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Lie Symmetries of Fundamental Solutions to the Leutwiler-Weinstein Equation

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Abstract
In this article, we study Lie symmetries to fundamental solutions to the Leutwiler-Weinstein equation

$$Lu := \Delta u + \frac{k}{x^n}\frac{\partial u}{\partial x^n} + \frac{\ell}{(x^n)^2}u = 0$$

in the upper half-space $\mathbb{R}^n_+$. Starting from the infinitesimal generators of the equation $Lu = 0$, we deduce symmetries of the equation $Lu = \delta(x - x_0)$, and using its invariant solutions, we construct a fundamental solution. As an application, we study a Green functions of the operator in the hyperbolic unit ball.

Keywords Leutwiler-Weinstein equation · Fundamental solution · Lie symmetries

Mathematics Subject Classification (2010) 22E70 · 35A08

1 Introduction

In 1953, Alexander Weinstein published his paper on axially symmetric potentials [25]. He started to study the problem

$$\Delta u + \frac{k}{x^n}\frac{\partial u}{\partial x^n} = 0$$

in the upper half-space $\mathbb{R}^n_+ = \{(x^1, ..., x^n) \in \mathbb{R}^n : x^n > 0\}$; which is now known as the Weinstein equation. The parameter k can be real or complex. The problem has mathematical significance. Indeed, it is a generalization of the Laplace equation, and it is maybe a one of the most simple partial differential equations with non-constant coefficients.

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In 1987, Heinz Leutwiler published the first paper in which he started to study the extension of the Weinstein equation with two parameters (see [21] and also [2])

$$Lu := \Delta u + \frac{k}{x^n}\frac{\partial u}{\partial x^n} + \frac{\ell}{(x^n)^2}u = 0. \tag{1}$$

This equation is called the Leutwiler-Weinstein equation. The parameters k and ℓ are considered as real numbers.

Finding a fundamental solution to the equation is an interesting and complicated task. It has already been studied in some special cases by Sirkka-Liisa Eriksson and the second author, for example in [11–15]. In these papers, the approach is based on differential equations and hyperbolic geometry.

The constructive approach to find a fundamental solution based on local Lie symmetries has been introduced by the first author in [5, 6]. The method itself is applicable, if a partial differential equation has enough symmetry. The method can be applied in different interesting cases; see for example [4, 7, 20].

In this paper, we give a general description of the method and the detailed construction of a fundamental solution to the Leutwiler-Weinstein equation. We hope that this gives the reader a good picture of the method itself and motivates the application of the algorithm in different cases.

2 Symmetries of Fundamental Solutions to Linear Partial Differential Equations

In [5] and [6], the first author introduced a constructive method to find fundamental solutions to linear partial equations of the form

$$Pu := \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha u = 0, \tag{2}$$

defined in an open set $\Omega \subset \mathbb{R}^n$. We assume that $a_\alpha \in C^\infty(\Omega)$. In Eq. 2, we use the standard multi-index notation $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + ... + \alpha_n$ and

$$D^\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n}.$$

Let us now describe the method of how to find fundamental solutions. We assume that the reader knows basics of the local symmetry theory of partial differential equations, what is represented, for example, in [19, 22, 24]. Let $\mathfrak{g}$ be a symmetry Lie algebra generated by the infinitesimal operators admitted by Eq. 2.

Proposition 1 [10] *The symmetry Lie algebra* $\mathfrak{g}$ *may be represented as the direct sum*

$$\mathfrak{g} = \mathfrak{g}_f \oplus \mathfrak{g}_\infty$$

where $\mathfrak{g}_f$ *is a finite dimensional Lie subalgebra generated by the infinitesimal generators of the form*

$$X = \sum_{j=1}^{n} \xi^j(x)\frac{\partial}{\partial x^j} + \eta(x)u\frac{\partial}{\partial u} \tag{3}$$

and $\mathfrak{g}_\infty$ *is an infinite dimensional Lie algebra generated by*

$$X_\infty = \beta(x)\frac{\partial}{\partial u},$$

where β *is an arbitrary smooth solution of* Eq. 2.

The explicit description of the symmetry Lie algebra of fundamental solutions is based on the existence of the canonical function $\theta(x)$ described in the next proposition.

Proposition 2 [5] *Let* $X \in \mathfrak{g}$ *be an infinitesimal generator. Then* $X \in \mathfrak{g}_f$ *if and only if there exists a function* $\theta \in C^\infty(\Omega)$ *satisfying the equation*

$$\underset{m}{X}(Lu) = \theta(x)Lu$$

for all $u \in C^\infty(\Omega)$*, where* $\underset{m}{X}$ *is the prolonged infinitesimal generator of* X.

Using the preceding function, we may give the following description for a symmetry Lie algebra of fundamental solutions.

Theorem 3 [5] *The symmetry Lie algebra* $\mathfrak{h}$ *of*

$$Pu = \delta(x - x_0)$$

is a subalgebra of $\mathfrak{g}$*, which may be represented as a direct sum*

$$\mathfrak{h} = \mathfrak{h}_f \oplus \mathfrak{g}_\infty.$$

The finite dimensional subalgebra $\mathfrak{h}_f$ *is a subalgebra of* $\mathfrak{g}_f$ *where the coefficients of infinitesimal generators* (3) *satisfy the system*

$$\xi^j(x_0) = 0, \;\; j = 1, ..., n, \tag{4}$$

$$\theta(x_0) + \sum_{j=1}^{n} \frac{\partial \xi^j(x_0)}{\partial x^j} = 0. \tag{5}$$

We observe, that if the Lie algebra $\mathfrak{h}_f$ is wide enough, we may try to construct a fundamental solution of operator P using invariants of the Lie algebra $\mathfrak{h}_f$. These observations allow us to formulate the following algorithm.

Remark 4 (An algorithm for finding fundamental solutions, [5, 6]) To do:

(a) Find the symmetry subalgebra $\mathfrak{g}_f$ of Eq. 2 and of the corresponding function $\theta(x)$ given in Proposition 2.
(b) Find the symmetry subalgebra $\mathfrak{h}_f$.
(c) Construct invariant fundamental solutions with the use of invariants of $\mathfrak{h}_f$.
(d) Obtain new fundamental solutions from known ones with the use of symmetries of the equation $Pu = \delta(x - x_0)$.

The preceding algorithm works in principle in every case, when the Lie subalgebra $\mathfrak{h}_f$ is wide enough, i.e., it allows us to construct invariant solutions. Examples of the use of the algorithm may be found in [3, 5, 20]. Step (d) is demonstrated for example in [8]. In this article, we will represented a comprehensive illustration of steps (a), (b), and (c) in the case of the Leutwiler-Weinstein equation.

3 Symmetries of the Equation $Lu = 0$

In this section, we compute the infinitesimal generators of the Lie symmetry subalgebra $\mathfrak{g}_f$ for Eq. 1. The second prolongation of the infinitesimal generator (3) is of the form

$$\underset{2}{X} = X + \zeta_n(x,u)\frac{\partial}{\partial u_{x^n}} + \sum_{j=1}^{n} \zeta_{jj}(x,u)\frac{\partial}{\partial u_{x^j x^j}}, \tag{6}$$

where the coeffients are given as

$$\zeta_j \;=\; D_j(u\eta) - \sum_{i=1}^{n} u_{x^i} D_j(\xi^i), \tag{7}$$

$$\zeta_{jj} \;=\; D_j(\zeta_j) - \sum_{i=1}^{n} u_{x^j x^i} D_j(\xi^i), \tag{8}$$

where the total derivative is of the form

$$D_j = \frac{\partial}{\partial x^j} + u_{x^j}\frac{\partial}{\partial u} + \sum_{i=1}^{n} u_{x^j x^i}\frac{\partial}{\partial u_{x^i}}, \tag{9}$$

see all details in [19, 22, 24]. Infinitesimal generators of $\mathfrak{g}_f$ may be obtained by solving the equation $\underset{2}{X}(Lu)|_{Lu=0} = 0$. The equivalent system is described in the following lemma.

Lemma 5 *The equation*

$$\underset{2}{X}(Lu)|_{Lu=0} = 0$$

is equivalent with the system

$$\sum_{j=1}^{n} \eta_{x^j x^j}(x) - \xi^n(x)\frac{2\ell}{(x^n)^3} + \eta_{x^n}(x)\frac{k}{x^n} + 2\xi^n_{x^n}(x)\frac{\ell}{(x^n)^2} = 0, \tag{10}$$

$$-\frac{k}{x^n}\xi^i_{x^n}(x) + 2\eta_{x^i}(x) - \sum_{j=1}^{n} \xi^i_{x^j x^j}(x) = 0,\ i = 1, ..., n-1, \tag{11}$$

$$-\xi^n(x)\frac{k}{(x^n)^2} + 2\eta_{x^n}(x) - \sum_{j=1}^{n} \xi^n_{x^j x^j}(x) + \xi^n_{x^n}(x)\frac{k}{x^n} = 0, \tag{12}$$

$$\xi^i_{x^j}(x) + \xi^j_{x^i}(x) = 0,\ i < j,\ i, j = 1, ..., n, \tag{13}$$

$$\xi^n_{x^n}(x) - \xi^j_{x^j}(x) = 0,\ j = 1, ..., n-1. \tag{14}$$

Proof The prolongation (6) acting on Eq. 1 gives

$$\underset{2}{X}(Lu) = -\xi^n(x)\Big(\tfrac{k}{(x^n)^2}u_{x^n} + \tfrac{2\ell}{(x^n)^3}u\Big) + \eta(x)\tfrac{\ell}{(x^n)^2}u$$
$$+\zeta_n(x,u)\tfrac{k}{x^n} + \sum_{j=1}^{n} \zeta_{jj}(x,u). \tag{15}$$

Using the total derivative (9), we obtain that the coefficients (7) and (8) take the form

$$\zeta_j = \eta_{x^j}(x)u + \eta(x)u_{x^j} - \sum_{i=1}^{n} \xi^i_{x^j}(x)u_{x^i}$$

and

$$\zeta_{jj} = \eta_{x^j x^j}(x)u + 2\eta_{x^j}(x)u_{x^j} - \sum_{i=1}^{n} \xi^i_{x^j x^j}(x)u_{x^i} + \eta(x)u_{x^j x^j} - 2\sum_{i=1}^{n} \xi^i_{x^j}(x)u_{x^j x^i}.$$

Using these formulas, the prolongation (15) takes the form

$$\begin{aligned} \underset{2}{X}(Lu) &= \eta(x)Lu - \xi^n(x)\Big(\frac{k}{(x^n)^2}u_{x^n} + \frac{2\ell}{(x^n)^3}u\Big) \\ &\quad + \eta_{x^n}(x)\frac{k}{x^n}u - \frac{k}{x^n}\sum_{i=1}^{n}\xi^i_{x^n}(x)u_{x^i} + \sum_{j=1}^{n}\eta_{x^j x^j}(x)u \\ &\quad + 2\sum_{j=1}^{n}\eta_{x^j}(x)u_{x^j} - \sum_{i,j=1}^{n}\xi^i_{x^j x^j}(x)u_{x^i} - 2\sum_{i,j=1}^{n}\xi^i_{x^j}(x)u_{x^j x^i}. \end{aligned} \tag{16}$$

We make the restriction on $Lu = 0$ by substituting

$$u_{x^n x^n} = -\sum_{j=1}^{n-1} u_{x^j x^j} - \frac{k}{x^n}u_{x^n} - \frac{\ell}{(x^n)^2}u,$$

and we obtain

$$\begin{aligned} \underset{2}{X}(Lu)|_{Lu=0} &= \Big(\sum_{j=1}^{n}\eta_{x^j x^j}(x) - \xi^n(x)\frac{2\ell}{(x^n)^3} + \eta_{x^n}(x)\frac{k}{x^n} + 2\xi^n_{x^n}(x)\frac{\ell}{(x^n)^2}\Big)u \\ &\quad + \sum_{i=1}^{n-1}\Big(-\frac{k}{x^n}\xi^i_{x^n}(x) + 2\eta_{x^i}(x) - \sum_{j=1}^{n}\xi^i_{x^j x^j}(x)\Big)u_{x^i} \\ &\quad + \Big(-\xi^n(x)\frac{k}{(x^n)^2} + 2\eta_{x^n}(x) - \sum_{j=1}^{n}\xi^n_{x^j x^j}(x) + \xi^n_{x^n}(x)\frac{k}{x^n}\Big)u_{x^n} \\ &\quad - 2\sum_{i<j}\big(\xi^i_{x^j}(x) + \xi^j_{x^i}(x)\big)u_{x^j x^i} + 2\sum_{j=1}^{n-1}\big(\xi^n_{x^n}(x) - \xi^j_{x^j}(x)\big)u_{x^j x^j}. \end{aligned}$$

Assuming u and its partial derivatives are linearly independent, we obtain the result. □

The solution of the system expressed in the preceding lemmas is the following.

Proposition 6 *The coefficients of the infinitesimal generators* (3) *are*

$$\begin{aligned} \xi^n(x) &= 2x^n\Big(\sum_{j=1}^{n-1}a_j x^j + b\Big), \\ \xi^i(x) &= -a_i\sum_{r=1}^{n}(x^r)^2 + 2x^i\Big(\sum_{j=1}^{n-1}a_j x^j + b\Big) + \sum_{s=1}^{n-1}e^i_s x^s + f^i, \\ \eta(x) &= -(k+n-2)\sum_{j=1}^{n-1}a_j x^j + c, \end{aligned}$$

where $e^i_j = -e^j_i$ *for* $k(2-k) + 4\ell \neq 0$ *and* $n \geq 3$*. Moreover* a_j, b, c, e^i_j, f^i *are real parameters.*

Proof We give a detailed proof in the Appendix A. □

Since

$$X=\sum_{j=1}^{n}\xi^j(x)\frac{\partial}{\partial x^j}+u\eta(x)\frac{\partial}{\partial u}$$

we may write it in the form

$$X=\sum_{j=1}^{n-1}a_jX_j+\sum_{i,j=1}^{n-1}e^i_jY_{ij}+\sum_{j=1}^{n-1}f_jZ_j+bU+cV.$$

Hence, we obtain the compete list of infinitesimal generators.

Theorem 7 *If* $k(2-k)+4\ell\neq 0$*, then the infinitesimal generators of the Lie subalgebra* $\mathfrak{g}_f$ *of* Eq. 1 *are*

$$\begin{aligned}
X_i &= 2x^ix^n\frac{\partial}{\partial x^n}-\sum_{r=1}^{n}(x^r)^2\frac{\partial}{\partial x^i}+2x^i\sum_{j=1}^{n-1}x^j\frac{\partial}{\partial x^j}-(k+n-2)x^iu\frac{\partial}{\partial u},\\
Y_{ij} &= x^i\frac{\partial}{\partial x^j}-x^j\frac{\partial}{\partial x^i},\\
Z_j &= \frac{\partial}{\partial x^j},\\
U &= \sum_{i=1}^{n}x^i\frac{\partial}{\partial x^i},\\
V &= u\frac{\partial}{\partial u},
\end{aligned}$$

for $i,j=1,...,n-1$.

Remark 8 In the Appendix A we give a detailed proof to find the preceding infinitesimal generators. There is also a another way to find them. If We make the substitution $u(x)=(x^n)^{-\frac{k}{2}}v(x)$, it transform Eq. 1 to the Helmholz equation with a singular potential

$$\Delta u+\frac{1}{4}(k(2-k)+4\ell)\frac{u}{(x^n)^2}=0.$$

Hence, if $k(2-k)+4\ell=0$ is just the Laplace equation and in other cases symmetry algebra is should be a subalgebra of that of Laplace equation (see, e.g., [19, 22, 24] or in above put $k=0$). Because the lack of the explicit dependence of x^n, most of the infinitesimal generators of the invariance algebra of the Laplace equation and and the preceding equation will remain the same. The only difference is the infinitesimal generator X_i. With these, we can proceed as follows. We start from the corresponding infinitesimal generator of the Laplace equation, and define

$$\widehat{X}_i=2x^ix^n\frac{\partial}{\partial x^n}-\sum_{r=1}^{n}(x^r)^2\frac{\partial}{\partial x^i}+2x^i\sum_{j=1}^{n-1}x^j\frac{\partial}{\partial x^j}+\alpha x^iu\frac{\partial}{\partial u},$$

for $\alpha\in\mathbb{R}$ as a parameter. Its second prolongation gives

$$\underset{2}{\widehat{X_i}}(Lu)=2(n+k-\alpha-2)u-(n+k+2)x_iLu$$

and we obtain $\underset{2}{\widehat{X_i}}(Lu)|_{Lu=0} = 0$ if and only if $\alpha = n + k - 2$.

4 Function $\theta(x)$ for the Leutwiler-Weinstein Equation

To compute infinitesimal generators of the Lie subalgebra $\mathfrak{h}_f$, we need to compute the function $\theta(x)$ described in Proposition 2. We need to substitute the coefficients given in Proposition 6 into the formula (16). We first observe, that Eq. 11 gives

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n} \xi^i_{x^j x^j}(x)u_{x^i} = -\frac{k}{x^n}\sum_{i=1}^{n-1}\xi^i_{x^n}(x)u_{x^i} + 2\sum_{i=1}^{n-1}\eta_{x^i}(x)u_{x^i}.$$

Then using Eq. 12, we obtain

$$\sum_{j=1}^{n}\xi^n_{x^j x^j}(x)u_{x^n} = -\xi^n(x)\frac{k}{(x^n)^2}u_{x^n} + 2\eta_{x^n}(x)u_{x^n} + \xi^n_{x^n}(x)\frac{k}{x^n}u_{x^n}$$

and these together gives

$$\sum_{i,j=1}^{n}\xi^i_{x^j x^j}(x)u_{x^i} = -\frac{k}{x^n}\sum_{i=1}^{n}\xi^i_{x^n}(x)u_{x^i} + 2\sum_{i=1}^{n}\eta_{x^i}(x)u_{x^i}$$
$$-\xi^n(x)\frac{k}{(x^n)^2}u_{x^n} + 2\xi^n_{x^n}(x)\frac{k}{x^n}u_{x^n}.$$

On the other hand, using (A.6), (A.7) and the information, that $e^j_i = -e^i_j$, we compute

$$\sum_{i,j=1}^{n}\xi^i_{x^j}(x)u_{x^j x^i} = \Big(\sum_{j=1}^{n-1} a_j x^j + b\Big)\Delta u.$$

Substituting these into (16), we obtain

$$\underset{2}{X}(Lu) = \eta(x)Lu - 2\xi^n_{x^n}(x)\frac{k}{x^n}u_{x^n} - 2\xi^n(x)\frac{\ell}{(x^n)^3}u$$
$$-4\Big(\sum_{j=1}^{n-1} a_j x^j + b\Big)\Delta u.$$

where we use the information $\eta_{x^n} = \eta_{x^i x^i} = 0$ for $i = 1, ..., n-1$. Substituting $\xi^n(x)$, we obtain

$$\underset{2}{X}(Lu) = \eta(x)Lu - 4\Big(\sum_{j=1}^{n-1} a_j x^j + b\Big)Lu.$$

This gives us the following result.

Proposition 9 *The function $\theta(x)$ for the Weinstein-Leutwiler equation is*

$$\theta(x) = -4b + c - (k + n + 2)\sum_{j=1}^{n-1} a_j x^j.$$

5 Symmetry of the Equation $Lu = \delta(x - x_0)$

Using the function $\theta(x)$, we can find the infinitesimal generators of the symmetry Lie algebra $\mathfrak{h}_f$. We observe, that since Eq. 1 is translation invariant with respect to the variables $\widetilde{x} = (x^1, ..., x^{n-1})$, it is enough to consider the symmetry for the equation

$$Lu = \delta(\widetilde{x})\delta(x^n - x_0^n). \tag{17}$$

In Theorem 3, we deduce, that the Lie algebra $\mathfrak{h}_f$ is generated by the infinitesimal generators (3) and they should satisfy the system

$$\xi^j(x_0) = 0, \ \ j = 1, ..., n,$$
$$\theta(x_0) + \sum_{j=1}^{n} \frac{\partial \xi^j(x_0)}{\partial x^j} = 0,$$

at the point $x_0 = (\widetilde{0}, x_0^n)$. In our case, the first equation gives

$$\begin{aligned} \xi^i(x_0) &= -a_i(x_0^n)^2 + f^i = 0 \ \Rightarrow \ f^i = a_i(x_0^n)^2, \ i = 1, ..., n-1, \\ \xi^n(x_0) &= 2x_0^n b = 0 \ \Rightarrow \ b = 0. \end{aligned}$$

Since $\xi^j_{x^j}(x) = \xi^n_{x^n}(x) = 2\sum_{j=1}^{n-1} a_j x^j$, the second equation gives

$$\theta(x_0) + \sum_{j=1}^{n-1} \frac{\partial \xi^j(x_0)}{\partial x^j} + \frac{\partial \xi^n(x_0)}{\partial x^n} = \theta(x_0) = c = 0.$$

Proposition 10 *If $k(2-k) + 4\ell \neq 0$, then the coefficients of the Lie subalgebra $\mathfrak{h}_f$ of* Eq. 17 *are*

$$\begin{aligned} \xi^i(x) &= -a_i \sum_{r=1}^{n} (x^r)^2 + 2x^i \sum_{j=1}^{n-1} a_j x^j + \sum_{s=1}^{n-1} e^i_s x^s + a_i(x_0^n)^2, \ i = 1, ..., n-1, \\ \xi^n(x) &= 2x^n \sum_{j=1}^{n-1} a_j x^j, \\ \eta(x) &= -(k + n - 2) \sum_{j=1}^{n-1} a_j x^j, \end{aligned}$$

where $e^i_j = -e^j_i$.

The general form of an infinitesimal generator of $\mathfrak{h}_f$ is

$$X = \sum_{j=1}^{n-1} a_j X_j + \sum_{i,j=1}^{n-1} e^i_j Y_{ij},$$

where infinitesimal generators X_j and Y_{ij} are given in Theorem 7. We obtain the following theorem.

Theorem 11 *If $k(2-k)+4\ell \neq 0$, then the infinitesimal generators of the Lie subalgebra $\mathfrak{h}_f$ of* Eq. 17 *are*

$$\begin{aligned} X_i &= 2x^i x^n \frac{\partial}{\partial x^n} + \Big((x_0^n)^2 - \sum_{r=1}^{n}(x^r)^2\Big)\frac{\partial}{\partial x^i} + 2x^i \sum_{j=1}^{n-1} x^j \frac{\partial}{\partial x^j} - (k+n-2)x^i u \frac{\partial}{\partial u}, \\ Y_{ij} &= x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \end{aligned}$$

where $i, j = 1, ..., n-1$.

6 Fundamental Invariants of the Equation *Lu* = 0

The notion "fundamental invariant" means an invariant solution of the equation $Lu = 0$ where the solution is invariant with respect to the Lie subalgebra $\mathfrak{h}_f$, depending on a point x_0. These invariants are natural candidates to build fundamental solutions. We start from the equation

$$X_i I = 0,$$

and we obtain the corresponding Lagrange-Charpit equations

$$\frac{dx^n}{2x^i x^n} = \frac{dx^i}{(x_0^n)^2 - \sum_{r=0}^{n}(x^r)^2 + 2(x^i)^2} = \frac{dx^j}{2x^i \sum_{j=1}^{n-1} x^j} = \frac{du}{-(k+n-2)x^i u}.$$

The first and last terms gives us

$$\frac{dx^n}{2x^n} = \frac{du}{-(k+n-2)u}.$$

Since this does not depend on x^i, for $i = 1, ..., n-1$, and we obtain

$$I = (x^n)^{\frac{k+n-2}{2}} u,$$

which is a fundamental invariant, since $Y_{ij} I = 0$. Now, if

$$Y_{ij} J = 0,$$

we observe, that $J = J(a, x^n)$ where $a = \sum_{j=1}^{n-1}(x^j)^2$. Then

$$\begin{aligned} X_i J &= 2x^i x^n \frac{\partial}{\partial x^n} J + \Big((x_0^n)^2 - \sum_{r=1}^{n}(x^r)^2\Big)\frac{\partial}{\partial x^i} J + 2x^i \sum_{j=1}^{n-1} x^j \frac{\partial}{\partial x^j} J \\ &= 2x^i x^n J_{x^n} + \big((x_0^n)^2 - (x^n)^2\big)2x^i J_a - 2x^i \sum_{r=1}^{n-1}(x^r)^2 J_a + 4x^i \sum_{j=1}^{n-1}(x^j)^2 J_a \\ &= 2x^i x^n J_{x^n} + \big((x_0^n)^2 - (x^n)^2\big)2x^i J_a + 2x^i \sum_{j=1}^{n-1}(x^j)^2 J_a = 0 \end{aligned}$$

if and only if

$$x^n J_{x^n} + ((x_0^n)^2 - (x^n)^2 + a) J_a = 0.$$

Using the classical method of characteristics for the first-order partial differential equations, we obtain the solution

$$J = \frac{(x_0^n)^2 + (x^n)^2 + a}{x^n}.$$

Proposition 12 *The fundamental invariants of* Eq. 1 *are*

$$\begin{aligned} I &= (x^n)^{\frac{k+n-2}{2}} u, \\ J &= \frac{\sum_{j=1}^n (x^j)^2 + (x_0^n)^2}{x^n}. \end{aligned}$$

7 Finding Invariant Solutions

Using fundamental invariants, we may construct an invariant solution for the equation $Lu = 0$ depending on point x_0. To obtain an invariant (fundamental) solution for $Lu = \delta(x - x_0)$, we use the form for the weak invariants of Berest, expressed in [9]. We make the Anzats

$$I = w(z)$$

where we denote

$$z = J/2 = \frac{|x|^2 + (x_0^n)^2}{2x^n} = \frac{|\widetilde{x}|^2 + (x^n)^2 + (x_0^n)^2}{2x^n}. \tag{18}$$

Then

$$u = (x^n)^{-\frac{k+n-2}{2}} w(z). \tag{19}$$

Let us next prove the following proposition.

Proposition 13 *Function* (19) *is a solution for* (1) *if and only if*

$$(z^2 - (x_0^n)^2) w''(z) + nzw'(z) + \frac{1}{4}\big(n^2 - k^2 + 2(k - n + 2\ell)\big) w(z) = 0.$$

Proof Using $\frac{\partial z}{\partial x^j} = \frac{x^j}{x^n}$ and $\frac{\partial z}{\partial x^n} = 1 - \frac{z}{x^n}$, we compute

$$\begin{aligned} \frac{\partial w}{\partial x^j} &= \frac{x^j}{x^n} w'(z), \\ \frac{\partial^2 w}{\partial (x^j)^2} &= \frac{(x^j)^2 w''(z) + x^n w'(z)}{(x^n)^2}, \end{aligned}$$

for $j = 1, ..., n-1$ and

$$\begin{aligned} \frac{\partial w}{\partial x^n} &= \frac{(x^n - z) w'(z)}{x^n}, \\ \frac{\partial^2 w}{\partial (x^n)^2} &= \frac{(x^n - z)^2 w''(z) + (2z - x^n) w'(z)}{(x^n)^2}. \end{aligned}$$

Then we compute

$$\begin{aligned} u_{x^j} &= (x^n)^{-\frac{k+n-2}{2}} \frac{x^j}{x^n} w'(z), \\ u_{x^n} &= (x^n)^{-\frac{k+n-2}{2}} \frac{(x^n - z) w'(z) - \frac{1}{2}(n + k - 2) w(z)}{x^n}, \end{aligned}$$

and

$$u_{x^jx^j} = (x^n)^{-\frac{k+n-2}{2}} \frac{(x^j)^2 w''(z) + x^n w'(z)}{(x^n)^2},$$
$$u_{x^nx^n} = (x^n)^{-\frac{k+n-2}{2}} \frac{(x^n - z)^2 w''(z) + \big((n+k)z - (n+k-1)x^n\big) w'(z) + \frac{1}{4}(n+k)(n+k-2)w(z)}{(x^n)^2}.$$

Using the preceding observations, we have

$$\begin{aligned}(x^n)^{\frac{k+n+2}{2}} \Delta u &= (x^n)^{\frac{k+n+2}{2}} \sum_{j=1}^{n} u_{x^jx^j} \\ &= \sum_{j=1}^{n-1} \big((x^j)^2 w''(z) + x^n w'(z)\big) + (x^n - z)^2 w''(z) \\ &\quad + \big((n+k)z - (n+k-1)x^n\big) w'(z) + \frac{1}{4}(n+k)(n+k-2)w(z) \\ &= \sum_{j=1}^{n-1} (x^j)^2 w''(z) + (n-1)x^n w'(z) + (x^n - z)^2 w''(z) \\ &\quad + \big((n+k)z - (n+k-1)x^n\big) w'(z) + \frac{1}{4}(n+k)(n+k-2)w(z).\end{aligned}$$

Substituting $\sum_{j=1}^{n-1} (x^j)^2 = 2x^n z - (x^n)^2 - (x_0^n)^2$, we obtain

$$\begin{aligned}(x^n)^{\frac{k+n+2}{2}} \Delta u &= \big(2x^n z - (x^n)^2 - (x_0^n)^2 + (x^n - z)^2\big) w''(z) + (n-1)x^n w'(z) \\ &\quad + \big((n+k)z - (n+k-1)x^n\big) w'(z) + \frac{1}{4}(n+k)(n+k-2)w(z) \\ &= \big(2x^n z - (x^n)^2 - (x_0^n)^2 + (x^n - z)\big) w''(z) \\ &\quad + \big((n+k)z - (n+k-1)x^n + (n-1)x^n\big) w'(z) + \frac{1}{4}(n+k)(n+k-2)w(z) \\ &= (z^2 - (x_0^n)^2) w''(z) \\ &\quad + \big((n+k)z - kx^n\big) w'(z) + \frac{1}{4}(n+k)(n+k-2)w(z).\end{aligned}$$

Then

$$\begin{aligned}(x^n)^{\frac{k+n+2}{2}} Lu &= (z^2 - (x_0^n)^2) w''(z) \\ &\quad + \big((n+k)z - kx^n\big) w'(z) + \frac{1}{4}(n+k)(n+k-2)w(z) \\ &\quad + k(x^n - z) w'(z) - \frac{k}{2}(n+k-2)w(z) + \ell w(z) \\ &= (z^2 - (x_0^n)^2) w''(z) + nzw'(z) + \frac{1}{4}\big(n^2 - k^2 + 2(k - n + 2\ell)\big) w(z).\end{aligned}$$

$\square$

We see, that the unknown function $w(z)$ may be found by the associated Legendre functions P_ν^μ and Q_ν^μ, which solves the associated Legendre equation

$$(y^2-1)v''(y)+2yv'(y)-\Big(\nu(\nu+1)+\frac{\mu^2}{y^2-1}\Big)v(y)=0, \tag{20}$$

where ν and μ are real or complex numbers.

Proposition 14 *Invariant solutions of* Eq. 1 *with respect to the Lie algebra* $\mathfrak{h}_f$ *are of the form*

$$\begin{aligned} u(x) \;=\; & C_1(x^n)^{-\frac{k}{2}-\mu}(z^2-(x_0^n)^2)^{-\mu/2}P_\nu^{-\mu}(z/x_0^n) \\ & +C_2(x^n)^{-\frac{k}{2}-\mu}(z^2-(x_0^n)^2)^{-\mu/2}Q_\nu^{-\mu}(z/x_0^n), \end{aligned}$$

where $z=\frac{\sum_{j=1}^{n}(x^j)^2+(x_0^n)^2}{2x^n}$ *with* $|z|\neq x_0^n$, *and the parameters* $\mu=\frac{n-2}{2}$ *and* $\nu=\frac{1}{2}(\sqrt{n(n-2)-4\gamma+1}-1)$ *with* $\gamma=\frac{1}{4}\big(n^2-k^2+2(k-n+2\ell)\big)$.

Proof Let us prove the equation given in the preceding proposition. First, we define new variables $z=yz_0^n$ and the function $g(y):=w(yx_0^n)$. Then

$$\begin{aligned} g'(y)=x_0^n w'(z) \;&\Leftrightarrow\; w'(z)=\frac{g'(y)}{x_0^n}, \\ g''(y)=(x_0^n)^2 w''(z) \;&\Leftrightarrow\; w''(z)=\frac{g''(y)}{(x_0^n)^2}. \end{aligned}$$

After the substitution, the equation in Proposition 13 takes the form

$$(y^2-1)g''(y)+nyg'(y)+\gamma g(y)=0,$$

where $\gamma=\frac{1}{4}\big(n^2-k^2+2(k-n+2\ell)\big)$. We make a substitution $g(y)=(y^2-1)^\alpha v(y)$, and the equation in the above takes the form

$$(y^2-1)v''(y)+(4\alpha+n)yv'(y)+\Big(2\alpha+\gamma+\frac{(4\alpha(\alpha-1)+2n\alpha)y^2}{y^2-1}\Big)v(y)=0,$$

assuming that $|y|\neq 1$. We compare this equation with Eq. 20 and obtain

$$4\alpha+n=2 \;\Leftrightarrow\; \alpha=\frac{2-n}{4}.$$

Putting $\delta=2\alpha+\gamma$ and $\epsilon=4\alpha(\alpha-1)+2n\alpha$, we have

$$(y^2-1)v''(y)+2yv'(y)+\Big(\delta+\epsilon+\frac{\epsilon}{y^2-1}\Big)v(y)=0, \tag{21}$$

where the coefficient of $v(y)$ is simplified using

$$\delta+\frac{\epsilon y^2}{y^2-1}=\delta+\frac{\epsilon y^2-\epsilon+\epsilon}{y^2-1}=\delta+\epsilon+\frac{\epsilon}{y^2-1}.$$

Comparing now Eq. 21 with the Legendre equation (20), we obtain

$$\nu(\nu+1)=-\delta-\epsilon=-\frac{1}{4}(n(2-n)+4\gamma) \;\Rightarrow\; \nu=\frac{1}{2}(\pm\sqrt{n(n-2)-4\gamma+1}-1),$$

$$\mu^2=-\epsilon=\frac{(n-2)^2}{4} \;\Rightarrow\; \mu=\pm\frac{n-2}{2}.$$

We see, that the general solution of the system is a linear combination of Legendre functions with the preceding coefficients ν and μ, i.e.,

$$\begin{aligned} w(z) \;=\; & C_1\big(z^2-(x_0^n)^2\big)^{\frac{2-n}{4}} P^{\pm\frac{n-2}{2}}_{\pm\frac{1}{2}\sqrt{n(n-2)-4\gamma+1}-\frac{1}{2}}(z/x_0^n) \\ & + C_2\big(z^2-(x_0^n)^2\big)^{\frac{2-n}{4}} Q^{\pm\frac{n-2}{2}}_{\pm\frac{1}{2}\sqrt{n(n-2)-4\gamma+1}-\frac{1}{2}}(z/x_0^n). \end{aligned}$$

We first observe, that using formulas 8.2.1-2 of [1], we may represent the functions $P^{\mu}_{-\kappa-\frac{1}{2}}$ and $Q^{\mu}_{-\kappa-\frac{1}{2}}$ using functions $P^{\mu}_{\kappa-\frac{1}{2}}$ and $Q^{\mu}_{\kappa-\frac{1}{2}}$. This means that we may choose, without a loss of generality, that

$$\nu = \frac{1}{2}\left(\sqrt{n(n-2)-4\gamma+1}-1\right).$$

Similarly, using formulas 8.2.5-6 of [1], we may represent P^{μ}_{ν} and Q^{μ}_{ν} using functions $P^{-\mu}_{\nu}$ and $Q^{-\mu}_{\nu}$. Hence, the general solution is always of the form

$$w(z) = C_1(z^2-(x_0^n)^2)^{-\mu/2} P^{-\mu}_{\nu}(z/x_0^n) + C_2(z^2-(x_0^n)^2)^{-\mu/2} Q^{-\mu}_{\nu}(z/x_0^n),$$

where $\mu = \frac{n-2}{2}$. Taking into account (19), we obtain the result. □

The associated Legendre functions in the above may be represented using the hypergeometric functions (see [1, 16])

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}, \;\; |x|<1,$$

where $(q)_0 = 1$ and $(q)_k = q(q+1)\cdots(q+k-1)$ for $k \geq 1$. Then, we obtain (see [1, 16])

$$P^{-\mu}_{\nu}(z/x_0^n) \;=\; \frac{1}{\Gamma(1+\mu)}\left(\frac{x_0^n-z}{x_0^n+z}\right)^{\mu/2} {}_2F_1\Big(-\nu,\nu+1;1+\mu;\frac{1-z/x_0^n}{2}\Big), \tag{22}$$

converging in $|1-z/x_0^n|<2$ and

$$\begin{aligned} Q^{-\mu}_{\nu}(z/x_0^n) \;=\; & \frac{\sqrt{\pi}e^{-i\pi\mu}}{2^{\nu+1}} \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+3/2)} \frac{\big((z/x_0^n)^2-1\big)^{-\mu/2}}{(z/x_0^n)^{\nu-\mu+1}} \times \\ & \times {}_2F_1\Big(\frac{\nu-\mu+2}{2},\frac{\nu-\mu+1}{2};\nu+\frac{3}{2};(x_0^n/z)^2\Big), \end{aligned} \tag{23}$$

converging in $|x_0^n/z|<1$.

In general, parameters ν and μ are arbitrary complex numbers. The behaviour of functions depends on their numerical values and relations. Gelfand and Shilov's method of analytic continuation allows us to study this dependence systematically, see e.g. [16] and their references.

We first denote the linearly independent functions in the general solution by

$$\mathcal{P}(x,x_0^n) = (x^n)^{-\frac{k}{2}-\mu}\big(z^2-(x_0^n)^2\big)^{-\mu/2} P^{-\mu}_{\nu}(z/x_0^n)$$

and

$$\mathcal{Q}(x,x_0^n) = (x^n)^{-\frac{k}{2}-\mu}\big(z^2-(x_0^n)^2\big)^{-\mu/2} Q^{\mu}_{-\nu}(z/x_0^n), \tag{24}$$

where $\nu = \frac{1}{2}(\sqrt{n(n-2)-4\gamma+1}-1)$ with $\gamma = \frac{1}{4}\big(n^2-k^2+2(k-n+2\ell)\big)$ and $\mu = \frac{n-2}{2}$. Then we may prove, that the first of these functions does not have singularity at the point $x_0 = (0, x_0^n)$.

Proposition 15 *The preceding function satisfies the equation*

$$L\mathcal{P}(x, x_0^n) = 0$$

in the neighbourhood $|x - (0, x_0^n)|^2 < 4x_0^n x^n$.

Proof The function $P_\nu^{-\mu}$ is defined, if $|x_0^n - z| < 2x_0^n$. Using Eq. 18, we obtain $|x - (0, x_0^n)|^2 < 4x_0^n x^n$. It is well known that $\mathcal{P}(x, x_0^n)$ is bounded when $z \to x_0^n$. □

8 Computing the Fundamental Solution

In the preceding section, we infer, that the fundamental solution may be founded by using the function $\mathcal{Q}(x; x_0^n)$. Since it is an invariant solution of Eq. 17, we infer $L\mathcal{Q}(x; x_0^n) = c\delta(x - x_0)$. In this section, we compute the constant c and obtain a fundamental solution.

First, we prove the following technical lemma.

Lemma 16 *If we define*

$$\widetilde{L}v = \Delta v + \frac{k(2-k)+4\ell}{4}\frac{v}{(x^n)^2},$$

we obtain

$$L((x^n)^{-\frac{k}{2}}v) = (x^n)^{-\frac{k}{2}}\widetilde{L}v.$$

Proof We first compute derivatives

$$\begin{aligned}
\frac{\partial}{\partial x^n}((x^n)^{-\frac{k}{2}}v) &= -\frac{k}{2}(x^n)^{-\frac{k}{2}-1}v + (x^n)^{-\frac{k}{2}}\frac{\partial v}{\partial x^n},\\
\frac{\partial^2}{\partial (x^n)^2}((x^n)^{-\frac{k}{2}}v) &= \frac{k(k+2)}{4}(x^n)^{-\frac{k}{2}-2}v - k(x^n)^{-\frac{k}{2}-1}\frac{\partial v}{\partial x^n} + (x^n)^{-\frac{k}{2}}\frac{\partial^2 v}{\partial (x^n)^2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
L((x^n)^{-\frac{k}{2}}v) &= (x^n)^{-\frac{k}{2}}\sum_{j=1}^{n-1}\frac{\partial^2 v}{\partial (x^j)^2} + \frac{k(k+2)}{4}(x^n)^{-\frac{k}{2}-2}v - k(x^n)^{-\frac{k}{2}-1}\frac{\partial v}{\partial x^n} + (x^n)^{-\frac{k}{2}}\frac{\partial^2 v}{\partial (x^n)^2}\\
&\quad -\frac{k^2}{2}(x^n)^{-\frac{k}{2}-2}v + k(x^n)^{-\frac{k}{2}-1}\frac{\partial v}{\partial x^n} + \ell(x^n)^{-\frac{k}{2}-2}v\\
&= (x^n)^{-\frac{k}{2}}\Delta v + \frac{k(k+2)}{4}(x^n)^{-\frac{k}{2}-2}v - \frac{k^2}{2}(x^n)^{-\frac{k}{2}-2}v + \ell(x^n)^{-\frac{k}{2}-2}v\\
&= (x^n)^{-\frac{k}{2}}\Big(\Delta v + \frac{k(2-k)+4\ell}{4(x^n)^2}v\Big).
\end{aligned}$$

□

Since the Dirac delta satisfies $f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)$ for all smooth functions f, we obtain the following corollary.

Corollary 17 *If* $\widetilde{L}v = \delta(x - x_0)$, *then* $L\big((\frac{x_0^n}{x^n})^{\frac{k}{2}}v\big) = \delta(x - x_0)$.

There preceding lemma shows, that it is enough to find the fundamental solution of $\widetilde{L}$. We will denote in this section $\lambda = z/x_0^n$. The preceding corollary motivates us to study the function

$$
\begin{aligned}
F(x; x_0^n) &= (x^n)^{-\mu}\big(z^2 - (x_0^n)^2\big)^{-\mu/2} Q_\nu^{-\mu}(\lambda) \\
&= \frac{1}{(x^n x_0^n)^\mu} \frac{Q_\nu^{-\mu}(\lambda)}{(\lambda^2-1)^{\mu/2}}.
\end{aligned}
$$

The preceding function multiplied with a constant give us the fundamental solution for $\widetilde{L}$.

Proposition 18 *We may write*

$$
F(x; x_0^n) = \frac{f(\lambda)}{(\lambda^2-1)^\mu}
$$

where

$$
f(\lambda) = \frac{\sqrt{\pi}}{2^{\nu+\mu}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu)\Gamma(\nu+3/2)} \Big(\frac{x_0^n}{x^n}\Big)^\mu \frac{1}{\lambda^{\nu-\mu+1}}\, {}_2F_1\Big(\frac{\nu-\mu+2}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{\lambda^2}\Big)
$$

and

$$
f(1) = 1.
$$

Proof We know that a hypergeometric functions ${}_2F_1(a, b; c; t)$ converges through the unit circle $|t| = 1$ if $\mathrm{Re}(c - a - b) > 0$, see [16]. We know that

$$
Q_\nu^{-\mu}(\lambda) = \frac{\sqrt{\pi} e^{-i\pi\mu}}{2^{\nu+1}} \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+3/2)} \frac{(\lambda^2-1)^{-\mu/2}}{\lambda^{\nu-\mu+1}}\, {}_2F_1\Big(\frac{\nu-\mu+2}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{\lambda^2}\Big)
$$

converges when $\lambda > 1$. A straight-forward computation shows, that in the preceding hypergeometric function $c - a - b = \mu > 0$, that is, we may compute its value at $\lambda = 1$. Recall Formula 9.131.2 of [16]

$$
{}_2F_1\Big(a, b; c; 1\Big) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.
$$

We put $a = \frac{\nu-\mu+2}{2}$, $b = \frac{\nu-\mu+1}{2}$ and $c = \nu+\frac{3}{2}$, and compute $c-a = \frac{\nu+\mu+1}{2}$, $c-b = \frac{\nu+\mu+2}{2}$ and $c - a - b = \mu$. The denominator can be simplified as follows. Recall the duplication formula 6.1.18 in [1]

$$
\Gamma(d)\Gamma\left(d + \frac{1}{2}\right) = 2^{1-2d}\sqrt{\pi}\,\Gamma(2d).
$$

In our case $d = \frac{\nu+\mu+1}{2}$, that is, $1 - 2d = -\nu - \mu$, and we compute

$$
\Gamma(c-a)\Gamma(c-b) = \frac{\sqrt{\pi}}{2^{\nu+\mu}}\Gamma(\nu+\mu+1).
$$

Using the preceding observations, we compute

$$
{}_2F_1\Big(\frac{\nu-\mu+2}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; 1\Big) = \frac{2^{\nu+\mu}}{\sqrt{\pi}} \frac{\Gamma(\nu+\frac{3}{2})\Gamma(\mu)}{\Gamma(\nu+\mu+1)}.
$$

We obtain

$$
\begin{aligned}
F(x; x_0^n) &= \frac{1}{(x^n x_0^n)^\mu} \frac{Q_\nu^{-\mu}(\lambda)}{(\lambda^2-1)^{\mu/2}} \\
&= \frac{\sqrt{\pi} e^{-i\pi\mu}}{2^{\nu+1}} \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+3/2)} \frac{1}{(x^n x_0^n)^\mu \lambda^{\nu-\mu+1}} \frac{1}{(\lambda^2-1)^\mu} \times \\
&\quad \times {}_2F_1\Big(\frac{\nu-\mu+2}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{\lambda^2}\Big) \\
&= \frac{f_0(\lambda)}{(\lambda^2-1)^\mu}
\end{aligned}
$$

where

$$
f_0(\lambda) = \frac{\sqrt{\pi} e^{-i\pi\mu}}{2^{\nu+1}} \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+3/2)} \frac{1}{(x^n x_0^n)^\mu \lambda^{\nu-\mu+1}} {}_2F_1\Big(\frac{\nu-\mu+2}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{\lambda^2}\Big).
$$

Then we compute

$$
\begin{aligned}
f_0(1) &= \frac{\sqrt{\pi} e^{-i\pi\mu}}{2^{\nu+1}} \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+3/2)} \frac{1}{(x_0^n)^{2\mu}} \frac{2^{\nu+\mu}}{\sqrt{\pi}} \frac{\Gamma(\nu+\frac{3}{2})\Gamma(\mu)}{\Gamma(\nu+\mu+1)} \\
&= e^{i\pi\mu} 2^{\mu-1} \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \frac{\Gamma(\mu)}{(x_0^n)^{2\mu}}.
\end{aligned}
$$

We can define the function $f(\lambda) = \frac{f_0(\lambda)}{f_0(1)}$ and we find

$$
f(\lambda) = \frac{\sqrt{\pi}}{2^{\nu+\mu}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu)\Gamma(\nu+3/2)} \Big(\frac{x_0^n}{x^n}\Big)^\mu \frac{1}{\lambda^{\nu-\mu+1}} {}_2F_1\Big(\frac{\nu-\mu+2}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{\lambda^2}\Big),
$$

completing the proof. □

The preceding function $F(x; x_0^n)$ is a candidate for the fundamental solution. Next we extend $\widetilde{L}$ to distributions by

$$
\langle \widetilde{L}F, \varphi \rangle = \langle F, \widetilde{L}\varphi \rangle,
$$

where $\varphi \in \mathcal{D}(\mathbb{R}_+^n)$ is a test function.

Proposition 19 (Green's formula) *Assume $\Omega \subset \mathbb{R}^n$ is a bounded set with a smooth enough boundary. If u and v are twice differentiable real-valued functions on an open set including Ω, we have*

$$
\int_\Omega (u\widetilde{L}v - v\widetilde{L}u)dx = \int_{\partial\Omega} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)dS,
$$

where dS is the Euclidean surface measure, n the outward unit normal on the boundary $\partial\Omega$ and

$$
\frac{\partial u}{\partial n} = \nabla u \cdot n.
$$

Proof The proposition follows from the classical Green's formula

$$
\int_\Omega (u\Delta v - v\Delta u)dx = \int_{\partial\Omega} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)dS
$$

by adding and subtracting the term $\frac{k(2-k)+4\ell}{4} \frac{uv}{(x^n)^2}$ in the integrand of the volume integral. □

Let us define the r-ball with the centre $(\widetilde{0}, x_0^n)$ by

$$B_r(x_0^n) = \{x \in \mathbb{R}^n : |\widetilde{x}|^2 + (x^n - x_0^n)^2 < r^2\}.$$

We will always assume, that $r > 0$ is defined such, that $B_r(x_0^n) \subset \mathbb{R}^n_+$. Let us compute the following crucial formula.

Lemma 20 *On $\partial B_r(x_0^n)$, we have*

$$\frac{\partial \lambda}{\partial n} = r \frac{x^n + x_0^n}{2(x^n)^2 x_0^n}.$$

Especially

$$\lim_{r \to 0} \frac{\partial \lambda}{\partial n} = 0$$

and

$$\lim_{r \to 0} \frac{1}{r} \frac{\partial \lambda}{\partial n} = \frac{1}{(x_0^n)^2}.$$

Proof We compute

$$\frac{\partial \lambda}{\partial x^j} = \frac{x^j}{x^n x_0^n}$$

for $j = 1, ..., n-1$ and

$$\begin{aligned} \frac{\partial \lambda}{\partial x^n} &= \frac{2x^n}{2x^n x_0^n} - \frac{|\widetilde{x}|^2 + (x^n)^2 + (x_0^n)^2}{2(x^n)^2 x_0^n} \\ &= \frac{(x^n)^2 - |\widetilde{x}|^2 - (x_0^n)^2}{2(x^n)^2 x_0^n}. \end{aligned}$$

Thus

$$\nabla \lambda = \frac{(\widetilde{x}, \frac{1}{2x^n}((x^n)^2 - |\widetilde{x}|^2 - (x_0^n)^2))}{x^n x_0^n}.$$

The unit normal on $\partial B_r(x_0^n)$ is

$$n = \frac{(\widetilde{x}, x^n - x_0^n)}{r}.$$

Using $|\widetilde{x}|^2 = r^2 - (x^n - x_0^n)^2$, we compute

$$\begin{aligned}
\nabla\lambda \cdot n &= \frac{(\widetilde{x}, \frac{1}{2x^n}((x^n)^2 - |\widetilde{x}|^2 - (x_0^n)^2)}{x^n x_0^n} \cdot \frac{(\widetilde{x}, x^n - x_0^n)}{r} \\
&= \frac{(\widetilde{x}, \frac{1}{2x^n}(2(x^n)^2 - r^2 - 2x^n x_0^n)}{x^n x_0^n} \cdot \frac{(\widetilde{x}, x^n - x_0^n)}{r} \\
&= \frac{(\widetilde{x}, x^n - x_0^n - \frac{r^2}{2x^n})}{x^n x_0^n} \cdot \frac{(\widetilde{x}, x^n - x_0^n)}{r} \\
&= \frac{|\widetilde{x}|^2 + (x^n - x_0^n)^2}{x^n x_0^n r} - \frac{\frac{r^2}{2x^n}(x^n - x_0^n)}{x^n x_0^n r} \\
&= \frac{r}{x^n x_0^n} - r\frac{x^n - x_0^n}{2(x^n)^2 x_0^n} \\
&= r\frac{2x^n}{2(x^n)^2 x_0^n} - r\frac{x^n - x_0^n}{2(x^n)^2 x_0^n} \\
&= r\frac{x^n + x_0^n}{2(x^n)^2 x_0^n},
\end{aligned}$$

completing the proof. □

Next, we recall the classical localization theorem.

Theorem 21 [17] *If $u : \Omega \to \mathbb{R}$ is a continuous function and $B_r(x_0^n) \subset \Omega$ for some $r > 0$, then*

$$\lim_{r\to 0} \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r(x_0^n)} u(x)dS(x) = u(\widetilde{0}, x_0^n),$$

where ω_{n-1} is the surface area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

Using the localization theorem, we can compute the following limits.

Lemma 22 *If F is the function defined above and $\varphi \in \mathcal{D}(\mathbb{R}^n_+)$ a test function, then*

$$\lim_{r\to 0} \int_{\partial B_r(x_0^n)} \varphi(x)\frac{\partial F}{\partial n}(x)dS(x) = -2\mu\omega_{n-1}(x_0^n)^{2\mu}\varphi(\widetilde{0}, x_n^n)$$

and

$$\lim_{r\to 0} \int_{\partial B_r(x_0^n)} F(x)\frac{\partial\varphi}{\partial n}(x)dS(x) = 0.$$

Proof Using the representation given in Proposition 18, we compute

$$\frac{\partial F}{\partial n} = \nabla F \cdot n = \tfrac{\nabla f(\lambda)\cdot n}{(\lambda^2-1)^\mu} + f(\lambda)\nabla\Big(\tfrac{1}{(\lambda^2-1)^\mu}\Big) \cdot n.$$

Since $\nabla g(\lambda) = \frac{dg}{d\lambda}\nabla\lambda$, we have

$$\begin{aligned}\frac{\partial F}{\partial n} &= \frac{f'(\lambda)\nabla\lambda\cdot n}{(\lambda^2-1)^\mu} + f(\lambda)\frac{d}{d\lambda}\Big(\frac{1}{(\lambda^2-1)^\mu}\Big)\nabla\lambda\cdot n\\ &= \frac{f'(\lambda)}{(\lambda^2-1)^\mu}\frac{\partial\lambda}{\partial n} - f(\lambda)\frac{2\mu\lambda}{(\lambda^2-1)^{\mu+1}}\frac{\partial\lambda}{\partial n}.\end{aligned}$$

Since $\lambda = \frac{r^2}{2x^n x_0^n} + 1$, we have

$$\begin{aligned}\lambda^2 - 1 &= (\frac{r^2}{2x^n x_0^n} + 1)^2 - 1\\ &= \frac{r^4}{4(x^n)^2(x_0^n)^2} + \frac{r^2}{x^n x_0^n}\\ &= r^2\Big(\frac{r^2}{4(x^n)^2(x_0^n)^2} + \frac{1}{x^n x_0^n}\Big)\\ &= r^2\frac{r^2+4x^n x_0^n}{4(x^n)^2(x_0^n)^2}.\end{aligned}$$

We can write $\lambda^2 - 1 = r^2\epsilon(\lambda)$, where

$$\epsilon(\lambda) = \frac{r^2+4x^n x_0^n}{4(x^n)^2(x_0^n)^2} \to \frac{1}{(x_0^n)^2}$$

for $\lambda \to 1$, or equivalently $r \to 0$ and especially then $x^n \to x_0^n$. Since $\mu = \frac{n-2}{2}$ we have

$$\begin{aligned}(\lambda^2-1)^\mu &= r^{n-2}\epsilon(\lambda)^\mu,\\ (\lambda^2-1)^{\mu+1} &= r^n\epsilon(\lambda)^{\mu+1}.\end{aligned}$$

Hence, using the localization Theorem 21 and Lemma 20, we obtain

$$\lim_{r\to 0}\int_{\partial B_r(x_0^n)}\varphi\frac{f'(\lambda)}{(\lambda^2-1)^\mu}\frac{\partial\lambda}{\partial n}\,dS = \lim_{r\to 0}\frac{1}{r^{n-1}}\int_{\partial B_r(x_0^n)} r\varphi\frac{f'(\lambda)}{\epsilon(\lambda)^\mu}\frac{\partial\lambda}{\partial n}\,dS = 0.$$

Similarly, we compute

$$\begin{aligned}&-2\mu\lim_{r\to 0}\int_{\partial B_r}\varphi\frac{\lambda f(\lambda)}{(\lambda^2-1)^{\mu+1}}\frac{\partial\lambda}{\partial n}\,dS\\ &= -2\mu\lim_{r\to 0}\int_{\partial B_r}\varphi\frac{\lambda f(\lambda)}{r^n\epsilon(\lambda)^{\mu+1}}\frac{\partial\lambda}{\partial n}\,dS\\ &= -2\mu\omega_{n-1}\lim_{r\to 0}\frac{1}{\omega_{n-1}r^{n-1}}\int_{\partial B_r}\varphi\frac{\lambda f(\lambda)}{\epsilon(\lambda)^{\mu+1}}\frac{1}{r}\frac{\partial\lambda}{\partial n}\,dS\\ &= -2\mu\omega_{n-1}(x_0^n)^{2\mu}\varphi(\widetilde{0}, x_n^n).\end{aligned}$$

We see that the first integral formula is true. To prove the second integral, we compute

$$\begin{aligned}\lim_{r\to 0}\int_{\partial B_r(x_0^n)} F\frac{\partial\varphi}{\partial n}dS &= \lim_{r\to 0}\int_{\partial B_r(x_0^n)}\frac{f(\lambda)}{(\lambda^2-1)^\mu}\frac{\partial\varphi}{\partial n}dS\\ &= \lim_{r\to 0}\frac{1}{r^{n-2}}\int_{\partial B_r}\frac{f(\lambda)}{\epsilon(\lambda)^\mu}\frac{\partial\varphi}{\partial n}dS\\ &= \lim_{r\to 0}\frac{1}{r^{n-1}}\int_{\partial B_r} r\frac{f(\lambda)}{\epsilon(\lambda)^\mu}\frac{\partial\varphi}{\partial n}dS = 0,\end{aligned}$$

since $\frac{\partial\varphi}{\partial n} \in \mathcal{D}(\mathbb{R}^n_+)$. □

Let us now define

$$G(x; x_0^n) = -\frac{F(x; x_0^n)}{2\mu\omega_{n-1}(x_0^n)^{2\mu}}.$$

Hence we obtain the following corollary.

Corollary 23 *If G is the function defined above and $\varphi \in \mathcal{D}(\mathbb{R}^n_+)$ a test function, then*

$$\lim_{r\to 0}\int_{\partial B_r(x_0^n)} \varphi(x)\frac{\partial G}{\partial n}(x)dS(x) = \varphi(\widetilde{0}, x_n^n)$$

and

$$\lim_{r\to 0}\int_{\partial B_r(x_0^n)} G(x)\frac{\partial\varphi}{\partial n}(x)dS(x) = 0.$$

Now we are ready to prove the following proposition.

Proposition 24 *For the preceding G, we have*

$$\widetilde{L}G = \delta(\widetilde{x})\delta(x^n - x_0^n).$$

Proof Assume $\varphi \in \mathcal{D}(\mathbb{R}^n_+)$ and $\mathrm{supp}(\varphi) \subset \Omega \subset \mathbb{R}^n_+$, where Ω is a set with a smooth enough boundary. Assume $(\widetilde{0}, x_0^n) \in \mathrm{supp}(\varphi)$ and take $B_r(x_0^n) \subset \Omega$ and $\Omega_r(x_0^n) := \Omega\backslash\overline{B}_r(x_0^n)$. Since $\partial\Omega_r(x_0^n) = \partial\Omega \cup (-\partial B_r(x_0^n))$, we compute by Green's formula

$$\begin{aligned}\int_{\Omega_r(x_0^n)}(G\widetilde{L}\varphi - \varphi\widetilde{L}G)dx &= \int_{\partial\Omega_r(x_0^n)}\Big(G\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial G}{\partial n}\Big)dS\\ &= \int_{\partial B_r(x_0^n)}\Big(\varphi\frac{\partial G}{\partial n} - G\frac{\partial\varphi}{\partial n}\Big)dS.\end{aligned}$$

In the last part, we use the information, that φ and $\frac{\partial\varphi}{\partial n}$ vanish in the boundary $\partial\Omega$. Since $\widetilde{L}G = 0$ in $\Omega_r(x_0^n)$, we have

$$\int_{\Omega_r(x_0^n)} G\widetilde{L}\varphi dx = \int_{\partial B_r(x_0^n)}\Big(\varphi\frac{\partial G}{\partial n} - G\frac{\partial\varphi}{\partial n}\Big)dS.$$

We observe, that since G is continuous outside of $\lambda = 1$ and $\widetilde{L}\varphi$ is smooth with compact support, then $G\widetilde{L}\varphi$ is locally integrable, and we can compute the limit $r \to 0$. Using the preceding corollary, we have

$$\begin{aligned}\langle\widetilde{L}G, \varphi\rangle &= \langle G, \widetilde{L}\varphi\rangle\\ &= \int_{\mathbb{R}^n_+} G(x; x_0^n)\widetilde{L}\varphi(x)dx\\ &= \int_{\Omega} G(x; x_0^n)\widetilde{L}\varphi(x)dx\\ &= \varphi(\widetilde{0}, x_n^n).\end{aligned}$$

□

We may give the following crucial result.

Theorem 25 *The function*

$$H(x;x_0^n) = \frac{h(\lambda)}{(\lambda^2-1)^{\mu}},$$

where

$$\begin{aligned} h(\lambda) &= -(\frac{x_0^n}{x^n})^{\frac{k}{2}} \frac{\sqrt{\pi}}{2^{\nu+\mu+1}\omega_{n-1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)\Gamma(\nu+3/2)} \frac{1}{(x^n x_0^n)^{\mu}\lambda^{\nu-\mu+1}} \times \\ &\quad \times {}_2F_1\Big(\frac{\nu-\mu+2}{2}, \frac{\nu-\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{\lambda^2}\Big). \end{aligned}$$

is a fundamental solution of the Weinstein-Leutwiler equation

$$\Delta u + \frac{k}{x^n}\frac{\partial u}{\partial x^n} + \frac{\ell}{(x^n)^2}u = 0$$

at the point $x_0 = (\widetilde{0}, x_0^n) \in \mathbb{R}^n_+$. *In the formula,* ω_{n-1} *is the surface area of the unit sphere* $S^{n-1} \subset \mathbb{R}^n$, $\lambda = \frac{|\widetilde{x}|^2+(x^n)^2+(x_0^n)^2}{2x^n x_0^n}$, $\nu = \frac{1}{2}(\sqrt{n(n-2)-4\gamma+1}-1)$ *with* $\gamma = \frac{1}{4}\big(n^2-k^2+2(k-n+2\ell)\big)$ *and* $\mu = \frac{n-2}{2}$.

Proof By virtue of Corollary 17 and the preceding proposition, we obtain that

$$H(x;x_0^n) = (\frac{x_0^n}{x^n})^{\frac{k}{2}} G(x;x_0^n),$$

which satisfies the equation $LH(x;x_0^n) = \delta(\widetilde{x})\delta(x^n - x_0^n)$. Using Proposition 18, we obtain

$$\begin{aligned} H(x;x_0^n) &= (\frac{x_0^n}{x^n})^{\frac{k}{2}} G(x;x_0^n) \\ &= -(\frac{x_0^n}{x^n})^{\frac{k}{2}} \frac{F(x;x_0^n)}{2\mu\omega_{n-1}(x_0^n)^{2\mu}} \\ &= -(\frac{x_0^n}{x^n})^{\frac{k}{2}} \frac{1}{2\mu\omega_{n-1}(x_0^n)^{2\mu}} \frac{f(\lambda)}{(\lambda^2-1)^{\mu}}. \end{aligned}$$

We define

$$h(\lambda) = -(\frac{x_0^n}{x^n})^{\frac{k}{2}} \frac{f(\lambda)}{2\mu\omega_{n-1}(x_0^n)^{2\mu}}.$$

The function $f(\lambda)$ is given in Proposition 18, and we have

$$\begin{aligned} h(\lambda) &= -(\frac{x_0^n}{x^n})^{\frac{k}{2}} \frac{1}{2\mu\omega_{n-1}(x_0^n)^{2\mu}} \frac{\sqrt{\pi}}{2^{\nu+\mu}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu)\Gamma(\nu+3/2)} \Big(\frac{x_0^n}{x^n}\Big)^{\mu} \frac{1}{\lambda^{\nu-\mu+1}} {}_2F_1\Big(\cdots; \frac{1}{\lambda^2}\Big) \\ &= -(\frac{x_0^n}{x^n})^{\frac{k}{2}} \frac{1}{\mu\omega_{n-1}} \frac{\sqrt{\pi}}{2^{\nu+\mu+1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu)\Gamma(\nu+3/2)} \frac{1}{(x^n x_0^n)^{\mu}\lambda^{\nu-\mu+1}} {}_2F_1\Big(\cdots; \frac{1}{\lambda^2}\Big). \end{aligned}$$

Using the formula 6.1.15 of [1], we have $\mu\Gamma(\mu) = \Gamma(\mu+1)$, that is,

$$h(\lambda) = -(\frac{x_0^n}{x^n})^{\frac{k}{2}} \frac{\sqrt{\pi}}{2^{\nu+\mu+1}\omega_{n-1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)\Gamma(\nu+3/2)} \frac{1}{(x^n x_0^n)^{\mu}\lambda^{\nu-\mu+1}} {}_2F_1\Big(\cdots; \frac{1}{\lambda^2}\Big).$$

□

We complete the paper by making the following remarks.

Remark 26 Since the Weinstein-Leutwiler equation is translation invariant with respect to transformations $\widetilde{x} \mapsto \widetilde{x} + \widetilde{x}_0$, we obtain a fundamental solution $H(x; x_0)$ at any point $x_0 \in \mathbb{R}^n_+$ just making the substitution.

Remark 27 The preceding fundamental solution $H(x; x_0^n)$ is not unique, since we can always add an arbitrary solution. We can say that all invariant fundamental solutions with respect to the Lie algebra $\mathfrak{h}_f$ are of the form $H(x; x_0^n) + c\mathcal{P}(x; x_0)$ (cf. Proposition 14), where $c \in \mathbb{R}$.

By representing the formula given in Theorem 25 using the Legendre function $Q_\nu^{-\mu}$, we obtain the main result of the study.

Theorem 28 *All invariant fundamental solutions of the Leutwiler-Weinstein equation in the neighbourhood of the point $x_0 \in \mathbb{R}^n_+$ are of the form*

$$H(x; x_0) = (x^n)^{-\frac{k}{2}-\mu}(x_0^n)^{\frac{k}{2}-\mu}\big(\lambda^2 - 1\big)^{-\mu/2}\big(cP_\nu^{-\mu}(\lambda) - \kappa(\mu, \nu)Q_\nu^{-\mu}(\lambda)\big)$$

for $c \in \mathbb{R}$. In the formula $\kappa(\mu, \nu) := \frac{e^{i\pi\mu}}{2^\mu \omega_{n-1}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)\Gamma(\nu-\mu+1)}$, ω_{n-1} is the surface area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$, $\lambda = \frac{|\widetilde{x}-\widetilde{x}_0|^2+(x^n)^2+(x_0^n)^2}{2x^n x_0^n}$, $\nu = \frac{1}{2}(\sqrt{n(n-2) - 4\gamma + 1} - 1)$ with $\gamma = \frac{1}{4}\big(n^2 - k^2 + 2(k - n + 2\ell)\big)$ and $\mu = \frac{n-2}{2}$.

9 Application: Mean Value Principle for the Hyperbolic Unit Ball at $(\widetilde{0}, x_0^n)$

The classical Dirichlet problem in Euclidean space is usually formulated as follows:

> *Given a function f that has values everywhere on the boundary of a region in $\mathbb{R}^n$, is there a unique continuous function u twice continuously differentiable in the interior and continuous on the boundary, such that u is harmonic in the interior and $u = f$ on the boundary?*

A solution to the problem depends on the geometry of the domain. For example, in unit ball $B(0, 1)$, a solution is given by the so-called Poisson integral formula

$$P[f](x) = \int_{S^{n-1}} f(y)P(x, y)dS(y),$$

where $P(x, y)$ is the so-called Poisson kernel and $S^{n-1} = \partial B(0, 1)$ is the unit sphere.

In this section, we consider a Dirichlet problem of the Leutwiler-Weinstein operator L. Recently, there has been a growing interest in such problems, see for example, [18, 23]. The Dirichlet problem is then

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ u = g, & \text{in } \partial\Omega. \end{cases}$$

As in the Euclidean case $k = \ell = 0$, to obtain an explicit representation formula, we need to restrict a geometrically suitable case. In this section, we consider the case where Ω is the so-called hyperbolic unit ball in the upper half-space, defined in the next section. Using the fundamental solution, we can find a Poisson-type kernel and general representation formula for the preceding Dirichlet problem at the origin of the ball. Unfortunately, a general formula is still an open question.

9.1 Poincaré Upper Half-Space

We assume, that in the upper half-space,

$$\mathbb{R}^n_+ = \{x = (x^1, ..., x^n) \in \mathbb{R}^n : x^n > 0\}$$

is endowed with the non-Euclidean metric

$$ds^2 = \frac{(dx^1)^2 + \cdots + (dx^n)^2}{(x^n)^2}.$$

The Riemannian manifold $(\mathbb{R}^n_+, ds^2)$ is called the hyperbolic Poincaré half-space. The straight lines in the preceding hyperbolic space are represented by circular arcs crossing perpendicular to the $x^n = 0$ plane. The distance between two points $x, x_0 \in \mathbb{R}^n_+$ with respect to the preceding metric is computed by

$$d_h(x, x_0) = \operatorname{arcosh}(\lambda(x, x_0)),$$

where the auxiliary function λ is

$$\lambda(x, x_0) = \frac{|\widetilde{x} - \widetilde{x}_0|^2 + (x^n)^2 + (x_0^n)^2}{2x^n x_0^n}.$$

These observations allow us to define balls in the upper half-space. We consider the r-ball, with the centre $(\widetilde{0}, x_0^n)$, and we denote

$$\begin{aligned} B_h(x_0^n, r) &= \{x \in \mathbb{R}^n_+ : 0 \leq d(x, x_0^n) < r\} \\ &= \{x \in \mathbb{R}^n_+ : 1 \leq \lambda(x, x_0^n) < R\} \end{aligned}$$

where $R = \cosh(r)$. For the unit ball $B_h(x_0^n) := B_h(x_0^n, 1)$, we denote $R_1 = \cosh(1) \approx 1.543...$

Geometrically, the preceding r-ball is just the Euclidean ball

$$B_e(z_e, r_e) = \{x \in \mathbb{R}^n : |x - z_e| < r_e\},$$

with the centre $z_e = (\widetilde{0}, x_0^n \cosh(r))$ and the radius $r_e = x_0^n \sinh(r)$. See all details of the preceding discussion and more, e.g., in [11–15].

9.2 Green Function of Unit Ball

A Green function on a domain $\Omega \subset \mathbb{R}^n_+$ is a function $G(x; x_0)$ satisfying

$$\begin{cases} LG(x; x_0) = \delta(x - x_0), & x \in \Omega, \\ G(x; x_0) = 0, & x \in \partial\Omega \end{cases}$$

for all $x, x_0 \in \Omega$. In general, such a function is not easy to find, since it depends on the shape of the Ω. Usually, to compute a Green's function, the set Ω should have enough symmetry. One of the cases with enough symmetry is a hyperbolic unit ball, which we consider next.

Assume $\Omega = B(x_0^n)$. Our starting point is the fundamental solution of the Leutwiler-Weinstein equation; see Theorem 28. Since $LH(x; x_0) = \delta(x - x_0)$, the preceding function is a candidate for a Green function, if the boundary values can be controlled properly.

Remark 29 All invariant fundamental solutions are the sum of a fundamental solution ($Q_\nu^{-\mu}$ part) and a null-solution ($P_\nu^{-\mu}$ part) of the operator L.

Moreover, we have the following useful transformation formula.

Proposition 30 (Symmetry property) *A fundamental solution satisfies*

$$(x^n)^k H(x; x_0) = (x_0^n)^k H(x_0; x).$$

Using the preceding information, we can compute the Green function at the origin $(\widetilde{0}, x_0^n)$ as follows.

Theorem 31 (Green function at the origin of the hyperbolic unit ball) *The function*

$$G(x; x_0^n) = (x^n)^{-\frac{k}{2}-\mu}(x_0^n)^{\frac{k}{2}-\mu}\kappa(\mu,\nu)\big(\lambda^2-1\big)^{-\mu/2}\Big(\frac{Q_\nu^{-\mu}(R_1)}{P_\nu^{-\mu}(R_1)}P_\nu^{-\mu}(\lambda) - Q_\nu^{-\mu}(\lambda)\Big)$$

satisfies

$$\begin{cases} LG(x; x_0^n) = \delta(x - (\widetilde{0}, x_0^n)), & x \in B_h(x_0^n), \\ G(x; x_0^n) = 0, & x \in \partial B_h(x_0^n). \end{cases}$$

Proof At the hyperbolic unit sphere $x \in \partial B_h(x_0^n)$, we have $H(x; x_0^n) = 0$ if and only if

$$cP_\nu^{-\mu}(R_1) - \kappa(\mu,\nu)Q_\nu^{-\mu}(R_1) = 0 \;\Leftrightarrow\; c = \kappa(\mu,\nu)\frac{Q_\nu^{-\mu}(R_1)}{P_\nu^{-\mu}(R_1)}.$$

The $P_\nu^{-\mu}$ part of the fundamental solution exists, if $|\lambda - 1| < 2$ or equivalently $-1 < \lambda < 3$. Especially, $R_1 < 3$, that is, the construction exists on the unit ball. □

Unfortunately, the preceding formula is not valid at every point of the unit ball, only at the origin. The usual technique used in the Euclidean case seems to be hard to apply directly. We leave this question open and just give the following conjecture.

Conjecture 32 (Green function of the hyperbolic unit ball) *There exists a Green function $G(x; x_0)$ with the symmetry property $(x^n)^k G(x; y) = (y^n)^k G(y; x)$ (maybe up to a constant) satisfying*

$$\begin{cases} LG(x; x_0) = \delta(x - x_0), & x \in B_h(x_0^n), \\ G(x; x_0) = 0, & x \in \partial B_h(x_0^n). \end{cases}$$

Using the classical methods of partial differential equations, one can prove that the preceding Green function exists. The symmetry property must also be true, since all Green functions are fundamental solutions.

9.3 Representation Formula for Solutions to the Dirichlet Problem

In this section, we derive an integral representation of solutions to the Dirichlet problem assuming that the Green function, given in Conjecture 32 exists. This motivates us to find an explicit expression for the Green function in future studies. Our problem is to study

$$\begin{cases} Lu = f, & \text{in } B_h(x_0^n), \\ u = g, & \text{in } \partial B_h(x_0^n). \end{cases}$$

The necessary condition for integral representations is the existence of the Green's type integral formula.

Proposition 33 (Green's integral formula for the Leutwiler-Weinstein operator) *Let u, v be a two times differentiable function defined in a neighbourhood of $\Omega \subset \mathbb{R}^n_+$, we have*

$$\int_{\Omega} \big(uLv - vLu\big)(x^n)^k dx = \int_{\partial\Omega} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)(x^n)^k dS,$$

where dx is the Euclidean volume measure and dS the Euclidean surface measure on a smooth enough $\partial\Omega$.

Proof Recall the classical Green's integral formula

$$\int_{\Omega} u\Delta v dx + \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n} dS.$$

Replacing u by $(x^n)^k u$, we have

$$\int_{\Omega} (x^n)^k u\Delta v dx + \int_{\Omega} (x^n)^k u\frac{k}{x^n}\frac{\partial v}{\partial x^n} dx + \int_{\Omega} (x^n)^k \nabla u \cdot \nabla v dx = \int_{\partial\Omega} (x^n)^k u\frac{\partial v}{\partial n} dS.$$

Changing the role of u and v, we have

$$\int_{\Omega} (x^n)^k v\Delta u dx + \int_{\Omega} (x^n)^k v\frac{\alpha}{x^n}\frac{\partial u}{\partial x^n} dx + \int_{\Omega} (x^n)^k \nabla v \cdot \nabla u dx = \int_{\partial\Omega} (x^n)^k v\frac{\partial u}{\partial n} dS.$$

Subtracting the preceding integrals from the upper on, we obtain

$$\int_{\Omega} (x^n)^k \Big(u\big(\Delta v + \frac{k}{x^n}\frac{\partial v}{\partial x^n}\big) - \big(\Delta u + \frac{k}{x^n}\frac{\partial u}{\partial x^n}\big)v\Big) dx = \int_{\partial\Omega} (x^n)^k \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big) dS.$$

We add and subtract the term $(x^n)^k \frac{\ell}{(x^n)^2} uv$ in the volume integral and we have

$$\int_{\Omega} (x^n)^k \Big(u\big(\Delta v + \frac{k}{x^n}\frac{\partial v}{\partial x^n} + \frac{\ell}{(x^n)^2}v\big) - \big(\Delta u + \frac{k}{x^n}\frac{\partial u}{\partial x^n} + \frac{\ell}{(x^n)^2}u\big)v\Big) dx = \int_{\partial\Omega} (x^n)^k \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big) dS$$

completing the proof. □

Using Green's formula, we can prove an integral representation formula for the Dirichlet problem. We will write L_x, if we want to emphasize the variable.

Theorem 34 *A solution to the Dirichlet problem*

$$\begin{cases} L_y u = f, & \text{in } B_h(x_0^n), \\ u = g, & \text{in } \partial B_h(x_0^n). \end{cases}$$

can be given by

$$u(y) = \int_{B_h(x_0^n)} G(y; x) f(x) dx + \int_{\partial B_h(x_0^n)} g(x)\frac{\partial G(x; y)}{\partial n}\Big(\frac{x^n}{y^n}\Big)^k dS(x).$$

Proof By Green's formula

$$\int_{B_h(x_0^n)} uLv(x^n)^k dx = \int_{B_h(x_0^n)} vLu(x^n)^k dx + \int_{\partial B_h(x_0^n)} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)(x^n)^k dS.$$

Taking $v = G(x; y)$, i.e., $LG(x; y) = \delta(x - y)$, we have

$$\int_{B_h(x_0^n)} uLG(x; y)(x^n)^k dx = \int_{B_h(x_0^n)} u\delta(x - y)(x^n)^k dx = u(y)(y^n)^k.$$

We have

$$\begin{aligned} u(y) &= \int_{B_h(x_0^n)} G(x; y)Lu(x)\Big(\frac{x^n}{y^n}\Big)^k dx + \int_{\partial B_h(x_0^n)} u\frac{\partial G(x; y)}{\partial n}\Big(\frac{x^n}{y^n}\Big)^k dS(x) \\ &\quad - \int_{\partial B_h(x_0^n)} G(x; y)\frac{\partial u}{\partial n}\Big(\frac{x^n}{y^n}\Big)^k dS(x). \end{aligned}$$

Since $G(x; y) = 0$ in $x \in \partial B(x_0^n)$, we have

$$u(y) = \int_{B_h(x_0^n)} G(x; y)Lu(x)\Big(\frac{x^n}{y^n}\Big)^k dx + \int_{\partial B_h(x_0^n)} u(x)\frac{\partial G(x; y)}{\partial n}\Big(\frac{x^n}{y^n}\Big)^k dS(x).$$

If $Lu = f$ in the interior and $u = g$ on a boundary, we have

$$u(y) = \int_{B_h(x_0^n)} G(x; y)f(x)\Big(\frac{x^n}{y^n}\Big)^k dx + \int_{\partial B_h(x_0^n)} g(x)\frac{\partial G(x; y)}{\partial n}\Big(\frac{x^n}{y^n}\Big)^k dS(x).$$

Using $(x^n)^k G(x; y) = (y^n)^k G(y; x)$, we have

$$u(y) = \int_{B_h(x_0^n)} G(y, x)f(x)dx + \int_{\partial B_h(x_0^n)} g(x)\frac{\partial G(x; y)}{\partial n}\Big(\frac{x^n}{y^n}\Big)^k dS(x).$$

□

If $f \equiv 0$, then we obtain the following Poisson-type representation formula.

Corollary 35 *A solution to the Dirichlet problem*

$$\begin{cases} L_y u = 0, & in\ B_h(x_0^n), \\ u = g, & in\ \partial B_h(x_0^n). \end{cases}$$

can be given by

$$u(y) = \int_{\partial B_h(x_0^n)} g(x)\frac{\partial G(x; y)}{\partial n}\Big(\frac{x^n}{y^n}\Big)^k dS(x),$$

where

$$y \mapsto \frac{\partial G(x; y)}{\partial n}\Big(\frac{x^n}{y^n}\Big)^k$$

belongs to the kernel of L_y.

Proof Formula $(x^n)^k G(x; y) = (y^n)^k G(y; x)$ implies

$$G(x; y) = \Big(\frac{y^n}{x^n}\Big)^k G(y; x),$$

and we have (the normal deriative act on x)

$$\begin{aligned} \frac{\partial G(x; y)}{\partial n}\Big(\frac{x^n}{y^n}\Big)^k &= \frac{\partial}{\partial n}\Big(\Big(\frac{y^n}{x^n}\Big)^k G(y; x)\Big)\Big(\frac{x^n}{y^n}\Big)^k \\ &= \frac{\partial}{\partial n}\Big(\frac{G(y; x)}{(x^n)^k}\Big)(x^n)^k \end{aligned}$$

and we observe that

$$y \mapsto \frac{\partial G(x;y)}{\partial n}\Big(\frac{x^n}{y^n}\Big)^k$$

belongs to the kernel of L_y and hence the preceding formula gives a solution $u(y)$. □

Thus, we can call the preceding formula a Poisson formula and $P(x;y) := \frac{\partial G(x;y)}{\partial n}\big(\frac{x^n}{y^n}\big)^k$ a Poisson kernel.

10 Conclusions

In this paper, we study the symmetries of fundamental solutions of the Leutwiler-Weinstein equation. The method is described by the first author in [5]. As a result, we compute detailed the fundamental solution and study, how to use it to find Green's function for the problem. In the future, we will complete this task and construct give a detailed construction for it. Also some other interesting linear partial differential equations with non-constant coefficients should be studied. We hope, that our text motivates researchers to apply the method in their studies.

Appendix A: Proof of Proposition 6

Using Eqs. 13 and 14, we have (for each $i \neq j$)

$$\xi^i_{x^j}(x) = -\xi^j_{x^i}(x) \;\Rightarrow\; \xi^i_{x^jx^j}(x) = -\xi^j_{x^jx^i}(x) = -\xi^i_{x^ix^i}(x)$$

and we may write Eqs. 11 and 12 of the form

$$-\tfrac{k}{x^n}\xi^i_{x^n}(x) + 2\eta_{x^i}(x) + (n-2)\xi^i_{x^ix^i}(x) = 0,\;\; i = 1, ..., n-1, \tag{A.1}$$

$$-\xi^n(x)\tfrac{k}{(x^n)^2} + 2\eta_{x^n}(x) + (n-2)\xi^n_{x^nx^n}(x) + \xi^n_{x^n}(x)\tfrac{k}{x^n} = 0. \tag{A.2}$$

From these, applying Eq. 13, we obtain the formulas

$$\begin{aligned}
2\sum_{i=1}^{n-1}\eta_{x^ix^i}(x) &= -(n-2)\sum_{i=1}^{n-1}\xi^i_{x^ix^ix^i}(x) + \frac{k(n-1)}{x^n}\xi^n_{x^nx^n}(x),\\
\frac{2k}{x^n}\eta_{x^n}(x) &= -\frac{k(n-2)}{x^n}\xi^n_{x^nx^n}(x) + \frac{k^2}{(x^n)^3}\xi^n(x) - \frac{k^2}{(x^n)^2}\xi^n_{x^n}(x),\\
2\eta_{x^nx^n}(x) &= -(n-2)\xi^n_{x^nx^nx^n}(x) - \frac{k}{x^n}\xi^n_{x^nx^n}(x) + \frac{2k}{(x^n)^2}\xi^n_{x^n}(x) - \frac{2k}{(x^n)^3}\xi^n(x).
\end{aligned}$$

Substituting the preceding formulas into Eq. 10 (multiplied by 2), we obtain the equation

$$-(n-2)\sum_{i=1}^{n}\xi^i_{x^ix^ix^i}(x) + \frac{2k-k^2+4\ell}{(x^n)^2}\xi^n_{x^n}(x) - \frac{2k-k^2+4\ell}{(x^n)^3}\xi^n(x) = 0.$$

We assume that $n \geq 3$ and $k(2-k) + 4\ell \neq 0$. Using Eqs. 13 and 14, we have

$$\xi^i_{x^ix^ix^i}(x) = \xi^n_{x^ix^ix^n}(x) = -\xi^i_{x^ix^nx^n}(x) = -\xi^n_{x^nx^nx^n}(x), \tag{A.3}$$

and

$$(n-2)^2\xi^n_{x^nx^nx^n}(x) + \frac{2k-k^2+4\ell}{(x^n)^2}\xi^n_{x^n}(x) - \frac{2k-k^2+4\ell}{(x^n)^3}\xi^n(x) = 0.$$

It is a third-order linear ordinary differential equation with respect to x^n, and it has three linearly independent solutions. We look for solutions of the form $\xi^n(x) = h(\widetilde{x})(x^n)^\alpha$, where $\widetilde{x} = (x^1, ..., x^{n-1})$. Making the substitution, we obtain the equation (putting $\beta = 2k - k^2 + 4\ell$)

$$(n-2)^2\alpha(\alpha-1)(\alpha-2) + \beta\alpha - \beta = 0$$
$$\Leftrightarrow \alpha = 1 \text{ or } (n-2)^2\alpha^2 - 2(n-2)^2\alpha + \beta = 0.$$

Assume that the root $\alpha \neq 1$. Then

$$\xi^n(x) = (x^n)^\alpha h(\widetilde{x})$$

and by Eq. 14

$$\xi^n_{x^n}(x) = \xi^j_{x^j}(x) = \alpha(x^n)^{\alpha-1}h(\widetilde{x})$$

and by Eq. 13

$$\xi^n_{x^j x^j} + \xi^j_{x^j x^n} = (x^n)^\alpha h_{x^j x^j}(\widetilde{x}) + \alpha(\alpha-1)(x^n)^{\alpha-2}h(\widetilde{x}) = 0,$$

that is, $h = 0$, and we see that these solutions do not give us a nontrivial symmetry.

Let us now look for symmetries for $\alpha = 1$, i.e.,

$$\xi^n(x) = h(\widetilde{x})x^n.$$

Then we have by (13) and (14) that

$$\xi^j_{x^j}(x) = \xi^n_{x^n}(x) = h(\widetilde{x}), \;\; j = 1, ..., n-1$$

and

$$\xi^n_{x^n x^n}(x) = \xi^j_{x^j x^n}(x) = -\xi^n_{x^j x^j}(x) = 0. \tag{A.4}$$

Since $\xi^n_{x^j x^j}(x) = 0$ for all $j = 1, ..., n$, it has to be of the form

$$\xi^n(x) = x^n\Big(\sum_{j=1}^{n-1} a_j x^j + b\Big)$$

and hence

$$h(\widetilde{x}) = \sum_{j=1}^{n-1} a_j x^j + b.$$

From (A.2), we infer

$$\eta_{x^n}(x) = 0,$$

that is, $\eta = \eta(\widetilde{x})$. Using (A.1), (A.3) and (A.4), we compute

$$\eta_{x^i x^i}(x) = 0,$$

for $i = 1, ..., n-1$. We infer that

$$\eta(x) = \sum_{j=1}^{n-1} g_j x^j + c.$$

Let us now compute the coefficients ξ^i. By Eq. 14, we obtain

$$\xi^i_{x^i}(x) = \xi^n_{x^n}(x) = h(\widetilde{x}) = a_i x^i + \sum_{\substack{j=1 \\ j\neq i}}^{n-1} a_j x^j + b,$$

for $i = 1, ..., n-1$ and we have

$$\xi^i(x) = \frac{a_i}{2}(x^i)^2 + x^i \sum_{\substack{j=1 \\ j \neq i}}^{n-1} a_j x^j + b x^i + c^i(x),$$

where $c^i(x) = c^i(x^1, ..., x^{i-1}, x^{i+1}, ..., x^n)$. If $i \neq p$ and

$$\xi^p(x) = \frac{a_p}{2}(x^p)^2 + x^p \sum_{\substack{j=1 \\ j \neq p}}^{n-1} a_j x^j + b x^p + c^p(x),$$

we have by Eq. 13 that

$$0 = \xi^i_{x^p}(x) + \xi^p_{x^i}(x) = x^i a_p + c^i_{x^p}(x) + x^p a_i + c^p_{x^i}(x),$$

then we obtain that

$$c^i_{x^p x^p} = -a_i \ \text{ and } \ c^i_{x^p x^p x^p} = 0$$

for any $p = 1, ..., n$. We have that

$$c^i(x) = \sum_{\substack{r,q=1 \\ r,q \neq i}}^{n} d^i_{rq} x^r x^q + \sum_{\substack{s=1 \\ s \neq i}}^{n} e^i_s x^s + f^i.$$

Since

$$\begin{aligned}
c^i_{x^p}(x) &= 2d^i_{pp} x^p + \sum_{\substack{q=1 \\ q \neq i,p}}^{n} d^i_{pq} x^q + \sum_{\substack{r=1 \\ r \neq i,p}}^{n} d^i_{rp} x^r + e^i_p, \\
c^i_{x^p x^p}(x) &= 2d^i_{pp} = -a^i,
\end{aligned}$$

we have that

$$c^i(x) = -\frac{a_i}{2} \sum_{\substack{r=1 \\ r \neq i}}^{n} (x^r)^2 + \sum_{\substack{r,q=1 \\ r,q \neq i \\ r \neq q}}^{n} d^i_{rq} x^r x^q + \sum_{\substack{s=1 \\ s \neq i}}^{n} e^i_s x^s + f^i.$$

Then we compute

$$\begin{aligned}
\xi^i(x) &= \frac{a_i}{2}(x^i)^2 + x^i \sum_{\substack{j=1\\ j\neq i}}^{n-1} a_j x^j + b x^i \\
&\quad - \frac{a_i}{2}\sum_{\substack{r=1\\ r\neq i}}^{n}(x^r)^2 + \sum_{\substack{r,q=1\\ r,q\neq i\\ r\neq q}}^{n} d^i_{rq} x^r x^q + \sum_{\substack{s=1\\ s\neq i}}^{n} e^i_s x^s + f^i \\
&= \frac{a_i}{2}(x^i)^2 + x^i \sum_{\substack{j=1\\ j\neq i}}^{n-1} a_j x^j + b x^i + \frac{a_i}{2}(x^i)^2 \\
&\quad - \frac{a_i}{2}(x^i)^2 - \frac{a_i}{2}\sum_{\substack{r=1\\ r\neq i}}^{n}(x^r)^2 + \sum_{\substack{r,q=1\\ r,q\neq i\\ r\neq q}}^{n} d^i_{rq} x^r x^q + \sum_{\substack{s=1\\ s\neq i}}^{n} e^i_s x^s + f^i \\
&= -\frac{a_i}{2}\sum_{r=1}^{n}(x^r)^2 + x^i\Big(\sum_{j=1}^{n-1} a_j x^j + b\Big) \\
&\quad + \sum_{\substack{r,q=1\\ r,q\neq i\\ r\neq q}}^{n} d^i_{rq} x^r x^q + \sum_{\substack{s=1\\ s\neq i}}^{n} e^i_s x^s + f^i.
\end{aligned}$$

Using Eq. 13, we obtain

$$\xi^i_{x^r x^q} = -\xi^q_{x^i x^r} = \xi^r_{x^i x^q}, \tag{A.5}$$

where we assume $i \neq r \neq q \neq i$. Since $\xi^i_{x^r x^q} = d^i_{rq} + d^i_{qr}$, Eq. A.5 gives

$$\begin{aligned}
d^i_{rq} + d^i_{qr} + d^r_{iq} + d^r_{qi} &= 0,\\
d^i_{rq} + d^i_{qr} - d^q_{ir} - d^q_{ri} &= 0.
\end{aligned}$$

Changing the role of r and q in the last equation, we obtain that

$$d^i_{rq} + d^i_{qr} = 0.$$

Then the term

$$\begin{aligned}
\sum_{\substack{r,q=1\\ r,q\neq i\\ r\neq q}}^{n} d^i_{rq} x^r x^q &= \sum_{\substack{r<q\\ r,q\neq i}} d^i_{rq} x^r x^q + \sum_{\substack{r>q\\ r,q\neq i}} d^i_{rq} x^r x^q \\
&= \sum_{\substack{r<q\\ r,q\neq i}} (d^i_{rq} + d^i_{qr}) x^r x^q = 0
\end{aligned}$$

and

$$\xi^i(x) = -\frac{a_i}{2}\sum_{r=1}^{n}(x^r)^2 + x^i\Big(\sum_{j=1}^{n-1} a_j x^j + b\Big) + \sum_{\substack{s=1\\ s\neq i}}^{n} e^i_s x^s + f^i.$$

Then, by Eq. 13, we have

$$0 = \xi^i_{x^j} + \xi^j_{x^i} = -a_i x^j + a_j x^i + e^i_j - a_j x^i + a_i x^j + e^j_i = e^i_j + e^j_i,$$

that is, $e^i_j = -e^j_i$ for $i \neq j$ and $e^j_j = 0$. We obtain

$$\xi^i(x) = -\frac{a_i}{2}\sum_{r=1}^{n}(x^r)^2 + x^i\Big(\sum_{j=1}^{n-1} a_j x^j + b\Big) + \sum_{s=1}^{n-1} e^i_s x^s + f^i,$$

where $e^i_j = -e^j_i$. Assume again, that $j \neq i$. Then we compute

$$\begin{aligned}
\xi^i_{x^n}(x) &= -a_i x^n,\\
\xi^i_{x^n x^n}(x) &= -a_i,\\
\eta_{x^i}(x) &= g_i,\\
\xi^i_{x^j}(x) &= -a_i x^j + x^i a_j + e^i_j, && \text{(A.6)}\\
\xi^i_{x^j x^j}(x) &= -a_i,\\
\xi^i_{x^i}(x) &= h(\widetilde{x}), && \text{(A.7)}\\
\xi^i_{x^i x^i}(x) &= a_i,
\end{aligned}$$

and Eq. 11 gives

$$k a_i + 2 g_i + (n-2) a_i = 0 \;\Leftrightarrow\; g_i = -(k+n-2)\frac{a_j}{2}$$

and

$$\eta(x) = -\frac{k+n-2}{2}\sum_{j=1}^{n-1} a_j x^j + c.$$

Scaling the coefficients a_i by 2, we have the solution

$$\begin{aligned}
\xi^n(x) &= 2x^n\Big(\sum_{j=1}^{n-1} a_j x^j + b\Big),\\
\xi^i(x) &= -a_i\sum_{r=1}^{n}(x^r)^2 + 2x^i\Big(\sum_{j=1}^{n-1} a_j x^j + b\Big) + \sum_{s=1}^{n-1} e^i_s x^s + f^i,\\
\eta(x) &= -(k+n-2)\sum_{j=1}^{n-1} a_j x^j + c,
\end{aligned}$$

where $e^i_j = -e^j_i$.

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