t)\| \leq 1$ for every $t \geq 0$ [90, Def. 3.1.12]. In this case, the semigroup generation of A is often obtained by the Hilbert space version of Lumer-Phillips theorem [90, Thm. 3.8.4], [48, Ch. 6].

Theorem 3.1.1 (Lumer–Phillips)**.** *Let A be a linear operator with domain $D(A)$ on a Hilbert space X. Then A is the infinitesimal generator of the contraction semigroup $T(t)$, $t \geq 0$ on X if and only if A is dissipative, i.e., $Re\,\langle Ax, x\rangle \leq 0$, $\forall x \in D(A)$, and $\mathcal{R}(I - A) = X$.*

In this work, we consider the situations where the actuators and the sensors are implemented at the same physical location which is also called as *collocated actuators and sensors.* This will lead to a special class of systems called *impedance passive linear systems* defined as follows.

Definition 3.1.2. *The system* (3.1) *is impedance passive if $U = Y$ and the solutions $x(t)$ satisfy*

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|_X^2 \leq Re\,\langle u(t), y(t)\rangle_{U,Y}\,, \quad t > 0.$$

The motivation to consider these class of systems is that Euler-Bernoulli beam models that we consider in this work have collocated actuators and sensors and they can be formulated as impedance passive abstract linear systems.

For models described by PDEs with non-homogeneous boundary conditions, the corresponding abstract systems will not naturally appear in the form (3.1). In what follows, we introduce *boundary control systems.*

Boundary Control Systems. For simplicity, assume that there are no external disturbances. If we have control inputs and observations at the boundary of the domain, then the abstract system corresponding to the given model is written in the form

$$\dot{x}(t) = \mathcal{A}x(t), \quad x(0) = x_0, \tag{3.2a}$$

$$\mathcal{B}x(t) = u(t), \tag{3.2b}$$

$$y(t) = \mathcal{C}x(t) \tag{3.2c}$$

where $\mathcal{A} : D(\mathcal{A}) \subset X \to X$, $\mathcal{B} : D(\mathcal{A}) \to U$ and $\mathcal{C} : D(\mathcal{A}) \to Y$ are linear operators. However, under some extra conditions it is possible to write the system model with boundary control and observation in the form (3.1).

Definition 3.1.3 (Boundary Control and Observation System [17, Def. 3.3.2], [48, Ch. 11])**.** *Let X, U and Y be Hilbert spaces. Then the system (3.2) is a boundary control and observation system if the following hold.*

1. *The operator $A : D(A) \subset X \to X$ with $D(A) = D(\mathcal{A}) \cap \mathcal{N}(\mathcal{B})$ and $Ax = \mathcal{A}x$ for $x \in D(A)$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$ on X.*
2. *There exists an operator $H \in \mathcal{L}(U, X)$ such that for all $u \in U$ we have $Hu \in D(\mathcal{A})$, $\mathcal{A}H \in \mathcal{L}(U, X)$ and $\mathcal{B}Hu = u$.*

In what follows, we introduce a result on how the equations (3.2a) and (3.2b) can be reformulated equivalently into the form (3.1).

Remark 3.1.4 ([90, Sec. 10.1])**.** *Let $(\mathcal{A}, \mathcal{B})$ be a boundary control system. Then there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$ such that $\mathcal{A} = A_{-1} + B\mathcal{B}$ on $D(\mathcal{A})$ and therefore* (3.2a) *and* (3.2b) *can be written as*

$$\dot{x}(t) = A_{-1}x(t) + Bu(t), \quad x(0) = x_0.$$

Abstract formulation of Euler-Bernoulli beam models can be found in [3, 4, 32, 66, 73, 96]. Euler-Bernoulli beams as boundary control systems have been studied in [90, Ch. 10] and [18, Ch. 10].

3.2 Solutions of Abstract Linear Systems

In this section, we focus on state trajectories x which are the solutions of (3.1a). The solution of (3.1a), if it exists, is given by

$$x(t) = T(t)x_0 + \int_0^t T(t-\tau)Bu(\tau)d\tau + \int_0^t T(t-\tau)B_d w_d(\tau)d\tau. \tag{3.3}$$

If the operators B and B_d are bounded, then the above solution belongs to the space X. If the operators B and B_d are unbounded, then the semigroup $T(t)$ in (3.3) is understood as the strongly continuous semigroup generated by the operator A_{-1} and the solution (3.3) belongs to the extended state space X_{-1}. In this work, we also study those unbounded operators B, B_d for which the above solution belongs to the space X. Such operators are called *admissible operators*. In what follows, we define admissibility of the control operator B. Admissible disturbance operator B_d is defined analogously.

Definition 3.2.1. *An operator $B \in \mathcal{L}(U, X_{-1})$ is called admissible control operator for the semigroup $T(t)$ if for some $t > 0$,*

$$\int_0^t T(t-\tau)Bu(\tau)d\tau \in X.$$

If the operators B and B_d are bounded, then they are admissible. Now we turn our attention to the output $y(t)$ of the system (3.1) when we have no control and disturbance inputs and $x_0 \in D(A)$. We define the concept of an admissible observation operator [90, Ch. 4.3].

Definition 3.2.2. *An operator $C \in \mathcal{L}(X_1, Y)$ is called an admissible observation operator for the semigroup $T(t)$ if for some $\tau > 0$, there exists a constant $k_\tau > 0$ such that*

$$\int_0^\tau \|CT(t)x_0\|_Y^2 dt \le k_\tau^2 \|x_0\|_X^2, \quad \forall\, x_0 \in D(A).$$

If $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator, then the output satisfies $y \in L^2(0, \tau; Y)$ [90, Prop. 4.3.2].

In general, if on any time interval $[0, \tau]$, for any initial condition $x_0 \in X$ and any input $u \in L^2(0, \tau; U)$ there exist continuous X-valued solutions x and the output satisfies $y \in L^2(0, \tau; Y)$, then the abstract linear system (3.1) is *well-posed* [91], [48, Ch. 13]. For an impedance passive linear system, well-posedness of the system can often be proved by showing that *transfer function* of the system is bounded on some complex right half plane [87]. In what follows, we introduce transfer functions of the systems (3.1) and (3.2).

Transfer Function. Let us assume that Laplace transforms of the input $u(t)$ and the output $y(t)$ exist and denote the Laplace transform of $u(t)$ by $\hat{u}(s)$ and the Laplace transform of $y(t)$ by $\hat{y}(s)$. Then $\hat{u}(s)$ and $\hat{y}(s)$ have the relation $\hat{y}(s) = G(s)\hat{u}(s)$ for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \beta$ for some real β, where $G(s)$ is the system transfer function, see for example [12], [17].

If the operators B, B_d and C are bounded, then the transfer function of (3.1) corresponding to the control input $u(t)$ is given by

$$G_c(s) = C(sI - A)^{-1}B, \quad s \in \rho(A)$$

and the transfer function of (3.1) corresponding to the disturbance input $w_d(t)$ is given by

$$G_d(s) = C(sI - A)^{-1}B_d, \quad s \in \rho(A).$$

For $s \in \rho(A)$, the transfer function $G(s)$ of the boundary control and observation system (3.2) is given by ([48, Ch. 12], [12])

$$G(s)u = \mathcal{C}x(s),$$

where $x(s)$ is the unique solution of

$$\begin{aligned} sx &= \mathcal{A}x, \\ \mathcal{B}x &= u. \end{aligned}$$

We note that for a boundary control and observation system, the transfer function can be computed directly from the system and there is no need to find the operator B in order to compute the system transfer function. Moreover, since we consider impedance passive linear systems it is sufficient to study transfer functions to verify well-posedness of the systems.

3.3 Stability and Stabilization

Stabilization is an important part of control design for output regulation. Stability of a system corresponds to the behaviour of the solutions with respect to time. In general, in the absence of external inputs, a system of the form (3.1) is stable if the solutions decay to zero asymptotically, i.e., $\|x(t)\| \to 0$ as $t \to \infty$ and a system is stabilizable if one can find a control input such that the corresponding solutions decay to zero asymptotically. Since the solution is written in terms of semigroup, then the stability corresponds to the behaviour of the semigroup when the time evolves. There are different types of stability concepts for strongly continuous semigroups, see for example [21, Ch. V.1]. In this thesis we focus on strong stability and exponential stability which are defined as follows.

Definition 3.3.1 ([21, Ch. V.1]). *A strongly continuous semigroup $T(t),\ t \geq 0$ is called*

1. *uniformly exponentially stable if there exists $\epsilon > 0$ such that*

$$\lim_{t\to\infty} e^{\epsilon t}\|T(t)\| = 0,$$

2. *strongly stable if*

$$\lim_{t\to\infty} \|T(t)x\| = 0 \quad \forall\ x \in X.$$

In this work since we have impedance passive linear systems, a negative output feedback stabilizes (strongly or exponentially) given system. Stability analysis of a system is not trivial. In this work, the exponential stability of the given system is proved using the following frequency domain criteria unless stability of the system is obtained from the existing literature.

Theorem 3.3.2 ([66, Cor. 3.36]). *Let $T(t)$ be a uniformly bounded strongly continuous semigroup, i.e., there exists $M > 0$ such that $\|T(t)\| \leq M$, on a Hilbert space X with generator A. Then $T(t)$ is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and $\sup_{\omega\in\mathbb{R}} \|(i\omega - A)^{-1}\| < \infty$.*

3.4 Robust Output Regulation

In this section, we introduce robust control designs for the output regulation of the system (3.1). For now, we assume that the operators B, B_d and C are bounded. The reference signals to be tracked and the disturbance signals to be rejected are of the form given by

$$y_{ref}(t) = a_0 + \sum_{k=1}^{q} a_k \cos(\omega_k t) + b_k \sin(\omega_k t) \tag{3.4}$$

$$w_d(t) = c_0 + \sum_{k=1}^{q} c_k \cos(\omega_k t) + d_k \sin(\omega_k t) \tag{3.5}$$

where $(\omega_k)_{k=1}^q$ are known frequencies and $(a_k)_{k=0}^q$, $(b_k)_{k=1}^q$, $(c_k)_{k=0}^q$ and $(d_k)_{k=1}^q$ are possibly unknown constant coefficients.

Our goal is to find a control input $u(t)$ such that the output $y(t)$ tracks given reference signal $y_{ref}(t)$ asymptotically despite parameter uncertainties and external disturbances in the system. In this work, the control inputs are produced by dynamic error feedback controllers [39], [75]. The controllers utilize the difference between the output $y(t)$ and the reference signal $y_{ref}(t)$ and they are of the form

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1 z(t) + \mathcal{G}_2(y(t) - y_{ref}(t)), \\ u(t) &= K z(t) - \kappa(y(t) - y_{ref}(t)) \end{aligned} \tag{3.6}$$

on a Hilbert space Z, where $\mathcal{G}_1 : D(\mathcal{G}_1) \subset Z \to Z$ generates a strongly continuous semigroup, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$, $K \in \mathcal{L}(Z, U)$ and $\kappa \in \mathcal{L}(Y, U)$.

Coupling of the system (3.1) with the controller (3.6) yields a closed-loop system which is depicted in Figure 3.1. Let us denote $X_e = X \times Z$ to be the extended state space, $x_e(t) = (x(t), z(t))^T$ to be the extended state and $u_e(t) = (w_d(t), y_{ref}(t))^T$ to be the extended input. Then the closed-loop system consisting of the system (3.1) and the controller (3.6) is given by

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e u_e(t), \quad x_e(0) = x_{e0}, \\ e(t) &= C_e x_e(t) + D_e u_e(t), \end{aligned} \tag{3.7}$$

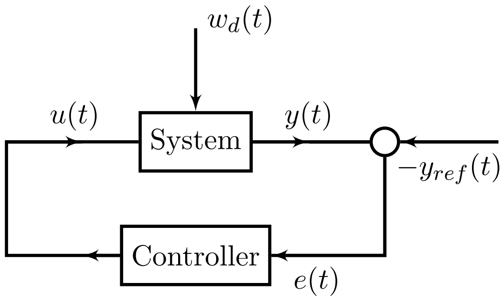


Figure 3.1 The closed-loop system interconnecting the system and the controller

where $C_e = \begin{bmatrix} C & 0 \end{bmatrix}, D_e = \begin{bmatrix} 0 & -I_Y \end{bmatrix}$,

$$A_e = \begin{bmatrix} A - B\kappa C & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 \end{bmatrix}, \quad B_e = \begin{bmatrix} B_d & B\kappa \\ 0 & -\mathcal{G}_2 \end{bmatrix},$$

and $D(A_e) = D(A) \times D(\mathcal{G}_1)$. The operator A_e generates a strongly continuous semigroup $T_e(t)$ on X_e.

The robust output regulation problem that we consider in this thesis is formulated as follows [75].

Robust Output Regulation Problem. Choose the controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K, \kappa)$ in such a way that the following hold.

(a) The closed-loop semigroup $T_e(t)$ generated by A_e is exponentially stable.

(b) There exist $\alpha, M_e > 0$ such that for all initial states $x_{e0} \in X_e$, for all reference signals and disturbance signals of the form (3.4) and (3.5), the regulation error $y(t) - y_{ref}(t)$ satisfies

$$e^{\alpha t}\|y(t) - y_{ref}(t)\| \to 0 \quad \text{as} \quad t \to \infty. \tag{3.8}$$

(c) If the operators (A, B, B_d, C) are perturbed in such a way that the perturbed closed-loop system is exponentially stable, then (b) continues to hold for some $\tilde{\alpha}, \tilde{M}_e > 0$.

In Publication I, we also consider the cases where the operators B, B_d and C are unbounded and the closed-loop semigroup $T_e(t)$ is not exponentially stable.

If the semigroup generated by A_e is not exponentially stable, then the decay rate in (3.8) is not exponentially fast [76, Sec. 4]. Moreover, if the input and the observation operators of the system are unbounded then the output $y(t)$ resulting from non-smooth initial states x_0 might not be continuous and we can not obtain pointwise convergence of the error as in (3.8), instead the regulation error satisfies [76, 81]

$$\int_t^{t+1} \|y(s) - y_{ref}(s)\| ds \to 0 \quad \text{as} \quad t \to \infty.$$

The controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K, \kappa)$ are chosen based on the *internal model principle* which states that a controller can solve the robust output regulation problem if its dynamics include copies of the frequencies from the reference and disturbance signals. In this work, we utilize control designs in [81], [75] and [76]. In [81], controllers were designed for output tracking and disturbance rejection of stable well-posed linear systems. In [75], three robust controllers were presented for the output regulation of regular linear systems [93], a special class of well-posed linear systems. In [76], passive controllers were presented for output regulation of passive linear systems.

3.5 Output Regulation for Systems with Input Saturation

In practice, it is natural to consider limitations in the actuators as amplitude of signals is limited to certain maximum level. This is known by the name of *input saturation.* Taking actuator limitations into account, the abstract linear system (3.1) has the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\phi(u(t)) + B_d w_d(t), \;\; x(0) = x_0, \\ y(t) &= Cx(t), \end{aligned} \tag{3.9}$$

where the operators A, B, B_d and C are defined as in Section 3.1 and ϕ is a saturation function where the input $u(t)$ takes values in an interval $[u_{min}, u_{max}]$. There are only few results studying output tracking of infinite-dimensional linear systems subject to input saturation. In [82], control designs for output regulation of finite-dimensional linear systems subject to input saturation have been presented. We focus on generalizing the control law presented in [82, Thm. 3.3.3] to a particular class of infinite-dimensional linear systems. Other existing results [24, 62, 63, 64] cover stable single-input single-output regular linear systems and output tracking of constant reference signals is achieved using integral controls.

In this work, we consider reference and disturbance signals of the form (3.4) and (3.5). These signals are assumed to be generated by an autonomous system called the *exosystem* given by

$$\begin{aligned} \dot{v}(t) &= Sv(t), \quad v(0) = v_0, \\ w_d(t) &= Ev(t), \\ y_{ref}(t) &= -Fv(t) \end{aligned} \tag{3.10}$$

on a finite-dimensional space $W = \mathbb{R}^q$, where $S \in \mathcal{L}(W)$, $E \in \mathcal{L}(W, U_d)$ and $F \in \mathcal{L}(W, Y)$. We assume that S has purely imaginary eigenvalues. In general, the exosystem is chosen in a way that the operator S contains frequencies from the signals and v_0, F and E are determined by amplitudes and phases of the signals.

In general, the output regulation problem for linear systems subject to input saturation is not solvable for all initial conditions of the exosystem ([82, Rem. 3.2.2]) in the sense that there exist initial conditions $v_0 \in W$ such that there is no input $u(t)$ or initial condition $x_0 \in X$ for which it holds that $\lim_{t\to\infty}(y(t) - y_{ref}(t)) = 0$. In this work, we consider the *semi-global output regulation problem*, where the initial conditions of the exosystem lie inside a given compact set. Assuming that the system (3.9) can be stabilized strongly by negative output feedback and the operators B, B_d and C are bounded, the semi-global output regulation problem that we consider is formulated as follows.

Semi-Global Output Regulation Problem. Consider the system (3.9), the exosystem (3.10) and a compact set $\mathcal{W}_0 \subset \mathbb{R}^q$. Find a linear output feedback control law of the form

$$u(t) = -\kappa y(t) + Lv(t), \tag{3.11}$$

such that $\kappa \in \mathcal{L}(Y, U)$, $L \in \mathcal{L}(W, U)$ and

(a) The origin of the system $\dot{x}(t) = Ax(t) + B\phi(-\kappa y(t))$, $x(0) = x_0$ is globally asymptotically stable.

(b) For all $x_0 \in X$ and $v_0 \in \mathcal{W}_0$, the error between the output $y(t)$ and the reference signal $y_{ref}(t)$ satisfies

$$\lim_{t \to \infty} \|y(t) - y_{ref}(t)\| = 0.$$

It is shown in [82] that solvability of a semi-global output regulation problem for finite-dimensional systems is equivalent to solvability of a pair of linear matrix equations with some additional assumption. Moreover, controllers that solve semi-global output regulation in [82] are internal model based controllers.

4 RESULTS AND DISCUSSION

The main objective of this thesis is to construct controllers for the robust output regulation of the selected Euler-Bernoulli beam models. Earlier results mainly focus on vibration suppression or stabilization problem. In this chapter, we discuss the results we obtained on

- Controller construction for robust output regulation of the cantilever beam.
- Controller construction for robust output regulation of the flexible satellite model.
- Generalization of output regulation theory for finite-dimensional linear systems subject to input saturation to infinite-dimensional linear systems.

We recall that the reference signals to be tracked and the disturbance signals to be rejected are of the form

$$\begin{aligned} y_{ref}(t) &= a_0 + \sum_{k=1}^{q} a_k \cos(\omega_k t) + b_k \sin(\omega_k t), \\ w_d(t) &= c_0 + \sum_{k=1}^{q} c_k \cos(\omega_k t) + d_k \sin(\omega_k t) \end{aligned} \tag{4.1}$$

and the dynamic error feedback controllers for the robust output regulation are of the form

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1 z(t) + \mathcal{G}_2 (y(t) - y_{ref}(t)), \\ u(t) &= K z(t) - \kappa (y(t) - y_{ref}(t)). \end{aligned} \tag{4.2}$$

More details on the signals and the controller can be found in Section 3.4.

4.1 Controller construction for the cantilever beam

Robust output regulation of the cantilever beam introduced in Section 2.1 is considered in Publication I. We consider physical parameters $\rho(\xi)$ and $EI(\xi)$ which satisfy the conditions

$$\rho(\cdot), EI(\cdot) \in C^4([0,1]), \qquad \rho(\xi), EI(\xi) > 0, \ \forall \xi \in [0,1] \tag{4.3}$$

As mentioned in Section 2.1, we consider two cases for the control input and the output and we assume that there are no external disturbances. In the first case, we consider shear force control input and velocity output at the free end of the beam. We formulate the beam system as an impedance passive abstract well-posed linear system, see sections 3.1 and 3.2. We show that the beam system can be stabilized exponentially by using negative output feedback $u(t) = -\kappa_1 y(t), \kappa_1 > 0$. Having exponentially stable well-posed linear system, we are able to construct a finite-dimensional dynamic error feedback controller controller based on [81] for the robust output tracking of the given sinusoidal reference signals of the form in (4.1).

In the second case, the control input is distributed inside the domain and the output is weighted average of velocities of the beam system in the domain. The beam system is formulated as an impedance passive abstract linear system with bounded input and output operators. We show that the beam system can be stabilized strongly by negative output feedback. Having passive and strongly stabilizable system, we are able to construct a controller based on [76]. Due to passivity of the system, the controller has similar structure as in the first case. However, in this case we only obtain strong stability of the closed-loop system and therefore slow decay rate of the regulation error as mentioned in Section 3.4.

The constructed controller is a dynamic error feedback controller (4.2) with state space $Z = \mathbb{R}^{2q}$ and the choice of parameters given by

$$\mathcal{G}_1 = \operatorname{diag}(G_1, G_2, \cdots, G_q), \quad G_k = \begin{bmatrix} 0 & \omega_k \\ -\omega_k & 0 \end{bmatrix}, \quad k = 1, 2, \cdots, q,$$

$$\mathcal{G}_2 = -[1, 0, \cdots, 1, 0]^T, \quad K = [2, 0, \cdots, 2, 0], \quad \kappa > \frac{1}{2},$$

where $\omega_k, k = 1, 2, \cdots, q$ are frequencies from the reference signal. From the above choice of parameters, we see that the controller does not depend on the physical parameters $\rho(\xi)$ and $EI(\xi)$ of the beam system. Moreover, the beam system can be stabilized exponentially or strongly for the given class of parameters (4.3). Therefore, the controller is robust with respect to the class of parameters (4.3).

The existing results [49], [38], [35],[34] consider Euler-Bernoulli beams with constant parameters and use infinite-dimensional controllers for output tracking. In comparison, our controller does not depend on the physical parameters of the beam system, therefore, the output regulation problem is solved for the cantilever beam with spatially varying parameters whereas the existing controllers are based on observers of the beam system and so the output tracking might be achieved only for a restrictive class of parameters. The reference [49] also use state feedback controller which utilizes state variables from the system for output tracking. However, in practice, the availability of full state information is not always possible. Since our controller utilizes only regulation error, there is no need find full state information.

4.2 Controller construction for the flexible satellite model

Robust output regulation of the flexible satellite model introduced in Section 2.2 is considered in Publications II and III. We consider force control inputs on the rigid body and the measured outputs are velocity and angular velocity of the center rigid body. We also consider external disturbances distributed in the solar panels and on the rigid body. As the first result, we formulate the satellite model as an abstract impedance passive linear system (3.1) with bounded input and output operators. The PDE and ODE systems in the model are formulated in the abstract form separately and then they are coupled using the coupling conditions. A detailed proof of exponential stability of the model using the frequency domain criteria is presented in Publication III. The exponential stability proof of the satellite model is one of the main contributions of the Publication III. As a part of the stability proof, we also show that the PDE system in the satellite model is not well-posed. In comparison, existing results [100] on stability of coupled PDE-ODE systems use controllability and observability results and only strong stability of the system is obtained. Moreover, those results consider a well-posed PDE system coupled with specific ODE structure. Due to the distributed viscous damping in the beam system, we are able to prove exponential stability of the satellite model.

Exponential stability of the satellite model enables us to construct a simple low-gain controller [75] for the robust output tracking of the given sinusoidal reference signals. The low-gain controller is a dynamic error feedback controller (4.2) with state space $Z = (\mathbb{C}^2)^{2q+1}$ with the controller parameters given by

$$\begin{aligned}
\mathcal{G}_1 &= \mathrm{diag}(-i\omega_q I_{\mathbb{C}^2}, \cdots, i\omega_0 I_{\mathbb{C}^2}, \cdots, i\omega_q I_{\mathbb{C}^2}),\\
K &= \epsilon(G(-i\omega_q)^\dagger, \cdots, G(i\omega_0)^\dagger, \cdots, G(i\omega_q)^\dagger), \quad \kappa = 0,\\
\mathcal{G}_2 &= (-I_{\mathbb{C}^2})_{-q}^{q},
\end{aligned}$$

where $I_{\mathbb{C}^2}$ is the identity matrix in $\mathbb{C}^2$, $G(\cdot)$ is the transfer function of the satellite system, $G(\cdot)^\dagger$ is the Moore-Penrose pseudoinverse of $G(\cdot)$ and the parameter $\epsilon > 0$ is a tuning parameter and it is chosen to be sufficiently small so that the closed-loop system is exponentially stable.

Due to passivity and boundedness of control and observation operators, we were able to construct a passive controller and an observer based controller [76], [75] for the robust output tracking of the given reference signals, see Publication III. In what follows, we present the observer based controller which is a PDE-ODE controller.

Choosing the state space to be $Z = (\mathbb{C}^2)^{2q+1} \times L^2(-1,0;\mathbb{R}^2) \times L^2(0,1;\mathbb{R}^2) \times \mathbb{R}^2$, then the PDE-ODE controller is given by

$$\begin{aligned}
&\dot{z}_1(t) = G_1 z_1(t) + G_2(y(t) - y_{ref}(t)),\\
&\rho\frac{\partial^2 \hat{w}_l}{\partial t^2}(\xi,t) = -EI\frac{\partial^4 \hat{w}_l}{\partial \xi^4}(\xi,t) - \gamma\frac{\partial \hat{w}_l}{\partial t}(\xi,t), \quad -1 < \xi < 0, \quad t > 0,\\
&\rho\frac{\partial^2 \hat{w}_r}{\partial t^2}(\xi,t) = -EI\frac{\partial^4 \hat{w}_r}{\partial \xi^4}(\xi,t) - \gamma\frac{\partial \hat{w}_r}{\partial t}(\xi,t), \quad 0 < \xi < 1, \quad t > 0,
\end{aligned}$$

$$
\begin{aligned}
&m\frac{d^2\hat{w}_c}{dt^2}(t) = EI\frac{\partial^3\hat{w}_l}{\partial\xi^3}(0,t) - EI\frac{\partial^3\hat{w}_r}{\partial\xi^3}(0,t) + u_1(t),\\
&I_m\frac{d^2\hat{\theta}_c}{dt^2}(t) = -EI\frac{\partial^2\hat{w}_l}{\partial\xi^2}(0,t) + EI\frac{\partial^2\hat{w}_r}{\partial\xi^2}(0,t) + u_2(t),\\
&\frac{\partial^2\hat{w}_l}{\partial\xi^2}(-1,t) = 0, \quad \frac{\partial^3\hat{w}_l}{\partial\xi^3}(-1,t) = 0,\\
&\frac{\partial^2\hat{w}_r}{\partial\xi^2}(1,t) = 0, \quad \frac{\partial^3\hat{w}_r}{\partial\xi^3}(1,t) = 0,\\
&\frac{\partial\hat{w}_l}{\partial t}(0,t) = \frac{\partial\hat{w}_r}{\partial t}(0,t) = \frac{d\hat{w}_c}{dt}(t),\\
&\frac{\partial^2\hat{w}_l}{\partial t\partial\xi}(0,t) = \frac{\partial^2\hat{w}_r}{\partial t\partial\xi}(0,t) = \frac{d\hat{\theta}_c}{dt}(t),\\
&u(t) = Kz(t), \quad z(t) = (z_1(t), z_2(t))^T,
\end{aligned}
$$

where $G_1 = \operatorname{diag}(-i\omega_q I_{\mathbb{C}^2}, \cdots i\omega_0 I_{\mathbb{C}^2}, \cdots, i\omega_q I_{\mathbb{C}^2})$,

$$G_2 = (G_2^k)_{k=-q}^q \in \mathcal{L}(\mathbb{R}^2, (\mathbb{C}^2)^{2q+1}),\ G_2^k = I_{\mathbb{C}^2},\ k = -q, \cdots, q,$$

$$z_2(t) = \left(\rho\frac{\partial\hat{w}_l}{\partial t}(\cdot,t), \frac{\partial^2\hat{w}_l}{\partial\xi^2}(\cdot,t), \rho\frac{\partial\hat{w}_r}{\partial t}(\cdot,t), \frac{\partial^2\hat{w}_r}{\partial\xi^2}(\cdot,t), \frac{d\hat{w}_c}{dt}(t), \frac{d\hat{\theta}_c}{dt}(t)\right)^T.$$

We note that due to technicality, we leave the choice of the operator $K \in \mathcal{L}(Z, \mathbb{R}^2)$, the details are given in Section 4.1.2 of Publication III. The performances of the passive and the observer-based controllers are demonstrated by numerical simulations in Publication III. It was noted that the finite-dimensional passive controller achieves a comparable performance to the infinite-dimensional observer based controller.

In comparison, existing results [2, 8, 40] consider similar satellite models with different control goals such as vibration control, stabilizability and output tracking of constant reference signals. However, the control goal in this work is to track reference signals of the form (3.4) and reject external disturbances of the form (3.5). Due to the properties such as exponential stability, passivity of the considered model and boundedness of the input, output operators, we are able to construct robust controllers that achieve the desired goal.

4.3 Output regulation of infinite-dimensional linear systems subject to input saturation

Output regulation of infinite-dimensional linear systems subject to input saturation is considered in Publication IV. We generalize the output regulation theory for finite-dimensional linear systems subject to input saturation to a particular class of infinite-dimensional linear systems. The considered class of systems are of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\phi(u(t)) + B_d w_d(t), \ x(0) = x_0, \\ y(t) &= B^* x(t),\end{aligned} \tag{4.4}$$

where A generates a strongly continuous semigroup of contractions on a real Hilbert space X, $B \in \mathcal{L}(\mathbb{R}, X)$ and $B_d \in \mathcal{L}(\mathbb{R}^{n_d}, X)$. We assume that the operator $A - \kappa BB^*$ generates a strongly stable contraction semigroup for any $\kappa > 0$. The motivation to consider such class of systems is that control problems of flexible structures with collocated actuators and sensors are often modelled as abstract linear systems with the above mentioned properties [18, 73, 99]. We consider a real-valued, uniformly Lipschitz continuous saturation function ϕ which has values $u(t)$ in the interval $[-1, 1]$ [51, Ch. 2]. We assume that the reference and disturbance signals (4.1) are generated by the exosystem (3.10) on a finite-dimensional space $W = \mathbb{R}^q$ with $S \in \mathbb{R}^{q\times q}$, $F \in \mathbb{R}^{1\times q}$ and $E \in \mathbb{R}^{n_d \times q}$ where n_d is the dimension of the disturbance input space. Moreover, we assume that $\sigma(S) \subset i\mathbb{R}$. The following main result provides solvability criteria for the semi-global output regulation problem introduced in Section 3.5.

Theorem 4.3.1. *Consider the systems* (4.4)*,* (3.10) *and the given compact set* $\mathcal{W}_0 \subset \mathbb{R}^q$*. Under the above assumptions, the semi-global output regulation problem is solvable if there exist* $\Pi \in \mathcal{L}(\mathbb{R}^q, X)$ *with* $\mathcal{R}(\Pi) \subset D(A)$ *and* $\Gamma \in \mathbb{R}^{1\times q}$ *such that they solve the regulator equations*

$$\begin{aligned}\Pi S &= A\Pi + B\Gamma + B_d E \\ 0 &= B^*\Pi + F\end{aligned} \tag{4.5}$$

and there exists a $\delta > 0$ *such that* $\sup_{t\geq 0} \|\Gamma v(t)\| \leq 1 - \delta$ *for all* $v(t) = e^{St} v_0$

with $v_0 \in \mathcal{W}_0$*. In this case, for any* $\kappa > 0$ *the feedback law*

$$u(t) = -\kappa y(t) + (\kappa B^*\Pi + \Gamma)v(t) \tag{4.6}$$

solves the semi-global output regulation problem.

As a new result, we solve the semi-global output regulation of the flexible satellite model subject to input saturation using the above result in Publication IV.

In comparison, the control law in (4.6) is a generalization and a simplified version of the one in [82, Thm. 3.3.3]. In [82, Thm. 3.3.3], solvability conditions and a low-gain-high-gain state feedback control design for the semi-global output regulation of finite-dimensional linear systems subject to input saturation have been presented. The low-gain parameter is to stabilize the system and high-gain parameter is to increase the control performance. In our work, since the considered class of systems are strongly stabilizable using negative output feedback, it is not necessary to find a control law for stabilization separately. Consequently, we have only one gain parameter in the control law (4.6) that corresponds to negative output feedback.

The existing results [62, 63, 64, 73] on the output regulation of infinite-dimensional linear systems subject to input saturation consider output tracking of constant reference signals using integral control input. The results [62, 63, 64] consider exponentially stable, single-input single-output exponentially stable linear systems where the transfer function $G(s)$ of the system satisfies $G(0) > 0$ and [73] consider strongly stable systems with strictly positive real transfer function $G(s)$ with certain Lipschitz nonlinearities. In our work, the considered class of systems is different from the existing ones and as the main novelty, we allow reference and disturbance signals to be linear combination of sinusoidal signals.

5 CONCLUSION AND FUTURE PERSPECTIVE

In this thesis, we have considered robust output regulation problem for the models consisting of Euler-Bernoulli beams. We solved the control problem for the cantilever beam and the flexible satellite model. We proposed practically implementable finite-dimensional controller for the robust output tracking of given sinusoidal reference signals for the cantilever beam. For the satellite model, we proposed three robust controllers, a low-gain controller, a passive controller and an observer based controller, for the robust output regulation. We noted that the finite-dimensional passive controller was able to achieve a comparable performance to the infinite-dimensional observer based controller. Stability of the models were analyzed in addition to controller construction for the output regulation. In particular, we proved the exponential stability of the satellite model.

We generalized output regulation theory for finite-dimensional systems subject to input saturation to the class of strongly stabilizable linear dissipative systems with collocated actuators and sensors. A linear output feedback law was proposed for the semi-global output regulation. Additionally, we solved the semi-global output regulation problem for the flexible satellite model subject to input saturation.

The advantages of control designs proposed in this work are that they are robust and they utilize measured output or regulation error and therefore no need to find information of the states. The proposed controllers, except the observer based controller, do not require any information from the system apart from certain properties such as passivity, well-posedness. On the other hand, the limitations of the proposed control designs include they are designed only for linear specified classes of systems.

Future Research Topics Related to Control of Beam Models

In this work, we considered flexible beam models with collocated inputs and outputs. It will be an interesting research topic to solve output regulation of beam models when the inputs and the outputs are non-collocated. In the satellite model, the beams are assumed to have viscous damping due to which we proved exponential stability of the model. Controllers designed for an undamped model work efficiently than those designed for damped models. So one could consider output regulation of undamped satellite model. In this case, finding a stabilizing controller can be challenging. Moreover, the controllers constructed for the satellite model are robust with respect to physical parameter uncertainties as long as the solar panels are identical. However, physical parameter uncertainties in the different panels can be different which we have not considered in this work. Taking this type of parameter uncertainties in to account in the robust output regulation of the satellite model can be interesting because these types of model uncertainties can affect stability of the system. Furthermore, considering distributed controls on the beams of the satellite model can also be an interesting problem. In addition, in this work we assumed that the external disturbances are known. However, in reality all the external disturbances are not known. Finding robust controllers that also reject the unknown disturbances will increase the implementation of the controllers in practice.

Future Research Topics Related to Systems with Input Saturation

Output regulation theory of finite-dimensional linear systems subject to input saturation has been generalized only for a particular class of infinite-dimensional systems with bounded control and observation. There are many possible research directions. Generalizing output regulation theory of finite-dimensional linear systems subject to input saturation to infinite-dimensional systems with unbounded control and observation operators will cover wider classes of systems. One can aim to construct controllers that are robust which we have not considered in this work. Solving an output regulation problem which includes robustness properties will be an interesting and challenging research topic.

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PUBLICATIONS

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A Finite-Dimensional Controller for Robust Output Tracking of an Euler-Bernoulli Beam

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A Finite-Dimensional Controller for Robust Output Tracking of an Euler–Bernoulli Beam*

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***Abstract*— In this paper, we consider robust output tracking problem of an undamped Euler-Bernoulli beam with boundary control and boundary observation. In particular, we study a cantilever beam which has control and observation at the free end. As our main result, we construct a finite-dimensional, internal model based controller for the output tracking of the beam system. In addition, we consider a case where the controller achieves the robust output tracking for the cantilever beam with distributed control and observation. Numerical simulations demonstrating the effectiveness of the controller are presented.**

I. Introduction

In this paper, we consider output tracking of an Euler Bernoulli beam with conservative clamped boundary conditions at one end and control at the other end. The beam system we study is given by

$$\begin{aligned} \rho(\xi)w_{tt}(\xi,t) + (EI(\xi)w_{\xi\xi})_{\xi\xi}(\xi,t) = 0,\ 0<\xi<1, t>0,\\ w(0,t)=0,\ w_\xi(0,t)=0,\\ (EI(\xi)w_{\xi\xi})(1,t)=0,\\ -(EI(\xi)w_{\xi\xi})_\xi(1,t)=u(t),\\ y(t)=w_t(1,t),\\ w(\xi,0)=w_0(\xi),\ w_t(\xi,0)=w_1(\xi),\ 0<\xi<1, \end{aligned} \tag{I.1}$$

where $w(\xi,t)$ is the transverse displacement of the beam at position ξ and time t, $w_t(\xi,t)$ and $w_\xi(\xi,t)$ denote time and spatial derivatives of $w(\xi,t)$, respectively, $\rho(\xi)$ and $EI(\xi)$ are linear density and flexural rigidity of the beam, respectively, $u(t)$ is an external boundary input and $y(t)$ is a boundary observation. The parameters $\rho(\xi)$ and $EI(\xi)$ satisfy the conditions

$$\rho(\cdot),\ EI(\cdot)\in C^4([0,1]),\quad \rho(\xi),\ EI(\xi)>0\quad \forall\,\xi\in[0,1]. \tag{I.2}$$

Our goal is to design a controller in such a way that the output $y(t)$ tracks a given reference signal $y_{ref}(t)$ asymptotically despite uncertainties and perturbations in the system. In other words, the objective is to find a controller that produces the input $u(t)$ such that

$$\int_t^{t+1} \|y(s)-y_{ref}(s)\|ds \to 0 \quad \text{as} \quad t\to\infty.$$

The reference signal to be considered is of the form

$$y_{ref}(t) = \sum_{k=1}^{q} a_k\cos(\omega_k t) + b_k\sin(\omega_k t) \tag{I.3}$$

where $(\omega_k)_{k=1}^q$ are known frequencies and $(a_k)_{k=1}^q$ and $(b_k)_{k=1}^q$ are possibly unknown constant coefficients.

This so-called *Robust Output Regulation Problem* has been studied widely in the literature for distributed parameter systems ([1], [2], [3], [4], [5], [6]), for regular and well-posed linear systems ([7], [8], [9]) and for boundary control systems ([10], [11]). The main key in the construction of robust regulating controllers is the *Internal model principle* which states that a controller can solve the robust output regulation problem if the dynamics of the controller contains copies of the frequencies from the reference signal. The internal model principle was introduced by Francis and Wonham in [12], [13] for finite-dimensional systems and since then it has been developed for infinite-dimensional systems by many authors, see for example, [5], [8], [11].

Robust output tracking of Euler-Bernoulli beam models has been studied recently in [14], [15] using infinite-dimensional controllers. In this paper, we solve the output tracking problem for the considered beam system (I.1) using a finite-dimensional dynamic error feedback controller.

As the main contribution, we construct a finite-dimensional, internal model based controller which achieves output tracking of given combination of sinusoidal signals as in (I.3). We formulate the beam system as an impedance passive well-posed linear system ([16], [17], [18]) and show that it can be stabilized exponentially using negative output feedback. The controller construction is based on the results for abstract well-posed linear systems [7]. As the main novelty compared to the recent articles [14] and [15] on output regulation of Euler-Bernoulli beam models, we consider spatially varying parameters in the beam system and solve the output tracking problem using a finite-dimensional controller.

As the second contribution, we consider a case where the cantilever beam (I.1) has distributed control and observation instead of boundary control and observation. We formulate the beam system as an impedance passive abstract linear system which can be stabilized strongly using negative output feedback. We show that the same finite-dimensional

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controller structure achieves robust output tracking of the given sinusoidal reference signals.

The paper is organized as follows. In Section II, we formulate the robust output regulation problem for the beam system. In Section III, we construct the controller for the robust output tracking of the reference signals. In addition, we present results related to stabilizability and well-posedness of the beam system. In Section IV, we consider the robust output tracking problem for the beam system with distributed control and observation. Section V is devoted to numerical simulations which demonstrate the performance of the controller for the robust output tracking of the beam system (I.1). In Section VI, we conclude our results.

A. Notation

For normed linear spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X to Y. For a linear operator A, $D(A), \mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the domain, range and the kernel of A, respectively. The resolvent and the spectrum of A are denoted by $\rho(A)$ and $\sigma(A)$, respectively. The resolvent operator is denoted by $R(\lambda, A) = (\lambda - A)^{-1}, \lambda \in \rho(A)$. We denote by X_{-1} the completion of X with respect to the norm $\|x\|_{-1} = \|(\beta I - A)^{-1}x\|,\ x \in X, \beta \in \rho(A)$ and by $A_{-1} \in \mathcal{L}(X, X_{-1})$ the extension of A to X_{-1}. For any $a \in \mathbb{R}$, $\mathbb{C}_a = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > a\}$.

II. Problem Formulation

In this section, we formulate the robust output regulation problem for the considered beam system (I.1). The dynamic error feedback controller to be constructed is of the form

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t),\ z(0) = z_0, \\ u(t) &= Kz(t) - k_1 e(t), \end{aligned} \tag{II.1}$$

where $z \in Z,\ Z = \mathbb{R}^{2q}$, $\mathcal{G}_1 \in \mathbb{R}^{2q\times 2q}$, $\mathcal{G}_2 \in \mathbb{R}^{2q\times 1}$, $K \in \mathbb{R}^{1\times 2q}$, $k_1 > 0$ and $e(t) = y(t) - y_{ref}(t)$ is the regulation error. Here q is the number of frequencies in the reference signal.

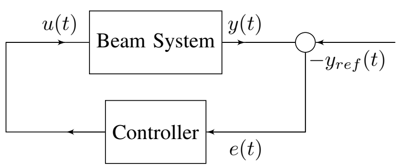

Fig. 1. The closed-loop system interconnecting the beam system and the controller

Robust Output Regulation Problem. Choose the controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K, k_1)$ in such a way that

(a) The closed-loop system in Figure 1 is exponentially stable in the sense that the closed-loop semigroup decays to zero exponentially.
(b) There exists $\alpha > 0$ such that for all reference signals of the form (I.3) and for all initial conditions $w_0(\xi), w_1(\xi)$ of the beam system and $z_0 \in Z$, the regulation error satisfies $e^{\alpha\cdot}e(\cdot) \in L^2([0,\infty), \mathbb{C})$.
(c) If (a) holds despite uncertainties, perturbations and disturbances in the system, then (b) is still satisfied for all initial conditions and some $\tilde{\alpha} > 0$.

III. Robust Output Regulation of the Cantilever Beam

In this section, we construct the controller for the robust output tracking of the sinusoidal reference signal y_{ref}. We start with presenting the controller. Based on [7], we choose the controller parameters as

$$\begin{aligned} \mathcal{G}_1 &= \operatorname{diag}(G_1, G_2, \cdots, G_q), \\ G_k &= \begin{bmatrix} 0 & \omega_k \\ -\omega_k & 0 \end{bmatrix}, \quad k = 1, 2, \cdots, q, \\ \mathcal{G}_2 &= -[1, 0, \cdots, 1, 0]^T, \\ K &= [2, 0, \cdots, 2, 0], \\ k_1 &> \frac{1}{2}. \end{aligned} \tag{III.1}$$

We note that the above choice of controller parameters does not depend on the coefficients a_k and b_k, $k = 1, 2, \cdots, q$ in the reference signal (I.3), a_k and b_k can possibly be unknown. The controller with the above choices of parameters solves the robust output regulation problem if the beam system is impedance passive, exponentially stabilizable using negative output feedback and well-posed linear system [16, Def. 1.1]. Therefore, in order to solve the output tracking problem, we need to verify the stabilizability of the beam system and formulate the beam system (I.1) as an impedance passive abstract well-posed linear system.

In the following, we present the abstract representation and stabilizability of the beam system followed by well-posedness results for the beam system. Afterward, we show that the controller presented in (II.1) and (III.1) solves the robust output tracking problem. Here we emphasize that the construction of the controller does not require the beam system as an abstract well-posed linear system. We will verify the above properties to prove that the controller in (II.1) and (III.1) solves the robust output regulation problem for the system (I.1).

A. Abstract Formulation of the Beam System

We formulate (I.1) in the state space $X = H_E^2(0,1) \times L^2(0,1)$ where $H_E^2(0,1) = \{f \in H^2(0,1) \mid f(0) = f'(0) = 0\}$. The norm on X is defined as

$$\|(f,g)^T\|_X^2 = \int_0^1 [\rho(\xi)|g(\xi)|^2 + EI(\xi)|f''(\xi)|^2]d\xi, \quad \forall (f,g)^T \in X.$$

The total energy of the beam system is given by

$$E(t) = \frac{1}{2}\int_0^1 [\rho(\xi)w_t^2(\xi,t) + EI(\xi)w_{\xi\xi}^2(\xi,t)]d\xi. \tag{III.2}$$

We define

$$x(t) = \begin{bmatrix} x_1(\cdot,t) \\ x_2(\cdot,t) \end{bmatrix} = \begin{bmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{bmatrix}.$$

Now (I.1) on X has the form

$$\begin{aligned}\frac{d}{dt}x(t) &= \mathcal{A}x(t), \quad x(0) = x_0,\\ \mathcal{B}x(t) &= u(t),\\ y(t) &= \mathcal{C}x(t),\end{aligned} \tag{III.3}$$

where $\mathcal{A} : D(\mathcal{A}) \subset X \to X$,

$$\mathcal{A}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2\\ \frac{-1}{\rho(\xi)}(EI(\xi)x_1''(\xi))''\end{bmatrix},$$

$$D(\mathcal{A}) = \{(x_1, x_2)^T \in [H^4(0,1) \cap H_E^2(0,1)] \times H_E^2(0,1) \mid x_1''(1) = 0\},$$

the operators $\mathcal{B} : D(\mathcal{A}) \to U$ and $\mathcal{C} : D(\mathcal{A}) \to Y$ with $U = \mathbb{C}$ and $Y = \mathbb{C}$ are given by

$$\mathcal{B}\begin{bmatrix}x_1\\x_2\end{bmatrix} = -(EI(\xi)x_1''(\xi))'(1,t),$$

$$\mathcal{C}\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2(1,t).$$

Let us introduce the operator $A = \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ with

$$D(A) = \{(f,g)^T \in [H^4(0,1) \cap H_E^2(0,1)] \times H_E^2(0,1) \mid f''(1) = f'''(1) = 0\}.$$

We have that A is a skew-adjoint operator with compact resolvent [19, Sec. 3]. This implies that A generates a unitary group on X. Moreover, we have that $\mathcal{N}(\mathcal{B}) = D(A)$. Therefore, $\mathcal{N}(\mathcal{B})$ is dense in X. Thus $(\mathcal{A},\mathcal{B},\mathcal{C})$ is a boundary control system in the sense of [20, Def. 10.1.1]. Next, we show that the boundary control system $(\mathcal{A},\mathcal{B},\mathcal{C})$ is impedance passive which is defined as follows.

Definition III.1. (*Impedance Passive System*). A boundary control system $(\mathcal{A},\mathcal{B},\mathcal{C})$ is an impedance passive system on (X,U,Y) if $U = Y$ and

$$\operatorname{Re}\langle \mathcal{A}x, x\rangle_X \le \operatorname{Re}\langle \mathcal{B}x, \mathcal{C}x\rangle_U, \quad x \in D(\mathcal{A}).$$

Lemma III.2. *The boundary control system $(\mathcal{A},\mathcal{B},\mathcal{C})$ in (III.3) is an impedance passive system.*

Proof. We have that for $x \in D(\mathcal{A})$,

$$\begin{aligned}\operatorname{Re}\langle \mathcal{A}x, x\rangle_X &= \operatorname{Re}\left\langle \begin{bmatrix}x_2\\ \frac{-1}{\rho(\xi)}(EI(\xi)x_1''(\xi))''\end{bmatrix}, \begin{bmatrix}x_1\\x_2\end{bmatrix}\right\rangle_X,\\ &= \operatorname{Re}\int_0^1 \rho(\xi)\frac{-1}{\rho(\xi)}(EI(\xi)x_1''(\xi))''\overline{x_2(\xi)}d\xi\\ &\quad + \operatorname{Re}\int_0^1 EI(\xi)\overline{x_1''(\xi)}x_2''(\xi)d\xi.\end{aligned}$$

Using integration by parts twice for the first term and applying boundary conditions, we obtain

$$\begin{aligned}&\operatorname{Re}\langle \mathcal{A}x, x\rangle_X\\ &= \operatorname{Re}\Big[-\overline{x_2}(1)(EI(\xi)x_1'')'(1) + \overline{x_2}(0)(EI(\xi)x_1'')'(0)\\ &\qquad + \overline{x_2}'(1)(EI(\xi)x_1'')(1) - \overline{x_2}'(0)(EI(\xi)x_1'')(0)\\ &\qquad - \int_0^1 EI(\xi)x_1''(\xi)\overline{x_2''(\xi)}d\xi + \int_0^1 EI(\xi)x_2''(\xi)\overline{x_1''(\xi)}d\xi\Big]\\ &= \operatorname{Re}[-\overline{x_2(1)}(EI(\xi)x_1'')'(1)]\\ &= \operatorname{Re}\mathcal{B}x\overline{\mathcal{C}x}\\ &= \operatorname{Re}\langle \mathcal{B}x, \mathcal{C}x\rangle_{\mathbb{C}}\end{aligned}$$

which implies that $(\mathcal{A},\mathcal{B},\mathcal{C})$ in (III.3) is impedance passive. □

B. Stabilization of the Beam

In [19, Thm. 2.5] it is shown that the beam (I.1) with output feedback $u(t) = -\kappa w_t(1,t),\ \kappa > 0$ is exponentially stable in the sense that the energy $E(t)$ of the solutions decays to zero exponentially. Here we note that $E(t) = \frac{1}{2}\|x(t)\|_X^2$. Therefore we have the following lemma.

Lemma III.3 ([19, Thm. 2.5]). *The beam (I.1) with new input $u(t) = \tilde{u}(t) - \kappa y(t),\ \kappa > 0$ is exponentially stable in the sense that for the semigroup $T(t)$ generated by $A_{cl} = \mathcal{A}|_{\mathcal{N}(\mathcal{B}+\kappa\mathcal{C})}$, there exist $\omega > 0$ and $M \ge 1$ such that*

$$\|T(t)\| \le Me^{-\omega t}, \quad t \ge 0.$$

C. Well-posedness of the Beam system

In this section, we present results related to the well-posedness ([18, Def. 3.1]) of the beam system.

Lemma III.4 ([19, Lem. 3.4]). *The eigenvalues $\{i\lambda_n, \overline{i\lambda_n}\}$ and the corresponding eigenfunctions $((i\lambda_n)^{-1}\phi_n, \phi_n)$ of A have the following asymptotic expressions*

$$\begin{aligned}i\lambda_n &= \frac{\mu_n^2}{h^2}, \quad h = \int_0^1 \left(\frac{\rho(s)}{EI(s)}\right)^{\frac{1}{4}}ds,\\ \mu_n &= \frac{1}{\sqrt{2}}(n+\frac{1}{2})\pi(1+i) + \mathcal{O}(\frac{1}{n}),\end{aligned} \tag{III.4}$$

as $n \to \infty$, n is a large positive integer and

$$\begin{aligned}\phi_n(\xi) &= e^{-\frac{1}{4}\int_0^z a(s)ds}\sqrt{2}(i-1)[\sin((n+\frac{\pi}{2})z)\\ &\quad - \cos((n+\frac{\pi}{2})z) + e^{-(n+\frac{1}{2})\pi z}\\ &\quad + (-1)^n e^{-(n+\frac{1}{2})\pi(1-z)} + \mathcal{O}(\frac{1}{n})]\end{aligned} \tag{III.5}$$

where

$$z = z(\xi) = \frac{1}{h}\int_0^\xi \left(\frac{\rho(s)}{EI(s)}\right)^{\frac{1}{4}}ds$$

$$\begin{aligned}a(z) &= \frac{3h}{2}\left(\frac{\rho(\xi)}{EI(\xi)}\right)^{-\frac{5}{4}}\frac{d}{d\xi}\left(\frac{\rho(\xi)}{EI(\xi)}\right)\\ &\quad + h\frac{2\frac{d}{d\xi}EI(\xi)}{EI(\xi)}\left(\frac{\rho(\xi)}{EI(\xi)}\right)^{-\frac{1}{4}}.\end{aligned}$$

Next, we show that the boundary control system $(\mathcal{A},\mathcal{B},\mathcal{C})$ in (III.3) defines a well-posed system node on (X,U,Y), where system node is defined in the sense of [17, Def. 2.1] or [21, Def. 2.1] and well-posed system node is defined in the sense of [17, Def. 2.6], [18].

Theorem III.5. *The boundary control system $(\mathcal{A},\mathcal{B},\mathcal{C})$ in (III.3) defines a well-posed system node on (X,U,Y).*

Proof. We have shown that the system $(\mathcal{A},\mathcal{B},\mathcal{C})$ is an impedance passive boundary control system. In addition, since A generates a unitary group, the boundary control system $(\mathcal{A},\mathcal{B},\mathcal{C})$ is internally well-posed in the sense of [21, Def. 1.1]. Therefore, by [21, Thm. 2.3], $(\mathcal{A},\mathcal{B},\mathcal{C})$ defines a system node $S=\begin{bmatrix}A\&B\\C\&D\end{bmatrix}:D(S)\subset X\times\mathbb{C}\to X\times\mathbb{C}$ and the system node is impedance passive [17, Thm. 4.2]. The system node S is defined by

$$\begin{bmatrix}A\&B\\C\&D\end{bmatrix}=\begin{bmatrix}A_{-1} & B\\ \mathcal{C} & 0\end{bmatrix}\Bigg|_{D(S)},$$

$$D(S)=\left\{\begin{bmatrix}x\\u\end{bmatrix}\in\begin{bmatrix}X\\U\end{bmatrix}\;\middle|\;A_{-1}x+Bu\in X\right\}$$

where $B\in\mathcal{L}(U,X_{-1})$ is uniquely determined by the relation $\mathcal{A}=A_{-1}+B\mathcal{B}$ on $D(\mathcal{A})$ [20, Prop. 10.1.2]. Next, we show that the transfer function of the system node S is bounded on some vertical line in the complex right half plane.

Using [20, Rem. 10.1.6], we obtain $B^*x=x_2(1)=Cx,\ x=(x_1,x_2)^T\in D(A^*)$, where $C=\mathcal{C}|_{\mathcal{N}(\mathcal{B})}$. The operator $B^*\in\mathcal{L}(D(A^*),U)$ is the adjoint of $B\in\mathcal{L}(U,X_{-1})$ in the sense that

$$\langle x,Bu\rangle_{D(A^*),X_{-1}}=\langle B^*x,u\rangle_{\mathbb{C}},\ x\in D(A^*),u\in U.$$

Therefore, (III.3) can be equivalently written as a second order system

$$\begin{aligned}w_{tt}(\cdot,t)+A_0w(\cdot,t)&=B_0u(t)\\ y(t)&=B_0^*w_t(\cdot,t)\end{aligned}\qquad\text{(III.6)}$$

where $A_0f=\frac{1}{\rho(\xi)}(EI(\xi)f'')''$ is a positive self-adjoint operator with $D(A_0)=\{f\in H^4(0,1)\cap H_E^2(0,1)\mid f''(1)=(EIf'')'(1)=0\}$ and $B_0=\delta(\cdot-1)$, $\delta(\cdot)$ is the Dirac delta distribution. Then λ_n^2 and ϕ_n from Lemma III.4 are the eigenvalues and the corresponding eigenfunctions of A_0.

From the expression (III.4), we have that $(\lambda_n)_{n\geq 1}$ are increasing. In addition, $|B_0^*\phi_n|=|\phi_n(1)|$ which from (III.5) is bounded for $n\geq 1$. This implies that B^* is admissible [20, Sec. 5.3], [22, Prop. 2]. By duality ([20, Sec. 4.4]), we have that B is admissible. Moreover, using [22, Rem. 4], we have that the eigenvalues of A_0 satisfies the spectral condition

$$\lambda_{n+1}-\lambda_n\geq\beta\lambda_{n+1}^{\gamma},\quad\forall\ n\text{ large},$$

for some $\beta,\gamma>0$. Therefore, using [22, Thm. 4], we conclude that the transfer function $s\mapsto G(s)=sB_0^*(s^2+A_0)^{-1}B_0\in\mathcal{L}(U)$ of (III.6) is bounded on some vertical line in the complex right half plane. Since

$$A=\begin{bmatrix}0 & I\\ -A_0 & 0\end{bmatrix},\ B=\begin{bmatrix}0\\B_0\end{bmatrix}\text{ and } C=\begin{bmatrix}0 & B_0^*\end{bmatrix},$$

we have that the transfer function G_S of the system node S which is given by [17, Def. 2.1], [18, Sec. 6]

$$\begin{aligned}G_S(s)u&=C\&D\begin{bmatrix}R(s,A_{-1})Bu\\u\end{bmatrix}=\mathcal{C}R(s,A_{-1})Bu\\&=sB_0^*(s^2+A_0)^{-1}B_0u\\&=G(s)u\end{aligned}$$

is bounded on $\mathbb{C}_0$. Therefore, by [17, Thm. 5.1], we conclude that the system node S is well-posed. ☐

Remark III.6. Since B is an admissible control operator, using [23, Thm. 2.7], we can deduce that

$$\lim_{s\to+\infty}G(s)=0,\quad s\in\mathbb{R}.$$

Since the above limit exists, we have that the beam system is a regular linear system [24].

D. Robust Regulating Controller for the Beam System

In this section, we show that the controller (II.1), (III.1) presented in Section II solves the robust output tracking problem.

We note that the transfer function $G(s)$ in Section III-C can also be written in terms of the solution of the elliptic problem corresponding to I.1 ([25, Sec. 12.1], [26])

$$\begin{aligned}\frac{1}{\rho(\xi)}(EI(\xi)\hat{w}_{\xi\xi})_{\xi\xi}&=-s^2\hat{w},\quad\xi\in[0,1],\\(EI(\xi)\hat{w}_{\xi\xi})_{\xi}(1)&=\hat{u},\\G(s)\hat{u}&=\hat{y}=s\hat{w}(1),\end{aligned}$$

for $(\hat{w},s\hat{w})\in D(\mathcal{A})$, $\hat{u},\hat{y}\in\mathbb{C}$ and $s\in\rho(A)$.

Theorem III.7. *Let $\omega_j\in\mathbb{R}$, $j=1,2,\cdots,q$ be the frequencies from the reference signal. Assume that $\operatorname{Re}G(i\omega_j)\hat{u}=\operatorname{Re}i\omega_j\hat{w}(1)\neq 0$ for all j. Then the controller (II.1), (III.1) solves the robust output regulation problem for (I.1).*

Proof. We consider the input $u(t)=Kz(t)-k_1e(t)$. Let us write $k_1=C_0+\kappa$, where $C_0\geq\frac{1}{2}$ and $\kappa>0$. Then we have $u(t)=Kz(t)-C_0e(t)-\kappa y(t)+\kappa y_{ref}(t)=u_1(t)-\kappa y(t)+\kappa y_{ref}(t)$ where $u_1(t)=Kz(t)-C_0e(t)$.

With this input, (III.3) can be written as

$$\begin{aligned}\frac{d}{dt}x(t)&=\mathcal{A}x(t),\quad x(0)=x_0,\\(\mathcal{B}+\kappa\mathcal{C})x(t)&=u_1(t)+\kappa y_{ref}(t),\\\mathcal{C}x(t)&=y(t).\end{aligned}\qquad\text{(III.7)}$$

From Lemma III.3, we have that the system $(\mathcal{A},\mathcal{B}+\kappa\mathcal{C},\mathcal{C})$ is exponentially stable and from Theorem III.5, we have that $(\mathcal{A},\mathcal{B},\mathcal{C})$ is a well-posed linear system since every well-posed system node defines a well-posed linear system ([18]). Moreover, due to Remark III.6, we have that κ is an admissible output feedback operator. This implies that the system $(\mathcal{A},\mathcal{B}+\kappa\mathcal{C},\mathcal{C})$ is a well-posed linear system

[24, Thm. 4.7]. Therefore, by considering $\kappa y_{ref}(t)$ as an external disturbance to the system (I.1), then we have that (III.7) is an exponentially stable well-posed linear system with input $u_1(t)$. In addition, the impedance passivity of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ implies that the transfer function $G(s)$ is positive, i.e., $\operatorname{Re} G(s) = \frac{1}{2}[G(s) + G(s^*)] \geq 0, \ \forall \ s \in \mathbb{C}_0$ ([17], [18]). This further implies that the transfer function $G_\kappa(s)$ of the system $(\mathcal{A}, \mathcal{B} + \kappa\mathcal{C}, \mathcal{C})$ is positive and the assumption $\operatorname{Re} G(i\omega_j) \neq 0, \ j = 1, 2, \cdots, q$ implies that $\operatorname{Re} G_\kappa(i\omega_j) \neq 0$ for all $j = 1, 2, \cdots, q$. Therefore, using [7, Thm. 3.4], a minimal realization of

$$C(s) = -C_0 - \sum_{j \in \mathcal{J}} \frac{1}{s - i\omega_j}, \tag{III.8}$$

where $C_0 \geq \frac{1}{2}$, $\mathcal{J} = \{-q, \cdots, -1, 1, \cdots, q\}$ and $\omega_{-j} = -\omega_j$, solves the robust output tracking problem and rejects the disturbance $\kappa y_{ref}(t)$.

It can be verified from (III.1) that $(\mathcal{G}_1, \mathcal{G}_2)$ is controllable, $(\mathcal{G}_1, K)$ is observable and the transfer function of $(\mathcal{G}_1, \mathcal{G}_2, K, -C_0)$ is given by (III.8). Therefore, the controller given in (II.1) and (III.1) is a minimal realization of (III.8). Combining the above arguments and using [7, Thm. 3.4], we have that the controller (II.1), (III.1) solves the robust tracking problem for (I.1). □

IV. A Robust Regulating Controller for an Euler-Bernoulli Beam with Distributed Control and Observation

In this section, we consider robust output tracking of a cantilever beam which has distributed control and observation. The beam system that we study is described by

$$\begin{aligned}
\rho(\xi) w_{tt}(\xi, t) &= -(EI(\xi) w_{\xi\xi})_{\xi\xi}(\xi, t) + b(\xi) u_2(t) \\
w(0, t) &= 0, \ w_\xi(0, t) = 0, \\
(EI(\xi) w_{\xi\xi})(1, t) &= 0, \ -(EI(\xi) w_{\xi\xi})_\xi(1, t) = 0, \\
w(\xi, 0) &= w_0(\xi), \ w_t(\xi, 0) = w_1(\xi), \\
y_2(t) &= \int_0^1 b(\xi) w_t(\xi, t) d\xi
\end{aligned} \tag{IV.1}$$

where $0 < \xi < 1$, $t > 0$, $u_2(t)$ and $y_2(t)$ are the external control input and observation respectively and $b(\cdot) \in L^2(0, 1)$ is a real-valued function. The parameters $\rho(\xi)$ and $EI(\xi)$ satisfy (I.2). The beam system (IV.1) cannot be stabilized exponentially [27, Cor. 3.58], [28, Sec. 8.4].

Assumption IV.1. Under negative output feedback $u_2(t) = -\kappa y_2(t)$, $\kappa > 0$, the solutions of the beam system (IV.1) satisfy

$$\|w(\cdot, t)\|_{L^2} + \|w_t(\cdot, t)\|_{L^2} \quad \to 0 \quad \text{as} \quad t \to \infty \tag{IV.2}$$

for any initial conditions.

Assumption IV.1 implies that the system (IV.1) can be stabilized strongly by negative output feedback.

Robust Output Regulation Problem (Strongly Stable Version). Choose $(\mathcal{G}_1, \mathcal{G}_2, K, k_1)$ in (II.1) such that

(a) The closed-loop system comprising the controller and the beam system (IV.1) is strongly stable.

(b) The regulation error $\tilde{e}(t) = y_2(t) - y_{ref}(t)$ satisfies

$$\int_t^{t+1} \|\tilde{e}(s)\| ds \to 0 \quad \text{as} \quad t \to \infty$$

for all initial conditions $w_0(\xi), w_1(\xi)$ and $z_0 \in Z$.

(c) If (a) holds despite uncertainties, perturbations and disturbances in the system, then (b) is still satisfied for all initial conditions.

Theorem IV.2. *Under the Assumption IV.1, the controller* (II.1) *and* (III.1) *solves the robust output regulation problem (Strongly Stable Version) for the beam system (IV.1).*

Proof. The system (IV.1) can be formulated as an abstract linear system

$$\begin{aligned}
\frac{d}{dt} x(t) &= A x(t) + \tilde{B} u_2(t), \quad x(0) = x_0, \\
y_2(t) &= \tilde{C} x(t)
\end{aligned}$$

in the state space $X = H_E^2(0, 1) \times L^2(0, 1)$ with state variable $x(t) = (w(\cdot, t), w_t(\cdot, t))^T$. The norm on X is defined as in Section III-A. The operator A corresponds to the skew-adjoint operator in Section III-A and the operators $\tilde{B} \in \mathcal{L}(\mathbb{C}, X)$ and $\tilde{C} \in \mathcal{L}(X, \mathbb{C})$ are given by

$$\begin{aligned}
\tilde{B} u_2 &= \begin{bmatrix} 0 \\ \tilde{B}_0 \end{bmatrix} u_2, \quad \tilde{B}_0 = \frac{b(\cdot)}{\rho(\cdot)}, \ u_2 \in \mathbb{C}, \\
\tilde{C} x &= \int_0^1 b(\xi) x_2(\xi) d\xi, \quad (x_1, x_2)^T \in X.
\end{aligned}$$

Here $\tilde{B}^* = \tilde{C}$.

By direct computation, we obtain

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|_X^2 = \operatorname{Re} \langle u_2(t), y_2(t) \rangle_{\mathbb{C}}.$$

This implies that the system $(A, \tilde{B}, \tilde{B}^*, 0)$ is an impedance passive system.

Now we have that the system (IV.1) is passive and assumed to be strongly stabilizable by negative output feedback. Therefore, by [9, Thm. 5.2], we conclude that the controller (II.1) and (III.1) solves the robust output tracking problem. □

V. Numerical Simulations

Simulations are carried out in Matlab for the beam system (I.1) with the following choices of parameters on the time interval [0,15]. We consider the case where $\rho(\xi) = 1$, $EI(\xi) = 1$. We aim to track the reference signal $y_{ref}(t) = \sin 2t + \cos t$. So, the frequencies are $\{2, 1\}$. We choose the beam initial state $w_0(\xi) = 0.1(\sin(\pi\xi) - \pi\xi)$, $w_1(\xi) = (1 + \frac{\pi^3}{60})\xi^2$ and the controller initial state $z_0 = 0$. The beam system is approximated using Legendre spectral Galerkin method [29]. The number of basis functions used for the approximation is 20. The controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ are chosen as in (III.1) with $k_1 = 6$. Figure 2 shows that the tracking of the given reference signal is achieved asymptotically. Velocity profile of the controlled beam is shown in Figure 3.

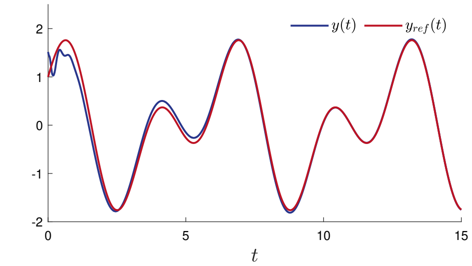

Fig. 2. Output tracking

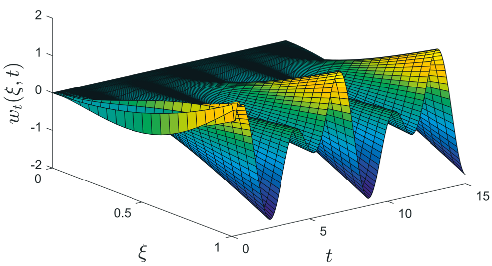

Fig. 3. Velocity profile of the controlled beam

VI. Conclusions

In this paper, we studied the robust output tracking of a cantilever beam. As the main problem, we considered the cantilever beam which has control and observation at the free end. In addition, we considered the case where the beam has distributed control and observation. We solved the output regulation problem using a finite-dimensional, internal model based controller. The advantage of using this controller is that the controller is simple and able to handle the spatially varying parameters in the beam system. Numerical simulations demonstrating the effectiveness of the controller were presented.

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II

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Robust Output Regulation of a Flexible Satellite ⋆

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Abstract: We consider a PDE-ODE model of a satellite and robust output regulation of the corresponding model. The satellite is composed of two flexible solar panels and a rigid center body. Exponential stability of the model is proved using passivity and resolvent estimates in the port-Hamiltonian framework. In addition, we construct a simple low-gain controller for robust output regulation of the satellite model.

Keywords: Port-Hamiltonian system, stability analysis, output regulation, distributed parameter systems.

1. INTRODUCTION

Flexible structures are widely used in the modern technology because of their advantages such as light weight, cost effectiveness and low energy consumption. Flexibility of these structures leads to problems of structural vibration and shape deformation, hence control problems of flexible systems have become a very interesting topic in research. Moreover, flexible structures are distributed parameter systems and they are often modeled as partial differential equations. Applications of flexible structures can be found, e.g., in robotics, satellites and wind turbines.

For the past few decades, satellite models have attracted many researchers in science and engineering as they are increasingly used, for instance, in communication systems, remote sensors, navigation and earth sciences. There are a number of satellites that are modeled as two flexible solar panels connected to a center rigid body. However, the flexibility of the panels affects the model dynamics such as shape deformation, which leads to challenges in controlling these type of systems. Control problems for satellite models can be found, for example, in Bontsema (1989), Aoues, Cardoso-Rebeiro, Matignon and Alazard (2018), Souza (2015) and Wei and Shuzhi Sam (2015). Robust output regulation of a coupled PDE-ODE system is considered, e.g., in Zhao and Weiss (2018). However, output regulation of satellite models has not been considered in the literature to our knowledge.

The goal of robust output regulation is designing a controller in such a way that the output of the controlled system converges to a given reference signal asymptotically despite perturbations, disturbances and uncertainties in the system.The main key in the construction of a robust regulating controller is the internal model principle which provides complete knowledge of the controllers and the ability to solve the robust output regulation problem. The investigation of robust output regulation theory was started in the 1970's for finite dimensional systems by Davison (1976), Francis and Wonham (1976), and Francis and Wonham (1975), and since then it has been developed for infinite dimensions by many authors, see for example, Paunonen and Pohjolainen (2014) and the references therein.

Many physical systems can be modeled as port-Hamiltonian systems(PHSs) (see Jacob, Zwart (2011)). The class of port-Hamiltonian systems includes a wide range of models including flexible structures, traveling waves in acoustics, heat exchangers, suspension systems and bio reactors. Moreover, several interconnected PHSs via standard feedback interconnection is again a PHS. Stability analysis of port-Hamiltonian systems is considered in Augner (2019), Augner and Jacob (2013) and Augner (2018). Robust output regulation problem of boundary controlled port-Hamiltonian systems can be found, e.g., in Humaloja and Paunonen (2018).

In this paper, we consider a satellite system that is composed of two symmetric flexible solar panels and a center rigid body. The panels are modeled as Euler-Bernoulli beams. In addition, it is assumed that the beams have distributed viscous damping. Both panels are modeled in the port-Hamiltonian framework and the passivity of the system is proved by computing the energy balance equation.

As the main contribution of the paper, a power-preserving interconnection is shown between the satellite panels and the center rigid body. This interconnection results in an impedance passive port-Hamiltonian system. We stabilize the rigid body by negative output feedback and we utilize the passivity property in proving that the satellite system generates an exponentially stable semigroup. Due to the exponential stability of the model, using the theories from

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Pohjolainen (1985) and Paunonen (2016), we construct a simple low-gain controller that solves the robust output regulation problem.

The paper is organized as follows. In section 2, we formulate our satellite model as an abstract PDE-ODE system and establish a power-preserving interconnection between the satellite panels and the center rigid body in the port-Hamiltonian framework. In section 3, we prove the exponential stability of the satellite system. In section 4, we consider robust output regulation of the satellite model and we construct a low-gain controller that achieves robust output regulation of the satellite model. In section 5, we conclude our work and present topics for future research.

1.1 Notation

For normed linear spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y. For a linear operator A, $D(A), \mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the domain, range and the kernel of A, respectively. The resolvent and the spectrum of A are denoted by $\rho(A)$ and $\sigma(A)$, respectively. The resolvent operator is denoted by $R(\lambda, A) = (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. We denote by X_{-1} the completion of X with respect to the norm $\|x\|_{-1} = \|((\beta I - A)^{-1}x)\|$, $x \in X, \beta \in \rho(A)$ and by $A_{-1} \in \mathcal{L}(X, X_{-1})$ the extension of A to X_{-1}. For $x(t,\xi) \in X$, $\dot{x}$ and x' denote time and spatial derivatives of x, respectively.

2. THE SATELLITE MODEL

We consider a dynamic model of a satellite composed of a center rigid body and two symmetric flexible solar panels. The panels are modeled as Euler-Bernoulli beams. Let us assume that both beams are of length 1 with cross sectional area a, mass density ρ, Young's modulus of elasticity E, second moment of area of the cross section I and the viscous damping coefficient γ.

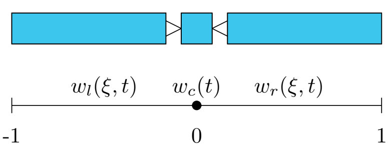

Fig. 1. Satellite with flexible solar panels

Let m and I_m denote the mass and the mass moment of inertia of the center rigid body. If $w_l(\xi,t)$ and $w_r(\xi,t)$ are the transverse displacements of the left and the right beam, respectively, and $w_c(t)$ and $\theta_c(t)$ are the linear and angular displacements of the rigid body respectively, then the governing equations of motion of the satellite are given by (similar models can be found in Bontsema (1989), Wei and Shuzhi Sam (2015)),

$$\ddot{w}_l(\xi,t) + \frac{EI}{\rho a} w_l''''(\xi,t) + \frac{\gamma}{\rho a}\dot{w}_l(\xi,t) = 0, \ -1 < \xi < 0, t > 0,$$

$$\ddot{w}_r(\xi,t) + \frac{EI}{\rho a} w_r''''(\xi,t) + \frac{\gamma}{\rho a}\dot{w}_r(\xi,t) = 0, \ 0 < \xi < 1, t > 0,$$

with the boundary conditions,

$$m\ddot{w}_c(t) = EIw_l'''(0,t) - EIw_r'''(0,t) + u_1(t),$$
$$I_m\ddot{\theta}_c(t) = -EIw_l''(0,t) + EIw_r''(0,t) + u_2(t),$$
$$w_l''(-1,t) = 0, \quad w_r''(1,t) = 0,$$
$$w_l'''(-1,t) = 0, \quad w_r'''(1,t) = 0,$$
$$\dot{w}_l(0,t) = \dot{w}_r(0,t) = \dot{w}_c(t),$$
$$\dot{w}_l'(0,t) = \dot{w}_r'(0,t) = \dot{\theta}_c(t),$$
$$y_1(t) = \dot{w}_c(t), \ y_2(t) = \dot{\theta}_c(t).$$

where $u_1(t)$ and $u_2(t)$ are external control inputs and $y_1(t)$ and $y_2(t)$ are outputs of the satellite model. Here $\dot{w}_c(t) = \dot{w}_l(\xi,t)|_{\xi=0} = \dot{w}_r(\xi,t)|_{\xi=0}$ and $\dot{\theta}_c(t) = \dot{w}_l'(\xi,t)|_{\xi=0} = \dot{w}_r'(\xi,t)|_{\xi=0}$ are the linear and the angular velocities of the rigid body respectively. We formulate this system as an abstract system of a PDE and an ODE in the port-Hamiltonian framework similarly as in Augner (2019).

2.1 Abstract Formulation of the Beams

The standard boundary control and boundary observation problem for port-Hamiltonian systems of order $N = 2$ on the spatial interval $[a,b]$ takes the form,

$$\dot{x}(t,\xi) = P_2(\mathcal{H}x)''(t,\xi) + P_1(\mathcal{H}x)'(t,\xi) + P_0(\mathcal{H}x)(t,\xi),$$
$$u(t) = \mathcal{B}x(t,\xi),$$
$$y(t) = \mathcal{C}x(t,\xi),$$

where, $P_0, P_1, P_2 \in \mathbb{R}^{n\times n}$, and $\mathcal{H} : [a,b] \to \mathbb{R}^{n\times n}$ is the Hamiltonian density matrix function.

Now, we formulate the beam systems in the satellite model as boundary controlled port-Hamiltonian systems of order $N = 2$.

The left beam in the satellite system can be modeled as a boundary controlled port-Hamiltonian system of order $N = 2$ on the energy space $X_l = L^2([-1,0];\mathbb{R}^2)$. The space X_l is a Hilbert space equipped with the energy norm $\|x_l(t)\|^2_{X_l} := \frac{1}{2}\langle x_l(t), \mathcal{H}_l x_l(t)\rangle_{L^2}$, $x_l \in X_l$, where $\mathcal{H}_l$ given in (2) is the Hamiltonian density matrix function associated with the left beam.

The left beam that we detach from the satellite system has $u_{l1}(t) = \dot{w}_l(0,t)$, $u_{l2}(t) = \dot{w}_l'(0,t)$ as boundary inputs and $y_{l1}(t) = -EIw_l'''(0,t)$, $y_{l2}(t) = EIw_l''(0,t)$ as outputs. Then choosing the energy state variable $x_l(t) = \begin{bmatrix}\rho a \dot{w}_l(\xi,t) \\ w_l''(\xi,t)\end{bmatrix}$, we have

$$\frac{d}{dt}x_l(t) = \mathcal{A}_l x_l(t), \ u_l(t) = \mathcal{B}_l x_l(t), \ y_l(t) = \mathcal{C}_l x_l(t), \quad (1)$$

where,

$$\mathcal{A}_l = \begin{bmatrix} -\gamma(\rho a)^{-1} & -EI\partial_{\xi\xi} \\ (\rho a)^{-1}\partial_{\xi\xi} & 0 \end{bmatrix},$$

$$\mathcal{B}_l x_l(t) = \begin{bmatrix}\dot{w}_l(0,t) \\ \dot{w}_l'(0,t)\end{bmatrix} \text{ and,}$$

$$\mathcal{C}_l x_l(t) = \begin{bmatrix} -EIw_l'''(0,t) \\ EIw_l''(0,t)\end{bmatrix}.$$

Here

$$P_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}, \ P_1 = 0, \ P_0 = \begin{bmatrix} -\gamma & 0 \\ 0 & 0\end{bmatrix},$$

$$\mathcal{H}_l = \begin{bmatrix} (\rho a)^{-1} & 0 \\ 0 & EI\end{bmatrix} \quad (2)$$

and

$$D(\mathcal{A}_l) = \{x_l \in X_l \mid \mathcal{H}_l x_l \in H^2([-1,0];\mathbb{R}^2), \ x_{l2}(-1) = x_{l2}'(-1) = 0\}.$$

The Hamiltonian i.e., energy for the left beam is given by,

$$H_l = \frac{1}{2}\|x_l\|^2_{X_l} = \frac{1}{2}\int_{-1}^{0} (\rho a|\dot{w}_l(t,\xi)|^2 + EI|w_l''(t,\xi)|^2)d\xi.$$

Differentiating,

$$\begin{aligned}\dot{H}_l &= \int_{-1}^{0} (\rho a\dot{w}_l(t,\xi)\ddot{w}_l(t,\xi) + EIw_l''(t,\xi)\dot{w}_l''(t,\xi))d\xi,\\ &= \int_{-1}^{0} \frac{\partial}{\partial\xi}(EIw_l''(t,\xi)\dot{w}_l'(t,\xi) - \dot{w}_l(t,\xi)EIw_l'''(t,\xi))d\xi\\ &\quad - \gamma\int_{-1}^{0} \dot{w}_l(t,\xi)^2 d\xi,\\ &\le EIw_l''(t,0)\dot{w}_l'(t,0) - \dot{w}_l(t,0)EIw_l'''(t,0),\\ &= u_l(t)^T y_l(t).\end{aligned}$$

This implies that the energy satisfies

$$\frac{1}{2}\frac{d}{dt}\|x_l(t)\|^2_{X_l} \le u_l(t)^T y_l(t).$$

Hence, the left beam is an impedance passive system on the Hilbert space $X_l = L^2([-1,0];\mathbb{R}^2)$, and thus, the operator $A_l = \mathcal{A}_l|_{\mathcal{N}(\mathcal{B}_l)}$ generates a contraction semigroup $T_l(t)$ on X_l. That is, $\|T_l(t)\| \le 1$ on X_l.

In the same way, the right beam that we detach from the satellite system can be modeled as a boundary controlled port-Hamiltonian system on the Hilbert space $X_r = L^2([0,1];\mathbb{R}^2)$ with $u_{r1}(t) = \dot{w}_r(0,t)$, $u_{r2}(t) = \dot{w}_r'(0,t)$ as boundary inputs and $y_{r1}(t) = EIw_r'''(0,t)$, $y_{r2}(t) = -EIw_r''(0,t)$ as outputs. Choosing the energy state variable $x_r(t) = \begin{bmatrix}\rho a\dot{w}_r(\xi,t)\\ w_r''(\xi,t)\end{bmatrix}$, we have

$$\frac{d}{dt}x_r(t) = \mathcal{A}_r x_r(t),\ u_r(t) = \mathcal{B}_r x_r(t),\ y_r(t) = \mathcal{C}_r x_r(t), \tag{3}$$

where,

$$\mathcal{A}_r = \begin{bmatrix}-\gamma(\rho a)^{-1} & -EI\partial_{\xi\xi}\\ (\rho a)^{-1}\partial_{\xi\xi} & 0\end{bmatrix},$$

$$\mathcal{B}_r x_r(t) = \begin{bmatrix}\dot{w}_r(0,t)\\ \dot{w}_r'(0,t)\end{bmatrix} \text{ and,}$$

$$\mathcal{C}_r x_r(t) = \begin{bmatrix}EIw_r'''(0,t)\\ -EIw_r''(0,t)\end{bmatrix}.$$

Here P_0, P_1, P_2 and $\mathcal{H}_r$ are defined the same as of the left beam and

$$D(\mathcal{A}_r) = \{x_r \in X_r \mid \mathcal{H}_r x_r \in H^2([0,1];\mathbb{R}^2),\ x_{r2}(1) = x_{r2}'(1) = 0\}.$$

Furthermore, it can be shown analogously to the case of the left beam that the energy satisfies

$$\frac{1}{2}\frac{d}{dt}\|x_r(t)\|^2_{X_r} \le u_r(t)^T y_r(t),$$

which shows that the right beam is also an impedance passive system on the Hilbert space $X_r = L^2([0,1];\mathbb{R}^2)$,thus, the operator $A_r = \mathcal{A}_r|_{\mathcal{N}(\mathcal{B}_r)}$ generates a contraction semigroup $T_r(t)$ on X_r.

2.2 Combined Beam System

The two beam systems (1) and (3) can be combined into a single open loop system as follows:

$$\frac{d}{dt}x(t) = \mathcal{A}x(t), \quad \hat{\mathcal{B}}x(t) = \hat{u}(t), \quad \hat{\mathcal{C}}x(t) = \hat{y}(t),$$

where

$$x(t) = \begin{bmatrix}x_l(t)\\ x_r(t)\end{bmatrix}, \quad \hat{u}(t) = \begin{bmatrix}u_l(t)\\ u_r(t)\end{bmatrix}, \quad \hat{y}(t) = \begin{bmatrix}y_l(t)\\ y_r(t)\end{bmatrix},$$

$$\mathcal{A} = \begin{bmatrix}\mathcal{A}_l & 0\\ 0 & \mathcal{A}_r\end{bmatrix}, \quad \hat{\mathcal{B}} = \begin{bmatrix}\mathcal{B}_l & 0\\ 0 & \mathcal{B}_r\end{bmatrix}, \quad \hat{\mathcal{C}} = \begin{bmatrix}\mathcal{C}_l & 0\\ 0 & \mathcal{C}_r\end{bmatrix},$$

and $D(\mathcal{A}) = D(\mathcal{A}_l) \times D(\mathcal{A}_r)$.

Using the boundary conditions $u_{l1}(t) = \dot{w}_l(0,t) = \dot{w}_r(0,t) = u_{r1}(t)$ and $u_{l2}(t) = \dot{w}_l'(0,t) = \dot{w}_r'(0,t) = u_{r2}(t)$, the energy of the combined system is given by,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 &= \frac{1}{2}\frac{d}{dt}\|x_l(t)\|^2_{X_l} + \frac{1}{2}\frac{d}{dt}\|x_r(t)\|^2_{X_r}\\ &\le u_l(t)^T y_l(t) + u_r(t)^T y_r(t),\\ &= u_l(t)^T (y_l(t) + y_r(t)).\end{aligned} \tag{4}$$

Let us define a new output function

$$\begin{aligned}y(t) = y_l(t) + y_r(t) &= \mathcal{C}_l x_l(t) + \mathcal{C}_r x_r(t)\\ &= (\mathcal{C}_l\ \mathcal{C}_r)\begin{pmatrix}x_l(t)\\ x_r(t)\end{pmatrix}\end{aligned}$$

and an input function

$$u(t) = \left(\frac{1}{2}\mathcal{B}_l\ \ \frac{1}{2}\mathcal{B}_r\right)\begin{pmatrix}x_l(t)\\ x_r(t)\end{pmatrix}.$$

With this input $u(t)$ and output $y(t)$, it follows from (4) that the system

$$\frac{d}{dt}x(t) = \mathcal{A}x(t), \quad \mathcal{B}x(t) = u(t), \quad \mathcal{C}x(t) = y(t) \tag{5}$$

is an impedance passive port-Hamiltonian system on $X = X_l \times X_r$ and $A = \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ generates a contraction semigroup $T(t)$ on X.

2.3 Abstract Formulation of the Rigid Body

The center rigid body that we detach from the satellite system has $u_{c1}(t) = EIw_l'''(0,t) - EIw_r'''(0,t)$ and $u_{c2}(t) = -EIw_l''(0,t) + EIw_r''(0,t)$ as inputs and $y_{c1}(t) = \dot{w}_c(t)$ and $y_{c2}(t) = \dot{\theta}_c(t)$ as outputs. Then, with the state variable $x_c(t) = \begin{bmatrix}m\dot{w}_c(t)\\ I_m\dot{\theta}_c(t)\end{bmatrix}$, the rigid body on the Hilbert space $X_c = \mathbb{R}^2$ can be written as,

$$\frac{d}{dt}\begin{bmatrix}m\dot{w}_c(t)\\ I_m\dot{\theta}_c(t)\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}m\dot{w}_c(t)\\ I_m\dot{\theta}_c(t)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_{c1}(t)\\ u_{c2}(t)\end{bmatrix},$$

$$y_c(t) = \begin{bmatrix}\dot{w}_c(t)\\ \dot{\theta}_c(t)\end{bmatrix}.$$

Equivalently,

$$\begin{aligned}\frac{d}{dt}x_c(t) &= A_c x_c(t) + B_c u_c(t),\\ y_c(t) &= C_c x_c(t),\end{aligned} \tag{6}$$

where,

$$A_c = 0,\ B_c = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ C_c = \begin{bmatrix}\frac{1}{m} & 0\\ 0 & \frac{1}{I_m}\end{bmatrix}, \text{ and}$$

$$u_c(t) = \begin{bmatrix}u_{c1}(t)\\ u_{c2}(t)\end{bmatrix}.$$

The Hamiltonian of the center rigid body is given by,

$$H_c = \frac{1}{2}m\dot{w}_c(t)^2 + \frac{1}{2}I_m\dot{\theta}_c(t)^2 = \frac{1}{2}\|x_c\|^2_{X_c}$$

Differentiating,

$$\dot{H}_c = \dot{w}_c(t)u_{c1}(t) + \dot{\theta}_c(t)u_{c2}(t) = u_c(t)^T y_c(t).$$

Equivalently,

$$\frac{1}{2}\frac{d}{dt}\|x_c(t)\|^2_{X_c} = u_c(t)^T y_c(t).$$

Hence, the rigid body is an impedance passive system on X_c.

2.4 The Satellite System as a Coupled PDE-ODE System

From the previous sections, we are able to write our satellite system as an abstract PDE-ODE system with the power-preserving interconnection $u(t) = y_c(t)$, $u_c(t) = -y(t)$ as follows:

$$\begin{aligned} \frac{d}{dt}x(t) &= \mathcal{A}x(t), \\ \frac{d}{dt}x_c(t) &= B_c u_c(t) + B_c u_{sat}(t), \\ \mathcal{B}x(t) &= C_c x_c(t), \\ u_c(t) &= -\mathcal{C}x(t), \end{aligned} \tag{7}$$

or equivalently,

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ x_c(t)\end{bmatrix} = \begin{bmatrix} \mathcal{A} & 0 \\ -B_c\mathcal{C} & 0\end{bmatrix}\begin{bmatrix} x(t) \\ x_c(t)\end{bmatrix} + \begin{bmatrix} 0 \\ B_c\end{bmatrix} u_{sat}(t),$$

$$[\mathcal{B}\ -C_c]\begin{bmatrix} x(t) \\ x_c(t)\end{bmatrix} = 0.$$

where $u_{sat}(t) = \begin{bmatrix} u_1(t) \\ u_2(t)\end{bmatrix}$.

The operator $\tilde{A}_{sat} := \begin{bmatrix} \mathcal{A} & 0 \\ -B_c\mathcal{C} & 0\end{bmatrix}$ with $D(\tilde{A}_{sat}) = \{(x, x_c) \in D(\mathcal{A}) \times X_c \ :\ \mathcal{B}x = C_c x_c\}$ is dissipative, since

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\| \begin{bmatrix} x(t) \\ x_c(t)\end{bmatrix} \right\|^2 &= \frac{1}{2}\frac{d}{dt}\|x(t)\|^2_X + \frac{1}{2}\frac{d}{dt}\|x_c(t)\|^2_{X_c} \\ &\le u(t)^T y(t) + u_c(t)^T y_c(t), \\ &= y_c(t)^T y(t) - y(t)^T y_c(t), \\ &= 0. \end{aligned}$$

and thus, according to Augner (2019)(see, example 3.4), $\tilde{A}_{sat}$ generates C_0-semigroup of contractions on the Hilbert space $X_{sat} = X \times X_c$.

3. STABILITY OF THE SATELLITE MODEL

An important step in constructing a robust regulating controller is to analyze the stability of the system. In this section, we analyze the stability of the satellite system (7).

3.1 Stabilization of the Finite Dimensional System

Since the eigenvalues of the rigid body are zeros, it is not asymptotically stable. We stabilize the rigid body by negative output feedback, hence the new input is given by $\tilde{u}_c(t) = u_c(t) - y_c(t)$. Now, from (6), we have,

$$\begin{aligned} \frac{d}{dt}x_c(t) &= B_c\tilde{u}_c(t), \\ &= B_c u_c(t) - B_c y_c(t), \\ &= B_c u_c(t) - B_c C_c x_c(t), \\ &= -B_c C_c x_c(t) + B_c u_c(t), \\ &= \tilde{A}_c x_c(t) + B_c u_c(t), \end{aligned}$$

where $\tilde{A}_c = -B_c C_c$. The stabilized rigid body is an impedance passive system. Hence the whole satellite system (7) can be written as,

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ x_c(t)\end{bmatrix} = \begin{bmatrix} \mathcal{A} & 0 \\ -B_c\mathcal{C} & -B_cC_c\end{bmatrix}\begin{bmatrix} x(t) \\ x_c(t)\end{bmatrix} + \begin{bmatrix} 0 \\ B_c\end{bmatrix} u_{sat}(t),$$

$$[\mathcal{B}\ -C_c]\begin{bmatrix} x(t) \\ x_c(t)\end{bmatrix} = 0. \tag{8}$$

where $A_{sat} := \begin{bmatrix} \mathcal{A} & 0 \\ -B_c\mathcal{C} & -B_cC_c\end{bmatrix}$ with $D(A_{sat}) = \{(x, x_c) \in D(\mathcal{A}) \times X_c \ :\ \mathcal{B}x = C_c x_c\}$ generates a contraction semigroup $T_{sat}(t)$ on X_{sat}.

3.2 Stability of the Beam System

Lemma 1. The left beam system is exponentially stable.

Proof. Let $x_l(t) \in D(\mathcal{A}_l)$ be the classical solution of the left beam. If $\mathcal{A}_0 = \begin{bmatrix} 0 & -EI\partial_{\xi\xi} \\ (\rho a)^{-1}\partial_{\xi\xi} & 0\end{bmatrix}$ and $C_0 = \begin{bmatrix}(\gamma(\rho a)^{-1})^{\frac{1}{2}} & 0\end{bmatrix}$, then $\mathcal{A}_l = \mathcal{A}_0 - C_0C_0^*$. Here $A_0 = \mathcal{A}_0|_{\mathcal{N}(\mathcal{B}_l)}$ generates a unitary group on X_l. It can be shown that (A_0, C_0) is exactly observable(see Ch.6, Tucsnak and Weiss (2009) for more details on exact observability).

Using the skew-adjoint property of the operator $\mathcal{A}_0$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|x_l(t)\|^2 &= \left\langle \frac{d}{dt}x_l(t), x_l(t)\right\rangle, \\ &= \langle \mathcal{A}_l x_l(t), x_l(t)\rangle, \\ &= \left\langle \begin{bmatrix} 0 & -EI\partial_{\xi\xi} \\ (\rho a)^{-1}\partial_{\xi\xi} & 0\end{bmatrix} x_l(t), x_l(t)\right\rangle \\ &\quad - \gamma(\rho a)^{-1}\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix} x_l(t), x_l(t)\right\rangle, \\ &= -\gamma\int_{-1}^{0} \dot{w}_l^2(t,\xi)d\xi. \end{aligned}$$

Now,

$$\begin{aligned} &\|x_l(T)\|^2 - \|x_l(0)\|^2 \\ &= \int_0^T \frac{d}{dt}\|x_l(t)\|^2 dt, \\ &= -2\gamma\int_0^T\int_{-1}^0 \dot{w}_l^2(t,\xi)d\xi dt, \\ &\le -2\gamma C_1\int_{-1}^0 (\rho a\dot{w}_l^2(0,\xi) + (w_l''(0,\xi))^2)d\xi, \end{aligned}$$

for some $0 < C_1 < 1$ where we used the exact observability of the pair (A_0, C_0). This yields,

$$\begin{aligned} \|x_l(T)\|^2 - \|x_l(0)\|^2 &\le -C_2\|x_l(0)\|^2, \quad 0 < C_2 < 1, \\ \|x_l(T)\|^2 &\le (1 - C_2)\|x_l(0)\|^2, \\ \|x_l(T)\| &\le C\|x_l(0)\|, \quad 0 < C < 1, \\ \Leftrightarrow \quad \|T_l(T)x_l(0)\| &\le C\|x_l(0)\|. \end{aligned}$$

That is, $\|T_l(T)\| < 1$ for some $T > 0$. We obtain, [Engel and Nagel (2000), Prop.V.1.7]

$$\|T_l(t)\| \le Me^{-\omega t}, \quad M \ge 1, \quad \omega > 0,$$

by which the left beam system (1) is exponentially stable.

Corollary 2. The beam system (5) is exponentially stable.

Proof. By symmetry, it follows from lemma 1 that the right beam system (3) is exponentially stable. Hence the semigroup $T(t)$ generated by $A = \mathcal{A}|_{\mathcal{N}(\mathcal{B})}$ is exponentially stable.

3.3 Stability of the satellite system

In this section, we sketch a proof for exponential stability of the satellite model. A detailed proof will be presented in a later paper.

Theorem 3. The satellite system (8) is exponentially stable.

Proof. By Gearheart-Greiner-Prüss theorem, the semigroup $T_{sat}(t)$ generated by A_{sat} is exponentially stable on a Hilbert space if and only if the spectrum of A_{sat} lies in the complex left half-plane and $\sup_{\omega\in\mathbb{R}}\|(i\omega - A_{sat})^{-1}\| < \infty$ (see, Engel and Nagel (2000), Thm.V.1.11). Since A_{sat} generates a contraction semigroup, the spectrum $\sigma(A_{sat})$ lies in the closed complex left-half plane. It remains to prove that the resolvent $R(i\omega, A_{sat})$ of the system exists and is uniformly bounded on the imaginary axis.

According to Tucsnak and Weiss (2009)(Prop.10.1.2), there exists a unique $B \in \mathcal{L}(U, X_{-1})$ such that the equations (8) of the satellite system can be written as,

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} = \begin{bmatrix} A_{-1} & BC_c \\ -B_c\mathcal{C} & -B_cC_c \end{bmatrix}\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_{sat}(t),$$

and the resolvent of the satellite system is given by,

$$R(i\omega, A_{sat}) = \begin{bmatrix} (i\omega - A_{-1}) & -BC_c \\ B_c\mathcal{C} & (i\omega + B_cC_c) \end{bmatrix}^{-1}.$$

Let $P(i\omega)$ and $P_c(i\omega)$ be the transfer functions of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $(\tilde{A}_c, B_c, C_c)$ respectively. Then the passivity of the systems implies that $\mathrm{Re}P(i\omega) \geq 0$ and $\mathrm{Re}P_c(i\omega) \geq 0$ for all $i\omega \in \rho(A)$ and $i\omega \in \rho(\tilde{A}_c)$. Also, it can be shown that $(I + P(i\omega)P_c(i\omega))$ and $(I + P_c(i\omega)P(i\omega))$ are boundedly invertible for all $\omega \in \mathbb{R}$. For more details on passive systems, see Paunonen (2017)(Appendix).

Using the Schur complement $S(i\omega) = [(i\omega + B_cC_c) + B_c\mathcal{C}(i\omega - A_{-1})^{-1}BC_c]^{-1}$, we obtain,

$$R(i\omega, A_{sat}) = \begin{bmatrix} R_{11}(i\omega, A_{sat}) & R_{12}(i\omega, A_{sat}) \\ R_{21}(i\omega, A_{sat}) & R_{22}(i\omega, A_{sat}) \end{bmatrix},$$

where,

$$\begin{aligned} R_{11}(i\omega, A_{sat}) &= R(i\omega, A) \\ &\quad - R(i\omega, A_{-1})BC_cS(i\omega)B_c\mathcal{C}R(i\omega, A), \\ R_{12}(i\omega, A_{sat}) &= R(i\omega, A_{-1})BC_cS(i\omega), \\ R_{21}(i\omega, A_{sat}) &= -S(i\omega)B_c\mathcal{C}R(i\omega, A), \\ R_{22}(i\omega, A_{sat}) &= S(i\omega). \end{aligned}$$

Using Kato perturbation formula, we have

$$\begin{aligned} &S(i\omega) \\ &= [(i\omega + B_cC_c) + B_c\mathcal{C}(i\omega - A_{-1})^{-1}BC_c]^{-1}, \\ &= R(i\omega, \tilde{A}_c) \\ &\quad - R(i\omega, \tilde{A}_c)B_cP(i\omega)(I + P(i\omega)P_c(i\omega))^{-1}C_cR(i\omega, \tilde{A}_c). \end{aligned}$$

From the stability of the beam system we have that $\|R(i\omega, A)\|$ is uniformly bounded and from the stability of the rigid body we have that $\|R(i\omega, \tilde{A}_c)\|$, $\|C_cR(i\omega, \tilde{A}_c)\|$, $\|R(i\omega, \tilde{A}_c)B_c\|$ and $\|P_c(i\omega)\|$ are all uniformly bounded and tend to zero as $|\omega| \to \infty$. Furthermore, $\|P_c(i\omega)\|$ tends to zero sufficiently fast such that $P(i\omega)P_c(i\omega)$ and $(I + P(i\omega)P_c(i\omega))^{-1}$ are uniformly bounded. This implies that the Schur complement $S(i\omega)$ is uniformly bounded. Moreover, $\|S(i\omega)\|$ tends to zero sufficiently fast as $|\omega| \to \infty$ such that $R_{11}(i\omega, A_{sat})$, $R_{12}(i\omega, A_{sat})$, and $R_{21}(i\omega, A_{sat})$ are also uniformly bounded. Hence, the resolvent $R(i\omega, A_{sat})$ is uniformly bounded and therefore A_{sat} generates an exponentially stable semigroup.

4. ROBUST OUTPUT REGULATION OF THE SATELLITE MODEL

In this section, we present the satellite system and the controller that solves the robust output regulation problem for the system. Our goal is to design a controller in such a way that the linear and angular velocities of the center rigid body converge to given reference signals of the form (11).

From the previous sections, the satellite system with control and observations on the rigid body is given by,

$$\begin{aligned} &\frac{d}{dt}\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & 0 \\ -B_c\mathcal{C} & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_{sat}(t), \\ &[\mathcal{B}\ -C_c]\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} = 0, \\ &y_{sat}(t) = [0\ C_c]\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}. \end{aligned} \tag{9}$$

We construct a dynamic error feedback controller of the form

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \quad z(0) = z_0, \\ u_{sat}(t) &= Kz(t) - y_c(t), \end{aligned} \tag{10}$$

on a Banach space Z, where $e(t) = y_{sat}(t) - y_{ref}(t)$, is the regulation error, $y_{ref}(t)$, a given reference signal, $\mathcal{G}_1 \in \mathcal{L}(Z)$, $\mathcal{G}_2 \in \mathcal{L}(Y_c, Z)$ and $K \in \mathcal{L}(Z, U_c)$, such that robust output regulation of the satellite system is achieved with a suitable choice of the parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$. Here U_c and Y_c are the input and the output spaces of the satellite system. The term $-y_c(t)$ appears in the controller (10) because it is used to stabilize the rigid body of the satellite system, see section 3.1. The reference signals to be tracked are of the form,

$$y_{ref}(t) = a_0 + \sum_{k=1}^{q} [a_k \cos(\omega_k t) + b_k \sin(\omega_k t)], \tag{11}$$

with $0 = \omega_0 < \omega_1 < \cdots < \omega_q$ as the known frequencies and $\{a_k\}_{k=0}^q$, $\{b_k\}_{k=1}^q \subset Y_c$ as the unknown coefficients.

The Robust Output Regulation Problem. Choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ in such a way that

(a) The closed loop semigroup $T_{cl}(t)$ comprised of the satellite system (9) and the controller (10) is exponentially stable.
(b) For all initial states $x(0) \in D(\mathcal{A})$ and $x_c(0) \in X_c$ satisfying $\mathcal{B}x(0) = C_cx_c(0)$, the regulation error $e(t)$ satisfies $e^{\alpha t}\|y_{sat}(t) - y_{ref}(t)\| \to 0$ as $t \to \infty$, for some $\alpha > 0$.
(c) If the system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, A_c, B_c, C_c)$ is perturbed in such a way that the perturbed closed loop system is still exponentially stable, the perturbed $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a boundary controlled impedance passive port-Hamiltonian system and the perturbed (A_c, B_c, C_c)

is an impedance passive system, then (b) continues to hold for some $\tilde{\alpha} > 0$.

4.1 Controller for the Satellite Model

Since the system is exponentially stable, using the theories in Rebarber and Weiss (2003), Paunonen (2016) and Pohjolainen (1985), a simple low-gain controller can be constructed for obtaining robust output regulation of the model with the following choices of parameters. Defining $Z = Y_c^{2q+1}$, and $\omega_{-k} = -\omega_k, k = 1, 2, \cdots, q$,

$\mathcal{G}_1 = \text{diag}(i\omega_{-q} I_{Y_c}, \cdots i\omega_0 I_{Y_c}, \cdots, i\omega_q I_{Y_c})$,

$K = \epsilon(K_0^{-q}, \cdots, K_0^0, \cdots, K_0^q)$, where, $K_0^k = P_{sat}(i\omega_k)^\dagger$,

$\mathcal{G}_2 = (-(P_{sat}(i\omega_k)K_0^k)^*)_{k=-q}^q$.

Here $P_{sat}(i\omega_k) = C_c S(i\omega_k) B_c, S(i\omega_k)$ is the Schur complement, is the transfer function of the satellite system (9) which can be obtained by frequency response measurement from the system, $P_{sat}(i\omega_k)^\dagger$ is the Moore-Penrose pseudoinverse of $P_{sat}(i\omega_k)$ and the tuning parameter $\epsilon > 0$ is to be chosen sufficiently small such that the closed loop system is exponentially stable.

5. CONCLUSION

We considered a PDE-ODE model of a flexible satellite. The model was formulated as an abstract system in the port-Hamiltonian framework and it was shown that there is a power-preserving interconnection between the satellite panels and the center rigid body of the model. The exponential stability of the model was proved using passivity and the resolvent estimate where we used Schur complement and Kato perturbation formula. Exponential stability of the satellite model enabled us to construct a simple low-gain controller for robust output regulation of the model.

Future works are possible for the same model. Numerical simulations testing the effectiveness of the controller and technical details in the proofs of exponential stability will be presented in a later paper. Since the model is an exponentially stable impedance passive system, a passive controller can be constructed for this model. In this paper, the beams are assumed to have damping, one could also consider an undamped model.

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ROBUST CONTROLLERS FOR A FLEXIBLE SATELLITE MODEL

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Abstract. We consider a PDE-ODE model of a flexible satellite that is composed of two identical flexible solar panels and a center rigid body. We prove that the satellite model is exponentially stable in the sense that the energy of the solutions decays to zero exponentially. In addition, we construct two internal model based controllers, a passive controller and an observer based controller, such that the linear and angular velocities of the center rigid body converge to the given sinusoidal signals asymptotically. A numerical simulation is presented to compare the performances of the two controllers.

1. **Introduction.** In this paper, we consider output tracking and disturbance rejection problem for a flexible satellite that is composed of two identical flexible solar panels and a center rigid body (Figure 1). Modeling the satellite panels as viscously damped Euler-Bernoulli beams of length 1, the satellite system we study is given by (similar models can be found in [6], [13])

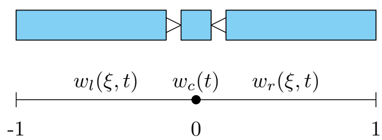

Figure 1. Satellite with flexible solar panels

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$$\begin{aligned}
\rho a\ddot{w}_l(\xi,t) + EIw_l''''(\xi,t) + \gamma\dot{w}_l(\xi,t) &= b_{d1}(\xi)w_{d1}(t),\ -1<\xi<0, t>0,\\
\rho a\ddot{w}_r(\xi,t) + EIw_r''''(\xi,t) + \gamma\dot{w}_r(\xi,t) &= b_{d2}(\xi)w_{d2}(t),\ 0<\xi<1, t>0,\\
m\ddot{w}_c(t) &= EIw_l'''(0,t) - EIw_r'''(0,t) + u_1(t) + w_{d3}(t),\\
I_m\ddot{\theta}_c(t) &= -EIw_l''(0,t) + EIw_r''(0,t) + u_2(t) + w_{d4}(t),\\
w_l''(-1,t) = 0, &\quad w_l'''(-1,t) = 0,\\
w_r''(1,t) = 0, &\quad w_r'''(1,t) = 0,\\
\dot{w}_l(0,t) &= \dot{w}_r(0,t) = \dot{w}_c(t),\\
\dot{w}_l'(0,t) &= \dot{w}_r'(0,t) = \dot{\theta}_c(t),
\end{aligned} \tag{1}$$

where $w_l(\xi,t)$ and $w_r(\xi,t)$ are the transverse displacements of the left and the right beam, respectively, $\dot{w}_l(\xi,t)$ and $w_l'(\xi,t)$ denote time and spatial derivatives of $w_l(\xi,t)$, respectively, $w_c(t)$ and $\theta_c(t)$ are the linear and angular displacements of the rigid body, respectively, $u_1(t)$ and $u_2(t)$ are external control inputs of the satellite model, $w_{d1}(t)$, $w_{d2}(t)$, $w_{d3}(t)$ and $w_{d4}(t)$ are external disturbances in the satellite model, $b_{d1}(\cdot)\in L^2(-1,0)$ and $b_{d2}(\cdot)\in L^2(0,1)$ are real-valued functions. Here $\dot{w}_c(t) = \dot{w}_l(\xi,t)|_{\xi=0} = \dot{w}_r(\xi,t)|_{\xi=0}$ and $\dot{\theta}_c(t) = \dot{w}_l'(\xi,t)|_{\xi=0} = \dot{w}_r'(\xi,t)|_{\xi=0}$ are linear and angular velocities of the rigid body, respectively. The parameters a, ρ, E, I and γ are cross sectional area, linear density, Young's modulus of elasticity, second moment of area of the cross section and the viscous damping coefficient of the beams, respectively, and m and I_m denote the mass and the mass moment of inertia of the center rigid body. Measurements that are the outputs of the model are taken on the center rigid body and are given by,

$$y_1(t) = \dot{w}_c(t),\ y_2(t) = \dot{\theta}_c(t). \tag{2}$$

The main control objective is to construct a dynamic error feedback controller such that the outputs, the linear and the angular velocities of the center rigid body, track given reference signals $y_{ref}(t)$ asymptotically. i.e.,

$$\|y(t) - y_{ref}(t)\| \to 0 \quad \text{as} \quad t\to\infty,$$

where $y(t) = (y_1(t), y_2(t))^T$ is the output of the satellite model. In addition, the proposed controller is required to be robust in the sense that it achieves output tracking despite perturbations, disturbances and uncertainties in the satellite system.

As the first main contribution of this paper, we present a detailed proof of uniform exponential stability of the satellite model in the sense that the energy of the solutions decay exponentially to zero. The stability proof is based on the results from C_0-semigroup theory. We write the satellite system as a coupled system of a PDE (two beams are combined into a single system) and an ODE (rigid body) via a power preserving interconnection. The main proof is divided naturally into two steps. In the first step, we show that the imaginary axis is included in the resolvent set of the satellite system operator. In the second step, we derive an explicit expression for the resolvent operator and show that it is uniformly bounded on the imaginary axis. The stability proof is challenging because the input and the output operators of the PDE are not admissible and its transfer function is not well-posed (in the sense that the input-output map of the PDE is unbounded).

As the second main contribution of this paper, we construct two robust controllers, a passive controller [20], [18] and an observer based controller [12], [17], for the robust output tracking of the satellite model. The proposed controller designs are based on the internal model principle [9], [10], [12], [17], [19]. Finally, simulation results testing the effectiveness of the controllers are presented.

There are several studies in the literature investigating control problems of satellite models. In [6], the stabilization problem of a flexible spacecraft has been investigated using frequency domain approach. In [13], dynamic modeling and vibration control of a flexible satellite has been considered and vibrations of the solar panels have been suppressed using the single-point control input on the center body. In [1], modeling and control of a rotating flexible spacecraft has been considered, a Proportional Derivative controller and a nonlinear controller have been presented to suppress elastic vibrations of the satellite model. References [13] and [1] use Lyapunov methods to prove the stability of the models. To the best of our knowledge, robust output tracking problem for flexible satellites has not been considered in the literature.

Stability of coupled PDE-ODE systems can often be obtained using controllability and observability results. In [24], controllability and observability results of a well-posed and strictly proper linear system coupled with a finite-dimensional linear system with an invertible first component in its feedthrough matrix were presented. In [25], using results from [24], strong stability of coupled impedance passive systems was shown and the results were applied to the SCOLE model to show that the SCOLE model coupled with tuned mass damper system is strongly stable. Moreover, the SCOLE model is not exactly controllable in the natural energy state space ([22, Sec. 1]) but it was shown in [22] that the SCOLE model is exactly controllable in a smoother state space. In [23], it was shown that a coupled system consisting of a well-posed and impedance passive linear system and an internal model based controller in a feedback connection is strongly stable. In our case, since the beam system in the satellite model is not well-posed on the natural energy state space and the rigid center body has no feedthrough term, the results of [24], [25] cannot be utilized in showing the exponential stability of the satellite system. Moreover, since our aim is to achieve exponential stability of the closed-loop system and one of the proposed controllers is infinite-dimensional, the results in [22], [24], [25] and [23] are not applicable in showing the exponential stability of the closed-loop system consisting of the satellite system and the controller. The results in the above mentioned references have unstable infinite-dimensional part and therefore only strong stability of the coupled system was obtained. In this work, since the beam system is exponentially stable due to the distributed damping, we are able to prove the exponential stability of the satellite system.

A preliminary version of these results has been presented in IFAC World Congress 2020 [11]. As the main novelty of this version with respect to [11], we present a detailed proof of the exponential stability of the satellite system. We present a passive controller and an observer based controller which also achieve the robust output tracking of the satellite model and reject external disturbances. In addition, simulation results showing the performances of the controllers are presented.

The paper is organized as follows. In Section 2, we present the abstract formulation of the satellite model. In Section 3, we present some technical lemmas and prove the exponential stability of the satellite model. In Section 4, we present the tracking problem, the reference signal to be tracked by the satellite model and the disturbance signals to be rejected. We present two internal model based controllers for the robust output tracking of the satellite model. In addition, simulation results are presented for particular choices of reference and disturbance signals. In Section 5, we conclude our results.

1.1. **Notation.** For normed linear spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from X to Y. For a linear operator A, $D(A), \mathcal{R}(A)$ and $\mathcal{N}(A)$ denote domain, range and kernel of A, respectively. The resolvent and the spectrum of A are denoted by $\rho(A)$ and $\sigma(A)$, respectively. The resolvent operator is denoted by $R(\lambda, A) = (\lambda - A)^{-1},\ \lambda \in \rho(A)$. We denote by X_{-1} the completion of X with respect to the norm $\|x\|_{-1} = \|(\beta I - A)^{-1}x\|,\ x \in X,\ \beta \in \rho(A)$ and by $A_{-1} \in \mathcal{L}(X, X_{-1})$, the extension of A to X_{-1}. For functions $f, g : I \subset \mathbb{R} \to \mathbb{R}_+$ and $f_k, g_k \geq 0$, we denote $f(x) \lesssim g(x)$ and $f_k \lesssim g_k$ if there exist $M_1, M_2 > 0$ such that $f(x) \leq M_1 g(x)$ and $f_k \leq M_2 g_k$ for all values of $x \in I$ and $k \in J \subset \mathbb{N}$.

2. **Abstract formulation of the satellite model.** In this section, we write our satellite model (1)-(2) in the state space form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_d w_d(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) \end{aligned} \tag{3}$$

where $x(t) \in X$ is the state variable and X is a Hilbert space, $u(t) \in U = \mathbb{R}^2$ is the control input, $w_d(t) \in U_d = \mathbb{R}^4$ is the external disturbance and $y(t) \in Y = \mathbb{R}^2$ is the output. The operator $A : D(A) \subset X \to X$ generates a strongly continuous semigroup on X and the operators $B \in \mathcal{L}(U, X)$, $B_d \in \mathcal{L}(U_d, X)$ and $C \in \mathcal{L}(X, Y)$ are bounded. The formulation (3) will be used in Section 4 in the construction of controllers for robust output regulation.

In order to write the satellite model (1)-(2) in the state space form, we decompose the satellite system into a PDE system (the two beams combined into a single system) coupled with an ODE system (center rigid body) where PDE interacts with ODE via boundary controls and boundary observations called "virtual boundary inputs" and "virtual boundary outputs", respectively. Figure 2 shows the boundary interconnections between the beams and the center rigid body. This type of decomposition has been considered, for example, in [22] for SCOLE model. As

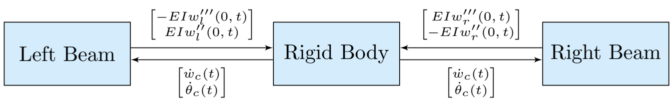

FIGURE 2. Coupling of the beams with the rigid body

the first step towards state space formulation, we write the PDE as an impedance passive abstract boundary control and observation system given by the following definitions.

Definition 2.1 (Boundary Control and Observation System [8, Def. 3.3.2], [14, Ch. 11])**.** Let $\hat{X}$, $\hat{U}$ and $\hat{Y}$ be Hilbert spaces. Consider the system

$$\dot{\hat{x}}(t) = \hat{\mathcal{A}}\hat{x}(t), \quad \hat{x}(0) = \hat{x}_0, \tag{4a}$$

$$\hat{\mathcal{B}}\hat{x}(t) = \hat{u}(t), \tag{4b}$$

$$\hat{y}(t) = \hat{\mathcal{C}}\hat{x}(t) \tag{4c}$$

where $\hat{\mathcal{A}} : D(\hat{\mathcal{A}}) \subset \hat{X} \to \hat{X}$, $\hat{\mathcal{B}} : D(\hat{\mathcal{B}}) \subset \hat{X} \to \hat{U}$ and $\hat{\mathcal{C}} : D(\hat{\mathcal{A}}) \to \hat{Y}$ are linear operators and $D(\hat{\mathcal{A}}) \subset D(\hat{\mathcal{B}})$. Then (4) is a boundary control and observation system if the following hold.

1. The operator $\hat{A} : D(\hat{A}) \to \hat{X}$ with $D(\hat{A}) = D(\hat{\mathcal{A}}) \cap \mathcal{N}(\hat{\mathcal{B}})$ and $\hat{A}\hat{x} = \hat{\mathcal{A}}\hat{x}$ for $\hat{x} \in D(\hat{A})$ is the infinitesimal generator of a C_0-semigroup $(\hat{T}(t))_{t\geq 0}$ on $\hat{X}$.
2. There exists an operator $\hat{H} \in \mathcal{L}(\hat{U}, \hat{X})$ such that for all $\hat{u} \in \hat{U}$ we have $\hat{H}\hat{u} \in D(\hat{\mathcal{A}})$, $\hat{\mathcal{A}}\hat{H} \in \mathcal{L}(\hat{U}, \hat{X})$ and $\hat{\mathcal{B}}\hat{H}\hat{u} = \hat{u}$, $\hat{u} \in \hat{U}$.

Remark 1. Let $(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ be a boundary control system. Then according to [21, Ch. 10], there exists a unique $\hat{B} \in \mathcal{L}(\hat{U}, \hat{X}_{-1})$ such that $\hat{\mathcal{A}} = \hat{A}_{-1} + \hat{B}\hat{\mathcal{B}}$ on $D(\hat{\mathcal{A}})$ and therefore (4a) and (4b) can be written as

$$\dot{\hat{x}}(t) = \hat{A}_{-1}\hat{x}(t) + \hat{B}\hat{u}(t), \quad \hat{x}(0) = \hat{x}_0.$$

Definition 2.2 (Impedance Passive System)**.** The system $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}})$ is called impedance passive if the solutions of (4) satisfy

$$\frac{1}{2}\frac{d}{dt}\|\hat{x}(t)\|^2_{\hat{X}} \leq \operatorname{Re}\langle \hat{u}(t), \hat{y}(t)\rangle_{\hat{U},\hat{Y}}, \quad t > 0.$$

We note that the above definition holds also for the systems in the state space form. Since we are interested in controlling velocities of the center rigid body, we use energy state space [14] instead of natural state space in order to write the PDE as an abstract system.

2.1. **Abstract formulation of the beams.** The left beam system that we extract from the the satellite system is described by,

$$\ddot{w}_l(\xi, t) + \frac{EI}{\rho a} w_l''''(\xi, t) + \frac{\gamma}{\rho a}\dot{w}_l(\xi, t) = 0, \tag{5a}$$

$$\dot{w}_l(0, t) = u_{l1}(t),\ \dot{w}_l'(0, t) = u_{l2}(t), \tag{5b}$$

$$w_l''(-1, t) = 0, \quad w_l'''(-1, t) = 0, \tag{5c}$$

$$y_{l1}(t) = -EIw_l'''(0, t),\ y_{l2}(t) = EIw_l''(0, t). \tag{5d}$$

where $-1 < \xi < 0, t > 0$ and $u_{l1}(t),\ u_{l2}(t)$ are the virtual boundary inputs and $y_{l1}(t),\ y_{l2}(t)$ are the virtual boundary outputs of the left beam (see Figure 2), respectively.

By choosing the state variable

$$x_l(t) = \begin{bmatrix} \rho a\dot{w}_l(\cdot, t) \\ w_l''(\cdot, t) \end{bmatrix},$$

where $\dot{w}_l(\cdot, t)$ and $w_l''(\cdot, t)$ are the velocity and the bending moment of the left beam, respectively, (5) can be written in boundary control and observation form on the state space $X_l = L^2([-1, 0]; \mathbb{R}^2)$ as

$$\dot{x}_l(t) = \mathcal{A}_l x_l(t), \tag{6a}$$

$$\mathcal{B}_l x_l(t) = u_l(t), \tag{6b}$$

$$y_l(t) = \mathcal{C}_l x_l(t), \tag{6c}$$

where

$$\mathcal{A}_l x_l(t) = \begin{bmatrix} -\gamma(\rho a)^{-1} & -EI\partial_{\xi\xi} \\ (\rho a)^{-1}\partial_{\xi\xi} & 0 \end{bmatrix} \begin{bmatrix} \rho a\dot{w}_l(\cdot, t) \\ w_l''(\cdot, t) \end{bmatrix},$$

with

$$D(\mathcal{A}_l) = \{x_l \in X_l \mid \mathcal{H}_l x_l \in H^2([-1,0];\mathbb{R}^2), x_{l2}(-1) = x'_{l2}(-1) = 0\},$$

$$\mathcal{H}_l = \begin{bmatrix} (\rho a)^{-1} & 0 \\ 0 & EI \end{bmatrix}, \ \mathcal{B}_l x_l(t) = \begin{bmatrix} \dot{w}_l(0,t) \\ \dot{w}'_l(0,t) \end{bmatrix} \text{ and } \mathcal{C}_l x_l(t) = \begin{bmatrix} -EIw'''_l(0,t) \\ EIw''_l(0,t) \end{bmatrix}.$$

The operators $\mathcal{B}_l : D(\mathcal{A}_l) \to U_l$ and $\mathcal{C}_l : D(\mathcal{A}_l) \to Y_l$ are the virtual control and observation operators with $U_l = \mathbb{R}^2$ and $Y_l = \mathbb{R}^2$. Here, it is noted that the equations (6a), (6b), (6c) and $D(\mathcal{A}_l)$ corresponds to (5a), (5b), (5d) and (5c), respectively. The space X_l is a Hilbert space equipped with the energy norm

$$\|x_l\|^2_{X_l} := \langle x_l, \mathcal{H}_l x_l \rangle_{L^2}, \ x_l \in X_l.$$

Here $\frac{1}{2}\|x_l\|^2_{X_l}$ is the sum of the kinetic and potential energies of the left beam. The above choice of the state variable corresponds to the port-Hamiltonian formulation of the Euler Bernoulli beam. More details can be found, for example, in [5], [3], and [4].

In the same way, the right beam can be written in boundary control and observation form on the Hilbert space $X_r = L^2([0,1];\mathbb{R}^2)$ with $u_{r1}(t) = \dot{w}_r(0,t), \ u_{r2}(t) = \dot{w}'_r(0,t)$ as virtual boundary inputs and $y_{r1}(t) = EIw'''_r(0,t)$, $y_{r2}(t) = -EIw''_r(0,t)$ as virtual outputs. We denote the input and output spaces of the right beam by $U_r = \mathbb{R}^2$ and $Y_r = \mathbb{R}^2$, respectively. Choosing the state variable $x_r(t) = \begin{bmatrix} \rho a \dot{w}_r(\cdot,t) \\ w''_r(\cdot,t) \end{bmatrix}$, we have

$$\begin{aligned} \dot{x}_r(t) &= \mathcal{A}_r x_r(t), \\ \mathcal{B}_r x_r(t) &= u_r(t), \\ y_r(t) &= \mathcal{C}_r x_r(t), \end{aligned} \tag{7}$$

where

$$\mathcal{A}_r = \begin{bmatrix} -\gamma(\rho a)^{-1} & -EI\partial_{\xi\xi} \\ (\rho a)^{-1}\partial_{\xi\xi} & 0 \end{bmatrix}, \mathcal{B}_r x_r(t) = \begin{bmatrix} \dot{w}_r(0,t) \\ \dot{w}'_r(0,t) \end{bmatrix}, \mathcal{C}_r x_r(t) = \begin{bmatrix} EIw'''_r(0,t) \\ -EIw''_r(0,t) \end{bmatrix}$$

$$\text{and} \quad D(\mathcal{A}_r) = \{x_r \in X_r \mid \mathcal{H}_r x_r \in H^2([0,1];\mathbb{R}^2), x_{r2}(1) = x'_{r2}(1) = 0\},$$

$\mathcal{H}_r = \begin{bmatrix} (\rho a)^{-1} & 0 \\ 0 & EI \end{bmatrix}$. The space X_r is equipped with the energy norm $\|x_r\|^2_{X_r} := \langle x_r, \mathcal{H}_r x_r \rangle_{L^2}, \ x_r \in X_r$.

Next, we combine the two beam systems (6) and (7) into a single open loop system on the Hilbert space $X_b = X_l \times X_r$ as follows. From the above formulation and from the boundary conditions in (1), it is clear that $u_l(t) = u_r(t)$. Now, in order to have the coupling between the beam system and the rigid body as in Figure 2, the input and the output of the combined beam system are chosen such that the output of the combined beam system is equal to the addition of the outputs of the left and the right beam systems and the input of the combined beam system is equal to the inputs of the left and the right beam systems. Therefore, denoting the input and output spaces of the combined system by U_b and Y_b, respectively, let us define a new virtual output function

$$y_b(t) = \begin{bmatrix} \mathcal{C}_l & \mathcal{C}_r \end{bmatrix} \begin{bmatrix} x_l(t) \\ x_r(t) \end{bmatrix}$$

and a virtual input function

$$u_b(t) = \begin{bmatrix} \frac{1}{2}\mathcal{B}_l & \frac{1}{2}\mathcal{B}_r \end{bmatrix} \begin{bmatrix} x_l(t) \\ x_r(t) \end{bmatrix}.$$

Then the combined system can be written as

$$\dot{x}_b(t) = \mathcal{A}_b x_b(t), \quad \mathcal{B}_b x_b(t) = u_b(t), \quad \mathcal{C}_b x_b(t) = y_b(t) \tag{8}$$

where

$$x_b(t) = \begin{bmatrix} x_l(t) \\ x_r(t) \end{bmatrix}, \; \mathcal{A}_b = \begin{bmatrix} \mathcal{A}_l & 0 \\ 0 & \mathcal{A}_r \end{bmatrix}, \; \mathcal{B}_b = \begin{bmatrix} \frac{1}{2}\mathcal{B}_l & \frac{1}{2}\mathcal{B}_r \end{bmatrix}, \; \mathcal{C}_b = \begin{bmatrix} \mathcal{C}_l & \mathcal{C}_r \end{bmatrix},$$

and $D(\mathcal{A}_b) = \{(x_l, x_r) \in D(\mathcal{A}_l) \times D(\mathcal{A}_r) : \mathcal{B}_l x_l = \mathcal{B}_r x_r\}$.

Lemma 2.3. *The beam system $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b)$ in (8) is an impedance passive system on (X_b, U_b, Y_b).*

Proof. From [11, Sec. 2.1], we have that the left beam $(\mathcal{A}_l, \mathcal{B}_l, \mathcal{C}_l)$ and the right beam $(\mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r)$ are impedance passive systems. Now, using the boundary condition $u_l(t) = u_r(t)$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|x_b(t)\|_{X_b}^2 &= \frac{1}{2}\frac{d}{dt}\|x_l(t)\|_{X_l}^2 + \frac{1}{2}\frac{d}{dt}\|x_r(t)\|_{X_r}^2, \\ &\leq \langle u_l(t), y_l(t)\rangle_{U_l,Y_l} + \langle u_r(t), y_r(t)\rangle_{U_r,Y_r} \\ &= \langle u_b(t), y_b(t)\rangle_{U_b,Y_b}, \end{aligned}$$

where $x_b(t),\ t > 0$ are solutions of (8). Therefore, (8) is an impedance passive system. □

Remark 2. The impedance passivity of the systems $(\mathcal{A}_l, \mathcal{B}_l, \mathcal{C}_l)$, $(\mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r)$ and $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b)$ imply that $A_l = \mathcal{A}_l|_{\mathcal{N}(\mathcal{B}_l)}$, $A_r = \mathcal{A}_r|_{\mathcal{N}(\mathcal{B}_r)}$ and $A_b = \mathcal{A}_b|_{\mathcal{N}(\mathcal{B}_b)}$ generate C_0-semigroups of contractions $T_l(t)$, $T_r(t)$ and $T_b(t)$ on X_l, X_r and X_b, respectively. Therefore, $(\mathcal{A}_l, \mathcal{B}_l, \mathcal{C}_l)$, $(\mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r)$ and $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b)$ are boundary control and observation systems [15, Sec. 4.2]. This implies from Remark 1 that there exist unique operators $B_l \in \mathcal{L}(U_l, X_{l_{-1}})$, $B_r \in \mathcal{L}(U_r, X_{r_{-1}})$ and $B_b \in \mathcal{L}(U_b, X_{b_{-1}})$ such that $\mathcal{A}_l = A_{l_{-1}} + B_l\mathcal{B}_l$ on $D(\mathcal{A}_l)$, $\mathcal{A}_r = A_{r_{-1}} + B_r\mathcal{B}_r$ on $D(\mathcal{A}_r)$ and $\mathcal{A}_b = A_{b_{-1}} + B_b\mathcal{B}_b$ on $D(\mathcal{A}_b)$, respectively.

2.2. **The rigid body.** Without external inputs, the center rigid body that we extract from the satellite system is given by

$$\begin{aligned} m\ddot{w}_c(t) &= u_{c1}(t), \\ I_m\ddot{\theta}_c(t) &= u_{c2}(t), \\ y_{c1}(t) &= \dot{w}_c(t), \\ y_{c2}(t) &= \dot{\theta}_c(t), \end{aligned} \tag{9}$$

where $u_{c1}(t),\ u_{c2}(t)$ are the virtual inputs and $y_{c1}(t),\ y_{c2}(t)$ are the outputs of the rigid body (see Figure 2), respectively. The state, input and output spaces of the rigid body are given by $X_c = \mathbb{R}^2,\ U_c = \mathbb{R}^2$ and $Y_c = \mathbb{R}^2$, respectively. Then, with the state variable $x_c(t) = \begin{bmatrix} \dot{w}_c(t) \\ \dot{\theta}_c(t) \end{bmatrix}$, the rigid body (9) on the Hilbert space X_c can be written as,

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c u_c(t), \\ y_c(t) &= C_c x_c(t), \end{aligned} \tag{10}$$

where,

$$A_c = 0,\ B_c = \begin{bmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{I_m} \end{bmatrix},\ C_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \text{and } u_c(t) = \begin{bmatrix} u_{c1}(t) \\ u_{c2}(t) \end{bmatrix}.$$

The space X_c is equipped with the energy norm

$$\|x_c\|^2_{X_c} = x_c^* \mathcal{H}_c x_c, \quad \text{where} \quad \mathcal{H}_c = \begin{bmatrix} m & 0 \\ 0 & I_m \end{bmatrix}.$$

It is straightforward to see that the rigid body is an impedance passive system on X_c (see [11, Sec. 2.3]). More details on the energy state space formulation of finite-dimensional systems can be found in [14, Ch. 2.3].

2.3. **The satellite system as a coupled PDE-ODE system.** From the equations (8) and (10), we are now ready to write our satellite system (1)-(2) as an abstract PDE-ODE system with the power-preserving interconnection $u_b(t) = y_c(t)$, $u_c(t) = -y_b(t)$ (see Figure 3) on the state space $X = X_b \times X_c$ as

$$\begin{aligned} \begin{bmatrix} \dot{x}_b(t) \\ \dot{x}_c(t) \end{bmatrix} &= \begin{bmatrix} \mathcal{A}_b & 0 \\ -B_c\mathcal{C}_b & 0 \end{bmatrix} \begin{bmatrix} x_b(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u(t) + \begin{bmatrix} B_{d0} & 0 \\ 0 & B_c \end{bmatrix} w_d(t), \\ y(t) &= \begin{bmatrix} 0 & C_c \end{bmatrix} \begin{bmatrix} x_b(t) \\ x_c(t) \end{bmatrix}, \end{aligned} \tag{11}$$

where $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$, $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, $w_d(t) = \begin{bmatrix} w_{d1}(t) & w_{d2}(t) & w_{d3}(t) & w_{d4}(t) \end{bmatrix}^T$ and $B_{d0} = \begin{bmatrix} b_{d1}(\cdot) & 0 \\ 0 & 0 \\ 0 & b_{d2}(\cdot) \\ 0 & 0 \end{bmatrix}$.

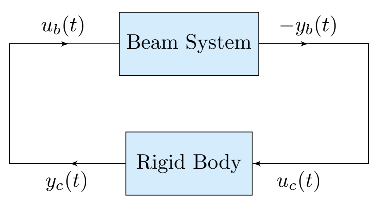

FIGURE 3. The interconnection between the beams and the rigid body

Equation (11) is in the form (3) with $A = \begin{bmatrix} \mathcal{A}_b & 0 \\ -B_c\mathcal{C}_b & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ B_c \end{bmatrix}$, $C = \begin{bmatrix} 0 & C_c \end{bmatrix}$, $B_d = \begin{bmatrix} B_{d0} & 0 \\ 0 & B_c \end{bmatrix}$ and $x(t) = \begin{bmatrix} x_b(t) \\ x_c(t) \end{bmatrix}$. The domain of A is given by

$$D(A) = \{(x_b, x_c) \in D(\mathcal{A}_b) \times X_c \ : \ \mathcal{B}_b x_b = C_c x_c\}.$$

The norm on X is defined as

$$\left\| \begin{bmatrix} x_b \\ x_c \end{bmatrix} \right\|^2_X = \|x_b\|^2_{X_b} + \|x_c\|^2_{X_c}, \quad x_b \in X_b,\ x_c \in X_c.$$

Remark 3. The operator A is dissipative, since using the power preserving interconnection, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\| \begin{bmatrix} x_b(t) \\ x_c(t) \end{bmatrix} \right\|_X^2 \leq 0.$$

Therefore, by [4, Theorem 3.5], A generates a C_0-semigroup of contractions on X.

3. **Stability of the satellite model.** In this section, we will show the exponential stability of the satellite system in the sense that the operator A defined in Section 2.3 generates an exponentially stable semigroup $T(t)$. Let us recall the operator A

$$\begin{aligned} A &= \begin{bmatrix} \mathcal{A}_b & 0 \\ -B_c\mathcal{C}_b & 0 \end{bmatrix}, \\ D(A) &= \{(x_b, x_c) \in D(\mathcal{A}_b) \times X_c \ : \ \mathcal{B}_b x_b = C_c x_c\}. \end{aligned} \tag{12}$$

Theorem 3.1. *The semigroup $T(t)$ generated by A in* (12) *is exponentially stable.*

We prove the theorem by using frequency domain criteria [16, Cor. 3.36] which states that the semigroup $T(t)$ generated by A is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and $\sup_{\omega\in\mathbb{R}} \|R(i\omega, A)\| < \infty$. We complete the proof in the following steps. Since the satellite system is a coupled system of the beam system and the center rigid body, we will first show that $i\mathbb{R} \subset \rho(A_b)$ and $\sup_{\omega\in\mathbb{R}} \|R(i\omega, A_b)\| < \infty$ where $A_b = \mathcal{A}_b|_{\mathcal{N}(\mathcal{B}_b)}$. As the second step, we will show that $i\mathbb{R} \subset \rho(A)$. In this step, we will obtain an expression for the resolvent operator $R(i\omega, A)$. Next, we will estimate upper bounds of the operators which appear in the resolvent expression. Finally, we will show that $R(i\omega, A)$ is uniformly bounded.

Lemma 3.2. *The operator A_b defined in Remark 2 satisfies $i\mathbb{R} \subset \rho(A_b)$ and $\sup_{\omega\in\mathbb{R}} \|R(i\omega, A_b)\| < \infty$.*

Proof. We show that the semigroup $T_b(t)$ generated by A_b is exponentially stable which guarantees $i\mathbb{R} \subset \rho(A_b)$ and uniform boundedness of the resolvent $R(i\omega, A_b)$. First we claim that the operator $A_r = \mathcal{A}_r|_{\mathcal{N}(\mathcal{B}_r)}$ corresponding to the right beam system (7) generates an exponentially stable semigroup $T_r(t), t \geq 0$. We use [7, Main Theorem 1]. We write A_r as $A_r = A_0 + B_0$ where

$$A_0 = \begin{bmatrix} 0 & -EI\partial_{\xi\xi} \\ (\rho a)^{-1}\partial_{\xi\xi} & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -\gamma(\rho a)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and $D(A_0) = D(A_r)$. We will show that the operators A_0 and B_0 satisfy the following conditions.

(c1) The operator A_0 is skew-adjoint and it has compact resolvent.

(c2) The spectrum of A_0 satisfies the gap property

$$\inf\{|\lambda_j - \lambda_k| \; |j, k = 1, 2, 3, \cdots, j \neq k\} > 0.$$

(c3) The operator B_0 is dissipative.

(c4) If any sequence $\{(x_{r_n}) \in X_r, \ n = 1, 2, \cdots\}$ satisfies

$$\lim_{n\to\infty} \operatorname{Re} \langle B_0 x_{r_n}, x_{r_n} \rangle_{X_r} = 0,$$

then $\lim_{n\to\infty} \|B_0 x_{r_n}\|_{X_r} = 0$.

(c5) There exists $\delta > 0$ such that $\|B_0 \phi_k\|_{X_r} \geq \delta$, where $\phi_k, \ k \in \mathbb{Z}$ is an orthonormal eigenvector of A_0.

We have $\operatorname{Re}\langle A_0 x_r, x_r\rangle = 0, x_r \in D(A_0)$. Therefore, by [5, Thm. 2.3], A_0 has compact resolvent. This implies that the operator A_0 is skew-adjoint.

By a direct computation, we can obtain eigenvalues $i\lambda_k$ of A_0 and orthonormal basis $\phi_k = (f_k, g_k)^T, k \in \mathbb{Z}$ consisting of eigenvectors of A_0. The eigenvalues and the eigenvectors are given by

$$\begin{aligned} i\lambda_k &= i\sqrt{\frac{EI}{\rho a}}[\pi(k-\frac{1}{2}) + \mathcal{O}(e^{-\pi(k-\frac{1}{2})})]^2, \\ f_k(\xi) &= \beta_k[(\cosh(\mu_k) + \cos(\mu_k))(\cosh(\mu_k\xi) - \cos(\mu_k\xi)) \\ &\quad - (\sinh(\mu_k) - \sin(\mu_k))(\sinh(\mu_k\xi) - \sin(\mu_k\xi))], \\ g_k(\xi) &= \frac{\beta_k}{i\sqrt{\rho a EI}}[(\cosh(\mu_k) + \cos(\mu_k))(\cosh(\mu_k\xi) + \cos(\mu_k\xi)) \\ &\quad - (\sinh(\mu_k) - \sin(\mu_k))(\sinh(\mu_k\xi) + \sin(\mu_k\xi))], \end{aligned} \tag{13}$$

where $\mu_k = (\frac{\rho a}{EI})^{\frac{1}{4}}\sqrt{\lambda_k}$ are the solutions of $\cosh(\mu_k)\cos(\mu_k) + 1 = 0$ and $\beta_k > 0$ are chosen such that $\|\phi_k\|_{X_r} = 1$. It is clear that the condition (c2) is satisfied since the gap between two successive eigenvalues satisfies $|\lambda_k - \lambda_{k+1}| \to \infty$ as $k \to \infty$. The operator B_0 is dissipative since

$$\operatorname{Re}\langle B_0 x_r, x_r\rangle_{X_r} = -\gamma(\rho a)^{-2}\|x_{r_1}\|_{L^2}^2 \le 0.$$

Also, $-\operatorname{Re}\langle B_0 x_r, x_r\rangle_{X_r} = \gamma^{-1}\rho a\|B_0 x_r\|_{X_r}^2$ holds. This implies that the conditions (c3) and (c4) are satisfied.

Next, we show that the condition (c5) is satisfied. The formulas for f_k and g_k in (13) can be used to show that

$$\lim_{|k|\to\infty} \frac{\|g_k\|_{L^2}}{\|f_k\|_{L^2}} = \frac{1}{\sqrt{\rho a EI}}. \tag{14}$$

Here we note that $f_k \neq 0,\ \forall k \in \mathbb{Z}$, since $(f_k, g_k)^T$ are eigenvectors and $f_k = 0$ would imply $g_k = \frac{(\rho a)^{-1}}{i\lambda_k} f_k'' = 0$. The equation (14) implies that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k \in \mathbb{Z}$ with $|k| \ge N$, we have

$$\left|\frac{\|g_k\|_{L^2}}{\|f_k\|_{L^2}}\right| \le \epsilon + \frac{1}{\sqrt{\rho a EI}}.$$

Thus

$$\frac{\|g_k\|_{L^2}}{\|f_k\|_{L^2}} \le \frac{C'}{\sqrt{\rho a EI}},\ \forall\ k \in \mathbb{Z},$$

where $C' = \max\{1 + \epsilon\sqrt{\rho a EI}, \sqrt{\rho a EI}\max_{|k|<N}\frac{\|g_k\|_{L^2}}{\|f_k\|_{L^2}}\}$. Now we obtain

$$\begin{aligned} \|B_0\phi_k\|_{X_r}^2 &= \gamma^2(\rho a)^{-3}\|f_k\|_{L^2}^2, \\ &\ge \frac{1}{2}\gamma^2(\rho a)^{-3}(\|f_k\|_{L^2}^2 + \frac{\rho a EI}{C'^2}\|g_k\|_{L^2}^2), \\ &\ge \frac{1}{2C'^2}\gamma^2(\rho a)^{-2}\|\phi_k\|_{X_r}^2, \\ &= \frac{1}{2C'^2}\gamma^2(\rho a)^{-2} \ge \delta^2 > 0,\ \forall k \in \mathbb{Z}. \end{aligned}$$

Now all the conditions (c1)-(c5) are satisfied. Hence by [7, Main Theorem 1], we have that A_r generates an exponentially stable semigroup $T_r(t)$.

Analogously, we have that $A_l = \mathcal{A}_l|_{\mathcal{N}(\mathcal{B}_l)}$ generates an exponentially stable semigroup $T_l(t),\ t > 0$. Thus A_b generates an exponentially stable semigroup $T_b(t)$ which completes the proof. □

Lemma 3.3. *Let $P_b(\cdot) = \mathcal{C}_b R(\cdot, A_{b_{-1}})B_b$ and $P_c(\cdot) = C_c R(\cdot, A_c)B_c$ be the transfer functions of the beam system $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b)$ and the center rigid body (A_c, B_c, C_c), respectively. Assume that $P_b(0)$ and $I + P_b(i\omega)P_c(i\omega)$, $\omega \in \mathbb{R}\backslash\{0\}$ are nonsingular. Then the operator A in (12) satisfies $i\mathbb{R} \subset \rho(A)$.*

Proof. We will show that the operator $i\omega - A$ is bijective. Let $\omega \in \mathbb{R}$ be arbitrary. We start by proving $i\omega - A$ is injective. Let $(x_b, x_c)^T \in D(A) = \{(x_b, x_c) \in D(\mathcal{A}_b) \times X_c \ :\ \mathcal{B}_b x_b = C_c x_c\}$ be such that $(i\omega - A)(x_b, x_c)^T = 0$. Then by using the structure of A, we obtain

$$\begin{bmatrix} (i\omega - \mathcal{A}_b)x_b \\ B_c\mathcal{C}_b x_b + i\omega x_c \end{bmatrix} = 0.$$

We have from Lemma 3.2 that $i\mathbb{R} \subset \rho(A_b)$. By using Remark 2, solving the above equation, we obtain

$$\begin{aligned} x_b &= R(i\omega, A_{b_{-1}})B_b C_c x_c, \\ [i\omega I_{X_c} &+ B_c P_b(i\omega) C_c] x_c = 0. \end{aligned} \tag{15}$$

We have that B_c, C_c are nonsingular and $P_b(0)$ and $I + P_b(i\omega)P_c(i\omega)$ are assumed to be nonsingular. Therefore, the function

$$S(i\omega) = \begin{cases} \frac{1}{i\omega} + \frac{1}{\omega^2} B_c P_b(i\omega)(I + P_b(i\omega)P_c(i\omega))^{-1} C_c, \ \omega \in \mathbb{R}\backslash\{0\}, \\ (B_c P_b(0) C_c)^{-1}, \ \omega = 0 \end{cases} \tag{16}$$

is well-defined for all $\omega \in \mathbb{R}$. A direct computation shows that $S(i\omega) = [i\omega I_{X_c} + B_c P_b(i\omega) C_c]^{-1}$ for all $\omega \in \mathbb{R}$. This implies by (15) that $(x_b, x_c) = 0$. Thus, the operator $i\omega - A$ is injective.

Now it remains to prove that $i\omega - A$ is surjective. For all $f_b \in X_b$ and $f_c \in X_c$, our aim is to find $(x_b, x_c)^T \in D(A)$ such that

$$\begin{bmatrix} f_b \\ f_c \end{bmatrix} = (i\omega - A)\begin{bmatrix} x_b \\ x_c \end{bmatrix} = \begin{bmatrix} (i\omega - \mathcal{A}_b)x_b \\ B_c\mathcal{C}_b x_b + i\omega x_c \end{bmatrix}. \tag{17}$$

Since $i\mathbb{R} \subset \rho(A_b)$, using Remark 2, the solution of (17) is given by

$$\begin{aligned} x_b &= [R(i\omega, A_b) - R(i\omega, A_{b_{-1}})B_b C_c S(i\omega) B_c C_b R(i\omega, A_b)] f_b \\ &\quad + R(i\omega, A_{b_{-1}}) B_b C_c S(i\omega) f_c \\ x_c &= -S(i\omega) B_c C_b R(i\omega, A_b) f_b + S(i\omega) f_c \end{aligned} \tag{18}$$

where $C_b = \mathcal{C}_b|_{\mathcal{N}(\mathcal{B}_b)}$. Moreover, for $(x_b, x_c) \in D(\mathcal{A}_b) \times X_c$, we have

$$\begin{aligned} &\begin{bmatrix} \mathcal{B}_b & -C_c \end{bmatrix} \begin{bmatrix} x_b \\ x_c \end{bmatrix} \\ &= \mathcal{B}_b R(i\omega, A_b) f_b - \mathcal{B}_b R(i\omega, A_{b_{-1}}) B_b C_c S(i\omega) B_c C_b R(i\omega, A_b) f_b \\ &\quad + \mathcal{B}_b R(i\omega, A_{b_{-1}}) B_b C_c S(i\omega) f_c + C_c S(i\omega) B_c C_b R(i\omega, A_b) f_b - C_c S(i\omega) f_c \\ &= 0 \end{aligned}$$

since $\mathcal{B}_b R(i\omega, A_{b_{-1}}) B_b = I$ [14, Prop. 10.1.2] and $\mathcal{R}(R(i\omega, A_b)) \subset D(A_b)$. Thus $(x_b, x_c)^T \in D(A)$. This implies that the operator $i\omega - A$ is surjective. Thus $i\omega - A,\ \omega \in \mathbb{R}$ has a bounded inverse, which completes the proof. □

Remark 4. From the equation (18) in Lemma 3.3, the resolvent operator $R(i\omega, A)$ has the form

$$R(i\omega, A) = \begin{bmatrix} R_{11}(i\omega) & R_{12}(i\omega) \\ R_{21}(i\omega) & R_{22}(i\omega) \end{bmatrix} \tag{19}$$

where

$$\begin{aligned} R_{11}(i\omega) &= R(i\omega, A_b) - R(i\omega, A_{b_{-1}})B_bC_cS(i\omega)B_cC_bR(i\omega, A_b), \\ R_{12}(i\omega) &= R(i\omega, A_{b_{-1}})B_bC_cS(i\omega), \\ R_{21}(i\omega) &= -S(i\omega)B_cC_bR(i\omega, A_b), \\ R_{22}(i\omega) &= S(i\omega). \end{aligned} \tag{20}$$

In the following we derive an upper bound for the transfer function $P_b(i\omega)$ and upper bounds for the operators $R(i\omega, A_{b_{-1}})B_b$, $C_bR(i\omega, A_b)$ and $(I + P_b(i\omega)P_c(i\omega))^{-1}$.

Lemma 3.4. *Let $P_b(\cdot)$ be the transfer function of the beam system $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b)$. Then there exists $M > 0$ such that $\|P_b(i\omega)\| \leq M(|\omega| + 1)$ for all $\omega \in \mathbb{R}$. Moreover, $P_b(0)$ is nonsingular.*

Proof. For $u_b \in U_b$, the transfer function of the beam system $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b)$ is given by

$$P_b(i\omega)u_b = P_l(i\omega)u_l + P_r(i\omega)u_r, \ \omega \in \mathbb{R},$$

where $P_l(i\omega)$ and $P_r(i\omega)$ are the transfer functions of the left and the right beam systems, respectively, and we will now derive an explicit expression for them. For $u_r \in U_r$, the transfer function $P_r(i\omega)$ of the right beam system $(\mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r)$ can be obtained as the unique solution of

$$\begin{aligned} (i\omega - \mathcal{A}_r)x_r &= 0, \\ \mathcal{B}_r x_r &= u_r \\ P_r(i\omega)u_r &= \mathcal{C}_r x_r \end{aligned}$$

with $x_r \in D(\mathcal{A}_r) = \{x_r = (f_r, g_r)^T \in X_r \mid \mathcal{H}_r x_r \in H^2([0,1]; \mathbb{R}^2), \ g_r(1) = g_r'(1) = 0\}$ ([14, Thm. 12.1.3]). Replacing the operators with the corresponding expressions, the above equations can be written as

$$\begin{aligned} (i\omega + \gamma(\rho a)^{-1})f_r + EIg_r'' &= 0, \\ -(\rho a)^{-1}f_r'' + i\omega g_r &= 0, \\ f_r(0) = \rho a \ u_{r1}, \ f_r'(0) &= \rho a \ u_{r2}, \\ EIg_r(1) = 0, \ EIg_r'(1) &= 0, \\ P_r(i\omega)u_r &= EI \begin{bmatrix} g_r'(0) \\ -g_r(0) \end{bmatrix}. \end{aligned} \tag{21}$$

We consider the case $\omega = 0$ separately. Solving (21) for $\omega = 0$, we obtain

$$\begin{aligned} f_r(\xi) &= \rho a(u_{r1} + \xi u_{r2}), \\ g_r(\xi) &= \frac{\gamma}{EI}\left[\left(\frac{-\xi^2}{2} + \xi - \frac{1}{2}\right)u_{r1} + \left(\frac{-\xi^3}{6} + \frac{\xi}{2} - \frac{1}{3}\right)u_{r2}\right] \end{aligned}$$

and therefore $P_r(0)$ is given by

$$P_r(0)u_r = EI \begin{bmatrix} g_r'(0) \\ -g_r(0) \end{bmatrix} = \gamma \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} u_{r1} \\ u_{r2} \end{bmatrix}.$$

Similarly, we obtain that $P_l(0)$ is given by

$$P_l(0)u_l = EI\begin{bmatrix} -g_l'(0) \\ g_l(0) \end{bmatrix} = \gamma \begin{bmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} u_{l1} \\ u_{l2} \end{bmatrix}.$$

Now using the boundary conditions $u_{l1} = u_{r1}, u_{l2} = u_{r2}$, the transfer function of the combined beam system is given by,

$$P_b(0)u_b = EI\begin{bmatrix} g_r'(0) - g_l'(0) \\ -g_r(0) + g_l(0) \end{bmatrix} = \gamma \begin{bmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{bmatrix} u_b. \tag{22}$$

Thus $P_b(0)$ is indeed nonsingular. For $\omega \in \mathbb{R}\backslash\{0\}$, solving (21), we obtain

$$\begin{aligned} f_r(\xi) &= \rho a\Big[(C_{1,\omega}u_{r1} - \frac{C_{3,\omega}}{\alpha(\omega)}u_{r2})f_1(\xi) + (C_{2,\omega}u_{r1} + \frac{C_{4,\omega}}{\alpha(\omega)}u_{r2})f_2(\xi) \\ &\qquad + \cos(\alpha(\omega)\xi)u_{r1} + \frac{\sin(\alpha(\omega)\xi)}{\alpha(\omega)}u_{r2}\Big], \\ g_r(\xi) &= \frac{\alpha(\omega)^2}{i\omega}\Big[(C_{1,\omega}u_{r1} - \frac{C_{3,\omega}}{\alpha(\omega)}u_{r2})g_1(\xi) + (C_{2,\omega}u_{r1} + \frac{C_{4,\omega}}{\alpha(\omega)}u_{r2})g_2(\xi) \\ &\qquad - \cos(\alpha(\omega)\xi)u_{r1} - \frac{\sin(\alpha(\omega)\xi)}{\alpha(\omega)}u_{r2}\Big], \end{aligned}$$

where

$$\begin{aligned} C_{1,\omega} &= \frac{C_1(\omega)}{C_2(\omega)},\ C_{2,\omega} = \frac{C_2(\omega)\cos(\alpha(\omega)) - C_1(\omega)C_5(\omega)}{C_2(\omega)C_3(\omega)}, \\ C_{3,\omega} &= \frac{C_4(\omega)}{C_2(\omega)},\ C_{4,\omega} = \frac{C_2(\omega)\sin(\alpha(\omega)) + C_4(\omega)C_5(\omega)}{C_2(\omega)C_3(\omega)}, \end{aligned} \tag{23}$$

$$\begin{aligned} f_1(\xi) &= \cosh(\alpha(\omega)\xi) - \cos(\alpha(\omega)\xi),\ f_2(\xi) = \sinh(\alpha(\omega)\xi) - \sin(\alpha(\omega)\xi), \\ g_1(\xi) &= \cosh(\alpha(\omega)\xi) + \cos(\alpha(\omega)\xi),\ g_2(\xi) = \sinh(\alpha(\omega)\xi) + \sin(\alpha(\omega)\xi) \end{aligned}$$

and

$$\begin{aligned} C_1(\omega) &= 1 + \cos(\alpha(\omega))\cosh(\alpha(\omega)) + \sin(\alpha(\omega))\sinh(\alpha(\omega)), \\ C_2(\omega) &= 2 + 2\cosh(\alpha(\omega))\cos(\alpha(\omega)), \\ C_3(\omega) &= \sinh(\alpha(\omega)) + \sin(\alpha(\omega)), \\ C_4(\omega) &= \cos(\alpha(\omega))\sinh(\alpha(\omega)) - \sin(\alpha(\omega))\cosh(\alpha(\omega)), \\ C_5(\omega) &= \cosh(\alpha(\omega)) + \cos(\alpha(\omega)), \\ \alpha(\omega) &= \left(\frac{\rho a}{EI}\right)^{\frac{1}{4}}(\omega^2 - i\gamma(\rho a)^{-1}\omega)^{\frac{1}{4}}. \end{aligned}$$

Therefore, the transfer function of the right beam can be written as,

$$P_r(i\omega)u_r = EI\begin{bmatrix} g_r'(0) \\ -g_r(0) \end{bmatrix}, \quad \omega \in \mathbb{R}\backslash\{0\}$$

where

$$\begin{aligned} g_r'(0) &= 2\frac{\alpha(\omega)^3}{i\omega}C_{2,\omega}u_{r1} + \frac{\alpha(\omega)^2}{i\omega}(2C_{4,\omega} - 1)u_{r2}, \\ g_r(0) &= \frac{\alpha(\omega)^2}{i\omega}(2C_{1,\omega} - 1)u_{r1} - 2\frac{\alpha(\omega)}{i\omega}C_{3,\omega}u_{r2}. \end{aligned}$$

In the same way, we can obtain the transfer function of the left beam which is given by,

$$P_l(i\omega)u_l = EI\begin{bmatrix}-g_l'(0)\\ g_l(0)\end{bmatrix}, \quad \omega\in\mathbb{R}\backslash\{0\}$$

where

$$\begin{aligned} g_l'(0) &= -2\frac{\alpha(\omega)^3}{i\omega}C_{2,\omega}u_{l1} + \frac{\alpha(\omega)^2}{i\omega}(2C_{4,\omega}-1)u_{l2},\\ g_l(0) &= \frac{\alpha(\omega)^2}{i\omega}(2C_{1,\omega}-1)u_{l1} + 2\frac{\alpha(\omega)}{i\omega}C_{3,\omega}u_{l2}. \end{aligned}$$

Thus, the transfer function of the combined beam system is given by,

$$P_b(i\omega)u_b = 4EI\frac{\alpha(\omega)}{i\omega}\begin{bmatrix}\alpha(\omega)^2C_{2,\omega} & 0\\ 0 & C_{3,\omega}\end{bmatrix}u_b, \quad \omega\in\mathbb{R}\backslash\{0\}. \tag{24}$$

Now, let us estimate the absolute values of $C_{2,\omega}$ and $C_{3,\omega}$ which contain trigonometric and hyperbolic terms. Writing $\alpha(\omega)$ in terms of its real and imaginary parts, we obtain

$$\begin{aligned} \alpha(\omega) &= \left(\frac{\rho a}{EI}\right)^{\frac{1}{4}}(\omega^2 - i\gamma(\rho a)^{-1}\omega)^{\frac{1}{4}},\\ &= |\alpha(\omega)|\Bigg(\cos\left(\frac{\theta(\omega)+2\pi k}{4}\right) + i\sin\left(\frac{\theta(\omega)+2\pi k}{4}\right)\Bigg),\ k=0,1,2,3, \end{aligned} \tag{25}$$

where

$$\begin{aligned} |\alpha(\omega)| &= \left(\frac{\rho a}{EI}|\omega|\sqrt{\omega^2+\gamma^2(\rho a)^{-2}}\right)^{\frac{1}{4}},\\ \theta(\omega) &= \tan^{-1}\left(\frac{-\gamma(\rho a)^{-1}}{\omega}\right). \end{aligned}$$

We have

$$\begin{aligned} \mathrm{Re}(\alpha(\omega)) &= \pm|\alpha(\omega)|\cos\left(\frac{\theta(\omega)}{4}\right) \text{ or } \mathrm{Re}(\alpha(\omega)) = \mp|\alpha(\omega)|\sin\left(\frac{\theta(\omega)}{4}\right),\\ \mathrm{Im}(\alpha(\omega)) &= \pm|\alpha(\omega)|\sin\left(\frac{\theta(\omega)}{4}\right) \text{ or } \mathrm{Im}(\alpha(\omega)) = \pm|\alpha(\omega)|\cos\left(\frac{\theta(\omega)}{4}\right). \end{aligned}$$

In addition, there exists $\omega_1 \geq \gamma(\rho a)^{-1} > 0$ such that $\tan^{-1}(\frac{\gamma(\rho a)^{-1}}{|\omega|}) \leq \frac{\gamma(\rho a)^{-1}}{|\omega|}$ for all $|\omega| > \omega_1$. Therefore, there exist $M_1, M_2, M_3, M_4 > 0$ and $\omega_2 > \omega_1$ such that

$$\begin{aligned} M_1\sqrt{|\omega|} &\leq |\alpha(\omega)|\left|\cos\left(\frac{\theta(\omega)}{4}\right)\right| \leq M_2\sqrt{|\omega|}\\ M_3\frac{1}{\sqrt{|\omega|}} &\leq |\alpha(\omega)|\left|\sin\left(\frac{\theta(\omega)}{4}\right)\right| \leq M_4\frac{1}{\sqrt{|\omega|}} \end{aligned}$$

for all $|\omega| \geq \omega_2$. Denoting $x_\omega = \mathrm{Re}(\alpha(\omega))$ and $y_\omega = \mathrm{Im}(\alpha(\omega))$, the above estimates imply that when $|x_\omega|$ grows at a rate of $\sqrt{|\omega|}$, $|y_\omega|$ decays at a rate of $\frac{1}{\sqrt{|\omega|}}$ or the other way around. Since $|C_{2,\omega}|$ and $|C_{3,\omega}|$ have similar terms for all the four roots of $\alpha(\omega)$, we restrict our analysis to the principal branch of the fourth root of $\alpha(\omega)$ and note that the other branches can be treated similarly.

The definition of $\alpha(\omega)$ and straightforward estimates can be used to verify that $|\cosh(\alpha(\omega))\cos(\alpha(\omega))| \to \infty$ as $|\omega| \to \infty$. Therefore, there exists $\omega_0 > \omega_2$ such that

$$|\cosh\alpha(\omega)\cos\alpha(\omega)| \geq 2 \tag{26}$$

for all $|\omega| \geq \omega_0$ and this further implies that

$$\begin{aligned}\left|\frac{\sin(\alpha(\omega))\sinh(\alpha(\omega))}{1+\cos(\alpha(\omega))\cosh(\alpha(\omega))}\right| &\leq \frac{|\sin(\alpha(\omega))\sinh(\alpha(\omega))|}{|\cos(\alpha(\omega))\cosh(\alpha(\omega))|-1}\\ &\leq 2|\tan(\alpha(\omega))||\tanh(\alpha(\omega))|\\ &\leq 2(|\coth(y_\omega)|+|\tanh(y_\omega)|)\\ &\qquad(|\tanh(x_\omega)|+|\coth(x_\omega)|)\end{aligned} \tag{27}$$

where the last inequality is obtained by separating real and imaginary parts of the second inequality and using straightforward estimates. Here we note that $|\tanh(x_\omega)|$, $|\coth(x_\omega)|$ and $|\tanh(y_\omega)|$ are all uniformly bounded for $|\omega| \geq \omega_0$ and since $|y_\omega|$ decays at a rate of $\frac{1}{\sqrt{|\omega|}}$, using Taylor series, we can estimate, there exists $M_0' > 0$ such that

$$|\coth(y_\omega)| = \left|y_\omega^{-1} + \frac{y_\omega}{3} - \frac{y_\omega^3}{45} + \cdots\right| \leq M_0'\sqrt{|\omega|}$$

for $|\omega| \geq \omega_0$. Therefore, from (27), we obtain

$$\left|\frac{\sinh(\alpha(\omega))\sin(\alpha(\omega))}{1+\cosh(\alpha(\omega))\cos(\alpha(\omega))}\right| \lesssim \sqrt{|\omega|} \tag{28}$$

for $|\omega| \geq \omega_0$. Moreover, $|\frac{\sin(x_\omega)\cosh(y_\omega)}{\sinh(x_\omega)\cos(y_\omega)}| \to 0$ as $|\omega| \to \infty$. This implies that there exists $M_0'' > 0$ such that

$$\begin{aligned}|\sinh(\alpha(\omega))+\sin(\alpha(\omega))| &\geq |\mathrm{Re}(\sinh(\alpha(\omega))+\sin(\alpha(\omega)))|\\ &= |\sinh(x_\omega)\cos(y_\omega)+\sin(x_\omega)\cosh(y_\omega)|\\ &= |\sinh(x_\omega)\cos(y_\omega)|\left|1+\frac{\sin(x_\omega)\cosh(y_\omega)}{\sinh(x_\omega)\cos(y_\omega)}\right|\\ &\geq M_0''|\sinh(x_\omega)\cos(y_\omega)|\end{aligned}$$

for $|\omega| \geq \omega_0$. Since $|\cos(\alpha(\omega))|$ and $|\tan(y_\omega)|$ are uniformly bounded for $|\omega| \geq \omega_0$, the above estimate implies that there exist $M_1' > 0$ and $M_1'' > 0$ such that

$$\left|\frac{\cos(\alpha(\omega))}{\sinh(\alpha(\omega))+\sin(\alpha(\omega))}\right| \leq M_1' \qquad \text{and} \tag{29}$$

$$\begin{aligned}\left|\frac{\cosh(\alpha(\omega))}{\sinh(\alpha(\omega))+\sin(\alpha(\omega))}\right| &\leq \frac{1}{M_0''}\left|\frac{\cosh(x_\omega)\cos(y_\omega)+i\sinh(x_\omega)\sin(y_\omega)}{\sinh(x_\omega)\cos(y_\omega)}\right|\\ &\leq \frac{1}{M_0''}[|\coth(x_\omega)|+|\tan(y_\omega)|] \leq M_1''\end{aligned} \tag{30}$$

for $|\omega| \geq \omega_0$. We note that $|\coth(x_\omega)|$ is uniformly bounded for $|\omega| \geq \omega_0$. Using the estimates (28), (29) and (30), from (23), we obtain

$$\begin{aligned}|C_{2,\omega}| &\leq \left|\frac{\cos(\alpha(\omega))}{\sinh(\alpha(\omega))+\sin(\alpha(\omega))}\right| \\ &\quad + \left|\frac{1}{2}+\frac{\sin(\alpha(\omega))\sinh(\alpha(\omega))}{2+2\cosh(\alpha(\omega))\cos(\alpha(\omega))}\right|\left|\frac{\cosh(\alpha(\omega))+\cos(\alpha(\omega))}{\sinh(\alpha(\omega))+\sin(\alpha(\omega))}\right| \\ &\lesssim \sqrt{|\omega|}\end{aligned} \tag{31}$$

for $|\omega| \geq \omega_0$. Again using (26), from (23), we can estimate

$$\begin{aligned}|C_{3,\omega}| &= \left|\frac{\cos(\alpha(\omega))\sinh(\alpha(\omega))-\sin(\alpha(\omega))\cosh(\alpha(\omega))}{2+2\cosh(\alpha(\omega))\cos(\alpha(\omega))}\right| \\ &\leq \left|\frac{\cos(\alpha(\omega))\sinh(\alpha(\omega))-\sin(\alpha(\omega))\cosh(\alpha(\omega))}{\cosh(\alpha(\omega))\cos(\alpha(\omega))}\right| \\ &\leq |\tanh(\alpha(\omega))| + |\tan(\alpha(\omega))| \\ &\leq |\tanh(x_\omega)| + |\coth(x_\omega)| + |\coth(y_\omega)| + |\tanh(y_\omega)|.\end{aligned}$$

Since $|\tanh(x_\omega)|$, $|\coth(x_\omega)|$ and $|\tanh(y_\omega)|$ are uniformly bounded and $|\coth(y_\omega)| \leq M_0'\sqrt{|\omega|}$ for $|\omega| \geq \omega_0$, we have

$$|C_{3,\omega}| \lesssim \sqrt{|\omega|} \tag{32}$$

for $|\omega| \geq \omega_0$. Finally, from the estimates (31), (32) and from equation (24) we obtain

$$\begin{aligned}\|P_b(i\omega)u_b\|^2 &\lesssim 16(EI)^2\left[\frac{|\alpha(\omega)|^6}{|\omega|^2}|\omega||u_{b1}|^2 + \frac{|\alpha(\omega)|^2}{|\omega|^2}|\omega||u_{b2}|^2\right], \\ &\lesssim (|\omega|+1)^2|u_b|^2\end{aligned}$$

for $|\omega| \geq \omega_0$. Hence $\|P_b(i\omega)\| \lesssim |\omega|+1$ for all $|\omega| \geq \omega_0$. Finally, by the continuity of the transfer function $P_b(\cdot)$ on $i\mathbb{R}$, we conclude that $\|P_b(i\omega)\| \lesssim |\omega|+1$ for all $\omega \in \mathbb{R}$. □

Lemma 3.5. *There exists $C' > 0$ such that $\|R(i\omega, A_{b_{-1}})B_b\| \leq C'\sqrt{|\omega|+1}$ for all $\omega \in \mathbb{R}$. Moreover, $I + P_b(i\omega)P_c(i\omega)$ is nonsingular for all $\omega \in \mathbb{R}\backslash\{0\}$.*

Proof. By using [21, Rem. 10.1.5], we have that for every $u_b \in U_b$, $i\omega \in \rho(A_{b_{-1}})$,

$$x_b = R(i\omega, A_{b_{-1}})B_b u_b = \begin{bmatrix} R(i\omega, A_{l_{-1}})B_l \\ R(i\omega, A_{r_{-1}})B_r \end{bmatrix} u_b \in D(\mathcal{A}_b)$$

where B_l, B_r and B_b are defined in Remark 2, is the unique solution of the abstract elliptic problem

$$\begin{aligned}(i\omega - \mathcal{A}_b)x_b &= 0, \\ \mathcal{B}_b x_b &= u_b.\end{aligned}$$

Assume that $|\omega| \geq 1$. Let us start by estimating the norm of $x_r = R(i\omega, A_{r_{-1}})B_r u_b$ which is the unique solution of $(i\omega - \mathcal{A}_r)x_r = 0$, $\mathcal{B}_r x_r = u_b$. If $x_r = (f_r, g_r)^T$, then using the expression for $\mathcal{A}_r$, we have

$$(i\omega + \gamma(\rho a)^{-1})f_r + EIg_r'' = 0, \tag{33}$$

$$-(\rho a)^{-1}f_r'' + i\omega g_r = 0, \tag{34}$$

$$f_r(0) = \rho a\ u_{b1},\ \ f_r'(0) = \rho a\ u_{b2}, \tag{35}$$

$$EIg_r(1) = 0,\ EIg_r'(1) = 0. \tag{36}$$

Taking L^2 inner product of (33) with $(\rho a)^{-1}f_r$ and L^2 inner product of (34) with EIg_r, respectively, we obtain

$$\begin{aligned}(\rho a)^{-1}(i\omega + \gamma(\rho a)^{-1})\|f_r\|_{L^2}^2 - EI(\rho a)^{-1}(\rho a)g_r'(0)u_{b1}& \\ -EI(\rho a)^{-1}\int_0^1 g_r'\bar{f}_r{}' d\xi = 0.&\end{aligned} \tag{37}$$

$$EI\bar{g_r}(0)u_{b2} + EI(\rho a)^{-1}\int_0^1 f_r'\bar{g}_r{}' d\xi + iEI\omega\|g_r\|_{L^2}^2 = 0. \tag{38}$$

Adding complex conjugate of (38) to (37), we obtain

$$(\rho a)^{-1}(i\omega + \gamma(\rho a)^{-1})\|f_r\|_{L^2}^2 - iEI\omega\|g_r\|_{L^2}^2 = \langle y_r, u_b\rangle\,. \tag{39}$$

Equating real and imaginary parts and using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\gamma(\rho a)^{-2}\|f_r\|_{L^2}^2 &\le \|P_r(i\omega)u_b\|\|u_b\| \\ EI\|g_r\|_{L^2}^2 &\le \left(\frac{\rho a}{\gamma} + \frac{1}{|\omega|}\right)\|P_r(i\omega)u_b\|\|u_b\|,\end{aligned}$$

where $P_r(\cdot)$ is the transfer function of the right beam system. Therefore,

$$\begin{aligned}\|x_r\|_{X_r}^2 &= (\rho a)^{-1}\|f_r\|_{L^2}^2 + EI\|g_r\|_{L^2}^2, \\ &\le \frac{\rho a}{\gamma}\|P_r(i\omega)u_b\|\|u_b\| + \left(\frac{\rho a}{\gamma} + \frac{1}{|\omega|}\right)\|P_r(i\omega)u_b\|\|u_b\|, \\ &\le \left(\frac{2\rho a}{\gamma} + 1\right)\|P_r(i\omega)u_b\|\|u_b\|.\end{aligned}$$

Since we have from Lemma 3.4 that $\|P_r(i\omega)\|$ can grow at most linearly, the above estimate implies that there exists $C_1 > 0$ such that $\|x_r\| = \|R(i\omega, A_{r_{-1}})B_r u_b\| \le C_1\sqrt{|\omega|+1}\|u_b\|,\ |\omega| \ge 1$. We can analogously show that there exists $C_2 > 0$ such that $\|R(i\omega, A_{l_{-1}})B_l u_b\| \le C_2\sqrt{|\omega|+1}\|u_b\|,\ |\omega| \ge 1$. Combining these estimates, we can see that $\|R(i\omega, A_{b_{-1}})B_b\| \lesssim \sqrt{|\omega|+1}$ for all $|\omega| \ge 1$. Finally, by continuity of $R(i\omega, A_{b_{-1}})B_b$ with respect to $i\omega$ on $i\mathbb{R}$, we have that $\|R(i\omega, A_{b_{-1}})B_b\| \lesssim \sqrt{|\omega|+1}$ for all $\omega \in \mathbb{R}$.

From equation (39), we observe that $\operatorname{Re} P_r(i\omega) > 0,\ \omega \in \mathbb{R}$. Indeed, from (39) we have

$$\operatorname{Re}\langle y_r, u_b\rangle = \operatorname{Re}\langle P_r(i\omega)u_b, u_b\rangle = \gamma(\rho a)^{-2}\|f_r\|_{L^2}^2.$$

Analogously, we have that $\operatorname{Re} P_l(i\omega) > 0,\ \omega \in \mathbb{R}$. This implies that $\operatorname{Re} P_b(i\omega) > 0,\ \omega \in \mathbb{R}$. In addition, from the transfer function

$$P_c(i\omega) = \frac{1}{i\omega}\begin{bmatrix}\frac{1}{m} & 0 \\ 0 & \frac{1}{I_m}\end{bmatrix},\ \omega \in \mathbb{R}\backslash\{0\} \tag{40}$$

of the rigid body, we see that $\operatorname{Re} P_c(i\omega) = 0,\ \omega \in \mathbb{R}\backslash\{0\}$. Consequently, we have that $I + P_b(i\omega)P_c(i\omega)$ is nonsingular for all $\omega \in \mathbb{R}\backslash\{0\}$. □

Lemma 3.6. *There exists $C'' > 0$ such that $\|C_b R(i\omega, A_b)\| \le C''\sqrt{|\omega|+1}$ for all $\omega \in \mathbb{R}$.*

Proof. First let us prove that $\|C_r R(i\omega, A_r)\|$, where $A_r = \mathcal{A}_r|_{\mathcal{N}(\mathcal{B}_r)}$, $C_r = \mathcal{C}_r|_{\mathcal{N}(\mathcal{B}_r)}$, grows at most at a rate of $\sqrt{|\omega|}$, $\omega \in \mathbb{R}$. Let us write A_r as bounded perturbation of a skew-adjoint operator. i.e., $A_r = A_0 + B_0$ where A_0 and B_0 are given as in Lemma 3.2 and $A_0^* = -A_0$, $D(A_0^*) = D(A_0)$ and $B_0^* = B_0$. Now, for the system (A_0, B_r, C_r), using duality between $D(A_0^*)$ and $X_{r_{-1}}$ (see [21, Sec. 2.10]), we have $B_r^* \in \mathcal{L}(D(A_0^*), U_r)$ is the adjoint of $B_r \in \mathcal{L}(U_r, X_{r_{-1}})$ in the sense that

$$\langle x_r, B_r u_r\rangle_{D(A_0^*),X_{r_{-1}}} = \langle B_r^* x_r, u_r\rangle_{U_r}, \quad x_r \in D(A_0^*),\ u_r \in U_r$$

and $A_{0_{-1}}$ is the adjoint of A_0^* in the sense that

$$\left\langle \psi_r, A_{0_{-1}} x_r\right\rangle_{D(A_0^*),X_{r_{-1}}} = \langle A_0^*\psi_r, x_r\rangle_{X_r}, \quad \psi_r \in D(A_0^*),\ x_r \in X_r.$$

Moreover, using [21, Rem.10.1.6], we have

$$\langle \mathcal{B}_r x_r, B_r^*\psi_r\rangle_{U_r} = \langle \mathcal{A}_0 x_r, \psi_r\rangle_{X_r} - \langle x_r, A_0^*\psi_r\rangle_{X_r}, \quad \psi_r \in D(A_0^*),\ x_r \in D(\mathcal{A}_0)$$

and by direct computation using integration by parts we obtain $B_r^* x_r = C_r x_r$ for $x_r \in D(A_0)$. Therefore, for all $x_r \in D(A_0)$, $u_r \in U_r$ and $i\omega \in \rho(A_0) \cap i\mathbb{R}$, we have

$$\begin{aligned}\left\langle x_r, R(i\omega, A_{0_{-1}})B_r u_r\right\rangle_{X_r} &= \left\langle R(\overline{i\omega}, A_0^*)x_r, B_r u_r\right\rangle_{D(A_0^*),X_{r_{-1}}}\\ &= \left\langle B_r^* R(\overline{i\omega}, A_0^*)x_r, u_r\right\rangle_{U_r},\\ &= -\left\langle C_r R(i\omega, A_0)x_r, u_r\right\rangle_{U_r}.\end{aligned}$$

Since $A_r = A_0 + B_0$ and $i\mathbb{R} \subset \rho(A_r)$, for $i\omega \in \rho(A_0) \cap i\mathbb{R}$, we obtain

$$\begin{aligned}&\left\langle x_r, R(i\omega, A_{r_{-1}})B_r u_r\right\rangle_{X_r}\\ &= \left\langle x_r, (I - R(i\omega, A_{0_{-1}})B_0)^{-1} R(i\omega, A_{0_{-1}})B_r u_r\right\rangle_{X_r},\\ &= \left\langle (I + B_0 R(i\omega, A_0))^{-1} x_r, R(i\omega, A_{0_{-1}})B_r u_r\right\rangle_{X_r},\\ &= -\left\langle C_r R(i\omega, A_0)(I + B_0 R(i\omega, A_0))^{-1} x_r, u_r\right\rangle_{U_r},\\ &= -\left\langle C_r R(i\omega, A_r)(I + 2B_0 R(i\omega, A_r))^{-1} x_r, u_r\right\rangle_{U_r}.\end{aligned}$$

Since $x_r \in X_r$ and $u_r \in U_r$ are arbitrary, we have

$$C_r R(i\omega, A_r) = -(R(i\omega, A_{r_{-1}})B_r)^*(I + 2B_0 R(i\omega, A_r)),\ i\omega \in \rho(A_0) \cap i\mathbb{R} \qquad (41)$$

where using Lemma 3.2, we have that $\sup_{\omega\in\mathbb{R}}\|I + 2B_0 R(i\omega, A_r)\| < \infty$. Since A_0 has discrete spectrum, the continuity of $R(i\omega, A_r)$, $C_r R(i\omega, A_r)$ and $R(i\omega, A_{r_{-1}})B_r$ with respect to $i\omega$ on $i\mathbb{R}$ imply that (41) holds for all $i\omega \in i\mathbb{R}$. Now, using Lemma 3.5, we have that there exists $C_0 > 0$ such that $\|C_r R(i\omega, A_r)\| \le C_0\sqrt{|\omega| + 1}$, $\omega \in \mathbb{R}$. We can analogously show that there exists $C_0' > 0$ such that $\|C_l R(i\omega, A_l)\| \le C_0'\sqrt{|\omega| + 1}$, $\omega \in \mathbb{R}$. Thus $\|C_b R(i\omega, A_b)\| \lesssim \sqrt{|\omega| + 1}$, $\omega \in \mathbb{R}$. □

Lemma 3.7. *Let $P_b(\cdot)$ and $P_c(\cdot)$ be the transfer functions of the beam system $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b)$ and the rigid body (A_c, B_c, C_c), respectively. Then there exist $\omega_0, \tilde{M} > 0$ such that $\|(I + P_b(i\omega)P_c(i\omega))^{-1}\| \le \tilde{M}$ for all $|\omega| \ge \omega_0$.*

Proof. From equation (24) in the proof of Lemma 3.4 and from equation (40) in the proof of Lemma 3.5, we have

$$I + P_b(i\omega)P_c(i\omega) = \begin{bmatrix} Q_1(\omega) & 0\\ 0 & Q_2(\omega)\end{bmatrix}, \quad \omega \in \mathbb{R}\backslash\{0\},$$

where

$$\begin{aligned}
Q_1(\omega) &= 1 - \frac{4EI}{m}\frac{\alpha(\omega)^3}{\omega^2}C_{2,\omega},\ Q_2(\omega) = 1 - \frac{4EI}{I_m}\frac{\alpha(\omega)}{\omega^2}C_{3,\omega},\\
\alpha(\omega) &= |\alpha(\omega)|\Bigg(\cos\left(\frac{\theta(\omega)+2\pi k}{4}\right) + i\sin\left(\frac{\theta(\omega)+2\pi k}{4}\right)\Bigg),\ k = 0,1,2,3,\\
|\alpha(\omega)| &= \left(\frac{\rho a}{EI}|\omega|\sqrt{\omega^2+\gamma^2(\rho a)^{-2}}\right)^{\frac{1}{4}},\ \theta(\omega) = \tan^{-1}(\frac{-\gamma(\rho a)^{-1}}{\omega}),
\end{aligned}$$

and $C_{2,\omega}$ and $C_{3,\omega}$ are defined in (23). We will show that there exist $\omega_0 > 0$ and $c_1, c_2 > 0$ such that $|Q_1(\omega)| > c_2$ and $|Q_2(\omega)| > c_1$ for all $|\omega| \geq \omega_0$. Since $|C_{2,\omega}|$ and $|C_{3,\omega}|$ have similar terms for all the four roots of $\alpha(\omega)$, we restrict our analysis to the principal branch of the fourth root of $\alpha(\omega)$ and analogous arguments can be used to show that the statement is also valid for the other roots of $\alpha(\omega)$.

We have from equation (32) that there exists $M_1, \omega_0 > 0$ such that $|C_{3,\omega}| \leq M_1\sqrt{|\omega|}$ for all $|\omega| \geq \omega_0$. Therefore, for $|\omega| \geq \omega_0$, we have that

$$|Q_2(\omega) - 1| = \left|\frac{4EI}{I_m}\frac{\alpha(\omega)}{\omega^2}C_{3,\omega}\right| \lesssim 4\frac{EI}{I_m}(\frac{\rho a}{EI})^{\frac{1}{4}}\frac{1}{|\omega|} \to 0$$

as $|\omega| \to \infty$. This implies that there exists $c_1 > 0$ such that $|Q_2(\omega)| > c_1$ for all $|\omega| \geq \omega_0$.

Now it remains to show that there exists $c_2 > 0$ such that $|Q_1(\omega)| \geq c_2$ for all $|\omega| \geq \omega_0$. We begin by showing that if we define $f(\omega) = \frac{2EI\alpha(\omega)^3}{m\omega^2}$ and $\tilde{Q}_1(\omega) = 1 + f(\omega)\tan(\alpha(\omega))$, then

$$\lim_{|\omega|\to\infty}|Q_1(\omega) - \tilde{Q}_1(\omega)| = 0. \tag{42}$$

This will imply that $|Q_1(\omega)|$ is uniformly bounded from below for $|\omega| \geq \omega_0$ if and only if the same is true for $|\tilde{Q}_1(\omega)|$. We have from equation (23) in Lemma 3.4 that

$$\begin{aligned}
C_{2,\omega} &= \frac{\cos(\alpha(\omega))}{\sinh(\alpha(\omega)) + \sin(\alpha(\omega))} - \frac{1}{2}(1 + C_{5,\omega})C_{6,\omega}\\
C_{5,\omega} &:= \frac{\sinh(\alpha(\omega))\sin(\alpha(\omega))}{1+\cosh(\alpha(\omega))\cos(\alpha(\omega))},\ C_{6,\omega} := \frac{\cosh(\alpha(\omega)) + \cos(\alpha(\omega))}{\sinh(\alpha(\omega)) + \sin(\alpha(\omega))}.
\end{aligned}$$

We have from equations (29) and (30) that $|\cos(\alpha(\omega))/(\sinh(\alpha(\omega)) + \sin(\alpha(\omega)))|$ and $|C_{6,\omega}|$ are uniformly bounded for $|\omega| \geq \omega_0$. Thus for all $|\omega| \geq \omega_0$, we have

$$\begin{aligned}
|Q_1(\omega) - \tilde{Q}_1(\omega)| &= |2f(\omega)C_{2,\omega} + f(\omega)\tan(\alpha(\omega))|\\
&\lesssim |f(\omega)| + |f(\omega)(C_{5,\omega}C_{6,\omega} - \tan(\alpha(\omega)))|\\
&\lesssim |f(\omega)| + |f(\omega)C_{5,\omega}(C_{6,\omega} - 1)|\\
&\quad + |f(\omega)\tan(\alpha(\omega))(\tanh(\alpha(\omega)) - 1)|\\
&\quad + |f(\omega)(C_{5,\omega} - \tanh(\alpha(\omega))\tan(\alpha(\omega)))|.
\end{aligned}$$

Using the definition of $\alpha(\omega)$, it is straightforward to show that $C_{6,\omega} \to 1$ and $\tanh(\alpha(\omega)) \to 1$ as $|\omega| \to \infty$. Moreover, as shown in (27) and (28), we have $|C_{5,\omega}| \lesssim \sqrt{|\omega|}$ and $|\tan(\alpha(\omega))| \leq |\coth(y_\omega)| + |\tanh(y_\omega)| \lesssim \sqrt{|\omega|}$ for $|\omega| \geq \omega_0$. Because of this, $|f(\omega)C_{5,\omega}|$ and $|f(\omega)\tan(\alpha(\omega))|$ are uniformly bounded for $|\omega| \geq \omega_0$, and therefore $|f(\omega)C_{5,\omega}(C_{6,\omega} - 1)| \to 0$ and $|f(\omega)\tan(\alpha(\omega))(\tanh(\alpha(\omega)) - 1)| \to 0$ as $|\omega| \to \infty$. Finally, the last term in the estimate for $|Q_1(\omega) - \tilde{Q}_1(\omega)|$ satisfies

$$|f(\omega)(C_{5,\omega} - \tanh(\alpha(\omega))\tan(\alpha(\omega)))|$$

$$= |f(\omega)|\left|\frac{\sinh(\alpha(\omega))\sin(\alpha(\omega))}{1+\cosh(\alpha(\omega))\cos(\alpha(\omega))} - \frac{\sinh(\alpha(\omega))\sin(\alpha(\omega))}{\cosh(\alpha(\omega))\cos(\alpha(\omega))}\right|$$
$$= |f(\omega)\tanh(\alpha(\omega))\tan(\alpha(\omega))|\left|\frac{1}{1+\cosh(\alpha(\omega))\cos(\alpha(\omega))}\right| \to 0$$

as $|\omega| \to \infty$, since $|f(\omega)\tanh(\alpha(\omega))\tan(\alpha(\omega))|$ is uniformly bounded for $|\omega| \geq \omega_0$, and $|\cosh(\alpha(\omega))\cos(\alpha(\omega))| \to \infty$ as $|\omega| \to \infty$. This finally shows that (42) holds.

We claim that there exists $c' > 0$ such that $|\tilde{Q}_1(\omega)| \geq c'$ for all $\omega \geq \omega_0$. The case where ω is negative can be proved analogously. We will use proof by contradiction. To this end we assume that no such $c' > 0$ exists. This implies that there exists a sequence $(\omega_k)_k \subset \mathbb{R}_+$ such that $\omega_k \to \infty$ as $k \to \infty$ and $|\tilde{Q}_1(\omega_k)| \to 0$ as $k \to \infty$. Separating real and imaginary parts of $\tilde{Q}_1(\omega_k)$ and denoting $x_k = \operatorname{Re}\alpha(\omega_k)$, $y_k = \operatorname{Im}\alpha(\omega_k)$, $R_{1,k} = \operatorname{Re}(f(\omega_k))\sin x_k$, $R_{2,k} = \operatorname{Re}(f(\omega_k))\cosh y_k$, $I_{1,k} = \operatorname{Im}(f(\omega_k))\cosh y_k$, $I_{2,k} = \operatorname{Im}(f(\omega_k))\sin x_k$, we obtain

$$\tilde{Q}_1(\omega_k) = 1 + \frac{R_{1,k}\cos x_k - I_{1,k}\sinh y_k}{\cos^2 x_k + \sinh^2 y_k} + i\frac{R_{2,k}\sinh y_k + I_{2,k}\cosh x_k}{\cos^2 x_k + \sinh^2 y_k}.$$

Since we consider the principal branch of the fourth root of $\alpha(\omega_k)$, we have that there exist $m_1, m_2, m_3, m_4 > 0$ and $N_1 \in \mathbb{N}$ such that

$$m_1\sqrt{\omega_k} \leq |x_k| \leq m_2\sqrt{\omega_k}$$
$$\frac{m_3}{\sqrt{\omega_k}} \leq |y_k| \leq \frac{m_4}{\sqrt{\omega_k}}$$

for all $k \geq N_1$. This implies that there exist $m_5, m_6 > 0$ and $N_2 \geq N_1$ such that $m_5\omega_k^{-1/2} \leq |\sinh y_k| \leq m_6\omega_k^{-1/2}$ for all $k \geq N_2$. Since $y_k \to 0$, we have $\cosh y_k \to 1$ as $k \to \infty$, and thus there exist $m_7, m_8, m_9, m_{10} > 0$ and $N_3 \geq N_2$ such that $m_7\omega_k^{-1/2} \leq |R_{2,k}| \leq m_8\omega_k^{-1/2}$ and $m_9\omega_k^{-3/2} \leq |I_{1,k}| \leq m_{10}\omega_k^{-3/2}$ for all $k \geq N_3$.

We will first show that $|\cos x_k| \to 0$ as $k \to \infty$. Indeed, we have

$$|R_{1,k}\cos x_k - I_{1,k}\sinh y_k| \leq |\operatorname{Re}(f(\omega_k))| + |\operatorname{Im}(f(\omega_k))||\sinh y_k||\cosh y_k|$$
$$\lesssim \frac{1}{\sqrt{\omega_k}}$$

for all $k \geq N_3$ and since the assumption $|\tilde{Q}_1(\omega_k)| \to 0$ implies $\operatorname{Re}\tilde{Q}_1(\omega_k) \to 0$, we must have $\cos^2 x_k + \sinh^2 y_k \to 0$ as $k \to \infty$. Thus $|\cos x_k| \to 0$ as $k \to \infty$, and consequently also $|\sin x_k| \to 1$ as $k \to \infty$. This further implies that there exist $m_{11}, m_{12}, m_{13}, m_{14} > 0$ and $N_4 \geq N_3$ such that $m_{11}\omega_k^{-1/2} \leq |R_{1,k}| \leq m_{12}\omega_k^{-1/2}$ and $m_{13}\omega_k^{-3/2} \leq |I_{2,k}| \leq m_{14}\omega_k^{-3/2}$ for all $k \geq N_4$. We consider the following cases.

Case 1 (fast decay of $|\cos x_k|$): Consider the subsequence of (ω_k) consisting of those elements ω_k which satisfy $|\cos x_k| \leq 1/\omega_k$. Then we have

$$\left|\frac{R_{1,k}\cos x_k - I_{1,k}\sinh y_k}{\cos^2 x_k + \sinh^2 y_k}\right| \leq \frac{|R_{1,k}\cos x_k - I_{1,k}\sinh y_k|}{\sinh^2 y_k}$$
$$\lesssim \frac{|\cos x_k|/\sqrt{\omega_k} + 1/\omega_k^2}{1/\omega_k} \lesssim \frac{1}{\sqrt{\omega_k}} + \frac{1}{\omega_k}$$

for all $k \geq N_4$. However, this implies $\tilde{Q}_1(\omega_k) \not\to 0$ as $k \to \infty$, since $\operatorname{Re}\tilde{Q}_1(\omega_k) \to 1$. This implies that the subsequence of $(\omega_k)_k$ consisting of elements such that $|\cos x_k| \leq 1/\omega_k$ must have at most finite number of elements.

Case 2 (slow decay of $|\cos x_k|$): As shown above, we necessarily have there exist $N_5 \geq N_4$ such that $|\cos x_k| > 1/\omega_k$ for all $k \geq N_5$, and we will now restrict our attention to this range of the indices k. Then

$$\left|\frac{R_{1,k}\cos x_k - I_{1,k}\sinh y_k}{R_{1,k}\cos x_k}\right| \to 1 \text{ and } \left|\frac{R_{2,k}\sinh y_k + I_{2,k}\cos x_k}{R_{2,k}\sinh y_k}\right| \to 1$$

as $k \to \infty$. In addition, for $k \geq N_5$, we have

$$\begin{aligned}|\mathrm{Im}(\tilde{Q}_1(w_k))| &= \frac{|R_{2,k}\sinh y_k + I_{2,k}\cos x_k|}{\cos^2 x_k + \sinh^2 y_k} \\ &= \left|\frac{R_{2,k}\sinh y_k + I_{2,k}\cos x_k}{R_{2,k}\sinh y_k}\right| \cdot \frac{|R_{2,k}\sinh y_k|}{\cos^2 x_k + \sinh^2 y_k} \\ &\gtrsim \frac{1/\omega_k}{\cos^2 x_k + \sinh^2 y_k}\end{aligned}$$

and

$$\begin{aligned}|\mathrm{Re}(\tilde{Q}_1(w_k)) - 1| &= \frac{|R_{1,k}\cos x_k - I_{1,k}\sinh y_k|}{\cos^2 x_k + \sinh^2 y_k} \\ &= \left|\frac{R_{1,k}\cos x_k - I_{1,k}\sinh y_k}{R_{1,k}\cos x_k}\right| \cdot \frac{|R_{1,k}\cos x_k|}{\cos^2 x_k + \sinh^2 y_k} \\ &\gtrsim \frac{|\cos x_k|/\sqrt{\omega_k}}{\cos^2 x_k + \sinh^2 y_k}.\end{aligned}$$

Using these estimates, we have that

$$\frac{1}{|\cos x_k|\sqrt{\omega_k}} = \frac{1/\omega_k}{\cos^2 x_k + \sinh^2 y_k} \cdot \frac{\cos^2 x_k + \sinh^2 y_k}{|\cos x_k|/\sqrt{\omega_k}} \to 0$$

as $k \to \infty$ since $|\mathrm{Re}\,\tilde{Q}_1(\omega_k) - 1| \to 1$ and $|\mathrm{Im}\,\tilde{Q}_1(\omega_k)| \to 0$ as $k \to \infty$. Because of this we also have

$$\left|\frac{\cos^2 x_k + \sinh^2 y_k}{\cos^2 x_k} - 1\right| = \frac{\sinh^2 y_k}{\cos^2 x_k} \lesssim \frac{1}{(\sqrt{\omega_k}\cos x_k)^2} \to 0$$

as $k \to \infty$. Finally, using this property we have that for all $k \geq N_5$

$$\begin{aligned}\left|\frac{R_{1,k}\cos x_k - I_{1,k}\sinh y_k}{\cos^2 x_k + \sinh^2 y_k}\right| &= \left|\frac{\cos^2 x_k}{\cos^2 x_k + \sinh^2 y_k}\right| \cdot \left|\frac{R_{1,k}\cos x_k - I_{1,k}\sinh y_k}{\cos^2 x_k}\right| \\ &\lesssim \frac{1}{|\cos x_k|\sqrt{\omega_k}} + \frac{1/\omega_k}{(\cos x_k\sqrt{\omega_k})^2}\end{aligned}$$

decays to zero as $k \to \infty$. However, this implies that $\mathrm{Re}\,\tilde{Q}_1(\omega_k) \to 1 \neq 0$ as $k \to \infty$ which contradicts the assumption that $|\tilde{Q}_1(\omega_k)| \to 0$ as $k \to \infty$. Hence there exists $c' > 0$ such that $|\tilde{Q}_1(\omega)| \geq c'$ for all $\omega \geq \omega_0$.

Finally, we have that there exist $\omega_0, c_1, c_2 > 0$ such that $|Q_1(\omega)| > c_2$ and $|Q_2(\omega)| > c_1$ for all $|\omega| \geq \omega_0$. This implies that $\|(I + P_b(i\omega)P_c(i\omega))^{-1}\|^2 \leq \frac{1}{c_2^2} + \frac{1}{c_1^2}$ for all $|\omega| \geq \omega_0$, which completes the proof. □

Having the above results, now we are ready to prove the main theorem.

Proof of Theorem 3.1. From Lemmas 3.4 and 3.5, we have that $P_b(0)$ and $I + P_b(i\omega)P_c(i\omega)$, $\omega \in \mathbb{R}\backslash\{0\}$ are nonsingular. These properties in Lemma 3.3 imply that the resolvent $R(i\omega, A)$ exists for all $\omega \in \mathbb{R}$ and is given by the equations

(19), (20) and (16). Therefore

$$\begin{aligned}\|R(i\omega, A)\|^2 \leq{}& \|R(i\omega, A_b) - R(i\omega, A_{b_{-1}})B_bC_cS(i\omega)B_cC_bR(i\omega, A_b)\|^2 \\ &+ \|R(i\omega, A_{b_{-1}})B_bC_cS(i\omega)\|^2 + \|S(i\omega)B_cC_bR(i\omega, A_b)\|^2 + \|S(i\omega)\|^2\end{aligned} \tag{43}$$

where $S(i\omega) = \frac{1}{i\omega} + \frac{1}{\omega^2}B_cP_b(i\omega)(I + P_b(i\omega)P_c(i\omega))^{-1}C_c$ for $\omega \in \mathbb{R}\backslash\{0\}$ and $S(0) = (B_cP_b(0)C_c)^{-1}$.

From Lemma 3.4, we have that there exists $M > 0$ such that $\|P_b(i\omega)\| \leq M(|\omega|+1)$ for all $\omega \in \mathbb{R}$. From Lemma 3.7, we have that there exist $\omega_0, \tilde{M} > 0$ such that $\|(I + P_b(i\omega)P_c(i\omega))^{-1}\| \leq \tilde{M}$ for all $|\omega| \geq \omega_0$. Moreover, from Lemma 3.2, we have that $\|R(i\omega, A_b)\|$ is uniformly bounded and from Lemmas 3.5 and 3.6, we have that there exist $C', C'' > 0$ such that $\|R(i\omega, A_{b_{-1}})B_b\| \leq C'\sqrt{|\omega| + 1}$ and $\|C_bR(i\omega, A_b)\| \leq C''\sqrt{|\omega| + 1}$ for all $\omega \in \mathbb{R}$. These estimates imply that there exists $M_1 > 0$ such that $\|S(i\omega)\| \leq M_1$ for all $|\omega| \geq \omega_0$ and this further from equation (43) implies that there exists $M_0 > 0$ such that $\|R(i\omega, A)\| \leq M_0$ for all $|\omega| \geq \omega_0$. Since from Lemma 3.3 we have $i\mathbb{R} \subset \rho(A)$, we conclude that $R(i\omega, A)$ is uniformly bounded, which completes the proof. □

4. **Robust output regulation of the satellite model.** In this section, we present two controllers that solve the robust output regulation problem for the satellite system. We start by formulating the robust output regulation problem followed by the controllers that achieve the robust output tracking of the given reference signals. In addition, we present simulation results demonstrating the effectiveness of the controllers.

From the previous sections, the satellite system with control and observations on the rigid body is given by,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + B_dw_d(t), \\ y(t) &= Cx(t).\end{aligned} \tag{44}$$

with $A = \begin{bmatrix} \mathcal{A}_b & 0 \\ -B_c\mathcal{C}_b & 0 \end{bmatrix}$, $D(A) = \{(x_b, x_c) \in D(\mathcal{A}_b) \times X_c \ : \ \mathcal{B}_bx_b = C_cx_c\}$, $B = \begin{bmatrix} 0 \\ B_c \end{bmatrix}$, $B_d = \begin{bmatrix} B_{d0} & 0 \\ 0 & B_c \end{bmatrix}$, $C = \begin{bmatrix} 0 & C_c \end{bmatrix}$, $x(t) = \begin{bmatrix} x_b(t) \\ x_c(t) \end{bmatrix}$. Here the operator A generates an exponentially stable semigroup.

The reference signals to be tracked and the disturbance signals to be rejected are of the form

$$y_{ref}(t) = a_0 + \sum_{k=1}^{q} [a_k \cos(\omega_kt) + b_k \sin(\omega_kt)], \tag{45}$$

$$w_d(t) = c_0 + \sum_{k=1}^{q} [c_k \cos(\omega_kt) + d_k \sin(\omega_kt)], \tag{46}$$

where $0 < \omega_1 < \omega_2 < \cdots < \omega_q$ are known frequencies and $\{a_k\}_{k=0}^q, \{b_k\}_{k=1}^q$, $\{c_k\}_{k=0}^q$, $\{d_k\}_{k=1}^q$ are possibly unknown constant coefficients.

We construct a dynamic error feedback controller of the form

$$\begin{aligned}\dot{z}(t) &= \mathcal{G}_1z(t) + \mathcal{G}_2e(t), \quad z(0) = z_0, \\ u(t) &= Kz(t) - \kappa e(t),\end{aligned} \tag{47}$$

on a Hilbert space Z, where $e(t) = y(t) - y_{ref}(t)$ is the regulation error, $y_{ref}(t)$ a given reference signal, $\mathcal{G}_1 : D(\mathcal{G}_1) \subset Z \to Z$ generates a strongly continuous semigroup on Z, $\mathcal{G}_2 \in \mathcal{L}(\mathbb{R}^2, Z)$, $K \in \mathcal{L}(Z, \mathbb{R}^2)$ and $\kappa \in \mathbb{R}^{2\times 2}$, such that robust output regulation of the satellite system is achieved with a suitable choice of the parameters $(\mathcal{G}_1, \mathcal{G}_2, K, \kappa)$.

Let us denote $X_e = X \times Z$ to be the extended state space and $x_e(t) = (x(t), z(t))^T$ be the extended state. Then the closed-loop system containing the satellite system (44) and the controller (47) is given by

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e u_e(t), \quad x_e(0) = x_{e0}, \\ e(t) &= C_e x_e(t) + D_e u_e(t), \end{aligned} \tag{48}$$

where $A_e = \begin{bmatrix} A - B\kappa C & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 \end{bmatrix}$, $B_e = \begin{bmatrix} B_d & B\kappa \\ 0 & -\mathcal{G}_2 \end{bmatrix}$, $C_e = \begin{bmatrix} C & 0 \end{bmatrix}$, $D_e = \begin{bmatrix} 0 & -I_Y \end{bmatrix}$ and $u_e(t) = \begin{bmatrix} w_d(t) \\ y_{ref}(t) \end{bmatrix}$. The operator A_e generates a strongly continuous semigroup $T_e(t)$ on X_e.

The Robust Output Regulation Problem. Choose the controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K, \kappa)$ in such a way that

(a) The closed-loop semigroup $T_e(t)$ generated by A_e is exponentially stable.

(b) There exists $\alpha_1 > 0$ such that for all initial states $x_{e0} \in X_e$, for the reference signal of the form (45) and for the disturbance signal of the form (46), the regulation error $e(t)$ satisfies

$$e^{\alpha_1 t}\|y(t) - y_{ref}(t)\| \to 0 \quad \text{as } t \to \infty.$$

(c) If the operators $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b, A_c, B_c, C_c)$ are perturbed in such a way that the perturbed closed-loop system is exponentially stable, the perturbed $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b)$ is an impedance passive boundary control system and the perturbed (A_c, B_c, C_c) is an impedance passive systems, then (b) continues to hold for some $\tilde{\alpha}_1 > 0$.

Remark 5. In the above, α_1 and $\tilde{\alpha}_1$ are determined by the stability margins of the closed-loop system and the perturbed closed-loop system, respectively.

Next, we show that the transfer function $P(i\omega)$ of the satellite system is nonsingular for all $\omega \in \mathbb{R}$. Because of this, we can track signals containing components at all frequencies ω.

Lemma 4.1. *On the imaginary axis, the transfer function of the satellite system (44) has the form $P(i\omega) = C_c S(i\omega) B_c$ and it is nonsingular for all $\omega \in \mathbb{R}$.*

Proof. The transfer function of (44) on the imaginary axis is given by $P(i\omega) = CR(i\omega, A)B$, where $R(i\omega, A)$ is the resolvent in (19) of A. Replacing the operators by their expressions, we obtain

$$\begin{aligned} P(i\omega) &= \begin{bmatrix} 0 & C_c \end{bmatrix} \begin{bmatrix} R_{11}(i\omega) & R_{12}(i\omega) \\ R_{21}(i\omega) & R_{22}(i\omega) \end{bmatrix} \begin{bmatrix} 0 \\ B_c \end{bmatrix} \\ &= C_c R_{22}(i\omega) B_c \\ &= C_c S(i\omega) B_c. \end{aligned}$$

Since B_c, C_c and $S(i\omega)$, $\omega \in \mathbb{R}$ are nonsingular, we have that $P(i\omega)$ is nonsingular for all $\omega \in \mathbb{R}$. □

4.1. **Robust controllers for the satellite system.** In this section, we present two internal model based controllers for the robust output regulation of the satellite system.

4.1.1. *A passive controller for the satellite model.* We have that the satellite system is (44) an impedance passive system and exponentially stable. Therefore, based on [20, Thm. 1.2] and [18, Def. 5.1], we can construct a passive controller for the robust output tracking of the given sinusoidal reference signals. We choose $Z = (\mathbb{R}^2)^{2q+1}$,

$$\begin{aligned}
&\mathcal{G}_1 = \operatorname{diag}(G_0, G_1, G_2, \cdots, G_q),\\
&G_0 = 0_Y,\ G_k = \begin{bmatrix} 0 & \omega_k I_Y \\ -\omega_k I_Y & 0 \end{bmatrix}, \quad k = 1, 2, \cdots, q,\\
&\mathcal{G}_2 = (\mathcal{G}_2^k)_{k=0}^q,\ \mathcal{G}_2^0 = -I_Y,\ \mathcal{G}_2^k = -c_1 \begin{bmatrix} I_Y \\ 0 \end{bmatrix},\ k = 1, 2, \cdots, q,\\
&K = -\mathcal{G}_2^*, \text{ and } \kappa = c_2 I_Y,
\end{aligned} \tag{49}$$

where $c_1, c_2 > 0$ affect the stability properties of the closed-loop system.

Theorem 4.2. *The controller* (47) *with the choices of parameters in* (49) *solves the robust output regulation problem for the satellite model.*

Proof. We have that the satellite system (44) is impedance passive and exponentially stable and the choices of parameters in (49) are adopted from [18, Def. 5.1]. Therefore, by [18, Thm. 5.2], the controller (47), (49) solves the robust output regulation problem. □

We note that the controller (47), (49) is the one given in [20, Thm. 1.2] when c_1 and c_2 are chosen such that (47), (49) is a minimal realization of

$$C(s) = -C_0 - \sum_{k=-q}^{q} \frac{I_Y}{s - i\omega_k}, \tag{50}$$

where $C_0 \geq \frac{1}{2} I_Y$ and $\omega_{-k} = -\omega_k$. The assumption $\operatorname{Re} P(i\omega_k)$ is nonsingular for all $k = 0, 1, 2, \cdots q$ in [20, Thm. 1.2] can be relaxed due to the fact that the feedthrough operator κ of the controller satisfies $\kappa > 0$ (see [18, sec. 5] for more details).

4.1.2. *An observer based controller for the satellite model.* Since the input operator B and the output operator C are bounded, we can construct an observer based controller based on [12] and [17, Sec. VI] for robust output tracking of the satellite system as follows.

We choose the state space of the controller as $Z = Z_0 \times X$, where $Z_0 = (\mathbb{R}^2)^{2q+1}$. The controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K, \kappa)$ of the dynamic error feedback controller (47) are given by,

$$\mathcal{G}_1 = \begin{bmatrix} G_1 & 0 \\ BK_1 & A + BK_2 \end{bmatrix},\ \mathcal{G}_2 = \begin{bmatrix} G_2 \\ 0 \end{bmatrix},\ K = \begin{bmatrix} K_1 & K_2 \end{bmatrix},\ \kappa = 0,$$

where $K_1 \in \mathcal{L}(Z_0, \mathbb{R}^2), K_2 \in \mathcal{L}(X, \mathbb{R}^2)$. The operators (G_1, G_2) are defined as

$$\begin{aligned}
&G_1 = \operatorname{diag}(i\omega_{-q} I_Y, \cdots i\omega_0 I_Y, \cdots, i\omega_q I_Y) \in \mathcal{L}(Z_0),\\
&G_2 = (G_2^k)_{k=-q}^q \in \mathcal{L}(\mathbb{R}^2, Z_0),\ G_2^k = I_Y,\ k = -q, \cdots, q.
\end{aligned}$$

We define an operator $H \in \mathcal{L}(X, Z_0)$ by $H = (H_k)_{k=-q}^q$ which is the solution of the Sylvester equation $G_1 H = HA + G_2 C$ and H_k can be obtained by solving the system

$$H_k = G_2^k C R(i\omega_k, A). \tag{51}$$

Then we define $B_1 = HB = (G_2^k P(i\omega_k))_{k=-q}^q \in \mathcal{L}(\mathbb{R}^2, Z_0)$. Finally, we choose $K_1 \in \mathcal{L}(Z_0, \mathbb{R}^2)$ in such a way that $G_1 + B_1 K_1 \in \mathcal{L}(Z_0)$ is Hurwitz and we define $K_2 = K_1 H$.

With the above parameters, the controller (47) can be written as,

$$\dot{z}_1(t) = G_1 z_1(t) + G_2 e(t), \tag{52}$$

$$\dot{z}_2(t) = BK_1 z_1(t) + (A + BK_2) z_2(t), \tag{53}$$

$$u(t) = Kz(t). \tag{54}$$

Here $z_1(t) \in Z_0,\ z_2(t) \in X(= X_b \times X_c)$. Equation (52) is the servocompensator on the state space Z_0 which contains internal model and it is an ODE system by construction. Equation (53) is an observer for the satellite system on the state space X and is given by,

$$\begin{aligned}
&\dot{\hat{x}}_{l1}(\xi,t) = -\gamma(\rho a)^{-1}\hat{x}_{l1}(\xi,t) - EI\hat{x}''_{l2}(\xi,t),\ -1 < \xi < 0,\\
&\dot{\hat{x}}_{l2}(\xi,t) = (\rho a)^{-1}\hat{x}''_{l1}(\xi,t),\ -1 < \xi < 0,\\
&\dot{\hat{x}}_{r1}(\xi,t) = -\gamma(\rho a)^{-1}\hat{x}_{r1}(\xi,t) - EI\hat{x}''_{r2}(\xi,t),\ 0 < \xi < 1,\\
&\dot{\hat{x}}_{r2}(\xi,t) = (\rho a)^{-1}\hat{x}''_{r1}(\xi,t),\ 0 < \xi < 1,\\
&\dot{\hat{x}}_{c1}(t,0) = EI\hat{x}'_{l2}(\xi,t)|_{\xi=0} - EI\hat{x}'_{r2}(\xi,t)|_{\xi=0} + u_1(t),\\
&\dot{\hat{x}}_{c2}(t,0) = -EI\hat{x}_{l2}(\xi,t)|_{\xi=0} + EI\hat{x}_{r2}(\xi,t)|_{\xi=0} + u_2(t),\\
&\hat{x}_{r2}(1,t) = \hat{x}'_{r2}(1,t) = 0,\ \hat{x}_{l2}(-1,t) = \hat{x}'_{l2}(-1,t) = 0,\\
&\hat{x}_{l1}(0,t) = \hat{x}_{r1}(0,t) = \hat{x}_{c1}(t),\ \hat{x}'_{l1}(0,t) = \hat{x}'_{r1}(0,t) = \hat{x}_{c2}(t),
\end{aligned}$$

where $\hat{x}_{l1}(\xi,t), \hat{x}_{l2}(\xi,t), \hat{x}_{r1}(\xi,t), \hat{x}_{r2}(\xi,t), \hat{x}_{c1}(\xi,t)$ and $\hat{x}_{c2}(\xi,t)$ are the estimates of $x_{l1}(\xi,t)$, $x_{l2}(t)$, $x_{r1}(\xi,t)$, $x_{r2}(\xi,t)$, $x_{c1}(\xi,t)$ and $x_{c2}(\xi,t)$, respectively, and $z_2(t)$ is given by $z_2(t) = (\hat{x}_{l1}(\cdot,t), \hat{x}_{l2}(\cdot,t), \hat{x}_{r1}(\cdot,t), \hat{x}_{r2}(\cdot,t), \hat{x}_{c1}(\cdot,t), \hat{x}_{c2}(\cdot,t))^T$. This shows that the controller (47) is a PDE-ODE system.

Theorem 4.3. *The controller* (47) *with the above choices of parameters solves the robust output regulation problem for the satellite system* (44).

Proof. Since the construction of the controller (47) with the above choices of parameters is adopted from [12, Sec. 7] and [17, Sec. VI], based on [17, Thm. 15], the controller solves the robust output regulation problem for the satellite system (44). □

4.2. **Robustness of closed-loop stability.** In the case of the passive controller, the controller parameters $\mathcal{G}_2$, K and κ depend on the parameters c_1, c_2 and therefore the closed-loop stability margin α_1 depends on the choice of the parameters c_1 and c_2. On the other hand, for the observer based controller the closed-loop stability margin is determined by the minimum of stability margins of A and $G_1 + B_1K_1$, respectively, see the proof of [17, Thm. 15] for more details. The stability margin of $G_1 + B_1K_1$ can be affected by adjusting the gain parameter K_1. This can be done for example by linear quadratic regulator design or pole placement.

From Section 4.1 we have that both controllers with suitable choices of parameters solve the robust output regulation problem. Therefore, A_e generates an exponentially stable semigroup $T_e(t)$ and there exist $\alpha_1 > 0$ and $M_e \geq 1$ depending on the controller and the chosen parameters such that $\|T_e(t)\| \leq M_e e^{-\alpha_1 t}$. If $\Delta \in \mathcal{L}(X_e)$ is a perturbation of A_e, where the perturbation is generated by the perturbations in $(\mathcal{A}_b, \mathcal{B}_b, \mathcal{C}_b, A_c, B_c, C_c)$, such that $\|\Delta\| < \alpha_1/M_e$, then $A_e+\Delta$ generates

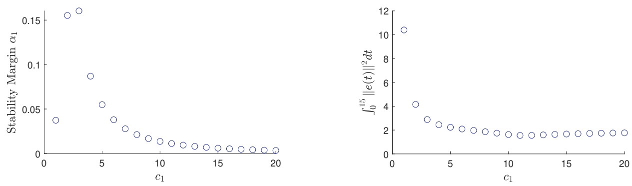

FIGURE 4. The closed-loop stability margin and $\int_0^{15} \|e(t)\|^2 dt$ for the passive controller with $c_2 = 4$

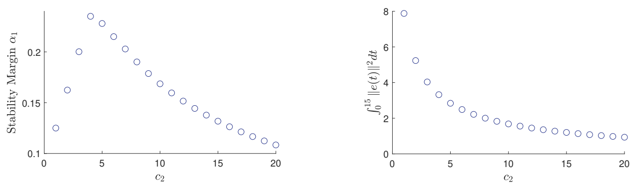

FIGURE 5. The closed-loop stability margin and $\int_0^{15} \|e(t)\|^2 dt$ for the passive controller with $c_1 = 2.5$

an exponentially stable semigroup $\tilde{T}_e(t)$ on X_e and $\|\tilde{T}_e(t)\| \leq M_e e^{(-\alpha_1 + M_e\|\Delta\|)t}$ for all $t \geq 0$. Therefore the stability margin $\tilde{\alpha}_1$ of the perturbed semigroup $\tilde{T}_e(t)$ satisfies $\tilde{\alpha}_1 \geq \alpha_1 - M_e\|\Delta\|$. In addition, the semigroup $T_e(t)$ may remain exponentially stable under perturbations with large norms in which cases the decay rates cannot be estimated explicitly by using the perturbation formula.

4.3. **Simulations.** Simulations are carried out in Matlab using passive and observer based controllers on the time interval $t = [0, 15]$. We choose $m = 1, I_m = 1, E = 1, I = 1, \rho = 1, a = 1$ and $\gamma = 5$. We track the reference signal $y_{ref}(t) = \begin{bmatrix} 1 + 3\cos(t) & 2 - \sin(5t) + 1.5\cos(2t) \end{bmatrix}^T$ and reject the disturbance signal $w_d(t) = \begin{bmatrix} 0 & 0 & 10 & 15 \end{bmatrix}^T$. Thus, the frequencies $\{\omega_k\}_{k=0}^q$ with $q = 3$ are $\{0, 1, 2, 5\}$. We choose the controller initial state as $z_0 = 0$ and the initial state for the satellite system as $x_0 = \begin{bmatrix} 0 & 4(1+\xi)^2 & 0 & 4(1-\xi)^2 & 0 & 0 \end{bmatrix}^T$. The solutions of the satellite system are approximated using Legendre spectral Galerkin method [2]. The number of basis functions used for the approximation is $N = 10$.

The controller parameters of the passive controller are chosen as in Section 4.1.1. To maximize the stability margin, ranges of values of the parameters c_1 and c_2 were tested. The closed-loop stability margin α_1 and $\int_0^{15} \|e(t)\|^2 dt$ for different parameter values are plotted in Figures 4 and 5, respectively. The figures indicate that smaller values of c_1 and c_2 result in larger closed-loop stability margin and larger transient errors. By choosing $c_1 = 2.5$ and $c_2 = 4$, the output tracking and the tracking errors are depicted in Figures 8 and 10, respectively.

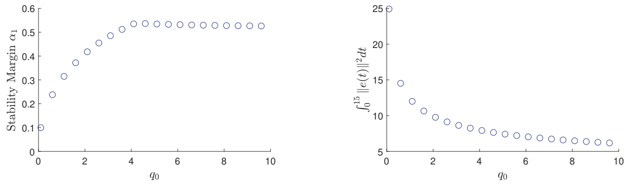

FIGURE 6. The closed-loop stability margin and $\int_0^{15} \|e(t)\|^2 dt$ for the observer based controller with $R = 0.1 I_2$

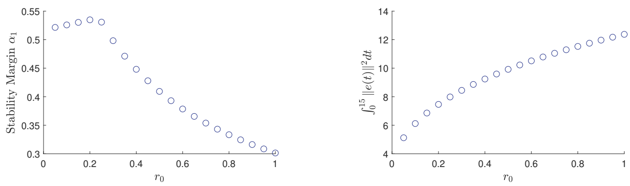

FIGURE 7. The closed-loop stability margin and $\int_0^{15} \|e(t)\|^2 dt$ for the observer based controller with $Q = 10 I_{Z_0}$

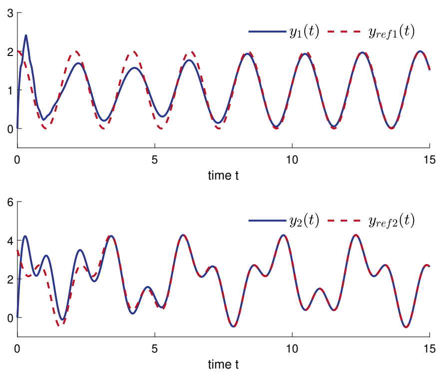

FIGURE 8. Output tracking using a passive controller

The components of the observer based controller are chosen as in Section 4.1.2. The matrix H is obtained by solving the system (51), where we use the approximations A^N and C^N in place of A and C, respectively. The gain matrix K_1 is obtained using Matlab lqr function with $Q = q_0 I_{Z_0},\ q_0 > 0$ and $R = r_0 I_2, r_0 > 0$. To maximize the stability margin, ranges of values of the parameters q_0 and r_0 were tested.

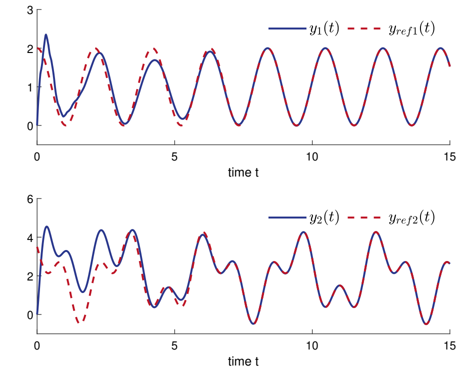

FIGURE 9. Output tracking using an observer based controller

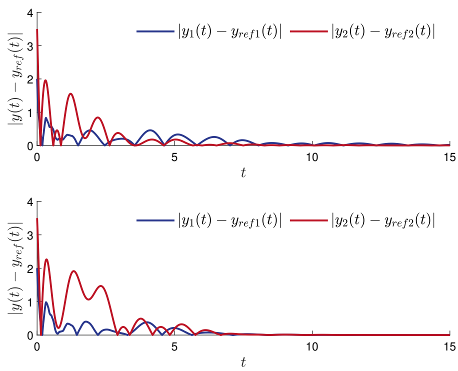

FIGURE 10. Tracking errors for passive(above) and observer based(below) controllers

The closed-loop stability margin α_1 and $\int_0^{15} \|e(t)\|^2 dt$ for different parameter values are plotted in Figures 6 and 7, respectively. It is observed that smaller control gains r_0 and larger q_0 result in larger closed-loop stability margin and smaller transient errors. By choosing $q_0 = 10$ and $r_0 = 0.1$, the output tracking and the tracking errors are depicted in Figures 9 and 10, respectively.

It can be seen from the figures that both controllers achieve tracking of the given reference signals asymptotically and the tracking error decays to zero at an exponential rate. Moreover, we can see that the observer based controller can achieve larger closed-loop stability margin and therefore the asymptotic error convergence for the observer based controller is faster than that for the passive controller. On the other hand, it is noted that even though the passive controller is a finite-dimensional controller and also the controller requires no information about the satellite system apart from passivity, it still achieves comparable performance to the infinite-dimensional observer based controller.

5. **Conclusion.** We investigated robust output tracking problem of a flexible satellite composed of two identical flexible solar panels and a center rigid body. A detailed proof of exponential stability of the model was presented. We constructed two robust controllers for the robust output tracking of the satellite model. Moreover, simulation results showing the performances of the controllers were presented.

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PUBLICATION IV

Saturated Output Regulation of Distributed Parameter Systems with Collocated Actuators and Sensors

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Saturated Output Regulation of Distributed Parameter Systems with Collocated Actuators and Sensors ⋆

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Abstract: This paper addresses the problem of output regulation of infinite-dimensional linear systems subject to input saturation. We focus on strongly stabilizable linear dissipative systems with collocated actuators and sensors. We generalize the output regulation theory for finite-dimensional linear systems subject to input saturation to the class of considered infinite-dimensional linear systems. The theoretic results are illustrated with an example where we consider the output regulation of a flexible satellite model that is composed of two identical flexible solar panels and a center rigid body.

Keywords: Distributed parameter systems, output regulation, input saturation, strongly stabilizable systems, collocated input and output.

1. INTRODUCTION

For the past few decades, there has been interest in studying linear systems subject to input saturation due to limitations on the control input. Stabilization and output regulation of such systems have been studied, for example, in Fuller (1969), Sontag and Sussmann (1990), Teel (1992), Saberi et al. (2003), Logemann et al. (1998), Slemrod (1989), Oostveen (2000), Lasiecka and Seidman (2003), Prieur et al. (2015), Marx et al. (2015), Mironchenko et al. (2021) and the references therein. However, there are only few results in the literature dealing with output regulation of infinite-dimensional linear systems subject to input saturation Logemann et al. (1998), Oostveen (2000), Fliegner et al. (2001).

In this paper, we study output regulation of infinite-dimensional abstract linear systems subject to input saturation. We focus on the class of abstract systems given by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\phi(u(t)) + B_d w_d(t), \quad x(0) = x_0,\\ y(t) &= B^* x(t)\end{aligned} \tag{1}$$

on a real Hilbert space X. Here $x(t) \in X$ is the state variable, $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}$ is the output, $w_d(t) \in \mathbb{R}^{n_d}$ is an external disturbance and ϕ is a saturation function. The saturation function ϕ is defined as

$$\phi(u) = \begin{cases} u, & |u| \le 1 \\ 1, & u > 1 \\ -1, & u < -1. \end{cases} \tag{2}$$

Our goal is to find a linear feedback control law such that the output $y(t)$ of the system (1) tracks the given reference signal $y_{ref}(t)$ asymptotically despite disturbances $w_d(t)$ in the system. The reference $y_{ref}(\cdot)$ and the disturbance $w_d(\cdot)$ signals are assumed to be generated by an exosystem

$$\begin{aligned}\dot{v}(t) &= Sv(t), \quad v(0) = v_0,\\ w_d(t) &= Ev(t),\\ y_{ref}(t) &= -Fv(t)\end{aligned} \tag{3}$$

on a finite-dimensional space $W = \mathbb{R}^q$. Here $S \in \mathbb{R}^{q\times q}$, $F \in \mathbb{R}^{1\times q}$ and $E \in \mathbb{R}^{n_d \times q}$. Furthermore, we make the following assumptions on the system (1) and the exosystem (3).

Assumption 1.1. (1) The operator A generates a C_0-semigroup $T(t)$ of contractions on X, $B \in \mathcal{L}(\mathbb{R}, X)$ and the operator $A - \kappa BB^*$ generates a strongly stable contraction semigroup $T_{-\kappa BB^*}(t)$ for any $\kappa > 0$.
(2) The spectrum $\sigma(S)$ of S lies on the imaginary axis.

As the main contribution, we extend the output regulation theory in (Saberi et al., 2003, Ch. 3) for finite-dimensional linear systems subject to input saturation to the class of systems in (1)-(2) under Assumption 1.1. The considered class of systems (1) arise in the study of systems with collocated actuators and sensors Oostveen (2000). We present a linear output feedback control law that solves the output regulation problem. In addition, we demonstrate the results on a flexible satellite model subject to input saturation.

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Stabilization is an important part of control design for the output regulation. Stabilization problem for infinite-dimensional linear systems subject to input saturation has been studied, for example in Slemrod (1989), Lasiecka and Seidman (2003), Prieur et al. (2015), Curtain and Zwart (2016), Marx et al. (2015) and Mironchenko et al. (2021). The output regulation of infinite-dimensional linear systems subject to input saturation has been studied, for example in Logemann et al. (1998), Logemann et al. (1999), Logemann and Adam (2001) and Fliegner et al. (2003) for exponentially stable single-input single-output regular linear systems and in Oostveen (2000) for strongly stable single-input single-output linear systems. The results in these references use integral control to achieve output tracking of constant reference signals. The key novelty in our work is that we allow the reference and disturbance signals to be combination of sinusoids. The output tracking is achieved by using a linear output feedback control law which is a generalization of the control law presented in (Saberi et al., 2003, Thm. 3.3.3).

The paper is organized as follows. In Section 2, we present preliminaries on semilinear systems and the output regulation problem. Section 3 is devoted to our main results where we present a linear feedback control law and the solvability conditions for the output regulation of the system (1). In Section 4, we present a numerical example where we consider output regulation of a flexible satellite model subject to input saturation. Concluding remarks and further research directions are presented in Section 5.

1.1 Notation

For normed linear spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from X to Y. For a linear operator A, $D(A)$, $\mathcal{R}(A)$ and $\sigma(A)$ denote the domain, the range and the spectrum of A, respectively.

2. PRELIMINARIES

In this section, we present definitions and lemmas that are used in proving the main results. Consider the system (1) on a real Hilbert space X with $A : D(A) \subset X \to X$, $B \in \mathcal{L}(\mathbb{R}, X)$ and $B_d \in \mathcal{L}(\mathbb{R}^{n_d}, X)$.

Definition 2.1. Let $G(\cdot) = B^*(\cdot I - A)^{-1}B$ be the transfer function of the system (A, B, B^*). Then $s \in \mathbb{C}$ is called a transmission zero if $G(s) = 0$.

Lemma 2.2. (Curtain and Zwart, 2020, Thm. 11.1.5). Consider the semilinear differential equation

$$\dot{x}(t) = Ax(t) + f(x(t)), \quad t \geq 0, \quad x(0) = x_0, \tag{4}$$

where A is the infinitesimal generator of the C_0-semigroup on the Hilbert space X. If $f : X \to X$ is uniformly Lipschitz continuous, then the system (4) has a unique mild solution on $[0, \infty)$ with the following properties:

(i) For $0 \leq t < \infty$ the solution depends continuously on the initial condition, uniformly on any bounded interval $[0, \tau] \subset [0, \infty)$.
(ii) If $x_0 \in D(A)$, then the mild solution is a classical solution on $[0, \infty)$.

Definition 2.3. (Curtain and Zwart, 2020, Def. 11.2.2). Consider the semilinear differential equation (4) on the Hilbert space X. Assume that $f : X \to X$ is locally Lipschitz continuous.

Then the origin of (4) is *stable* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\|x_0\| < \delta$ there exists a solution $x(t)$ of (4) on $[0, \infty)$ satisfying $\|x(t)\| < \epsilon$ for all $t \geq 0$. If, in addition, there exists $\gamma > 0$ such that $\|x_0\| < \gamma$ implies that $\|x(t)\| \to 0$ as $t \to \infty$, then the origin is said to be *asymptotically stable*. The origin is said to be *globally asymptotically stable* if for every $x_0 \in X$ we have $\|x(t)\| \to 0$ as $t \to \infty$.

From the theory of output regulation of finite-dimensional linear systems subject to input saturation we know that the output regulation problem, in general, is not solvable for all initial conditions $v_0 \in \mathbb{R}^q$ of the exosystem (Saberi et al., 2003, Rem. 3.2.2). However, if we restrict to the initial conditions v_0 of the exosystem lying inside a given compact set, then the output regulation problem is solvable. In this work, we focus on this semi-global output regulation problem of (1).

Semi-Global Output Regulation Problem. Consider the systems (1)-(3) and a compact set $\mathcal{W}_0 \subset \mathbb{R}^q$. Find a linear output feedback control law in the form

$$u(t) = -\kappa y(t) + Lv(t) \tag{5}$$

such that $\kappa > 0,\ L \in \mathbb{R}^{1\times q}$ and

(1) The origin of the system $\dot{x}(t) = Ax(t) + B\phi(-\kappa y(t))$, $x(0) = x_0$ is globally asymptotically stable.
(2) For all $x_0 \in X$ and $v_0 \in \mathcal{W}_0$, the error between the output $y(t)$ and the reference signal $y_{ref}(t)$ satisfies
$$\lim_{t\to\infty} y(t) - y_{ref}(t) = 0.$$

3. MAIN RESULTS

In this section, we present our main theorem which provides the solvability conditions and the control law for the semi-global output regulation of the system (1). The theorem is an infinite-dimensional generalization of (Saberi et al., 2003, Thm. 3.3.3) where a low-and-high-gain state feedback control design is used to achieve semi-global output regulation of finite-dimensional linear systems subject to input saturation. In our case, since the considered class of systems can be stabilized strongly using negative output feedback, it is not necessary to find a stabilizing state feedback law separately. Consequently, there is no low-gain requirement on the stabilizing feedback law and there is only one gain parameter that corresponds to negative output feedback. So, the strong stabilizability property of the system (1) by output feedback enables simplifying the control design compared to the original one in (Saberi et al., 2003, Thm. 3.3.3). Our approach for showing the asymptotic convergence of the regulation error is motivated by the techniques in (Curtain and Zwart, 2020, Thm. 11.2.11).

Theorem 3.1. Consider the systems (1), (3) and the given compact set $\mathcal{W}_0 \subset \mathbb{R}^q$. Under the Assumption 1.1, the semi-global output regulation problem is solvable if there exist $\Pi \in \mathcal{L}(\mathbb{R}^q, X)$ with $\mathcal{R}(\Pi) \subset D(A)$ and $\Gamma \in \mathbb{R}^{1\times q}$ such that they solve the regulator equations

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + B_d E \\ 0 &= B^*\Pi + F \end{aligned} \tag{6}$$

and there exists a $\delta > 0$ such that $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1-\delta$ for all $v(t) = e^{St}v_0$ with $v_0 \in \mathcal{W}_0$. In this case, for any $\kappa > 0$ the feedback law

$$u(t) = -\kappa y(t) + (\kappa B^*\Pi + \Gamma)v(t) \tag{7}$$

solves the semi-global output regulation problem.

Proof. By Assumption 1.1, we have that $A - \kappa BB^*$ generates a strongly stable contraction semigroup for any $\kappa > 0$. In addition, the saturation function ϕ is uniformly Lipschitz continuous on $\mathbb{R}$, $\phi(0) = 0$ and

$$\begin{aligned}\langle u, \phi(u)\rangle_{\mathbb{R}} &= u^2, \quad \text{if } |u| \leq 1,\\ \langle u, \phi(u)\rangle_{\mathbb{R}} &> 1, \quad \text{if } u > 1,\\ \langle u, \phi(u)\rangle_{\mathbb{R}} &> 1, \quad \text{if } u < -1.\end{aligned}$$

Therefore, by (Curtain and Zwart, 2020, Thm. 11.2.11), we have that the origin of

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\phi(-\kappa y(t)),\\ x(0) &= x_0\end{aligned}$$

is globally asymptotically stable.

Next, using the feedback law (7), we will show that $y(t) - y_{ref}(t) \to 0$ as $t \to \infty$. Assume that $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1-\delta$. Let us introduce a new variable $\xi(t) = x(t) - \Pi v(t)$ which is the mild solution of

$$\begin{aligned}\dot{\xi}(t) &= A\xi(t) + B[\phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t)]\\ \xi(0) &= \xi_0.\end{aligned} \tag{8}$$

on X, where we have used $u(t) = -\kappa y(t) + (\kappa B^*\Pi + \Gamma)v(t) = -\kappa B^*\xi(t) + \Gamma v(t)$. We will begin by showing that the mild solution $\xi(t)$ of (8) exists for $t \in [0,\infty)$. Let us consider the composite system

$$\begin{aligned}\dot{\xi}_e(t) &= A_e\xi_e(t) + f_e(\xi_e(t))\\ \xi_e(0) &= \xi_{e0}\end{aligned} \tag{9}$$

on $X \times \mathbb{R}^q$ where

$$\xi_e(t) = \begin{bmatrix}\xi(t)\\ v(t)\end{bmatrix},\ \xi_{e0} = \begin{bmatrix}\xi_0\\ v_0\end{bmatrix},\ A_e = \begin{bmatrix}A & 0\\ 0 & S\end{bmatrix},$$

$$f_e(\xi_e(t)) = \begin{bmatrix}B[\phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t)]\\ 0\end{bmatrix}.$$

Here the operator A_e generates a C_0-semigroup (since it is block-diagonal and A generates a C_0-semigroup) and since ϕ is uniformly Lipschitz continuous and B and Γ are bounded linear operators, we have that f_e is uniformly Lispschitz continuous. In fact, using $\|\phi(u_1) - \phi(u_2)\| \leq \|u_1 - u_2\|$ for $u_1, u_2 \in \mathbb{R}$, for $\xi_{e1} = (\xi_1, v_1)^T, \xi_{e2} = (\xi_2, v_2)^T \in X \times \mathbb{R}^q$, we have

$$\begin{aligned}&\|f_e(\xi_{e1}) - f_e(\xi_{e2})\|\\ &\leq \|B\|\|\phi(-\kappa B^*\xi_1 + \Gamma v_1) - \phi(-\kappa B^*\xi_2 + \Gamma v_2)\|\\ &\qquad + \|B\|\|\Gamma\|\|v_1 - v_2\|\\ &\leq \|B\|\kappa\|B^*\|\|\xi_1 - \xi_2\| + 2\|B\|\|\Gamma\|\|v_1 - v_2\|\\ &\leq C\|\xi_{e1} - \xi_{e2}\|,\end{aligned}$$

where $C = \max\{\kappa\|B\|^2, 2\|B\|\|\Gamma\|\}$. Thus by Lemma 2.2, the system (9) has a unique mild solution $\xi_e(t)$ for $t \in [0,\infty)$. The mild solution $\xi_e(t)$ satisfies

$$\xi_e(t) = \begin{bmatrix}T(t) & 0\\ 0 & e^{St}\end{bmatrix}\xi_{e0} + \int_0^t \begin{bmatrix}T(t-s) & 0\\ 0 & e^{S(t-s)}\end{bmatrix} f_e(\xi_e(s))ds.$$

Furthermore, if $\xi_{e0} \in D(A_e)$, then $\xi_e(t)$ is a classical solution for $t \in [0,\infty)$. In particular, we have

$$\begin{aligned}\xi(t) &= T(t)\xi_0\\ &\quad + \int_0^t T(t-s)B[\phi(-\kappa B^*\xi(s) + \Gamma v(s)) - \Gamma v(s)]ds\end{aligned}$$

which is the mild solution for the system (8) and if $\xi_0 \in D(A)$, then $\xi(t)$ is a classical solution for $t \in [0,\infty)$.

Next, we show that the solution $\xi(t)$ is uniformly bounded. For $\xi_0 \in D(A)$, we have

$$\begin{aligned}&\frac{d}{dt}\|\xi(t)\|^2 = 2\left\langle \dot{\xi}(t), \xi(t)\right\rangle\\ &= 2\left\langle A\xi(t) + B[\phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t)], \xi(t)\right\rangle_X\\ &\leq 2\left\langle B[\phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t)], \xi(t)\right\rangle_X\\ &= 2\left\langle \phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t), B^*\xi(t)\right\rangle_{\mathbb{R}}\end{aligned} \tag{10}$$

where we have used the contractivity of A. Now by using the definition (2) of the saturation function ϕ and the assumption $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1-\delta$, we show that the right hand side of (10) is always non-positive. If we consider those $t \geq 0$ such that $|-\kappa B^*\xi(t) + \Gamma v(t)| \leq 1$, then

$$\begin{aligned}&\langle \phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}}\\ &\quad = \langle -\kappa B^*\xi(t) + \Gamma v(t) - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}}\\ &\quad = -\kappa\|B^*\xi(t)\|^2 \leq 0.\end{aligned}$$

If we consider those $t \geq 0$ such that $-\kappa B^*\xi(t) + \Gamma v(t) > 1$, then $-\kappa B^*\xi(t) > 1 - \Gamma v(t) > 0$. This implies that $B^*\xi(t) < 0$. Therefore

$$\begin{aligned}&\langle \phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}}\\ &\quad = \langle 1 - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}} \leq 0.\end{aligned}$$

Finally, if we consider those $t \geq 0$ such that $-\kappa B^*\xi(t) + \Gamma v(t) < -1$, then $-\kappa B^*\xi(t) < -1 - \Gamma v(t) < 0$. This implies that $B^*\xi(t) > 0$. Therefore

$$\begin{aligned}&\langle \phi(-\kappa B^*\xi(t) + \Gamma v(t)) - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}}\\ &\quad = \langle -1 - \Gamma v(t), B^*\xi(t)\rangle_{\mathbb{R}} \leq 0.\end{aligned}$$

Therefore $\frac{d}{dt}\|\xi(t)\|^2 \leq 0$. Integrating (10), we obtain for all $t \geq 0$

$$\begin{aligned}&\|\xi(t)\|^2 \leq \|\xi_0\|^2\\ &\qquad + 2\int_0^t \langle \phi(-\kappa B^*\xi(s) + \Gamma v(s)) - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds\\ &\qquad \leq \|\xi_0\|^2.\end{aligned} \tag{11}$$

By the continuity of $\xi(t)$ with respect to the initial conditions, the above inequality holds for all $\xi_0 \in X$. This implies that for all $\xi_0 \in X$, $\xi(t)$ is bounded uniformly in t on $[0,\infty)$. Next, we show that the mild solution $\xi(t)$ converges to zero as $t \to \infty$. Let us reformulate the system (8) as

$$\begin{aligned}\dot{\xi}(t) &= (A - \kappa BB^*)\xi(t)\\ &\qquad - B[-\kappa B^*\xi(t) + \Gamma v(t) - \phi(-\kappa B^*\xi(t) + \Gamma v(t))]\\ \xi(0) &= \xi_0.\end{aligned}$$

Denote $\hat{u}(t) := -\kappa B^*\xi(t) + \Gamma v(t) - \phi(-\kappa B^*\xi(t) + \Gamma v(t))$. Since $B\hat{u} \in L^1_{loc}(0,\infty;X)$, the solution of the above system is given by

$$\xi(t) = T_{-\kappa BB^*}(t)\xi_0 - \int_0^t T_{-\kappa BB^*}(t-s)B\hat{u}(s)ds. \tag{12}$$

We will first show that $\hat{u} \in L^2(0,\infty;\mathbb{R})$. We will begin by splitting the interval $[0,\infty)$ into three parts. Let $\Omega_1 := \{t \in [0,\infty) \mid -\kappa B^*\xi(t) + \Gamma v(t) > 1\}$, $\Omega_2 := \{t \in [0,\infty) \mid -$

$\kappa B^*\xi(t) + \Gamma v(t) < -1\}$ and $\Omega_3 := \{t \in [0,\infty) \mid |-\kappa B^*\xi(t) + \Gamma v(t)| \leq 1\}$. Then using the definition of ϕ, the assumption $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1-\delta$ and $\kappa B^*\xi(t) < \Gamma v(t) - 1$ on Ω_1, we obtain

$$\begin{aligned}
&\int_{\Omega_1} \langle \phi(-\kappa B^*\xi(s) + \Gamma v(s)) - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds \\
&= \int_{\Omega_1} \langle 1 - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds \\
&\leq \int_{\Omega_1} \left\langle 1 - \Gamma v(s), \frac{\Gamma v(s) - 1}{\kappa} \right\rangle_{\mathbb{R}} ds \\
&= -\frac{1}{\kappa}\int_{\Omega_1} \|1 - \Gamma v(s)\|^2 ds \\
&\leq -\frac{\delta^2}{\kappa}\nu(\Omega_1),
\end{aligned}$$

where ν is a Lebesgue measure. Moreover, from (11), we have

$$\int_0^\infty \langle \phi(-\kappa B^*\xi(s) + \Gamma v(s)) - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds < \infty \tag{13}$$

which implies that Ω_1 has finite measure. Similarly, using the definition of ϕ and $-\kappa B^*\xi(t) < -1 - \Gamma v(t)$ on Ω_2, we obtain

$$\begin{aligned}
&\int_{\Omega_2} \langle \phi(-\kappa B^*\xi(s) + \Gamma v(s)) - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds \\
&= \int_{\Omega_2} \langle -1 - \Gamma v(s), B^*\xi(s)\rangle_{\mathbb{R}}\, ds \\
&\leq \int_{\Omega_2} \left\langle 1 + \Gamma v(s), \frac{-\Gamma v(s) - 1}{\kappa} \right\rangle_{\mathbb{R}} ds \\
&= -\frac{1}{\kappa}\int_{\Omega_2} \|1 + \Gamma v(s)\|^2 ds \\
&\leq -\frac{\delta^2}{\kappa}\nu(\Omega_2)
\end{aligned}$$

which from (13) implies that Ω_2 has finite measure. Furthermore, by using the definition of ϕ, we obtain

$$\begin{aligned}
&\int_{\Omega_3} |\hat{u}(s)|^2 ds \\
&= \int_{\Omega_3} |-\kappa B^*\xi(s) + \Gamma v(s) - \phi(-\kappa B^*\xi(s) + \Gamma v(s))|^2 ds \\
&= 0.
\end{aligned}$$

Since $B^* \in \mathcal{L}(X, \mathbb{R})$, $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1-\delta$ and $\xi(t)$ is uniformly bounded, we have that $\hat{u}(t)$ is uniformly bounded and therefore by using the above arguments, we obtain

$$\int_0^\infty |\hat{u}(s)|^2 ds = \int_{\Omega_1} |\hat{u}(s)|^2 ds + \int_{\Omega_2} |\hat{u}(s)|^2 ds < \infty.$$

Thus $\hat{u} \in L^2(0,\infty;\mathbb{R})$. By Assumption 1.1, A generates a contraction semigroup $T(t)$ and $B \in \mathcal{L}(\mathbb{R}, X)$. Therefore, by (Curtain and Zwart, 2020, Thm. 6.4.4), we have that the system $(A - \kappa BB^*, B, B^*, 0)$ is input stable, i.e., there exists a constant $\beta > 0$ such that for all $t > 0$ and $\tilde{u} \in L^2(0,\infty;\mathbb{R})$, we have

$$\|\int_0^t T_{-\kappa BB^*}(t-s)B\tilde{u}(s)ds\|^2 \leq \beta^2 \int_0^t \|\tilde{u}(s)\|^2 ds.$$

Moreover, by Assumption 1.1, $T_{-\kappa BB^*}(t)$ is strongly stable. Since $\hat{u} \in L^2(0,\infty;\mathbb{R})$, (Curtain and Zwart, 2020, Thm. 5.2.3) implies that $\xi(t) \to 0$ as $t \to \infty$.

Finally, using $\xi(t) = x(t) - \Pi v(t)$ and $B^*\Pi + F = 0$ from (6), we obtain

$$\begin{aligned}
y(t) - y_{ref}(t) &= B^*x(t) - y_{ref}(t) \\
&= B^*(\xi(t) + \Pi v(t)) + Fv(t) \\
&= B^*\xi(t) + (B^*\Pi + F)v(t) \\
&= B^*\xi(t) \to 0
\end{aligned}$$

as $t \to \infty$ which completes the proof.

Remark 3.2. From the proof of Theorem 3.1, we note that the control law (7) can be written as $u(t) = -\kappa B^*\xi(t) + \Gamma v(t)$. Now we can see that the system (1) asymptotically operates in the linear region of saturation since $\xi(t) \to 0$ as $t \to \infty$ and $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1-\delta$.

Based on (Byrnes et al., 2000, Sec. V), the solvability conditions for the regulator equations are given in the following lemma.

Lemma 3.3. (Byrnes et al., 2000, Sec. V). Let $A - \kappa BB^*$ generate a strongly stable contraction semigroup for any $\kappa > 0$. Assume that $\sigma(S) \subset i\mathbb{R}$ and S has no nontrivial Jordan blocks. Then the regulator equations (6) are solvable if and only if no eigenvalue of S coincides with a transmission zero of the system (1). In this case, Π and Γ are given by

$$\begin{aligned}
\Pi\Phi_k &= (i\omega_k - A)^{-1}(B\Gamma + B_dE)\Phi_k \\
\Gamma\Phi_k &= -G(i\omega_k)^{-1}(B^*(i\omega_k - A)^{-1}B_dE + F)\Phi_k,
\end{aligned}$$

$k = 1, 2, \cdots, q$, where $i\omega_k$ and Φ_k are the eigenvalues and the corresponding orthonormal eigenvectors of S, respectively and $G(\cdot) = B^*(\cdot I - A)^{-1}B$ is the transfer function of the system (1).

Corollary 3.4. Let the assumptions of Lemma 3.3 hold and no eigenvalue of S coincides with a transmission zero of the system (1). Let $i\omega_k, k = 1, 2, \cdots q$ be the eigenvalues and $\{\Phi_k\}_{k=1}^q$ be the corresponding orthonormal basis of S. Then for any $v_0 \in \mathcal{W}_0$, the control input (7) has the representation

$$\begin{aligned}
u(t) = -\kappa y(t) &- \sum_{k=1}^q e^{i\omega_k t}\langle v_0, \Phi_k\rangle (\kappa + G(i\omega_k)^{-1})F\Phi_k \\
&- \sum_{k=1}^q e^{i\omega_k t}\langle v_0, \Phi_k\rangle G(i\omega_k)^{-1}B^*(i\omega_k - A)^{-1}B_dE\Phi_k.
\end{aligned}$$

Remark 3.5. Since the expressions for Γ and Π use information from the exosystem and the exosystem is determined by the reference and disturbance signals, we can derive expressions for $\Gamma v(t)$ and $B^*\Pi v(t)$ in terms of frequency, phase and amplitude of the reference and disturbance signals. This is illustrated in the following.

For simplicity, we assume that $y_{ref}(t) = a\cos(\omega t + \varphi)$ and $w_d(t) \equiv 0$. Then the exosystem can be chosen as

$$\begin{aligned}
\dot{v}(t) = Sv(t),\ v(0) = v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ S = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \\
F = -a\left[\cos(\varphi)\ \sin(\varphi)\right],\ E = 0.
\end{aligned}$$

Moreover, the eigenvalues of S are $\pm i\omega$ and the corresponding orthonormal eigenvectors of S are given by $\frac{1}{\sqrt{2}}\left\{ \begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\}$.

Now, substituting the above information and the expression for $\Gamma\Phi_k$ from Lemma 3.3 in

$$\Gamma v(t) = \sum_{k=1}^{2} e^{i\omega_k t} \langle v_0, \Phi_k \rangle \Gamma \Phi_k$$

we obtain $\Gamma v(t) = a|G(i\omega)^{-1}| \cos(\omega t + \varphi + \theta)$, where $\theta = \tan^{-1}(\beta/\alpha)$, $\alpha = \mathrm{Re}(G(i\omega)), \beta = -\mathrm{Im}(G(i\omega))$. Similarly, we obtain $B^*\Pi v(t) = a \cos(\omega t + \varphi)$.

Furthermore, the condition $\sup_{t\geq 0} \|\Gamma v(t)\| \leq 1 - \delta$ in Theorem 3.1 can be reformulated as $|aG(i\omega)^{-1}| \leq 1 - \delta$ and the control input (7) can be written as

$$\begin{aligned} u(t) = -\kappa y(t) + \kappa a \cos(\omega t + \varphi) \\ + a|G(i\omega)^{-1}| \cos(\omega t + \varphi + \theta). \end{aligned} \tag{14}$$

This implies that the above feedback law (14) solves the semi-global output regulation problem provided that the frequency ω from the reference signal satisfies $G(i\omega) \neq 0$ and $|aG(i\omega)^{-1}| \leq 1-\delta$. This shows that it is not necessary to formulate the exosystem in order to solve the semi-global output regulation problem.

4. NUMERICAL EXAMPLE

In this section, we illustrate our main results in Section 3 on a flexible satellite model that is composed of two symmetrical flexible solar panels and a center rigid body (Bontsema et al. (1988), He and Ge (2015)). Modeling the satellite panels as viscously damped Euler-Bernoulli beams of length 1, the satellite model that we consider is described by (Govindaraj et al. (2020))

$$\begin{aligned} \ddot{w}_l(\xi,t) + w_l''''(\xi,t) + 5\dot{w}_l(\xi,t) = 0, \ -1 < \xi < 0, t > 0, \\ \ddot{w}_r(\xi,t) + w_r''''(\xi,t) + 5\dot{w}_r(\xi,t) = 0, \ 0 < \xi < 1, t > 0, \\ \ddot{w}_c(t) = w_l'''(0,t) - w_r'''(0,t) + \phi(u(t)) + w_d(t), \\ \ddot{\theta}_c(t) = -w_l''(0,t) + w_r''(0,t), \\ w_l''(-1,t) = 0, \qquad w_l'''(-1,t) = 0, \\ w_r''(1,t) = 0, \qquad w_r'''(1,t) = 0, \\ \dot{w}_l(0,t) = \dot{w}_r(0,t) = \dot{w}_c(t), \\ \dot{w}_l'(0,t) = \dot{w}_r'(0,t) = \dot{\theta}_c(t), \\ y(t) = \dot{w}_c(t), \end{aligned} \tag{15}$$

where $w_l(\xi,t)$ and $w_r(\xi,t)$ are the transverse displacements of the left and the right beam, respectively, $\dot{w}_l(\xi,t)$ and $w_l'(\xi,t)$ denote time and spatial derivatives of $w_l(\xi,t)$, respectively, $w_c(t)$ and $\theta_c(t)$ are the linear and angular displacements of the rigid body, respectively, the function $\phi(u(t))$ is the saturated external control input defined in (2) and $w_d(t)$ is an external disturbance. Here $\dot{w}_c(t) = \dot{w}_l(\xi,t)|_{\xi=0} = \dot{w}_r(\xi,t)|_{\xi=0}$ and $\dot{\theta}_c(t) = \dot{w}_l'(\xi,t)|_{\xi=0} = \dot{w}_r'(\xi,t)|_{\xi=0}$ are linear and angular velocities of the rigid body, respectively.

As shown in Govindaraj et al. (2023), the satellite model (15) can be written in the form (1) and the operator A generates an exponentially stable contraction semigroup on the state space $X = L^2(-1,0;\mathbb{R}^2) \times L^2(0,1;\mathbb{R}^2) \times \mathbb{R}^2$. It can be also verified that $A - \kappa BB^*$ generates an exponentially stable contraction semigroup on X for any $\kappa > 0$ (Govindaraj et al., 2020, Sec. 3). This implies that Assumption 1.1(1) is satisfied.

Our goal is to track the reference signal $y_{ref}(t) = 0.09\sin(1.5t)$ and reject the disturbance $w_d(t) \equiv 0.08$. Motivated by this, we choose the exosystem

$$\dot{v}(t) = Sv(t), \quad v(0) = \begin{bmatrix} 0 \\ 0.09 \\ 0.08 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1.5 & 0 \\ -1.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $F = [1\ 0\ 0]$, $E = [0\ 0\ 1]$. The eigenvalues of S are given by $\{0, \pm 1.5i\}$ and therefore, Assumption 1.1(2) is satisfied. Moreover, it can be verified that the system (1) does not have any transmission zeros at 0, $1.5i$ and $-1.5i$ (Govindaraj et al., 2023, Lem. 4.1) implying that the regulator equations are solvable.

The control input from Section 3 is given by $u(t) = -\kappa y(t) + (\kappa B^*\Pi + \Gamma)v(t)$. The control parameters Γ and Π can be obtained by using Lemma 3.3 as in Remark 3.5 and they are given by $\Gamma v(t) = 0.09|G(1.5i)^{-1}| \sin(1.5t + \theta) + 0.08$, $\theta = \tan^{-1}(\beta/\alpha)$, $\alpha = \mathrm{Re}(G(i\omega)), \beta = -\mathrm{Im}(G(i\omega))$ and $B^*\Pi v(t) = 0.09\sin(1.5t)$ where $G(\cdot)$ is the transfer function of the satellite system (A, B, B^*). Simulations are carried out in Matlab with $\kappa = 100$ on the time interval $[0, 15]$. The solutions of the satellite system (15) are approximated by using Legendre Spectral Galerkin method with number of basis functions $N = 10$ Asti (2020). Figure 1 shows that after the transient period the controller operates in the linear region of the saturation function and $\sup_{t\geq 0}\|\Gamma v(t)\| \leq 1 - \delta$. The output tracking and the tracking error are depicted in Figures 2 and 3 respectively and the velocity profile of the right solar panel is depicted in Figure 4.

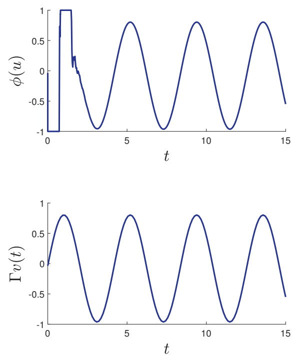

Fig. 1. Behaviour of saturated control input $\phi(u)$ (above) and $\Gamma v(t)$ (below) over the time interval [0,15]

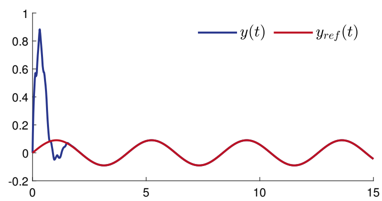

Fig. 2. Output tracking

5. CONCLUSION

We considered output regulation problem for the class of strongly stabilizable infinite-dimensional linear systems

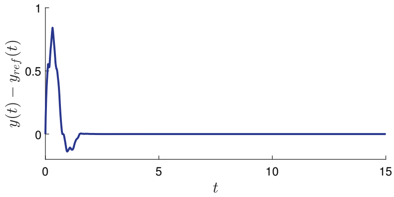

Fig. 3. Tracking error

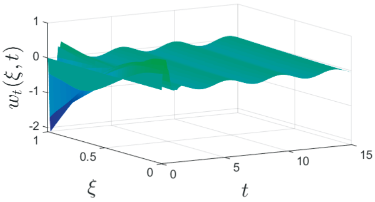

Fig. 4. Velocity profile of the right solar panel

with collocated actuators and sensors subject to input saturation. Strong stabilization of the system enabled us to construct a linear feedback control law that solves the semi-global output regulation problem. The results were illustrated on a flexible satellite model subject to input saturation where output tracking of a given sinusoidal reference signal and rejection of a constant disturbance signal were achieved by using the proposed control law.

Many future research directions are possible. In this work, we considered a particular class of infinite-dimensional systems with bounded input and output operators. So, the theory can be developed for wider class of systems, for example, for the systems with unbounded input and output operators and multi-input multi-output systems.

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