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Properties of BLUEs in full versus small linear models

Stephen J. Haslett[a,b,c], Augustyn Markiewicz[d], and Simo Puntanen[e]

[a]Research School of Finance, Actuarial Studies and Statistics, The Australian National University, Canberra, Australia; [b]Centre for Public Health Research, Massey University, Wellington, New Zealand; [c]School of Fundamental Sciences, Massey University, Palmerston North, New Zealand; [d]Department of Mathematical and Statistical Methods, Poznań University of Life Sciences, Poznań, Poland; [e]Faculty of Information Technology and Communication Sciences, Tampere University, Tampere, Finland

ABSTRACT

In this article we consider the partitioned linear model $\mathscr{M}_{12} = \{\mathbf{y},\ \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2,\ \mathbf{V}\}$ and the corresponding small model $\mathscr{M}_1 = \{\mathbf{y},\ \mathbf{X}_1\boldsymbol{\beta}_1,\ \mathbf{V}\}$. We focus on comparing the best linear unbiased estimators, BLUEs, of $\mathbf{X}_1\boldsymbol{\beta}_1$ under $\mathscr{M}_{12}$ and $\mathscr{M}_1$. In other words, we are interested in the effect of adding regressors on the BLUEs. Particular attention is paid on the consistency of the model, that is, whether the realized value of the response vector $\mathbf{y}$ belongs to the column space of $(\mathbf{X}_1 : \mathbf{V})$ or $(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$.

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1. Introduction

In this article we consider the partitioned linear model $\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}$ and so-called small model (submodel) $\mathbf{y} = \mathbf{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}$, or shortly

$$\mathscr{M}_{12} = \{\mathbf{y},\ \mathbf{X}\boldsymbol{\beta},\ \mathbf{V}\} = \{\mathbf{y},\ \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2,\ \mathbf{V}\},\ \mathscr{M}_1 = \{\mathbf{y},\ \mathbf{X}_1\boldsymbol{\beta}_1,\ \mathbf{V}\}.$$

Here $\mathbf{y}$ is an n-dimensional observable response variable, and $\boldsymbol{\varepsilon}$ is an unobservable random error with a known covariance matrix $\operatorname{cov}(\boldsymbol{\varepsilon}) = \mathbf{V} = \operatorname{cov}(\mathbf{y})$ and expectation $\operatorname{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$. The matrix $\mathbf{X}$ is a known $n \times p$ matrix, that is, $\mathbf{X} \in \mathbb{R}^{n\times p}$, partitioned columnwise as $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$, $\mathbf{X}_i \in \mathbb{R}^{n\times p_i}$, $i = 1, 2$. Vector $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')' \in \mathbb{R}^p$ is a vector of fixed (but unknown) parameters; symbol $'$ stands for the transpose.

As for notation, $\operatorname{r}(\mathbf{A})$, $\mathbf{A}^-$, $\mathbf{A}^+$, $\mathscr{C}(\mathbf{A})$, $\mathscr{N}(\mathbf{A})$, and $\mathscr{C}(\mathbf{A})^\perp$, denote, respectively, the rank, a generalized inverse, the (unique) Moore–Penrose inverse, the column space, the null space, and the orthogonal complement of the column space of the matrix $\mathbf{A}$. By $\mathbf{A}^\perp$ we denote any matrix satisfying $\mathscr{C}(\mathbf{A}^\perp) = \mathscr{C}(\mathbf{A})^\perp$. Furthermore, we will write $\mathbf{P}_\mathbf{A} = \mathbf{P}_{\mathscr{C}(\mathbf{A})} = \mathbf{A}\mathbf{A}^+ = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$ to denote the orthogonal projector onto $\mathscr{C}(\mathbf{A})$. The orthogonal projector onto $\mathscr{C}(\mathbf{A})^\perp$ is denoted as $\mathbf{Q}_\mathbf{A} = \mathbf{I}_a - \mathbf{P}_\mathbf{A}$, where $\mathbf{I}_a$ is the $a \times a$ identity

CONTACT Simo Puntanen simo.puntanen@tuni.fi Faculty of Information Technology and Communication Sciences, Tampere University, FI-33014 Tampere, Finland.

matrix and a is the number of rows of $\mathbf{A}$. We write shortly

$$\mathbf{M} = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}}, \quad \mathbf{M}_i = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}_i}, \; i = 1, \; 2.$$

One obvious choice for $\mathbf{X}^{\perp}$ is $\mathbf{M}$.

When using generalized inverses it is essential to know whether the expressions are independent of the choice of the generalized inverses involved. The following lemma gives an important invariance condition; cf. Rao and Mitra (1971, Lemma 2.2.4)

Lemma 1.1. *For nonnull matrices* $\mathbf{A}$ *and* $\mathbf{C}$ *the following holds*:

$$\mathbf{AB}^{-}\mathbf{C} = \mathbf{AB}^{+}\mathbf{C} \text{ for all } \mathbf{B}^{-} \iff \mathscr{C}(\mathbf{C}) \subseteq \mathscr{C}(\mathbf{B}) \quad \& \quad \mathscr{C}(\mathbf{A}') \subseteq \mathscr{C}(\mathbf{B}').$$

For a given linear model $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$, let the set $\mathcal{W}(\mathscr{M})$ of nonnegative definite matrices be defined as

$$\mathcal{W}(\mathscr{M}) = \{\mathbf{W} \in \mathbb{R}^{n \times n} : \mathbf{W} = \mathbf{V} + \mathbf{XUU}'\mathbf{X}', \; \mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V})\}. \tag{1.1}$$

In (1.1), $\mathbf{U}$ can be any matrix comprising p rows as long as $\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X} : \mathbf{V})$ is satisfied. Lemma 1.2 collects together some important properties of the class $\mathcal{W}(\mathscr{M})$; see, for example, Puntanen, Styan, and Isotalo (2011, Prop. 12.1 and 15.2).

Lemma 1.2. *Consider the model* $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$ *and let* $\mathbf{W} = \mathbf{V} + \mathbf{XUU}'\mathbf{X}' \in \mathcal{W}(\mathscr{M})$. *Then*

$$\begin{aligned} \mathbf{G}_{12} &= \mathbf{X}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X})^{-}\mathbf{X}'\mathbf{W}^{+} = \mathbf{P}_{\mathbf{W}} - \mathbf{VM}(\mathbf{MVM})^{-}\mathbf{MP}_{\mathbf{W}} \\ &= \mathbf{P}_{\mathbf{W}} - \mathbf{VM}(\mathbf{MVM})^{+} = \mathbf{P}_{\mathbf{W}} - \mathbf{VM}(\mathbf{MVM})^{+}\mathbf{M}. \end{aligned} \tag{1.2}$$

Moreover, the following statements are equivalent:

(a) $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{W})$,
(b) $\mathscr{C}(\mathbf{X}) \subseteq \mathscr{C}(\mathbf{W})$,
(c) $\mathbf{X}'\mathbf{W}^{-}\mathbf{X}$ is invariant for any choice of $\mathbf{W}^{-}$,
(d) $\mathscr{C}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X}) = \mathscr{C}(\mathbf{X}')$ for any choice of $\mathbf{W}^{-}$,
(e) $\mathbf{X}(\mathbf{X}'\mathbf{W}^{-}\mathbf{X})^{-}\mathbf{X}'\mathbf{W}^{-}\mathbf{X} = \mathbf{X}$ for any choices of $\mathbf{W}^{-}$ and $(\mathbf{X}'\mathbf{W}^{-}\mathbf{X})^{-}$.

It is noteworthy that the matrix $\mathbf{G}_{12}$ in (1.2) is invariant for the choice of the generalized inverses denoted as "$-$", and it is independent of any choice of $\mathbf{W} \in \mathcal{W}(\mathscr{M})$. Notice also that the invariance properties in (d) and (e) in Lemma 1.2 are valid for all choices of $\mathbf{W} \in \mathcal{W}(\mathscr{M})$. It is clear that $\mathbf{V} \in \mathcal{W}(\mathscr{M})$ if and only if $\mathscr{C}(\mathbf{X}) \subseteq \mathscr{C}(\mathbf{V})$.

In Lemma 1.2, the matrix $\mathbf{W}$ is nonnegative definite, denoted as $\mathbf{W} \geq_{\mathrm{L}} \mathbf{0}$. A corresponding version of Lemma 1.2 can be presented for $\mathbf{W} = \mathbf{V} + \mathbf{XTX}'$ which may not be symmetric but satisfies $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{W})$.

Corresponding to (1.1), we will say that $\mathbf{W}_i \in \mathcal{W}(\mathscr{M}_i)$ if there exist $\mathbf{U}_i$ such that

$$\mathbf{W}_i = \mathbf{V} + \mathbf{X}_i\mathbf{U}_i\mathbf{U}_i'\mathbf{X}_i', \quad \mathscr{C}(\mathbf{W}_i) = \mathscr{C}(\mathbf{X}_i : \mathbf{V}), \quad i = 1, 2. \tag{1.3}$$

For the partitioned linear model $\mathscr{M}_{12}$ we will say that $\mathbf{W} \in \mathcal{W}(\mathscr{M}_{12})$ if

$$\mathbf{W} = \mathbf{V} + \mathbf{X}_1\mathbf{U}_1\mathbf{U}_1'\mathbf{X}_1' + \mathbf{X}_2\mathbf{U}_2\mathbf{U}_2'\mathbf{X}_2',$$

where $\mathbf{U}_1$ and $\mathbf{U}_2$ are defined as in (1.3). For our considerations the actual choice of $\mathbf{U}_1$ and $\mathbf{U}_2$ does not matter as long as they satisfy (1.3).

By the consistency of the model $\mathscr{M}$ it is meant that $\mathbf{y}$ lies in $\mathscr{C}(\mathbf{X} : \mathbf{V})$ with probability 1. Hence we assume that under the consistent model $\mathscr{M}$ the observed numerical value of $\mathbf{y}$ satisfies

$$\mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{X} : \mathbf{VM}) = \mathscr{C}(\mathbf{X}) \oplus \mathscr{C}(\mathbf{VM}) = \mathscr{C}(\mathbf{X}) \boxplus \mathscr{C}(\mathbf{MV}),$$

where "$\oplus$" refers to the direct sum and "$\boxplus$" refers to the direct sum of orthogonal subspaces. For the equality $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{X} : \mathbf{VM})$, see Rao (1974, Lemma 2.1).

For parts (a) and (b) of Lemma 1.3, see, for example, Puntanen, Styan, and Isotalo (2011, Th. 8). and for part (c), see the rank rule of the matrix product of Marsaglia and Styan (1974, Cor. 6.2). Claim (d) is straightforward to confirm.

Lemma 1.3. *Consider* $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$ *and let* $\mathbf{M}_2 = \mathbf{I}_n - \mathbf{P}_{\mathbf{X}_2}$. *Then*

(a) $\mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2) = \mathscr{C}(\mathbf{X}_1 : \mathbf{M}_1\mathbf{X}_2)$,
(b) $\mathbf{M} = \mathbf{I}_n - \mathbf{P}_{(\mathbf{X}_1:\mathbf{X}_2)} = \mathbf{I}_n - (\mathbf{P}_{\mathbf{X}_2} + \mathbf{P}_{\mathbf{M}_2\mathbf{X}_1}) = \mathbf{M}_2\mathbf{Q}_{\mathbf{M}_2\mathbf{X}_1} = \mathbf{Q}_{\mathbf{M}_2\mathbf{X}_1}\mathbf{M}_2$,
(c) $\mathrm{r}(\mathbf{M}_2\mathbf{X}_1) = \mathrm{r}(\mathbf{X}_1) - \dim\mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2)$,
(d) $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V}) \iff \mathscr{C}(\mathbf{M}_1\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{M}_1\mathbf{V})$.

For Lemma 1.4, see, for example, Puntanen, Styan, and Isotalo (2011, p. 152).

Lemma 1.4. *For conformable matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *the following three statements are equivalent:*

$$\text{(a) } \mathbf{P}_\mathbf{A} - \mathbf{P}_\mathbf{B} \text{ is an orth. projector}, \quad \text{(b) } \mathbf{P}_\mathbf{A} - \mathbf{P}_\mathbf{B} \geq_\mathrm{L} \mathbf{0}, \quad \text{(c) } \mathscr{C}(\mathbf{B}) \subseteq \mathscr{C}(\mathbf{A}).$$

If any of the above conditions holds then

$$\mathbf{P}_\mathbf{A} - \mathbf{P}_\mathbf{B} = \mathbf{P}_{\mathscr{C}(\mathbf{A}) \cap \mathscr{C}(\mathbf{B})^\perp} = \mathbf{P}_{(\mathbf{I}-\mathbf{P}_\mathbf{B})\mathbf{A}}.$$

Let $\mathbf{A}$ and $\mathbf{B}$ be arbitrary $m \times n$ matrices. Then, in the consistent linear model $\mathscr{M}$, the estimators $\mathbf{Ay}$ and $\mathbf{By}$ are said to be equal (with probability 1) if

$$\mathbf{Ay} = \mathbf{By} \quad \text{for all } \mathbf{y} \in \mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathscr{C}(\mathbf{X} : \mathbf{VM}) = \mathscr{C}(\mathbf{W}), \tag{1.4}$$

where $\mathbf{W} \in \mathcal{W}(\mathscr{M})$. Thus, if $\mathbf{A}$ and $\mathbf{B}$ satisfy (1.4), then $\mathbf{A} - \mathbf{B} = \mathbf{CQ}_\mathbf{W}$ for some matrix $\mathbf{C}$. It is crucial to notice that in (1.4) we are dealing with the "statistical" equality of the estimators $\mathbf{Ay}$ and $\mathbf{By}$. In (1.4) $\mathbf{y}$ refers to a vector in $\mathbb{R}^n$. Thus we do not make any notational difference between a random vector and its observed value.

According to the well-known fundamental BLUE-equation, see Lemma 2.1 in Section 2, $\mathbf{Ay}$ is the BLUE of $\mathbf{X}\boldsymbol{\beta}$ if and only if

$$\mathbf{A}(\mathbf{X} : \mathbf{VM}) = (\mathbf{X} : \mathbf{0}).$$

Obviously $(\mathbf{A} + \mathbf{NQ}_\mathbf{W})\mathbf{y}$ is another representation of BLUE for *any* $n \times n$ matrix $\mathbf{N}$. However, the equality

$$\mathbf{Ay} = (\mathbf{A} + \mathbf{NQ}_\mathbf{W})\mathbf{y} \quad \text{for all } \mathbf{y} \in \mathscr{C}(\mathbf{W})$$

holds when the model is consistent in the sense that $\mathbf{y} \in \mathscr{C}(\mathbf{W})$. The properties of the BLUE deserve particular attention when $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathbb{R}^n$ does not hold: then there is an infinite number of multipliers $\mathbf{B}$ such that $\mathbf{By}$ is BLUE but for all such multipliers the vector $\mathbf{By}$ itself is unique once the response $\mathbf{y}$ has been observed. The case of two linear models, $\mathscr{B}_i = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}_i\}$, $i = 1, 2$, is extensively studied by Mitra and Moore (1973). They ask, for example, when is a specific linear representation of the BLUE of $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$

under $\mathscr{B}_1$ also a BLUE under $\mathscr{B}_2$, and when is the BLUE of $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ under $\mathscr{B}_1$ irrespective of the linear representation used in its expression, also a BLUE under $\mathscr{B}_2$.

The purpose of this paper is to consider the models $\mathscr{M}_1$ and $\mathscr{M}_{12}$ in the spirit of Mitra and Moore (1973). We pick up particular fixed representations for the BLUE s of $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ under these two models, say $\mathbf{G}_1\mathbf{y}$ and $\mathbf{G}_{1\#}\mathbf{y}$, and study the conditions under which they are equal for all values of $\mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$ or $\mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{V})$, that is,

$$\mathbf{G}_1\mathbf{W}_1 = \mathbf{G}_{1\#}\mathbf{W}_1, \quad \text{or} \quad \mathbf{G}_1\mathbf{W} = \mathbf{G}_{1\#}\mathbf{W}. \tag{1.5}$$

Moreover, we review the conditions under which (1.5) holds for *all* representations of the BLUE s, not only for fixed $\mathbf{G}_1$ and $\mathbf{G}_{1\#}$. Some related considerations were made by Haslett, Markiewicz, and Puntanen (2020) when these models are supplemented with the new unobservable random vector $\mathbf{y}_*$, coming from $\mathbf{y}_* = \mathbf{K}\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_*$, where the covariance matrix of $\mathbf{y}_*$ is known as well as the cross-covariance matrix between $\mathbf{y}_*$ and $\mathbf{y}$.

The well-known (or pretty well-known) results are given as Lemmas, while the new (or at least not so well-known) results are represented as Propositions.

2. The fundamental BLUE equations

A linear statistic $\mathbf{By}$ is said to be linear unbiased estimator, LUE, for the parametric function $\mathbf{K}\boldsymbol{\beta}$ in $\mathscr{M}_{12}$ if its expectation is equal to $\mathbf{K}\boldsymbol{\beta}$, which happens if and only if $\mathbf{K}' = \mathbf{X}'\mathbf{B}'$; in this case $\mathbf{K}\boldsymbol{\beta}$ is said to be estimable. The LUE $\mathbf{By}$ is the best linear unbiased estimator, BLUE, of estimable $\mathbf{K}\boldsymbol{\beta}$ if $\mathbf{By}$ has the smallest covariance matrix in the Löwner sense among all LUEs of $\mathbf{K}\boldsymbol{\beta}$:

$$\operatorname{cov}(\mathbf{By}) \leq_{\mathrm{L}} \operatorname{cov}(\mathbf{B}_{\#}\mathbf{y}) \quad \text{for all } \mathbf{B}_{\#} : \mathbf{B}_{\#}\mathbf{X} = \mathbf{K}.$$

It is well known that $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is estimable under $\mathscr{M}_{12}$ if and only if

$$\mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2) = \{\mathbf{0}\}, \quad \text{i.e.,} \quad \mathrm{r}(\mathbf{M}_2\mathbf{X}_1) = \mathrm{r}(\mathbf{X}_1).$$

For Lemma 2.1, characterizing the BLUE, see, for example, Rao (1973, p. 282).

Lemma 2.1. *Consider the model $\mathscr{M}_{12}$ where $\boldsymbol{\eta} = \mathbf{K}\boldsymbol{\beta}$ is estimable. Then*

(a) $\mathbf{Ay} = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}) \iff \mathbf{A}(\mathbf{X} : \mathbf{VM}) = (\mathbf{X} : \mathbf{0})$, *that is,* $\mathbf{A} \in \{\mathbf{P}_{\boldsymbol{\mu}|\mathscr{M}_{12}}\}$,
(b) $\mathbf{By} = \mathrm{BLUE}(\mathbf{K}\boldsymbol{\beta}) \iff \mathbf{B}(\mathbf{X} : \mathbf{VM}) = (\mathbf{K} : \mathbf{0})$, *that is,* $\mathbf{B} \in \{\mathbf{P}_{\boldsymbol{\eta}|\mathscr{M}_{12}}\}$.
In particular, if $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is estimable,
(c) $\mathbf{Cy} = \mathrm{BLUE}(\boldsymbol{\mu}_1) \iff \mathbf{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{X}_1 : \mathbf{0} : \mathbf{0})$, *that is,* $\mathbf{C} \in \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_{12}}\}$.

Of course, under the model $\mathscr{M}_1$ we have

$$\mathbf{Dy} = \mathrm{BLUE}(\boldsymbol{\mu}_1) \iff \mathbf{D}(\mathbf{X}_1 : \mathbf{VM}_1) = (\mathbf{X}_1 : \mathbf{0}), \text{ i.e., } \mathbf{D} \in \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_1}\}.$$

To indicate that $\mathbf{A} \in \{\mathbf{P}_{\boldsymbol{\mu}|\mathscr{M}_{12}}\}$ we will also use notations

$$\mathbf{Ay} = \tilde{\boldsymbol{\mu}}(\mathscr{M}_{12}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta} \mid \mathscr{M}_{12}), \quad \mathbf{Ay} \in \{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta} \mid \mathscr{M}_{12})\}.$$

Using Lemma 1.2 we can obtain, for example, the following well-known solution to $\mathbf{A}$ in Lemma 2.1:

$$\mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^- \in \{\mathbf{P}_{\boldsymbol{\mu}|\mathscr{M}_{12}}\},$$

where $\mathbf{W} \in \mathcal{W}(\mathscr{M}_{12})$ and we can freely choose the generalized inverses involved. Expression $\mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X}')^-\mathbf{X}'\mathbf{W}^-$ is not necessarily unique with respect to the choice of $\mathbf{W}^-$ but by Lemma 1.2, the matrix

$$\mathbf{G}_{12} = \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X}')^-\mathbf{X}'\mathbf{W}^+ = \mathbf{P_W} - \mathbf{VM}(\mathbf{MVM})^-\mathbf{MP_W}$$

is unique whatever choices of $\mathbf{W}^-$ and $(\mathbf{X}'\mathbf{W}^-\mathbf{X}')^-$ we have and moreover, $\mathbf{G}_{12}$ does not depend on the choice of $\mathbf{W} \in \mathcal{W}(\mathscr{M}_{12})$. The *general* solution for $\mathbf{A}$ in Lemma 2.1, can be expressed, for example, as

$$\mathbf{G}_0 = \mathbf{G}_{12} + \mathbf{NQ_W}, \qquad \text{where } \mathbf{N} \in \mathbb{R}^{n\times n} \text{ is free to vary,}$$

and $\mathbf{Q_W} = \mathbf{I}_n - \mathbf{P_W}$. Thus the solution for $\mathbf{A}$ (as well as for $\mathbf{B}$ and $\mathbf{C}$) in Lemma 2.1 is unique if and only if $\mathscr{C}(\mathbf{X} : \mathbf{V}) = \mathbb{R}^n$.

Consider then the estimation of $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ under $\mathscr{M}_{12}$ assuming that $\boldsymbol{\mu}_1$ is estimable. Premultiplying the model $\mathscr{M}_{12}$ by $\mathbf{M}_2$ yields the reduced model

$$\mathscr{M}_{12\cdot 2} = \{\mathbf{M}_2\mathbf{y},\ \mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1,\ \mathbf{M}_2\mathbf{VM}_2\}.$$

Now the well-known Frisch–Waugh–Lovell theorem, see, for example, Groß and Puntanen (2000, Sec. 6), states that the BLUE s of $\boldsymbol{\mu}_1$ under $\mathscr{M}_{12}$ and $\mathscr{M}_{12\cdot 2}$ coincide. To obtain an explicit expression for the BLUE of $\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$ under $\mathscr{M}_{12\cdot 2}$ we need a $\mathbf{W}$-matrix in $\mathscr{M}_{12\cdot 2}$. Now any matrix of the form

$$\mathbf{M}_2\mathbf{VM}_2 + \mathbf{M}_2\mathbf{X}_1\mathbf{T}_1\mathbf{T}_1'\mathbf{X}_1'\mathbf{M}_2 = \mathbf{M}_2(\mathbf{V} + \mathbf{X}_1\mathbf{T}_1\mathbf{T}_1'\mathbf{X}_1')\mathbf{M}_2$$

satisfying

$$\mathscr{C}[\mathbf{M}_2(\mathbf{V} : \mathbf{X}_1\mathbf{T}_1)] = \mathscr{C}[\mathbf{M}_2(\mathbf{V} : \mathbf{X}_1)] = \mathscr{C}(\mathbf{M}_2\mathbf{W}_1), \tag{2.1}$$

is a $\mathbf{W}$-matrix in $\mathscr{M}_{12\cdot 2}$. Choosing $\mathbf{T}_1 = \mathbf{U}_1$ as in (1.3) we have

$$\mathbf{M}_2\mathbf{WM}_2 = \mathbf{M}_2\mathbf{W}_1\mathbf{M}_2 \in \mathcal{W}(\mathscr{M}_{12\cdot 2}).$$

Thus the BLUE of $\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$ under $\mathscr{M}_{12\cdot 2}$ can be expressed as

$$\mathrm{BLUE}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{M}_{12\cdot 2}) = \mathbf{M}_2\mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{y},$$

where $\dot{\mathbf{M}}_2 = \mathbf{M}_2(\mathbf{M}_2\mathbf{W}_1\mathbf{M}_2)^-\mathbf{M}_2$.

We observe that (2.1) holds for $\mathbf{T}_1 = \mathbf{0}$ if and only if $\mathscr{C}(\mathbf{M}_2\mathbf{X}_1) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{V})$, that is, see part (d) of Lemma 1.3,

$$\mathscr{C}(\mathbf{X}_1) \subseteq \mathscr{C}(\mathbf{X}_2 : \mathbf{V}). \tag{2.2}$$

Our conclusion: If (2.2) holds, then the BLUE of $\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1$ under $\mathscr{M}_{12\cdot 2}$ can be expressed as

$$\mathrm{BLUE}(\mathbf{M}_2\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{M}_{12\cdot 2}) = \mathbf{M}_2\mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_{2V}\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_{2V}\mathbf{y}, \tag{2.3}$$

where $\dot{\mathbf{M}}_{2V} = \mathbf{M}_2(\mathbf{M}_2\mathbf{VM}_2)^-\mathbf{M}_2$. Actually, it can be shown that (2.2) is also a necessary condition for (2.3). It is obvious that under the estimability of $\boldsymbol{\mu}_1$ we have

$$\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_{12\cdot 2}) = \mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_{12}) = \mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{y}, \tag{2.4a}$$

$$\mathrm{BLUE}(\boldsymbol{\mu}_2 \mid \mathscr{M}_{12\cdot 1}) = \mathrm{BLUE}(\boldsymbol{\mu}_2 \mid \mathscr{M}_{12}) = \mathbf{X}_2(\mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{X}_2)^-\mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{y}, \tag{2.4b}$$

where $\dot{\mathbf{M}}_i = \mathbf{M}_i(\mathbf{M}_i\mathbf{WM}_i)^-\mathbf{M}_i, i = 1, 2$.

An alternative expression for the BLUE of $\boldsymbol{\mu}_1$ can be obtained by premultiplying the fundamental BLUE-equation

$$\mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{0})$$

by $\mathbf{M}_2$, yielding

$$(\mathbf{M}_2\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{M}_2\mathbf{X}_1 : \mathbf{0} : \mathbf{0}). \tag{2.5}$$

Because $\mathrm{r}(\mathbf{M}_2\mathbf{X}_1) = \mathrm{r}(\mathbf{X}_1)$, we can, by the rank cancelation rule of Marsaglia and Styan (1974), cancel $\mathbf{M}_2$ in (2.5) and thus an alternative expression for (2.4a) is

$$\tilde{\boldsymbol{\mu}}_1(\mathscr{M}_{12}) = (\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^-\mathbf{y}.$$

Now we should pay attention to numerous generalized inverses appearing in the representations of the BLUEs. Namely, when the observable response $\mathbf{y}$ belongs to a "correct" subspace of $\mathbb{R}^n$, then there is no problem with the generalized inverses. In the next section we will consider particular unique representations of the multipliers of $\mathbf{y}$ and study the equality of the relevant estimators taking the space where $\mathbf{y}$ belongs into account.

3. Some useful matrix results

Let us denote

$$\mathbf{G}_{1\#} = \mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_2, \quad \mathbf{D}_{1\#} = (\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+,$$
$$\mathbf{G}_{2\#} = \mathbf{X}_2(\mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{X}_2)^-\mathbf{X}_2'\dot{\mathbf{M}}_1, \quad \mathbf{D}_{2\#} = (\mathbf{0} : \mathbf{X}_2)(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+,$$

where $\dot{\mathbf{M}}_1$ and $\dot{\mathbf{M}}_2$ are now *unique* (once $\mathbf{W}$ is given) matrices defined as

$$\dot{\mathbf{M}}_1 = \mathbf{M}_1(\mathbf{M}_1\mathbf{W}\mathbf{M}_1)^+\mathbf{M}_1 = \mathbf{M}_1(\mathbf{M}_1\mathbf{W}_2\mathbf{M}_1)^+\mathbf{M}_1,$$
$$\dot{\mathbf{M}}_2 = \mathbf{M}_2(\mathbf{M}_2\mathbf{W}\mathbf{M}_2)^+\mathbf{M}_2 = \mathbf{M}_2(\mathbf{M}_2\mathbf{W}_1\mathbf{M}_2)^+\mathbf{M}_2.$$

It is noteworthy that the following types of equalities hold:

$$\mathbf{M}_1(\mathbf{M}_1\mathbf{W}\mathbf{M}_1)^+\mathbf{M}_1 = \mathbf{M}_1(\mathbf{M}_1\mathbf{W}\mathbf{M}_1)^+ = (\mathbf{M}_1\mathbf{W}\mathbf{M}_1)^+.$$

Now under the estimability of $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ we have

$$\tilde{\boldsymbol{\mu}}_1(\mathscr{M}_{12}) = \mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{y} = (\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+\mathbf{y},$$
$$\tilde{\boldsymbol{\mu}}_2(\mathscr{M}_{12}) = \mathbf{X}_2(\mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{X}_2)^-\mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{y} = (\mathbf{0} : \mathbf{X}_2)(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+\mathbf{y},$$

and

$$\tilde{\boldsymbol{\mu}}(\mathscr{M}_{12}) = (\mathbf{G}_{1\#} + \mathbf{G}_{2\#})\mathbf{y} = (\mathbf{D}_{1\#} + \mathbf{D}_{2\#})\mathbf{y} \text{ for all } \mathbf{y} \in \mathscr{C}(\mathbf{W}).$$

Because $\mathbf{G}_{1\#}$ and $\mathbf{D}_{1\#}$ belong to $\{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_{12}}\}$, they satisfy the equation

$$\mathbf{G}_{1\#}\mathbf{W} = \mathbf{D}_{1\#}\mathbf{W}. \tag{3.3}$$

Next we show that we also have

$$\mathbf{G}_{1\#}\mathbf{Q}_\mathbf{W} = \mathbf{D}_{1\#}\mathbf{Q}_\mathbf{W}. \tag{3.4}$$

We immediately observe that $\mathbf{D}_{1\#}\mathbf{Q}_\mathbf{W} = \mathbf{0}$ and what remains is to show that $\mathbf{G}_{1\#}\mathbf{Q}_\mathbf{W} = \mathbf{0}$. Now the equation

$$\mathbf{G}_{1\#}\mathbf{Q}_\mathbf{W} = \mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{Q}_\mathbf{W} = \mathbf{0}$$

holds if and only if

$$\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{Q}_\mathbf{W} = \mathbf{0}, \quad \text{i.e.,} \quad \mathscr{C}(\dot{\mathbf{M}}_2\mathbf{X}_1) \subseteq \mathscr{C}(\mathbf{W}). \tag{3.5}$$

Clearly (3.5) holds because

$$\mathscr{C}(\dot{\mathbf{M}}_2\mathbf{X}_1) \subseteq \mathscr{C}(\dot{\mathbf{M}}_2) = \mathscr{C}\left[(\mathbf{M}_2\mathbf{W}_1\mathbf{M}_2)^+\right] = \mathscr{C}(\mathbf{M}_2\mathbf{W}_1) \subseteq \mathscr{C}(\mathbf{W}),$$

where the last inclusion follows from

$$\mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V}) = \mathscr{C}[\mathbf{X}_2 : \mathbf{M}_2(\mathbf{X}_1 : \mathbf{V})] = \mathscr{C}(\mathbf{X}_2 : \mathbf{M}_2\mathbf{W}_1).$$

Combining (3.3) and (3.4) gives the following result.

Proposition 3.1. *Assume that $\boldsymbol{\mu}_1$ is estimable under $\mathscr{M}_{12}$. Then*

$$\mathbf{G}_{1\#} = \mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_2 = (\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+ = \mathbf{D}_{1\#}, \tag{3.6}$$

where $\dot{\mathbf{M}}_2 = \mathbf{M}_2(\mathbf{M}_2\mathbf{W}_1\mathbf{M}_2)^+\mathbf{M}_2$. Moreover, the expressions in (3.6) are invariant for any choices of generalized inverses $(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-$, $\mathbf{W}^-$, and $(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-$ as well as for the choice of $\mathbf{W} \in \mathcal{W}(\mathscr{M}_{12})$. Corresponding equality holds between $\mathbf{G}_{2\#}$ and $\mathbf{D}_{2\#}$. Moreover,

$$\mathbf{G}_{12} = \mathbf{X}(\mathbf{X}'\mathbf{W}^-\mathbf{X}')^-\mathbf{X}'\mathbf{W}^+ = \mathbf{G}_{1\#} + \mathbf{G}_{2\#} = \mathbf{D}_{1\#} + \mathbf{D}_{2\#}.$$

We will also need the following proposition.

Proposition 3.2. *Denote*

$$\mathbf{G}_{1\#} = \mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_2,$$

where $\dot{\mathbf{M}}_2 = \mathbf{M}_2(\mathbf{M}_2\mathbf{W}_1\mathbf{M}_2)^+\mathbf{M}_2$. Then

(a) $\mathscr{C}(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{W}) = \mathscr{C}(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1) = \mathscr{C}(\mathbf{X}_1'\mathbf{M}_2)$,
(b) $\mathrm{r}(\mathbf{W}_1\dot{\mathbf{M}}_2\mathbf{X}_1) = \mathrm{r}(\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1) = \mathrm{r}(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1) = \mathrm{r}(\mathbf{M}_2\mathbf{X}_1)$,
(c) $\mathscr{C}(\mathbf{W}\mathbf{G}_{1\#}') = \mathscr{C}[\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'] = \mathscr{C}(\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1)$,
(d) $\mathrm{C}(\mathbf{G}_{1\#}\mathbf{W}) = \mathrm{C}(\mathbf{X}_1{}'\mathbf{M}_2)$.
In particular, when $\boldsymbol{\mu}_1$ is estimable under $\mathscr{M}_{12}$, we have
(e) $\mathscr{C}(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{W}) = \mathscr{C}(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1) = \mathscr{C}(\mathbf{G}_{1\#}\mathbf{W}) = \mathscr{C}(\mathbf{X}_1')$.

Proof. Property (b) comes from the following:

$$\begin{aligned} \mathrm{r}(\mathbf{M}_2\mathbf{X}_1) &\geq \mathrm{r}(\mathbf{W}_1\dot{\mathbf{M}}_2\mathbf{X}_1) = \mathrm{r}[\mathbf{W}_1\mathbf{M}_2(\mathbf{M}_2\mathbf{W}_1\mathbf{M}_2)^-\mathbf{M}_2\mathbf{X}_1] \\ &\geq \mathrm{r}[\mathbf{M}_2\mathbf{W}_1\mathbf{M}_2(\mathbf{M}_2\mathbf{W}_1\mathbf{M}_2)^-\mathbf{M}_2\mathbf{X}_1] \\ &= \mathrm{r}(\mathbf{M}_2\mathbf{X}_1). \end{aligned} \tag{3.7}$$

The last equality in (3.7) follows from the fact that $\mathscr{C}(\mathbf{M}_2\mathbf{X}_1) \subseteq \mathscr{C}(\mathbf{M}_2\mathbf{W}_1)$. The other statements can be confirmed in the corresponding way. □

Proposition 3.3 appears to be useful for our BLUE-considerations and it also provides some interesting linear algebraic matrix results. By $\mathbf{A}^{1/2}$ we refer to the nonnegative definite square root of a nonnegative definite matrix $\mathbf{A}$ and $\mathbf{A}^{+1/2} = (\mathbf{A}^{1/2})^+$ so that $\mathbf{A}^{1/2}\mathbf{A}^{+1/2} = \mathbf{P}_\mathbf{A}$.

Proposition 3.3. *The following five statements hold*:

(a) $\mathscr{C}(\mathbf{W}^{+}\mathbf{X})^{\perp} = \mathscr{C}(\mathbf{WM} : \mathbf{Q_W}) = \mathscr{C}(\mathbf{VM} : \mathbf{Q_W})$,
(b) $\mathscr{C}(\mathbf{W}_1^{+}\mathbf{X}_1)^{\perp} = \mathscr{C}(\mathbf{W}_1\mathbf{M}_1 : \mathbf{Q}_{\mathbf{W}_1}) = \mathscr{C}(\mathbf{VM}_1 : \mathbf{Q}_{\mathbf{W}_1})$,
(c) $\mathbf{P}_{\mathbf{W}^{1/2}\mathbf{M}_2} = \mathbf{P_W} - \mathbf{P}_{\mathbf{W}^{+1/2}\mathbf{X}_2}$,
(d) $\mathbf{P_W}\dot{\mathbf{M}}_2\mathbf{P_W} = \mathbf{W}^{+} - \mathbf{W}^{+}\mathbf{X}_2(\mathbf{X}_2'\mathbf{W}^{+}\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{W}^{+}$,
(e) $\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1 = \mathbf{W}_1\dot{\mathbf{M}}_2\mathbf{X}_1 = [\mathbf{I}_n - \mathbf{X}_2(\mathbf{X}_2'\mathbf{W}^{+}\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{W}^{+}]\mathbf{X}_1$.
The following three statements are equivalent:
(f) $\mathrm{r}(\mathbf{X}_2) = \dim\mathscr{C}(\mathbf{W}_1)\cap\mathscr{C}(\mathbf{X}_2) + \dim\mathscr{C}(\mathbf{W}_1)^{\perp}\cap\mathscr{C}(\mathbf{X}_2)$,
(g) $\mathrm{r}(\mathbf{W}_1) = \mathrm{r}(\mathbf{W}_1\mathbf{M}_2) + \mathrm{r}(\mathbf{W}_1\mathbf{X}_2)$,
(h) $\mathbf{P}_{\mathbf{W}_1^{1/2}\mathbf{M}_2} = \mathbf{P}_{\mathbf{W}_1} - \mathbf{P}_{\mathbf{W}_1^{+1/2}\mathbf{X}_2}$.
If any of the conditions (f)–(h) *holds, then*
(i) $\mathbf{P}_{\mathbf{W}_1}\dot{\mathbf{M}}_2\mathbf{P}_{\mathbf{W}_1} = \mathbf{W}_1^{+} - \mathbf{W}_1^{+}\mathbf{X}_2(\mathbf{X}_2'\mathbf{W}_1^{+}\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{W}_1^{+}$,
(j) $\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1 = \mathbf{W}_1\dot{\mathbf{M}}_2\mathbf{X}_1 = [\mathbf{P}_{\mathbf{W}_1} - \mathbf{P}_{\mathbf{W}_1}\mathbf{X}_2(\mathbf{X}_2'\mathbf{W}_1^{+}\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{W}_1^{+}]\mathbf{X}_1$.
If $\mathscr{C}(\mathbf{X}_2)\subseteq\mathscr{C}(\mathbf{W}_1)$, *then*
(k) $\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1 = \mathbf{W}_1\dot{\mathbf{M}}_2\mathbf{X}_1 = [\mathbf{I}_n - \mathbf{X}_2(\mathbf{X}_2'\mathbf{W}_1^{+}\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{W}_1^{+}]\mathbf{X}_1$.

Proof. The first five statements (a)–(e) appear in Markiewicz and Puntanen (2019, Sec. 4). The claim (h), that is,

$$\mathbf{P}_{\mathbf{W}_1^{1/2}\mathbf{M}_2} = \mathbf{P}_{\mathbf{W}_1} - \mathbf{P}_{\mathbf{W}_1^{+1/2}\mathbf{X}_2},$$

holds if and only if, see Lemma 1.4,

$$\mathscr{C}(\mathbf{W}_1^{1/2}\mathbf{M}_2) = \mathscr{C}(\mathbf{W}_1^{+1/2}\mathbf{X}_2 : \mathbf{Q}_{\mathbf{W}_1})^{\perp} = \mathscr{C}(\mathbf{W}_1^{+1/2}\mathbf{X}_2)^{\perp}\cap\mathscr{C}(\mathbf{W}_1). \tag{3.8}$$

Now (3.8) holds if and only if

$$\mathrm{r}(\mathbf{W}_1^{1/2}\mathbf{M}_2) = n - \mathrm{r}(\mathbf{W}_1^{+1/2}\mathbf{X}_2 : \mathbf{Q}_{\mathbf{W}_1}),$$

that is,

$$\mathrm{r}(\mathbf{W}_1) = \mathrm{r}(\mathbf{W}_1\mathbf{M}_2) + \mathrm{r}(\mathbf{W}_1\mathbf{X}_2),$$

which further is equivalent to (f). Clearly (f) holds, for example, when $\mathscr{C}(\mathbf{X}_2)\subseteq\mathscr{C}(\mathbf{W}_1)$.

Assuming that (f) holds we can write

$$\begin{aligned}\mathbf{P}_{\mathbf{W}_1}\dot{\mathbf{M}}_2\mathbf{P}_{\mathbf{W}_1} &= \mathbf{P}_{\mathbf{W}_1}\mathbf{M}_2(\mathbf{M}_2\mathbf{W}_1\mathbf{M}_2)^{+}\mathbf{M}_2\mathbf{P}_{\mathbf{W}_1}\\ &= \mathbf{W}_1^{+1/2}\mathbf{P}_{\mathbf{W}_1^{1/2}\mathbf{M}_2}\mathbf{W}_1^{+1/2}\\ &= \mathbf{W}_1^{+1/2}(\mathbf{P}_{\mathbf{W}_1} - \mathbf{P}_{\mathbf{W}_1^{+1/2}\mathbf{X}_2})\mathbf{W}_1^{+1/2}\\ &= \mathbf{W}_1^{+} - \mathbf{W}_1^{+}\mathbf{X}_2(\mathbf{X}_2'\mathbf{W}_1^{+}\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{W}_1^{+}.\end{aligned} \tag{3.9}$$

From (3.9) it follows that

$$\begin{aligned}\mathbf{W}_1\dot{\mathbf{M}}_2\mathbf{X}_1 &= \mathbf{W}_1[\mathbf{W}_1^{+} - \mathbf{W}_1^{+}\mathbf{X}_2(\mathbf{X}_2'\mathbf{W}_1^{+}\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{W}_1^{+}]\mathbf{X}_1\\ &= [\mathbf{P}_{\mathbf{W}_1} - \mathbf{P}_{\mathbf{W}_1}\mathbf{X}_2(\mathbf{X}_2'\mathbf{W}_1^{+}\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{W}_1^{+}]\mathbf{X}_1,\end{aligned}$$

and hence, supposing that $\mathscr{C}(\mathbf{X}_2)\subseteq\mathscr{C}(\mathbf{W}_1)$, we obtain (k):

$$\mathbf{W}_1\dot{\mathbf{M}}_2\mathbf{X}_1 = [\mathbf{I}_n - \mathbf{X}_2(\mathbf{X}_2'\mathbf{W}_1^+\mathbf{X}_2)^-\mathbf{X}_2'\mathbf{W}_1^+]\mathbf{X}_1.$$

Thus the proof is completed. □

4. Difference of the BLUEs under the full and small model

Next we introduce a particular expression for the difference $(\mathbf{G}_1 - \mathbf{G}_{1\#})\mathbf{y}$ which is valid for all $\mathbf{y} \in \mathscr{C}(\mathbf{W})$.

Proposition 4.1. *Consider the models $\mathscr{M}_{12}$ and $\mathscr{M}_1$ and suppose that $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is estimable under $\mathscr{M}_{12}$. Using the earlier notation, we have for all $\mathbf{y} \in \mathscr{C}(\mathbf{W})$:*

$$\begin{aligned}(\mathbf{G}_1 - \mathbf{G}_{1\#})\mathbf{y} &= \mathbf{G}_1\mathbf{G}_{2\#}\mathbf{y} \\ &= \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{W}_1^+ \cdot \mathbf{X}_2(\mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{X}_2)^-\mathbf{X}_2'\dot{\mathbf{M}}_1\mathbf{y}.\end{aligned} \tag{4.1}$$

Proof. It is clear that $\mathbf{G}_1\mathbf{G}_{1\#} = \mathbf{G}_{1\#}$. Premultiplying

$$\mathbf{G}_{12} = \mathbf{P}_\mathbf{W} - \mathbf{VM}(\mathbf{MVM})^-\mathbf{MP}_\mathbf{W}$$

by $\mathbf{G}_1$ we observe that $\mathbf{G}_1\mathbf{G}_{12} = \mathbf{G}_1$ as $\mathbf{G}_1\mathbf{VM} = \mathbf{0}$. Thus we have

$$\mathbf{G}_1 - \mathbf{G}_{1\#} = \mathbf{G}_1(\mathbf{G}_{12} - \mathbf{G}_{1\#}) = \mathbf{G}_1\mathbf{G}_{2\#}. \tag{4.2}$$

The claim (4.1) follows from (4.2). □

Proposition 4.1 was proved by Haslett and Puntanen (2010, Lemma 3.1) in the situation when

$$\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V}) = \mathscr{C}(\mathbf{W}_1).$$

Using different formulation and proof, it appears also in Werner and Yapar (1996, Th. 2.3). See also Sengupta and Jammalamadaka (2003, Ch. 9) and Güler, Puntanen, and Özdemir (2014). In the full rank model, that is, when $\mathbf{X}$ has full column rank and $\mathbf{V}$ is positive definite, it appears, for example, in Haslett (1996).

Remark 4.1. We might be tempted to express the equality $\mathbf{G}_1\mathbf{y} = \mathbf{G}_{1\#}\mathbf{y}$ as

$$\tilde{\boldsymbol{\mu}}_1(\mathscr{M}_1) = \tilde{\boldsymbol{\mu}}_1(\mathscr{M}_{12}), \quad \text{i.e.,} \quad \mathrm{BLUE}(\boldsymbol{\mu}_1|\mathscr{M}_1) = \mathrm{BLUE}(\boldsymbol{\mu}_1|\mathscr{M}_{12}). \tag{4.3}$$

However, the notation used in (4.3) can be problematic when the possible values of the response vector $\mathbf{y}$ are taken into account. It is clear that $\mathbf{G}_1\mathbf{y}$ is the BLUE of $\boldsymbol{\mu}_1$ under $\mathscr{M}_1$ and we may write shortly $\mathbf{G}_1\mathbf{y} = \tilde{\boldsymbol{\mu}}_1(\mathscr{M}_1)$. Now, there might be another estimator $\mathbf{Ay}$ for which we can also write $\mathbf{Ay} = \tilde{\boldsymbol{\mu}}_1(\mathscr{M}_1)$ but, however, $\mathbf{Ay}$ and $\mathbf{G}_1\mathbf{y}$ may have different numerical observed values. The numerical value of the BLUE under $\mathscr{M}_1$ is unique if and only if $\mathbf{y}$ lies in $\mathscr{C}(\mathbf{W}_1)$. □

Notice that in above considerations all the matrices $\mathbf{G}_1$, $\mathbf{G}_{12}$ and so on. are fixed. Let us check whether (4.1) holds for *arbitrary* $\mathbf{H}_1 \in \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_1}\}$, $\mathbf{H}_{12} \in \{\mathbf{P}_{\boldsymbol{\mu}|\mathscr{M}_{12}}\}$ and so on.

Corollary 4.1. *Let us denote*

$$\begin{aligned}\mathbf{H}_1 = \mathbf{G}_1 + \mathbf{N}_1\mathbf{Q}_{\mathbf{W}_1}, \quad \mathbf{H}_{12} = \mathbf{G}_{12} + \mathbf{N}_2\mathbf{Q}_\mathbf{W}, \\ \mathbf{H}_{1\#} = \mathbf{G}_{1\#} + \mathbf{N}_3\mathbf{Q}_\mathbf{W}, \quad \mathbf{H}_{2\#} = \mathbf{G}_{2\#} + \mathbf{N}_4\mathbf{Q}_\mathbf{W},\end{aligned}$$

where the matrices $\mathbf{N}_1, \ldots, \mathbf{N}_4$ are free to vary. Then

(a) $(\mathbf{H}_1 - \mathbf{H}_{1\#})\mathbf{y} = \mathbf{G}_1\mathbf{G}_{2\#}\mathbf{y} + \mathbf{N}_1\mathbf{Q}_{\mathbf{W}_1}\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W})$,
(b) $(\mathbf{H}_1 - \mathbf{H}_{1\#})\mathbf{y} = \mathbf{H}_1\mathbf{H}_{2\#}\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W})$.
Moreover, the following two statements are equivalent:
(c) $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{W}_1)$,
(d) $(\mathbf{H}_1 - \mathbf{H}_{1\#})\mathbf{y} = \mathbf{G}_1\mathbf{G}_{2\#}\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W})$.

Proof. In view of

$$\begin{aligned}(\mathbf{H}_1 - \mathbf{H}_{1\#})\mathbf{W} &= (\mathbf{G}_1 + \mathbf{N}_1\mathbf{Q}_{\mathbf{W}_1} - \mathbf{G}_{1\#} - \mathbf{N}_3\mathbf{Q}_{\mathbf{W}})\mathbf{W}\\ &= (\mathbf{G}_1 + \mathbf{N}_1\mathbf{Q}_{\mathbf{W}_1} - \mathbf{G}_{1\#})\mathbf{W}\\ &= (\mathbf{G}_1 - \mathbf{G}_{1\#})\mathbf{W} + \mathbf{N}_1\mathbf{Q}_{\mathbf{W}_1}\mathbf{W}\\ &= \mathbf{G}_1\mathbf{G}_{2\#}\mathbf{W} + \mathbf{N}_1\mathbf{Q}_{\mathbf{W}_1}\mathbf{W},\end{aligned}$$

the statement (a) holds. We observe that

$$\begin{aligned}\mathbf{H}_1\mathbf{H}_{2\#}\mathbf{W} &= (\mathbf{G}_1 + \mathbf{N}_1\mathbf{Q}_{\mathbf{W}_1})\mathbf{G}_{2\#}\mathbf{W}\\ &= \mathbf{G}_1\mathbf{G}_{2\#}\mathbf{W} + \mathbf{N}_1\mathbf{Q}_{\mathbf{W}_1}\mathbf{G}_{2\#}\mathbf{W}.\end{aligned}$$

Thus the statement (b), that is, the equality $(\mathbf{H}_1 - \mathbf{H}_{1\#})\mathbf{W} = \mathbf{H}_1\mathbf{H}_{2\#}\mathbf{W}$ holds if and only if

$$\mathbf{Q}_{\mathbf{W}_1}\mathbf{W} = \mathbf{Q}_{\mathbf{W}_1}\mathbf{G}_{2\#}\mathbf{W}. \tag{4.5}$$

Replacing $\mathbf{W}$ with $(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM})$ in (4.5) we observe that (4.5) indeed holds. The equivalence of (c) and (d) is obvious. □

Proposition 4.2. *Consider the models* $\mathscr{M}_{12}$ *and* $\mathscr{M}_1$ *and suppose that* $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ *is estimable under* $\mathscr{M}_{12}$*. Then the following statements are equivalent:*

(a) $\mathbf{G}_1\mathbf{y} = \mathbf{G}_{1\#}\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W})$, *that is,* $\mathbf{G}_1\mathbf{W} = \mathbf{G}_{1\#}\mathbf{W}$,
(b) $\mathbf{G}_1\mathbf{y} = \mathbf{G}_{1\#}\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2)$,
(c) $\mathbf{G}_1\mathbf{y} = \mathbf{G}_{1\#}\mathbf{y}$ *for all* $\mathbf{y} \in \mathbb{R}^n$, *that is,* $\mathbf{G}_1 = \mathbf{G}_{1\#}$,
(d) $\mathbf{G}_1 \in \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_{12}}\}$, *that is,* $\mathbf{G}_1\mathbf{y} \in \{\mathrm{BLUE}(\boldsymbol{\mu}_1|\mathscr{M}_{12})\}$,
(e) $\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_2 = \mathbf{0}$,
(f) $\mathbf{G}_1\mathbf{X}_2 = \mathbf{0}$,
(g) $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{W}_1^+\mathbf{X}_1)^\perp = \mathscr{C}(\mathbf{W}_1\mathbf{M}_1 : \mathbf{Q}_{\mathbf{W}_1}) = \mathscr{C}(\mathbf{VM}_1 : \mathbf{Q}_{\mathbf{W}_1})$.

Proof. Consider the statement (a) which is obviously equivalent to (d):

$$\mathbf{G}_1(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = \mathbf{G}_{1\#}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}). \tag{4.6}$$

Now $\mathbf{G}_1\mathbf{VM} = \mathbf{G}_1\mathbf{VM}_1\mathbf{Q}_{\mathbf{M}_1\mathbf{X}_2} = \mathbf{0}$ and hence (4.6) holds if and only if

$$(\mathbf{X}_1 : \mathbf{G}_1\mathbf{X}_2 : \mathbf{0}) = (\mathbf{X}_1 : \mathbf{0} : \mathbf{0}), \tag{4.7}$$

that is,

$$\mathbf{G}_1\mathbf{X}_2 = \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^-\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_2 = \mathbf{0},$$

which is equivalent to $\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_2 = \mathbf{0}$. The equivalence between (a) and (b) follows from the equivalence between (4.6) and (4.7).

To prove that (a) and (c) are equivalent we need to show that

$$\mathbf{G}_1\mathbf{Q}_{\mathbf{W}} = \mathbf{G}_{1\#}\mathbf{Q}_{\mathbf{W}}.$$

It is clear that $\mathbf{G}_1\mathbf{Q}_\mathbf{W} = \mathbf{0}$. Similarly, $\mathbf{G}_{1\#}\mathbf{Q}_\mathbf{W} = \mathbf{D}_{1\#}\mathbf{Q}_\mathbf{W} = \mathbf{0}$. Thus (a) is equivalent to (c). The claim (g) follows from part (b) of Proposition 3.3. □

Remark 4.2. Clearly (a) in Proposition 4.2 is equivalent to

$$\text{(i)}\ \mathbf{G}_1(\mathbf{X}_1 : \mathbf{X}_2) = \mathbf{G}_{1\#}(\mathbf{X}_1 : \mathbf{X}_2) = (\mathbf{X}_1 : \mathbf{0}) \quad \text{and} \quad \text{(ii)}\ \mathbf{G}_1\mathbf{V} = \mathbf{G}_{1\#}\mathbf{V},$$

that is, (i) $\mathbf{G}_1\mathbf{X}_2 = \mathbf{0}$ and (ii) $\mathbf{G}_1\mathbf{V} = \mathbf{G}_{1\#}\mathbf{V}$. Here is a question: where does the condition (ii) vanish in Proposition 4.2?

In view of Proposition 4.2, the condition (i) implies that $\mathbf{G}_1 = \mathbf{G}_{1\#}$, and hence trivially (ii) holds, that is, $\mathbf{G}_1\mathbf{V} = \mathbf{G}_{1\#}\mathbf{V}$. However, (ii) does not imply (i). Moreover, the condition (ii) implies that $\text{cov}(\mathbf{G}_1\mathbf{y}) = \text{cov}(\mathbf{G}_{1\#}\mathbf{y})$ which by Proposition 4.3 (see below) is equivalent to $\mathbf{X}_1'\mathbf{W}^+\mathbf{X}_2 = \mathbf{0}$. Thus we can conclude that $\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_2 = \mathbf{0} \Rightarrow \mathbf{X}_1'\mathbf{W}^+\mathbf{X}_2 = \mathbf{0}$. □

In Propositions 4.3–4.5 we assume that $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ is estimable under $\mathscr{M}_{12}$.

Proposition 4.3. *The following statements are equivalent:*

(a) $\mathbf{G}_1\mathbf{y} = \mathbf{G}_{1\#}\mathbf{y}$ *for all* $\mathbf{y} \in \mathscr{C}(\mathbf{W}_1)$, *that is,* $\mathbf{G}_1\mathbf{W}_1 = \mathbf{G}_{1\#}\mathbf{W}_1$,
(b) $\mathbf{G}_{1\#} \in \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_1}\}$, *that is,* $\mathbf{G}_{1\#}\mathbf{y} \in \{\text{BLUE}(\boldsymbol{\mu}_1|\mathscr{M}_1)\}$,
(c) $\{\text{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_{12})\} \subseteq \{\text{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_1)\}$, *that is,* $\{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_{12}}\} \subseteq \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_1}\}$,
(d) $(\mathbf{H}_1 - \mathbf{H}_{1\#})\mathbf{W}_1 = \mathbf{0}$ *for all* $\mathbf{H}_1 \in \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_1}\}$, $\mathbf{H}_{1\#} \in \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_{12}}\}$,
(e) $\mathbf{G}_{1\#}\mathbf{V}\mathbf{M}_1 = \mathbf{0}$,
(f) $\mathscr{C}(\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1) = \mathscr{C}(\mathbf{W}_1\dot{\mathbf{M}}_2\mathbf{X}_1) = \mathscr{C}(\mathbf{X}_1)$,
(g) $\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1 = \mathbf{W}_1\dot{\mathbf{M}}_2\mathbf{X}_1 = \mathbf{X}_1$,
(h) $\mathbf{X}_1'\mathbf{W}^+\mathbf{X}_2 = \mathbf{0}$,
(i) $\mathbf{G}_1\mathbf{V} = \mathbf{G}_{1\#}\mathbf{V}$,
(j) $\text{cov}(\mathbf{G}_{1\#}\mathbf{y} - \mathbf{G}_1\mathbf{y}) = \mathbf{0}$,
(k) $\text{cov}(\mathbf{G}_1\mathbf{y}) = \text{cov}(\mathbf{G}_{1\#}\mathbf{y})$.
Moreover, we always have
(l) $\text{cov}(\mathbf{G}_{1\#}\mathbf{y} - \mathbf{G}_1\mathbf{y}) = \text{cov}(\mathbf{G}_{1\#}\mathbf{y}) - \text{cov}(\mathbf{G}_1\mathbf{y})$,
(m) $\text{cov}(\mathbf{G}_1\mathbf{y}) \leq_L \text{cov}(\mathbf{G}_{1\#}\mathbf{y})$,
(n) $\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_2 = \mathbf{0} \Rightarrow \mathbf{X}_1'\mathbf{W}^+\mathbf{X}_2 = \mathbf{0}$.

Proof. It is clear that (b) is simply an alternative expression for (a) and similarly (d) for (c). The claim (a) holds if and only if

$$\mathbf{G}_1(\mathbf{X}_1 : \mathbf{V}\mathbf{M}_1) = \mathbf{G}_{1\#}(\mathbf{X}_1 : \mathbf{V}\mathbf{M}_1) = (\mathbf{X}_1 : \mathbf{0}),$$

which gives (e): $\mathbf{G}_{1\#}\ \mathbf{V}\mathbf{M}_1 = \mathbf{0}$, that is,

$$\mathbf{X}_1(\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{X}_1)^-\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{V}\mathbf{M}_1 = \mathbf{0}. \tag{4.8}$$

Premultiplying (4.8) by $\mathbf{X}_1'\dot{\mathbf{M}}_2$ yields

$$\mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{V}\mathbf{M}_1 = \mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{W}\mathbf{M}_1 = \mathbf{X}_1'\dot{\mathbf{M}}_2\mathbf{W}_1\mathbf{M}_1 = \mathbf{0},$$

that is, $\mathscr{C}(\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1) \subseteq \mathscr{C}(\mathbf{X}_1)$. In view of Proposition 3.2, we have $\text{r}(\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1) = \text{r}(\mathbf{X}_1)$ and hence $\mathscr{C}(\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1) \subseteq \mathscr{C}(\mathbf{X}_1)$ becomes

$$\mathscr{C}(\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1) = \mathscr{C}(\mathbf{W}_1\dot{\mathbf{M}}_2\mathbf{X}_1) = \mathscr{C}(\mathbf{X}_1). \tag{4.9}$$

Thus we have shown that (e) and (f) are equivalent. Equality (4.9) implies

$$\mathbf{X}_1'\mathbf{W}^+\mathbf{X}_2 = \mathbf{0}, \tag{4.10}$$

that is, (f) implies (h). In view of part (e) of Proposition 3.3 we have

$$\mathbf{W}\dot{\mathbf{M}}_2\mathbf{X}_1 = \left[\mathbf{I}_n - \mathbf{X}_2(\mathbf{X}_2'\mathbf{W}^+\mathbf{X}_2)^-\mathbf{X}_2'\mathbf{W}^+\right]\mathbf{X}_1. \tag{4.11}$$

Substituting (4.10) into (4.11) we observe that (h) implies (g), and so far we have confirmed the equivalence between (a) and any of (e)–(h).

The statement (c) holds if and only if

$$(\mathbf{G}_{1\#} + \mathbf{N}_2\mathbf{Q}_\mathbf{W})(\mathbf{X}_1 : \mathbf{VM}_1) = (\mathbf{X}_1 : \mathbf{0}) \quad \text{for all } \mathbf{N}_2 \in \mathbb{R}^{n\times n},$$

that is,

$$(\mathbf{G}_{1\#} + \mathbf{N}_2\mathbf{Q}_\mathbf{W})\mathbf{VM}_1 = \mathbf{0} \quad \text{for all } \mathbf{N}_2 \in \mathbb{R}^{n\times n},$$

which holds if and only if $\mathbf{G}_{1\#}\ \mathbf{VM}_1 = \mathbf{0}$. Thus (c) and (e) are equivalent.

The claim (a) holds if and only if $\mathbf{G}_1(\mathbf{X}_1 : \mathbf{V}) = \mathbf{G}_{1\#}(\mathbf{X}_1 : \mathbf{V})$, which is precisely (l): $\mathbf{G}_1\mathbf{V} = \mathbf{G}_{1\#}\mathbf{V}$. It is clear that (i) is equivalent to (j). Consider then

$$\text{cov}(\mathbf{G}_{1\#} - \mathbf{G}_1)\mathbf{y} = \mathbf{G}_{1\#}\mathbf{VG}_{1\#}' + \mathbf{G}_1\mathbf{VG}_1' - \mathbf{G}_{1\#}\mathbf{VG}_1' - \mathbf{G}_1\mathbf{VG}_{1\#}'.$$

Notice that $\mathbf{G}_1\mathbf{T}_1 = \mathbf{G}_{1\#}\mathbf{T}_1 = \mathbf{T}_1$, where $\mathbf{T}_1 = \mathbf{X}_1\mathbf{U}_1\mathbf{U}_1'\mathbf{X}_1'$ and hence

$$\begin{aligned}\mathbf{G}_1\mathbf{VG}_1' &= \mathbf{G}_1(\mathbf{W}_1 - \mathbf{T}_1)\mathbf{G}_1' = \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{G}_1' - \mathbf{T}_1\\ &= \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_1)^-\mathbf{X}_1' - \mathbf{T}_1,\end{aligned}$$

and

$$\begin{aligned}\mathbf{G}_1\mathbf{VG}_{1\#}' &= \mathbf{G}_1(\mathbf{W}_1 - \mathbf{T}_1)\mathbf{G}_{1\#}' = \mathbf{G}_1\mathbf{W}_1\mathbf{G}_{1\#}' - \mathbf{T}_1\\ &= \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^-\mathbf{X}_1)^-\mathbf{X}_1'\mathbf{G}_{1\#}' - \mathbf{T}_1\\ &= \mathbf{X}_1(\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_1)^-\mathbf{X}_1' - \mathbf{T}_1 = \mathbf{G}_1\mathbf{VG}_1'.\end{aligned}$$

Thus $\text{cov}(\mathbf{G}_{1\#}\mathbf{y} - \mathbf{G}_1\mathbf{y}) = \text{cov}(\mathbf{G}_{1\#}\mathbf{y}) - \text{cov}(\mathbf{G}_1\mathbf{y})$, and so (l) and (m) hold. Statement (l) obviously confirms the equivalence between (j) and (k). Property (n) is obvious. See also Remark 4.1. □

Next we consider the condition under which an arbitrary matrix from the set $\{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_1}\}$ provides the BLUE for $\boldsymbol{\mu}_1$ under $\mathscr{M}_{12}$.

Proposition 4.4. *The following statements are equivalent:*

(a) $\{\text{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_1)\} \subseteq \{\text{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_{12})\}$, *that is,* $\{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_1}\} \subseteq \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_{12}}\}$,
(b) $(\mathbf{H}_1 - \mathbf{H}_{1\#})\mathbf{W} = \mathbf{0}$ *for all* $\mathbf{H}_1 \in \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_1}\}$, $\mathbf{H}_{1\#} \in \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_{12}}\}$,
(c) $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{VM}_1)$, *that is,* $\mathscr{C}(\mathbf{W}_1) = \mathscr{C}(\mathbf{W})$, *and* $\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_2 = \mathbf{0}$,
(d) $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{VM}_1)$,
(e) $\mathscr{C}(\mathbf{X}_2 : \mathbf{VM}) \subseteq \mathscr{C}(\mathbf{VM}_1)$,
(f) $\{\text{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_1)\} = \{\text{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_{12})\}$, *that is,* $\{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_1}\} = \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_{12}}\}$,
(g) $\mathscr{C}(\mathbf{X}_2 : \mathbf{VM}) = \mathscr{C}(\mathbf{VM}_1)$.

Proof. Notice first that (b) is simply an alternative way to express (a). The statement (a) holds if and only if

$$(\mathbf{G}_1 + \mathbf{N}_1\mathbf{Q}_{\mathbf{W}_1})(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{VM}) = (\mathbf{X}_1 : \mathbf{0} : \mathbf{0}) \quad \text{for all } \mathbf{N}_1 \in \mathbb{R}^{n\times n},$$

that is,

$$(\mathbf{G}_1 + \mathbf{N}_1\mathbf{Q}_{\mathbf{W}_1})\mathbf{X}_2 = \mathbf{0} \quad \text{for all } \mathbf{N}_1 \in \mathbb{R}^{n\times n},$$

which holds if and only if $\mathbf{Q}_{\mathbf{W}_1}\mathbf{X}_2 = \mathbf{0}$ and $\mathbf{G}_1\mathbf{X}_2 = \mathbf{0}$, which is precisely (c). Moreover, (c) implies that

$$\mathbf{X}_2 = \mathbf{X}_1\mathbf{A} + \mathbf{V}\mathbf{M}_1\mathbf{B} \tag{4.12}$$

for some $\mathbf{A}$ and $\mathbf{B}$ and

$$\mathbf{X}_1'\mathbf{W}_1^+(\mathbf{X}_1\mathbf{A} + \mathbf{V}\mathbf{M}_1\mathbf{B}) = \mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_1\mathbf{A} = \mathbf{0}. \tag{4.13}$$

Now (4.13) implies that $\mathbf{W}_1^+\mathbf{X}_1\mathbf{A} = \mathbf{0}$, which further implies that $\mathbf{X}_1\mathbf{A} = \mathbf{0}$, so that by (4.12) we get (d). The claim (d) obviously implies (c). The equivalence between (d) and (e) is obvious because $\mathscr{C}(\mathbf{VM}) \subseteq \mathscr{C}(\mathbf{VM}_1)$.

It is clear that (f) implies (b). Thus to confirm the equivalence of (b) and (f) we have to show that

$$\text{(b)} \Rightarrow \{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_{12})\} \subseteq \{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_1)\}. \tag{4.14}$$

This follows at once from Proposition 4.3 by noting that the right-hand side of (4.14) means that $(\mathbf{H}_1 - \mathbf{H}_{1\#})\mathbf{W}_1 = \mathbf{0}$. The equivalence between (f) and (g) follows by combining part (d) of Proposition 4.4 and (k) of Proposition 4.3. □

Our next task is to find necessary and sufficient conditions for

$$\mathbf{G}_1\mathbf{y} = \mathbf{G}_{1\#}\mathbf{y} \quad \text{for all } \mathbf{y} \in \mathscr{C}(\mathbf{W})$$

when the inclusion $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V})$ holds.

Proposition 4.5. *Consider the models $\mathscr{M}_{12}$ and $\mathscr{M}_1$ and suppose that*

$$\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{X}_1 : \mathbf{V}) = \mathscr{C}(\mathbf{W}_1), \ \textit{i.e.}, \ \mathscr{C}(\mathbf{W}_1) = \mathscr{C}(\mathbf{W}). \tag{4.15}$$

Then the following statements are equivalent:

(a) $\mathbf{G}_1\mathbf{W}_1 = \mathbf{G}_{1\#}\mathbf{W}_1$,
(b) $\mathbf{H}_1\mathbf{W}_1 = \mathbf{H}_{1\#}\mathbf{W}_1$ *for all* $\mathbf{H}_1$ *and* $\mathbf{H}_{1\#}$,
(c) $\{\mathrm{BLUE}(\boldsymbol{\mu}_1|\mathscr{M}_{12})\} \subseteq \{\mathrm{BLUE}(\boldsymbol{\mu}_1|\mathscr{M}_1)\}$, *that is,* $\{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_{12}}\} \subseteq \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_1}\}$,
(d) $\{\mathrm{BLUE}(\boldsymbol{\mu}_1|\mathscr{M}_1)\} \subseteq \{\mathrm{BLUE}(\boldsymbol{\mu}_1|\mathscr{M}_{12})\}$, *that is,* $\{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_1}\} \subseteq \{\mathbf{P}_{\boldsymbol{\mu}_1|\mathscr{M}_{12}}\}$,
(e) $\mathrm{BLUE}(\boldsymbol{\mu}_1|\mathscr{M}_1) = \mathrm{BLUE}(\boldsymbol{\mu}_1|\mathscr{M}_{12})$ *with probability* 1,
(f) $\mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_2 = \mathbf{0}$,
(g) $\mathscr{C}(\mathbf{X}_2) \subseteq \mathscr{C}(\mathbf{VM}_1)$,
(h) $\mathbf{X}_1\mathbf{C}^{12}\mathbf{X}_2' = \mathbf{0}$, *where* $\mathbf{C}^{12}$ *is defined as*

$$(\mathbf{X}'\mathbf{W}_1^+\mathbf{X})^+ = \begin{pmatrix} \mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_1 & \mathbf{X}_1'\mathbf{W}_1^+\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{W}_1^+\mathbf{X}_1 & \mathbf{X}_2'\mathbf{W}_1^+\mathbf{X}_2 \end{pmatrix}^+ = \begin{pmatrix} \mathbf{C}^{11} & \mathbf{C}^{12} \\ \mathbf{C}^{21} & \mathbf{C}^{22} \end{pmatrix}. \tag{4.16}$$

Proof. The equivalence between (a)–(g) is obvious. Consider then part (h). Now we have

$$\mathbf{D}_{1\#} = (\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}^-\mathbf{X})^-\mathbf{X}'\mathbf{W}^+ = (\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}_1^-\mathbf{X})^-\mathbf{X}'\mathbf{W}_1^+. \tag{4.17}$$

Hence (a) holds, under (4.15), if and only if $\mathbf{G}_1\mathbf{W}_1 = \mathbf{D}_{1\#}\mathbf{W}_1$, that is,

$$\mathbf{G}_1(\mathbf{X}_1 : \mathbf{VM}_1) = \mathbf{D}_{1\#}(\mathbf{X}_1 : \mathbf{VM}_1) = (\mathbf{X}_1 : \mathbf{0}),$$

that is,

$$\mathbf{D}_{1\#}\mathbf{VM}_1 = \mathbf{D}_{1\#}\mathbf{W}_1\mathbf{M}_1 = \mathbf{0}. \tag{4.18}$$

Using (4.17) the equality (4.18) becomes

$$\begin{aligned}
\mathbf{D}_{1\#}\mathbf{VM}_1 &= (\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}_1^-\mathbf{X})^-\mathbf{X}'\mathbf{W}_1^+\mathbf{W}_1\mathbf{M}_1 \\
&= (\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}_1^-\mathbf{X})^-\mathbf{X}'\mathbf{P}_{\mathbf{W}_1}\mathbf{M}_1 \\
&= (\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}_1^-\mathbf{X})^-\mathbf{X}'\mathbf{M}_1 \\
&= (\mathbf{X}_1 : \mathbf{0})(\mathbf{X}'\mathbf{W}_1^-\mathbf{X})^-(\mathbf{0} : \mathbf{M}_1\mathbf{X}_2)' \\
&= \mathbf{X}_1\mathbf{C}^{12}\mathbf{X}_2'\mathbf{M}_1 = \mathbf{0},
\end{aligned} \tag{4.19}$$

where $\mathbf{C}^{12}$ is defined in (4.16). In light of $\mathrm{r}(\mathbf{X}_2'\mathbf{M}_1) = \mathrm{r}(\mathbf{X}_2)$, we can cancel $\mathbf{M}_1$ in the last expression in (4.19). This proves the equivalence between (a) and (h). □

5. Conclusions

In this article we consider the partitioned linear model $\mathscr{M}_{12} = \{\mathbf{y},\ \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2,\ \mathbf{V}\}$ and the corresponding small model $\mathscr{M}_1 = \{\mathbf{y},\ \mathbf{X}_1\boldsymbol{\beta}_1,\ \mathbf{V}\}$. We focus on comparing the BLUEs of $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ under $\mathscr{M}_{12}$ and $\mathscr{M}_1$. The observed numerical value of the BLUE is unique under the model $\mathscr{M}_1$ if the $\mathscr{M}_1$ is consistent in the sense that $\mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{V})$ and the same uniqueness concerns the full model in the respective way. But now there may be some problems if we write

$$\mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{M}_1) = \mathrm{BLUE}(\mathbf{X}_1\boldsymbol{\beta}_1 \mid \mathscr{M}_{12}). \tag{5.1}$$

What is the meaning of the above equality? It is not fully clear because we know that under $\mathscr{M}_1$ the values of $\mathbf{y}$ vary over $\mathscr{C}(\mathbf{X}_1 : \mathbf{V})$ but under $\mathscr{M}_{12}$ the values of $\mathbf{y}$ vary over $\mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$ and these column spaces may be different. However, if $\mathscr{C}(\mathbf{X}_1 : \mathbf{V}) = \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$ there is no difficulties to interpret the equality (5.1), which means that

$$\mathbf{Ay} = \mathbf{By} \quad \text{for all } \mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{V}),$$

where $\mathbf{Ay} \in \{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_1)\}$ and $\mathbf{By} \in \{\mathrm{BLUE}(\boldsymbol{\mu}_1 \mid \mathscr{M}_{12})\}$.

We consider the resulting problems by picking up particular fixed expressions for the BLUE s of $\boldsymbol{\mu}_1 = \mathbf{X}_1\boldsymbol{\beta}_1$ under these two models, and study the conditions under which they are equal for all values of $\mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{V})$ or $\mathbf{y} \in \mathscr{C}(\mathbf{X}_1 : \mathbf{V})$. Moreover, we review the conditions under which *all* representations of the BLUE s in one model continue to be valid in the other model. Some related considerations, using different approach, have been made by Lu et al. (2015), Tian (2013), and Tian and Zhang (2016).

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References

Groß, J., and S. Puntanen. 2000. Estimation under a general partitioned linear model. *Linear Algebra and Its Applications* 321, 131–44. doi:10.1016/S0024-3795(00)00028-8.

Güler, N., S. Puntanen, and H. Özdemir. 2014. On the BLUEs in two linear models via C.R. Rao's Pandora's Box. *Communications in Statistics - Theory and Methods*, 5, 43, 921–31. doi:10.1080/03610926.2013.826366.

Haslett, S. J. 1996. Updating linear models with dependent errors to include additional data and/or parameters. *Linear Algebra and Its Applications* 237(238):329–49.

Haslett, S. J., A. Markiewicz, and S. Puntanen. 2020. Properties of BLUEs and BLUPs in full vs. small linear models with new observations. In *Recent developments in multivariate and random matrix analysis: Festschrift in honour of Dietrich von Rosen*, eds. T. Holgersson and M. Singull, 123–46. Cham: Springer.

Haslett, S. J., and S. Puntanen. 2010. Effect of adding regressors on the equality of the BLUEs under two linear models. *Journal of Statistical Planning and Inference* 140, 104–10. doi:10.1016/j.jspi.2009.06.010.

Lu, C., S. Gan, and Y. Tian. 2015. Some remarks on general linear model with new regressors. *Statistics & Probability Letters*, 97, 16–24. doi:10.1016/j.spl.2014.10.015.

Markiewicz, A, and S. Puntanen. 2019. Further properties of the linear sufficiency in the partitioned linear model. In *Matrices, statistics and big data*, eds. S. E. Ahmed, F. Carvalho and S. Puntanen, 1–22. Cham: Springer.

Marsaglia, G, and G. P. H. Styan. 1974. Equalities and inequalities for ranks of matrices. *Linear Multilinear Algebra* 2:269–92.

Mitra, S. K, and B. J. Moore. 1973. Gauss–Markov estimation with an incorrect dispersion matrix. *Sankhyā Series A* 35:139–52.

Puntanen, S., G. P. H. Styan, and J. Isotalo. 2011. *Matrix tricks for linear statistical models: our personal top twenty*. Heidelberg: Springer.

Rao, C. R. 1973. Representations of best linear estimators in the Gauss–Markoff model with a singular dispersion matrix. *Journal of Multivariate Analysis* 3, 276–92. doi:10.1016/0047-259X(73)90042-0.

Rao, C. R. 1974. Projectors, generalized inverses and the BLUE's. *Journal of the Royal Statistical Society: Series B* 36:442–8.

Rao, C. R, and S. K. Mitra. 1971. *Generalized inverse of matrices and its applications*. New York: Wiley.

Sengupta, D, and S. R. Jammalamadaka. 2003. *Linear models: An integrated approach*. River Edge: World Scientific.

Tian, Y. 2013. On properties of BLUEs under general linear regression models. *Journal of Statistical Planning and Inference* 43, 771–82. doi:10.1016/j.jspi.2012.10.005.

Tian, Y., and X. Zhang. 2016. On connections among OLSEs and BLUEs of whole and partial parameters under a general linear model. *Statistics & Probability Letters* 112, 105–12. doi:10.1016/j.spl.2016.01.019.

Werner, H.J., and C. Yapar. 1996. A BLUE decomposition in the general linear regression model. *Linear Algebra and its Applications* 237-238, 395–404. doi:10.1016/0024-3795(95)00542-0.