Properties of BLUEs and BLUPs in full vs. small linear models with new observations

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Abstract In this article we consider the partitioned linear model $\mathscr{M}_{12} = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}\}$, where $\boldsymbol{\mu} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2$, and the corresponding small model $\mathscr{M}_1 = \{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1, \mathbf{V}\}$, where $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$. These models are supplemented with the new unobservable random vector $\mathbf{y}_*$, coming from $\mathbf{y}_* = \mathbf{K}\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_*$, where the covariance matrix of $\mathbf{y}_*$ is known as well as the cross-covariance matrix between $\mathbf{y}_*$ and $\mathbf{y}$. We focus on comparing the BLUEs of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}$, and BLUPs of $\mathbf{y}_*$ and $\boldsymbol{\varepsilon}_*$ under $\mathscr{M}_{12}$ and $\mathscr{M}_1$.

Key words and phrases: Best linear unbiased estimator, BLUE, best linear unbiased predictor, BLUP, linear model with new observations, Löwner ordering, partitioned linear model.

1 Introduction

In this paper we consider the partitioned linear model $\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}$ and so-called small model (submodel) $\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}$, or shortly

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$$\mathscr{A}_{12} = \{\mathbf{y},\, \mathbf{X}\boldsymbol{\beta},\, \mathbf{V}\} = \{\mathbf{y},\, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2,\, \mathbf{V}\}\,, \quad \mathscr{A}_1 = \{\mathbf{y},\, \mathbf{X}_1\boldsymbol{\beta}_1,\, \mathbf{V}\}\,. \tag{1}$$

Here $\mathbf{y}$ is an n-dimensional observable response variable, and $\boldsymbol{\varepsilon}$ is an unobservable random error with a known covariance matrix $\operatorname{cov}(\boldsymbol{\varepsilon}) = \mathbf{V} = \operatorname{cov}(\mathbf{y})$ and expectation $\mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$. The matrix $\mathbf{X}$ is a known $n \times p$ matrix, i.e., $\mathbf{X} \in \mathbb{R}^{n\times p}$, partitioned columnwise as $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$, $\mathbf{X}_i \in \mathbb{R}^{n\times p_i}$, $i = 1, 2$. Vector $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')' \in \mathbb{R}^p$ is a vector of fixed (but unknown) parameters; here symbol $'$ stands for the transpose.

Let the new unknown q-dimensional future response $\mathbf{y}_*$ be

$$\mathbf{y}_* = \mathbf{X}_*\boldsymbol{\beta} + \boldsymbol{\varepsilon}_* = \mathbf{K}\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_*\,, \quad \mathbf{X}_* = (\mathbf{K} : \mathbf{0})\,, \quad \mathbf{K} \in \mathbb{R}^{q\times p_1}, \tag{2}$$

and

$$\operatorname{cov}\begin{pmatrix}\mathbf{y}\\ \mathbf{y}_*\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{V}_{12}\\ \mathbf{V}_{21} & \mathbf{V}_{22}\end{pmatrix}. \tag{3}$$

Of course, the word "new" need not be taken here literally. Putting $\mathscr{A}_{12}$, $\mathscr{A}_1$ and (2) together, we can denote the models shortly as

$$\mathscr{M}_1 = \left\{\begin{pmatrix}\mathbf{y}\\ \mathbf{y}_*\end{pmatrix}, \begin{pmatrix}\mathbf{X}_1\\ \mathbf{K}\end{pmatrix}\boldsymbol{\beta}_1, \begin{pmatrix}\mathbf{V} & \mathbf{V}_{12}\\ \mathbf{V}_{21} & \mathbf{V}_{22}\end{pmatrix}\right\}, \tag{4a}$$

$$\begin{aligned}\mathscr{M}_{12} &= \left\{\begin{pmatrix}\mathbf{y}\\ \mathbf{y}_*\end{pmatrix}, \begin{pmatrix}\mathbf{X}\\ \mathbf{X}_*\end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix}\mathbf{V} & \mathbf{V}_{12}\\ \mathbf{V}_{21} & \mathbf{V}_{22}\end{pmatrix}\right\}\\ &= \left\{\begin{pmatrix}\mathbf{y}\\ \mathbf{y}_*\end{pmatrix}, \begin{pmatrix}\mathbf{X}_1 & \mathbf{X}_2\\ \mathbf{K} & \mathbf{0}\end{pmatrix}\boldsymbol{\beta}, \begin{pmatrix}\mathbf{V} & \mathbf{V}_{12}\\ \mathbf{V}_{21} & \mathbf{V}_{22}\end{pmatrix}\right\}. \end{aligned}\tag{4b}$$

Thus: $\mathscr{M}_{12}$ is the full model with new observations and $\mathscr{M}_1$ is the small model with new observations. We may drop off the subscripts from $\mathscr{M}_{12}$ if the partitioning is not essential in the context. We are interested in estimating $\boldsymbol{\mu}_* = \mathbf{X}_*\boldsymbol{\beta}$ and predicting $\mathbf{y}_*$ and $\boldsymbol{\varepsilon}_*$ on the basis of $\mathbf{y}$.

As for notations, the symbols $\mathrm{r}(\mathbf{A})$, $\mathbf{A}^-$, $\mathbf{A}^+$, $\mathscr{C}(\mathbf{A})$, and $\mathscr{C}(\mathbf{A})^\perp$, denote, respectively, the rank, a generalized inverse, the (unique) Moore–Penrose inverse, the column space, and the orthogonal complement of the column space of the matrix $\mathbf{A}$. By $\mathbf{A}^\perp$ we denote any matrix satisfying $\mathscr{C}(\mathbf{A}^\perp) = \mathscr{C}(\mathbf{A})^\perp$. Furthermore, we will write $\mathbf{P}_\mathbf{A} = \mathbf{P}_{\mathscr{C}(\mathbf{A})} = \mathbf{A}\mathbf{A}^+ = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ to denote the orthogonal projector (with respect to the standard inner product) onto $\mathscr{C}(\mathbf{A})$. The orthogonal projector onto $\mathscr{C}(\mathbf{A})^\perp$ is denoted as $\mathbf{Q}_\mathbf{A} = \mathbf{I}_a - \mathbf{P}_\mathbf{A}$, where $\mathbf{I}_a$ refers to the $a \times a$ identity matrix and a is the number of rows of $\mathbf{A}$. It appears convenient to use the short notations

$$\mathbf{M} = \mathbf{I}_n - \mathbf{P}_\mathbf{X}\,, \quad \mathbf{M}_i = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}_i}\,,\ i = 1, 2. \tag{5}$$

One obvious choice for $\mathbf{X}^\perp$ is $\mathbf{M}$.

When using generalized inverses it is important to know whether the expressions are independent of the choice of the generalized inverses involved. Lemma 1 below gives some invariance conditions; cf. Rao & Mitra (1971, Lemma 2.2.4).

Lemma 1 *For nonnull matrices* $\mathbf{A}$ *and* $\mathbf{C}$ *the following holds:*

(a) $\mathbf{AB^-C} = \mathbf{AB^+C}$ *for all* $\mathbf{B}^-$ $\iff$ $\mathscr{C}(\mathbf{C}) \subset \mathscr{C}(\mathbf{B})$ & $\mathscr{C}(\mathbf{A}') \subset \mathscr{C}(\mathbf{B}')$.
(b) $\mathbf{AA^-C} = \mathbf{C}$ *for some (and hence for all)* $\mathbf{A}^-$ $\iff$ $\mathscr{C}(\mathbf{C}) \subset \mathscr{C}(\mathbf{A})$.

Let the set $\mathcal{W}$ of nonnegative definite matrices be defined as

$$\mathcal{W} = \{\mathbf{W} \in \mathbb{R}^{n\times n} : \mathbf{W} = \mathbf{V} + \mathbf{XUU'X'},\ \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V})\}\,. \tag{6}$$

In (6), $\mathbf{U}$ can be any matrix comprising p rows as long as $\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V})$ is satisfied. Lemma 2 collects together some important properties of the class $\mathcal{W}$; see, e.g., Baksalary et al. (1990, Th. 2) and Puntanen et al. (2011, Sect. 12.3).

Lemma 2 *Let* $\mathbf{V}$ *be an* $n \times n$ *nonnegative definite matrix, let* $\mathbf{X}$ *be an* $n \times p$ *matrix, and define* $\mathbf{W}$ *as* $\mathbf{W} = \mathbf{V} + \mathbf{XUU'X'}$, *where* $\mathbf{U}$ *is a* $p \times p$ *matrix, i.e.,* $\mathbf{W} \in \mathcal{W}$. *Then the following statements are equivalent:*

(a) $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{W})$,
(b) $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{W})$,
(c) $\mathscr{C}(\mathbf{X'W^-X}) = \mathscr{C}(\mathbf{X}')$ *for any choice of* $\mathbf{W}^-$,
(d) $\mathbf{X}(\mathbf{X'W^-X})^-\mathbf{X'W^-X} = \mathbf{X}$ *for any choices of* $\mathbf{W}^-$ *and* $(\mathbf{X'W^-X})^-$.

For the partitioned linear model $\mathscr{M}_{12}$ we will say that $\mathbf{W} \in \mathcal{W}$ if the following properties hold:

$$\mathbf{W} = \mathbf{V} + \mathbf{XUU'X'} = \mathbf{V} + \mathbf{X}_1\mathbf{U}_1\mathbf{U}_1'\mathbf{X}_1' + \mathbf{X}_2\mathbf{U}_2\mathbf{U}_2'\mathbf{X}_2', \tag{7a}$$

$$\mathbf{W}_i = \mathbf{V} + \mathbf{X}_i\mathbf{U}_i\mathbf{U}_i'\mathbf{X}_i', \quad i = 1, 2, \tag{7b}$$

$$\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V}), \quad \mathscr{C}(\mathbf{W}_i) = \mathscr{C}(\mathbf{X}_i : \mathbf{V}), \quad i = 1, 2. \tag{7c}$$

The particular choice of $\mathbf{U} = (\mathbf{X}_1 : \mathbf{U}_2)$ does not matter in our considerations and for simplicity we have put $\mathbf{U}_1'\mathbf{U}_2 = \mathbf{0}$.

By the consistency of the model $\mathscr{M}$ it is meant that $\mathbf{y}$ lies in $\mathscr{C}(\mathbf{X} : \mathbf{V})$ with probability 1; see, e.g., Baksalary et al. (1992). Hence we assume that under the consistent model $\mathscr{M}$ the observed numerical value of $\mathbf{y}$ satisfies

$$\mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{X} : \mathbf{VX}^\perp) = \mathscr{C}(\mathbf{X} : \mathbf{VM}) = \mathscr{C}(\mathbf{X}) \oplus \mathscr{C}(\mathbf{VM}), \tag{8}$$

where "$\oplus$" refers to the direct sum, implying that $\mathscr{C}(\mathbf{X}) \cap \mathscr{C}(\mathbf{VX}^\perp) = \{\mathbf{0}\}$. For the equality $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{X} : \mathbf{VM})$, we refer to Rao (1974, Lemma 2.1). There is a related decomposition, see, e.g., Puntanen et al. (2011, Th. 8): for any conformable matrices $\mathbf{A}$ and $\mathbf{B}$ we have

$$\mathscr{C}(\mathbf{A} : \mathbf{B}) = \mathscr{C}(\mathbf{A} : \mathbf{Q_A B}), \text{ and thereby } \mathbf{P}_{(\mathbf{A}:\mathbf{B})} = \mathbf{P_A} + \mathbf{P}_{\mathbf{Q_A B}}\,. \tag{9}$$

Thus we can obtain part (a) of Lemma 3 below.

Lemma 3 *Consider* $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$ *and let* $\mathbf{M}_2 = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}_2}$. *Then*

(a) $\mathbf{M} = \mathbf{I}_n - \mathbf{P}_{(\mathbf{X}_1:\mathbf{X}_2)} = \mathbf{I}_n - (\mathbf{P}_{\mathbf{X}_2} + \mathbf{P}_{\mathbf{M}_2\mathbf{X}_1}) = \mathbf{M}_2\mathbf{Q}_{\mathbf{M}_2\mathbf{X}_1} = \mathbf{Q}_{\mathbf{M}_2\mathbf{X}_1}\mathbf{M}_2$,
(b) $\mathscr{C}(\mathbf{X}_2) \subset \mathscr{C}(\mathbf{X}_1 : \mathbf{V}) \iff \mathscr{C}(\mathbf{M}_1\mathbf{X}_2) \subset \mathscr{C}(\mathbf{M}_1\mathbf{V})$.

For the following lemma, see, e.g., Isotalo et al. (2008a), Puntanen et al. (2011, Prop. 15.2) and Markiewicz & Puntanen (2019, Sec. 4).

Lemma 4 *Consider the partitioned linear model* $\{\mathbf{y}, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2, \mathbf{V}\}$*, let* $\mathbf{W} = \mathbf{V} + \mathbf{XUU}'\mathbf{X}' \in \mathcal{W}$ *and denote* $\mathbf{M}_1 = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}_1}$ *and*

$$\dot{\mathbf{M}} = \mathbf{M}(\mathbf{MVM})^{-}\mathbf{M}, \quad \dot{\mathbf{M}}_1 = \mathbf{M}_1(\mathbf{M}_1\mathbf{VM}_1)^{-}\mathbf{M}_1 . \tag{10a}$$

Then the following equalities hold:

(a) $$\begin{aligned}\mathbf{X}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X})^{-}\mathbf{X}'\mathbf{W}^{+} &= \mathbf{P}_{\mathbf{W}} - \mathbf{VM}(\mathbf{MVM})^{-}\mathbf{MP}_{\mathbf{W}} \\ &= \mathbf{P}_{\mathbf{W}} - \mathbf{V}\dot{\mathbf{M}}\mathbf{P}_{\mathbf{W}} \\ &= \mathbf{P}_{\mathbf{W}} - \mathbf{VM}(\mathbf{MVM})^{+} \\ &= \mathbf{P}_{\mathbf{W}} - \mathbf{VM}(\mathbf{MVM})^{+}\mathbf{M} .\end{aligned}$$

(b) *Replacing* $\mathbf{X}, \mathbf{W}, \mathbf{M}$ *and* $\dot{\mathbf{M}}$ *with* $\mathbf{X}_1, \mathbf{W}_1, \mathbf{M}_1$ *and* $\dot{\mathbf{M}}_1$ *in* (a)*, the corresponding expressions for* $\mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^{-}\mathbf{X}_1)^{-}\mathbf{X}_1'\mathbf{W}_1^{+}$ *can be obtained.*

A couple of clarifying words about Lemma 4 may be in place. We observe that

$$\begin{aligned}\mathbf{VM}(\mathbf{MVM})^{-}\mathbf{MP}_{\mathbf{W}} &= \mathbf{VM}(\mathbf{MVM})^{-}\mathbf{P}_{\mathbf{MV}} \\ &= \mathbf{VM}(\mathbf{MVM})^{+}\mathbf{P}_{\mathbf{MV}} \\ &= \mathbf{VM}(\mathbf{MVM})^{+},\end{aligned} \tag{11}$$

where we have used Lemma 1 and Lemma 3, which gives

$$\mathbf{MP}_{\mathbf{W}} = \mathbf{M}(\mathbf{P}_{\mathbf{X}} + \mathbf{P}_{\mathbf{MV}}) = \mathbf{MP}_{\mathbf{MV}} = \mathbf{P}_{\mathbf{MV}} . \tag{12}$$

In addition, it is noteworthy that the matrix $\dot{\mathbf{M}} = \mathbf{M}(\mathbf{MVM})^{-}\mathbf{M}$ is unique with respect to the choice of $(\mathbf{MVM})^{-}$ if and only if $\mathbb{R}^n = \mathscr{C}(\mathbf{X} : \mathbf{V})$, see Isotalo et al. (2008a, p. 1439). For the Moore–Penrose inverse the following holds:

$$\mathbf{M}(\mathbf{MVM})^{+}\mathbf{M} = (\mathbf{MVM})^{+}\mathbf{M} = \mathbf{M}(\mathbf{MVM})^{+} = (\mathbf{MVM})^{+}. \tag{13}$$

Let $\mathbf{A}$ and $\mathbf{B}$ be arbitrary $m \times n$ matrices. Then, in the consistent linear model $\mathscr{M}$, the estimators $\mathbf{Ay}$ and $\mathbf{By}$ are said to be equal with probability 1 if

$$\mathbf{Ay} = \mathbf{By} \quad \text{for all } \mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{W}), \tag{14}$$

where $\mathbf{W} \in \mathcal{W}$. Thus, if $\mathbf{A}$ and $\mathbf{B}$ satisfy (14), then $\mathbf{A} - \mathbf{B} = \mathbf{CQ}_{\mathbf{W}}$ for some matrix $\mathbf{C}$. When talking about the equality of estimators like $\mathbf{Ay} = \mathbf{By}$, we often drop off the phrase "with probability 1".

The properties of the BLUE deserve particular attention when $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathbb{R}^n$ does not hold: then there is an infinite number of multipliers $\mathbf{B}$ such that $\mathbf{By}$ is BLUE but for all such multipliers the vector $\mathbf{By}$ itself is unique (with probability 1, which is the phrase in this context). In the case of two linear models, $\mathscr{B}_i = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_i\}$, $i = 1, 2$, Mitra & Moore (1973) divide the problems into three questions:

(a) When is a specific linear representation of the BLUE of $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ under $\mathscr{B}_1$ also a BLUE under $\mathscr{B}_2$?

(b) When does $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ have a common BLUE under $\mathscr{B}_1$ and $\mathscr{B}_2$?
(c) When is the BLUE of $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ under $\mathscr{B}_1$ irrespective of the linear representation used in its expression, also a BLUE under $\mathscr{B}_2$?

The purpose of this paper is to do considerations in the spirit of Mitra & Moore (1973) regarding the models $\mathscr{M}_{12}$ and $\mathscr{M}_1$. We pick up particular fixed representations for the BLUEs and BLUPs under these two models, study the conditions under which they are equal for all values of $\mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$ or $\mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{V})$. Moreover, we review the conditions under which *all* representations of the BLUEs and BLUPs in one model continue to be valid in the other model. Corresponding relations between the covariance matrices of the BLUEs, BLUPs and prediction errors are characterized. The well-known (or pretty well-known) results are given as Lemmas, while the new (or at least not so well-known) results are represented as Propositions. As this paper is more like a review-type, though providing some new characterizations, we provide a reasonable background and matrix tools, to make the article more self-contained, i.e., easier to read. Most of this backgound material is in the first two sections.

2 Fundamental BLUE and BLUP equations

A linear statistic $\mathbf{By}$ is said to be linear unbiased estimator (LUE) for $\boldsymbol{\mu}_* = \mathbf{X}_*\boldsymbol{\beta}$ in $\mathscr{M}_{12}$ if its expectation is equal to $\boldsymbol{\mu}_*$, which happens if and only if $\mathbf{X}_*' = \mathbf{X}'\mathbf{B}'$. For our purposes, the parametric function $\boldsymbol{\mu}_* = \mathbf{K}\boldsymbol{\beta}_1$ must be estimable in $\mathscr{M}_{12}$ and $\mathscr{M}_1$ as well. Now, see, e.g., Groß & Puntanen (2000, Lemma 1),

$$\boldsymbol{\mu}_* = \mathbf{X}_*\boldsymbol{\beta} = (\mathbf{K} : \mathbf{0})\boldsymbol{\beta} = \mathbf{K}\boldsymbol{\beta}_1 \quad \text{is estimable under } \mathscr{M}_{12} \tag{15}$$

if and only if $\mathscr{C}(\mathbf{K}') \subset \mathscr{C}(\mathbf{X}_1'\mathbf{M}_2)$, i.e., $\mathbf{K} = \mathbf{J}\mathbf{M}_2\mathbf{X}_1$. Thus

$$\mathbf{X}_* = (\mathbf{K} : \mathbf{0}) = (\mathbf{J}\mathbf{M}_2\mathbf{X}_1 : \mathbf{0}) = \mathbf{J}\mathbf{M}_2\mathbf{X} = \mathbf{L}\mathbf{X}, \text{ where } \mathbf{L} = \mathbf{J}\mathbf{M}_2 . \tag{16}$$

This means that for our purpose it is essential to consider the best LUE, i.e., the BLUE of $\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$. Obviously (16) means that $\mathbf{K}\boldsymbol{\beta}_1$ is estimable also under $\mathscr{M}_1$.

The LUE $\mathbf{By}$ is the best linear unbiased estimator, BLUE, of estimable $\mathbf{X}_*\boldsymbol{\beta}$ if $\mathbf{By}$ has the smallest covariance matrix in the Löwner sense among all LUEs of $\mathbf{X}_*\boldsymbol{\beta}$:

$$\operatorname{cov}(\mathbf{By}) \leq_{\mathrm{L}} \operatorname{cov}(\mathbf{B}_\#\mathbf{y}) \quad \text{for all } \mathbf{B}_\# : \mathbf{B}_\#\mathbf{X} = \mathbf{X}_* . \tag{17}$$

Correspondingly, the linear predictor $\mathbf{Ay}$ is said to be unbiased for $\mathbf{y}_*$ if the expected prediction error is zero, i.e., $\mathrm{E}(\mathbf{y}_* - \mathbf{Ay}) = \mathbf{0}$ for all $\boldsymbol{\beta} \in \mathbb{R}^p$, which happens if and only if $\mathbf{X}_*' = \mathbf{X}'\mathbf{A}'$. When $\mathscr{C}(\mathbf{X}_*') \subset \mathscr{C}(\mathbf{X}')$ holds, we will say that $\mathbf{y}_*$ is predictable under $\mathscr{M}$. Now a linear unbiased predictor $\mathbf{Ay}$ is the best linear unbiased predictor, BLUP, for $\mathbf{y}_*$, if we have the Löwner ordering

$$\operatorname{cov}(\mathbf{y}_* - \mathbf{Ay}) \leq_{\mathrm{L}} \operatorname{cov}(\mathbf{y}_* - \mathbf{A}_\#\mathbf{y}) \quad \text{for all } \mathbf{A}_\# : \mathbf{A}_\#\mathbf{X} = \mathbf{X}_* . \tag{18}$$

Consider then the BLUP of $\boldsymbol{\varepsilon}_*$. Obviously $\mathbf{Dy}$ is an unbiased predictor for $\boldsymbol{\varepsilon}_*$ if and only if $\mathbf{DX} = \mathbf{0}$, i.e., $\mathbf{D} = \mathbf{FM}$ for some $\mathbf{L}$. Thus the unbiased $\mathbf{Dy}$ is the BLUP for $\boldsymbol{\varepsilon}_*$ if and only if

$$\operatorname{cov}(\boldsymbol{\varepsilon}_* - \mathbf{Dy}) \leq_{\mathrm{L}} \operatorname{cov}(\boldsymbol{\varepsilon}_* - \mathbf{FMy}) \quad \text{for all } \mathbf{F} \in \mathbb{R}^{q\times n}. \tag{19}$$

For Lemma 5, characterizing the BLUE, see, e.g., Rao (1973, p. 282), and the BLUP, see, e.g., Christensen (2011, p. 294), and Isotalo & Puntanen (2006, p. 1015). For part (d), see Isotalo et al. (2018, Th. 3.1). For the general reviews of the BLUP-properties, see, e.g., Tian (2015a,b), Haslett & Puntanen (2017), and Markiewicz & Puntanen (2018).

Lemma 5 *Consider the linear model with new observations defined as $\mathscr{M}_{12}$ where $\mathscr{C}(\mathbf{X}_*') \subset \mathscr{C}(\mathbf{X}')$, i.e., $\mathbf{y}_*$ is predictable. Then the following statements hold:*

(a) $\mathbf{Ay} = \operatorname{BLUP}(\mathbf{y}_*) \iff \mathbf{A}(\mathbf{X} : \mathbf{VX}^{\perp}) = (\mathbf{X}_* : \mathbf{V}_{21}\mathbf{X}^{\perp})$, *i.e.,* $\mathbf{A} \in \{\mathbf{P}_{\mathbf{y}_*|\mathscr{M}_{12}}\}$.
(b) $\mathbf{By} = \operatorname{BLUE}(\mathbf{X}_*\boldsymbol{\beta}) \iff \mathbf{B}(\mathbf{X} : \mathbf{VX}^{\perp}) = (\mathbf{X}_* : \mathbf{0})$, *i.e.,* $\mathbf{B} \in \{\mathbf{P}_{\mathbf{X}_*|\mathscr{M}_{12}}\}$.
(c) $\mathbf{Cy} = \operatorname{BLUE}(\mathbf{X}\boldsymbol{\beta}) \iff \mathbf{C}(\mathbf{X} : \mathbf{VX}^{\perp}) = (\mathbf{X} : \mathbf{0})$, *i.e.,* $\mathbf{C} \in \{\mathbf{P}_{\mathbf{X}|\mathscr{M}_{12}}\}$.
(d) $\mathbf{Dy} = \operatorname{BLUP}(\boldsymbol{\varepsilon}_*) \iff \mathbf{D}(\mathbf{X} : \mathbf{VX}^{\perp}) = (\mathbf{0} : \mathbf{V}_{21}\mathbf{X}^{\perp})$, *i.e.,* $\mathbf{D} \in \{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_{12}}\}$.

The sets $\{\mathbf{P}_{\mathbf{y}_*|\mathscr{M}_1}\}$, $\{\mathbf{P}_{\mathbf{X}_1|\mathscr{M}_1}\}$ and $\{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_1}\}$ are defined in the corresponding way. Putting (b) and (d) of Lemma 5 together yields

$$\begin{pmatrix} \mathbf{B} \\ \mathbf{D} \end{pmatrix} (\mathbf{X} : \mathbf{VX}^{\perp}) = \begin{pmatrix} \mathbf{X}_* & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{21}\mathbf{X}^{\perp} \end{pmatrix}, \tag{20}$$

which implies that

$$\mathbf{A}(\mathbf{X} : \mathbf{VX}^{\perp}) = (\mathbf{B} + \mathbf{D})(\mathbf{X} : \mathbf{VX}^{\perp}) = (\mathbf{X}_* : \mathbf{V}_{21}\mathbf{X}^{\perp}), \tag{21}$$

and thereby $(\mathbf{B} + \mathbf{D})\mathbf{y}$ is the BLUP for $\mathbf{y}_*$ and we have

$$\operatorname{BLUP}(\mathbf{y}_*) = \operatorname{BLUE}(\mathbf{X}_*\boldsymbol{\beta}) + \operatorname{BLUP}(\boldsymbol{\varepsilon}_*), \ \text{ i.e., } \tilde{\mathbf{y}}_* = \tilde{\boldsymbol{\mu}}_* + \tilde{\boldsymbol{\varepsilon}}_*. \tag{22}$$

Using Lemma 2 we can obtain, for example, the following well-known solutions to $\mathbf{B}$ and $\mathbf{C}$ in Lemma 5:

$$\mathbf{X}_*(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^- \in \{\mathbf{P}_{\mathbf{X}_*|\mathscr{M}}\}, \quad \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^- \in \{\mathbf{P}_{\mathbf{X}|\mathscr{M}}\}, \tag{23}$$

where $\mathbf{W} \in \mathcal{W}$ and we can freely choose the generalized inverses involved. Expression $\mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X}')^-\mathbf{X}'\mathbf{W}^-$ is not necessarily unique with respect to the choice of $\mathbf{W}^-$ but

$$\mathbf{G} = \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X}')^-\mathbf{X}'\mathbf{W}^+ \in \{\mathbf{P}_{\mathbf{X}|\mathscr{M}}\} \tag{24}$$

is unique whatever choice of $\mathbf{W}^-$ we have. The *general* solution for $\mathbf{C}$ in Lemma 5, can be expressed, for example, as

$$\mathbf{P}_{\mathbf{X}|\mathscr{M}} = \mathbf{G} + \mathbf{NQ}_{\mathbf{W}}, \quad \text{where } \mathbf{N} \in \mathbb{R}^{n\times n} \text{ is free to vary}, \tag{25}$$

and $\mathbf{Q_W} = \mathbf{I}_n - \mathbf{P_W}$. Thus the solution for $\mathbf{C}$ is unique if and only if $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathbb{R}^n$.

In particular, in view of Lemma 4, we have the following:

$$\begin{aligned}\mathbf{G} &= \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+ = \mathbf{P_W} - \mathbf{VM}(\mathbf{MVM})^-\mathbf{MP_W} \\ &= \mathbf{P_W} - \mathbf{VM}(\mathbf{MVM})^+ = \mathbf{P_W} - \mathbf{VM}(\mathbf{MVM})^+\mathbf{M}, \end{aligned} \tag{26}$$

and thus

$$\mathbf{I}_n - \mathbf{G} = \mathbf{Q_W} + \mathbf{VM}(\mathbf{MVM})^-\mathbf{MP_W} = \mathbf{Q_W} + \mathbf{VM}(\mathbf{MVM})^+\mathbf{M}\,. \tag{27}$$

Corresponding expressions for $\mathbf{G}_1$ can be obtained by replacing $\mathbf{X}$, $\mathbf{M}$ and $\mathbf{W}$ with $\mathbf{X}_1$, $\mathbf{M}_1$ and $\mathbf{W}_1$, respectively, in (26). Premultiplying (26) by $\mathbf{P_X}$ gives

$$\mathbf{G} = \mathbf{P_X} - \mathbf{P_X}\mathbf{VM}(\mathbf{MVM})^+\mathbf{M}, \tag{28}$$

and correspondingly,

$$\mathbf{G}_1 = \mathbf{P}_{\mathbf{X}_1} - \mathbf{P}_{\mathbf{X}_1}\mathbf{VM}_1(\mathbf{M}_1\mathbf{VM}_1)^+\mathbf{M}_1\,. \tag{29}$$

Notice that by Lemma 1,

- $\mathbf{G} = \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+$ is unique for any choice of $\mathbf{W}^-$ and $(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-$,
- $\mathbf{VM}(\mathbf{MVM})^-\mathbf{MP_W}$ is unique for any choice of $(\mathbf{MVM})^-$,
- $\mathbf{VM}(\mathbf{MVM})^-\mathbf{M}$ is unique for any choice of $(\mathbf{MVM})^-$ if and only if $\mathrm{r}(\mathbf{MV}) = \mathrm{r}(\mathbf{M})$, i.e., $\mathrm{r}(\mathbf{X} : \mathbf{V}) = n$.

In order to consider the $\mathrm{BLUP}(\boldsymbol{\varepsilon}_*)$ under $\mathscr{M}_{12}$, we observe that

$$\mathbf{V}_{21}\mathbf{V}^+(\mathbf{I}_n - \mathbf{G})(\mathbf{X} : \mathbf{VM}) = (\mathbf{0} : \mathbf{V}_{21}\mathbf{M}), \tag{30}$$

and thus $\mathbf{V}_{21}\mathbf{V}^+(\mathbf{I}_n - \mathbf{G}) \in \{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_{12}}\}$. On the other hand, in view of (27),

$$\mathbf{V}_{21}\mathbf{V}^+(\mathbf{I}_n - \mathbf{G}) = \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^-\mathbf{MP_W} = \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^+\mathbf{M}, \tag{31}$$

and so

$$\begin{aligned}\tilde{\boldsymbol{\varepsilon}}_* &= \mathrm{BLUP}(\boldsymbol{\varepsilon}_* \mid \mathscr{M}_{12}) = \mathbf{V}_{21}\mathbf{V}^+(\mathbf{I}_n - \mathbf{G})\mathbf{y} \\ &= \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^-\mathbf{MP_W}\mathbf{y} \\ &= \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^+\mathbf{My} \\ &= \mathbf{Ey}, \end{aligned} \tag{32}$$

where we have denoted

$$\mathbf{E} = \mathbf{V}_{21}\mathbf{V}^+(\mathbf{I}_n - \mathbf{G}) = \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^+\mathbf{M} \in \{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_{12}}\}\,. \tag{33}$$

The equation (32) holds for *any* $\mathbf{y} \in \mathbb{R}^n$. In particular, if $\mathbf{y} \in \mathscr{C}(\mathbf{W})$, then we can replace $(\mathbf{MVM})^+$ with any $(\mathbf{MVM})^-$. In the case of the small model we denote

$$\mathbf{E}_1 = \mathbf{V}_{21}\mathbf{V}^{+}(\mathbf{I}_n - \mathbf{G}_1) = \mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{V}\mathbf{M}_1)^{+}\mathbf{M}_1 \in \{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_1}\}\,. \tag{34}$$

Moreover, it can be observed that

$$\mathbf{E} = \mathbf{V}_{21}\mathbf{W}^{+}(\mathbf{I}_n - \mathbf{G})\,, \quad \mathbf{E}_1 = \mathbf{V}_{21}\mathbf{W}_1^{+}(\mathbf{I}_n - \mathbf{G}_1)\,. \tag{35}$$

Let us denote $\mathbf{L} = \mathbf{J}\mathbf{M}_2$ and $\mathbf{S} = \mathbf{L} - \mathbf{V}_{21}\mathbf{V}^{+}$. Then we can write

$$\begin{aligned}\tilde{\mathbf{y}}_* &= \mathrm{BLUP}(\tilde{\mathbf{y}}_* \mid \mathscr{M}_{12}) \\ &= \mathbf{LGy} + \mathbf{V}_{21}\mathbf{V}^{+}(\mathbf{y} - \mathbf{Gy}) = \mathbf{LGy} + \mathbf{Ey} \\ &= (\mathbf{L} - \mathbf{V}_{21}\mathbf{V}^{+})\mathbf{Gy} + \mathbf{V}_{21}\mathbf{V}^{+}\mathbf{y} = \mathbf{SGy} + \mathbf{V}_{21}\mathbf{V}^{+}\mathbf{y} \\ &= \mathbf{Ty},\end{aligned} \tag{36}$$

where

$$\begin{aligned}\mathbf{T} &= \mathbf{LG} + \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^{-}\mathbf{M}\mathbf{P}_{\mathbf{W}} = \mathbf{LG} + \mathbf{E} \\ &= \mathbf{SG} + \mathbf{V}_{21}\mathbf{V}^{+} \in \{\mathbf{P}_{\mathbf{y}_*|\mathscr{M}_{12}}\}\,.\end{aligned} \tag{37}$$

Let us put our results together:

Lemma 6 *Let* $\mathbf{y}_*$ *be predictable under* $\mathscr{M}_{12}$*, so that*

$$\mathbf{X}_* = (\mathbf{K} : \mathbf{0}) = \mathbf{J}\mathbf{M}_2\mathbf{X} = \mathbf{LX} = (\mathbf{J}\mathbf{M}_2\mathbf{X}_1 : \mathbf{0}) = (\mathbf{L}\mathbf{X}_1 : \mathbf{0}) \tag{38}$$

for some $\mathbf{J} \in \mathbb{R}^{q\times n}$*,* $\mathbf{L} = \mathbf{J}\mathbf{M}_2$*,* $\mathbf{S} = \mathbf{L} - \mathbf{V}_{21}\mathbf{V}^{+}$ *and*

$$\mathbf{G} = \mathbf{X}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X}')^{-}\mathbf{X}'\mathbf{W}^{+}, \qquad \mathbf{G}_1 = \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}^{-}\mathbf{X}_1)^{-}\mathbf{X}_1'\mathbf{W}^{+}, \tag{39a}$$

$$\mathbf{E} = \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^{+}\mathbf{M}, \qquad \mathbf{E}_1 = \mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{V}\mathbf{M}_1)^{-}\mathbf{M}_1\,. \tag{39b}$$

Then the $\mathrm{BLUP}(\mathbf{y}_*)$ *under* $\mathscr{M}_{12}$ *can be written as*

$$\begin{aligned}\mathrm{BLUP}(\mathbf{y}_* \mid \mathscr{M}_{12}) &= \tilde{\mathbf{y}}_* \\ &= \mathbf{LGy} + \mathbf{V}_{21}\mathbf{V}^{+}(\mathbf{I}_n - \mathbf{G})\mathbf{y} \\ &= \mathbf{LGy} + \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^{-}\mathbf{M}\mathbf{P}_{\mathbf{W}}\mathbf{y} \\ &= \mathbf{LGy} + \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^{+}\mathbf{My} \\ &= \mathbf{LGy} + \mathbf{Ey} \\ &= (\mathbf{L} - \mathbf{V}_{21}\mathbf{V}^{+})\mathbf{Gy} + \mathbf{V}_{21}\mathbf{V}^{+}\mathbf{y} \\ &= \mathbf{SGy} + \mathbf{V}_{21}\mathbf{V}^{+}\mathbf{y} \\ &= \mathbf{Ty},\end{aligned} \tag{40}$$

or shortly,

$$\tilde{\mathbf{y}}_* = \tilde{\boldsymbol{\mu}}_* + \tilde{\boldsymbol{\varepsilon}}_*\,. \tag{41}$$

Corresponding expressions for the $\mathrm{BLUP}(\mathbf{y}_*)$ *under* $\mathscr{M}_1$*, i.e., for* $\tilde{\mathbf{y}}_{*1} = \mathbf{T}_1\mathbf{y}$ *can be obtained by replacing* $\mathbf{G}$*,* $\mathbf{X}$*,* $\mathbf{M}$ *and* $\mathbf{W}$ *with* $\mathbf{G}_1$*,* $\mathbf{X}_1$*,* $\mathbf{M}_1$ *and* $\mathbf{W}_1$*, respectively, in*

(40)*; shortly,*

$$\tilde{\mathbf{y}}_{*1} = \tilde{\boldsymbol{\mu}}_{*1} + \tilde{\boldsymbol{\varepsilon}}_{*1}\,. \tag{42}$$

For the covariance matrices we get

$$\operatorname{cov}(\tilde{\boldsymbol{\mu}}) = \operatorname{cov}(\mathbf{Gy}) = \mathbf{GVG}' = \mathbf{V} - \mathbf{VM}(\mathbf{MVM})^{-}\mathbf{MV}, \tag{43a}$$

$$\operatorname{cov}(\tilde{\boldsymbol{\mu}}_*) = \operatorname{cov}(\mathbf{LGy}) = \mathbf{LGVG}'\mathbf{L}', \tag{43b}$$

$$\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_*) = \operatorname{cov}(\mathbf{Ey}) = \mathbf{EVE}' = \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^{-}\mathbf{MV}_{12}\,. \tag{43c}$$

For an extensive review of the BLUE's covariance matrix, see Isotalo et al. (2008b). The random vectors $\tilde{\boldsymbol{\mu}}_*$ and $\tilde{\boldsymbol{\varepsilon}}_*$ are uncorrelated,

$$\operatorname{cov}(\tilde{\boldsymbol{\mu}}_*, \tilde{\boldsymbol{\varepsilon}}_*) = \operatorname{cov}(\mathbf{LGy}, \mathbf{Ey}) = \mathbf{LGVM}(\mathbf{MVM})^{+}\mathbf{M} = \mathbf{0}\,, \tag{44}$$

where we have used the fact that $\mathbf{GVM} = \mathbf{0}$ and thereby

$$\operatorname{cov}(\tilde{\mathbf{y}}_*) = \operatorname{cov}(\tilde{\boldsymbol{\mu}}_*) + \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_*) = \mathbf{LGVG}'\mathbf{L}' + \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^{-}\mathbf{MV}_{12}\,. \tag{45}$$

3 Equalities of the BLUEs under the full and small models

Let us start by considering the equality between $\operatorname{cov}(\mathbf{Gy})$ and $\operatorname{cov}(\mathbf{G}_1\mathbf{y})$. In light of $\mathbf{GG}_1 = \mathbf{G}_1$, we have

$$\operatorname{cov}(\tilde{\boldsymbol{\mu}}_1 \mid \mathscr{M}_1) = \mathbf{G}_1\mathbf{VG}_1' = \mathbf{GG}_1\mathbf{VG}_1'\mathbf{G}', \tag{46}$$

and thus

$$\begin{aligned}\operatorname{cov}(\tilde{\boldsymbol{\mu}} \mid \mathscr{M}_{12}) - \operatorname{cov}(\tilde{\boldsymbol{\mu}}_1 \mid \mathscr{M}_1) &= \mathbf{GVG}' - \mathbf{GG}_1\mathbf{VG}_1'\mathbf{G}' \\ &= \mathbf{G}(\mathbf{V} - \mathbf{G}_1\mathbf{VG}_1')\mathbf{G}' \\ &= \mathbf{GVM}_1(\mathbf{M}_1\mathbf{VM}_1)^{-}\mathbf{M}_1\mathbf{VG}' \\ &= \mathbf{GV}\dot{\mathbf{M}}_1\mathbf{VG}',\end{aligned} \tag{47}$$

where we have used

$$\operatorname{cov}(\mathbf{G}_1\mathbf{y}) = \mathbf{G}_1\mathbf{VG}_1' = \mathbf{V} - \mathbf{VM}_1(\mathbf{M}_1\mathbf{VM}_1)^{-}\mathbf{M}_1\mathbf{V}. \tag{48}$$

Clearly $\mathbf{GV}\dot{\mathbf{M}}_1\mathbf{VG}'$ is nonnegative definite and thereby

$$\operatorname{cov}(\tilde{\boldsymbol{\mu}}_1 \mid \mathscr{M}_1) \leq_{\mathrm{L}} \operatorname{cov}(\tilde{\boldsymbol{\mu}} \mid \mathscr{M}_{12})\,. \tag{49}$$

It is obvious that the equality $\operatorname{cov}(\tilde{\boldsymbol{\mu}}_1 \mid \mathscr{M}_1) = \operatorname{cov}(\tilde{\boldsymbol{\mu}} \mid \mathscr{M}_{12})$ holds if and only if

$$\mathbf{GVM}_1 = \mathbf{0}\,. \tag{50}$$

Actually, the above equality (50) is a necessary and sufficient condition for $\mathbf{Gy}$ being the BLUE for $\boldsymbol{\mu}_1$ under the small model $\mathscr{M}_1$. Recall that the fundamental BLUE equation in this case is

$$\mathbf{G}(\mathbf{X}_1 : \mathbf{VM}_1) = (\mathbf{X}_1 : \mathbf{0}), \tag{51}$$

where the left-hand part $\mathbf{GX}_1 = \mathbf{X}_1$ trivially holds. Thus (50) is equivalent to $\mathbf{G} \in \{\mathbf{P}_{\mathbf{X}_1|\mathscr{M}_1}\}$. The general expression for a member of the class $\{\mathbf{P}_{\mathbf{X}|\mathscr{M}}\}$ is

$$\mathbf{P}_{\mathbf{X}|\mathscr{M}} = \mathbf{G} + \mathbf{NQ_W}, \quad \text{where } \mathbf{N} \text{ is free to vary.} \tag{52}$$

It is easy to confirm that $\{\mathbf{P}_{\mathbf{X}|\mathscr{M}}\} \subset \{\mathbf{P}_{\mathbf{X}_1|\mathscr{M}_1}\}$ if and only if $\mathbf{GVM}_1 = \mathbf{0}$. In other words, every representation of the BLUE of $\boldsymbol{\mu}$ under $\mathscr{M}_{12}$ is BLUE also under $\mathscr{M}_1$, for which we can use notation

$$\{\mathrm{BLUE}(\boldsymbol{\mu} \mid \mathscr{M}_{12})\} \subset \{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_1)\}, \quad \text{i.e.,} \quad \{\mathbf{P}_{\mathbf{X}|\mathscr{M}_{12}}\} \subset \{\mathbf{P}_{\mathbf{X}_1|\mathscr{M}_1}\}. \tag{53}$$

It may be mentioned that writing up the condition $\mathbf{GVM}_1 = \mathbf{0}$ we obtain

$$\mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X}')^-\mathbf{X}'\mathbf{W}^+\mathbf{VM}_1 = \mathbf{0}, \tag{54}$$

which is equivalent to

$$\mathbf{X}'\mathbf{W}^+\mathbf{VM}_1 = \mathbf{0}. \tag{55}$$

Consider then the covariance matrix of $\mathbf{Gy} - \mathbf{G}_1\mathbf{y}$:

$$\mathrm{cov}(\mathbf{Gy} - \mathbf{G}_1\mathbf{y}) = \mathbf{GVG}' + \mathbf{G}_1\mathbf{VG}_1' - \mathbf{GVG}_1' - \mathbf{G}_1\mathbf{VG}'. \tag{56}$$

In view of $\mathbf{G}_1 = \mathbf{P}_{\mathbf{W}_1} - \mathbf{VM}_1(\mathbf{M}_1\mathbf{VM}_1)^+\mathbf{M}_1$, we have

$$\begin{aligned}
\mathbf{G}_1\mathbf{VG}' &= [\mathbf{P}_{\mathbf{W}_1} - \mathbf{VM}_1(\mathbf{M}_1\mathbf{VM}_1)^+\mathbf{M}_1]\mathbf{VG}' \\
&= [\mathbf{V} - \mathbf{VM}_1(\mathbf{M}_1\mathbf{VM}_1)^+\mathbf{M}_1\mathbf{V}]\mathbf{G}' \\
&= \mathbf{G}_1\mathbf{VG}_1'\mathbf{G}' = \mathbf{G}_1\mathbf{VG}_1',
\end{aligned} \tag{57}$$

where we have used $\mathbf{GG}_1 = \mathbf{G}_1$ and $\mathbf{G}_1\mathbf{VG}_1' = \mathbf{V} - \mathbf{V}\dot{\mathbf{M}}_1\mathbf{V}$. Thus

$$\begin{aligned}
\mathrm{cov}(\mathbf{Gy} - \mathbf{G}_1\mathbf{y}) &= \mathrm{cov}(\mathbf{Gy}) - \mathrm{cov}(\mathbf{G}_1\mathbf{y}) \\
&= \mathbf{GVM}_1(\mathbf{M}_1\mathbf{VM}_1)^-\mathbf{M}_1\mathbf{VG}' \\
&= \mathrm{cov}\,|\mathbf{GVM}_1(\mathbf{M}_1\mathbf{VM}_1)^-\mathbf{M}_1\mathbf{y}].
\end{aligned} \tag{58}$$

Notice that the matrix $\mathbf{GV}\dot{\mathbf{M}}_1$ may not be unique but $\mathrm{cov}(\mathbf{GV}\dot{\mathbf{M}}_1\mathbf{y})$ is unique with respect to the choice of $(\mathbf{M}_1\mathbf{VM}_1)^-$ in $\dot{\mathbf{M}}_1 = \mathbf{M}_1(\mathbf{M}_1\mathbf{VM}_1)^-\mathbf{M}_1$.

We can now put our findings together:

Proposition 1 *The following statements are equivalent:*

(a) $\mathbf{Gy} = \mathbf{G}_1\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W}_1) = \mathscr{C}(\mathbf{X}_1 : \mathbf{VM}_1)$,
(b) $\mathrm{cov}(\tilde{\boldsymbol{\mu}}_1 \mid \mathscr{M}_1) = \mathrm{cov}(\tilde{\boldsymbol{\mu}} \mid \mathscr{M}_{12})$,
(c) $\mathbf{GVM}_1 = \mathbf{0}$,

(d) $\mathbf{X}'\mathbf{W}^{+}\mathbf{V}\mathbf{M}_1 = \mathbf{0}$,
(e) $\mathbf{G} \in \{\mathbf{P}_{\mathbf{X}_1 \mid \mathscr{M}_1}\}$,
(f) $\{\mathbf{P}_{\mathbf{X} \mid \mathscr{M}_{12}}\} \subset \{\mathbf{P}_{\mathbf{X}_1 \mid \mathscr{M}_1}\}$, *i.e.,* $\{\mathrm{BLUE}(\boldsymbol{\mu} \mid \mathscr{M}_{12})\} \subset \{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_1)\}$.

Moreover, the following properties hold:

(g) $\mathbf{G}_1\mathbf{V}\mathbf{G}' = \mathbf{G}_1\mathbf{V}\mathbf{G}_1'$,
(h) $\mathrm{cov}(\mathbf{G}\mathbf{y} - \mathbf{G}_1\mathbf{y}) = \mathrm{cov}(\mathbf{G}\mathbf{y}) - \mathrm{cov}(\mathbf{G}_1\mathbf{y}) = \mathrm{cov}(\mathbf{G}\mathbf{V}\dot{\mathbf{M}}_1\mathbf{y})$,
(i) $\mathrm{cov}(\mathbf{G}_1\mathbf{y}) \leq_{\mathrm{L}} \mathrm{cov}(\mathbf{G}\mathbf{y})$.

What about the equality

$$\mathbf{G}\mathbf{y} = \mathbf{G}_1\mathbf{y} \quad \text{for all } \mathbf{y} \in \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V}\mathbf{M}), \tag{59}$$

i.e.,

$$\mathbf{G}_1(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V}\mathbf{M}) = (\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{0}). \tag{60}$$

Noting that $\mathbf{G}_1\mathbf{X}_1 = \mathbf{X}_1$ and $\mathbf{G}_1\mathbf{V}\mathbf{M} = \mathbf{G}_1\mathbf{V}\mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2} = \mathbf{0}$, we conclude that (60) holds, i.e., $\mathbf{G}_1 \in \{\mathbf{P}_{\mathbf{X} \mid \mathscr{M}_{12}}\}$, if and only if

$$\mathbf{G}_1\mathbf{X}_2 = \mathbf{X}_2, \quad \text{i.e.,} \quad \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^{-}\mathbf{X}_1')^{-}\mathbf{X}_1'\mathbf{W}_1^{+}\mathbf{X}_2 = \mathbf{X}_2. \tag{61}$$

It is clear that (61) implies

$$\mathscr{C}(\mathbf{X}_2) \subset \mathscr{C}(\mathbf{X}_1). \tag{62}$$

On the other hand, if (62) holds then $\mathbf{X}_2 = \mathbf{X}_1\mathbf{A}$ for some $\mathbf{A}$ which further implies (61). Assuming that (62) holds, i.e., $\mathscr{C}(\mathbf{X}) = \mathscr{C}(\mathbf{X}_1)$, we can use (28) and (29) and conclude that $\mathbf{G} = \mathbf{G}_1$ holds if and only if (62) holds.

Let us consider the general expression for the member of the class $\{\mathbf{P}_{\mathbf{X}_1 \mid \mathscr{M}_1}\}$:

$$\mathbf{P}_{\mathbf{X}_1 \mid \mathscr{M}_1} = \mathbf{G}_1 + \mathbf{N}\mathbf{Q}_{\mathbf{W}_1} \quad \text{for some } \mathbf{N}. \tag{63}$$

The equality

$$\mathbf{G}\mathbf{y} = (\mathbf{G}_1 + \mathbf{N}\mathbf{Q}_{\mathbf{W}_1})\mathbf{y} \quad \text{for all } \mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V}\mathbf{M}), \tag{64}$$

i.e., $(\mathbf{G}_1 + \mathbf{N}\mathbf{Q}_{\mathbf{W}_1})(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V}\mathbf{M}) = (\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{0})$, simplifies to

$$\mathbf{G}_1\mathbf{X}_2 + \mathbf{N}\mathbf{Q}_{\mathbf{W}_1}\mathbf{X}_2 = \mathbf{X}_2. \tag{65}$$

Requesting (65) to hold for *any* $\mathbf{N}$ yields $\mathscr{C}(\mathbf{X}_2) \subset \mathscr{C}(\mathbf{W}_1)$, and consequently, $\mathbf{G}_1\mathbf{X}_2 = \mathbf{X}_2$. Thus we conclude the following:

$$\{\mathbf{P}_{\mathbf{X}_1 \mid \mathscr{M}_1}\} \subset \{\mathbf{P}_{\mathbf{X} \mid \mathscr{M}_{12}}\} \iff \mathscr{C}(\mathbf{X}_2) \subset \mathscr{C}(\mathbf{X}_1). \tag{66}$$

Moreover, the inclusion $\mathscr{C}(\mathbf{X}_2) \subset \mathscr{C}(\mathbf{X}_1)$ implies that $\mathbf{G}\mathbf{V}\mathbf{M}_1 = \mathbf{0}$, which further, by Proposition 1, implies that $\{\mathbf{P}_{\mathbf{X} \mid \mathscr{M}_{12}}\} \subset \{\mathbf{P}_{\mathbf{X}_1 \mid \mathscr{M}_1}\}$.

We can also pose a question under which the set $\{\mathbf{P}_{\mathbf{X} \mid \mathscr{M}_{12}}\} \cap \{\mathbf{P}_{\mathbf{X}_1 \mid \mathscr{M}_1}\}$ is not empty. This happens whenever the equation

$$\mathbf{A}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM} : \mathbf{X}_1 : \mathbf{VM}_1) = (\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{0} : \mathbf{X}_1 : \mathbf{0})\,, \tag{67}$$

i.e., $\mathbf{A}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}_1) = (\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{0})$ has a solution for $\mathbf{A}$. This happens if and only if

$$\mathscr{C}\begin{pmatrix}\mathbf{X}'\\ \mathbf{0}\end{pmatrix} \subset \mathscr{C}\begin{pmatrix}\mathbf{X}'\\ \mathbf{M}_1\mathbf{V}\end{pmatrix}, \tag{68}$$

which can be expressed equivalently as $\mathscr{C}(\mathbf{X}) \cap \mathscr{C}(\mathbf{VM}_1) = \{\mathbf{0}\}$; see Puntanen et al. (2011, Ch. 16).

Thus we have proved the following:

Proposition 2 *The following statements are equivalent:*

(a) $\mathbf{Gy} = \mathbf{G}_1\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM})$,
(b) $\mathbf{Gy} = \mathbf{G}_1\mathbf{y}$ *for all* $\mathbf{y} \in \mathbb{R}^n$,
(c) $\mathbf{G}_1\mathbf{X}_2 = \mathbf{X}_2$,
(d) $\mathbf{G}_1 \in \{\mathbf{P}_{\mathbf{X}\mid\mathscr{M}_{12}}\}$,
(e) $\mathscr{C}(\mathbf{X}_2) \subset \mathscr{C}(\mathbf{X}_1)$,
(f) $\{\mathbf{P}_{\mathbf{X}_1\mid\mathscr{M}_1}\} \subset \{\mathbf{P}_{\mathbf{X}\mid\mathscr{M}_{12}}\}$, *i.e.*, $\{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_1)\} \subset \{\mathrm{BLUE}(\boldsymbol{\mu} \mid \mathscr{M}_{12})\}$,
(g) $\{\mathbf{P}_{\mathbf{X}_1\mid\mathscr{M}_1}\} = \{\mathbf{P}_{\mathbf{X}\mid\mathscr{M}_{12}}\}$, *i.e.*, $\{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_1)\} = \{\mathrm{BLUE}(\boldsymbol{\mu} \mid \mathscr{M}_{12})\}$.

Moreover,

(g) $\{\mathbf{P}_{\mathbf{X}\mid\mathscr{M}_{12}}\} \cap \{\mathbf{P}_{\mathbf{X}_1\mid\mathscr{M}_1}\} \neq \{\emptyset\} \iff \mathscr{C}(\mathbf{X}) \cap \mathscr{C}(\mathbf{VM}_1) = \{\mathbf{0}\}$.

In this context it is convenient to refer to the possible equality of $\tilde{\boldsymbol{\mu}}_1(\mathscr{M}_1)$ and $\tilde{\boldsymbol{\mu}}_1(\mathscr{M}_{12})$. Notice that in this section we have put our attention on the equality between $\tilde{\boldsymbol{\mu}}_1(\mathscr{M}_1)$ and $\tilde{\boldsymbol{\mu}}(\mathscr{M}_{12})$. Haslett & Puntanen (2010b) showed that if $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is estimable under $\mathscr{M}_{12}$ and $\mathscr{C}(\mathbf{X}_2) \subset \mathscr{C}(\mathbf{W}_1)$, then

$$\tilde{\boldsymbol{\mu}}_1(\mathscr{M}_{12}) = \tilde{\boldsymbol{\mu}}_1(\mathscr{M}_1) - \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^-\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{W}_1^+\,\tilde{\boldsymbol{\mu}}_2(\mathscr{M}_{12})\,. \tag{69}$$

From (69) it can be concluded that $\tilde{\boldsymbol{\mu}}_1(\mathscr{M}_{12}) = \tilde{\boldsymbol{\mu}}_1(\mathscr{M}_1)$ holds if and only if $\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_2 = \mathbf{0}$; see, e.g., Markiewicz & Puntanen (2019, Sec. 4). Some related considerations, using different approach, appear in Lu et al. (2015) and Tian & Zhang (2016).

4 Equalities of the BLUPs of the error term

The covariance matrices for $\tilde{\boldsymbol{\varepsilon}}_1$ and $\tilde{\boldsymbol{\varepsilon}}_{*1}$ are

$$\mathrm{cov}(\tilde{\boldsymbol{\varepsilon}}_*) = \mathbf{EVE}' = \mathbf{V}_{21}\mathbf{M}(\mathbf{MW}_1\mathbf{M})^-\mathbf{MV}_{12}\,, \tag{70a}$$

$$\mathrm{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1}) = \mathbf{E}_1\mathbf{VE}_1' = \mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{W}_1\mathbf{M}_1)^-\mathbf{M}_1\mathbf{V}_{12}\,. \tag{70b}$$

Thus the the difference $\mathrm{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1}) - \mathrm{cov}(\tilde{\boldsymbol{\varepsilon}}_*)$ can be expressed as

$$\begin{aligned}\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1}) - \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*}) &= \mathbf{V}_{21}\mathbf{W}_1^{+1/2}(\mathbf{P}_{\mathbf{W}_1^{1/2}\mathbf{M}_1} - \mathbf{P}_{\mathbf{W}_1^{1/2}\mathbf{M}})\mathbf{W}_1^{+1/2}\mathbf{V}_{12} \\ &= \mathbf{V}_{21}\mathbf{W}_1^{+1/2}\mathbf{P}_{\mathbf{A}}\mathbf{W}_1^{+1/2}\mathbf{V}_{12}\,, \end{aligned} \tag{71}$$

where

$$\mathscr{C}(\mathbf{A}) = \mathscr{C}(\mathbf{W}_1^{1/2}\mathbf{M}_1) \cap \mathscr{C}(\mathbf{W}_1^{1/2}\mathbf{M})^{\perp}. \tag{72}$$

In (71) $\mathbf{W}_1^{1/2}$ refers to the nonnegative definite square root of $\mathbf{W}_1$ and $\mathbf{W}_1^{+1/2}$ is the Moore–Penrose inverse of $\mathbf{W}_1^{1/2}$, and so $\mathbf{W}_1^{1/2}\mathbf{W}_1^{+1/2} = \mathbf{P}_{\mathbf{W}_1}$. For the difference of two orthogonal projectors, see, e.g., Puntanen et al. (2011, Ch. 7).

In light of (71), we observe that $\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*}) \leq_{\mathrm{L}} \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1})$, while $\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*}) = \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1})$ holds if and only if

$$\mathscr{C}(\mathbf{W}_1^{+1/2}\mathbf{V}_{12}) \subset \mathscr{C}(\mathbf{A})^{\perp} = \mathscr{C}[(\mathbf{W}_1^{1/2}\mathbf{M}_1)^{\perp} : \mathbf{W}_1^{1/2}\mathbf{M}]\,. \tag{73}$$

In view of Lemma 4 of Markiewicz & Puntanen (2019),

$$\mathscr{C}(\mathbf{W}_1^{1/2}\mathbf{M}_1)^{\perp} = \mathscr{C}(\mathbf{W}_1^{+1/2}\mathbf{X}_1 : \mathbf{Q}_{\mathbf{W}_1})\,. \tag{74}$$

Thus the equality $\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*}) = \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1})$ holds if and only if

$$\begin{aligned}\mathscr{C}(\mathbf{W}_1^{+1/2}\mathbf{V}_{12}) &\subset \mathscr{C}[(\mathbf{W}_1^{1/2}\mathbf{M}_1)^{\perp} : \mathbf{W}_1^{1/2}\mathbf{M}] \\ &= \mathscr{C}(\mathbf{W}_1^{+1/2}\mathbf{X}_1 : \mathbf{Q}_{\mathbf{W}_1} : \mathbf{W}_1^{1/2}\mathbf{M})\,. \end{aligned} \tag{75}$$

Premultiplying (75) by $\mathbf{W}_1^{1/2}$ yields

$$\mathscr{C}(\mathbf{V}_{12}) \subset \mathscr{C}(\mathbf{X}_1 : \mathbf{W}_1\mathbf{M}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{V}\mathbf{M})\,, \tag{76}$$

which is a necessary and sufficient condition for the equality $\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*}) = \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1})$.

We can further show the following:

$$\begin{aligned}\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1}, \tilde{\boldsymbol{\varepsilon}}_{*}) &= \mathbf{E}_1\mathbf{V}\mathbf{E}' \\ &= \mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{V}\mathbf{M}_1)^{+}\mathbf{M}_1\mathbf{V}\mathbf{M}(\mathbf{M}\mathbf{V}\mathbf{M})^{+}\mathbf{M}\mathbf{V}_{12} \\ &= \mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{V}\mathbf{M}_1)^{+}\mathbf{M}_1\mathbf{V}\mathbf{M}_1\mathbf{M}(\mathbf{M}\mathbf{V}\mathbf{M})^{+}\mathbf{M}\mathbf{V}_{12} \\ &= \mathbf{V}_{21}\mathbf{M}_1\mathbf{M}(\mathbf{M}\mathbf{V}\mathbf{M})^{+}\mathbf{M}\mathbf{V}_{12} \\ &= \mathbf{E}\mathbf{V}\mathbf{E}' = \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*})\,, \end{aligned} \tag{77}$$

where we have used $\mathbf{M} = \mathbf{M}_1\mathbf{M}$ and $\mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{V}\mathbf{M}_1)^{+}\mathbf{M}_1\mathbf{V}\mathbf{M}_1 = \mathbf{V}_{21}\mathbf{M}_1$. Thus we have proved the following:

Proposition 3 *Denote*

$$\mathbf{E} = \mathbf{V}_{21}\mathbf{M}(\mathbf{M}\mathbf{V}\mathbf{M})^{+}\mathbf{M}\,, \quad \mathbf{E}_1 = \mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{V}\mathbf{M}_1)^{+}\mathbf{M}_1\,. \tag{78}$$

The following statements are equivalent:

(a) $\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*}) = \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1})$*, i.e.,* $\mathbf{E}\mathbf{V}\mathbf{E}' = \mathbf{E}_1\mathbf{V}\mathbf{E}_1'$,

(b) $\mathscr{C}(\mathbf{V}_{12}) \subset \mathscr{C}(\mathbf{X}_1 : \mathbf{VM})$.

Moreover, the following statements hold:

(c) $\mathbf{E}_1\mathbf{VE}' = \mathbf{EVE}'$,
(d) $\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1} - \tilde{\boldsymbol{\varepsilon}}_*) = \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1}) - \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_*)$,
(e) $\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_*) \leq_{\mathrm{L}} \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1})$.

Here is an extended version of Proposition 3.

Proposition 4 *The following statements are equivalent:*

(a) $\mathbf{Ey} = \mathbf{E}_1\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W}_1) = \mathscr{C}(\mathbf{X}_1 : \mathbf{VM}_1)$,
(b) $\mathbf{EVM}_1 = \mathbf{V}_{21}\mathbf{M}_1$,
(c) $\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_*) = \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1})$, *i.e.,* $\mathbf{EVE}' = \mathbf{E}_1\mathbf{VE}_1'$,
(d) $\mathscr{C}(\mathbf{V}_{12}) \subset \mathscr{C}(\mathbf{X}_1 : \mathbf{VM})$,
(e) $\mathbf{E} \in \{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_1}\}$,
(f) $\{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_{12}}\} \subset \{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_1}\}$, *i.e.,* $\{\mathrm{BLUP}(\boldsymbol{\varepsilon}_* \mid \mathscr{M}_{12})\} \subset \{\mathrm{BLUP}(\boldsymbol{\varepsilon}_* \mid \mathscr{M}_1)\}$,
(g) $\mathscr{C}\begin{pmatrix}\mathbf{VM}_1 \\ \mathbf{V}_{21}\mathbf{M}_1\end{pmatrix} \subset \mathscr{C}\begin{pmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \mathbf{VM} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{21}\mathbf{M}\end{pmatrix}$.

Proof Consider the equality $\mathbf{Ey} = \mathbf{E}_1\mathbf{y}$ for all $\mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{VM}_1)$. Now

$$\mathbf{E}(\mathbf{X}_1 : \mathbf{VM}_1) = \mathbf{E}_1(\mathbf{X}_1 : \mathbf{VM}_1) = (\mathbf{0} : \mathbf{V}_{21}\mathbf{M}_1) \tag{79}$$

holds if and only if

$$\mathbf{EVM}_1 = \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^+\mathbf{MVM}_1 = \mathbf{V}_{21}\mathbf{M}_1 \,. \tag{80}$$

Postmultiplying (80) by $(\mathbf{M}_1\mathbf{VM}_1)^-\mathbf{M}_1\mathbf{V}_{12}$ yields

$$\mathbf{EVE}_1' = \mathbf{EVE}' = \mathbf{E}_1\mathbf{VE}_1', \tag{81}$$

i.e., $\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_*) = \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1})$. On the other hand, suppose that the equality $\operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_*) = \operatorname{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1})$ holds. Then by part (b) of Proposition 3,

$$\mathbf{V}_{12} = \mathbf{X}_1\mathbf{A} + \mathbf{VMB} \quad \text{for some } \mathbf{A} \text{ and } \mathbf{B} \,. \tag{82}$$

Straightforward calculation shows that (82) implies (80). Thus we have shown the equivalence of (a), . . . , (e).

An arbitrary member of $\{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_{12}}\}$ can be expressed as $\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_{12}} = \mathbf{E} + \mathbf{NQ_W}$, where $\mathbf{N}$ is free to vary. Clearly

$$(\mathbf{E} + \mathbf{NQ_W})(\mathbf{X}_1 : \mathbf{VM}_1) = (\mathbf{0} : \mathbf{V}_{21}\mathbf{M}_1) \tag{83}$$

for any $\mathbf{N}$ if and only if $\mathbf{EVM}_1 = \mathbf{V}_{21}\mathbf{M}_1$. This proves the equivalence between (b) and (f).

Obviously $\mathscr{C}(\mathbf{VM}_1) \subset \mathscr{C}(\mathbf{W})$ and so

$$\mathbf{VM}_1 = (\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM})\mathbf{D}, \tag{84}$$

for some $\mathbf{D} = (\mathbf{A}' : \mathbf{B}' : \mathbf{C}')'$. Thus $\mathbf{EVM}_1 = \mathbf{V}_{21}\mathbf{M}_1$ gets the form

$$\begin{aligned}\mathbf{EVM}_1 &= \mathbf{E}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM})\mathbf{D} = (\mathbf{0} : \mathbf{0} : \mathbf{V}_{21}\mathbf{M})\mathbf{D} \\ &= \mathbf{V}_{21}\mathbf{MC} = \mathbf{V}_{21}\mathbf{M}_1 \,, \end{aligned} \tag{85}$$

and thereby (g) is a necessary condition for (b). Its sufficiency follows by postmultiplying

$$\mathbf{E}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{0} : \mathbf{0} : \mathbf{V}_{21}\mathbf{M}) \tag{86}$$

by

$$\begin{pmatrix} \mathbf{I}_{p_1} & \mathbf{A} \\ \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix} \tag{87}$$

which yields $\mathbf{E}(\mathbf{X}_1 : \mathbf{VM}_1) = (\mathbf{0} : \mathbf{V}_{21}\mathbf{M}_1)$.

Consider then the equality

$$\mathbf{Ey} = \mathbf{E}_1\mathbf{y} \quad \text{for all } \mathbf{y} \in \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) \,, \tag{88}$$

i.e.,

$$\mathbf{E}_1(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{0} : \mathbf{0} : \mathbf{V}_{21}\mathbf{M}) \,. \tag{89}$$

In view of $\mathbf{M} = \mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2}$, we have

$$\mathbf{E}_1\mathbf{VM} = \mathbf{E}_1\mathbf{VM}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2} = \mathbf{V}_{21}\mathbf{M}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2} = \mathbf{V}_{21}\mathbf{M} \,, \tag{90}$$

and thus (89) becomes

$$\mathbf{E}_1\mathbf{X}_2 = \mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{VM}_1)^{+}\mathbf{M}_1\mathbf{X}_2 = \mathbf{0} \,. \tag{91}$$

In this context we may mention that requesting

$$\mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{VM}_1)^{-}\mathbf{M}_1\mathbf{X}_2 = \mathbf{0} \quad \text{for any } (\mathbf{M}_1\mathbf{VM}_1)^{-}, \tag{92}$$

yields $\mathscr{C}(\mathbf{M}_1\mathbf{X}_2) \subset \mathscr{C}(\mathbf{M}_1\mathbf{V})$, i.e., $\mathscr{C}(\mathbf{X}_2) \subset \mathscr{C}(\mathbf{X}_1 : \mathbf{V})$, where we have used Lemma 3.

An arbitrary member of the class $\{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_1}\}$ can be expressed as

$$\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_1} = \mathbf{E}_1 + \mathbf{NQ}_{\mathbf{W}_1} \,, \tag{93}$$

where $\mathbf{N}$ is free to vary. Thereby, if the equality

$$(\mathbf{E}_1 + \mathbf{NQ}_{\mathbf{W}_1})(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{0} : \mathbf{0} : \mathbf{V}_{21}\mathbf{M}) \tag{94}$$

holds for any matrix $\mathbf{N}$, then necessarily

$$\mathscr{C}(\mathbf{X}_2) \subset \mathscr{C}(\mathbf{W}_1) \,, \tag{95}$$

in which case (94) simplifies to

$$\mathbf{E}_1\mathbf{X}_2 = \mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{V}\mathbf{M}_1)^+\mathbf{M}_1\mathbf{X}_2 = \mathbf{0}\,. \tag{96}$$

In view of (95), we have

$$\mathbf{X}_2 = \mathbf{X}_1\mathbf{A} + \mathbf{V}\mathbf{M}_1\mathbf{B} \tag{97}$$

for some $\mathbf{A}$ and $\mathbf{B}$. Substituting (97) into (96) yields

$$\begin{aligned}\mathbf{E}_1\mathbf{X}_2 &= \mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{V}\mathbf{M}_1)^+\mathbf{M}_1\mathbf{V}\mathbf{M}_1\mathbf{B}\\ &= \mathbf{V}_{21}\mathbf{M}_1\mathbf{B}\\ &= \mathbf{0}\,. \end{aligned} \tag{98}$$

Putting (97) and (98) together, gives

$$\mathscr{C}\begin{pmatrix}\mathbf{X}_2\\ \mathbf{0}\end{pmatrix} \subset \mathscr{C}\begin{pmatrix}\mathbf{X}_1 & \mathbf{V}\mathbf{M}_1\\ \mathbf{0} & \mathbf{V}_{21}\mathbf{M}_1\end{pmatrix}. \tag{99}$$

Thus (99) is a necessary condition for $\{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_1}\} \subset \{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_{12}}\}$. Its sufficiency is straightforward to show. Thus we have proved the following:

Proposition 5 *The following statements are equivalent:*

(a) $\mathbf{E}\mathbf{y} = \mathbf{E}_1\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V}\mathbf{M})$,
(b) $\mathbf{E}_1\mathbf{X}_2 = \mathbf{0}$.

Moreover, the following statements are equivalent:

(c) $\mathbf{E}\mathbf{y} = (\mathbf{E}_1 + \mathbf{N}\mathbf{Q}_{\mathbf{W}_1})\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W})$ *and for all* $\mathbf{N}$,
(d) $\mathbf{E}_1\mathbf{X}_2 = \mathbf{0}$ *and* $\mathscr{C}(\mathbf{X}_2) \subset \mathscr{C}(\mathbf{W}_1)$,
(e) $\mathbf{E}_1\mathbf{X}_2 = \mathbf{V}_{21}\mathbf{M}_1(\mathbf{M}_1\mathbf{V}\mathbf{M}_1)^-\mathbf{M}_1\mathbf{X}_2 = \mathbf{0}$ *for all* $(\mathbf{M}_1\mathbf{V}\mathbf{M}_1)^-$,
(f) $\{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_1}\} \subset \{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_{12}}\}$,
(g) $\mathscr{C}\begin{pmatrix}\mathbf{X}_2\\ \mathbf{0}\end{pmatrix} \subset \mathscr{C}\begin{pmatrix}\mathbf{X}_1 & \mathbf{V}\mathbf{M}_1\\ \mathbf{0} & \mathbf{V}_{21}\mathbf{M}_1\end{pmatrix}$.

The following result can be straightforwardly confirmed.

Proposition 6 *The following statements are equivalent:*

(a) $\{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_1}\} = \{\mathbf{P}_{\boldsymbol{\varepsilon}_*|\mathscr{M}_{12}}\}$,
(b) $\mathscr{C}\begin{pmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \mathbf{V}\mathbf{M}\\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{21}\mathbf{M}\end{pmatrix} = \mathscr{C}\begin{pmatrix}\mathbf{X}_1 & \mathbf{V}\mathbf{M}_1\\ \mathbf{0} & \mathbf{V}_{21}\mathbf{M}_1\end{pmatrix}$.

5 Properties of the BLUPs of the future response

Let us recall the notations $\tilde{\mathbf{y}}_* = \mathbf{T}\mathbf{y}$ and $\tilde{\mathbf{y}}_{*1} = \mathbf{T}_1\mathbf{y}$, where

$$\mathbf{T} = \mathbf{LG} + \mathbf{E} = \mathbf{SG} + \mathbf{V}_{21}\mathbf{V}^+, \tag{100a}$$

$$\mathbf{T}_1 = \mathbf{LG}_1 + \mathbf{E}_1 = \mathbf{SG}_1 + \mathbf{V}_{21}\mathbf{V}^+, \tag{100b}$$

and $\mathbf{S} = \mathbf{L} - \mathbf{V}_{21}\mathbf{V}^+$. Thus $\tilde{\mathbf{y}}_* = \mathbf{Ty}$ is one representation for the $\mathrm{BLUP}(\mathbf{y}_* \mid \mathscr{M}_{12})$ and $\mathbf{y}_* - \tilde{\mathbf{y}}_*$ is the corresponding prediction error. In this section we pay particular attention on the covariance matrices of $\tilde{\mathbf{y}}_*$ and $\tilde{\mathbf{y}}_{*1}$ and of the corresponding prediction errors.

In view of (45), we have

$$\mathrm{cov}(\tilde{\mathbf{y}}_*) = \mathrm{cov}(\tilde{\boldsymbol{\mu}}_*) + \mathrm{cov}(\tilde{\boldsymbol{\varepsilon}}_*) = \mathbf{LGVG'L'} + \mathbf{EVE'}, \tag{101a}$$

$$\mathrm{cov}(\tilde{\mathbf{y}}_{*1}) = \mathrm{cov}(\tilde{\boldsymbol{\mu}}_{*1}) + \mathrm{cov}(\tilde{\boldsymbol{\varepsilon}}_{*1}) = \mathbf{LG}_1\mathbf{VG}_1'\mathbf{L}' + \mathbf{E}_1\mathbf{VE}_1'\,. \tag{101b}$$

Moreover,

$$\begin{aligned}\mathrm{cov}(\tilde{\mathbf{y}}_*, \tilde{\mathbf{y}}_{*1}) &= \mathrm{cov}(\mathbf{LGy} + \mathbf{Ey},\ \mathbf{LG}_1\mathbf{y} + \mathbf{E}_1\mathbf{y}) \\ &= \mathbf{LGVG}_1\mathbf{L}' + \mathbf{LGVE}_1' + \mathbf{EVG}_1'\mathbf{L}' + \mathbf{EVE}_1' \\ &= \mathbf{LG}_1\mathbf{VG}_1\mathbf{L}' + \mathbf{EVE}' + \mathbf{LGVE}_1'\,,\end{aligned} \tag{102}$$

where we have used $\mathbf{GVG}_1' = \mathbf{G}_1\mathbf{VG}_1'$, $\mathbf{EVE}_1' = \mathbf{EVE}'$, and

$$\begin{aligned}\mathbf{EVG}_1' &= \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^+\mathbf{MVW}_1^+\mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_1)^-\mathbf{X}_1' \\ &= \mathbf{V}_{21}\mathbf{M}(\mathbf{MVM})^+\mathbf{MW}_1\mathbf{W}_1^+\mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_1)^-\mathbf{X}_1' \\ &= \mathbf{0}\,.\end{aligned} \tag{103}$$

Thus the equality

$$\begin{aligned}\mathrm{cov}(\tilde{\mathbf{y}}_* - \tilde{\mathbf{y}}_{*1}) &= \mathrm{cov}(\tilde{\mathbf{y}}_*) + \mathrm{cov}(\tilde{\mathbf{y}}_{*1}) - \mathrm{cov}(\tilde{\mathbf{y}}_*, \tilde{\mathbf{y}}_{*1}) - \mathrm{cov}(\tilde{\mathbf{y}}_{*1}, \tilde{\mathbf{y}}_1) \\ &= \mathrm{cov}(\tilde{\mathbf{y}}_*) - \mathrm{cov}(\tilde{\mathbf{y}}_{*1}) + [2\,\mathrm{cov}(\tilde{\mathbf{y}}_{*1}) - \mathrm{cov}(\tilde{\mathbf{y}}_*, \tilde{\mathbf{y}}_{*1}) - \mathrm{cov}(\tilde{\mathbf{y}}_{*1}, \tilde{\mathbf{y}}_1)] \\ &= \mathrm{cov}(\tilde{\mathbf{y}}_*) - \mathrm{cov}(\tilde{\mathbf{y}}_{*1})\,,\end{aligned} \tag{104}$$

holds if and only if $2\,\mathrm{cov}(\tilde{\mathbf{y}}_{*1}) = \mathrm{cov}(\tilde{\mathbf{y}}_*, \tilde{\mathbf{y}}_{*1}) + \mathrm{cov}(\tilde{\mathbf{y}}_{*1}, \tilde{\mathbf{y}}_1)$, i.e.,

$$2(\mathbf{LG}_1\mathbf{VG}_1'\mathbf{L}' + \mathbf{E}_1\mathbf{VE}_1') = 2(\mathbf{LG}_1\mathbf{VG}_1\mathbf{L}' + \mathbf{EVE}') + \mathbf{LGVE}_1' + \mathbf{E}_1\mathbf{VG'L'}, \tag{105}$$

which further can be written as

$$2(\mathbf{E}_1\mathbf{VE}_1' - \mathbf{EVE}') = \mathbf{LGVE}_1' + \mathbf{E}_1\mathbf{VG'L'}. \tag{106}$$

In passing we may mention that it be shown that

$$\mathbf{GVE}_1' = \mathbf{V}(\mathbf{E}_1 - \mathbf{E})'. \tag{107}$$

Let us calculate $\mathrm{cov}(\tilde{\mathbf{y}}_* - \tilde{\mathbf{y}}_{*1})$ in another way. In view of

$$\tilde{\mathbf{y}}_* - \tilde{\mathbf{y}}_{*1} = \mathbf{Ty} - \mathbf{T}_1\mathbf{y} = \mathbf{S}(\mathbf{G} - \mathbf{G}_1)\mathbf{y}, \tag{108}$$

and (58), we have

$$\mathrm{cov}(\tilde{\mathbf{y}}_* - \tilde{\mathbf{y}}_{*1}) = \mathbf{SGVM}_1(\mathbf{M}_1\mathbf{VM}_1)^-\mathbf{M}_1\mathbf{VG}'\mathbf{S}'. \tag{109}$$

We observe from (109) that $\mathrm{cov}(\tilde{\mathbf{y}}_* - \tilde{\mathbf{y}}_{*1}) = \mathbf{0}$ if and only if $\mathbf{SGVM}_1 = \mathbf{0}$. Moreover, the equality $\tilde{\mathbf{y}}_* = \tilde{\mathbf{y}}_{*1}$ holds for all $\mathbf{y} \in \mathscr{C}(\mathbf{W}_1)$ if and only if

$$\mathbf{SG}(\mathbf{X}_1 : \mathbf{VM}_1) = \mathbf{SG}_1(\mathbf{X}_1 : \mathbf{VM}_1) = \mathbf{S}(\mathbf{X}_1 : \mathbf{0}), \tag{110}$$

i.e., $\mathbf{SGVM}_1 = \mathbf{0}$. Correspondingly, the equality $\tilde{\mathbf{y}}_* = \tilde{\mathbf{y}}_{*1}$ for all $\mathbf{y} \in \mathscr{C}(\mathbf{W})$ yields the requirement $\mathbf{SG}_1\mathbf{X}_2 = \mathbf{SX}_2$.

Following Sengupta & Jammalamadaka (2003, p. 292) and Haslett et al. (2014, p. 553), we can write the prediction errors as

$$\mathbf{y}_* - \tilde{\mathbf{y}}_* = \mathbf{y}_* - \mathbf{Ty} = (\mathbf{y}_* - \mathbf{V}_{21}\mathbf{V}^+\mathbf{y}) - \mathbf{SGy}, \tag{111a}$$

$$\mathbf{y}_* - \tilde{\mathbf{y}}_{*1} = \mathbf{y}_* - \mathbf{T}_1\mathbf{y} = (\mathbf{y}_* - \mathbf{V}_{21}\mathbf{V}^+\mathbf{y}) - \mathbf{SG}_1\mathbf{y}. \tag{111b}$$

The random vectors $\mathbf{y}_* - \mathbf{V}_{21}\mathbf{V}^+\mathbf{y}$ and $\mathbf{SGy}$ are uncorrelated,

$$\mathrm{cov}(\mathbf{SGy}, \mathbf{y}_* - \mathbf{V}_{21}\mathbf{V}^+\mathbf{y}) = \mathbf{SGV}_{12} - \mathbf{SGVV}^+\mathbf{V}_{12} = \mathbf{0}, \tag{112}$$

and hence

$$\begin{aligned}\mathrm{cov}(\mathbf{y}_* - \tilde{\mathbf{y}}_*) &= \mathrm{cov}(\mathbf{y}_* - \mathbf{V}_{21}\mathbf{V}^+\mathbf{y}) + \mathrm{cov}(\mathbf{SGy}) \\ &= \mathbf{V}_{22} - \mathbf{V}_{21}\mathbf{V}^+\mathbf{V}_{12} + \mathbf{SGVG}'\mathbf{S}' \\ &= \mathbf{\Sigma}_{22\cdot1} + \mathbf{SGVG}'\mathbf{S}'.\end{aligned} \tag{113}$$

The first term $\mathbf{\Sigma}_{22\cdot1} = \mathbf{V}_{22} - \mathbf{V}_{21}\mathbf{V}^+\mathbf{V}_{12}$ in (113) is the Schur complement of $\mathbf{V}$ in

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{V} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{pmatrix}, \tag{114}$$

and as Sengupta & Jammalamadaka (2003, p. 293) point out, it is the covariance matrix of the prediction error associated with the best linear predictor (supposing that $\mathbf{X}\boldsymbol{\beta}$ were known) while the second term represents the increase in the covariance matrix of the prediction error due to estimation of $\mathbf{X}\boldsymbol{\beta}$.

Remark 5.1. Suppose that $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta} = \mathrm{E}(\mathbf{y})$ and $\boldsymbol{\mu}_* = \mathbf{X}_*\boldsymbol{\beta} = \mathrm{E}(\mathbf{y}_*)$ are known. Then the Best Linear Predictor of $\mathbf{y}_*$ on the basis of $\mathbf{y}$ is the following:

$$\mathrm{BLP}(\mathbf{y}_* \mid \mathbf{y}) = \boldsymbol{\mu}_* - \mathbf{V}_{21}\mathbf{V}^+(\mathbf{y} - \boldsymbol{\mu}). \tag{115}$$

The prediction error is

$$\mathbf{e}_{\mathbf{y}_*|\mathbf{y}} = \mathbf{y}_* - \mathrm{BLP}(\mathbf{y}_* \mid \mathbf{y}) = \mathbf{y}_* - [\boldsymbol{\mu}_* - \mathbf{V}_{21}\mathbf{V}^+(\mathbf{y} - \boldsymbol{\mu})], \tag{116}$$

with

$$\mathrm{cov}(\mathbf{e}_{\mathbf{y}_*|\mathbf{y}}) = \mathbf{\Sigma}_{22\cdot1} = \mathbf{V}_{22} - \mathbf{V}_{21}\mathbf{V}^+\mathbf{V}_{12}. \tag{117}$$

Notice that $\mathbf{\Sigma}_{22\cdot 1} \leq_{\mathrm{L}} \operatorname{cov}(\mathbf{y}_* - \mathbf{N}\mathbf{y})$ for all $\mathbf{N}$.

Next we consider the difference between the covariance matrices of the prediction errors. We have

$$\operatorname{cov}(\mathbf{y}_* - \tilde{\mathbf{y}}_*) = \mathbf{\Sigma}_{22\cdot 1} + \mathbf{SGVG'S'} =: \mathbf{C}_{12}\,, \tag{118a}$$

$$\operatorname{cov}(\mathbf{y}_* - \tilde{\mathbf{y}}_{*1}) = \mathbf{\Sigma}_{22\cdot 1} + \mathbf{SG}_1\mathbf{VG}_1'\mathbf{S}' =: \mathbf{C}_1\,, \tag{118b}$$

and thereby, on account of (47),

$$\begin{aligned}\mathbf{C}_{12} - \mathbf{C}_1 &= \mathbf{SGVG'S'} - \mathbf{SG}_1\mathbf{VG}_1'\mathbf{S}' \\ &= \mathbf{SGVG'S'} - \mathbf{SGG}_1\mathbf{VG}_1'\mathbf{G'S'} \\ &= \mathbf{SGVM}_1(\mathbf{M}_1\mathbf{VM}_1)^-\mathbf{M}_1\mathbf{VG'S'} \\ &= \operatorname{cov}(\tilde{\mathbf{y}}_* - \tilde{\mathbf{y}}_{*1})\,. \end{aligned} \tag{119}$$

Obviously $\mathbf{C}_1 \leq_{\mathrm{L}} \mathbf{C}_{12}$ and $\mathbf{C}_{12} = \mathbf{C}_1$ if and only if $\mathbf{SGVM}_1 = \mathbf{0}$. Conclusion: If we add $\mathbf{X}_2$ into the model, observe the resulting $\mathbf{y}$, predict $\mathbf{y}_*$ on the basis of this particular $\mathbf{y}$, the resulting prediction error has a bigger covariance matrix (in the Löwner sense) than that error which is based on $\mathbf{X}_1$ only.

We omit the consideration of the inclusion of the type $\{\mathrm{BLUP}(\mathbf{y}_* \mid \mathscr{M}_1)\} \subset \{\mathrm{BLUP}(\mathbf{y}_* \mid \mathscr{M}_{12})\}$. Some related results appear in Haslett & Puntanen (2010a, 2013).

The proposition below collects together the results obtained in this section.

Proposition 7 *Denote* $\tilde{\mathbf{y}}_* = \mathbf{Ty}$ *and* $\tilde{\mathbf{y}}_{*1} = \mathbf{T}_1\mathbf{y}$*, where* $\mathbf{T}$ *and* $\mathbf{T}_1$ *are defined as in* (100a)–(100b). *Then the following statements hold:*

(a) $\operatorname{cov}(\tilde{\mathbf{y}}_* - \tilde{\mathbf{y}}_{*1}) = \mathbf{SGV}\dot{\mathbf{M}}_1\mathbf{VG'S'}$,
(b) $\operatorname{cov}(\mathbf{y}_* - \tilde{\mathbf{y}}_*) - \operatorname{cov}(\mathbf{y}_* - \tilde{\mathbf{y}}_{*1}) = \operatorname{cov}(\tilde{\mathbf{y}}_* - \tilde{\mathbf{y}}_{*1})\,,$
(c) $\operatorname{cov}(\mathbf{y}_* - \tilde{\mathbf{y}}_{*1}) \leq_{\mathrm{L}} \operatorname{cov}(\mathbf{y}_* - \tilde{\mathbf{y}}_*)\,.$

The following statements are equivalent:

(d) $\operatorname{cov}(\tilde{\mathbf{y}}_* - \tilde{\mathbf{y}}_{*1}) = \mathbf{0}\,,$
(e) $\operatorname{cov}(\mathbf{y}_* - \tilde{\mathbf{y}}_{*1}) = \operatorname{cov}(\mathbf{y}_* - \tilde{\mathbf{y}}_*)\,,$
(f) $\tilde{\mathbf{y}}_* = \tilde{\mathbf{y}}_{*1}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W}_1) = \mathscr{C}(\mathbf{X}_1 : \mathbf{VM}_1)\,,$
(g) $\mathbf{SGVM}_1 = \mathbf{0}\,.$

Moreover, the following statements are equivalent:

(h) $\operatorname{cov}(\tilde{\mathbf{y}}_* - \tilde{\mathbf{y}}_{*1}) = \operatorname{cov}(\tilde{\mathbf{y}}_*) - \operatorname{cov}(\tilde{\mathbf{y}}_{*1})\,,$
(i) $2(\mathbf{E}_1\mathbf{VE}_1' - \mathbf{EVE}') = \mathbf{LGVE}_1' + \mathbf{E}_1\mathbf{VG'L'}$.

6 Conclusions

In this article we consider the partitioned linear model $\mathscr{M}_{12} = \{\mathbf{y},\, \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2,\, \mathbf{V}\}$ and the small model $\mathscr{M}_1 = \{\mathbf{y},\, \mathbf{X}_1\boldsymbol{\beta}_1,\, \mathbf{V}\}$. Both models are supplemented with the new unobservable random vector $\mathbf{y}_*$, coming from $\mathbf{y}_* = \mathbf{K}\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_*$, where $\mathbf{K}\boldsymbol{\beta}_1$ is estimable under both models. The covariance matrix of $\mathbf{y}_*$ is known as well as the cross-covariance matrix between $\mathbf{y}_*$ and $\mathbf{y}$.

Our aim to predict $\mathbf{y}_*$ on the basis of $\mathscr{M}_{12}$ and $\mathscr{M}_1$ and consider the resulting differences in the BLUEs and BLUPs. We consider the situation using given fixed multipliers of the response $\mathbf{y}$ yielding the BLUEs and BLUPs, and in addition, we characterize the whole class of multipliers in one model yielding the BLUEs and BLUPs that continue providing the BLUEs and BLUPs in the other model. Corresponding relations between the covariance matrices of the BLUEs, BLUPs and prediction errors are characterized. Particular attention is paid on the cases whether the response $\mathbf{y}$ lies in $\mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$ or in $\mathscr{C}(\mathbf{X}_1 : \mathbf{V})$.

We may mention, see, e.g., Isotalo et al. (2018), that the results regarding the model $\mathscr{M}_{12}$ with new observations can be applied to the mixed linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e}$, where $\mathbf{X}_{n\times p}$ and $\mathbf{Z}_{n\times q}$ are known matrices, $\boldsymbol{\beta} \in \mathbb{R}^p$ is a vector of unknown fixed effects, $\mathbf{u}$ is an unobservable vector (q elements) of *random effects* with $\mathrm{E}(\mathbf{u}) = \mathbf{0}$, $\mathrm{cov}(\mathbf{u}) = \boldsymbol{\Delta}$, $\mathbf{e}$ is a random error vector with $\mathrm{E}(\mathbf{e}) = \mathbf{0}$, $\mathrm{cov}(\mathbf{e}) = \boldsymbol{\Phi}$, and $\mathrm{cov}(\mathbf{e}, \mathbf{u}) = \mathbf{0}$. Denoting $\mathbf{g} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u}$, we have $\mathrm{cov}(\mathbf{y}) = \mathrm{cov}(\mathbf{Z}\mathbf{u} + \mathbf{e}) = \mathbf{Z}\boldsymbol{\Delta}\mathbf{Z}' + \boldsymbol{\Phi}$, and the mixed linear model can be expressed as a version of the model with "new observations" corresponding $\mathbf{y}_*$ in (2) being in $\mathbf{g} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u}$.

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