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ABSTRACT

Adaptive expertise is a highly sought after, but difficult to achieve, outcome of mathematics education. Many teaching methods appear to support the development of adaptive expertise only in a small proportion of students. Game-based learning environments may be useful for supporting adaptive expertise. Therefore, we carried out a quasi-experimental classroom intervention to examine the possibility of using a game-based learning environment to promote adaptive rational number knowledge, a potential indicator of adaptive expertise in the domain of rational numbers. The Number Line Elaboration and Exploration learning environment relied on the increasing elaboration of the number line analogy as a means for students to explore connections between multiple aspects of rational number knowledge. Our results show the game-based learning environment was successful in promoting adaptive rational number knowledge. These results provide directions for the development and examination of how learning environments may be able to support adaptive expertise.

The development of mathematics knowledge that can be used to solve novel problems is a highly sought-after goal of mathematics education (Finnish National Agency for Education, 2014; National Council Of Teachers Of Mathematics, 2000). Unfortunately, evidence suggests that often the majority of students are unable to use their knowledge outside of well-rehearsed tasks (Lobato et al., 2003; McMullen et al., 2020). Thus, mathematics educators and researchers continue to seek out ways to support the type of transferable knowledge and skills that allow students to use their knowledge in future learning and everyday situations. One recent promising approach to this problem has been the marriage of theories of adaptive expertise with game-based learning environments (Brezovszky et al., 2019; Mercier & Higgins, 2013).

Adaptive expertise in school mathematics is an ideal outcome of mathematics instruction as the cumulative nature of mathematics requires the remixing and reapplication of existing knowledge in novel contexts (Hatano & Oura, 2012). Adaptive expertise also requires "building links between conceptual knowledge and procedural knowledge" (Baroody, 2003, p. 14). These links also support the invention and use of novel strategies in new situations; for example, being able to recognize the relevance of shortcut strategies such as commutativity in three-step addition (e.g. first adding together the first and third numbers in 43 + 15 + 17) and apply them procedurally, without explicit guidance to do so (Gaschler et al., 2013). In contrast, routine expertise involves static and less connected knowledge, which can only be used on well-practiced tasks; for example, only being able to use commutativity in well-rehearsed tasks such as 14 + 26 = 26 + 14. Adaptive expertise in mathematics has been

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indicated by high levels of various mathematical skills and knowledge (Torbeyns et al., 2006), including adaptive rational number knowledge (McMullen et al., 2020).

Adaptive rational number knowledge is defined as a well-integrated network of knowledge of rational number characteristics and arithmetic relations (McMullen et al., 2020). A key feature of high levels of adaptive rational number knowledge is that it can be flexibly applied in solving novel tasks. For example, students with high levels of adaptive rational number knowledge were more able to use multiple representations of rational numbers (i.e. fractions and decimals) in solving a novel arithmetic task than their peers. Even among students with high levels of other aspects of rational number knowledge, such as arithmetic procedural knowledge, only a small proportion develop exceptional adaptive rational number knowledge (McMullen et al., 2020). In order to support the development of adaptive rational number knowledge, we designed a game-based learning environment elaborating on the basic representation of the number line to continuously produce novel tasks that allowed students to explore connections between various features of rational numbers. In the present study, we aimed to examine if a game-based learning environment would support middle school students' adaptive rational number knowledge.

Promoting adaptive rational number knowledge

Adaptive rational number knowledge can be distinguished from routine rational number knowledge by the level of integration of different aspects of procedural and conceptual knowledge (McMullen et al., 2020). High levels of adaptive rational number knowledge require integrating multiple features of rational number knowledge, including arithmetic procedures and conceptual knowledge of fraction and decimal magnitudes, representations, and operations. This well-integrated knowledge should support, for instance, recognizing that when mentally adding $\frac{1}{4}$ and $\frac{2}{5}$, the decimal notations of .25 and .4 may be more efficient. This process requires recognizing the relation between different representations of rational numbers (fractions and decimals are interchangeable), their magnitudes ($\frac{1}{4} = .25$; $\frac{2}{5} = .4$), and identifying the most effective notation for adding in this situation (adding these decimals may be easier than adding the fractions), as well as the procedural means needed to complete the task.

In their study of middle school students, McMullen et al. (2020) found that many students appeared capable of solving well-rehearsed tasks that required the use of isolated aspects of rational number knowledge, such as arithmetic procedures or fraction magnitude comparisons. This indicates that these students had, at the very least, above-average levels of routine rational number knowledge. Yet, even among the top 45% of performers on measures of routine rational number knowledge, there appeared to be a group of students who were more capable of combining different aspects of their rational number knowledge to solve novel tasks. These students can be said to have higher levels of adaptive rational number knowledge. However, it is not known to what extent adaptive rational number knowledge can be promoted in school-age children.

According to Feltovich et al. (1997), learning environments that promote adaptive expertise should include less-structured non-routine tasks, be conceptually rich, and require meaning-making (Feltovich et al., 1997). As well, educational environments that involve "meeting varied and changing demands" should especially support adaptive expertise (Hatano, 2003, p. xii). Extensive practice with varying tasks reflects a kind of deliberate practice that should support adaptive expertise, which can be difficult to organize in classroom teaching (Lehtinen et al., 2017). For instance, Baroody (2003) notes the extensive difficulties in developing curricular resources to achieve a balance between promoting conceptual and procedural knowledge in an integrated format that would meet these theoretical criteria for supporting adaptive expertise in school mathematics. Though many mathematical reform curricula aim to develop more integrated instruction (e.g., Finnish National Agency for Education, 2014), this often causes difficulties for classroom teachers to implement sustainably. For instance, instructional material often still provides mathematical content as discrete units with little connections

drawn between different topics (Sievert et al., 2019). These difficulties are precisely the type of challenge that new instructional techniques, such as game-based learning, should aim to address (Devlin, 2011).

In particular, game-based learning environments can present opportunities to engage with mathematical content in ways that may be difficult otherwise (Hamilton et al., 2016). For example, the Number Navigation Game, has been shown to promote adaptive number knowledge with whole numbers (Brezovszky et al., 2019). In this game-based learning environment, players must navigate an archipelago in a ship using the hundreds-square and whole number arithmetic. Players navigate from one point to another using arithmetic operations based on their current position and their target position. The game encourages players to use multiplicative relations in their arithmetic operations to maximize points. This is expected to support the development of more well-connected knowledge of numerical characteristics and arithmetic relations (Brezovszky et al., 2015; Lehtinen et al., 2015). Indeed, the Number Navigation Game was found to support adaptive number knowledge in a large-scale classroom experiment (Brezovszky et al., 2019). The present study aims to apply a similar approach to examine whether it is possible to support the development of adaptive rational number knowledge using a digital game-based learning environment.

The number line elaboration and exploration learning environment

Based on the above-mentioned features of learning environments that promote adaptive expertise, we developed a digital game-based learning environment, The Number Line Elaboration and Exploration learning environment (NLEE), to support the development of students' adaptive rational number knowledge. The number line has been shown to be an effective representation and basis for core game mechanics for promoting numerical magnitude knowledge (Riconscente, 2013; Siegler & Ramani, 2009). As well, the number line is a core mental representation of numerical magnitude (Göbel et al., 2011; Siegler, 2016), which may make it a powerful foundation for designing game-based learning environments (e.g., Riconscente, 2013). The NLEE relies on two general principles for the design of the learning environment: (a) extensive practice with constantly elaborated tasks and (b) possibilities for exploration. The NLEE elaborates on the number line metaphor to continuously produce novel learning tasks, which involve increasingly complex rational number concepts. This continuous elaboration encourages players to connect different aspects of rational number knowledge and skills, including the relations between numerical magnitudes, different representations of rational numbers (e.g. non-symbolic, fraction, decimal), and arithmetic outcomes, in solving novel tasks.

Table 1 outlines the task types, descriptions, and demands within the NLEE. The first game of the NLEE, Number Trace (e.g., Kiili et al., 2021; Koskinen et al., 2022), and its predecessor, Semideus (Kiili et al., 2018; Ninaus et al., 2017), has been widely used to support students' ability to estimate the magnitudes of rational numbers on a number line. The player controlled the movement of an animated dog to move along a number line. Students were tasked with interpreting the location of a bone (the given rational number) as a position on a number line. In the NLEE we redesigned the game to align with the principle of constant elaboration of rational number concepts and procedures. Thus, the game began with basic estimation tasks using an empty number line with only endpoints (0 and 1, 3, or 5) marked and with a bone location given as a fraction, decimal, or a non-symbolic quantity, such as a fraction circle (Figure 1a). The first elaboration was to utilize unbounded number lines (Figure 1b), in which the endpoint of the number line was not specified and varied between tasks. An intermediate point on the number line was shown and the student had to determine the location of the target magnitude in relation to this given magnitude. The use of unbounded number lines explicitly highlights the relational nature of rational numbers, in contrast to a direct link to a specific length. Further elaborations involved using multiple rational number representations on a single task, for example, using a decimal to denote the intermediate point, but representing the target number as a fraction (as in Figure 1b). Players were also tasked with estimating a target number given as a one-step arithmetic problem (e.g., $\frac{1}{2} + \frac{1}{4}$). Finally, on some levels, players could only navigate by

Table 1. Task type, description, and demands included in the NLEE.

Task Type	Task Description	Task demands
Number Trace		
Basic Estimation	Estimate the location of the magnitude of a single representation (fraction, decimal, non-symbolic) on a bounded number line. Number lines marked with 0 at the start and either 1, 3, or 5 at the end. Points awarded based on accuracy of estimation.	Translating magnitude from one rational number representation to position on number line.
Unbounded estimation	Number lines marked with 0 and an intermediate value not at the endpoint (e.g., on a number line that in theory ends at 1.5, $\frac{3}{5}$ is marked and $\frac{6}{5}$ must be estimated). Intermediate given value and target value are the same notation (e.g., both are fractions)	Determining proportional relation between target and given rational numbers. Translating magnitude to proportion/length of number line.
Cross-notation unbounded estimation	Given value and target value are different notations (e.g., 0.5 is marked on the number line and $\frac{1}{4}$ should be estimated)	Converting between notations. Determining proportional relation between target and given rational numbers. Translating magnitude to proportion/length of number line.
Simple arithmetic estimation	Estimating the outcome of single step arithmetic on bounded number line	Determining sum, difference, or product of arithmetic operation. Translating magnitude to proportion/length of number line.
Fixed-jump length	Movement along the number line is limited to jumps of a specific length (e.g., $\frac{1}{4}$). In initial tasks length is shown, in later tasks, lengths must be identified by the player.	Calculate relation between jump length (e.g., $\frac{1}{4}$) and target magnitude.
NanoRoboMath		
Missing value arithmetic "power"	Inputting arithmetic operation and second operand to navigate from current location on number line to target location on number line (multiple steps were allowed). Points awarded based on operands used (smaller operands provides more points).	Identifying arithmetic relations between current and target location. Emphasis on identifying multiplicative relations between values, especially inverse multiplicative relations (e.g. operating with values between 0 and 1).
Missing value arithmetic "time"	Input of operation and operand to navigate to a target interval around the target position (e.g., ±0.5 from 1.47). Points awarded for accuracy and speed.	Identifying approximate arithmetic relations (e.g., using 1.5 as a target rather than 1.47)
Cross-notation missing value arithmetic	Current location, target location, and or input values are different notations (e.g., current location is 3/4, target location is $\frac{1}{5}$, input values must be in decimal format)	All task demands of power or time as above. Converting between notations
Missing value arithmetic "approaching"	Inputting arithmetic operation and second operand to navigate as near as possible to the target without touching it or passing it. Points awarded based on the number of moves.	Task demands of power. Identifying rational numbers close to a given rational number (e.g., 0.499 is close to 1/2).

making jumps of a fixed length. The length of the jump was shown as a fraction in the first tasks of this type; later tasks required players to estimate the lengths of the jumps themselves with help of fraction or decimal numbers shown on the number line and then use this jump length to estimate the target fraction.

In the second game, NanoRoboMath, the number line exploration activities were expanded. NanoRoboMath has been previously shown to support different features of primary school students' rational number knowledge, including conceptual knowledge of rational number representations and operations (Kärki et al., 2021, 2022). In the game, students explore a dynamic, continuous number line – which stretches, zooms, and shifts depending on players' position relative to their target – by creating arithmetic operations with rational numbers. Their task was to move a nanorobot to a target position to destroy bacteria, viruses, or parasites (Figure 1c). The current position of the nanorobot was used as the first operand in an arithmetic sentence (e.g., $4\frac{1}{5}$). To move the nanorobot, the player should choose one of the four basic arithmetic operations and input a second operand (e.g., "$+ 4\frac{4}{5}$"). The nanorobot then moved to the resulting value's location on the number line (e.g., 9). Moving on the number line with several steps was allowed in order to enhance exploration and different solution strategies.

Figure 1. Gameplay images for Number Trace (a and b) and NanoRoboMath (c and d). Figure 1a shows a cross-notation number line estimation task. Figure 1b shows an unbounded cross-notation number line estimation task. Figure 1c shows a missing value "power" task with a current location of $4\frac{1}{5}$ and a target value of 9. Figure 1d shows a missing value "time" task with a current location of 11.125 and a target value of $6\frac{1}{2}$, input values were required to be in fraction form.

In the NLEE, different game modes of NanoRoboMath were designed to highlight particular features that would promote adaptive rational number knowledge. In the *power mode*, using smaller magnitude numbers as the second operand produced higher scores (e.g., using $4\frac{1}{5} \div \frac{7}{15}$ would provide the highest points for a direct move between $4\frac{1}{5}$ and 9). This encouraged players to consider the use of both additive and multiplicative operations in their solutions, especially multiplicative inverses. The *time mode* (Figure 1d) required players to quickly move toward an interval target, which allowed for estimation. Points were awarded based on speed and accuracy, which created a trade-off between the two (e.g., quickly estimating an approximate solution, or slowly estimating an accurate solution were both possible strategies). Cross-notation tasks were also included in NanoRoboMath, such as needing to use a decimal as a second operand when the first (given) operand and/or the target is a fraction and vice versa. The *approach mode* required moving with as few operations as possible very near a given target without touching it (e.g. approaching $\frac{1}{2}$ by navigating to 0.499).

As a whole, the NLEE is expected to promote adaptive rational number knowledge by providing practice in making connections between different characteristics of rational numbers and their arithmetic relations. The two games were adapted from previous versions to align with each other in providing extensive practice with constantly elaborated tasks. These tasks increase in complexity from basic number line estimation, through, for example, cross-notation unbounded tasks, to complex arithmetic tasks with fractions and decimals. In doing so, they provide students with explicit opportunities to continuously elaborate the connections between the different rational number procedures and concepts targeted by the tasks, as described in Table 1. This constant elaboration is based on the principle that in order to promote adaptive expertise students must be provided with opportunities to engage in deliberate practice with tasks of varied and changing demands (Hatano, 2003; Lehtinen

et al., 2017). Importantly, players were able to purposefully enhance their game performance by replaying tasks and attempting more sophisticated solution strategies.

The versions of the games used in NLEE were designed to support students' own exploration of the relations between arithmetic operations and magnitudes within and across rational number notations. For example, by allowing players to dictate the arithmetic operation they are using in making a move in NanoRoboMath, players have the opportunity to explore a variety of arithmetic relations between different rational numbers in a goal-oriented, but self-guided manner. This is in line with previous game-based learning environments that support adaptive number knowledge with whole numbers (e.g. Brezovszky et al., 2019). These opportunities for exploration are based on the principle that in order to promote adaptive expertise students must be provided with opportunities to engage in non-routine, conceptually-rich tasks (Feltovich et al., 1997; Holbert & Wilensky, 2019).

The present study

The main aim of the present study was to examine the effects of the NLEE on students' adaptive rational number knowledge. Previous evidence suggests only a small proportion of students develop exceptional adaptive rational number knowledge (McMullen et al., 2020) and it is not clear how malleable this knowledge is, as even routine rational number knowledge is difficult to promote through intervention (Braithwaite & Siegler, 2021). We examined the effectiveness of the NLEE within naturalistic classroom settings, in which teachers are given advice, but the freedom, to use the NLEE as they see fit.

We, therefore, asked: *What is the effect the NLEE has on students' adaptive and routine rational number knowledge?* We expected that playing the games would provide practice in making connections between rational number characteristics and their arithmetic relations. Thus, we hypothesized that students' adaptive rational number knowledge would improve after playing the NLEE. Based on previous studies (Brezovszky et al., 2019), we also expected students' routine rational number knowledge would improve after playing the NLEE. To control for re-test effects, we compare their performance with students in classrooms who did not play the NLEE. As well, we aimed to examine the specific impact of students' performance in the NLEE on students' learning gains during the same period. Since both games included in the NLEE were designed to integrate the learning content into the gaming activities of the player, we expected that students' performance during the games would be positively related to learning gains from pre- to posttest.

Methods

Participants

A total of fourteen seventh-grade classrooms (mean age = 13.2 years, SD = 0.32 years) from 8 schools with 191 students (92 female, 93 male, 6 no response) participated in the study. The schools were from varying socio-economic status areas from a city located in southern Finland. Seven of the classrooms were randomly assigned to the Game condition (103 students) and six classrooms continued regular classroom instruction as a control condition (88 students). All tests were carried out in the instructional language of the classroom, Finnish.

The permission to conduct the study was received from the municipality, school principals, and participating teachers. Permission to take part in the study was asked in written form from the caretakers of the students. At the start of each testing, students were informed of their right to stop taking part in the study at any point. Those without parental permission were not included in the analyses. Ethical permission was granted from the University of Helsinki ethical board and the ethical guidelines of the Finnish national board on research integrity (TENK) were followed.

Design and procedure

We conducted a quasi-experimental classroom intervention with assignment to conditions by classroom. While all participating teachers had volunteered to participate in the study, assignment was stratified randomization, done to ensure a balance in the timing of rational number instruction and textbook use across conditions. Power analysis of a two-level hierarchical design with covariates with effect size of d = 0.5 (based on Brezovszky et al., 2019), power >.8, correlations within clusters $R_w^2 = .5$, and among clusters $R_s^2 = .8$, and intraclass correlations =.1 (Yang Hansen et al., 2014) revealed the need for 8 classrooms per condition (Hedges & Rhoads, 2010). There were in total 8 classrooms included in the NLEE condition and 6 in the control condition revealing less than ideal power (.73), thus any null effects should be treated cautiously.

Figure 2 illustrates the progression of the study, which ran over an eight week-period in Fall 2019. All participants first completed a pretest of their rational number knowledge. The Game condition classrooms then used the NLEE within a six week period during their scheduled mathematics classes. In control classes, teaching followed the national core curriculum. In seventh grade, topics in rational numbers include strengthening calculation skills with fractions and learning how to multiply and divide by fractions, deepening the skills to operate with decimals, as well as conceptual knowledge of percentages. Rational number instruction in the textbooks used in these classrooms approached these topics in sequential fashion, covering one concept or procedure at a time with worked examples and routine practice problems. We cannot ensure that students did not receive explicit instruction that aligns with principles of adaptive expertise, thus, we do not consider the NLEE in comparison to this instructional approach per se, but rather use the control condition to examine the effects of the NLEE on adaptive rational number knowledge in comparison to any potential test-retest effects.

Prior to playing the games, during the first week of the study, NLEE condition teachers participated in a two-hour training session led by the research team, which provided information on the main aim of the study, the basic instructional mechanics of the games, the specific learning objectives of the NLEE, and how to use the Number Trace portion of the NLEE in their classroom. In particular, we encouraged teachers to support struggling students and use the games as avenues for supporting whole-class discussion, as the NLEE was meant to be embedded in typical classroom practices. However, in line with principles of teacher autonomy within the Finnish educational system, teachers were free to use the NLEE as they saw fit. Teachers were asked to have their students play the Number Trace portion of the NLEE during three normal mathematics lessons in weeks two and three. Lessons were 45 minutes in length, with typically shorter periods of actual playing time (see below). Students then completed a mid-test on week four, which was not intended to be included in the present analysis, as it aimed to examine specific effects of the games rather than the overall effectiveness of the combined learning environment. Additionally, due to a technical issue in the testing platform, a large number of students' responses were not recorded on the mid-test.

Teachers again participated in a 1.5-hour training session led by the research team, which focused on the NanoRoboMath portion of the NLEE and provided specific guidance for how to use the game to elicit students' deep engagement with the rational number content. Special emphasis was placed on using the first portion of lessons involving NanoRoboMath to elicit classroom discussion of alternative strategies to solve a particular level and encourage students to explore these alternative strategies to improve their scores. After an autumn break in week six, the teachers were asked to have their students play the NanoRoboMath game for five lessons over a two-week period. Following the final game
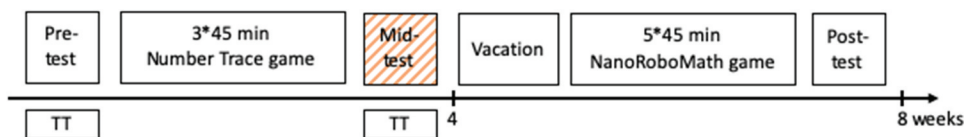

Figure 2. The structure of the NLEE intervention. TT refers to teacher training. Mid-test is not included in the present analysis.

session, students completed a posttest of their rational number knowledge, which is used as the outcome measure in the present study. In line with examining the use of the NLEE in a naturalistic classroom environment, all students who completed at least one level of the NLEE were included in the analysis. According to the game log data, the mean playtime for Number Trace was 1 hour 44 minutes (SD = 27 minutes, Range = 36–208 minutes; median = 1 h 43 min) and NanoRoboMath 1 hours 24 minutes (SD = 58 minutes; Range = 0–268 minutes; median = 1 h 26 min). On average the players competed 18.88 (SD = 3.64, range = 5–21, median = 21) levels out of the 21 levels of Number Trace. The Pearson correlation between the Number Trace playing time and completed levels was $r = .56$.On average the players competed 12.1 (SD = 8.6; range = 0–33; median = 10) levels out of the 50 levels of NanoRoboMath. The Pearson correlation between NanoRoboMath playing time and completed levels was $r = .81$.

Measures

Students' rational number knowledge was measured using computer-based tests, in their regular mathematics classrooms, led by a member of the research team. All test instructions were read aloud and presented on the computer screen. The same items were used on pre- and posttests; students were not allowed to use calculators. The testing sessions lasted approximately 40 minutes and were administered in the following order: arithmetic sentence production task, number line estimation, rational number sets, mental arithmetic calculation, mental cross-notation conversion, and magnitude ordering.

Adaptive rational number knowledge

The arithmetic sentence production task has been previously validated as a measure of adaptive number knowledge with whole number and rational number arithmetic (McMullen et al., 2020). Participants had two minutes to form as many valid arithmetic sentences as they can that equal a target number, using a set of five given numbers. First, participants completed a practice item with whole numbers (target number of 6 and given numbers of 1, 2, 3, and 4). After completing the practice item, participants were given the opportunity to ask questions. Then participants completed four test items. Each test item included two pairs of equivalent fractions and decimals (e.g., $\frac{1}{2}$ and 0.5, $\frac{1}{4}$ and 0.25), a single whole number (e.g., 4), and a target number (e.g., 1). Participants could use each number and operation repeatedly. It was not possible to make literal repetitions of a previous answer. Answers were counted as correct if they were mathematically valid arithmetic sentences, including equivalent solutions (e.g., $\frac{1}{2} + \frac{1}{2}$ and $.5 + .5$) and commutative equivalents (e.g., $\frac{1}{4} + \frac{1}{2}$ and $\frac{1}{2} + \frac{1}{4}$). The average number of correct responses across all four items was calculated. Reliability was good at both time points: pretest Cronbach's α = 0.87and posttest Cronbach's α = 0.88.

Routine rational number knowledge

We assessed students' routine rational number knowledge using a series of tasks that required using knowledge of fractions and decimals in ways that were routinely practiced by students during typical classroom instruction. This involved knowledge about the magnitude of rational numbers, representations of rational numbers, and simple mental calculations with rational numbers. Across all 36 items measuring these aspects of knowledge reliability was good at the pretest (Cronbach's α = .82). To account for different scales in the creation of an overall measure of routine rational number knowledge, we created standardized scores for each task and summed these standardized scores.

Magnitude knowledge. Knowledge of rational number magnitudes was measured with two tasks: an ordering task and a number line estimation task (Schneider & Siegler, 2010; Stafylidou & Vosniadou, 2004; Van Hoof et al., 2015). The ordering task required students to put three to four numbers in ascending order and included two decimal items, two fraction items, and two items with mixed

decimals and fractions (e.g., "Put the numbers in order from smallest to largest:" 0.5; $\frac{1}{4}$; $\frac{5}{7}$; 0.356.). One point was given for each correct answer with a maximum of 6 points.

Number line estimation was assessed with four items on a 0–1 number line (0.6, $\frac{1}{5}$, $\frac{3}{7}$, and 0.42), and four items on a 0–5 number line ($\frac{11}{7}$, 3.7, $\frac{9}{2}$, and 0.83). Percent absolute error (the distance between the estimated value and the correct value divided by the total length of the line) was calculated for measuring accuracy (Siegler et al., 2009).

Rational number representation knowledge. Knowledge of rational number representations was assessed with two tasks: an adapted Number Sets Test (Geary et al., 2009; Mazzocco & Hanich, 2010) and a conversion task. In the Number Sets Test, students had one minute to identify as many symbolic and non-symbolic representations as possible that equal a given rational number ($\frac{1}{2}$ or 0.9). Each item had 18 alternative answers, with 11 correct matches per item. One point was awarded for each correct answer and one point subtracted for each incorrect answer. The average score for the two items was recorded with a maximum of 11 points.

The conversion task included four items of fraction to decimal conversions ($\frac{3}{5}$, $\frac{19}{100}$, $\frac{1}{4}$, $\frac{3}{8}$) and four items of decimal to fraction conversions (0.4, 0.81, 0.75, 0.125). One point was given for each correct answer with a maximum of eight points.

Mental rational number calculation. Proficiency with mentally calculating rational number arithmetic was measured with 12 mental arithmetic problems. These included six fraction items ($\frac{1}{4} \times \frac{1}{4}$; $\frac{1}{2} + \frac{1}{2}$; $\frac{1}{2} \div 2$; $\frac{3}{4} - \frac{1}{4}$; $2 \div \frac{1}{2}$; $2 \times \frac{1}{4}$), five decimal items (0.25 × 4; 0.5 + 0.5; 0.5 ÷ 2; 0.75–0.25; 4 ÷ 0.5) and one mixed item ($\frac{1}{2} \times 0.5$). One point was given for each correct answer with a maximum of 12 points.

Game performance

For each game, we calculated a measure of game performance based on students' gaming activities and outcomes. These measures were chosen to capture both the quantity (i.e., amount of the game played) and quality (i.e., success in the game) of students' game performance. For Number Trace, we calculated students' performance based on the number of stars they earned. Students received 1 point for each star they received (out of a possible three per level). For NanoRoboMath, we calculated students' performance based on the type and number of medals they earned. Students received 1 point for each bronze medal, 2 points for each silver medal, 3 points for each gold medal, and 4 points for each diamond medal. To create a composite measure of game performance across games, students' game scores were standardized for each game separately and added together ($r = .29$, $p = .002$).

Analysis

The study was carried out in a naturalistic setting, with randomization at the classroom level, thus there were possible classroom-level effects that should be taken into account in the analysis. Intra-Class Correlation (ICC) levels (within class variance divided by total variance) for classroom effects on pretest scores were as follows: Adaptive Rational Number Knowledge ICC =.03; Routine Rational Number Knowledge ICC =.14. This suggests that multi-level modeling of individual and classroom effects may be appropriate in the present study.

In order to take into account the hierarchical nature of the study design, we estimated linear mixed models using SPSS version 24 (Heck et al., 2010; West, 2009). We compared two models estimating the effects of the intervention on each dependent variable, using a likelihood-ratio test (West, 2009). First, we estimated a model with fixed effects to explore the effect of condition and pretest scores on posttest scores. The second model included a random intercept and slopes in order to estimate potential classroom effects.

In each model, level 1 included observations for individual students nested within the classes (level 2). Since we were interested in examining the effects of the intervention on individuals, we added the treatment/control variable at level 1 as a fixed covariate, along with individuals' pretest scores. In

the random model, along with the fixed model structure, at level 2, random intercepts and slopes of classroom membership were included. Covariance structures were "variance components" for level 2 variables.

Results

Intervention effects

According to the values reported in Table 2, the inclusion of random effects did not improve the model fit at a statistically significant level, all changes in −2 times log-likelihood were less than the critical value 5.99 of chi-squared distribution ($p < .05$) for $\Delta df = 2$. Therefore only the fixed effects of experimental condition membership and pretest scores were used as predictors of posttest performance.

The effects of the intervention on adaptive rational number knowledge was more appropriately modeled using only fixed effects, indicating that there were similar effects of the intervention across participating classrooms. Unsurprisingly, there was a large effect of pretest adaptive rational number knowledge on posttest performance: $\beta = 1.06$, SE =.06, $t(162) = 16.07$, $p < .001$ (See Table 3). As well, there was a moderate effect of condition on posttest adaptive rational number knowledge: $\beta = .70$, SE =.24, $t(162) = 2.92$, $p = .004$. Students in the NLEE condition outperformed those in the control condition on the posttest after taking into account pretest scores. In total, these results suggest that the game-based learning environment had a positive impact on students' adaptive rational number knowledge.

Routine rational number knowledge was also best modeled with only fixed effects, indicating that there were similar effects of the intervention across participating classrooms for routine knowledge as well. Again, there was a large effect of pretest scores on posttest routine rational number knowledge: $\beta = .88$, SE =.04, $t(162) = 21.67$, $p < .001$. Finally, there was a small, but statistically significant, effect of condition on posttest routine rational number knowledge: $\beta = .23$, SE =.07, $t(162) = 3.54$ $p = .001$. Students in the NLEE condition outperformed those in the control condition on the posttest routine knowledge test, after taking into account their pretest knowledge. Thus, the NLEE appeared to have a positive impact on students' routine rational number knowledge as well. However, this effect appears substantially weaker than the effect of the NLEE on adaptive rational number knowledge.

Table 2. −2 times log-likelihood for linear mixed models with fixed or random intercept and slopes for each dependent variable. Lower values indicate better fit. Best fitting model in bold.

	Fixed Intercept Fixed Slope (4 parameters)	Random Intercept Random Slope (6 parameters)
Adaptive rational number knowledge	**611.09**	606.23†
Routine rational number knowledge	**181.54**	178.26†

Critical value of chi-squared distribution ($p < .05$) for $\Delta df = 2$ is 5.99. †Hessian matrix not positive definite.

Table 3. Results of linear mixed models estimating effects of intervention on rational number knowledge. Descriptive statistics for pre- and posttest scores by condition. Estimates of effects are the associated standardized coefficients from each linear model for the intercept, pretest scores, and condition. Standard errors for all values reported in parentheses.

	Pretest Score Means and (St. Dev)		Posttest Score Means and (St. Dev)		Estimate of Fixed Effects (SE)		
	Game	Business-as-usual	Game	Business-as-usual	Intercept	Pretest	Condition
Adaptive rational number knowledge	3,40 (1.87)	2,86 (1.88)	5.08 (2.63)	3.88 (2.20)	1.66*** (.27)	1.06 (.06)***	.70** (.24)
Routine rational number knowledge	.11 (.68)	−.21 (.72)	.13 (.72)	−.29 (.77)	−.12** (.04)	.88 (.04)***	.23** (.07)

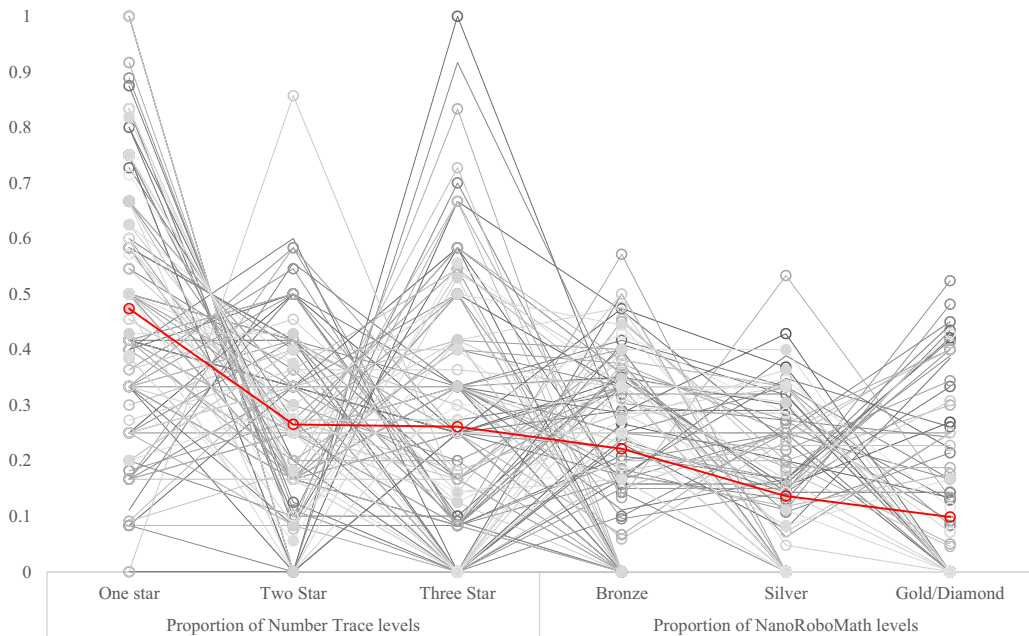†$p > .05$, *$p < .05$, **$p < .01$, ***$p < .001$.

Figure 3. Proportion of Number Trace and NanoRoboMath levels on which each participant received particular stars or medals. Each gray-scale line represents one participant, the thick red line represents the mean proportion of levels across all participants. Players earned one star upon completion of a level in Number Trace. However, it was possible to not receive a medal in NanoRoboMath even after completing a level.

Effects of game performance

First, to determine the general patterns of performance by the students who participated in the NLEE we examined the distribution of stars and medals that students received across the two games. Figure 3 shows the proportion of levels in which students received one, two, or three stars in Number Trace and bronze, silver, and gold or diamond medals in NanoRoboMath. These performance levels are particularly valuable for understanding students' strategies in NanoRoboMath as they align with the use of multiplicative (gold) and inverse multiplicative (diamond) strategies on power levels. There were substantial inter- and intra-individual differences in game performance. There were differences in the variation of performance across the two games and students used a variety of strategies to solve the tasks. In general, there appeared to be more students who earned high proportions of one or three stars in Number Trace, while there were few students who earned a single type of medal on more than half of the NanoRoboMath levels.

To examine the specific effects of game performance on the positive intervention effects, multiple hierarchical linear regression models were analyzed with posttest performance of adaptive and routine rational number knowledge as the dependent variables. Independent variables were the corresponding pretest measure for that posttest performance (e.g. only pretest adaptive rational number knowledge was used in the regression predicting posttest adaptive rational number knowledge) and the game play score for Number Trace and NanoRoboMath.

As can be seen in Table 4, even after controlling for prior knowledge of the same type, game performance predicted posttest scores of adaptive and routine rational number knowledge. In fact, despite prior knowledge being a strong predictor of posttest scores, students' quality and quantity of gaming explained 12% of variance in posttest adaptive rational number knowledge, but only 4% of variance in routine rational number knowledge. These results suggest that game play is directly related to the positive learning outcomes found in the NLEE condition.

Table 4. Hierarchical linear regression analyses: impact of game performance on the learning outcomes in the game condition (*n* = 116).

		Posttest			
		Adaptive rational number knowledge		Routine rational number knowledge	
		β	Δ*R*²	β	Δ*R*²
1	Pretest	.62***	.34***	.73***	.43***
2	Game performance	.36***	.12***	.23**	.04**
	Total *R*²		.67		.73
	F (2, 85)		85.99***		111.57***

1. **p* < .01, ***p* < .001. Pretest = Corresponding pretest variable to the posttest variable.
2. *ΔR2* derived from entering predictor as last step.

Discussion

Summary of key findings

It appears possible to promote adaptive rational number knowledge using a game-based learning environment. Students who were in the classrooms that used the game-based NLEE improved on a multi-faceted test of their rational number knowledge, including adaptive rational number knowledge. Crucially, the quality and quantity of students' playing activities predicted their learning gains, suggesting that playing the games had a direct effect on learning outcomes. The explorative and elaborative opportunities with conceptually rich tasks facilitated by the NLEE may be particularly valuable for promoting integrated rational number knowledge that is applicable in novel situations, which is needed for adaptive expertise with rational numbers. The success of the learning activities facilitated by the NLEE suggests that similar opportunities for exploration and elaboration may be useful when implemented in other digital tools and even non-digital classroom instructional settings.

Digital learning environments for adaptive expertise

The present findings expand on previous research on the design of learning environments that support features of adaptive expertise (Feltovich et al., 1997; Hatano, 2003). One crucial feature of adaptive expertise is a high level of integration of knowledge across multiple aspects, such as different concepts and/or procedures (Baroody, 2003). The present study provides evidence that game-based learning environments can be designed in a way to support students in making connections across different concepts and that this can lead to improvements in their adaptive rational number knowledge. Previous evidence has indicated that working with a single learning environment that supported integrating different aspects of arithmetic knowledge led to improvements in adaptive number knowledge with whole numbers (Brezovszky et al., 2019). However, more diverse connections are most likely needed for supporting adaptive rational number knowledge due to the complexity of rational numbers and their arithmetic procedures. Creating learning environments that progress through a series of novel tasks may be particularly useful for supporting the development of these complex connections. Therefore, the integration of two complementary digital games into a single coherent learning environment may be one of the crucial strengths of the NLEE. Not only did students experience novel tasks within a game, but also across multiple related games.

The elaboration that was central to the design of the NLEE may be valuable for the design of future learning environments to promote adaptive expertise in mathematics. The elaborative structure of the learning environment meant that students were almost constantly interacting with increasingly complex rational number concepts through novel gaming activities – often, as mentioned previously, multiple concepts at once. This is in contrast with many experiences in the mathematics classroom that involve dealing with mathematical concepts and objects in discrete, disconnected units or digital games that merely replace existing instructional activities with digital forms (Devlin, 2011). An explicit attempt to increase the connections between different mathematical concepts, using a consistent

representational anchor (in this case the number line) aligns closely with the types of activities that are expected to promote adaptive expertise (e.g., Feltovich et al., 1997)

As well, the dynamic and interactive features that allowed for exploration using various strategies could be valuable for promoting adaptive expertise. Important for the intrinsic integration of the gameplay, the dynamism and interactivity in the games are directly related to the mathematical content and not surface features. In moving their characters along number lines, the students are not only interacting with the mathematical objects they are learning about but are also experiencing the magnitudes of rational numbers in the game world. Thus, students had some degree of control over how they interacted with the learning content. As in previous games aimed at supporting adaptive expertise in mathematics (e.g., Brezovszky et al., 2019), the actual gameplay in NanoRoboMath was constantly changing based on the previous actions of the player. The outcomes of mathematical operations were directly represented by dynamic changes in the game, as players' previous moves dictate the position of the nanorobot and offer subsequent possibilities for reaching the solution. This is especially valuable for creating moments for graceful failure and learning through trial by error. As, for instance, in the case of NanoRoboMath, an operation going awry does not lead to the "wrong" answer, but merely moves the robot to an unexpected, and perhaps disadvantageous, location.

The NLEE learning environment was less-structured than typical classroom tasks; students could approach most tasks in multiple ways as evident from the fairly large variance in game performance measures. This leads to possibilities for creating (a) appropriate levels of challenge for students by, for example, allowing for multiple solution strategies, and (b) appropriate scaffolding by, for example, modeling and discussing these strategies in whole-class activities around the game. For example, in the cross-notation unbounded number line tasks in Number Trace (Figure 1b), students could, among other strategies, (a) convert the fraction to a decimal, (b) convert the decimal to a fraction, (c) examine the multiplicative relation of the target ($\frac{2}{5}$) and given (0.3) values and work from there, or (d) use an additive strategy by estimating multiples of the length of the given value (e.g., 0.1) to find an easier value to compare with the target. Similarly, a student could use multiple strategies to approach a target in NanoRoboMath. In addition to straightforward additive and precise moves, a player could use more sophisticated strategies with multiplicative relations, inverse operations, or approximate calculations. Sometimes also the representation type of the rational numbers in the mathematical expressions could be chosen freely. More evidence of students' particular strategies in these games would be valuable to better understand the support mechanisms that lead to the learning gains found in the present study.

This openness in individual tasks allowed students to engage with the task at different levels of complexity and difficulty, reflecting the possibilities to match the challenge level with students' abilities. However, the overall design of the NLEE across levels and games still followed a carefully developed learning trajectory. This flexibility within tasks, combined with a well-structured developmental trajectory across tasks, allows students to practice the skills that are on the edge of their abilities, in line with theories of deliberate practice (Ericsson et al., 1993). This bares out in the finding that there were multiple benefits for students, in both routine knowledge and adaptive knowledge.

The strongest contribution of the present study to understanding how to design learning environments to support adaptive expertise may be the combination of all of these features into a coherent design across multiple digital games. While conceptually grounded in the same representation (i.e. the number line), the multiple game and task designs, and level structures consistently challenge students to apply their knowledge in novel contexts. This higher level of dynamism may be particularly challenging to implement consistently in the mathematics classroom, given the constant shifting from topic to topic that is encouraged in most textbooks.

Educational implications

Students face a great deal of difficulty in learning about rational numbers and only a small portion of students who have strong routine knowledge develop high levels of adaptive rational number knowledge through standard instructional practices (McMullen et al., 2020). The developed game-based

learning environment appears to support both basic, routine knowledge of rational numbers and adaptive rational number knowledge. While recent studies have found some success in promoting routine knowledge of rational numbers (Braithwaite & Siegler, 2021), to our knowledge, the NLEE is one of the first to provide proven support for the development of both routine and adaptive rational number knowledge. Importantly, evidence from game performance records supports the notion that game play was directly responsible for at least part of the learning gains found in the present study. In short, while teacher support may be valuable for implementing these games in the classroom (Kapon et al., 2019), we show here that more time playing the game and performing well while playing both led to learning gains.

Understanding the relation between rational number magnitudes and the effects of operations on them appears to be one of the most difficult aspects of rational numbers for students and is also crucial for developing adaptive expertise with rational number arithmetic (McMullen et al., 2020; Siegler & Lortie-Forgues, 2015). The success of the NLEE in supporting this understanding likely benefited from the elaborative, but consistent, use of the number line as the core representation of rational number magnitude in the included games. The present findings suggest that designing learning environments, whether digital games or not, that require students to gradually integrate their magnitude, representations, and arithmetic knowledge into a single problem-solving process may be beneficial for their adaptive rational number knowledge.

Limitations and future directions

The main issue in the design of the present study is the use of a control condition. This limits the interpretations of the results of the study to the conclusion that it appears possible to promote adaptive rational number knowledge with the NLEE, not how effective it is in doing so, and does not allow for conclusions regarding the nature of this effectiveness. In other words, we cannot determine from the present study the mechanisms by which the NLEE was effective. For example, future studies should examine the role of the teacher in, for example, providing support for struggling students while they played the games or using the NLEE to foster classroom discussions about strategies when playing NanoRoboMath. Additionally, since the control group received no special instruction, there is a potential novelty effect of a new learning environment being captured in the results of the present study. Future research should include an active comparison group to account for this possibility, for example, using a game-based learning environment that is not designed to promote adaptive expertise. Despite these limitations, we believe this to be an important first step in understanding the nature of promoting adaptive expertise using game-based learning environments, and future studies that contrast different features of the games are now warranted.

Additional clarity regarding the effectiveness of the intervention, and its long-term impact on rational number knowledge would come from including (a) a more standardized assessment of rational number knowledge, (b) more indicators of adaptive rational number knowledge, (c) further transfer tasks, such as other aspects of adaptive expertise, and (d) a delayed posttest. However, we do note that the measures of routine size, operations, and representations of conceptual knowledge all draw from well-established measures of rational number knowledge (Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004; Van Hoof et al., 2015), which were not designed by the authors. Nonetheless, exploring the boundaries and other potential (long-term) effects of game playing would be needed.

Conclusions

In total, these results provide further evidence that designing digital game-based learning environments to promote adaptive expertise is an achievable and worthwhile goal and that open exploration and constant elaboration may be valuable design principles for promoting adaptive expertise in mathematics. Providing students with opportunities to explore mathematical ideas in less-

structured, dynamic, interactive, and conceptually rich environments is possible in game-based environments. The potential cost-effectiveness of scaling up these interventions, the ease of access to the environments even during extraordinary circumstances, and the malleability of the environment to specific needs both within and across classrooms all suggest that these environments are worthy of consideration for researchers and educators worldwide.

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