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# SYMMETRIES IN QUATERNIONIC ANALYSIS

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**Abstract.** This survey-type paper deals with the symmetries related to quaternionic analysis. The main goal is to formulate an $SU(2)$ invariant version of the theory. First, we consider the classical Lie groups related to the algebra of quaternions. After that, we recall the classical Spin(4) invariant case, that is Cauchy–Riemann operators, and recall their basic properties. We define the $SU(2)$ invariant operators called the Coifman–Weiss operators. Then we study their relations with the classical Cauchy–Riemann operators and consider the factorization of the Laplace operator. Using $SU(2)$ invariant harmonic polynomials, we obtain the Fourier series representations for quaternionic valued functions studying in detail the matrix coefficients.

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## 1. Introduction

The roots of quaternionic analysis come from the works of Moisil and Théodoresco (see [1]), and Fueter (see [2, 3]). Deavours published the first survey [4] of Fueter's works at the beginning of 70s. The idea is to found a well-defined and explicit counter-part for complex function theory on the plane. A function class in quaternionic analysis corresponding to complex holomorphic functions, called regular functions, is characterized by so-called Cauchy–Riemann operators. It seems that Sudbery's survey paper [5] was one of the starting points for modern research of the area, making the theory again visible. Around the same time in Gent, Richard Delangle and his students Fred Brackx and Frank Sommen started to study function theory on Clifford numbers, called Clifford analysis (see [6]). The quaternionic analysis is a natural special case for this theory still having its own special features. See [7] as a modern introduction to the topic.

Cauchy–Riemann operators are well-studied and known. Their counter-part in Clifford analysis is the so-called Dirac operator, whose kernel is used to define a higher dimensional analogy for complex holomorphic functions, called monogenic functions. The theory of monogenic functions is well-known and offers a nice approach for function theory in higher dimensions. The symmetry of the Dirac operator is also well-studied (see, e.g., [8, 9]).

Quaternionic analysis symmetries are not so well eloborated, although the starting point is fascinating. Many classical Lie groups are associated with quaternions, indeed $SU(2)$, Spin(3), Spin(4), $SO(3)$ and $SO(4)$. In the spirit of Clifford analysis, Cauchy–Riemann operators are Spin(4) invariant under $L$ and $H$ actions (see Section 3). However, they are *not* invariant under canonical actions

$$R[u]f(x) = f(\overline{u}x) \text{ and } S[u]f(x) = f(xu), \tag{1}$$

where $u \in SU(2)$. This action is mentioned and studied in the book by Coifman and Weiss [10]. In this survey-type paper, our aim is to complete their studies and give a modern representation of their theory using quaternions instead of matrices. The fundamental tools are $SU(2)$ invariant operators under the preceding actions. We are call them Coifman–Weiss operators. Coifman–Weiss operators give also a decomposition for the Laplacian, such as the Cauchy–Riemann operator and its conjugate. The results given in this paper offer a new way to look at quaternionic analysis via symmmetry and many possibilities to continue research in this direction.

The structure of the paper is the following. Section 2 is completely algebraic: we recall all needed tools and more as a starting point for further needs. In Section 3, we recall the Cauchy–Riemann operators and their classical actions. In Section 4, we define the Coifman–Weiss operators by $SU(2)$ invariant differential operators and represent them by the Cauchy–Riemann operators and derive some fundamental formulas. In Section 5, we find the Fourier series representation for a quaternion valued functions using $SU(2)$ invariant harmonic polynomials.

## 2. Quaternions

In this section, we recall Quanternions and classical Lie groups related to them. All theory is completely known, and we use references [11, 12]. This section is the starting point for the following ones, and for this reason we want to give a detailed representation.

**2.1. Algebra of Quaternions.** The associative division algebra of quaternions is generated in $\mathbb{R}^4$ with the basis $\{e_0, e_1, e_2, e_3\}$ putting

$$e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1.$$

The algebra is denoted by $\mathbb{H}$. In the above, we denote the identity by $e_0 = 1$. We also denote basis quaternions by

$$i = e_1, \quad j = e_2, \quad k = e_3.$$

A general quaternion is

$$x = x_0 + \underline{x}$$

where the $x_0$ is the scalar part of $x$ and the vector part is respectively

$$\underline{x} = x_1e_1 + x_2e_2 + x_3e_3.$$

The conjugate is defined by

$$\overline{x} = x_0 - \underline{x}.$$

The conjugation is an anti-involution, that is

$$\overline{xy} = \overline{y}\,\overline{x}$$

for all $x, y \in \mathbb{H}$. A quaternion and its conjugate satisfies the relation

$$\sum_{j=0}^{3} e_j x e_j = -2\overline{x}. \tag{2}$$

The real and vector part of a quaternion $x$ may be computed by

$$\mathrm{Re}(x) := x_0 = \frac{1}{2}(x + \overline{x}), \quad \mathrm{Vec}(x) := \underline{x} = \frac{1}{2}(x - \overline{x}).$$

The norm in $\mathbb{H}$ is defined by

$$|x|^2 = \overline{x}x = x\overline{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

and it is multiplicative, that is, for all $x, y \in \mathbb{H}$ we have

$$|xy| = |x||y|.$$

This means, that the unit sphere

$$S^3 = \{x \in \mathbb{H} : |x| = 1\}$$

admits the group structure. The definition of the norm gives us the explicit formulas for inverse elements

$$x^{-1} = \frac{\overline{x}}{|x|^2}$$

for non-zero quaternions. For an element of the unit sphere $x \in S^3$, the inverse is just $x^{-1} = \overline{x}$.

The inner product of quaternions $x, y \in \mathbb{H}$ may be computed by

$$\langle x, y \rangle = \mathrm{Re}(\overline{x}y) = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3.$$

If $a \in \mathbb{H}$, we have

$$\langle ax, y \rangle = \langle x, \overline{a}y \rangle. \tag{3}$$

The real part satisfies $\mathrm{Re}(xy) = \mathrm{Re}(yx)$ and

$$\mathrm{Re}(xyz) = \mathrm{Re}(zxy) = \mathrm{Re}(yzx) \tag{4}$$

for all $x, y, z \in \mathbb{H}$.

In this paper, we often use so-called complex or polar representations for quaternions, defined by

$$x = X_1 + X_2 j,$$

where $X_1 = x_0 + x_1 i$ and $X_2 = x_2 + x_3 i$ are complex numbers. Then the conjugation is

$$\overline{x} = \overline{X}_1 - X_2 j \tag{5}$$

and the multiplication

$$xy = (X_1Y_1 - X_2\overline{Y}_2) + (X_1Y_2 + X_2\overline{Y}_1)j. \tag{6}$$

This gives

$$|x|^2 = |X_1|^2 + |X_2|^2.$$

**2.2.** $SU(2) \cong S^3$. The special unitar group $SU(2)$ is defined by

$$SU(2) = \left\{ \begin{pmatrix} X_1 & X_2 \\ -\overline{X}_2 & \overline{X}_1 \end{pmatrix} \in \mathbb{C}^{2\times 2} : |X_1|^2 + |X_2|^2 = 1 \right\}.$$

Since

$$\begin{pmatrix} X_1 & X_2 \\ -\overline{X}_2 & \overline{X}_1 \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \\ -\overline{Y}_2 & \overline{Y}_1 \end{pmatrix} = \begin{pmatrix} \overline{X_1Y_1 - X_2\overline{Y}_2} & \overline{X_1Y_2 + X_2\overline{Y}_1} \\ -(X_1Y_2 + X_2\overline{Y}_1) & X_1Y_1 - X_2\overline{Y}_2 \end{pmatrix},$$

we observe comparing this with the formula (6), that

$$SU(2) \cong S^3.$$

Let us define the mapping

$$\rho_u(x) = u\underline{x}\overline{u}$$

where $u \in S^3$ and $\underline{x} \in \mathbb{R}^3 = \mathrm{Vec}(\mathbb{H})$. Since $\overline{\underline{x}} = -\underline{x}$, we have

$$\mathrm{Re}(u\underline{x}\overline{u}) = \frac{1}{2}(u\underline{x}\overline{u} + \overline{u\underline{x}\overline{u}}) = \frac{1}{2}(u\underline{x}\overline{u} - u\underline{x}\overline{u}) = 0,$$

and observe that for all $u \in S^3$ we have

$$\rho_u : \mathbb{R}^3 \to \mathbb{R}^3.$$

Since $|u\underline{x}\overline{u}| = |\underline{x}|$, we obtain $\rho_u \in O(3)$ for any $u \in S^3$. Since

$$\rho_u(\underline{x} \times \underline{y}) = \frac{1}{2}u(\underline{x}\underline{y} - \underline{y}\underline{x})\overline{u} = \frac{1}{2}(u\underline{x}\overline{u}u\underline{y}\overline{u} - u\underline{y}\overline{u}u\underline{x}\overline{u}) = \rho_u(\underline{x}) \times \rho_u(\underline{y})$$

i.e., the mapping $\rho_u$ preserves the orientation, so we have $\rho_u \in SO(3)$, or more precisely,

$$u \mapsto \rho_u; \quad S^3 \to SO(3)$$

is a group homomorphims. We know, that any rotation in $\mathbb{R}^3$ is the composite of two plane reflections. Let $\underline{u} \in S^3 \cap \mathbb{R}^3$ and decompose $\underline{x} = \lambda\underline{u} + \underline{v}$, where $\underline{v} \perp \underline{u}$. Since $\rho_{\underline{u}}(\underline{u}) = \underline{u}$ and $\rho_{\underline{u}}(\underline{v}) = -\underline{v}$, we have

$$\rho_{\underline{u}}(\underline{x}) = \lambda\underline{u} - \underline{v}$$

i.e., $\rho_{\underline{u}}$ is the reflection of the plane $\mathrm{Span}(\underline{u})^{\perp}$. Then for each rotation, we may find reflections i.e., vectors $\underline{u}_1$ and $\underline{u}_2$, such that $\rho_{\underline{u}_1\underline{u}_2}$ is the wanted rotation. We conclude that

$$u \mapsto \rho_u; \quad S^3 \to SO(3)$$

is surjection. Since the mapping is $2-1$, we observe that $S^3$ is a two fold cover group of $SO(3)$, i.e., $S^3 \cong \mathrm{Spin}(3)$.

**2.3. Rotations in $\mathbb{R}^4$.** If $u, v \in S^3$, then our first observation is that the mappings

$$u_L : x \mapsto ux, \quad v_R : x \mapsto xv$$

are orthogonal mappings.

**Proposition 2.1** [11, 12]. *The preceding mappings belongs to $SO(4)$.*

Proof. We need to prove that their determinant is one. Let now $u \in S^3$ and $\underline{x} \in \mathbb{R}^3$ such that $\underline{u} \perp \underline{x}$. Then

$$u\underline{x} = u_0\underline{x} + \underline{u}\underline{x} = u_0\underline{x} + \underline{u} \times \underline{x} \in \mathbb{R}^3.$$

Since $|u\underline{x}| = |\underline{x}|$, by the preceding section, there exists $v \in S^3$ such that

$$u\underline{x} = \rho_v(\underline{x}) = v\underline{x}\overline{v}.$$

Then $u = v\underline{x}\overline{v}\underline{x}^{-1}$ and

$$\begin{aligned}\det(u_L) &= \det(v_L)\det(\underline{x}_L)\det(\overline{v}_L)\det(\underline{x}_L^{-1}) \\ &= \det(v_L)\det(\underline{x}_L)\det(v_L)^{-1}\det(\underline{x}_L)^{-1} = 1. \quad \square\end{aligned}$$

Let us now define the mapping $\rho_{u,v}(x) = ux\overline{v}$, where $u, v \in S^3$. Obviously $\rho_{u,v} \in SO(4)$.

**Lemma 2.2** [11, 12]. *The mapping*

$$\rho : S^3 \times S^3 \to SO(4)$$

*is surjection.*

Proof. For the sake of completeness, we recall the proof. Let $Q \in SO(4)$ be an arbitrary rotation and let $u = Q(1)$. Then $|u| = 1$ and $x \mapsto \overline{u}Q(x)$ belongs to $SO(4)$ leaving the $x_0$-axis invariant, indeed $\overline{u}Q(1) = \overline{u}u = 1$. Hence $\overline{u}Q(\underline{x})$ is a rotation in $\mathbb{R}^3$ and there exists $v \in S^3$ satisfying

$$\overline{u}Q(\underline{x}) = \rho_v(\underline{x}) = v\underline{x}\overline{v}.$$

We compute

$$\overline{u}Q(x) = \overline{u}Q(x_0) + \overline{u}Q(\underline{x}) = x_0 + v\underline{x}\overline{v} = vx\overline{v}.$$

Hence $Q(x) = uvx\overline{v} = \rho_{uv,v}(x)$, completing the proof. $\square$

Since the kernel of the mapping $\rho$ is $\{(1,1),(-1,-1)\}$, we have that

$$S^3 \times S^3 \cong \mathrm{Spin}(4).$$

## 3. Cauchy–Riemann operators

In this section, we represent Cauchy–Riemann operators and recall their invariance properties.

**3.1. Definition and basic properties.** The Cauchy–Riemann operator and its conjugate is defined by

$$\partial_x = \partial_{x_0} + \partial_{\underline{x}}$$

and

$$\partial_{\overline{x}} = \partial_{x_0} - \partial_{\underline{x}},$$

where $\partial_{\underline{x}} = e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3}$ is called the Dirac operator. They factorize the Laplacian by

$$\partial_x\partial_{\overline{x}} = \partial_{\overline{x}}\partial_x = \Delta_x = \partial^2_{x_0} + \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3}.$$

It is well-known (see, e.g., [13]) that the Laplacian is $SO(4)$ invariant, that is, if we define an action by $T[u,v]f(x) = f(\overline{u}xv)$, where $u, v \in SU(2)$, then

$$[\Delta_x, T[u,v]] = 0.$$

As a special case of this, we have

$$[\Delta_x, R[u]] = [\Delta_x, S[u]] = 0,$$

that is, the Laplacian is also left and right $SU(2)$ invariant under actions (1).

**3.2. On the invariance of the Cauchy–Riemann operator.** Let us now study invariance of the Cauchy–Riemann operator. For this, we need the following lemma.

**Lemma 3.2.** *If $u, v \in S^3$, then*

$$\partial_{uxv} = u\partial_x v.$$

Proof. A proof may be found in [8, p. 222] in the Clifford analysis level. But for the sake of completeness and since the situation is a little bit different, we want to give a detailed proof. Let us first consider $Q \in SO(4)$ and $y = Q(x)$. Hence

$$Q(e_k) = \sum_{j=0}^{3} Q_{jk}e_j.$$

Consider the Cauchy–Riemann operators

$$\partial_x = \sum_{k=0}^{3} e_k\partial_{x_k} \quad \text{and} \quad \partial_y = \sum_{j=0}^{3} e_j\partial_{y_j}$$

Since $x = Q^T(y)$, we find

$$x_k = \sum_{j=0}^{3} Q_{jk}y_j$$

and $\frac{\partial x_k}{\partial y_j} = Q_{jk}$. Then we compute using the chain rule

$$\frac{\partial f}{\partial y_j} = \sum_{k=0}^{3} \frac{\partial f}{\partial x_k}\frac{\partial x_k}{\partial y_j} = \sum_{k=0}^{3} Q_{jk}\frac{\partial f}{\partial x_k}.$$

Then we obtain

$$\partial_y = \sum_{j=0}^{3} e_j\partial_{y_j} = \sum_{j,k=0}^{3} e_jQ_{jk}\partial_{x_k} = \sum_{k=0}^{3}\left(\sum_{j=0}^{3} Q_{jk}e_j\right)\partial_{x_k} = \sum_{k=0}^{3} Q(e_k)\partial_{x_k}.$$

Now, choosing $Q(x) = uxv$, we obtain the result. □

Using the preceding lemma, we deduce that the Cauchy–Riemann operator is not $R$ or $S$ invariant.

**Proposition 3.2.** *For the Cauchy–Riemann operator*
(a) $R[u]\partial_x = \overline{u}\partial_x R[u]$,
(b) $S[u]\partial_x = \partial_x u S[u]$.

REMARK 3.3. The preceding proposition may be found in the book by Coifman and Weiss [10. p. 118], but they prove it only for differentiable functions $f : \mathbb{H} \to \mathbb{R}$ using a different technique.

Although the Cauchy–Riemann operator is not $SU(2)$ invariant under actions $R$ and $S$, we can make it invariant by defining actions

$$L[u]f(x) = uf(\overline{u}xu)$$

and

$$H[u]f(x) = uf(\overline{u}xu)\overline{u}.$$

Indeed, using Lemma 3.1, we compute

$$L[u]\partial_x f(x) = u\partial_{\overline{u}xu} f(\overline{u}xu) = u\overline{u}\partial_x uf(\overline{u}xu) = \partial_x L[u]f(x)$$

and

$$H[u]\partial_x f(x) = u\partial_{\overline{u}xu} f(\overline{u}xu)\overline{u} = u\overline{u}\partial_x uf(\overline{u}xu)\overline{u} = \partial_x L[u]f(x),$$

since $u\overline{u} = 1$.

**Proposition 3.4.** *For each $u \in SU(2)$, we have*

$$[\partial_x, L[u]] = [\partial_{\overline{x}}, L[u]] = [\partial_x, H[u]] = [\partial_{\overline{x}}, H[u]] = 0.$$

The $L$ and $H$ operators are defined and studied for the first time in context of the Clifford Dirac operator by Frank Sommen in his paper [14–16], but of course the formulas work also for the Cauchy–Riemann operator in quaternionic analysis.

## 4. Coifman–Weiss operators

The Lie algebra $su(2) \cong \mathbb{H}$ is generated by $\{e_0, e_1, e_2, e_3\}$ and the exponential mapping

$$e^{(\cdot)} : su(2) \to SU(2)$$

is defined via matrix exponentiation (see [10, 17]). We define left $SU(2)$ invariant derivatives by

$$\partial\!\!\!/^R_{x_j} f(x) = \lim_{t\to 0} \frac{f(xe^{te_j}) - f(x)}{t}.$$

Extending $f$ outside of $S^3$, we may write $e^{te_j} = 1 + te_j + o(t^2)$, that is,

$$\partial\!\!\!/^R_{x_j} f(x) = \lim_{t\to 0} \frac{f(x + txe_j) - f(x)}{t} = \langle xe_j, \partial_x\rangle f(x).$$

Obviously $\partial\!\!\!/^R_{x_j} R_u = R_u \partial\!\!\!/^R_{x_j}$.

Similarly, we define left $SU(2)$ invariant derivatives by

$$\partial\!\!\!/^S_{x_j} f(x) = \lim_{t\to 0} \frac{f(e^{te_j}x) - f(x)}{t}.$$

Extending $f$ outside of $S^3$, we may write $e^{te_j} = 1 + te_j + o(t^2)$, that is,

$$\partial\!\!\!/^S_{x_j} f(x) = \lim_{t\to 0} \frac{f(x+te_jx)-f(x)}{t} = \langle e_j x, \partial_x\rangle f(x).$$

Obviously $\partial\!\!\!/^S_{x_j} S_u = S_u \partial\!\!\!/^S_{x_j}$.

We define the left and right $SU(2)$ invariant *Coifman–Weiss operators* and their conjugations by

$$\partial\!\!\!/^R_x = \sum_{j=0}^{3} e_j \partial\!\!\!/^R_{x_j} \quad\text{and}\quad \partial\!\!\!/^R_{\overline{x}} = \sum_{j=0}^{3} \overline{e}_j \partial\!\!\!/^R_{x_j}.$$

and

$$\partial\!\!\!/^S_x = \sum_{j=0}^{3} e_j \partial\!\!\!/^S_{x_j} \quad\text{and}\quad \partial\!\!\!/^S_{\overline{x}} = \sum_{j=0}^{3} \overline{e}_j \partial\!\!\!/^S_{x_j}.$$

Let us first prove the following representation result. The first of them can be found in [10].

**Proposition 4.1.** *Left invariant Coifman–Weiss operators satisfy*

$$\partial\!\!\!/^R_x f(x) = \overline{x}\partial_x f(x)$$

*and*

$$\partial\!\!\!/^R_{\overline{x}} f(x) = \dot{\partial}_{\overline{x}} x \dot{f}(x).$$

Proof. In [10], the proof is based on matrix operators. We give here a direct proof with quaternions. Let $f$ be a differentiable function. Using (3), we have

$$\partial\!\!\!/^R_{x_j} = \langle xe_j, \partial_x\rangle = \langle e_j, \overline{x}\partial_x\rangle.$$

Hence

$$\partial\!\!\!/^R_x f(x) = \sum_{j=0}^{3} e_j \partial\!\!\!/^R_{x_j} f(x) = \sum_{j=0}^{3} e_j \langle e_j, \overline{x}\partial_x\rangle f(x) = \overline{x}\partial_x f(x).$$

Using (4), we obtain

$$\partial\!\!\!/^R_{x_j} = \langle xe_j, \partial_x\rangle = \mathrm{Re}(\overline{xe_j}\partial_x) = \mathrm{Re}(\overline{e}_j\overline{x}\partial_x) = \mathrm{Re}(\overline{\partial_{\overline{x}} x}\overline{e}_j) = \langle \overline{e}_j, \partial_{\overline{x}} x\rangle$$

and

$$\partial\!\!\!/^R_{\overline{x}} f(x) = \sum_{j=0}^{3} \overline{e}_j \partial\!\!\!/^R_{x_j} f(x) = \sum_{j=0}^{3} \overline{e}_j \langle \overline{e}_j, \partial_{\overline{x}} x\rangle f(x) = \dot{\partial}_{\overline{x}} x \dot{f}(x). \quad \square$$

**Proposition 4.2.** *Right invariant Coifman–Weiss operators satisfy*

$$\partial\!\!\!/^S_x f(x) = \dot{\partial}_x \overline{x} \dot{f}(x)$$

*and*

$$\partial\!\!\!/^S_{\overline{x}} f(x) = x\partial_{\overline{x}} f(x).$$

Proof. Let $f$ be a differentiable function. Using (4), we have

$$\partial\!\!\!/^S_{x_j} = \langle e_j x, \partial_x\rangle = \mathrm{Re}(\overline{e_j x}\partial_x) = \mathrm{Re}(\overline{x}\,\overline{e}_j\partial_x) = \mathrm{Re}(\overline{e}_j\partial_x\overline{x}) = \langle e_j, \partial_x\overline{x}\rangle.$$

Hence

$$\partial\!\!\!/_x^S f(x) = \sum_{j=0}^{3} e_j \partial\!\!\!/_{x_j}^S f(x) = \sum_{j=0}^{3} e_j \langle e_j, \dot{\partial}_x \overline{x} \rangle \dot{f}(x) = \dot{\partial}_x \overline{x} \dot{f}.$$

In the above, we have

$$\partial\!\!\!/_{x_j}^S = \mathrm{Re}(\overline{e}_j \partial_x \overline{x}) = \langle \overline{\partial_x \overline{x}}, \overline{e}_j \rangle = \langle \overline{e}_j, x\partial_{\overline{x}} \rangle.$$

Hence

$$\partial\!\!\!/_{\overline{x}}^S f(x) = \sum_{j=0}^{3} \overline{e}_j \partial\!\!\!/_{x_j} f(x) = \sum_{j=0}^{3} \overline{e}_j \langle \overline{e}_j, x\partial_{\overline{x}} \rangle f(x) = x\partial_{\overline{x}} f. \quad \square$$

Using the preceding formulas, we can give direct proof for invariance.

**Proposition 4.3.** *If $u \in SU(2)$, then*

$$\big[\partial\!\!\!/_x^R, R[u]\big] = \big[\partial\!\!\!/_{\overline{x}}^R, R[u]\big] = \big[\partial\!\!\!/_x^S, S[u]\big] = \big[\partial\!\!\!/_{\overline{x}}^S, S[u]\big] = 0.$$

Proof. Let $u \in S^3$. Using Lemma 3.1, we compute

$$S[u]\big(\partial\!\!\!/_x^S f(x)\big) = \dot{\partial}_{xu} \overline{xu} \dot{f}(xu) = \dot{\partial}_x u\overline{u}\, \overline{x} \dot{f}(xu) = \dot{\partial}_x \overline{x} S[u] \dot{f}(x) = \partial\!\!\!/_x^S (S[u]f(x)).$$

Other formulas can be proved by a similar technique. $\square$

Remark 4.4. Coifman–Weiss operators obey the quaternionic conjugation law if we allow that they act also from the right, that is, $\overline{\partial\!\!\!/_x^R f} = \overline{f} \partial\!\!\!/_{\overline{x}}^R$ and $\overline{\partial\!\!\!/_x^S f} = \overline{f} \partial\!\!\!/_{\overline{x}}^S$. Thus, Cauchy–Riemann and Coifman–Weiss operators behave algebraically similarly.

Let us next prove the following decomposition formulas for the Laplacian. One of these formulas can be found in [10]. Next, we formulate both and give a direct proof using quaternions.

**Proposition 4.5.** $\partial\!\!\!/_x^R\big(\partial\!\!\!/_{\overline{x}}^R + 2\big) = \big(\partial\!\!\!/_{\overline{x}}^R + 2\big)\partial\!\!\!/_x^R = |x|^2 \Delta_x$, $\big(\partial\!\!\!/_x^S + 2\big)\partial\!\!\!/_{\overline{x}}^S = \partial\!\!\!/_{\overline{x}}^S\big(\partial\!\!\!/_x^S + 2\big) = |x|^2 \Delta_x$.

Proof. We compute

$$\partial\!\!\!/_x^S\big(\partial\!\!\!/_{\overline{x}}^S f\big) = |x|^2 \Delta_x + \sum_{j=0}^{3} e_j \overline{x} \frac{\partial x}{\partial x_j} \partial_{\overline{x}} f.$$

Using (2), we compute

$$\partial\!\!\!/_x^S\big(\partial\!\!\!/_{\overline{x}}^S f\big) = |x|^2 \Delta_x f + \sum_{j=0}^{3} e_j \overline{x} e_j \partial_{\overline{x}} f = |x|^2 \Delta_x f - 2x\partial_{\overline{x}} f = |x|^2 \Delta_x f - 2\partial\!\!\!/_{\overline{x}}^S f.$$

Similarly, we compute

$$\partial\!\!\!/_{\overline{x}}^S\big(\partial\!\!\!/_x^S f\big) = |x|^2 \Delta_x f - 2\partial\!\!\!/_{\overline{x}}^S f, \quad \partial\!\!\!/_x^R\big(\partial\!\!\!/_{\overline{x}}^R f\big) = |x|^2 \Delta_x f - 2\partial\!\!\!/_x^R f,$$
$$\partial\!\!\!/_{\overline{x}}^R\big(\partial\!\!\!/_x^R f\big) = |x|^2 \Delta_x f - 2\partial\!\!\!/_x^R f.$$

The proof follows from these. $\square$

These formulas indicate the commutativity of Coifman–Weiss operators and their adjungated versions.

**Corollary 4.6.** $\partial\!\!\!/_x^R \partial\!\!\!/_{\overline{x}}^R = \partial\!\!\!/_{\overline{x}}^R \partial\!\!\!/_x^R$, $\partial\!\!\!/_x^S \partial\!\!\!/_{\overline{x}}^S = \partial\!\!\!/_{\overline{x}}^S \partial\!\!\!/_x^S$.

## 5. Fourier series on $SU(2)$

In this section, we find the Fourier series expansion for quaternionic valued functions $L^2(SU(2))$ and extend them to the whole space. This job is already mostly done, e.g., in the book by Ruzhansky and Turunen [17]. Using the Fourier series, it is easy to obtain a series representation for the quaternionic valued function on spherical domains.

**5.1. Homogeneous spherical harmonic polynomials.** Let us consider integrable functions $f, g : S^3 \to \mathbb{C}$. We define the innerproduct

$$(f,g)_{L^2(S^3)} := \frac{1}{\omega_3} \int_{S^3} f(x)\overline{g(x)}\, dS(x)$$

and in the usual way this leads us to the space $L^2(SU(2))$, square integrable complex valued functions on $S^3$. We define the left and right $SU(2)$ action on $L^2(SU(2))$ by (1). The Lie group $SU(2) \cong S^3$ admits a natural biaxial nature, i.e., every point is of the form $y = Y_1 + Y_2 j$, where $Y_1 = y_0 + y_1 i$ and $Y_2 = y_2 + y_3 i$.

We look for irreducible representations in the usual manner. We take a space of $2\ell$-homogeneous complex valued polynomials $V_\ell$ with an orthonormal basis

$$p_{\ell n}(y) = p_{\ell n}(Y_1, Y_2) = \frac{Y_1^{\ell-n} Y_2^{\ell+n}}{\sqrt{(\ell-n)!(\ell+n)!}},$$

for $n \in \{-\ell, -\ell+1, \ldots, \ell-1, \ell\}$ and $\ell \in \frac{1}{2}\mathbb{N}_0$. The dimension of the space $V_\ell$ is $2\ell+1$ and $\ell - n, \ell + n \in \mathbb{Z}$. See all details in [10, 17].

The matrix elements $\{{}^R t^\ell_{mn}\}$ and $\{{}^S t^\ell_{mn}\}$, with respect to the actions (1), are defined by

$$R_x p_{\ell n}(Y_1, \overline{Y}_2) = \sum_k {}^R t^\ell_{kn}(x) p_{\ell k}(Y_1, \overline{Y}_2)$$

and

$$S_x p_{\ell n}(Y_1, Y_2) = \sum_k {}^S t^\ell_{kn}(x) p_{\ell k}(Y_1, Y_2),$$

for $x \in SU(2)$. For the left action, we need to use conjugates in the second variable to complete the calculations successfully.

**Proposition 5.1.** *Matrix elements admits to representations*

$${}^R t^\ell_{mn}(x) = \frac{1}{2\pi} \sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}} \int_{-\pi}^{\pi} (\overline{z}_1 e^{i\theta} + z_2 e^{-i\theta})^{\ell-n} (z_1 e^{-i\theta} - \overline{z}_2 e^{i\theta})^{\ell+n} e^{i2m\theta}\, d\theta$$

*and*

$${}^S t^\ell_{mn}(x) = \frac{1}{2\pi} \sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}} \int_{-\pi}^{\pi} (z_1 e^{i\theta} - \overline{z}_2 e^{-i\theta})^{\ell-n} (\overline{z}_1 e^{-i\theta} + z_2 e^{i\theta})^{\ell+n} e^{i2m\theta}\, d\theta$$

*for* $x = z_1 + z_2 j$.

Proof. The right invariant version of the formulas can be found in [10] in a slightly different form.

Left invariant case. Consider the homogeneous polynomials

$$p_{\ell n}(y) = p_{\ell n}(Y_1, \overline{Y}_2) = \frac{Y_1^{\ell-n}\overline{Y}_2^{\ell+n}}{\sqrt{(\ell-n)!(\ell+n)!}}.$$

Using (5) and (6), we have

$$R_x p_{\ell n}(y) = p_{\ell n}(\overline{x}y) = p_{\ell n}(\overline{z}_1 Y_1 + z_2\overline{Y}_2, \overline{\overline{z}_1 Y_2 - z_2\overline{Y}_1}) = p_{\ell n}(\overline{z}_1 Y_1 + z_2\overline{Y}_2, z_1\overline{Y}_2 - \overline{z}_2 Y_1).$$

Hence

$$\frac{(\overline{z}_1 Y_1 + z_2\overline{Y}_2)^{\ell-n}(z_1\overline{Y}_2 - \overline{z}_2 Y_1)^{\ell+n}}{\sqrt{(\ell-n)!(\ell+n)!}} = \sum_k {}^R t^{\ell}_{kn}(x)\frac{Y_1^{\ell-k}\overline{Y}_2^{\ell+k}}{\sqrt{(\ell-k)!(\ell+k)!}}.$$

To find an explicit formula for a matrix coefficient, we first substitute $y = \frac{1}{\sqrt{2}}(e^{i\theta} + e^{i\theta}j)$, i.e., we obtain

$$\frac{(\overline{z}_1 e^{i\theta} + z_2 e^{-i\theta})^{\ell-n}(z_1 e^{-i\theta} - \overline{z}_2 e^{i\theta})^{\ell+n}}{\sqrt{(\ell-n)!(\ell+n)!}} = \sum_k {}^R t^{\ell}_{kn}(x)\frac{e^{-i2k\theta}}{\sqrt{(\ell-k)!(\ell+k)!}}.$$

We multiply both sides by $e^{i2m\theta}$ and use the fact

$$\int_{-\pi}^{\pi} e^{i2(m-k)\theta}\, d\theta = \begin{cases} 2\pi, & \text{if } m = k, \\ 0, & \text{other integers.} \end{cases}$$

Then we obtain

$${}^R t^{\ell}_{mn}(x) = \frac{1}{2\pi}\sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}}\int_{-\pi}^{\pi}(\overline{z}_1 e^{i\theta} + z_2 e^{-i\theta})^{\ell-n}(z_1 e^{-i\theta} - \overline{z}_2 e^{i\theta})^{\ell+n} e^{i2m\theta}\, d\theta.$$

Right invariant case. Using (6), we have

$$S_x p_{\ell n}(y) = p_{\ell n}(yx) = p_{\ell n}(Y_1 z_1 - Y_2\overline{z}_2, Y_1 z_2 + Y_2\overline{z}_1).$$

Hence

$$\frac{(Y_1 z_1 - Y_2\overline{z}_2)^{\ell-n}(Y_1 z_2 + Y_2\overline{z}_1)^{\ell+n}}{\sqrt{(\ell-n)!(\ell+n)!}} = \sum_k {}^S t^{\ell}_{kn}(x)\frac{Y_1^{\ell-k}Y_2^{\ell+k}}{\sqrt{(\ell-k)!(\ell+k)!}}.$$

We substitute $y = \frac{1}{\sqrt{2}}(e^{i\theta} + e^{-i\theta}j)$ and we have

$$\frac{(z_1 e^{i\theta} - \overline{z}_2 e^{-i\theta})^{\ell-n}(z_2 e^{i\theta} + \overline{z}_1 e^{-i\theta})^{\ell+n}}{\sqrt{(\ell-n)!(\ell+n)!}} = \sum_k {}^S t^{\ell}_{kn}(x)\frac{e^{-i2k\theta}}{\sqrt{(\ell-k)!(\ell+k)!}}.$$

Multiplying both sides by $e^{i2m\theta}$ and integrating as in the above, we obtain

$${}^S t^{\ell}_{mn}(x) = \frac{1}{2\pi}\sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}}\int_{-\pi}^{\pi}(z_1 e^{i\theta} - \overline{z}_2 e^{-i\theta})^{\ell-n}(z_2 e^{i\theta} + \overline{z}_1 e^{-i\theta})^{\ell+n} e^{i2m\theta}\, d\theta. \qquad \square$$

REMARK 5.2. For example, in the left invariant case, the integral can be written by standard change of variables

$$\int_{-\pi}^{\pi} (\overline{z}_1 e^{i\theta} + z_2 e^{-i\theta})^{\ell-n}(z_1 e^{-i\theta} - \overline{z}_2 e^{i\theta})^{\ell+n} e^{i2m\theta}\, d\theta$$

$$= \int_{-\pi}^{\pi} (\overline{z}_1 e^{i2\theta} + z_2)^{\ell-n}(z_1 - \overline{z}_2 e^{i2\theta})^{\ell+n} e^{i2(m-\ell)\theta}\, d\theta$$

$$= 2\int_{0}^{2\pi} (\overline{z}_1 e^{i\theta} + z_2)^{\ell-n}(z_1 - \overline{z}_2 e^{i\theta})^{\ell+n} e^{i(m-\ell)\theta}\, d\theta.$$

Thus, we may represent the matrix elements in a coordinate independent way by

$${}^R t^{\ell}_{mn}(x) = -\frac{i}{\pi}\sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}} \int_{|w|=1} (\overline{z}_1 w + z_2)^{\ell-n}(z_1 - \overline{z}_2 w)^{\ell+n} \frac{dw}{w^{\ell-m+1}}$$

and

$${}^S t^{\ell}_{mn}(x) = -\frac{i}{\pi}\sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}} \int_{|w|=1} (z_1 w - \overline{z}_2)^{\ell-n}(\overline{z}_1 + z_2 w)^{\ell+n} \frac{dw}{w^{\ell-m+1}}.$$

The preceding formulas allow us to extend the matrix coefficients ${}^R t^{\ell}_{mn}(x)$ and ${}^S t^{\ell}_{mn}(x)$ to the whole space, i.e., we may assume $x \in \mathbb{H}$. Let us recall that if $f : S^3 \to \mathbb{C}$ is a restriction of a harmonic function $F : \mathbb{H} \to \mathbb{C}$, it is called a *spherical harmonics.*

**Proposition 5.3.** *Matrix coefficients are* $2\ell$*-homogeneous spherical harmonics.*

PROOF. Let $x = z_1 + z_2 j$. In a left invariant matrix coeffient, the kernel is given in the preceding remark by

$$k_\ell(x) = k_\ell(z_1, z_2) = (\overline{z}_1 w + z_2)^{\ell-n}(z_1 - \overline{z}_2 w)^{\ell+n}$$

for $|w| = 1$. If $x = z_1 + z_2 j$, we put

$$\partial_{z_1} = \partial_{x_0} + i\partial_{x_1} \quad \text{and} \quad \partial_{z_2} = \partial_{x_2} + i\partial_{x_3}$$

with their conjugations

$$\partial_{\overline{z}_1} = \partial_{x_0} - i\partial_{x_1} \quad \text{and} \quad \partial_{\overline{z}_2} = \partial_{x_2} - i\partial_{x_3}.$$

Hence $\partial_{z_p} z_p = \partial_{\overline{z}_p}\overline{z}_p = 0$ and $\partial_{\overline{z}_p} z_p = \partial_{z_p}\overline{z}_p = 2$ for $p = 1, 2$. It is easy to see, that

$$\Delta_x = \partial_{\overline{z}_1}\partial_{z_1} + \partial_{\overline{z}_2}\partial_{z_2}.$$

It is straightforward to compute

$$\partial_{\overline{z}_1}\partial_{z_1} k_\ell = 4(\ell-n)(\ell+n) w k_{\ell-1}, \quad \partial_{\overline{z}_2}\partial_{z_2} k_\ell = -4(\ell-n)(\ell+n) w k_{\ell-1},$$

that is

$$\Delta_x k_\ell = 0.$$

The right invariant case is similar. $\square$

**5.2. Fourier series representation.** Let $t^\ell_{nm}$ be either the right or left invariant matrix coefficient. Hence, any $f \in L^2(SU(2))$ admits the Fourier series representation

$$f(x) = \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1) \sum_{m,n} \big(f, t^\ell_{nm}\big)_{L^2(S^3)} t^\ell_{nm}(x),$$

where the summation is taken over $m, n$ satisfying $-\ell \le m, n \le \ell$ and $\ell - m, \ell - n \in \mathbb{Z}$ (see details in [10, 17]). If we define $x = ru$, $r = |x|$ and $u = \frac{x}{r} \in S^3$, we may represent any $f : \Omega \to \mathbb{C}$, where $\Omega \in \mathbb{H}$ is a spherical neighborhood and $f|_{S^3} \in L^2(SU(2))$ by the series

$$f(x) = \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1) r^{2\ell} \sum_{m,n} \big(f, t^\ell_{mn}\big)_{L^2(S^3)} t^\ell_{mn}(u).$$

This allows us to find an explicit series expansion also for quaternion valued functions.

**Theorem 5.4.** *Let $f = f_1 + f_2 j$ be a quaternion valued function and $f_1$ and $f_2$ be complex valued functions defined on any spherical neighborhood $\Omega \subset \mathbb{H}$ and $f_1|_{S^3}, f_2|_{S^3} \in L^2(SU(2))$. Then*

$$f(x) = \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1) r^{2\ell} \sum_{m,n} \Big(\big(f_1, t^\ell_{mn}\big)_{L^2(S^3)} + (-1)^{m-n} \big(f_2, t^\ell_{-m,-n}\big)_{L^2(S^3)} j\Big) t^\ell_{mn}(u),$$

*where $x = ru$, $r = |x|$ and $u = \frac{x}{r} \in S^3$.*

Proof. We consider the left invariant case. Our first observation is

$$\overline{{}^R t^\ell_{mn}(z_1, z_2)} = \frac{1}{2\pi} \sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}} \int_{-\pi}^{\pi} (\overline{z}_1 e^{i\theta} - z_2 e^{-i\theta})^{\ell+n} \times (z_1 e^{-i\theta} + \overline{z}_2 e^{i\theta})^{\ell-n} e^{-i2m\theta}\, d\theta = {}^R t^\ell_{-m,-n}(z_1, -z_2).$$

Recall the coordinate invariant form of the matrix coefficient given in Remark 5.2

$${}^R t^\ell_{mn}(z_1, z_2) = -\frac{i}{\pi} \sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}} \int_{|w|=1} (\overline{z}_1 w + z_2)^{\ell-n} (z_1 - \overline{z}_2 w)^{\ell+n} \frac{dw}{w^{\ell-m+1}}.$$

We assume $z_1 z_2 \neq 0$ and make the change of variables given by Coifman and Weiss [10, p. 108]

$$\overline{z}_1 \overline{z}_2 w = t - |z_2|^2 + |z_1|^2$$

and we have

$$\overline{z}_1 w + z_2 = \frac{t - |z_1|^2}{\overline{z}_2} \quad \text{and} \quad z_1 - \overline{z}_2 w = \frac{|z_2|^2 - t}{\overline{z}_1}.$$

We define $\Gamma = \{t = \overline{z}_1\overline{z}_2 w + |z_2|^2 - |z_1|^2 : |w| = 1\}$. Hence the matrix coefficient takes the form

$$
\begin{aligned}
{}^R t^{\ell}_{mn}(z_1, z_2) = -\frac{i}{\pi}\sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}} \\
\times \int_{\Gamma} \left(\frac{t-|z_1|^2}{\overline{z}_2}\right)^{\ell-n} \left(\frac{|z_2|^2-t}{\overline{z}_1}\right)^{\ell+n} \frac{\overline{z}_1^{\ell-m+1}\overline{z}_2^{\ell-m+1}}{(t-|z_2|^2+|z_1|^2)^{\ell-m+1}} \frac{dt}{\overline{z}_1\overline{z}_2} \\
= -\frac{i}{\pi}\sqrt{\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}} \overline{z}_1^{-n-m}\overline{z}_2^{n-m} \\
\times \int_{\Gamma} (t-|z_1|^2)^{\ell-n}(|z_2|^2-t)^{\ell+n} \frac{dt}{(t-|z_2|^2+|z_1|^2)^{\ell-m+1}}.
\end{aligned}
$$

The path $\Gamma$ is the same if we choose $\pm z_2$. Hence the integral above is invariant under substitution by $-z_2$ and we obtain ${}^R t^{\ell}_{mn}(z_1, -z_2) = (-1)^{n-m} {}^R t^{\ell}_{mn}(z_1, z_2)$. Especially

$$\overline{{}^R t^{\ell}_{mn}(z_1, z_2)} = (-1)^{m-n} {}^R t^{\ell}_{-m,-n}(z_1, z_2).$$

The proof is similar for the right invariant matrix coefficients. Hence

$$t^{\ell}_{mn}(u) j = j\overline{t^{\ell}_{mn}(u)} = (-1)^{m-n} j t^{\ell}_{-m,-n}(u),$$

and we obtain

$$
\begin{aligned}
f(x) = f_1(x) + f_2(x) j = \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1) r^{2\ell} \sum_{m,n} \big(f_1, t^{\ell}_{mn}\big)_{L^2(S^3)} t^{\ell}_{mn}(u) \\
+ \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1) r^{2\ell} \sum_{m,n} \big(f_2, t^{\ell}_{mn}\big)_{L^2(S^3)} t^{\ell}_{mn}(u) j \\
= \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1) r^{2\ell} \sum_{m,n} \big(f_1, t^{\ell}_{mn}\big)_{L^2(S^3)} t^{\ell}_{mn}(u) \\
+ \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1) r^{2\ell} \sum_{m,n} \big(f_2, t^{\ell}_{-n,-m}\big)_{L^2(S^3)} j (-1)^{m-n} t_{mn}(u) \\
= \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1) r^{2\ell} \sum_{m,n} \Big(\big(f_1, t^{\ell}_{mn}\big)_{L^2(S^3)} + (-1)^{m-n} (f_2, t^{\ell}_{-m,-n})_{L^2(S^3)} j\Big) t^{\ell}_{mn}(u). \quad \square
\end{aligned}
$$

**Conclusions.** In this paper, we recall the $SU(2)$ invariant version of quaternionic analysis and write all proofs using modern tools. The fundamental objective is to find an $SU(2)$ invariant Dirac type operators, which are called Coifman–Weiss operators, after the founders of the theory. In addition, we discuss the $SU(2)$ invariant Fourier series and recall nice and explicit formulas for matrix coefficients.

This paper allows us to continue studies with Coifman–Weiss operators in different directions.

There are still open questions in the basic theory. The connection of the representation of the Coifman and Weiss matrix coefficients that we get in this paper, and the representation of Ruzhansky and Turunen [17], should be studied. Also, the connection to Jacobi polynomials should be recorded explicitly for all cases. We leave these to the interested reader as an exercise.

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