Tampere University

MARKUS KLEMETTI

Generalized Persistence and Graded Structures

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Responsible supervisor and Custos	Professor Eero Hyry Tampere University Finland	
Pre-examiners	Professor Peter Bubenik University of Florida USA	Professor Kevin Knudson University of Florida USA
Opponent	Professor Ran Levi University of Aberdeen UK	

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ABSTRACT

The correspondence theorem of Carlsson and Zomorodian, which states that one can view persistence modules as modules over a polynomial ring of one variable, opened the graded perspective in topological data analysis. In this thesis, we want to propose a new generic theoretical framework for understanding generalized persistence modules from this perspective by considering monoid actions on preordered sets. Secondly, in the case the indexing set is a poset, we introduce a new tameness condition for a generalized persistence module by defining the notion of S-determinacy, where S is a subposet containing all the 'births' and the 'deaths'.

We first focus on the correspondence between generalized persistence modules and graded modules in the case the indexing set has a monoid action. We introduce the notion of an action category over a monoid graded ring. We show that the category of additive functors from this category to the category of Abelian groups is isomorphic to the category of modules graded over the set with a monoid action, and to the category of unital modules over a certain smash product.

In the case S is finite, our notion of S-determinacy leads to a new characterization for a generalized persistence module being finitely presented. Moreover, we show that after adding 'infinitary points' to $\mathbb{Z}^n$, 'S-determined' is equivalent to 'finitely determined' as defined by Miller.

PREFACE

Returning home after a late military service, I had to make a big decision of whether I should pursue a doctorate or some other career. The answer came quite naturally (and without too much thought), when I was offered a full-time position of a doctoral student at the then University of Tampere. Back then, topological data analysis, which is the subject of this dissertation, was a brand new research area for everyone at the faculty, so there was some groundwork to cover. Unbeknownst to me at the time, the years writing this dissertation would be filled not only with learning experiences, but also with some highs and lows of life. It feels like an understatement, but getting to this point was not easy, so I would like to acknowledge some of the people that made it possible.

First of all, I want to thank my supervisor, Professor Eero Hyry, for his continued support and guidance during the process, as well as for his firm belief in my capability to finalize this dissertation. For all these years it has been a pleasure to work with and to learn from Eero. His expertise has been invaluable. I also wish to thank the pre-examiners of this dissertation, Professor Peter Bubenik and Professor Kevin Knudson, for their good work and inspiring commentary, and Professor Ran Levi for agreeing to be the opponent in my thesis defence.

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Lastly, I thank my family and friends, who have tried their best to keep me sane during this time, and of course my girlfriend Heidi. Kiitos!

CONTENTS

Introduction 9

1 Preliminaries 17
- 1.1 Basic properties of RC-modules 17
- 1.2 Additive and R-linear functors 18
- 1.3 Order theory 20
- 1.4 Colimits and limits 21
- 1.5 Finiteness conditions 24
- 1.6 Kan extensions 26

2 Modules over a monoid act 31
- 2.1 Monoid action 31
- 2.2 Automorphic actions 33
- 2.3 Action categories over a graded ring 37
- 2.4 Category algebras and smash products 40
- 2.5 Kan extensions and action categories 44

3 Finitely presented RC-modules 49
- 3.1 S-presented and S-generated RC-modules 49
- 3.2 Births and deaths relative to S 54
- 3.3 S-splitting 56
- 3.4 Minimality of births and deaths 60

4 Presentations with finite support 65
- 4.1 S-determined RC-modules 65
- 4.2 Minimal upper bounds 66
- 4.3 Finitely presented RC-modules in mub-complete posets 68
- 4.4 Pointwise stabilizing direct systems 70

5 Strongly bounded posets 79
5.1 Modules over strongly bounded posets 79
5.2 Modules determined by cartesian sets 82
5.3 Finitely determined modules 87

Bibliography 95

INTRODUCTION

One of the main methods of topological data analysis is persistent homology. In the simplest case, data is encoded in an increasing nested sequence

$$\emptyset = X_0 \subseteq X_1 \subset \ldots \subseteq X_r = X$$

of simplicial complexes. This filtration reflects the topological and geometric structure of the data at different scales. By taking homology with coefficients in a field k, one obtains for every i a sequence of vector spaces and linear maps

$$0 = H_i(X_0) \to H_i(X_1) \to \ldots \to H_i(X_r) = H_i(X).$$

This sequence is called a persistence module. Intuitively, persistent homology is a tool to track down how topological features are born and die throughout the filtration. If a non-zero homology class in $H_i(X_j)$ is not in the image of $H_i(X_{j-1})$, it is said to be born at the step j of the filtration. It dies at the step $j+1$ if its image in $H_i(X_{j+1})$ is zero. Otherwise, the homology class is said to persist. Carlsson and Zomorodian [33, p. 259, Thm. 3.1 (Correspondence)] realized that one can view persistence modules as modules over a polynomial ring of one variable. The above persistence module then corresponds to the graded $k[x]$-module

$$M := \sum_{j \geq 0} H_i(X_j).$$

The variable x acts on M by means of the maps $H_i(X_j) \to H_i(X_{j+1})$. One can now use the structure theorem of finitely generated modules over a principal ideal domain to observe that we have the decomposition

$$M = \bigoplus_i x^{a_i} k[x] / x^{b_i} k[x] \oplus \bigoplus_i x^{c_i} k[x],$$

where $a_i, b_i, c_i \in \mathbb{N}$ and $a_i < b_i$ The numbers a_i and b_i record the birth and death of a homology class, respectively, whereas the homology class born at the step c_i lives forever. The intervals $[a_i, b_i[$ and $[c_i, \infty[$ thus represent homological properties that span over a certain range of scales. The collection of these intervals is called a barcode, and it is a complete and discrete invariant for persistence modules ([5]).

Considering filtrations indexed by $\mathbb{N}^n$ leads to the so called multipersistence. In [5, p. 78, Thm. 1], Carlsson and Zomorodian showed that multipersistence modules now correspond to modules over a polynomial ring of n variables. More generally, one can start from a filtration of a topological space indexed by a preordered set. However, the resulting generalized persistence modules do not necessarily have an immediate expression as a module over a graded ring.

The correspondences by Carlsson and Zomorodian opened the graded perspective in topological data analysis, leading many researchers to utilize graded module theory in their investigations (see, for example, [6], [19], [4], [12], [14], [16], [29]). The most general cases of modules over a ring in this line of research are modules graded over Abelian groups with monoids as their positive cones, and modules canonically graded over cancellative monoids. In this thesis, we want to propose a new generic theoretical framework for understanding generalized persistence modules under the lens of graded algebra by considering monoid actions on preordered sets. Secondly, we want to investigate finitely presented generalized persistence modules. In particular, we will give a certain subclass of preordered sets over which finite presentation can be characterized by a suitable 'tameness' condition.

We now want to explain this in more detail. Using the language of category theory, it is convenient to define a generalized persistence module as a functor from a preordered set P to the category of k-vector spaces, where k is a field. In representation theory, given a commutative ring R and a small category $\mathcal{C}$, a functor $\mathcal{C} \to R\text{-}\mathbf{Mod}$ is called an $R\mathcal{C}$-module. In this terminology, a generalized persistence module is then a kP-vector space. Following Mitchell ([20]), we also regard a small preadditive category $\mathcal{A}$ as a 'ring with several objects', and an additive functor $\mathcal{A} \to \mathbf{Ab}$ as an $\mathcal{A}$-module. The $R\mathcal{C}$-modules may then be seen as modules over the linearization $R\mathcal{C}$, where $R\mathcal{C}$ is a preadditive category with the same objects as $\mathcal{C}$ and morphisms $R[\mathrm{Mor}_{\mathcal{C}}(c, d)]$, where $c, d \in \mathrm{Ob}\,\mathcal{C}$ (for any set S, we denote by $R[S]$ the free R-module generated by S).

Suppose now that G is a monoid. To any G-act (or G-set) A, we can associate an action category $G\int A$, whose objects are the elements of A, and for any $a, b \in A$ the morphisms $a \to b$ are pairs (a, g), where $g \in G$ with $b = ga$. It is easy to see that the category of $R(G\int A)$-modules is now equivalent to the category of A-graded $R[G]$-modules. Note that if the action of G on A is free, then simply $G\int A \cong A$, where A is the thin category whose morphisms are given by G. Moreover, if in this situation A is connected, G is both commutative and cancellative, and the action is automorphic, then A is in fact isomorphic to the Grothendieck group G^{gp} (see Corollary 2.8).

Given any G-graded ring S, this leads us is to investigate the relationship between A-graded S-modules and modules over A in general. We define the action category over S, denoted by $G\int_S A$, with objects A and morphisms

$$s \in \bigoplus_{g \in G,\ ga=b} S_g,$$

where $a, b \in A$. In the case $S = R[G]$, $G\int_S A$ is just the linearization of $G\int A$. Our first main result, Theorem 2.13, then says that the categories of A-graded S-modules and $G\int_S A$-modules are isomorphic.

We can also look at the category algebra $R[G\int A]$. If $\mathcal{C}$ is any category, then the category algebra $R[\mathcal{C}]$ is defined as the free R-module with a basis consisting of the morphisms of $\mathcal{C}$, and the product of two basis elements is given by their composition, if defined, and is zero otherwise. It now turns out in Proposition 2.18 that the category algebra $R[G\int A]$ coincides with the smash product $R[G]\#A$, which has been much studied in ring theory (see [22]). This leads us to Theorem 2.20, where we identify $G\int_S A$-modules with the category of unital $S\#A$-modules i.e. the category of $S\#A$-modules M with $M = (S\#A)M$.

A morphism of G-acts $\varphi\colon A \to B$ defines an obvious functor of action categories $\varphi\colon G\int A \to G\int B$. The restriction functor res_φ from the category of $R(G\int B)$-modules to the category of $R(G\int A)$-modules has the left adjoint ind_φ, the induction functor, and the right adjoint coind_φ, the coinduction functor, to the opposite direction. In Proposition 2.23 and Proposition 2.24 we examine the reindexing of $R(G\int A)$-modules and $R(G\int B)$-modules by means of the functors ind_φ and coind_φ.

We then turn to consider finitely presented generalized persistence modules. Note that being finitely presented is a categorical property, so an equivalence between generalized persistence modules and graded modules preserves this property. Recall first that an $R\mathcal{C}$-module M is finitely presented if there exists an exact sequence

$$\bigoplus_{j\in J} R[\mathrm{Mor}_{\mathcal{C}}(d_j,-)] \to \bigoplus_{i\in I} R[\mathrm{Mor}_{\mathcal{C}}(c_i,-)] \to M \to 0,$$

where I and J are finite sets, and $c_i, d_j \in \mathcal{C}$ for all $i \in I, j \in J$. We will look at posets $\mathcal{C}$ which are weakly bounded from above and mub-complete. By 'weakly bounded' we mean that every finite subset $S \subseteq \mathcal{C}$ has a finite number of minimal upper bounds in $\mathcal{C}$, whereas $\mathcal{C}$ is mub-complete if given a finite non-empty subset $S \subseteq \mathcal{C}$ and an upper bound c of S, there exists a minimal upper bound s of S such that $s \leq c$. In our Theorem 4.15, we characterize finitely presented generalized persistence modules in this situation. More precisely, we can show that an $R\mathcal{C}$-module M is finitely presented if and only if the R-modules $M(c)$ are finitely presented for all $c \in \mathcal{C}$, and M is S-determined for some finite set $S \subseteq \mathcal{C}$.

Given $S \subseteq \mathcal{C}$, we call an $R\mathcal{C}$-module M S-determined if $\mathrm{Supp}(M) \subseteq {\uparrow}S$ and the implication

$$S \cap {\downarrow}c = S \cap {\downarrow}d \;\Rightarrow\; \text{the morphism } M(c \leq d) \text{ is an isomorphism}$$

holds for every $c \leq d$ in $\mathcal{C}$. Here $\mathrm{Supp}(M) := \{c \in \mathcal{C} \mid M(c) \neq 0\}$ denotes the support of M, and for any $T \subseteq \mathcal{C}$, we use the usual notations

$${\uparrow}T := \{c \in \mathcal{C} \mid t \leq c \text{ for some } t \in T\}$$

and

$${\downarrow}T := \{c \in \mathcal{C} \mid c \leq t \text{ for some } t \in T\}$$

for the upset generated and the downset cogenerated by T, respectively. Our intuition for this definition comes from topological data analysis, where one tracks how the elements of each $M(c)$ evolve in the morphisms $M(c \leq c')$ $(c, c' \in \mathcal{C})$. One says that an element $m \in M(c)$ is born at c if it is not in the image of any morphism $M(c' \leq c)$, where $c' < c$, and dies at c'' if $M(c \leq c'')(m) = 0$ and $M(c \leq c')(m) \neq 0$ for all $c \leq c' < c''$. Suppose that there exists a set S such that all births and deaths occur inside S. The condition $S \cap {\downarrow}c = S \cap {\downarrow}d$ then implies that looking down from both c and d, we see the same deaths and births. In particular, the morphism $M(c \leq d)$ must be an isomorphism.

Our proof for Theorem 4.15 starts from the fact that an $R\mathcal{C}$-module M is finitely presented if and only if the R-modules $M(c)$ are finitely presented for all $c \in \mathcal{C}$ and M is S-presented for some finite subset $S \subseteq \mathcal{C}$. Here S-presented means the existence of a set $S \subseteq \mathcal{C}$ and an exact sequence of the type

$$\bigoplus_{s\in S} B_s[\mathrm{Mor}_{\mathcal{C}}(s,-)] \to \bigoplus_{s\in S} A_s[\mathrm{Mor}_{\mathcal{C}}(s,-)] \to M \to 0,$$

where A_s and B_s are R-modules for all $s \in S$. It is easily seen that if M is S-presented, then M is S-determined. We denote the set of minimal upper bounds of non-empty subsets of a finite set $S \subseteq \mathcal{C}$ by

$$\hat{S} := \bigcup_{\emptyset \neq S' \subseteq S} \mathrm{mub}_{\mathcal{C}}(S').$$

In Corollary 4.13 we now make the crucial observation that M is $\hat{\hat{S}}$-presented if $S \subseteq \mathcal{C}$ is a finite set such that M is S-determined.

As a useful tool we introduce the sets of births and and deaths relative to S by

$$B_S(M) := \{c \in \mathcal{C} \mid \operatorname*{colim}_{s<c,\ s\in S} M(s) \to M(c) \text{ is a non-epimorphism}\}$$

and

$$D_S(M) := \{c \in \mathcal{C} \mid \operatorname*{colim}_{s<c,\ s\in S} M(s) \to M(c) \text{ is a non-monomorphism}\}.$$

An $R\mathcal{C}$-module M is known to be S-presented if and only if the natural morphism $\mathrm{ind}_S \operatorname{res}_S M \to M$ is an isomorphism. Here res_S denotes the restriction functor from the category of $R\mathcal{C}$-modules to the category of RS-modules, and ind_S its left adjoint, the induction functor. Note the pointwise formula

$$(\mathrm{ind}_S \operatorname{res}_S M)(c) = \operatorname*{colim}_{s<c,\ s\in S} M(s)$$

for all $c \in \mathcal{C}$. We observe in Proposition 3.9 that the module M is S-presented if and only if $B_S(M) \cup D_S(M) \subseteq S$. Interestingly, if S is Artinian, then $B_S(M) \cup D_S(M)$ is the minimal subset $T \subseteq S$ such that M is T-presented (see Proposition 3.25). Suppose that $\mathcal{C} = \mathbb{Z}^n$, $R = k$ is a field and

$$0 \to L \to N \xrightarrow{f} M \to 0,$$

is an exact sequence, where N is a free module and f a minimal epimorphism. In this case our Theorem 3.28 says that $D_S(M) = B_S(L)$ confirming the intuition that deaths should correspond to 'relations'.

Since the category $R\mathcal{C}$-**Mod** of $R\mathcal{C}$-modules is a locally finitely generated Grothendieck category, we know that an $R\mathcal{C}$-module M is finitely presented if and only if the functor $\mathrm{Hom}_{R\mathcal{C}}(M, -)$ preserves colimits of direct systems. One says that a direct system $(M_i)_{i\in I}$ is pointwise stabilizing if for all $c \in \mathcal{C}$ there exists $i_c \in I$ such that

$$i_c \le i \le j \;\Rightarrow\; \varphi_{ij}\colon M_i(c) \to M_j(c) \text{ is an isomorphism.}$$

We prove in Proposition 4.25 that M is S-presented for some finite subset $S \subseteq \mathcal{C}$ if and only if $\mathrm{Hom}_{R\mathcal{C}}(M, -)$ preserves colimits of pointwise stabilizing direct systems. This result is due to Djament, but is given without a proof in [9, p. 14, Remarque 2.15].

This thesis unifies several earlier results. In the context of topological data analysis, monoid actions have been considered by Bubenik et al. in their article [3], where they looked at the action on any preordered set given by the monoid of its translations. In the article [7] of de Silva et al., an indexing category with an additional structure of a $[0, \infty)$-action is called a category with a coherent flow. Recently, Bubenik and Milicevic considered modules graded over Abelian groups with monoids as their positive cones ([4]).

We have in particular been motivated by the article [6] of Corbet and Kerber, who generalized the result of Carlsson and Zomorodian to the case where the indexing set is a so called good monoid. We point out that if G is a monoid, then RG-modules of finitely presented type of Corbet and Kerber ([6, p. 19, Def. 15]) are the same thing as finitely presented RG-modules. The set $\hat{S}$ is a framing set in the sense of [6, p. 19, Def. 15].

Our sets of births and and deaths relative to S are related to the invariants ξ_0 and ξ_1 studied by Carlsson and Zomorodian in [5], and also by Knudson in [14]. For a finitely generated $\mathbb{Z}^n$-graded $k[X_1, \ldots, X_n]$-module M, the invariants $\xi_0(M)$ and $\xi_1(M)$ are multisets indicating the degrees of minimal generators and minimal relations of M equipped with the multiplicities they occur. The underlying sets of $\xi_0(M)$ and $\xi_1(M)$ are now $B_S(M)$ and $D_S(M)$.

Miller introduces in [19, p. 31, Def. 5.1] another notion of tameness. He defines an $R\mathbb{Z}^n$-module M to be *finitely determined*, if there is a closed interval $[a, b] \subseteq \mathbb{Z}^n$

such that the morphisms $M(c \leq c+e_i)$ are isomorphisms for all $i = 1, \ldots, n$ whenever c_i lies outside $[a_i, b_i]$. Here $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{Z}^n$. It is obvious that if M is $[a, b]$-determined, then M is finitely determined with respect to the interval $[a - u, b]$, where $u = (1, \ldots, 1)$. In general, finitely determined modules do not, of course, fill the requirement that $\mathrm{Supp}(M) \subseteq \uparrow S$ for some finite set S. However, we can save the situation by adding some infinitary points. This idea is due to Perling (see [23, pp. 16-19, Ch. 3.1]). Set $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty\}$. It is easy to see that $\overline{\mathbb{Z}}^n$ inherits the poset structure from $\mathbb{Z}^n$. Any $R\mathbb{Z}^n$-module M may be naturally extended to an $R\overline{\mathbb{Z}}^n$-module $\overline{M}$ by setting

$$\overline{M}(c) = \lim_{d \geq c,\ d \in \mathbb{Z}^n} M(d)$$

for all $c \in \overline{\mathbb{Z}}^n$. In Theorem 5.19, we show that an $R\mathbb{Z}^n$-module M is finitely determined if and only if $\overline{M}$ is S-determined for some finite $S \subseteq \overline{\mathbb{Z}}^n$. We also show that the notion of an S-determined module M is compatible with that of an M-admissible poset S defined in [23, p. 18, Def. 3.4].

1 PRELIMINARIES

Throughout this thesis, let $\mathcal{C}$ be a small category, R a commutative ring (with unit), and G a monoid. If S is a set, we use the notation $R[S]$ for the free R-module with the basis S. In particular, we can write the elements of $R[S]$ uniquely in the form $\sum_{s\in S} r_s e_s$, where $\{e_s \mid s \in S\}$ is the basis of $R[S]$.

1.1 Basic properties of $R\mathcal{C}$-modules

We shall assume that the reader is familiar with the basics of category theory. For a more detailed reference, see, for example, [18].

A functor from the small category $\mathcal{C}$ to the category R-**Mod** of R-modules is called an *$R\mathcal{C}$-module*. A morphism between two $R\mathcal{C}$-modules M and N is a natural transformation $\mu\colon M \to N$. More explicitly, a morphism of $R\mathcal{C}$-modules $\mu\colon M \to N$ is a collection of R-homomorphisms $\mu_c\colon M(c) \to N(c)$, where $c \in \mathcal{C}$, such that the diagram

$$\begin{array}{ccc} M(c) & \xrightarrow{M(u)} & M(d) \\ {\scriptstyle \mu_c}\downarrow & & \downarrow{\scriptstyle \mu_d} \\ N(c) & \xrightarrow{N(u)} & N(d) \end{array}$$

commutes for all morphisms $u\colon c \to d$ in $\mathcal{C}$. If M is an $R\mathcal{C}$-module, then the *support of M* is the set

$$\mathrm{Supp}(M) := \{c \in \mathcal{C} \mid M(c) \neq 0\}.$$

The category $R\mathcal{C}$-**Mod** of $R\mathcal{C}$-modules is an Abelian category with kernels, images, products and coproducts computed objectwise. For example, if $\mu\colon M \to N$ is a morphism of $R\mathcal{C}$-modules, then $\mathrm{Ker}\,\mu$ is defined by $(\mathrm{Ker}\,\mu)(c) = \mathrm{Ker}\,\mu_c$ for all $c \in \mathcal{C}$.

Let I be a collection of objects of $\mathcal{C}$. Such a collection gives rise to a functor $I\colon \mathrm{Ob}\,\mathcal{C} \to$ **Set**, which we call an *$\mathrm{Ob}\,\mathcal{C}$-set*. Here $\mathrm{Ob}\,\mathcal{C}$ is considered to be the category with the same objects as $\mathcal{C}$ and with identities being the only morphisms.

More precisely, for an object $c \in \mathcal{C}$, the set $I(c)$ has the same cardinality as the set of elements in I that are equal to c. A morphism $\lambda\colon I \to J$ of $\mathrm{Ob}\,\mathcal{C}$-sets is just a set of functions $I(c) \to J(c)$, where $c \in \mathcal{C}$. We denote by U the forgetful functor from $R\mathcal{C}$-modules to $\mathrm{Ob}\,\mathcal{C}$-sets.

An $R\mathcal{C}$-module F is *free with the basis* I if there exists a morphism of $\mathrm{Ob}\,\mathcal{C}$-sets $j\colon I \to UF$ that satisfies the following universal property: For a morphism $f\colon I \to UM$ of $\mathrm{Ob}\,\mathcal{C}$-sets, where M is an $R\mathcal{C}$-module, there exists a unique morphism of $R\mathcal{C}$-modules, $\mu\colon F \to M$, such that the diagram

$$\begin{tikzcd} I \arrow[r, "j"] \arrow[dr, "f"'] & UF \arrow[d, "U\mu"] \\ & UM \end{tikzcd}$$

commutes. Since the free $R\mathcal{C}$-module with the basis I solves a universal problem, it is unique up to isomorphism. It follows in particular that every free $R\mathcal{C}$-module with the basis I is isomorphic to the $R\mathcal{C}$-module $\bigoplus_{c\in I} R[\mathrm{Mor}_{\mathcal{C}}(c,-)]$.

For more details on $R\mathcal{C}$-modules, we refer to [17] and [32].

1.2 Additive and R-linear functors

A category $\mathcal{A}$ is called *preadditive* if

1) the set of morphisms $\mathrm{Mor}_{\mathcal{A}}(a,b)$ has the structure of an Abelian group for all $a, b \in \mathcal{A}$;
2) the composition operation of morphisms

$$\mathrm{Mor}_{\mathcal{A}}(a,b) \times \mathrm{Mor}_{\mathcal{A}}(b,c) \to \mathrm{Mor}_{\mathcal{A}}(a,c)$$

is bilinear for all $a, b, c \in \mathcal{A}$.

A functor $F\colon \mathcal{A} \to \mathcal{B}$ between preadditive categories is called *additive* if the function $\varphi_{a,b}\colon \mathrm{Mor}_{\mathcal{A}}(a,b) \to \mathrm{Mor}_{\mathcal{B}}(F(a),F(b))$ is a group homomorphism for all $a, b \in \mathcal{A}$.

Furthermore, if the groups of morphisms of the preadditive category $\mathcal{A}$ are also R-modules, then $\mathcal{A}$ is an *R-linear category*. A functor $F\colon \mathcal{A} \to \mathcal{B}$ between two R-linear categories is *R-linear* if the function $\mathrm{Mor}_{\mathcal{A}}(a,b) \to \mathrm{Mor}_{\mathcal{B}}(F(a),F(b))$ is a homomorphism of R-modules for all $a, b \in \mathcal{A}$.

Let $\mathcal{C}$ be a small category. We say that an R-linear category $\mathcal{L}$ with a morphism $j\colon \mathcal{C} \to \mathcal{L}$ is an *R-linearization of $\mathcal{C}$*, if the following universal property is satisfied: For any functor $f\colon \mathcal{C} \to \mathcal{A}$, where $\mathcal{A}$ is an R-linear category, there exists a unique R-linear functor $g\colon \mathcal{L} \to \mathcal{A}$ such that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "j"] \arrow[dr, "f"'] & \mathcal{L} \arrow[d, "g"] \\ & \mathcal{A} \end{tikzcd}$$

commutes. The standard argument of universal properties suffices to show that the linearization, if it exists, is unique up to isomorphism.

We define the *canonical R-linearization of $\mathcal{C}$* as the category $R\mathcal{C}$, with the set of objects $\operatorname{Ob}\mathcal{C}$, and for all $c, d \in R\mathcal{C}$ the group of morphisms

$$\operatorname{Mor}_{R\mathcal{C}}(c, d) = R[\operatorname{Mor}_{\mathcal{C}}(c, d)].$$

Proposition 1.1. *The category $R\mathcal{C}$ is an R-linearization of $\mathcal{C}$.*

Proof. Obviously $R\mathcal{C}$ is an R-linear category. Let $i\colon \mathcal{C} \to R\mathcal{C}$ be the embedding where $c \mapsto c$ for all $c \in \mathcal{C}$, and $u \mapsto e_u$ for all morphisms $u\colon c \to d$ in $\mathcal{C}$. Suppose $f\colon \mathcal{C} \to \mathcal{A}$ is a functor, where $\mathcal{A}$ is an R-linear category. Let us define a functor $g\colon R\mathcal{C} \to \mathcal{A}$ by setting $g(c) = f(c)$ and $g(e_u) = f(u)$ for all $c \in \mathcal{C}$ and $u \in \operatorname{Mor}_{\mathcal{C}}(c, d)$. Clearly then g is R-linear with $g \circ i = f$. The uniqueness of g immediately follows from the requirement that g has to be linear. Thus $R\mathcal{C}$ is an R-linearization of $\mathcal{C}$. □

If $\mathcal{A}$ is a small preadditive category, following Mitchell ([20]), an additive functor $M\colon \mathcal{A} \to \mathbf{Ab}$ is called an *$\mathcal{A}$-module*. A morphism of $\mathcal{A}$-modules $\mu\colon M \to N$ is a natural transformation, where each μ_c, $c \in \mathcal{C}$, is a group homomorphism. The category of $\mathcal{A}$-modules is denoted $\mathcal{A}$-**Mod**. Like $R\mathcal{C}$-**Mod**, the category $\mathcal{A}$-**Mod** of $\mathcal{A}$-modules is an Abelian category with kernels, images, products and coproducts computed objectwise.

Note that an $R\mathcal{C}$-module could either be a functor from $\mathcal{C}$ to R-**Mod** or an additive functor from $R\mathcal{C}$ to **Ab**. This ambiguity is covered next.

Example 1.2. Let M be a functor $\mathcal{C} \to R$-**Mod**. The R-linearization of $\mathcal{C}$ extends M to an R-linear functor $M'\colon R\mathcal{C} \to R$-**Mod**, where $M' \circ i = M$. Obviously an

R-linear functor is also additive.

On the other hand, let N be an additive functor $R\mathcal{C} \to \mathbf{Ab}$. We may define an R-module structure on $N(c)$ for all $c \in \mathcal{C}$ by setting

$$rn := N(r \cdot e_{\mathrm{id}_c})(n)$$

for all $r \in R$, $c \in \mathcal{C}$ and $n \in N(c)$. This yields a functor from $\mathcal{C}$ to R-**Mod**.

In Example 1.2, we have essentially proven

Proposition 1.3. *The following categories are isomorphic:*

1) *Functors $\mathcal{C} \to R$-**Mod**;*
2) *Additive functors $R\mathcal{C} \to$ **Ab**;*
3) *R-linear functors $R\mathcal{C} \to R$-**Mod**.*

1.3 Order theory

Let P be a set, and let $\leq$ be a binary relation on P. We say that P is a *preordered set* if the relation $\leq$ is both reflexive and transitive. If the relation $\leq$ is also antisymmetric, then P is a *partially ordered set*, or a *poset* for short.

Let $\mathcal{C}$ be a poset. Given a subset $S \subseteq \mathcal{C}$, an element $c \in \mathcal{C}$ is an *upper bound of S*, if $s \leq c$ for all $s \in S$. We say that $\mathcal{C}$ is *(upward) directed*, if there exists an upper bound for every finite subset $S \subseteq \mathcal{C}$. An element $c \in \mathcal{C}$ is said to be *minimal*, if for every $d \in \mathcal{C}$, we have

$$d \leq c \;\Rightarrow\; c = d.$$

If for any $c, d \in \mathcal{C}$ there exists a unique minimal upper bound in $\mathcal{C}$, we say that $\mathcal{C}$ is a *join-semilattice*. We denote this unique minimal upper bound by $c \vee d$, and call it the *join* of c and d. A join-semilattice is *bounded* if it has a unique minimal element.

Remark 1.4. We could equivalently define a join-semilattice as a set P with a binary operation $\vee$ such that

- $p \vee (q \vee r) = (p \vee q) \vee r$;
- $p \vee q = q \vee p$;
- $p \vee p = p$

for all $p, q, r \in P$. This operation induces a partial order on P by setting $p \leq q$ if and only if $p \vee q = q$ for $p, q \in P$. Defined in this way, $p \vee q$ then coincides with the other definition. The above axioms also imply that the notation $\bigvee S$ makes sense for finite non-empty subsets $S \subseteq \mathcal{C}$.

Let L be a join-semilattice. If $S \subseteq L$ is a join-semilattice such that

$$\bigvee_L T = \bigvee_S T$$

for all finite non-empty subsets $T \subseteq S$, then we call S a *join-sublattice* of L.

All of the above definitions have dual versions, obtained by changing $\leq$ to $\geq$. These are, respectively, *lower bounds*, *downward directed sets*, *maximal elements*, *meet-semilattices*, *meets*, and *meet-sublattices*.

The poset $\mathcal{C}$ is called a *lattice* if it is both a join-semilattice and a meet-semilattice. Given a subset $S \subseteq \mathcal{C}$, we will use the notations

$$\uparrow S := \{c \in \mathcal{C} \mid s \leq c \text{ for some } s \in S\}$$

for the *upset generated by* S, and

$$\downarrow S := \{c \in \mathcal{C} \mid c \leq s \text{ for some } s \in S\}$$

for the *downset cogenerated by* S.

1.4 Colimits and limits

Let $\mathcal{A}$ be a preadditive category, I a small category, and $F\colon I \longrightarrow \mathcal{A}$ a functor. Assume that A is an object in $\mathcal{A}$, and that for all $i \in I$, we are given a morphism $\alpha_i\colon F(i) \longrightarrow A$. Then the family $(\alpha_i)_{i\in I}$ is called a *cone* from F to A, if for every morphism $u\colon i \longrightarrow j$ in I, we have $\alpha_j = \alpha_i F(u)$. A *colimit* of the functor F is an object $\operatorname{colim}_I F \in \mathcal{A}$ together with a cone $(\mu_i)_{i\in I}$ from F to $\operatorname{colim}_I F$, satisfying the following universal property: If $(\lambda_i)_{i\in I}$ is a cone from F to L, where $L \in \mathcal{A}$, then there exists a unique morphism $f\colon \operatorname{colim}_I F \longrightarrow L$ such that $f\mu_i = \lambda_i$ for all $i \in I$. In other words, $(\mu_i)_{i\in I}$ is the initial cone (from F).

If the colimit $\operatorname{colim}_I F$ exists for every functor $I \longrightarrow \mathcal{A}$, where I is a small category, then the category $\mathcal{A}$ is called *cocomplete*.

Proposition 1.5. [30, p. 99, Prop. IV.8.1] *If $\mathcal{A}$ is cocomplete, then* colim_I *is a functor from category of functors $I \to \mathcal{A}$ to the category $\mathcal{A}$.*

Let I be a preordered set. Note that a functor $F\colon I \to \mathcal{A}$ may be described by the family $(F(i))_{i\in I}$ and the morphisms $F(i \le j)$ for all $i \le j$ in I. With the data given in this form, we can also denote the colimit $\operatorname{colim}_I F$ more concretely by $\operatorname{colim}_{i\in I} F(i)$, and say that it is the colimit of the family $(F(i))_{i\in I}$.

If I is a directed poset and $\mathcal{C}$ is any category, then a functor $F\colon I \to \mathcal{C}$ is called a *direct system* in $\mathcal{C}$.

The next lemma is a basic property of the colimit that will be used later. The proof is straightforward and follows from the universal property of the colimit.

Lemma 1.6. [13, p. 40] *Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between cocomplete preadditive categories. There exists a canonical morphism*

$$\theta\colon \operatorname*{colim}_{i\in I}(\varphi \circ F)(i) \to \varphi(\operatorname*{colim}_{i\in I} F(i)).$$

For an easy reference, we gather some elementary results and facts about colimits in the next remark.

Remark 1.7. Let I be a small category.

- If $M\colon I \to R$-**Mod** is a functor, then $\operatorname{colim}_I M$ may be constructed as follows: For the R-module $N := \bigoplus_{i\in I} M(i)$ and its submodule

  $$J := \langle M(u)(x) - x \mid u\colon i \to j \text{ is a morphism in } I,\ x \in M(i)\rangle,$$

  we have $\operatorname{colim}_I M \cong N/J$.
- If I be a discrete category, i.e., a category with only identity morphisms, then $\operatorname*{colim}_{i\in I} F \cong \coprod_{i\in I} F(i)$.
- Let I be a poset. A subset $J \subseteq I$ is said to be *final*, if for every $i \in I$ there exists an element $j \in J$ such that $i \le j$. Let $\mathcal{A}$ be cocomplete. If I is directed and $J \subseteq I$ is final, then

  $$\operatorname*{colim}_I F \cong \operatorname*{colim}_J F.$$

  In particular, if I has a maximum element, then $\{\max(I)\} \subseteq I$ is final, and

  $$\operatorname*{colim}_I F \cong F(\max(I)).$$

The limit of a functor $F\colon I \to \mathcal{A}$ is defined in a dual fashion. Assume that A is an object in $\mathcal{A}$, and that for all $i \in I$, we are given a morphism $\beta_i\colon A \to F(i)$. Then the family $(\beta_i)_{i\in I}$ is called a *cone* from A to F, if for every morphism $u\colon i \to j$ in I, we have $\beta_j = F(u)\beta_i$. The *limit* of F is then an object $\lim_I F \in \mathcal{A}$ together with a cone $(\nu_i)_{i\in I}$ from $\lim_I F$ to F that satisfies the following universal property: If $(\lambda_i)_{i\in I}$ is a cone from L to F, where $L \in \mathcal{A}$, then there exists a unique morphism $g\colon L \to \lim_I F$ such that $\nu_i g = \lambda_i$ for all $i \in I$. In other words, $(\nu_i)_{i\in I}$ is the terminal cone (to F).

If the limit $\lim_I F$ exists for every functor $I \to \mathcal{A}$, where I is a small category, then the category $\mathcal{A}$ is called *complete*. We can also state the dual versions of Proposition 1.5, Lemma 1.6 and Remark 1.7.

Proposition 1.8. *If $\mathcal{A}$ is complete, then $\lim_I$ is a functor from category of functors $I \to \mathcal{A}$ to the category $\mathcal{A}$.*

Lemma 1.9. *Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between complete preadditive categories. There exists a canonical morphism*

$$\theta\colon \varphi(\lim_{i\in I} F(i)) \to \lim_{i\in I}(\varphi \circ F)(i).$$

Remark 1.10. Let I be a small category.

- If $M\colon I \to R\text{-}\mathbf{Mod}$ is a functor, then $\lim_I M$ may be constructed as follows: For the R-module $N := \prod_{i\in I} M(i)$ and its submodule

$$J := \{(x_i)_{i\in I} \in N \mid M(u)(x_i) = x_j \text{ for all morphisms } u\colon i \to j \text{ in } I\},$$

  we have $\lim_I M \cong J$.
- If I is a discrete category, i.e., a category with only identity morphisms, then $\lim_{i\in I} F \cong \prod_{i\in I} F(i)$.
- Let I be a poset. A subset $J \subseteq I$ is said to be *initial*, if for every $i \in I$ there exists an element $j \in J$ such that $i \geq j$. Let $\mathcal{A}$ be complete. If I is downward directed and $J \subseteq I$ is initial, then

$$\lim_I F \cong \lim_J F.$$

In particular, if I has a minimum element, then $\{\min(I)\} \subseteq I$ is initial, and

$$\lim_{I} F \cong F(\min(I)).$$

1.5 Finiteness conditions

Let $\mathcal{A}$ be a preadditive category. In the following, we use the notation $\mathrm{Hom}_{\mathcal{A}}$ for morphisms in $\mathcal{A}$-**Mod** and $\mathrm{Mor}_{\mathcal{A}}$ for morphisms in $\mathcal{A}$.

Let $\mathcal{G}$ be an Abelian category with arbitrary coproducts, and in which taking colimits of direct systems is exact. Recall that an element $G \in \mathcal{G}$ is a *generator* for $\mathcal{G}$, if for all $M \in \mathcal{G}$ there exists an epimorphism

$$\coprod_{j \in J} G \to M$$

for some set J. A family $(G_i)_{i \in I}$ of objects of $\mathcal{G}$ is a *generating family* for $\mathcal{G}$ if and only if $\coprod_{i \in I} G_i$ is a generator for $\mathcal{G}$. If $(G_i)_{i \in I}$ indeed is a generating family, then for every object $M \in \mathcal{G}$ there exists an epimorphism

$$\coprod_{j \in J_M} G_j \to M$$

where J_M is a collection of elements of I. If $\mathcal{G}$ has a generating family, it is called a *Grothendieck category*.

Example 1.11. The category R-**Mod** is a Grothendieck category with a generator R.

Let $\mathcal{G}$ be a Grothendieck category, and $M \in \mathcal{G}$.

- M is *finitely generated* if for every direct system $(M_i)_{i \in I}$ of subobjects of M

$$M = \sum_{i \in I} M_i \quad \Rightarrow \quad \text{There exists } i_0 \in I \text{ such that } M = M_{i_0}.$$

- $\mathcal{G}$ is *locally finitely generated* if it has a generating family consisting of finitely generated objects.
- M is *finitely presented* if M is finitely generated and for every epimorphism

$\varphi \colon L \to M$

$$L \text{ is finitely generated} \quad \Rightarrow \quad \operatorname{Ker}\varphi \text{ is finitely generated.}$$

Let $\mathcal{G}$ be a locally finitely generated Grothendieck category with a generating family $(G_i)_{i\in I}$. Let $M \in \mathcal{G}$. It is well known ([25, p. 710, Prop. E.1.13]) that M is finitely generated if and only if there exists an epimorphism

$$\coprod_{j\in J} G_j \to M,$$

where J is a finite collection of elements of I. Similarly, M is finitely presented if and only if there exists an exact sequence

$$\coprod_{k\in K} G_k \to \coprod_{j\in J} G_j \to M \to 0,$$

where K and J are finite collections of elements of I ([24, p. 95, Prop. 5.13]). We also have the following equivalent characterizations of finitely presented and finitely generated objects:

Proposition 1.12. [30, p. 122, Prop. V.3.2, V.3.4] *Let $\mathcal{G}$ be a locally finitely generated Grothendieck category. An object $M \in \mathcal{G}$ is*

- *finitely generated if and only if the functor* $\operatorname{Hom}_{\mathcal{G}}(M, -)$ *preserves the colimits of direct systems with monomorphisms;*
- *finitely presented if and only if the functor* $\operatorname{Hom}_{\mathcal{G}}(M, -)$ *preserves the colimits of direct systems.*

Example 1.13. The category $\mathcal{A}$-**Mod** of $\mathcal{A}$-modules is a locally finitely generated Grothendieck category with a generating family $(\operatorname{Mor}_{\mathcal{A}}(a, -))_{a\in\mathcal{A}}$.

We note that being finitely presented is a categorical property, which is a well known fact, but in lack of suitable reference, we present the proof here.

Proposition 1.14. *Let $F \colon \mathcal{G} \to \mathcal{H}$ be an equivalence of Grothendieck categories, and let $M \in \mathcal{G}$. If M is finitely presented in $\mathcal{G}$, then FM is finitely presented in $\mathcal{H}$.*

Proof. Let $(N_i)_{i\in I}$ be a direct system in $\mathcal{H}$, and $(M_i)_{i\in I}$ the corresponding direct system in $\mathcal{G}$. Since F preserves direct limits (as a left adjoint), we see that

$$\begin{aligned}\operatorname{Hom}_{\mathcal{H}}(FM, \operatorname*{colim}_{i\in I} N_i) &\cong \operatorname{Hom}_{\mathcal{H}}(FM, \operatorname*{colim}_{i\in I} FM_i)\\ &\cong \operatorname{Hom}_{\mathcal{H}}(FM, F(\operatorname*{colim}_{i\in I} M_i)).\end{aligned}$$

But F is an equivalence, so we have

$$\operatorname{Hom}_{\mathcal{H}}(FM, F(\operatorname*{colim}_{i\in I} M_i)) \cong \operatorname{Hom}_{\mathcal{G}}(M, \operatorname*{colim}_{i\in I} M_i).$$

Because M is finitely presented, Proposition 1.12 yields

$$\operatorname{Hom}_{\mathcal{G}}(M, \operatorname*{colim}_{i\in I} M_i) \cong \operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathcal{G}}(M, M_i).$$

Finally, since F is an equivalence,

$$\begin{aligned}\operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathcal{G}}(M, M_i) &\cong \operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathcal{H}}(FM, FM_i)\\ &\cong \operatorname*{colim}_{i\in I} \operatorname{Hom}_{\mathcal{H}}(FM, N_i).\end{aligned}$$

□

1.6 Kan extensions

In the following section, we assume that $\mathcal{C}$ and $\mathcal{D}$ are small categories. Let $\alpha\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{C} \to \mathcal{X}$ be functors. The *left Kan extension of F along α* is a pair (L, μ), where

- $L\colon \mathcal{D} \to \mathcal{X}$ is a functor;
- $\mu\colon F \to L \circ \alpha$ is a natural transformation.

This pair has the universal property saying that for all pairs (H, ν), where $H\colon \mathcal{D} \to \mathcal{X}$ is a functor and $\nu\colon F \to H \circ \alpha$ is a natural transformation, there is a unique natural transformation $\rho\colon L \to H$ with the property that $\rho_\alpha \circ \mu = \nu$. Here ρ_α is the natural transformation $L \circ \alpha \to H \circ \alpha$.

Dually, the *right Kan extension of F along α* is a pair (R, η), where

- $R\colon \mathcal{D} \to \mathcal{X}$ is a functor;
- $\eta\colon R \circ \alpha \to F$ is a natural transformation.

This pair has the universal property saying that for all pairs (H, ν), where $H \colon \mathcal{D} \to \mathcal{X}$ is a functor and $\nu \colon H \circ \alpha \to F$ is a natural transformation, there is a unique natural transformation $\tau \colon H \to R$ with the property that $\eta \circ \tau_\alpha = \nu$. Here τ_α is the natural transformation $H \circ \alpha \to R \circ \alpha$.

In general, Kan extensions do not necessarily exist. Before discussing their existence, we introduce the following categories.

Definition 1.15. Let $d \in \mathcal{D}$.

1) The *slice category* (d/α) has the objects

$$\mathrm{Ob}(d/\alpha) = \{(c, u) \mid c \in \mathcal{C},\ u \colon d \to \alpha(c) \text{ is a morphism in } \mathcal{D}\}.$$

For $(c, u), (c', u') \in (d/\alpha)$, the morphisms $(c, u) \to (c', u')$ are those morphisms $f \colon c \to c'$ in $\mathcal{C}$ for which the diagram

$$\begin{tikzcd} d \arrow[r, "u"] \arrow[dr, "u'"'] & \alpha(c) \arrow[d, "\alpha(f)"] \\ & \alpha(c') \end{tikzcd}$$

commutes.

2) The *slice category* (α/d) has the objects

$$\mathrm{Ob}(\alpha/d) = \{(c, u) \mid c \in \mathcal{C},\ u \colon \alpha(c) \to d \text{ is a morphism in } \mathcal{D}\}.$$

For $(c, u), (c', u') \in (\alpha/d)$, the morphisms $(c, u) \to (c', u')$ are those morphisms $f \colon c \to c'$ in $\mathcal{C}$ for which the diagram

$$\begin{tikzcd} \alpha(c) \arrow[r, "u"] \arrow[d, "\alpha(f)"'] & d \\ \alpha(c') \arrow[ur, "u'"'] & \end{tikzcd}$$

commutes.

From Definition 1.15, it is clear that for every $d \in \mathcal{D}$ there are canonical projection functors

$$p_d\colon (\alpha/d) \to \mathcal{C},\ \ (c,u) \mapsto c,\ f \mapsto f,$$
$$q_d\colon (d/\alpha) \to \mathcal{C},\ \ (c,u) \mapsto c,\ f \mapsto f.$$

We write $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ for functors from $\mathcal{C}$ to $\mathcal{D}$.

Proposition 1.16. [26, pp. 60, 65, Thm. 4.1.4, Thm. 4.2.2] *Let $M\colon \mathcal{C} \to \mathcal{X}$ be a functor.*

1) *If $\mathcal{X}$ be cocomplete, then M has a left Kan extension $\mathrm{LKan}_\alpha M$, defined on objects by*
$$(\mathrm{LKan}_\alpha M)(d) = \operatorname*{colim}_{(\alpha/d)}(M \circ p_d)$$
for all $d \in \mathcal{D}$. This defines a functor
$$\mathrm{LKan}_\alpha\colon\ \mathrm{Fun}(\mathcal{C},\mathcal{X}) \to \mathrm{Fun}(\mathcal{D},\mathcal{X}).$$

2) *Let $\mathcal{X}$ be complete. If $M\colon \mathcal{C} \to \mathcal{X}$ is a functor, then M has a right Kan extension $\mathrm{RKan}_\alpha M$ defined on objects by*
$$(\mathrm{RKan}_\alpha M)(d) = \lim_{(d/\alpha)}(M \circ q_d)$$
for all $d \in \mathcal{D}$. This defines a functor
$$\mathrm{RKan}_\alpha\colon\ \mathrm{Fun}(\mathcal{C},\mathcal{X}) \to \mathrm{Fun}(\mathcal{D},\mathcal{X}).$$

Definition 1.17. The *restriction functor* $\alpha^*\colon\ \mathrm{Fun}(\mathcal{D},\mathcal{X}) \to \mathrm{Fun}(\mathcal{C},\mathcal{X})$ maps any functor $M\colon \mathcal{D} \to \mathcal{X}$ to the functor
$$\alpha^* M = M\alpha\colon \mathcal{C} \to X.$$

If $\varphi\colon M \to N$ is a morphism in $\mathrm{Fun}(\mathcal{D},\mathcal{X})$, then the natural transformation
$$\alpha^*\varphi\colon M\alpha \to N\alpha$$

is defined by
$$(\alpha^*\varphi)_c = \varphi_{\alpha(c)}$$

for all $c \in \mathcal{C}$.

Proposition 1.18. [26, p. 64, Thm. 4.1.11] *If $\mathcal{X}$ is cocomplete, the restriction functor α^* has a left adjoint given by the left Kan extension. Dually, if $\mathcal{X}$ is complete, the restriction functor α^* has a right adjoint given by the right Kan extension.*

Definition 1.19. If $\mathcal{X} = R$-**Mod**, we write $\mathrm{res}_\alpha := \alpha^*$. We call the left and right adjoints of res_α induction and coinduction, respectively, and denote them by

$$\mathrm{ind}_\alpha := \mathrm{LKan}_\alpha \quad \text{and} \quad \mathrm{coind}_\alpha := \mathrm{RKan}_\alpha .$$

Example 1.20. For any $R\mathcal{C}$-module M, and any object $d \in \mathcal{D}$, we have the pointwise formulas

$$(\mathrm{ind}_\alpha M)(d) = \operatorname*{colim}_{(c,u)\in(\alpha/d)} M(c) \quad \text{and} \quad (\mathrm{coind}_\alpha M)(d) = \lim_{(c,u)\in(d/\alpha)} M(c).$$

If $\mathcal{C}$ and $\mathcal{D}$ are posets, these formulas yield

$$(\mathrm{ind}_\alpha M)(d) = \operatorname*{colim}_{c\in\mathcal{C},\ \alpha(c)\leq d} M(c) \quad \text{and} \quad (\mathrm{coind}_\alpha M)(d) = \lim_{c\in\mathcal{C},\ d\leq\alpha(c)} N(c).$$

2 MODULES OVER A MONOID ACT

2.1 Monoid action

In this section we recall some basic properties of monoid actions. Let G be a monoid and let A be a set. If there exists an operation $\cdot\colon G \times A \to A$ such that $(gh)a = g(ha)$ and $1_G \cdot a = a$ for all $g, h \in G$ and $a \in A$, we say that A is a *(left) G-act*. A morphism of G-acts $f : A \to B$ is a function, where $f(ga) = gf(a)$ for all $g \in G$ and $a \in A$.

Given a G-act A, we get a preorder on A by setting $a \leq b$ if $b = ga$ for some $g \in G$. The action naturally gives rise to two categories having A as the set of objects.

First, we have a small thin category A, where for all $a, b \in A$ there exists a unique morphism $a \to b$ if $a \leq b$ in the preorder. By abuse of notation we write $a \leq b$ for this morphism. Recall that in general a category is *thin* if there exists at most one morphism between any two objects.

Secondly, there is an *action category* $G\int A$, where morphisms $a \to b$ are pairs (a, g) such that $b = ga$ for some $g \in G$. If there is no possibility of confusion, we sometimes denote the morphism (a, g) by g. Composition of morphisms in $G\int A$ is defined by the multiplication of G:

$$(ga, h) \circ (a, g) = (a, hg).$$

There is an obvious functor $G\int A \to A$ where

$$a \mapsto a \quad \text{and} \quad (a, g) \mapsto (a \leq ga).$$

This functor is an isomorphism if and only if the G-action on A is *free*, i.e. for all $g, h \in G$,

$$ga = ha \text{ for some } a \in A \;\Rightarrow\; g = h.$$

Remark 2.1. We often consider the monoid G itself as a G-act, so it gives rise to a thin category G and the action category $G \int G$. Sometimes, the monoid G is viewed as a category BG with a single object.

Example 2.2. Let $N: \mathcal{C} \to \mathbf{Set}$ be a functor. The *category of elements* of N, denoted $\int N$, has pairs (c, x), where $c \in \mathcal{C}$ and $x \in N(c)$, as objects. A morphism $(c, x) \to (c', x')$ in $\int N$ is a morphism $u: c \to c'$ in $\mathcal{C}$ such that $N(u)(x) = x'$.

If we view the monoid G as the single object category BG, then a G-act A could equivalently be defined as a functor $A_G: BG \to \mathbf{Set}$. As shown in [27, p. 66, Ex. 2.4.10], the corresponding action category $G \int A$ coincides with the category of elements $\int A_G$.

Example 2.3. An abelian group G is called *partially ordered* if it is equipped with a partial order $\leq$ such that $g \leq g'$ implies $g + h \leq g' + h$ for all $g, g', h \in G$. If $G_+ = \{g \in G \mid g \geq 0\}$ is its *positive cone*, then $g \leq g'$ is equivalent to $g' - g \in G_+$. The action of the monoid G_+ on G is free, because G is an Abelian group. Thus, as stated above, we may identify the action category $G_+ \int G$ with the poset G.

A *translation* is an order-preserving function $F: P \to P$ on a preordered set P that satisfies the condition $p \leq F(p)$ for all $p \in P$. The translations of P form a monoid $\mathrm{Trans}(P)$ with composition as the operation.

Let A be a G-act. The action by an element $g \in G$ now determines a translation on A if and only if $a \leq b$ implies $ga \leq gb$. If this implication holds for all $g \in G$, then we say that A is an *order-preserving G-act*. Note that any G-act A is order-preserving if G is commutative. For an order-preserving G-act A, we get a monoid homomorphism φ from G into the monoid of translations $\mathrm{Trans}(A)$. This induces a monoid embedding $\hat{\varphi}: G/\mathrm{Ker}\,\varphi \to \mathrm{Trans}(A)$, where $\mathrm{Ker}\,\varphi$ is the congruence relation defined by

$$(g, h) \in \mathrm{Ker}\,\varphi \iff ga = ha \text{ for all } a \in A.$$

In particular, φ is an embedding if and only if the G-action on A is *faithful*: for all $g, h \in G$, $ga = ha$ for all $a \in A$ implies that $g = h$.

We next give a slight generalization of [11, p. 4, Thm. 2.2].

Proposition 2.4. *For any preordered set P, there exists a monoid G and a G-act A such that P and A are isomorphic as thin categories.*

Proof. We present the proof here for the convenience of the reader. Let G denote the submonoid of the monoid of all functions $P \to P$ consisting of the functions $g: P \to P$ for which $a \leq_P g(a)$ for all $a \in P$. Define the G-action on $A := P$ by setting $g \cdot a = g(a)$ for all $g \in G$ and $a \in A$. Then A is a G-act. It remains to show that $a \leq_P b$ if and only if $a \leq_A b$.

Assume first that $a \leq_A b$. By definition, there exists an element $g \in G$ such that $b = ga$. But this means that $a \leq_P g(a) = ga = b$. Conversely, if $a \leq_P b$, we define a function $g \colon P \to P$ by setting

$$g(p) = \begin{cases} b, & \text{if } p = a; \\ p, & \text{otherwise.} \end{cases}$$

We then immediately see that $g \in G$ and $ga = g(a) = b$, so that $a \leq_A b$ □

Remark 2.5. The monoid G in Proposition 2.4 does not need to be unique: For example, the one element set $P = \{*\}$ has the trivial monoid action for any monoid G. Note that if G is the monoid of the proof of Proposition 2.4, then the order-preserving elements in G are exactly the translations on A, so that $\mathrm{Trans}(A) \subseteq G$.

2.2 Automorphic actions

Often we would like to work with group indexed structures instead of monoid indexed structures. In this section we explore certain circumstances in which this change is possible.

Let G be a commutative cancellative monoid and let A be a G-act. We assume that the action of G on A is *automorphic*. That is, for every $g \in G$ and $a \in A$, there exists a unique $a' \in A$ such that $ga' = a$, so that multiplication by g is an automorphism in A. We introduce a more specific notation by setting $T(g, a) := a'$. Since G is commutative and cancellative, it can be embedded into the Grothendieck group G^{gp}, which consists of elements g/h, where $g, h \in G$. Note that G is a poset (with the natural order) if and only if there are no non-trivial invertible elements in G.

We may extend A into a G^{gp}-act by setting

$$\frac{g}{h} \cdot a = gT(h, a).$$

Before proving this, we introduce some elementary properties of the notation introduced above in the next lemma.

Lemma 2.6. *For every $g, h \in G$ and $a \in A$, we have*

1) $T(gh, a) = T(g, T(h, a)) = T(h, T(g, a))$;

2) $T(g, ha) = hT(g, a)$.

Proof. Let $g, h \in G$ and $a \in A$. To show 1), we first note that by definition $ghT(gh, a) = a$. On the other hand,

$$ghT(g, T(h, a)) = hT(h, a) = a = gT(g, a) = ghT(h, T(g, a)).$$

From the uniqueness of $T(gh, a)$, we see that 1) holds.

For 2), note that $ghT(g, a) = ha = gT(g, ha)$. Thus, again by the uniqueness of $T(g, ha)$, 2) holds. □

To prove that A is a G^{gp}-act, let us first show that the operation is well defined. Suppose that $g/h = g'/h'$. Then $gh' = g'h$. Furthermore,

$$ghT(h, a) = ga = gh'T(h', a) = g'hT(h', a).$$

Since G is commutative and cancellative, we see that $gT(h, a) = g'T(h', a)$. That is, $(g/h) \cdot a = (g'/h') \cdot a$, so the operation is well defined.

Suppose next that $g/h, g'/h' \in G^{\mathrm{gp}}$ and $a \in A$. We have

$$\left(\frac{g}{h} \cdot \frac{g'}{h'}\right) a = \frac{gg'}{hh'} \cdot a = gg'T(hh', a).$$

Using Lemma 2.6, we may write this as

$$gg'T(hh', a) = gg'T(h, T(h', a)) = gT(h, g'T(h', a)) = \frac{g}{h}\left(\frac{g'}{h'} \cdot a\right),$$

showing us that

$$\left(\frac{g}{h} \cdot \frac{g'}{h'}\right) a = \frac{g}{h}\left(\frac{g'}{h'} \cdot a\right),$$

as required. Of course, we also have $(1/1)a = 1 \cdot T(1, a) = a$. Thus A is a G^{gp}-act.

Since A is a G^{gp}-act, we get for every $a \in A$ the obvious group act map

$$\varphi_a \colon G^{\mathrm{gp}} \longrightarrow A,\ \frac{g}{h} \mapsto \frac{g}{h} \cdot a.$$

The natural question we ask here is how closely does A resemble G^{gp} and does A possibly have a group structure? We go through some properties of these maps in the next proposition. Before that, we go through some terminology.

Given $a \in A$, we say that the G-action is *free on a* if $ga = ha$ implies $g = h$ for all

$g, h \in G$. The G-act A is *connected* if the thin category A (or equivalently, the action category $G \int A$) is connected. Recall that a category $\mathcal{C}$ is connected if for each $c, d \in \mathcal{C}$ there exists a sequence $c = c_0, \ldots, c_n = d$ of elements of $\mathcal{C}$ with a morphism $c_i \to c_{i+1}$ or $c_{i+1} \to c_i$ for all $i \in \{0, \ldots, n-1\}$.

Proposition 2.7. *Let G be a commutative cancellative monoid and A a G-act, where the action is automorphic. Fix $a \in A$, so that we get a morphism φ_a as defined above.*

1) *The morphism φ_a is injective if and only if the G-action is free on a.*
2) *The morphism φ_a is surjective if and only if the G-act A is connected.*
3) *The morphism φ_a is bijective if and only if the G-act A is connected and the G-action is free on a' for some $a' \in A$.*

Proof. Let us show 1) first. Suppose that the G-action is free on a. Let $g, g', h, h' \in G$ such that $(g/h)a = (g'/h')a$. Multiplying this equation by hh' yields $gh'a = hg'a$. By the freeness of the action on a, we have $gh' = g'h$, and also $g/h = g'/h'$. Thus φ_a is injective. Conversely, suppose that φ_a is injective. If $ga = ha$ for some $g, h \in G$, we have

$$\varphi_a\left(\frac{g}{1}\right) = ga = ha = \varphi_a\left(\frac{h}{1}\right),$$

and $g = h$ follows from injectivity of φ_a.

Next, we will prove 2). Suppose that the G-act A is connected. That is, for any $b \in A$, there exist elements $a = a_0, a_1, \ldots, a_n = b$ in A such that

$$a_{i+1} = g_i a_i \quad \text{or} \quad a_{i+1} = (1/g_i)a_i.$$

for every $i \in \{0, \ldots, n-1\}$. From these equations, it is clear that $b = (g/h)a = \varphi_a(g/h)$ for some $g, h \in G$. Conversely, if φ_a is surjective, then for any $b \in A$ there exists $g, h \in G$ such that $(g/h)a = b$. That is,

$$a \leq ga \geq \frac{1}{h}ga = \frac{g}{h}a = b,$$

so b is connected to a. Since b was arbitrary, the G-act A is connected.

Finally, we will prove 3). Suppose that A is connected and the G-action is free on a' for some $a' \in A$. Since A is connected, φ_a is an epimorphism by 2), and there exist $k, l \in G$ such that $a' = (k/l)a$. Let $ga = ha$. Multiplication by k gives

$$gla' = gk\frac{l}{k}a' = gka = hka = hk\frac{l}{k}a' = hla'.$$

By the freeness of the action on a', we now see that $gl = hl$. Because G is cancellative, this implies $g = h$. Thus φ_a is injective by 1). The other direction follows immediately from 1) and 2). $\square$

Let $\mathcal{C}$ be a small category. If $\mathcal{C}$ is not connected, it can be written as a disjoint union $\coprod_{i\in I} \mathcal{C}_i$ of its connected components ([18, p. 90, Ex. 7]). Then each $R\mathcal{C}$-module M is equivalent to a family $(M_i)_{i\in I}$, where M_i is an $R\mathcal{C}_i$-module for all $i \in I$. That is, we have an isomorphism of categories

$$R\mathcal{C}\text{-}\mathbf{Mod} \cong \prod_{i\in I} R\mathcal{C}_i\text{-}\mathbf{Mod}.$$

If $\mathcal{C} = G\int A$, the connected components $\mathcal{C}_i$ are also G-acts, so we can write $\mathcal{C}_i = G\int A_i$, where $A_i = \operatorname{Ob}\mathcal{C}_i$, for all $i \in I$. Therefore

$$R(G\textstyle\int A)\text{-}\mathbf{Mod} \cong \displaystyle\prod_{i\in I} R(G\textstyle\int A_i)\text{-}\mathbf{Mod}.$$

Finally, we use Proposition 2.7 3) to sum up this discussion.

Corollary 2.8. *Let G be a commutative cancellative monoid and A a G-act, where the action is automorphic. Let A_i, where $i \in I$ be the connected components of A. If for each $i \in I$ there exists an $a_i \in A_i$ such that G acts freely on a_i, then*

$$R(G\textstyle\int A)\text{-}\boldsymbol{Mod} \cong \displaystyle\prod_{i\in I} RG^{\mathrm{gp}}\text{-}\boldsymbol{Mod}.$$

Furthermore, if the G-action on A is free, then

$$RA\text{-}\boldsymbol{Mod} \cong \prod_{i\in I} RG^{\mathrm{gp}}\text{-}\boldsymbol{Mod}.$$

2.3 Action categories over a graded ring

Theorem 2.13 will generalize the equivalence of the correspondence theorem of Carlsson and Zomorodian [33, p. 259, Thm. 3.1] mentioned in the Introduction. Moreover, it generalizes the multi-parameter version of the theorem by Carlsson and Zomorodian ([5, p. 78, Thm. 1]) as well as the generalization given by Corbet and Kerber ([6, p. 18, Lemma 14]). For a discussion on related finiteness conditions, see [6, p. 3] and Remark 4.16.

Let G be a monoid. Recall that a ring S is *G-graded*, if

1) $S = \bigoplus_{g \in G} S_g$, where S_g is an additive subgroup of S for all $g \in G$;
2) $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$.

Let A be a G-act, and let $S := \oplus_{g \in G} S_g$ be a G-graded ring. We say that a (left) S-module M is *A-graded*, if

1) $M = \bigoplus_{a \in A} M_a$, where M_a is an Abelian group for all $a \in A$;
2) $S_g M_a \subseteq M_{ga}$ for all $g \in G$ and $a \in A$.

A morphism of A-graded S-modules $f \colon M \to N$ is an S-module homomorphism such that $f(M_a) \subseteq N_a$ for all $a \in A$. The category of A-graded S-modules is locally finitely generated Grothendieck category with a generating family $(S(a))_{a \in A}$, where the free S-module $S(a)$ generated by $a \in A$ is defined by

$$S(a)_b = \bigoplus_{g \in G,\ ga = b} S_g$$

for all $a, b \in A$.

We begin by defining a certain preadditive category.

Definition 2.9. Let A be a G-act, and let $S := \oplus_{g \in G} S_g$ be a G-graded ring. The *action category over S*, denoted $G \int_S A$, is the category with the set A as objects, and morphisms $(a, s) \colon a \to b$, where $a, b \in A$ and

$$s \in \bigoplus_{g \in G,\ ga = b} S_g.$$

Composition for morphisms $(a, s) : a \to ga$ and $(ga, t) : ga \to hga$ is defined by

$$(ga, t) \circ (a, s) = (a, ts).$$

Remark 2.10. Keeping a close eye on the domains, we may write $s := (a, s)$. With this notation, composition is just the multiplication in S.

Example 2.11. Let A be a G-act. If R is a commutative ring, then the action category $G\int_{R[G]} A$ over the monoid ring $R[G]$ coincides with the linearized action category $R(G\int A)$. Indeed, by definition $\operatorname{Ob} R(G\int A) = A$, and

$$\operatorname{Hom}_{R(G\int A)}(a, b) = R[\{(a, g) \mid g \in G \text{ and } ga = b\}].$$

for all $a, b \in A$.

Example 2.12. If G is an Abelian group and $S := \bigoplus_{g \in G} S_g$ is a G-graded ring, the category $G\int_S G$ is called in [8, p. 358, Def. 2.1] a *companion category*. In this case, we may identify $\operatorname{Hom}_{G\int_S G}(g, h)$ with S_{h-g}.

Preparing for Theorem 2.13, we will now define two functors, Φ and Ψ, that connect A-graded S-modules to $(G\int_S A)$-modules.

Let M be a $G\int_S A$-module. By setting $sm = M(s)(m)$ for all $g \in G$, $s \in S_g$ and $m \in M(a)$, we can define an A-graded S-module

$$\Phi M := \bigoplus_{a \in A} M(a).$$

A morphism $f : M \to N$ of $G\int_S A$-modules consists of homomorphisms of Abelian groups $f_a : M(a) \to N(a)$ with commutative diagrams

$$\begin{tikzcd}
M(a) \arrow[r, "M(s)"] \arrow[d, "f_a"'] & M(ga) \arrow[d, "f_{ga}"] \\
N(a) \arrow[r, "N(s)"'] & N(ga)
\end{tikzcd}$$

for all $a \in A$, $g \in G$ and $s \in S_g$. These homomorphisms and diagrams obviously give rise to a homomorphism $\Phi f : \Phi M \to \Phi N$ of A-graded S-modules with $(\Phi f)_a = f_a$ for all $a \in A$.

Next, let Q be an A-graded S-module. We set $(\Psi Q)(a) = Q_a$ for all $a \in A$. If $(a, s)\colon a \to ga$ is a morphism, where $a \in A$, $g \in G$ and $s \in S_g$, we can define a homomorphism

$$(\Psi Q)((a, s))\colon (\Psi Q)(a) \to (\Psi Q)(ga)$$

by setting $(\Psi Q)((a, s))(q) = s \cdot q$ for all $q \in Q_a$. It is clear that ΨQ is an additive functor $G \int_S A \to \mathbf{Ab}$, i.e., a $G \int_S A$-module. Moreover, if $h\colon Q \to P$ is a homomorphism of A-graded S-modules, we have a morphism of $G \int_S A$-modules $\Psi h\colon \Psi Q \to \Psi P$ given by $(\Psi h)_a = h_a$ for all $a \in A$.

We are now ready to state

Theorem 2.13. *Let A be a G-act, and let $S := \oplus_{g \in G} S_g$ be a G-graded ring. The above functors Φ and Ψ give an isomorphism of categories*

$$(G \textstyle\int_S A)\text{-}\boldsymbol{Mod} \cong A\text{-}gr\ S\text{-}\boldsymbol{Mod}.$$

Proof. It remains to prove that $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$, which is straightforward. $\square$

Combining this theorem with Example 2.11 gives

Corollary 2.14. *Let A be a G-act, and let R be a commutative ring. There is an isomorphism of categories*

$$R(G \textstyle\int A)\text{-}\boldsymbol{Mod} \cong A\text{-}gr\ R[G]\text{-}\boldsymbol{Mod}.$$

In particular, if the G-action on A is free, we obtain an isomorphism

$$RA\text{-}\boldsymbol{Mod} \cong A\text{-}gr\ R[G]\text{-}\boldsymbol{Mod}.$$

Example 2.15. If $A = \{e\}$ is a one object set, Theorem 2.13 gives us an isomorphism $G \int_S \{e\}$-**Mod** $\cong S$-**Mod**. In the case $S = R[G]$, where R is a commutative ring, this means that RG-**Mod** $\cong R[G]$-**Mod**, where RG is the linearization of the 1-object category G.

Example 2.16. Let G be a partially ordered Abelian group with the positive cone G_+ (see Example 2.3). If R is a commutative ring, then by Corollary 2.14 the categories RG-**Mod** and G-gr $R[G_+]$-**Mod** are isomorphic.

2.4 Category algebras and smash products

Let $\mathcal{C}$ be a small category, and let R be a commutative ring. A *category algebra* $R[\mathcal{C}]$ is the free R-module with the basis consisting of the elements e_u, where $u\colon c \longrightarrow d$ is a morphism in $\mathcal{C}$, and with multiplication defined by

$$e_v \cdot e_u = \begin{cases} e_{vu}, & \text{if } c' = d; \\ 0, & \text{otherwise} \end{cases}$$

for morphisms $u\colon c \longrightarrow d$ and $v\colon c' \longrightarrow d'$ in $\mathcal{C}$. Equipped with this product, $R[\mathcal{C}]$ becomes a ring that has a unit if $\mathcal{C}$ is finite.

Let A be a G-act and S a G-graded ring. We recall (see [21, p. 390]) that a *smash product* $S\#A$ is the free (left) S-module with the basis $\{p_a \mid a \in A\}$, and with multiplication defined by the bilinear extension of

$$(s_g p_a)(t_h p_b) = \begin{cases} (s_g t_h) p_b, & \text{if } hb = a; \\ 0, & \text{otherwise} \end{cases}$$

where $g, h \in G$, $s_g \in S_g$, $t_h \in S_h$ and $a, b \in A$. Equipped with this multiplication, $S\#A$ is a non-unital ring, i.e. a ring possibly without identity. However, $S\#A$ has *local units*. This means that every finite subset of $S\#A$ is contained in a subring of the form $w(S\#A)w$, where w is an idempotent of $S\#A$. More precisely, let $T := \{t_1, \ldots, t_n\}$ be a finite subset of $S\#A$. We may assume that $t_i = s_i p_{a_i}$, where $g_i \in G$, $a_i \in A$ and $s_i \in S_{g_i}$ for all $i \in \{1, \ldots, n\}$. We denote

$$B := \{a \in A \mid a = a_i \text{ or } a = g_i a_i \text{ for some } i \in \{1, \ldots, n\}\}$$

and $w := \sum_{a \in B} p_a$. It is now straightforward to see that w is idempotent and $wt_iw = wt_i = t_i$ for all $i \in \{1, \ldots, n\}$.

Let R' be a non-unital ring. An R'-module M is *unital* if it satisfies the condition $M = R'M$.

The next proposition and its proof are inspired by [2, p. 221, Cor. 2.4].

Proposition 2.17. *Let M be an $S\#A$-module. Then M is unital if and only if for every finite subset $N \subseteq M$ there exists a finite subset $B \subseteq A$ such that $wn = n$ for all $n \in N$, where $w := \sum_{a \in B} p_a$.*

Proof. Assume first that M is unital. Let $N := \{n_1, \dots, n_p\} \subseteq M$ be a finite set. Now, for all $i \in \{1, \dots, p\}$, the element n_i may be written as

$$n_i = \sum_{j=1}^{q} s_{i,j} n_{i,j},$$

where $s_{i,j} \in S\#A$ and $n_{i,j} \in M$ for all $j \in \{1, \dots, q\}$. This gives us a finite set

$$T = \{s_{i,j} \mid i \in \{1, \dots, p\},\ j \in \{1, \dots, q\}\} \subseteq S\#A.$$

As stated above, we then have a finite subset $B \subseteq A$ such that $w = ws$ for all $s \in T$, where $w := \sum_{a \in B} p_a$. Thus for all $i \in \{1, \dots, p\}$,

$$wn_i = w(\sum_{i=1}^{q} s_{i,j} n_{i,j}) = \sum_{i=1}^{q} (ws_{i,j}) n_{i,j} = \sum_{i=1}^{q} s_{i,j} n_{i,j} = n_i.$$

Conversely, suppose that for every finite subset $N \subseteq M$ there exists a finite subset $B \subseteq A$ such that $wn = n$ for all $n \in N$, where $w := \sum_{a \in B} p_a$. Taking $N = \{m\}$ for $m \in M$, we get $m = wm \in S\#A$. □

Proposition 2.18. *Let R be a commutative ring, G a monoid, and A a G-act. There exists an isomorphism of non-unital rings*

$$\varphi \colon R[G \int A] \to R[G]\#A$$

defined by $e_{(a,g)} \mapsto e_g p_a$ for all $a \in A$ and $g \in G$.

Proof. It is easy to see that φ is an isomorphism of R-modules. It is also a ring homomorphism, since for all $a, b \in A$ and $g, h \in G$,

$$\begin{aligned}
\varphi(e_{(b,h)} e_{(a,g)}) &= \begin{cases} \varphi(e_{(a,hg)}), & \text{if } b = ga; \\ 0, & \text{else} \end{cases} \\
&= \begin{cases} e_{hg} p_a, & \text{if } b = ga; \\ 0, & \text{else} \end{cases} \\
&= (e_h p_b)(e_g p_a) \\
&= \varphi(e_{(b,h)}) \varphi(e_{(a,g)}).
\end{aligned}$$

□

Proposition 2.19. *Let M be an $S\#A$-module. Then $M = \bigoplus_{a\in A} p_a M$ if and only if M is unital.*

Proof. Assume first that $M = \bigoplus_{a\in A} p_a M$. Let $N := \{n_1, \dots, n_p\} \subseteq M$. Since for all $i \in \{1, \dots, p\}$, the element n_i may be written as

$$n_i = \sum_{j=1}^{q} p_{a_{i,j}} n_{i,j},$$

where $a_{i,j} \in A$ and $n_{i,j} \in M$ for all $j \in \{1, \dots, q\}$, there exists a finite subset

$$B := \{a_{i,j} \mid i \in \{1, \dots, p\},\ j \in \{1, \dots, q\}\}$$

of A. Let $w := \sum_{a\in B} p_a$. Then

$$wn_i = w\left(\sum_{j=1}^{q} p_{a_{i,j}} n_{i,j}\right) = \sum_{j=1}^{q} wp_{a_{i,j}} n_{i,j} = n_i,$$

so M is unital by Proposition 2.17.

Assume next that M is unital. Let $m \in M$. By Proposition 2.17, we may write $m = wm$ for some $w = \sum_{a\in B} p_a$, where $B \subseteq A$ is finite. Thus

$$m = (\sum_{a\in B} p_a)m = \sum_{a\in B} p_a m,$$

so that $M = \sum_{a\in A} p_a M$. Furthermore, since the elements p_a are orthogonal, the sum is direct. □

Let us denote by $S\#A$-**Mod** the category of unital $S\#A$-modules. We will now define two functors, Γ and Λ, that connect unital $(S\#A)$-modules to $(G\textstyle\int_S A)$-modules. Let M be a $G\textstyle\int_S A$-module. Set

$$\Gamma M := \bigoplus_{a\in A} M(a).$$

It is not difficult to check that by setting $(sp_a)m = M((a,s))(m_a)$ for all $g \in G$, $s \in S_g$, $a \in A$ and $m := \sum_{b\in A} m_b \in \Gamma M$, ΓM becomes an $S\#A$-module. To show unitality, notice that $p_a(\Gamma M) = M(a)$ for all $a \in A$, which implies that $\Gamma M = \bigoplus_{a\in A} p_a(\Gamma M)$. Thus ΓM is unital by Proposition 2.19. If $f : M \to N$ is a morphism of $G\textstyle\int_S A$-

modules, we can define a homomorphism $\Gamma f\colon \Gamma M \to \Gamma N$ of $(S\#A)$-modules by setting

$$(\Gamma f)(m) = \sum_{a\in A} f_a(m_a)$$

for all $m = \sum_{a\in A} m_a \in \Gamma M$.

Next, let Q be a unital $S\#A$-module. We define a $G\int_S A$-module ΛQ by first setting $(\Lambda Q)(a) = p_a Q$ for all $a \in A$. Let $a \in A$ and $g \in G$, $s \in S_g$. Given a morphism $(a, s)\colon a \to ga$, we then have a homomorphism of Abelian groups

$$(\Lambda Q)((a, s))\colon (\Lambda Q)(a) \to (\Lambda Q)(ga), q \mapsto (sp_a)q.$$

Finally, for a homomorphism $h\colon Q \to P$ of $S\#A$-modules, there is a morphism of $G\int_S A$-modules $\Lambda h\colon \Lambda Q \to \Lambda P$ with $(\Lambda h)_a(q) = h(q)$ for all $a \in A$ and $q \in (\Lambda Q)(a)$.

Theorem 2.20. *Let A be a G-act, and let $S := \oplus_{g\in G} S_g$ be a G-graded ring. The functors Γ and Λ give an isomorphism of categories*

$$(G\textstyle\int_S A)\text{-}\mathbf{Mod} \cong S\#A\text{-}\mathbf{Mod}.$$

Proof. We need to show that $\Gamma\Lambda = \mathrm{id}$ and $\Lambda\Gamma = \mathrm{id}$.

Let Q be a unital $S\#A$-module. By Proposition 2.19 we then have

$$(\Gamma\Lambda)Q = \bigoplus_{a\in A} (\Lambda Q)(a) = \bigoplus_{a\in A} p_a Q = Q.$$

Moreover, the $S\#A$-module structures of Q and $(\Gamma\Lambda)Q$ are the same. Indeed, writing $*$ for the multiplication by $S\#A$ on $(\Gamma\Lambda)Q$, we get

$$(sp_a) * q = (\Lambda Q)((a, s))(p_a q_a) = (sp_a)(p_a q_a) = (sp_a)q$$

for all $a \in A$, $g \in G$, $s \in S_g$ and $q := \sum_{a\in A} p_a q_a \in Q$.

On the other hand, let M be a $G\int_S A$-module. For an object $a \in A$,

$$((\Lambda\Gamma)M)(a) = p_a(\Gamma M) = M(a).$$

Furthermore, if $(a, s)\colon a \to ga$ is a morphism in $G\int_S A$, then

$$((\Lambda\Gamma)M)((a, s))(m) = (sp_a)m = M((a, s))(m)$$

for all $m \in M(a)$, so that $((\Lambda\Gamma)M)((a,s)) = M((a,s))$. $\square$

Corollary 2.21. *Let A be a G-act, and let R be a commutative ring. There exists an isomorphism of categories between the category of $R(G\int A)$-modules and the category of unital $R[G\int A]$-modules.*

Proof. This follows from Proposition 2.18 and Theorem 2.20. $\square$

2.5 Kan extensions and action categories

Let G be a commutative monoid. A morphism of G-acts $\varphi\colon A \to B$ defines a functor of action categories

$$\varphi\colon G\textstyle\int A \to G\int B,$$

where $a \mapsto \varphi(a)$ and $(a,g) \mapsto (\varphi(a),g)$ for all $a \in A, g \in G$. It is important to be able to reindex $R(G\int A)$-modules as $R(G\int B)$-modules, and conversely. In this section, we do this by means of adjoint pairs $(\mathrm{ind}_\varphi, \mathrm{res}_\varphi)$ and $(\mathrm{res}_\varphi, \mathrm{coind}_\varphi)$.

We first remark that the notions of final and initial subsets from Examples 1.7 and 1.10 can be generalized. Let $\alpha\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. We say that α is *final* if the slice category (d/α) is non-empty and connected for every $d \in \mathcal{D}$. Dually, α is *initial* if (α/d) is non-empty and connected for every $d \in \mathcal{D}$. Part 1) of the following proposition appears in [18], and part 2) is the dual result.

Proposition 2.22. [18, p. 217, Thm. 1] *Let $\alpha\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories, $\mathcal{A}$ a preadditive category, and $F\colon \mathcal{D} \to \mathcal{A}$ a functor.*

1) *If α is final and* $\mathrm{colim}_{\mathcal{C}}(F \circ \alpha)$ *exists, then* $\mathrm{colim}_{\mathcal{D}} F$ *exists, and the canonical morphism* $\mathrm{colim}_{\mathcal{C}}(F \circ \alpha) \to \mathrm{colim}_{\mathcal{D}} F$ *is an isomorphism.*
2) *If α is initial and* $\lim_{\mathcal{C}}(F \circ \alpha)$ *exists, then* $\lim_{\mathcal{D}} F$ *exists, and the canonical morphism* $\lim_{\mathcal{D}} F \to \lim_{\mathcal{C}}(F \circ \alpha)$ *is an isomorphism.*

Before continuing, we will define two functors, L_φ and R_φ, from the category of $R(G\int A)$-modules to the category of $R(G\int B)$-modules. Let M be an $R(G\int A)$-module. We want to define $R(G\int B)$-modules $L_\varphi M$ and $R_\varphi M$. First, for $b \in B$, we set

$$(L_\varphi M)(b) = \bigoplus_{a \in \varphi^{-1}(b)} M(a) \quad \text{and} \quad (R_\varphi M)(b) = \prod_{a \in \varphi^{-1}(b)} M(a).$$

Secondly, given a morphism $(b, h)\colon b \to hb$ in $G\int B$, we have $ha \in \varphi^{-1}(hb)$ for all $a \in \varphi^{-1}(b)$. We define a morphism

$$(L_\varphi M)((b, h))\colon (L_\varphi M)(b) \to (L_\varphi M)(hb)$$

as follows: If $m := \sum_{a\in\varphi^{-1}(b)} m_a \in (L_\varphi M)(b)$, then

$$(L_\varphi M)((b, h))(m) = \sum_{a\in\varphi^{-1}(b)} M((a, h))(m_a).$$

Similarly, a morphism

$$(R_\varphi M)((b, h))\colon (R_\varphi M)(b) \to (R_\varphi M)(hb)$$

is defined for all $n := (n_a)_{a\in\varphi^{-1}(b)} \in (R_\varphi M)(b)$ by the formula

$$(R_\varphi M)((b, h))(n) = (M((a, h))(n_a))_{a\in\varphi^{-1}(b)}.$$

Finally, let $\mu\colon M \to N$ be a morphism of $R(G \int A)$-modules. We will define natural transformations

$$L_\varphi\mu\colon L_\varphi M \to L_\varphi N \quad \text{and} \quad R_\varphi\mu\colon R_\varphi M \to R_\varphi N.$$

Let $b \in B$. If

$$m := \sum_{a\in\varphi^{-1}(b)} m_a \in (L_\varphi N)(b) \quad \text{and} \quad n := (n_a)_{a\in\varphi^{-1}(b)} \in (R_\varphi N)(b),$$

we set

$$(L_\varphi\mu)_b(m) = \sum_{a\in\varphi^{-1}(b)} \mu_a(m_a) \quad \text{and} \quad (R_\varphi\mu)_b(n) = (\mu_a(n_a))_{a\in\varphi^{-1}(b)}.$$

From the definitions of L_φ and R_φ, it is clear that we have a natural inclusion

$$i : L_\varphi \to R_\varphi.$$

Proposition 2.23. *Let $\varphi\colon A \to B$ be a morphism of G-acts. Then*

$$\operatorname{ind}_\varphi \cong L_\varphi.$$

Proof. Let $b \in B$, and let M be an $R(G\int A)$-module. Recall that by definition

$$(\operatorname{ind}_\varphi M)(b) = \operatorname*{colim}_{(a,g)\in(\varphi/b)} M(a).$$

Let I be the full subcategory of (φ/b) consisting of all elements of the form $(a,1)$. We note that I is a discrete category. Indeed, if there is a morphism $h\colon (a,1) \to (a',1)$, we immediately see that $a' = ha$ and $h = 1$. We wish to show that the inclusion functor $i\colon I \to (\varphi/b)$ is final. For this, fix $(a,g) \in (\varphi/b)$. We must prove that the slice category $((a,g)/i)$ is non-empty and connected.

First, since $g\varphi(a) = 1\cdot\varphi(ga)$, we note that $(ga,1) \in (\varphi/b)$, and there is a morphism $g\colon (a,g) \to (ga,1)$. Hence $((ga,1),g) \in ((a,g)/i)$, so that the slice category is non-empty. Secondly, we will show that (φ/b) consists of only one element. Suppose that $((a',1),h) \in ((a,g)/i)$. Then $(a',1) \in (\varphi/b)$ and there is a morphism $h\colon (a,g) \to (a',1)$. This means that the diagram

$$\begin{tikzcd}
\varphi(a) \arrow[r, "g"] \arrow[d, "h"'] & b \\
\varphi(a') \arrow[ur, "1"'] &
\end{tikzcd}$$

commutes, implying $h = g$, and furthermore, $a' = ga$. Thus $((ga,1),g)$ is the only element of $((a,g)/i)$. Since i is final and I is discrete, we have

$$(\operatorname{ind}_\varphi M)(b) \cong \operatorname*{colim}_{(a,1)\in I} M(a) \cong \bigoplus_{a\in\varphi^{-1}(b)} M(a) = (L_\varphi M)(b).$$

by Proposition 2.22 1). It is now straightforward to see that these isomorphisms yield an isomorphism of $R(G\int B)$-modules $\operatorname{ind}_\varphi M \to L_\varphi M$, and moreover, a natural isomorphism $\operatorname{ind}_\varphi \to L_\varphi$. $\square$

If G is an Abelian group, we get a dual result.

Proposition 2.24. *Assume that G is an Abelian group. Let $\varphi\colon A \to B$ be a morphism* *of G-acts. Then*

$$\mathrm{coind}_\varphi \cong R_\varphi.$$

Proof. Let $b \in B$, and let M be an $R(G\int A)$-module. By definition,

$$(\mathrm{coind}_\varphi M)(b) = \lim_{(a,g)\in(b/\varphi)} M(a).$$

Let I be the full subcategory of (b/φ) consisting of all elements of the form $(a,1)$. We note that I is a discrete category, because if there is a morphism $h\colon (a,1) \to (a',1)$, we immediately see that $a' = ha$ and $h = 1$. We wish to show that the inclusion functor $i\colon I \to (b/\varphi)$ is initial. To do this, fix $(a,g) \in (b/\varphi)$. We must prove that the slice category $(i/(a,g))$ is non-empty and connected.

First, since $1 \cdot b = g^{-1}\varphi(a) = \varphi(g^{-1}a)$, we note that $(g^{-1}a,1) \in (b/\varphi)$ and there is a morphism $g\colon (g^{-1}a,1) \to (a,g)$. Hence $((g^{-1}a,1),g) \in (i/(a,g))$, so that the slice category is non-empty. Secondly, we will show that (b/φ) consists of only one element. Suppose that $((a',1),h) \in (i/(a,g))$. Then $(a',1) \in (b/\varphi)$ and there is a morphism $h\colon (a',1) \to (a,g)$. This means that the diagram

$$\begin{tikzcd} b \arrow[r, "1"] \arrow[dr, "g"'] & \varphi(a') \arrow[d, "h"] \\ & \varphi(a) \end{tikzcd}$$

commutes, implying $h = g$, and furthermore, $a' = g^{-1}a$. Thus $((g^{-1}a,1),g)$ is the only element of $((a,g)/i)$. Since i is initial and I is discrete, we have

$$(\mathrm{coind}_\varphi M)(b) \cong \lim_{(a,1)\in I} M(a) \cong \prod_{a\in\varphi^{-1}(b)} M(a) \cong (R_\varphi M)(b).$$

by Proposition 2.22 2). It is now straightforward to see that these isomorphisms yield an isomorphism of $R(G\int B)$-modules $\mathrm{coind}_\varphi M \to R_\varphi M$, and moreover, a natural isomorphism $\mathrm{coind}_\varphi \to R_\varphi$. $\square$

Corollary 2.25. *Assume that G is an Abelian group. Let $\varphi\colon A \to B$ be a morphism of G-acts. Then there is a natural transformation*

$$\rho\colon \mathrm{ind}_\varphi \to \mathrm{coind}_\varphi$$

that is an isomorphism if and only if $\varphi^{-1}(b)$ is finite for all $b \in B$. If M is an $R(G\int A)$-module, then ρ_M is an isomorphism if and only if $\mathrm{Supp}(M) \cap \varphi^{-1}(b)$ *is finite for all $b \in B$.*

Proof. We define a natural transformation $\rho\colon \mathrm{ind}_\varphi \to \mathrm{coind}_\varphi$ as the composition

$$\mathrm{ind}_\varphi \xrightarrow{\cong} L_\varphi \xrightarrow{i} R_\varphi \xrightarrow{\cong} \mathrm{coind}_\varphi .$$

Here the first isomorphism is from Proposition 2.23, i is the natural inclusion, and the last isomorphism is given in Proposition 2.24. We immediately see that ρ is an isomorphism if and only if direct sums and direct products indexed by $\varphi^{-1}(b)$ coincide for all $b \in B$, and this is equivalent to $\varphi^{-1}(b)$ being finite for all $b \in B$.

For the last part of the proposition, let M be an $R(G\int A)$-module, and let $b \in B$. Then $\mathrm{Supp}(M)\cap\varphi^{-1}(b)$ is finite if and only if $M(a) \neq 0$ for finitely many $a \in \varphi^{-1}(b)$, and these conditions are true if and only if $\rho_{M,b}$ is an isomorphism. This implies the claimed equivalence. □

Remark 2.26. By Corollary 2.14, the category of $R(G\int A)$-modules is isomorphic to the category of A-graded $R[G]$-modules. Let $\varphi\colon A \to B$ be a morphism of G-acts. Then the graded module versions of the functors

$$\begin{aligned} &\mathrm{ind}_\varphi\colon R(G\textstyle\int A)\text{-}\mathbf{Mod} \to R(G\textstyle\int B)\text{-}\mathbf{Mod} \quad \text{and} \\ &\mathrm{res}_\varphi\colon R(G\textstyle\int B)\text{-}\mathbf{Mod} \to R(G\textstyle\int A)\text{-}\mathbf{Mod} \end{aligned}$$

appear in [28] with several different names:

- If φ is surjective, the functor ind_φ is called *φ-coarsening* and res_φ is *φ-refinement*.
- If φ is injective, the functor ind_φ is called *φ-extension* and res_φ is *φ-restriction*.

3 FINITELY PRESENTED $R\mathcal{C}$-MODULES

We will assume in the following that $\mathcal{C}$ is a small category and R a commutative ring. Recall from Section 1.5 that an $R\mathcal{C}$-module M is

- *finitely generated* if there exists an epimorphism

$$\bigoplus_{i\in I} R[\mathrm{Mor}_{\mathcal{C}}(c_i, -)] \to M$$

  where I is a finite set, and $c_i \in \mathcal{C}$ for all $i \in I$;

- *finitely presented* if there exists an exact sequence

$$\bigoplus_{j\in J} R[\mathrm{Mor}_{\mathcal{C}}(d_j, -)] \to \bigoplus_{i\in I} R[\mathrm{Mor}_{\mathcal{C}}(c_i, -)] \to M \to 0,$$

  where I and J are finite sets, and $c_i, d_j \in \mathcal{C}$ for all $i \in I$ and $j \in J$.

3.1 $\mathcal{S}$-presented and $\mathcal{S}$-generated $R\mathcal{C}$-modules

Let $\mathcal{S} \subseteq \mathcal{C}$ be a full subcategory. The notions of $\mathcal{S}$-generated and $\mathcal{S}$-presented modules will play an important role in the rest of this thesis. Before going into details, we will recall some facts about the restriction and induction functors along the inclusion $i\colon \mathcal{S} \subseteq \mathcal{C}$ from Section 1.6.

The restriction $\mathrm{res}_{\mathcal{S}}\colon R\mathcal{C}\text{-}\mathbf{Mod} \to R\mathcal{S}\text{-}\mathbf{Mod}$ is defined by precomposition with i, and the induction $\mathrm{ind}_{\mathcal{S}}\colon R\mathcal{S}\text{-}\mathbf{Mod} \to R\mathcal{C}\text{-}\mathbf{Mod}$ is its left Kan extension along i. The induction is the left adjoint of the restriction. Note, in particular, that it thus commutes with colimits. The counit of this adjunction gives us for every $R\mathcal{C}$-module M the *canonical morphism*

$$\mu_M\colon\ \mathrm{ind}_{\mathcal{S}}\,\mathrm{res}_{\mathcal{S}}\, M \to M,$$

which we will use frequently.

Let A be an R-module and $c \in \mathcal{C}$. We define an $R\mathcal{C}$-module

$$A[\mathrm{Mor}_{\mathcal{C}}(c, -)] := A \otimes_R R[\mathrm{Mor}_{\mathcal{C}}(c, -)]$$

by taking a pointwise tensor product. We note that the functor R-**Mod** $\to R\mathcal{C}$-**Mod** that sends A to $A[\mathrm{Mor}_{\mathcal{C}}(c, -)]$ is right exact for all $c \in \mathcal{C}$.

Proposition 3.1. *Let $S \subseteq \mathcal{C}$ be a full subcategory, A an R-module, and $s \in S$. Then*

$$\mathrm{ind}_S \,\mathrm{res}_S\, A[\mathrm{Mor}_{\mathcal{C}}(s, -)] \cong A[\mathrm{Mor}_{\mathcal{C}}(s, -)].$$

Proof. By Yoneda's lemma and the aforementioned adjunction, we have the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{R\mathcal{C}}(R[\mathrm{Mor}_{\mathcal{C}}(s, -)], M) &\cong M(s) \\
&\cong \mathrm{Hom}_{RS}(R[\mathrm{Mor}_S(s, -)], \mathrm{res}_S\, M) \\
&\cong \mathrm{Hom}_{R\mathcal{C}}(\mathrm{ind}_S\, R[\mathrm{Mor}_S(s, -)], M).
\end{aligned}$$

This shows us that $\mathrm{ind}_S \,\mathrm{res}_S\, R[\mathrm{Mor}_{\mathcal{C}}(s, -)] \cong R[\mathrm{Mor}_{\mathcal{C}}(s, -)]$. In particular

$$\operatorname*{colim}_{(t,u) \in (i/d)} R[\mathrm{Mor}_{\mathcal{C}}(s, t)] \cong R[\mathrm{Mor}_{\mathcal{C}}(s, d)]$$

for $d \in \mathcal{C}$. Since tensoring commutes with colimits, we see that for all $d \in \mathcal{C}$,

$$\begin{aligned}
(\mathrm{ind}_S \,\mathrm{res}_S\, A[\mathrm{Mor}_{\mathcal{C}}(s, -)])(d) &= \operatorname*{colim}_{(t,u) \in (i/d)} A[\mathrm{Mor}_{\mathcal{C}}(s, t)] \\
&\cong A \otimes_R \operatorname*{colim}_{(t,u) \in (i/d)} R[\mathrm{Mor}_{\mathcal{C}}(s, t)] \\
&\cong A \otimes_R R[\mathrm{Mor}_{\mathcal{C}}(s, d)] \\
&\cong A[\mathrm{Mor}_{\mathcal{C}}(s, d)].
\end{aligned}$$

Therefore $\mathrm{ind}_S \,\mathrm{res}_S\, A[\mathrm{Mor}_{\mathcal{C}}(s, -)] \cong A[\mathrm{Mor}_{\mathcal{C}}(s, -)]$ as wanted. □

An $R\mathcal{C}$-module M is said to be S-*generated* if the natural morphism

$$\bigoplus_{s \in S} M(s)[\mathrm{Mor}_{\mathcal{C}}(s, -)] \to M$$

is an epimorphism. Since this morphism factors through the canonical morphism $\mu_M\colon \operatorname{ind}_S \operatorname{res}_S M \to M$, we see that M is S-generated if and only if μ_M is an epimorphism.

Proposition 3.2. *Let $S \subseteq \mathcal{C}$ be a full subcategory. Assume that M is an S-generated $R\mathcal{C}$-module, so that we have an exact sequence of $R\mathcal{C}$-modules*

$$0 \to K \to \bigoplus_{s \in S} M(s)[\mathrm{Mor}_{\mathcal{C}}(s, -)] \to M \to 0.$$

Then the following are equivalent:

1) *The canonical morphism $\mu_M\colon \operatorname{ind}_S \operatorname{res}_S M \to M$ is an isomorphism;*
2) *If there exists an exact sequence of $R\mathcal{C}$-modules*

   $$0 \to L \to N \to M \to 0,$$

   where N is S-generated, then L is S-generated;
3) *K is S-generated;*
4) *The sequence*

   $$\bigoplus_{s \in S} K(s)[\mathrm{Mor}_{\mathcal{C}}(s, -)] \to \bigoplus_{s \in S} M(s)[\mathrm{Mor}_{\mathcal{C}}(s, -)] \to M \to 0$$

   is exact;
5) *For each $s \in S$, there exist R-modules A_s and B_s such that the sequence*

   $$\bigoplus_{s \in S} B_s[\mathrm{Mor}_{\mathcal{C}}(s, -)] \to \bigoplus_{s \in S} A_s[\mathrm{Mor}_{\mathcal{C}}(s, -)] \to M \to 0$$

   is exact.

*When these equivalent conditions hold, we say that M is S-*presented*.*

Proof. We will show that 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) $\Rightarrow$ 1). Assume first that 1) holds, and that there is an exact sequence of $R\mathcal{C}$-modules

$$0 \to L \to N \to M \to 0.$$

Since the functor res_S is exact and the functor ind_S right exact, we get a commutative diagram with exact rows

$$\begin{tikzcd}
 & \mathrm{ind}_S \, \mathrm{res}_S \, L \arrow[r] \arrow[d, "\mu_L"'] & \mathrm{ind}_S \, \mathrm{res}_S \, N \arrow[r] \arrow[d, "\mu_N"'] & \mathrm{ind}_S \, \mathrm{res}_S \, M \arrow[r] \arrow[d, "\mu_M"'] & 0 \arrow[d] \\
0 \arrow[r] & L \arrow[r] & N \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where μ_M is an isomorphism and μ_N is an epimorphism. An easy diagram chase shows us that μ_L is an epimorphism, so 2) holds.

The implication 2) $\Rightarrow$ 3) is trivial. Assume next that 3) holds. Now the morphism $\bigoplus_{s\in S} K(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)] \to K$ is an epimorphism, so the images of K and $\bigoplus_{s\in S} K(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)]$ are the same in $\bigoplus_{s\in S} M(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)]$. The required exactness then follows immediately.

Trivially 4) implies 5). Finally, let us assume that 5) holds. By Proposition 3.1, we get a commutative diagram with exact rows

$$\begin{tikzcd}
\bigoplus_{s\in S} B_s[\mathrm{Mor}_{\mathcal{C}}(s,-)] \arrow[r] \arrow[d, "\cong"'] & \bigoplus_{s\in S} A_s[\mathrm{Mor}_{\mathcal{C}}(s,-)] \arrow[r] \arrow[d, "\cong"'] & \mathrm{ind}_S \, \mathrm{res}_S \, M \arrow[r] \arrow[d, "\mu_M"'] & 0 \arrow[d] \\
\bigoplus_{s\in S} B_s[\mathrm{Mor}_{\mathcal{C}}(s,-)] \arrow[r] & \bigoplus_{s\in S} A_s[\mathrm{Mor}_{\mathcal{C}}(s,-)] \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

from which we can see that μ_M is an isomorphism by the five lemma. □

Remark 3.3. Proposition 3.2 is due to Djament [9, p. 11, Prop. 2.14]. The reader should be cautious, since we use the term 'support' in a different meaning as in [9].

A small category is said to be *locally finite*, if its every morphism set is finite. The following proposition is a special case of [10, p. 83, Prop.]. For the sake of clarity, we present a proof using our notation.

Proposition 3.4. *Let $\mathcal{C}$ be locally finite. An $R\mathcal{C}$-module M is finitely presented if and only if there exists a finite full subcategory $S \subseteq \mathcal{C}$ such that*

1) $M(s)$ is finitely presented for all $s \in S$;

2) M is S-presented.

Proof. Assume first that M is finitely presented, so that there exists an exact sequence

$$\bigoplus_{j\in J} R[\mathrm{Mor}_{\mathcal{C}}(b_j,-)] \to \bigoplus_{i\in I} R[\mathrm{Mor}_{\mathcal{C}}(a_i,-)] \to M \to 0,$$

where I and J are finite sets, and $a_i, b_j \in \mathcal{C}$ for all $i \in I$ and $j \in J$. Evaluating this at point $c \in \mathcal{C}$ gives us an exact sequence

$$R^{m_c} \to R^{n_c} \to M(c) \to 0$$

for some $m_c, n_c \in \mathbb{N}$, so that 1) holds. For 2), by setting

$$S := \{a_i \mid i \in I\} \cup \{b_j \mid j \in J\}$$

we immediately see that M is S-presented by Proposition 3.2 5).

Assume next that there exists a finite full subcategory $S \subseteq \mathcal{C}$ such that 1) and 2) hold. Now M is S-generated, so the natural morphism $\bigoplus_{s\in S} M(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)] \to M$ is an epimorphism. Since $M(s)$ is finitely generated for all $s \in S$, there exists an epimorphism $R^{n_s} \to M(s)$ for all $s \in S$, where $n_s \in \mathbb{N}$. Combining these epimorphisms, we get an epimorphism

$$\bigoplus_{t\in S} R^{n_t}[\mathrm{Mor}_{\mathcal{C}}(t,-)] \to \bigoplus_{t\in S} M(t)[\mathrm{Mor}_{\mathcal{C}}(t,-)] \to M$$

and an exact sequence

$$0 \to N \to \bigoplus_{t\in S} R^{n_t}[\mathrm{Mor}_{\mathcal{C}}(t,-)] \to M \to 0.$$

Because M is S-presented, N must be S-generated by Proposition 3.2 2), so there exists an epimorphism $\bigoplus_{t\in S} N(t)[\mathrm{Mor}_S(t,-)] \to N$. On the other hand, $M(s)$ is finitely presented, so $N(s)$ is finitely generated for all $s \in S$. Thus there exists an epimorphism $R^{m_s} \to N(s)$ for all $s \in S$, where $m_s \in \mathbb{N}$. Hence we get an exact sequence

$$\bigoplus_{t\in S} R^{m_t}[\mathrm{Mor}_{\mathcal{C}}(t,-)] \to \bigoplus_{t\in S} R^{n_t}[\mathrm{Mor}_{\mathcal{C}}(t,-)] \to M \to 0.$$

□

From the proof of Proposition 3.4 we immediately get the following corollary:

Corollary 3.5. *An $R\mathcal{C}$-module M is finitely generated if and only if there exists a finite full subcategory $S \subseteq \mathcal{C}$ such that*

1) $M(s)$ is finitely generated for all $s \in S$;

2) M is S-generated.

3.2 Births and deaths relative to S

From now on, we will assume that $\mathcal{C}$ is a poset.

Let M be an $R\mathcal{C}$-module, $S \subseteq \mathcal{C}$ a subset, and $c \in \mathcal{C}$. Write $S' := S\backslash\{c\}$. We note that

$$\operatorname*{colim}_{d<c,\ d\in S} M(d) = \operatorname*{colim}_{d\leq c,\ d\in S'} M(d) = (\mathrm{ind}_{S'}\,\mathrm{res}_{S'}\,M)(c).$$

Since $\mathrm{res}_{S'}$ is exact and $\mathrm{ind}_{S'}$ is right exact, we then see that the functor

$$R\mathcal{C}\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod},\ M \mapsto \operatorname*{colim}_{d<c,\ d\in S} M(d).$$

is also right exact.

Definition 3.6. Let $\mathcal{C}$ be a poset, M an $R\mathcal{C}$-module, $S \subseteq \mathcal{C}$ a subset and $c \in \mathcal{C}$. Let

$$\lambda_{M,c}\colon \operatorname*{colim}_{d<c,\ d\in S} M(d) \to M(c)$$

be the natural homomorphism. We define the *set of births relative to S* by

$$B_S(M) := \{c \in \mathcal{C} \mid \lambda_{M,c} \text{ is a non-epimorphism}\}$$

and the *set of deaths relative to S* by

$$D_S(M) := \{c \in \mathcal{C} \mid \lambda_{M,c} \text{ is a non-monomorphism}\}.$$

Remark 3.7. Note that $\lambda_{M,c}$ is an epimorphism if and only if the natural homomorphism $\bigoplus_{d<c,\ d\in S} M(d) \to M(c)$ is an epimorphism. This implies that if $T \subseteq S \subseteq \mathcal{C}$, then $B_S(M) \subseteq B_T(M)$.

Example 3.8. Let $\mathcal{C}$ be a poset. Let I be an *interval* of $\mathcal{C}$ i.e. a non-empty subset of $\mathcal{C}$ satisfying the condition that if $a, b \in I$, $c \in \mathcal{C}$ and $a \leq c \leq b$, then $c \in I$. Let R_I be the $R\mathcal{C}$-module defined on objects by

$$R_I(c) = \begin{cases} R, & \text{when } c \in I; \\ 0, & \text{otherwise,} \end{cases}$$

and with identity morphisms inside the interval. Then the sets of births $B_{\mathcal{C}}(R_I)$ and $B_I(R_I)$ both consist of the minimal elements of I. To find the deaths, we note that R_I is $\uparrow I$-presented, so deaths must either be inside I or above it (see Remark 4.2).

First, let $c \in S_1 := (\uparrow I)\backslash I$. Now $R_I(c) = 0$. Since $S_1 \subseteq \uparrow \mathrm{Supp}(R_I)$, we see that $\mathrm{colim}_{d<c,\ d\in I} R_I(d) \neq 0$. Thus $c \in D_I(R_I)$, so $S_1 \subseteq D_I(R_I)$. Furthermore, it is clear that $\mathrm{colim}_{d<c} R_I(d) \neq 0$ if and only if c is minimal in S_1. This implies that exactly the minimal elements of S_1 are in $D_{\mathcal{C}}(R_I)$.

Secondly, let $c \in I$. It is straightforward to see that $c \in D_I(R_I)$ if and only if the set $(I \cap \downarrow c)\backslash\{c\}$ is not connected as a poset. This applies also to $D_{\mathcal{C}}(R_I)$. Set

$$S_2 := \{c \in I \mid (I \cap \downarrow c)\backslash\{c\} \text{ is not connected }\}.$$

We conclude that $D_I(R_I) = S_1 \cup S_2$, while $D_{\mathcal{C}}(R_I)$ is the union of the set of the minimal elements of S_1, and the set S_2.

Proposition 3.9. *Let $\mathcal{C}$ be a poset, M an $R\mathcal{C}$-module, and $S \subseteq \mathcal{C}$ a subset. Then*

1) M is S-generated if and only if $B_S(M) \subseteq S$;

2) M is S-presented if and only if $B_S(M) \cup D_S(M) \subseteq S$.

Proof. Both 1) and 2) are proved similarly. We only prove 2) here. Directly from the definitions,

$$\begin{aligned} & M \text{ is } S\text{-presented} \\ \Leftrightarrow\ & \mu_{M,c} \colon \operatorname*{colim}_{d\leq c,\ d\in S} M(d) \to M(c) \text{ is an isomorphism for all } c \in \mathcal{C} \\ \Leftrightarrow\ & \lambda_{M,c} \colon \operatorname*{colim}_{d< c,\ d\in S} M(d) \to M(c) \text{ is an isomorphism for all } c \in \mathcal{C}\backslash S \\ \Leftrightarrow\ & B_S(M) \cup D_S(M) \subseteq S. \end{aligned}$$

□

Let $S \subseteq \mathcal{C}$. We may think of $B_{\mathcal{C}}(M)$ as the set of ‘real’ births of M. The following proposition shows that for S-generated modules we may focus only on births relative to S.

Proposition 3.10. *Let M be an S-generated $R\mathcal{C}$-module. Then $B_{\mathcal{C}}(M) = B_S(M)$.*

Proof. By Remark 3.7 it is enough to show that $B_S(M) \subseteq B_{\mathcal{C}}(M)$. Let $c \in \mathcal{C} \backslash B_{\mathcal{C}}(M)$, so that the natural homomorphism $\bigoplus_{d<c} M(d) \to M(c)$ is an epimorphism. Since M is S-generated, there is an epimorphism

$$\bigoplus_{d' \leq d,\ d' \in S} M(d') \to M(d)$$

for all $d < c$. We may combine these epimorphisms to get an epimorphism

$$\bigoplus_{d<c,\ d \in S} M(d) \to M(c),$$

implying that $c \in \mathcal{C} \backslash B_S(M)$. □

3.3 S-splitting

Definition 3.11. Let $S \subseteq \mathcal{C}$ a subset and $c \in \mathcal{C}$. If M is an $R\mathcal{C}$-module, denote by $S_{S,c}M$ the R-module defined by the exact sequence

$$\operatorname*{colim}_{d<c,\ d \in S} M(d) \xrightarrow{\lambda_{M,c}} M(c) \xrightarrow{\pi_{M,c}} S_{S,c}M \to 0,$$

where $\pi_{M,c}$ is the canonical epimorphism. If $\varphi \colon M \to N$ is a morphism of $R\mathcal{C}$-modules, we have a commutative diagram

$$\begin{tikzcd}
\operatorname*{colim}_{d<c,\ d \in S} M(d) \arrow[r, "\lambda_{M,c}"] \arrow[d] & M(c) \arrow[r, "\pi_{M,c}"] \arrow[d, "\varphi_c"'] & S_{S,c}M \arrow[r] \arrow[d, dashed, "S_{S,c}\varphi"] & 0 \\
\operatorname*{colim}_{d<c,\ d \in S} N(d) \arrow[r, "\lambda_{N,c}"'] & N(c) \arrow[r, "\pi_{N,c}"'] & S_{S,c}N \arrow[r] & 0
\end{tikzcd}$$

where the existence of $S_{S,c}\varphi$ follows from the the universal property of cokernels. This gives rise to a functor $S_{S,c}$, the *S-splitting functor at c*. More explicitly,

$$S_{S,c}M = M(c)/\operatorname{Im}(\lambda_{M,c}).$$

Remark 3.12. The concept of a splitting functor is due to Lück ([17, p. 156]). For $c \in \mathcal{C}$, the original splitting functor $S_c \colon R\mathcal{C}$-**Mod** $\to R$-**Mod** is the case $S = \mathcal{C}$ of the S-splitting functor. That is, if M is an $R\mathcal{C}$-module, then

$$S_c M := M(c)/\mathrm{Im}(\eta_{M,c}),$$

where $\eta_{M,c}$ is the canonical morphism $\mathrm{colim}_{d<c} M(d) \to M(c)$.

The S-splitting functor at c could equivalently be defined as the composition of the splitting functor $S_c \colon R(S \cup \{c\})$-**Mod** $\to R$-**Mod** and the restriction functor $\mathrm{res}_{S\cup\{c\}}$ by setting

$$S_{S,c} = S_c \circ \mathrm{res}_{S\cup\{c\}}\,.$$

Since both S_c and $\mathrm{res}_{S\cup\{c\}}$ are left adjoints, we see that $S_{S,c}$ is a left adjoint, and thus additive.

The basic example for us is the following:

Example 3.13. Let k be a field, $S \subseteq \mathbb{Z}^n$ a subset, and M a $k(\mathbb{N}^n \int \mathbb{Z}^n)$-module. We identify M with the corresponding $\mathbb{Z}^n$-graded $k[X_1, \dots, X_n]$-module. Denote by $m := \langle X_1, \dots, X_n \rangle$ the maximal homogeneous ideal of $k[X_1, \dots, X_n]$. If N is the homogeneous submodule of M generated by the union of M_s, where $s \in S$, we notice that

$$(M/mN)_c = M_c/(mN)_c = M(c)/\mathrm{Im}(\lambda_{M,c}) = S_{S,c}M$$

for all $c \in \mathbb{Z}^n$. In particular, this yields an isomorphism of k-vector spaces,

$$M/mN \cong \bigoplus_{c\in\mathbb{Z}^n} S_{S,c}M.$$

Remark 3.14. Let M be an $R\mathcal{C}$-module and $S \subseteq \mathcal{C}$ a subset. Note that for all $c \in \mathcal{C}$, we have $c \in B_S(M)$ if and only if $S_{S,c}M \neq 0$.

Remark 3.15. Let A be an R-module, $S \subseteq \mathcal{C}$ a subset, $s \in S$, and $c \in \mathcal{C}$. Let $S' := S \backslash \{c\}$. If $s \neq c$, we see that

$$\operatorname*{colim}_{d<c,\ d\in S} A[\mathrm{Mor}_{\mathcal{C}}(s,d)] = \operatorname*{colim}_{d\leq c,\ d\in S'} A[\mathrm{Mor}_{\mathcal{C}}(s,d)] \cong A[\mathrm{Mor}_{\mathcal{C}}(s,c)]$$

by Proposition 3.1. If $s = c$, then obviously $\mathrm{colim}_{d<c,\ d\in S} A[\mathrm{Mor}_{\mathcal{C}}(s,d)] = 0$. In

particular

$$S_{S,c}(A[\mathrm{Mor}_{\mathcal{C}}(s,-)]) = \begin{cases} A, & \text{when } s = c \\ 0, & \text{otherwise.} \end{cases}$$

Next, we prove a version of Nakayama's lemma (cf. [31, p. 12, Lemma 6.2]).

Lemma 3.16. *Let M be an $R\mathcal{C}$-module and $S \subseteq \mathcal{C}$ a subset. If $\mathrm{Supp}(M) \cap S$ has a minimal element c, then $S_{S,c}M \neq 0$.*

Proof. Assume that $c \in \mathrm{Supp}(M) \cap S$ is minimal. Then $M(d) = 0$ for all $d \in S$ with $d < c$. In particular, $\mathrm{colim}_{d<c,\ d\in S}\, M(d) = 0$. Thus $S_{S,c}M \neq 0$. □

Recall that a poset P is called *Artinian*, if there are no infinite strictly descending chains of elements of P, or equivalently, if every non-empty subset $S \subseteq P$ has a minimal element.

Proposition 3.17. *Let $f\colon L \to M$ be a morphism of $R\mathcal{C}$-modules, where M is S-generated with an Artinian $S \subseteq \mathcal{C}$. If $S_{S,c}f\colon S_{S,c}L \to S_{S,c}M$ is an epimorphism for all $c \in B_S(M)$, then f is an epimorphism.*

Proof. We first note that $\mathrm{Coker} f$ is S-generated, since M is S-generated. Suppose that f is not an epimorphism. Then $\mathrm{Coker} f \neq 0$, so there exists $s \in S$ such that $(\mathrm{Coker} f)(s) \neq 0$. Hence $\mathrm{Supp}(\mathrm{Coker} f) \cap S$ has a minimal element c by the Artinian property. Now $S_{S,c}(\mathrm{Coker} f) \neq 0$ by Lemma 3.16, which implies that $c \in B_S(\mathrm{Coker} f) \subseteq B_S(M)$. Since $S_{S,c}$ is right exact, we get $\mathrm{Coker}\, S_{S,c}f \neq 0$, so $S_{S,c}f$ is not an epimorphism. □

Lemma 3.18. *Let $S \subseteq \mathcal{C}$ be a subset and*

$$0 \to L \xrightarrow{j} N \xrightarrow{f} M \to 0$$

an exact sequence of $R\mathcal{C}$-modules. The following are equivalent for all $c \in \mathcal{C}$:

1) $(\mathrm{Ker} f)(c) \subseteq \mathrm{Im}\, \lambda_{N,c}$;

2) $S_{S,c}(j) = 0$;

3) $S_{S,c}(f)$ *is a monomorphism;*

4) $S_{S,c}(f)$ *is an isomorphism.*

Proof. The equivalence of 1) and 3) immediately follows from the fact that

$$\operatorname{Ker} S_{S,c}(f) = ((\operatorname{Ker} f)(c) + \operatorname{Im} \lambda_{N,c})/\operatorname{Im} \lambda_{N,c}.$$

Since $S_{S,c}$ is right exact, we have $\operatorname{Ker} S_{S,c}(f) = \operatorname{Im} S_{S,c}(j)$. Therefore 2) is equivalent to 3). The equivalence of 3) and 4) holds, because $S_{S,c}$ preserves epimorphisms. □

We recall that an epimorphism of $R\mathcal{C}$-modules $f\colon N \to M$ is called *minimal*, if for all morphisms $g\colon L \to N$, fg is an epimorphism if and only if g is an epimorphism. It is known that an epimorphism f is minimal if and only if for all submodules $N' \subseteq N$

$$N' + \operatorname{Ker} f = N \;\Rightarrow\; N' = N.$$

A minimal epimorphism $f\colon N \to M$, where N is projective, is called a *projective cover* of M (see e.g. [1, p. 28]).

Remark 3.19. Let A be an R-module and $c \in \mathcal{C}$. Then A may be thought of as an $R\{c\}$-module, and we note that $A[\operatorname{Mor}_{\mathcal{C}}(c,-)] \cong \operatorname{ind}_{\{c\}} A$. In particular, the functor $A \mapsto A[\operatorname{Mor}_{\mathcal{C}}(c,-)]$ preserves projectives, since it is the left adjoint of the exact functor $\operatorname{res}_{\{c\}}$.

Proposition 3.20. *Let $f\colon N \to M$ be an epimorphism of S-generated $R\mathcal{C}$-modules, where $S \subseteq \mathcal{C}$ is Artinian. If $(\operatorname{Ker} f)(c) \subseteq \operatorname{Im} \lambda_{N,c}$ for all $c \in \mathcal{C}$, then f is minimal. The converse implication holds if $S_{S,c}M$ is projective for all $c \in S$.*

Proof. Let $(\operatorname{Ker} f)(c) \subseteq \operatorname{Im} \lambda_{N,c}$ for all $c \in \mathcal{C}$. Suppose that $N' + \operatorname{Ker} f = N$ for some submodule $N' \subseteq N$. We note that for all $c \in \mathcal{C}$, $(N')(c) + \operatorname{Im} \lambda_{N,c} = N(c)$. This implies that $S_{S,c}N' = S_{S,c}N$ for all $c \in \mathcal{C}$. Since S is Artinian, we may use Proposition 3.17 to conclude that $N' = N$, so f is minimal.

Next, let f be minimal, and let $S_{S,c}M$ be projective for all $c \in S$. Thus we can find sections $S_{S,c}M \to M(c)$ for all $c \in S$. These induce a morphism

$$h\colon \bigoplus_{s\in S} S_{S,s}M[\operatorname{Mor}_{\mathcal{C}}(s,-)] \to \bigoplus_{s\in S} M(s)[\operatorname{Mor}_{\mathcal{C}}(s,-)] \to M.$$

Remark 3.15 now implies that $S_{S,c}h = \operatorname{id}_{S_{S,c}M}$ for all $c \in S$, so h is an epimorphism by Proposition 3.17.

Since $S_{S,c}M$ is projective for all $c \in S$, we see that $\bigoplus_{s\in S} S_{S,s}M[\operatorname{Mor}_{\mathcal{C}}(s,-)]$ is also projective by Remark 3.19 (as a sum of projectives). Thus the morphism h factors through f, and we get a diagram

$$\begin{tikzcd}
\bigoplus_{s\in S} S_{S,s}M[\mathrm{Mor}_c(s,-)] \arrow[r, "g"] \arrow[dr, "h"'] & N \arrow[d, "f"] \\
& M
\end{tikzcd}$$

that commutes. Now f is minimal, so g is an epimorphism. Applying functor $S_{S,c}$, where $c \in S$, on the diagram, we see that $S_{S,c}f \circ S_{S,c}g = \mathrm{id}$, which implies that $S_{S,c}g$ is a monomorphism, and therefore an isomorphism. Hence $S_{S,c}f$ is an isomorphism for all $c \in S$. This is equivalent to $(\mathrm{Ker} f)(c) \subseteq \mathrm{Im}\,\lambda_{N,c}$ for all $c \in \mathcal{C}$ by Lemma 3.18. □

Remark 3.21. Let M be an S-generated $R\mathcal{C}$-module, where S is Artinian. If $S_{S,c}M$ is projective for all $c \in S$, the morphism $h\colon \bigoplus_{s\in S} S_{S,s}M[\mathrm{Mor}_c(s,-)] \to M$ induced by sections $S_{S,s}M \to M(s)$ is a projective cover of M.

Indeed, as noted earlier, h is an epimorphism with $S_{S,c}h = \mathrm{id}$ for all $c \in S$. Then Lemma 3.18 implies that $(\mathrm{Ker} f)(c) \subseteq \mathrm{Im}\,\lambda_{N,c}$ for all $c \in \mathcal{C}$, and the rest follows from Proposition 3.20.

3.4 Minimality of births and deaths

We will now show how the sets of births and deaths relative to a subset $S \subseteq \mathcal{C}$ are in a sense minimal if the module is S-generated or S-presented.

Proposition 3.22. *Let M be an S-generated $R\mathcal{C}$-module, where $S \subseteq \mathcal{C}$ is Artinian. Then M is $B_S(M)$-generated. Furthermore, $B_S(M)$ is the minimum element of the set $\{T \subseteq S \mid M \text{ is } T\text{-generated}\}$.*

Proof. Let ρ be the natural morphism $\rho\colon \bigoplus_{s\in B_S(M)} M(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)] \to M$. Remark 3.15 shows us that applying the S-splitting functor at $c \in S$ yields the canonical epimorphism $S_{S,c}\rho = \pi\colon M(c) \to S_{S,c}M$. Thus ρ is an epimorphism by Proposition 3.17.

To show the claimed minimality: If M is T-generated for some $T \subseteq S$, we have $B_S(M) \subseteq B_T(M) \subseteq T$ by Proposition 3.9 and Remark 3.7. □

Next, we introduce a technical lemma.

Lemma 3.23. *Assume that we have a commutative diagram of R-modules with exact rows*

$$\begin{tikzcd} & L \arrow[r] \arrow[d, "f"'] & N \arrow[r] \arrow[d, "g"'] & M \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & L' \arrow[r] & N' \arrow[r] & M' & \end{tikzcd}$$

where g is a monomorphism. If f is an epimorphism, then h is a monomorphism. The converse holds if either the natural morphism $\operatorname{Coker} g \to \operatorname{Coker} h$ *is a monomorphism or g is an epimorphism.*

Proof. The snake lemma gives us an exact sequence

$$\operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h \to \operatorname{Coker} f \to \operatorname{Coker} g \to \operatorname{Coker} h,$$

where $\operatorname{Ker} g = 0$. If $\operatorname{Coker} f = 0$, we get $\operatorname{Ker} h = 0$. If $\operatorname{Coker} g = 0$, we have $\operatorname{Ker} h \cong \operatorname{Coker} f$, and we are done. If $\operatorname{Coker} g \to \operatorname{Coker} h$ is a monomorphism, we see that $\operatorname{Coker} f$ maps to 0, so $\operatorname{Ker} h \to \operatorname{Coker} f$ is an epimorphism. Since $\operatorname{Ker} g = 0$, the morphism $\operatorname{Ker} h \to \operatorname{Coker} f$ is also a monomorphism. □

Lemma 3.24. *Let M be an S-presented RC-module, where $S \subseteq C$ is Artinian. Assume that we have an exact sequence of RC-modules*

$$0 \to L \to N \xrightarrow{f} M \to 0,$$

where N is S-generated and $D_S(N) = \emptyset$. Then $D_S(M) \subseteq B_S(L)$. Furthermore, if N is $B_S(M)$-generated, we have $B_S(L) \subseteq B_S(M) \cup D_S(M)$.

Proof. Let $c \in C$. Applying $\operatorname{colim}_{d<c,\ d\in S}$ to the exact sequence above, we get a diagram with exact rows

$$\begin{tikzcd} & \operatorname*{colim}\limits_{d<c,\ d\in S} L(d) \arrow[r] \arrow[d, "\lambda_{L,c}"'] & \operatorname*{colim}\limits_{d<c,\ d\in S} N(d) \arrow[r] \arrow[d, "\lambda_{N,c}"'] & \operatorname*{colim}\limits_{d<c,\ d\in S} M(d) \arrow[r] \arrow[d, "\lambda_{M,c}"'] & 0 \\ 0 \arrow[r] & L(c) \arrow[r] & N(c) \arrow[r, "f_c"] & M(c) \arrow[r] & 0 \end{tikzcd}$$

that commutes. Here $\lambda_{N,c}$ is a monomorphism, because $D_S(N) = \emptyset$. To show that $D_S(M) \subseteq B_S(L)$, suppose that $c \notin B_S(L)$. In this case $\lambda_{L,c}$ is an epimorphism, so $\lambda_{M,c}$ is a monomorphism by Lemma 3.23. Thus $c \notin D_S(M)$.

Assume that N is $B_S(M)$-generated. Since f is an epimorphism, Proposition 3.22 implies that $B_S(M) = B_S(N)$. Suppose that $c \notin B_S(M) \cup D_S(M)$. Now $\lambda_{N,c}$ is an epimorphism, since $c \notin B_S(N) = B_S(M)$. Moreover, $\lambda_{M,c}$ is a monomorphism because $c \notin D_S(M)$. It follows from Lemma 3.23 that $\lambda_{L,c}$ is an epimorphism, so $c \notin B_S(L)$. Thus $B_S(L) \subseteq B_S(M) \cup D_S(M)$. □

Proposition 3.25. *Let M be an S-presented $R\mathcal{C}$-module, where $S \subseteq \mathcal{C}$ is Artinian. Then M is $B_S(M) \cup D_S(M)$-presented. Furthermore, $B_S(M) \cup D_S(M)$ is the minimum element of the set $\{T \subseteq S \mid M \text{ is } T\text{-presented}\}$.*

Proof. Let us examine an exact sequence

$$0 \to L \to N \to M \to 0,$$

with N of the form $N = \bigoplus_{s \in B_S(M)} A_s[\mathrm{Mor}_{\mathcal{C}}(s,-)]$, where A_s is an R-module for all $s \in B_S(M)$. Note that such N always exists by Proposition 3.22. Since M is S-presented, Proposition 3.2 2) implies that L is S-generated. Using Proposition 3.2 5), we notice that if L is T-generated for some $T \subseteq S$, then M is $(B_S(M) \cup T)$-presented. Now L is $B_S(L)$-generated by Proposition 3.22, so we deduce that M is $(B_S(M) \cup B_S(L))$-presented. We can now use Lemma 3.24 to see that then M is$(B_S(M) \cup D_S(M))$-presented.

Suppose next that M is also T-presented for some $T \subseteq S$. As in the proof of Lemma 3.24, we note that $B_S(M) = B_S(N)$. The minimality of $B_S(M)$ in Proposition 3.22 implies that $B_S(N) = B_S(M) \subseteq T$, so N is T-generated by Proposition 3.22. Thus L is T-generated by Proposition 3.2 2). Therefore we must have

$$B_S(M) \subseteq B_T(M) \subseteq T \text{ and } B_S(L) \subseteq B_T(L) \subseteq T,$$

by Proposition 3.9 1) and Remark 3.7. We use Lemma 3.24 to conclude that

$$B_S(M) \cup D_S(M) = B_S(L) \cup B_S(M) \subseteq T.$$

□

Remark 3.26. Assume that M is an S-presented $R\mathcal{C}$-module, where $S \subseteq \mathcal{C}$ is Artinian. Let $f : N \to M$ be a projective cover. Then $S_{S,c}f$ is an isomorphism for all $c \in \mathcal{C}$ if and only if $S_{S,c}M$ is projective for all $c \in \mathcal{C}$.

To see this, first suppose that $S_{S,c}f$ is an isomorphism for all $c \in \mathcal{C}$. Since $S_{S,c}$ preserves projectives for all $c \in \mathcal{C}$, we see that $S_{S,c}N$ is projective, and thus $S_{S,c}M$ is projective.

Conversely, suppose that $S_{S,c}M$ is projective for all $c \in \mathcal{C}$. We may now apply Proposition 3.20 and Lemma 3.18 to get isomorphisms $S_{S,c}f \colon S_{S,c}N \to S_{S,c}M$ for all $c \in \mathcal{C}$.

Remark 3.27. In [5], Carlsson and Zomorodian define multiset-valued invariants ξ_0 and ξ_1 for a finitely generated $\mathbb{Z}^n$-graded $k[X_1, \dots, X_n]$-module M, where k is a field. The multisets $\xi_0(M)$ and $\xi_1(M)$ indicate the degrees in $\mathbb{Z}^n$ where the elements of M are born and where they die, respectively. In more algebraic terms, $\xi_0(M)$ and $\xi_1(M)$ consist of the degrees of minimal generators and minimal relations of M equipped with the multiplicities they occur. Consider an exact sequence

$$0 \to L \to N \xrightarrow{f} M \to 0,$$

where N is a free module and f a minimal homomorphism. Since M is S-presented for some finite $S \subseteq \mathbb{Z}^n$, it is easy to see that $\xi_0(M)$ is a multiset where the underlying set is $B_S(M)$ and the multiplicity of $c \in B_S(M)$ is the dimension of $M(c)$. Note that the choice of S does not matter here, since $B_S(M) = B_{\mathcal{C}}(M)$ by Proposition 3.10. We note that L is S-generated by Proposition 3.2 3), so we may apply a similar argument to conclude that $\xi_1(M)$ is a multiset with $B_S(L)$ as the underlying set and the dimension of $L(c)$ as the multiplicity of $c \in B_S(L)$. The next theorem will show that $D_S(M)$ is the underlying set of $\xi_1(M)$.

Theorem 3.28. *Let M be an S-presented $R\mathcal{C}$-module, where $S \subseteq \mathcal{C}$ is Artinian. Assume that $S_{S,c}M$ is projective for all $c \in B_S(M)$. If*

$$0 \to L \to N \xrightarrow{f} M \to 0$$

is an exact sequence where f is a projective cover, then $B_S(L) = D_S(M)$.

Proof. By Lemma 3.24, it is enough to show that $B_S(L) \subseteq D_S(M)$. If $c \notin B_S(M)$, then $c \in B_S(L)$ implies $c \in D_S(M)$ by Lemma 3.24. Let $c \in B_S(M)$. Suppose that $c \notin D_S(M)$. Then $\lambda_{M,c}$ is a monomorphism. Since f is minimal, by Proposition 3.20

and Lemma 3.18, there exists a natural isomorphism

$$S_{S,c}f : S_{S,c}N = \operatorname{Coker} \lambda_{N,c} \to \operatorname{Coker} \lambda_{M,c} = S_{S,c}M.$$

It now follows from Lemma 3.23 that $\lambda_{L,c}$ is an epimorphism, which is equivalent to $c \notin B_S(L)$. □

4 PRESENTATIONS WITH FINITE SUPPORT

In this chapter we will prove our main result, Theorem 4.15, which gives a characterization for finitely presented modules. We will assume that $\mathcal{C}$ is a poset and R a commutative ring.

4.1 S-determined $R\mathcal{C}$-modules

Let M be an $R\mathcal{C}$-module. If $S \subseteq \mathcal{C}$ is a finite set such that M is S-presented, we say that S is a *finite support of a presentation (FSP)* of M. In what follows, we are trying to find a condition equivalent for M having an FSP.

Definition 4.1. An $R\mathcal{C}$-module M is *S-determined* if there exists a subset $S \subseteq \mathcal{C}$ such that $\operatorname{Supp}(M) \subseteq \uparrow S$, and for every $c \leq d$ in $\mathcal{C}$

$$S \cap \downarrow c = S \cap \downarrow d \;\Rightarrow\; M(c \leq d) \text{ is an isomorphism.}$$

Remark 4.2. Let M be an $R\mathcal{C}$-module and $S \subseteq \mathcal{C}$ a subset. Denote $T := \uparrow S$. Then the condition $\operatorname{Supp}(M) \subseteq T$ of Definition 4.1 is equivalent to the following conditions:

1) M is T-generated;
2) M is T-presented;
3) If $S \cap \downarrow c = \emptyset$, then $M(c) = 0$.

To show this, we first note that 1) implies 3), because $\uparrow S = T$. Taking the contraposition of 3), we get $\operatorname{Supp}(M) \subseteq T$. Next, note that below every $c \in D_T(M)$ there must be some $d \in \operatorname{Supp}(M)$ such that $d < c$. Thus $D_T(M) \subseteq \uparrow \operatorname{Supp}(M)$. Obviously also $B_T(M) \subseteq \operatorname{Supp}(M)$. We now observe that if $\operatorname{Supp}(M) \subseteq T$, we get

$$B_T(M) \cup D_T(M) \subseteq \uparrow \operatorname{Supp}(M) \subseteq \uparrow T = T,$$

This means that $\operatorname{Supp}(M) \subseteq T$ implies 2) by Proposition 3.9. Finally, 1) trivially follows from 2).

Proposition 4.3. *Let M be an S-presented $R\mathcal{C}$-module, where $S \subseteq \mathcal{C}$. Then M is S-determined.*

Proof. Trivially $\mathrm{Supp}(M) \subseteq \uparrow S$. If $c \leq d$ in $\mathcal{C}$, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_{e\leq c,\ e\in S} M(e) \arrow[r] \arrow[d, "\cong"'] & \operatorname{colim}_{e\leq d,\ e\in S} M(e) \arrow[d, "\cong"] \\
M(c) \arrow[r, "M(c\leq d)"'] & M(d)
\end{tikzcd}$$

with the vertical isomorphisms being components of the canonical isomorphism of Proposition 3.2, 1). This immediately shows us that M is S-determined. □

4.2 Minimal upper bounds

Let $S \subseteq \mathcal{C}$. We would like to find conditions under which S-determined implies S-presented. In general this is false (see Example 4.17), so we first need to apply some technical limitations on the poset $\mathcal{C}$ to guarantee that it is “small” enough.

Notation 4.4. Let $S \subseteq \mathcal{C}$ be a subset. We denote the set of minimal upper bounds of S by $\mathrm{mub}_{\mathcal{C}}(S)$.

Definition 4.5. The poset $\mathcal{C}$ is *weakly bounded from above* if every finite $S \subseteq \mathcal{C}$ has a finite number of minimal upper bounds in $\mathcal{C}$.

Definition 4.6. The poset $\mathcal{C}$ is *mub-complete* if given a finite non-empty subset $S \subseteq \mathcal{C}$ and an upper bound c of S, there exists a minimal upper bound s of S such that $s \leq c$.

Remark 4.7. A poset that is weakly bounded from above and mub-complete is called a poset with property $\mathcal{M}$ in [15]. In contrast to [15], we do not require the empty set to have minimal upper bounds for a poset to be mub-complete.

Example 4.8. If L is a lattice, then L is weakly bounded from above and mub-complete.

A ‘good’ monoid G in [6] is a cancellative monoid that is weakly bounded from above as a poset (with the natural order). If G is also commutative, we get the following description of mub-completeness.

Proposition 4.9. *Let G be a commutative cancellative monoid that is weakly bounded from above as a poset (with the natural order). Then G is mub-complete if and only if there exists a maximal common divisor for each $g, h \in G$.*

Proof. Assume first that G is mub-complete. Let $g, h \in G$. Since gh is an upper bound of g and h, there exists a minimal upper bound $j \in G$ of g and h such that $lj = gh$ for some $l \in G$. We claim that l is a maximal common divisor of g and h. We may write $j = ag = bh$, where $a, b \in G$. Now

$$gh = lj = lag = lbh,$$

so that $g = lb$ and $h = la$ by cancellativity. Thus l is a common divisor of g and h. For the maximality, let $k \in G$ be another common divisor of g and h such that l divides k. We may then write $k = k'l$, where $k' \in G$. Furthermore, we have $g = ck$ and $h = dk$ for some $c, d \in G$. Combining these equations, we get

$$lj = gh = ck'lh = gdk'l.$$

Cancelling l, we see that $j = k'ch = k'dg$. Furthermore, cancelling k' yields $ch = dg$, another upper bound for g and h. Since j is a minimal upper bound of g and h, we must have $k' = 1$, proving the maximality of l.

For the other direction, assume that each pair $g, h \in G$ has a maximal common divisor. Let $H := \{h_1, \ldots, h_n\} \subseteq G$ be a finite non-empty set, and let d be an upper bound of H. We now have

$$d = g_1 h_1 = \cdots = g_n h_n$$

for some $g_1, \ldots, g_n \in G$. Let $g' \in G$ be a maximal common divisor of $g_1, \ldots, g_n$. Hence there exists $g_i' \in G$ such that $g_i = g_i' g'$ for all $i \in \{1, \ldots, n\}$. Also, $d = d'g'$ for some $d' \in G$. It is now easy to see that the maximal common divisor of $g_1', \ldots, g_n'$ is 1, and that d' is a minimal upper bound of H. $\square$

Notation 4.10. Let $S \subseteq \mathcal{C}$ be a finite subset. We denote the set of minimal upper bounds of non-empty subsets of S by

$$\hat{S} := \bigcup_{\emptyset \neq S' \subseteq S} \mathrm{mub}_{\mathcal{C}}(S').$$

We notice that if C is weakly bounded from above, then $\hat{S}$ is finite.

Using the terminology from [6], a set $S \subseteq C$ is *a framing set of* M if every $c \in {\uparrow}\operatorname{Supp}(M)$ has an element $s \in S \cap {\downarrow}c$, called a *frame of* c, such that $M(s \leq c')$ is an isomorphism for all $s \leq c' \leq c$.

Lemma 4.11. *If an* RC*-module* M *has a framing set* S*, then* M *is* S*-determined.*

Conversely, if C *is weakly bounded from above and mub-complete, and* M *is* S*-determined for some finite set* $S \subseteq C$*, then* $\hat{S}$ *is a finite framing set of* M*. In particular, if* $c \in C$*, then there exists a frame* $s \in \operatorname{mub}(S \cap {\downarrow}c) \subseteq \hat{S}$ *of* c *such that* $S \cap {\downarrow}c = S \cap {\downarrow}s$*.*

Proof. Assume first that S is a framing set for M. If $c \in \operatorname{Supp}(M)$, then there exists a frame $s \in S$ of c, and therefore $c \in {\uparrow}S$. Thus $\operatorname{Supp}(M) \subseteq {\uparrow}S$. Let $c \leq d$ in C such that $S \cap {\downarrow}c = S \cap {\downarrow}d$. If $d \notin {\uparrow}\operatorname{Supp}(M)$, we have $M(c) = M(d) = 0$, and we are done. Otherwise, there exists a frame $s \in S$ of d. Since $S \cap {\downarrow}c = S \cap {\downarrow}d$, we see that $s \leq c \leq d$. Therefore $M(c \leq d)$ is an isomorphism.

Assume next that C is weakly bounded from above and mub-complete, and M is S-determined for some finite set S. Since C is weakly bounded from above, $\hat{S}$ is finite. Let $c \in {\uparrow}\operatorname{Supp}(M)$. Now there exists an element $b \leq c$ such that $M(b) \neq 0$. Since M is S-determined, we see that $S \cap {\downarrow}c \supseteq S \cap {\downarrow}b \neq \emptyset$. Thus c is an upper bound of the non-empty set $S \cap {\downarrow}c$, so by mub-completeness there exists a minimal upper bound $s \in \operatorname{mub}(S \cap {\downarrow}c) \subseteq \hat{S}$ such that $s \leq c$. It follows that $S \cap {\downarrow}c \subseteq S \cap {\downarrow}s$. Obviously $s \leq c$ implies $S \cap {\downarrow}s \subseteq S \cap {\downarrow}c$. Hence $S \cap {\downarrow}s = S \cap {\downarrow}c$. If $s \leq c' \leq c$, then trivially $S \cap {\downarrow}s = S \cap {\downarrow}c'$, so $M(s \leq c')$ is an isomorphism. □

4.3 Finitely presented RC-modules in mub-complete posets

In the next proposition we find out how the minimal upper bounds connect to births and deaths relative to S. This allows us to prove our main result, Theorem 4.15.

Proposition 4.12. *Let* C *be weakly bounded from above and mub-complete. Let* M *be an* RC*-module that is* S*-determined for some finite* $S \subseteq C$*. If* $B_{\hat{S}}(M) \subseteq S$*, then* $D_{\hat{S}}(M) \subseteq \hat{S}$*.*

Proof. Suppose that $B_{\hat{S}}(M) \subseteq S$. This implies that M is $\hat{S}$-generated by Proposition 3.9, so M is $B_{\hat{S}}(M)$-generated by Proposition 3.22. Let $c \in D_{\hat{S}}(M)$, so that $\lambda_{M,c}$ is not a monomorphism. This means that there exist $d_1, \ldots, d_n \in \hat{S}$ such that

$d_i < c$ for all $i = \{1, \ldots, n\}$, and a non-zero sum of $x_1 \in M(d_1), \ldots, x_n \in M(d_n)$, for which $\sum_{i=1}^n M(d_i \leq c)(x_i) = 0$. If there exists $c' \in \hat{S}$ such that $d_i \leq c' < c$ for all $i \in \{1, \ldots, n\}$, we may assume that $\sum_{i=1}^n M(d_i \leq c')(x_i) \neq 0$, since the homomorphism

$$\bigoplus_{i=1}^{n} M(d_i) \to \operatorname*{colim}_{d<c,\ d\in\hat{S}} M(d)$$

factors through $M(c')$. Because M is $B_{\hat{S}}(M)$-generated, we may also assume that $d_i \in B_{\hat{S}}(M) \subseteq S$ for all $i \in \{1, \ldots, n\}$.

On the other hand, by Lemma 4.11, we have $S \cap \downarrow s = S \cap \downarrow c$ for some frame $s \in \hat{S}$ of c. This implies that $d_i \leq s$ for all $i \in \{1, \ldots, n\}$. If $s < c$, we get a contradiction $\sum_{i=1}^n M(d_i \leq s)(x_i) = 0$. Therefore $c = s \in \hat{S}$. □

Corollary 4.13. *Let $\mathcal{C}$ be weakly bounded from above and mub-complete. Let M be an S-determined $R\mathcal{C}$-module, where $S \subseteq \mathcal{C}$ is a finite subset. Then $\hat{\hat{S}}$ is an FSP of M.*

Proof. Obviously M is $\hat{S}$-determined, since $S \subseteq \hat{S}$. By Proposition 4.12, it is enough to show that $B_{\hat{\hat{S}}}(M) \subseteq \hat{S}$, because then $D_{\hat{\hat{S}}}(M) \subseteq \hat{\hat{S}}$. The rest now follows from Proposition 3.9. Since $\hat{S} \subseteq \hat{\hat{S}}$, by Remark 3.7 we have $B_{\hat{\hat{S}}}(M) \subseteq B_{\hat{S}}(M)$. Let $c \in \mathcal{C}$. If $c \notin \hat{S}$, then c has a frame $s \in \hat{S}$ by Lemma 4.11. This means that $c \notin B_{\hat{S}}(M)$, and thus $B_{\hat{S}}(M) \subseteq \hat{S}$. □

We sum up Proposition 4.3 and Corollary 4.13 in the next corollary.

Corollary 4.14. *Let $\mathcal{C}$ be weakly bounded from above and mub-complete. An $R\mathcal{C}$-module M has an FSP if and only if M is S-determined for some finite $S \subseteq \mathcal{C}$.*

Finally, we get our new characterization of finitely presented modules.

Theorem 4.15. *Let M be an $R\mathcal{C}$-module. If M is finitely presented, then*

1) $M(c)$ is finitely presented for all $c \in \mathcal{C}$;

2) M is S-determined for some finite $S \subseteq \mathcal{C}$.

Furthermore, if $\mathcal{C}$ is weakly bounded from above and mub-complete, and M satisfies conditions 1) and 2), then M is finitely presented.

Proof. Using Proposition 3.4, the first part of the theorem immediately follows from Proposition 4.3 and the second part from Corollary 4.13. □

Remark 4.16. Let G be a monoid. Theorem 4.15 and Lemma 4.11 show us that the RG-modules of finitely presented type of Corbet and Kerber ([6, p. 19, Def. 15]) are the same thing as finitely presented RG-modules.

Furthermore, let A be a free G-act that is mub-complete and weakly bounded from above as a poset. Starting from the isomorphism RA-**Mod** $\cong$ A-gr $R[G]$-**Mod** of Corollary 2.14, and using the fact that being finitely presented is a categorical property, we get an isomorphism between finitely presented RA-modules and finitely presented A-graded $R[G]$-modules. Taking $A = G$ now gives the commutative case of [6, p. 25, Thm. 21].

Example 4.17. Let $\mathcal{C} = \{a, b\} \cup \mathbb{Z}$, where $a < n$ and $b < n$ for all $n \in \mathbb{Z}$, and $\mathbb{Z}$ has the usual ordering. Then the $R\mathcal{C}$-module $M := R[\mathrm{Mor}_{\mathcal{C}}(a, -)] \oplus R[\mathrm{Mor}_{\mathcal{C}}(b, -)]$ is obviously finitely presented, but M does not have a finite framing set even though it is $\{a, b\}$-determined. Caution is required here: If we define an $R\mathcal{C}$-module N by

$$N(a) = N(b) = R \quad \text{and} \quad N(n) = R^3,$$

for all $n \in \mathbb{Z}$, then N satisfies the conditions 1) and 2) in Theorem 4.15 but is not finitely presented. This follows from the fact that $\mathcal{C}$ is not mub-complete.

4.4 Pointwise stabilizing direct systems

Definition 4.18. Let I be a directed set, and let $(M_i)_{i\in I}$ be a direct system of $R\mathcal{C}$-modules with morphisms $\varphi_{ij} \colon M_i \to M_j$ for all $i \leq j$ in I. The system $(M_i)_{i\in I}$ is *pointwise stabilizing* if for all $c \in \mathcal{C}$ there exists an element $i_c \in I$ such that

$$i_c \leq i \leq j \implies \varphi_{ij} \colon M_i(c) \to M_j(c) \text{ is an isomorphism.}$$

Remark 4.19. For a pointwise stabilizing direct system $(M_i)_{i\in I}$ of $R\mathcal{C}$-modules as above, if $c \in \mathcal{C}$, then

$$(\operatorname*{colim}_{i\in I} M_i)(c) \cong (\operatorname*{colim}_{i\geq i_c} M_i)(c) \cong M_{i_c}(c).$$

This follows from Remark 1.7, since the set $\{i \in I \mid i \geq i_c\}$ is final in I.

Let M be an $R\mathcal{C}$-module. The result we are aiming for, Theorem 4.25, states that M has an *FSP* if and only if the functor $\mathrm{Hom}_{R\mathcal{C}}(M, -)$ preserves the colimits of pointwise stabilizing direct systems. This result is mentioned without a proof in [9, p. 14, Remarque 2.15]. Note the similarity to Proposition 1.12, which states that M is finitely presented if and only if $\mathrm{Hom}_{RC}(M, -)$ preserves the colimits of direct systems. We will first present a few lemmas.

Lemma 4.20. *Denote by $\mathcal{C}_f$ the set of finite subposets of $\mathcal{C}$. Then the direct system $(\mathrm{ind}_S \,\mathrm{res}_S M)_{S\in\mathcal{C}_f}$ with the natural morphisms is pointwise stabilizing, and*

$$\operatorname*{colim}_{S\in\mathcal{C}_f} \mathrm{ind}_S \,\mathrm{res}_S M \cong M.$$

Proof. Given $c \in \mathcal{C}$ and $S \in \mathcal{C}_f$, we have $S \subseteq S \cup \{c\}$, which implies that the set

$$\mathcal{C}_{f,c} := \{T \in \mathcal{C}_f \mid c \in T\}$$

is final in $\mathcal{C}_f$. Thus, by Remark 1.7,

$$\operatorname*{colim}_{S\in\mathcal{C}_f} \mathrm{ind}_S \,\mathrm{res}_S M \cong \operatorname*{colim}_{S\in\mathcal{C}_{f,c}} \mathrm{ind}_S \,\mathrm{res}_S M.$$

On the other hand, we see that if $S \in \mathcal{C}_{f,c}$, then trivially

$$(\mathrm{ind}_S \,\mathrm{res}_S M)(c) = \operatorname*{colim}_{d\leq c,\ d\in S} M(d) \cong M(c).$$

This shows us that the direct system $(\mathrm{ind}_S \,\mathrm{res}_S M)_{S\in\mathcal{C}_f}$ is pointwise stabilizing.

Next, we have the canonical morphisms $\mathrm{ind}_S \,\mathrm{res}_S M \to M$ for all $S \in \mathcal{C}_f$ that form a cone from the direct system $(\mathrm{ind}_S \,\mathrm{res}_S M)_{S\in\mathcal{C}_f}$ to M. Thus, by the universal property of colimits, there exists a natural morphism

$$\operatorname*{colim}_{S\in\mathcal{C}_f} \mathrm{ind}_S \,\mathrm{res}_S M \to M.$$

This morphism is in fact an isomorphism: Since colimits are calculated pointwise, we get

$$\begin{aligned}(\operatorname*{colim}_{S\in\mathcal{C}_f} \mathrm{ind}_S \,\mathrm{res}_S M)(c) &= (\operatorname*{colim}_{S\in\mathcal{C}_{f,c}} \mathrm{ind}_S \,\mathrm{res}_S M)(c)\\ &\cong \operatorname*{colim}_{S\in\mathcal{C}_{f,c}} (\mathrm{ind}_S \,\mathrm{res}_S M)(c)\\ &\cong \operatorname*{colim}_{S\in\mathcal{C}_{f,c}} M(c)\end{aligned}$$

$$\cong M(c)$$

for all $c \in \mathcal{C}$. We conclude that $\operatorname{colim}_{S \in \mathcal{C}_f} \operatorname{ind}_S \operatorname{res}_S M \cong M$. $\square$

Lemma 4.21. *Let A be an R-module, and $c \in \mathcal{C}$. Then $A[\operatorname{Mor}_{\mathcal{C}}(c, -)]$ preserves colimits of pointwise stabilizing direct systems.*

Proof. Let $(N_i)_{i \in I}$ be a pointwise stabilizing direct system of $R\mathcal{C}$-modules, where the morphisms are denoted by $\varphi_{ij} \colon N_i \to N_j$ for all $i \leq j$ in I. Given an $R\mathcal{C}$-module M, by the adjointness of functors $A \otimes_R -$ and $\operatorname{Hom}_R(A, -)$, and by Yoneda's lemma, we see that

$$\begin{aligned}
\operatorname{Hom}_{R\mathcal{C}}(A[\operatorname{Mor}_{\mathcal{C}}(c, -)], M) &= \operatorname{Hom}_{R\mathcal{C}}(A \otimes_R R[\operatorname{Mor}_{\mathcal{C}}(c, -)], M) \\
&\cong \operatorname{Hom}_{R\mathcal{C}}(R[\operatorname{Mor}_{\mathcal{C}}(c, -)], \operatorname{Hom}_R(A, M)) \\
&\cong \operatorname{Hom}_R(A, M(c)).
\end{aligned}$$

On the other hand, since the direct system $(N_i)_{i \in I}$ is pointwise stabilizing, there exists an element $i_c \in I$ such that

$$i \geq i_c \implies N_i(c) \cong N_{i_c}(c).$$

Because the set $\{i \geq i_c \mid i \in I\}$ is final in I, Remark 1.7 shows us that

$$\begin{aligned}
\operatorname*{colim}_{i \in I} \operatorname{Hom}_R(A, N_i(c)) &\cong \operatorname*{colim}_{i \geq i_c} \operatorname{Hom}_R(A, N_i(c)) \\
&\cong \operatorname*{colim}_{i \geq i_c} \operatorname{Hom}_R(A, N_{i_c}(c)) \\
&\cong \operatorname{Hom}_R(A, N_{i_c}(c)) \\
&\cong \operatorname{Hom}_R(A, \operatorname*{colim}_{i \in I} N_i(c)).
\end{aligned}$$

Combining these results, we notice that in the commutative diagram

$$\begin{tikzcd}
\operatorname*{colim}_{i \in I} \operatorname{Hom}_{R\mathcal{C}}(A[\operatorname{Mor}_{\mathcal{C}}(c, -)], N_i) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Hom}_{R\mathcal{C}}(A[\operatorname{Mor}_{\mathcal{C}}(c, -)], \operatorname*{colim}_{i \in I} N_i) \arrow[d, "\cong"] \\
\operatorname*{colim}_{i \in I} \operatorname{Hom}_R(A, N_i(c)) \arrow[r, "\cong"'] & \operatorname{Hom}_R(A, \operatorname*{colim}_{i \in I} N_i(c))
\end{tikzcd}$$

both vertical morphisms and the lower horizontal morphism are isomorphisms.

Thus the upper horizontal morphism is also an isomorphism. □

Remark 4.22. Finite coproducts of modules of the form $A[\mathrm{Mor}_{\mathcal{C}}(c,-)]$ also preserve the colimits of pointwise stabilizing direct systems. To show this, consider an $R\mathcal{C}$-module $A := \bigoplus_{s\in S} A_s[\mathrm{Mor}_{\mathcal{C}}(s,-)]$, where S is a finite set, and A_s is an R-module for all $s \in S$. Given a pointwise stabilizing direct system $(N_i)_{i\in I}$, we have

$$\mathrm{Hom}_{R\mathcal{C}}(A, \operatorname*{colim}_{i\in I} N_i) \cong \bigoplus_{s\in S} \mathrm{Hom}_{R\mathcal{C}}(A_s[\mathrm{Mor}_{\mathcal{C}}(s,-)], \operatorname*{colim}_{i\in I} N_i)$$

since S is finite. Lemma 4.21 now shows us that

$$\mathrm{Hom}_{R\mathcal{C}}(A, \operatorname*{colim}_{i\in I} N_i) \cong \bigoplus_{s\in S} \operatorname*{colim}_{i\in I} \mathrm{Hom}_{R\mathcal{C}}(A_s[\mathrm{Mor}_{\mathcal{C}}(s,-)], N_i).$$

Finally, since colimits commute with each other, using the finiteness of S again, we get

$$\begin{aligned}\mathrm{Hom}_{R\mathcal{C}}(A, \operatorname*{colim}_{i\in I} N_i) &\cong \operatorname*{colim}_{i\in I} \bigoplus_{s\in S} \mathrm{Hom}_{R\mathcal{C}}(A_s[\mathrm{Mor}_{\mathcal{C}}(s,-)], N_i)\\ &\cong \operatorname*{colim}_{i\in I} \mathrm{Hom}_{R\mathcal{C}}(A, N_i),\end{aligned}$$

as desired.

We still need two more lemmas concerning S-presented and S-generated $R\mathcal{C}$-modules.

Lemma 4.23. *Let $0 \to L \to N \to M \to 0$ be an exact sequence of $R\mathcal{C}$-modules. If N is S-presented and L is S-generated, then M is S-presented.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} & \mathrm{ind}_S \mathrm{res}_S L \arrow[r] \arrow[d, "\mu_L"'] & \mathrm{ind}_S \mathrm{res}_S N \arrow[r] \arrow[d, "\mu_N"'] & \mathrm{ind}_S \mathrm{res}_S M \arrow[r] \arrow[d, "\mu_M"'] & 0 \\ 0 \arrow[r] & L \arrow[r] & N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

where μ_L is an epimorphism and μ_N is an isomorphism. The five lemma now states that μ_M is an isomorphism. □

Lemma 4.24. *Let M be an $R\mathcal{C}$-module such that $M = M_1 \bigoplus M_2$.*

1) If M is S-generated, then M_1 is S-generated.

2) If M is S-presented, then M_1 is S-presented.

Proof. To prove 1), let M be S-generated. We have a commutative diagram

$$\begin{tikzcd}
\bigoplus_{s\in S} M(s)[\mathrm{Mor}_C(s,-)] \arrow[r] \arrow[d, "\varphi_M"'] & \bigoplus_{s\in S} M_1(s)[\mathrm{Mor}_C(s,-)] \arrow[d, "\varphi_{M_1}"] \\
M \arrow[r] & M_1
\end{tikzcd}$$

where the horizontal morphisms are epimorphisms, and the natural morphism φ_M is an epimorphism because M is S-generated. It now follows from the commutativity of the diagram that φ_{M_1} is an epimorphism, so that M_1 is S-generated.

Next, for 2), suppose that M is S-presented. Note that we have an exact sequence

$$0 \to M_2 \to M \to M_1 \to 0.$$

where M_2 is S-generated by the first statement. Thus we immediately see that M_1 is S-presented by Lemma 4.23. □

We are now ready to prove

Theorem 4.25. *Let M be an RC-module. Then M has an FSP if and only if the functor* $\mathrm{Hom}_{RC}(M,-)$ *preserves the colimits of pointwise stabilizing direct systems.*

Proof. Assume first that S is an FSP of M. Let $(N_i)_{i\in I}$ be a pointwise stabilizing direct system of RC-modules. By Proposition 3.2 5), we have an exact sequence

$$\bigoplus_{s\in S} B_s[\mathrm{Mor}_C(s,-)] \to \bigoplus_{s\in S} A_s[\mathrm{Mor}_C(s,-)] \to M \to 0,$$

where A_s and B_s are R-modules for all $s \in S$. Let us denote

$$A := \bigoplus_{s\in S} A_s[\mathrm{Mor}_C(s,-)] \quad \text{and} \quad B := \bigoplus_{s\in S} B_s[\mathrm{Mor}_C(s,-)].$$

Recall that taking colimits of direct systems is exact, and the contravariant Hom-functor is left exact. Applying these functors on the exact sequence above, and using the morphism from Lemma 1.6, we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname*{colim}_{i\in I}\operatorname{Hom}_{RC}(M, N_i) \arrow[r] \arrow[d] & \operatorname*{colim}_{i\in I}\operatorname{Hom}_{RC}(A, N_i) \arrow[r] \arrow[d, "\cong"] & \operatorname*{colim}_{i\in I}\operatorname{Hom}_{RC}(B, N_i) \arrow[d, "\cong"] \\
0 \arrow[r] & \operatorname{Hom}_{RC}(M, \operatorname*{colim}_{i\in I} N_i) \arrow[r] & \operatorname{Hom}_{RC}(A, \operatorname*{colim}_{i\in I} N_i) \arrow[r] & \operatorname{Hom}_{RC}(B, \operatorname*{colim}_{i\in I} N_i)
\end{tikzcd}$$

where the rightmost two vertical morphisms are isomorphisms because of Remark 4.22. It now follows from the five lemma that the leftmost vertical morphism is also an isomorphism.

Assume next that $\operatorname{Hom}_{RC}(M, -)$ preserves the colimits of pointwise stabilizing direct systems. Denote by $\mathcal{C}_f$ the set of finite subposets of $\mathcal{C}$. Lemma 4.20 tells us that $(\operatorname{ind}_S \operatorname{res}_S M)_{S\in\mathcal{C}_f}$ with the natural morphisms is a pointwise stabilizing direct system, and M is the colimit of this system. Combined with the assumption about $\operatorname{Hom}_{RC}(M, -)$, this implies that

$$\begin{aligned}
\operatorname{Hom}_{RC}(M, M) &\cong \operatorname{Hom}_{RC}(M, \operatorname*{colim}_{S\in\mathcal{C}_f} \operatorname{ind}_S \operatorname{res}_S M) \\
&\cong \operatorname*{colim}_{S\in\mathcal{C}_f} \operatorname{Hom}_{RC}(M, \operatorname{ind}_S \operatorname{res}_S M).
\end{aligned}$$

In particular, the identity morphism id_M factors through $\operatorname{ind}_S \operatorname{res}_S M$ for some $S \in \mathcal{C}_f$. By the splitting lemma, we then have a split exact sequence

$$0 \to M \to \operatorname{ind}_S \operatorname{res}_S M \to M' \to 0,$$

so that M is a direct summand of $\operatorname{ind}_S \operatorname{res}_S M$. Finally, using Lemma 4.24, we conclude that M has an FSP S. □

We may also use a similar argument to characterize S-generated modules, where S is finite.

Proposition 4.26. *Let M be an RC-module. Then M is S-generated for some finite subset $S \subseteq \mathcal{C}$ if and only if $\operatorname{Hom}_{RC}(M, -)$ preserves the colimits of pointwise stabilizing direct systems of monomorphisms.*

Proof. Suppose first that M is S-generated for some finite subset $S \subseteq \mathcal{C}$, so that we have an epimorphism

$$\rho\colon \bigoplus_{s\in S} M(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)] \to M.$$

Let $(N_i)_{i\in I}$ be a pointwise stabilizing direct system of monomorphisms. Since taking colimits over direct systems is exact, and the contravariant hom-functor is left exact, by Lemma 1.6 we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname*{colim}_{i\in I} \mathrm{Hom}_{R\mathcal{C}}(M, N_i) \arrow[r] \arrow[d, "\theta"] & \operatorname*{colim}_{i\in I} \mathrm{Hom}_{R\mathcal{C}}\left(\bigoplus_{s\in S} M(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)], N_i\right) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{R\mathcal{C}}(M, \operatorname*{colim}_{i\in I} N_i) \arrow[r] & \mathrm{Hom}_{R\mathcal{C}}\left(\bigoplus_{s\in S} M(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)], \operatorname*{colim}_{i\in I} N_i\right)
\end{tikzcd}$$

Note that the rightmost vertical morphism is an isomorphism by Lemma 4.22. It is obvious from the diagram that θ is a monomorphism. We will show that θ is also an epimorphism. Let $f\colon M \to \mathrm{colim}_{i\in I} N_i$ be a morphism of $R\mathcal{C}$-modules. Since M is S-generated, we get a morphism

$$g\colon \bigoplus_{s\in S} M(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)] \to \operatorname*{colim}_{i\in I} N_i$$

such that $g = f\rho$. By Lemma 4.21, g factors through N_j for some $j \in I$. Note that N_j is a submodule of $\mathrm{colim}_{i\in I} N_i$, because $(N_i)_{i\in I}$ is a direct system of monomorphisms. Thus the image of g is in N_j. This implies that the image of f is also in N_j, so f factors through N_j, which was required.

Conversely, suppose that $\mathrm{Hom}_{R\mathcal{C}}(M,-)$ preserves the colimits of pointwise stabilizing direct systems of monomorphisms. Denote by $\mathcal{C}_f$ the set of finite subposets of $\mathcal{C}$. For every $S \subseteq \mathcal{C}$ we have a natural morphism

$$\rho_S\colon \bigoplus_{s\in S} M(s)[\mathrm{Mor}_{\mathcal{C}}(s,-)] \to M.$$

We now note that if $S \subseteq T \subseteq \mathcal{C}$, then $\mathrm{Im}\,\rho_S \subseteq \mathrm{Im}\,\rho_T$. Thus we get a direct system of monomorphisms $(\mathrm{Im}\,\rho_S)_{S\in\mathcal{C}_f}$ with inclusions as morphisms. It is easy to see (compare to Lemma 4.20) that this system is pointwise stabilizing with a colimit M. By the assumption about $\mathrm{Hom}_{RC}(M, -)$, we get

$$\mathrm{Hom}_{RC}(M, M) \cong \mathrm{Hom}_{RC}(M, \operatorname*{colim}_{S \in \mathcal{C}_f} \mathrm{Im}\, \rho_S) \cong \operatorname*{colim}_{S \in \mathcal{C}_f} \mathrm{Hom}_{RC}(M, \mathrm{Im}\, \rho_S).$$

In particular, the identity morphism id_M factors through $\mathrm{Im}\, \rho_S$ for some $S \in \mathcal{C}_f$. Therefore, we have a split exact sequence

$$0 \to M \to \mathrm{Im}\, \rho_S \to M' \to 0,$$

so M is a direct summand of $\mathrm{Im}\, \rho_S$, again by the splitting lemma. Using Lemma 4.24, we conclude that M is S-generated. □

5 STRONGLY BOUNDED POSETS

5.1 Modules over strongly bounded posets

Definition 5.1. The poset $\mathcal{C}$ is *strongly bounded from above* if every finite $S \subseteq \mathcal{C}$ has a unique minimal upper bound in $\mathcal{C}$. We denote this unique minimal upper bound by $\mathrm{mub}(S)$.

Remark 5.2. The condition of $\mathcal{C}$ being strongly bounded from above is equivalent to $\mathcal{C}$ being a bounded join-semilattice. Also note that if $\mathcal{C}$ is strongly bounded from above, then $\mathcal{C}$ is weakly bounded from above and mub-complete.

Let $\mathcal{C}$ be strongly bounded from above, and let $S \subseteq \mathcal{C}$ be a finite set. In this section we consider $\mathrm{mub}(S)$ as an element of $\mathcal{C}$, and not as a (one element) set as in Notation 4.4. In particular, every element of $\hat{S}$ is then of the form $\mathrm{mub}(S')$, where $S' \subseteq S$ is a non-empty subset. Viewing $\mathcal{C}$ as a join-semilattice, we have the join-operation

$$a \vee b := \mathrm{mub}(a, b) := \mathrm{mub}(\{a, b\}).$$

Extending this operation to finite sets, we get an operation that coincides with minimal upper bounds.

Lemma 5.3. *Let $\mathcal{C}$ be strongly bounded from above, and let $S \subseteq \mathcal{C}$ be a finite subset. Then $\hat{\hat{S}} = \hat{S}$.*

Proof. An element $s \in \hat{\hat{S}}$ may be written in the form

$$s = \mathrm{mub}(\mathrm{mub}(S_1), \ldots, \mathrm{mub}(S_n)),$$

where $S_1, \ldots, S_n$ are (finite) non-empty subsets of S. Since the join-operation is associative in join-semilattices, we see that

$$s = \bigvee_{i=1}^{n} (\bigvee S_i) = \bigvee (\bigcup_{i=1}^{n} S_i).$$

This implies that $s = \mathrm{mub}(S_1 \cup \cdots \cup S_n)$, which belongs to $\hat{S}$ by definition. $\square$

Assume that $\mathcal{C}$ is strongly bounded from above. Then $\mathcal{C}$ has a minimum element $\min(\mathcal{C}) = \mathrm{mub}(\emptyset)$. Let $S \subseteq \mathcal{C}$ be a finite subset. Denote

$$\tilde{S} := \hat{S} \cup \{\min(\mathcal{C})\}.$$

We define a poset morphism $\alpha_S \colon \mathcal{C} \to \tilde{S}$ by setting

$$\alpha(c) = \mathrm{mub}(S \cap {\downarrow}c)$$

for every $c \in \mathcal{C}$. In other words, α_S maps each $c \in \mathcal{C}$ to the minimal upper bound of the elements of S below it. To show that α actually is a poset morphism, suppose that $c \leq d$ in $\mathcal{C}$. Then $S \cap {\downarrow}c \subseteq S \cap {\downarrow}d$, which implies that $\alpha(c) \leq \alpha(d)$.

Proposition 5.4. *Let $\mathcal{C}$ be strongly bounded from above, and let $S \subseteq \mathcal{C}$ be a finite subset. Then $\alpha_S = \alpha_{\hat{S}} = \alpha_{\tilde{S}}$.*

Proof. Using Lemma 5.3, we first note that $\hat{\tilde{S}} = \tilde{S}$ and $\tilde{\tilde{S}} = \tilde{S}$. Let $c \in \mathcal{C}$. We claim that

$$\mathrm{mub}(S \cap {\downarrow}c) = \mathrm{mub}(\hat{S} \cap {\downarrow}c) = \mathrm{mub}(\tilde{S} \cap {\downarrow}c).$$

The latter equation follows from the fact that for all subsets $T \subseteq \mathcal{C}$, we have $\mathrm{mub}(T) = \mathrm{mub}(T \cup \{\min(\mathcal{C})\})$. (In particular $\mathrm{mub}(T) = \min(\mathcal{C})$, if $T = \emptyset$.)

For the first equation, since $S \subseteq \hat{S}$, we have $\mathrm{mub}(S \cap {\downarrow}c) \leq \mathrm{mub}(\hat{S} \cap {\downarrow}c)$. On the other hand, $\hat{S} \cap {\downarrow}c$ is a subset of $\hat{S}$. Thus $\mathrm{mub}(\hat{S} \cap {\downarrow}c) \in \hat{\hat{S}} = \hat{S}$, where the equation follows from Lemma 5.3. By the definition of $\hat{S}$, we may now write

$$\mathrm{mub}(\hat{S} \cap {\downarrow}c) = \mathrm{mub}(s_1, \ldots, s_n),$$

where $s_1, \ldots, s_n \in S$. Furthermore, $\mathrm{mub}(\hat{S} \cap {\downarrow}c) \leq c$, so we also have $s_1, \ldots, s_n \leq c$. This implies that

$$\mathrm{mub}(s_1, \ldots, s_n) \leq \mathrm{mub}(S \cap {\downarrow}c),$$

which completes the proof. $\square$

Encouraged by Proposition 5.4, we just write α instead of α_S, if there is no risk of confusion.

Lemma 5.5. *Let $\mathcal{C}$ be strongly bounded from above, and let $S \subseteq \mathcal{C}$ be a finite subset. Then $\hat{S} \cap \downarrow\alpha(c) = \hat{S} \cap \downarrow c$ for all $c \in \mathcal{C}$.*

Proof. Let $c \in \mathcal{C}$. We immediately see that $\hat{S} \cap \downarrow\alpha(c) \subseteq \hat{S} \cap \downarrow c$, because $\alpha(c) \leq c$. Suppose that $d \in \hat{S} \cap \downarrow c$. We need to show that $d \leq \alpha(c)$. This follows from Proposition 5.4, because now

$$\alpha(c) = \alpha_{\hat{S}}(c) = \mathrm{mub}(\hat{S} \cap \downarrow c).$$

$\square$

Let $\mathcal{C}$ be strongly bounded from above, let $S \subseteq \mathcal{C}$ be a finite subset, and M an $R\mathcal{C}$-module. The morphism α gives rise to a natural transformation

$$T_\alpha\colon\ \mathrm{res}_\alpha \mathrm{res}_{\hat{S}} M \to M,$$

where for any $c \in \mathcal{C}$, $T_{\alpha,c}$ is the morphism

$$M(\alpha(c) \leq c)\colon (\mathrm{res}_\alpha \mathrm{res}_{\hat{S}} M)(c) = M(\alpha(c)) \to M(c).$$

Proposition 5.6. *Let $\mathcal{C}$ be strongly bounded from above, let $S \subseteq \mathcal{C}$ be a finite subset, and M an $R\mathcal{C}$-module. Then M is S-determined if and only if* $\mathrm{Supp}(M) \subseteq \uparrow S$ *and $T_\alpha\colon \mathrm{res}_\alpha \mathrm{res}_{\hat{S}} M \to M$ is an isomorphism.*

Proof. Suppose first that M is S-determined. Let $c \in \mathcal{C}$. Then, by definition, $\mathrm{Supp}(M) \subseteq \uparrow S$. Also, M is $\hat{\hat{S}}$-presented by Corollary 4.13. Lemma 5.3 now tells us that M is $\hat{S}$-presented, so we have $M \cong \mathrm{ind}_{\hat{S}} \mathrm{res}_{\hat{S}} M$ by Proposition 3.2. This implies that

$$(\mathrm{res}_\alpha \mathrm{res}_{\hat{S}} M)(c) = M(\alpha(c)) \cong \operatorname*{colim}_{d \leq \alpha(c),\ d \in \hat{S}} M(d).$$

Furthermore, by Lemma 5.5, we get

$$\operatorname*{colim}_{d \leq \alpha(c),\ d \in \hat{S}} M(d) = \operatorname*{colim}_{d \leq c,\ d \in \hat{S}} M(d) = (\mathrm{ind}_{\hat{S}} \mathrm{res}_{\hat{S}} M)(c) \cong M(c).$$

Suppose then that $\mathrm{Supp}(M) \subseteq \uparrow S$ and $T_\alpha \colon \operatorname{res}_\alpha \operatorname{res}_{\widetilde{S}} M \to M$ is an isomorphism. Let $c \leq d$ in $\mathcal{C}$ such that $S \cap \downarrow c = S \cap \downarrow d$. This immediately implies that $\alpha(c) = \alpha(d)$. Thus

$$M(c) \cong M(\alpha(c)) = M(\alpha(d)) \cong M(d),$$

so $M(c \leq d)$ is an isomorphism. □

5.2 Modules determined by cartesian sets

One approach to understanding $R\mathbb{Z}^n$-modules better is to expand the set $\mathbb{Z}^n$ to include points at infinity. This idea has been utilized by Perling in [23]. Set $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty\}$. It is easy to see that $\overline{\mathbb{Z}}^n$ inherits the poset structure from $\mathbb{Z}^n$. Any $R\mathbb{Z}^n$-module M may be naturally extended to an $R\overline{\mathbb{Z}}^n$-module $\overline{M}$ by setting

$$\overline{M}(c) = \lim_{d \geq c,\ d \in \mathbb{Z}^n} M(d)$$

for all $c \in \overline{\mathbb{Z}}^n$. More formally, this is the coinduction of M with respect to the inclusion $\mathbb{Z}^n \to \overline{\mathbb{Z}}^n$. The functor $M \mapsto \overline{M}$ establishes an equivalence of categories between $R\mathbb{Z}^n$-**Mod** and its essential image in $R\overline{\mathbb{Z}}^n$-**Mod**.

Let $S \subseteq \overline{\mathbb{Z}}^n$ be a finite non-empty subset. We denote by $\mathrm{mlb}(S)$ the (unique) maximal lower bound of S.

Proposition 5.7. *Let $p_i \colon \overline{\mathbb{Z}}^n \to \overline{\mathbb{Z}}$ be the canonical projection for every $i \in \{1, \dots, n\}$, and let $S \subseteq \overline{\mathbb{Z}}^n$ be a finite non-empty subset. Then*

1) $\mathrm{mub}(S) = (\max(p_1(S)), \dots, \max(p_n(S)))$;

2) $\mathrm{mlb}(S) = (\min(p_1(S)), \dots, \min(p_n(S)))$.

Proof. Both 1) and 2) are proved in the same way. We will present the proof of 1) here. Let $i \in \{1, \dots, n\}$. The existence of $\max(p_i(S))$ follows from the fact that $p_i(S)$ is non-empty, linearly ordered and finite. Write

$$d = (d_1, \dots, d_n) := \mathrm{mub}(S).$$

We will show that $d_i = \max(p_i(S))$. First, since d is an upper bound of S and the canonical projection p_i preserves order, we see that

$$d_i = p_i(d) \geq \max(p_i(S)).$$

Secondly, if $\max(p_i(S)) < d_i$, then

$$d' := (d_1, \dots, d_{i-1}, \max(p_i(S)), d_{i+1}, \dots, d_n)$$

is an upper bound of S such that $d' < d$, contradicting the minimality of d. Thus $d_i = \max(p_i(S))$. $\square$

Let $S \subseteq \overline{\mathbb{Z}}^n$ be a non-empty subset. We say that the subset S is *cartesian*, if it is of the form

$$S = S_1 \times \cdots \times S_n,$$

where $S_1, \dots, S_n$ are subsets of $\overline{\mathbb{Z}}$. In this situation, we write

$$\overline{S} = \tilde{S_1} \times \cdots \times \tilde{S_n},$$

where $\widetilde{S_i} = S_i \cup \{-\infty\} \subseteq \overline{\mathbb{Z}}$ for all $i \in \{1, \dots, n\}$. Note that if S is finite, then $\overline{S}$ is finite.

Example 5.8. Let $a \leq b$ in $\mathbb{Z}^n$, and write $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$. For the closed interval $[a, b]$, we have

$$\begin{aligned} [a, b] &= \{c \in \mathbb{Z}^n \mid a \leq c \leq b\} \\ &= [a_1, b_1] \times \cdots \times [a_n, b_n]. \end{aligned}$$

Therefore

$$\begin{aligned} \overline{[a, b]} &= \widetilde{[a_1, b_1]} \times \cdots \times \widetilde{[a_n, b_n]} \\ &= \{(c_1, \dots, c_n) \mid a_i \leq c_i \leq b_i \text{ or } c_i = -\infty \ \ (i \in \{1, \dots, n\})\}. \end{aligned}$$

Lemma 5.9. *Let $S := S_1 \times \cdots \times S_n \subseteq \overline{\mathbb{Z}}^n$ be a finite cartesian subset, and let $T \subseteq S$ be a finite non-empty subset. Then*

1) $\operatorname{mub}(T) \in S$*;*

2) $\operatorname{mlb}(T) \in S$*;*

3) $\overline{\overline{S}} = \overline{S}$*.*

Proof. To prove 1), let p_i be the canonical projection $\overline{\mathbb{Z}}^n \to \overline{\mathbb{Z}}$ for all $i \in \{1, \dots, n\}$.

From Proposition 5.7 1), we get that

$$\mathrm{mub}(T) = (\max(p_1(T)), \ldots, \max(p_n(T))).$$

Thus $\mathrm{mub}(T) \in S$, because $p_i(T) \subseteq p_i(S) = S_i$ for all $i \in \{1, \ldots, n\}$.

Next, the proof for 2) is done in the same way as 1), this time using Proposition 5.7 2).

Finally, for 3), we note that $\overline{S}$ is finite and cartesian, so 1) implies $\hat{\overline{S}} = \overline{S}$. Since $\overline{S}$ already contains the minimum element of $\overline{\mathbb{Z}}^n$, we get

$$\begin{aligned} \tilde{\overline{S}} &= \hat{\overline{S}} \cup \{(-\infty, \ldots, -\infty)\} \\ &= \overline{S} \cup \{(-\infty, \ldots, -\infty)\} \\ &= \overline{S}. \end{aligned}$$

□

Let $S := S_1 \times \cdots \times S_n \subseteq \overline{\mathbb{Z}}^n$ be a finite cartesian subset. Since $\tilde{\overline{S}} = \overline{S}$ by Lemma 5.9 3), we have a poset morphism $\alpha := \alpha_{\overline{S}} \colon \overline{\mathbb{Z}}^n \to \overline{S}$, where

$$\alpha(c) = \mathrm{mub}(\overline{S} \cap {\downarrow} c)$$

for all $c \in \overline{\mathbb{Z}}^n$. By Lemma 5.9 2), we may define a "dual" poset morphism $\beta := \beta_S \colon \overline{S} \to S$, with

$$\beta(c) = \mathrm{mlb}(S \cap {\uparrow} c)$$

for all $c \in \overline{S}$. Here the set $S \cap {\uparrow} c$ is always non-empty, because S is final in $\overline{S}$.

Proposition 5.10. *We write $\alpha_i := \alpha_{\overline{S_i}}$ and $\beta_i := \beta_{S_i}$ for all $i \in \{1, \ldots, n\}$. For $c := (c_1, \ldots, c_n) \in \overline{\mathbb{Z}}^n$, we have*

1) $\alpha(c) = (\alpha_1(c_1), \ldots, \alpha_n(c_n))$;

2) if $c \in \overline{S}$, then $\beta(c) = (\beta_1(c_1), \ldots, \beta_n(c_n))$.

Proof. To prove 1), we will first show that

$$p_i(\overline{S} \cap {\downarrow} c) = \overline{S_i} \cap {\downarrow} c_i,$$

where $p_i \colon \overline{\mathbb{Z}}^n \to \overline{\mathbb{Z}}$ is the canonical projection for all $i \in \{1, \ldots, n\}$. Since $p_i(\overline{S}) = \overline{S_i}$ and $p_i({\downarrow} c) = {\downarrow} c_i$, we see that $p_i(\overline{S} \cap {\downarrow} c) \subseteq \overline{S_i} \cap {\downarrow} c_i$. For the other direction, suppose that $d \in \overline{S_i} \cap \downarrow c_i$. Then $d \le c_i$, so we have an element

$$d' := (-\infty, \ldots, -\infty, d, -\infty, \ldots, -\infty) \in \overline{S} \cap \downarrow c$$

such that $p_i(d') = d$. Hence $p_i(\overline{S} \cap \downarrow c) = \overline{S_i} \cap \downarrow c_i$. Now, using this result and Proposition 5.7 1), we get

$$\begin{aligned} \alpha(c) &= \operatorname{mub}(\overline{S} \cap \downarrow c) \\ &= (\max(\overline{S_1} \cap \downarrow c_1), \ldots, \max(\overline{S_n} \cap \downarrow c_n)) \\ &= (\alpha_1(c_1), \ldots, \alpha_n(c_n)). \end{aligned}$$

For 2), the proof is similar. Let $c \in \overline{S}$. We will first show that

$$p_i(S \cap \uparrow c) = S_i \cap \uparrow c.$$

From $p_i(S) = S_i$ and $p_i(\uparrow c) = \uparrow c_i$, we see that $p_i(S \cap \uparrow c) \subseteq S_i \cap \uparrow c_i$. Next, suppose that $d \in S_i \cap \uparrow c_i$. Since $c \in \overline{S}$, there is an element $s := (s_1, \ldots, s_n) \in S$ such that $s \ge c$. Because $d \ge c_i$ and S is cartesian, we again have an element

$$d' := (s_1, \ldots, s_{i-1}, d, s_{i+1}, \ldots, s_n) \in S \cap \uparrow c$$

such that $p_i(d') = d$. Thus $p_i(S \cap \uparrow c) = S_i \cap \uparrow c$. To finish the proof, we use Proposition 5.7 2):

$$\begin{aligned} \beta(c) &= \operatorname{mlb}(S \cap \uparrow c) \\ &= (\min(S_1 \cap \uparrow c_1), \ldots, \min(S_n \cap \uparrow c_n)) \\ &= (\beta_1(c_1), \ldots, \beta_n(c_n)). \end{aligned}$$

□

We note that α and $\beta \circ \alpha$ are "continuous" in the following sense.

Proposition 5.11. *Let $c := (c_1, \ldots, c_n) \in \overline{\mathbb{Z}}^n$.*

1) If N is an $R\overline{S}$-module, then

$$\lim_{d \ge c,\ d \in \mathbb{Z}^n} N(\alpha(d)) \cong N(\alpha(c)).$$

2) If Q is an RS-module, then

$$\lim_{d \geq c,\ d \in \mathbb{Z}^n} Q((\beta \circ \alpha)(d)) \cong Q((\beta \circ \alpha)(c)).$$

Proof. To show 1), suppose that N is an $R\overline{S}$-module. We define an element $c' := (c'_1, \dots, c'_n) \in \overline{\mathbb{Z}}^n$ as follows: For any $i \in \{1, \dots, n\}$, we set $a_i = \min(S_i \cap \mathbb{Z})$, if it exists, and

$$c'_i := \begin{cases} \max(c_i, 0), & \text{if } S_i \cap \mathbb{Z} = \emptyset; \\ \max(c_i, a_i - 1), & \text{otherwise.} \end{cases}$$

This guarantees that we always have $c \leq c'$ and $c' \in \mathbb{Z}^n$. With the notation from Proposition 5.10, we may write

$$\alpha(c) = (\alpha_1(c_1), \dots, \alpha_n(c_n)).$$

Let $i \in \{1, \dots, n\}$. If $S_i \cap \mathbb{Z} = \emptyset$, then $\alpha_i(c'_i) = -\infty = \alpha_i(c_i)$. Similarly, if $c'_i = a_i - 1$, then $\alpha_i(c'_i) = -\infty = \alpha_i(c_i)$. Thus $\alpha(c) = \alpha(c')$ in all cases. Since α is a poset morphism, we see that for all $d \in \mathbb{Z}^n$ such that $c \leq d \leq c'$,

$$\alpha(c) = \alpha(d) = \alpha(c'),$$

and therefore

$$N(\alpha(c)) = N(\alpha(d)) = N(\alpha(c')).$$

Furthermore, because the set $\{d \in \mathbb{Z}^n \mid c \leq d \leq c'\}$ is an initial subset of $\{d \in \mathbb{Z}^n \mid c \leq d\}$, Remark 1.10 shows us that

$$\lim_{d \geq c,\ d \in \mathbb{Z}^n} N(\alpha(d)) \cong \lim_{c \leq d \leq c',\ d \in \mathbb{Z}^n} N(\alpha(d)) \cong N(\alpha(c)).$$

Next, for 2), let Q be an RS-module. Now $\operatorname{res}_\beta Q$ is an $R\overline{S}$-module, so by 1), we have

$$\lim_{d \geq c,\ d \in \mathbb{Z}^n} (\operatorname{res}_\beta Q)(\alpha(d)) \cong (\operatorname{res}_\beta Q)(\alpha(c)).$$

On the other hand, by definition, for all $e \in \overline{\mathbb{Z}}^n$,

$$(\operatorname{res}_\beta Q)(\alpha(e)) = Q(\beta(\alpha(e))) = Q((\beta \circ \alpha)(e)).$$

This means that we may write the above isomorphism as

$$\lim_{d \geq c,\ d \in \mathbb{Z}^n} Q((\beta \circ \alpha)(d)) \cong Q((\beta \circ \alpha)(c)).$$

□

Corollary 5.12. *Let N be an $R\overline{\mathbb{Z}}^n$-module, and let $c \in \overline{\mathbb{Z}}^n$. Then*

1) $\lim_{d \geq c,\ d \in \mathbb{Z}^n} N(\alpha(d)) \cong N(\alpha(c))$*;*

2) $\lim_{d \geq c,\ d \in \mathbb{Z}^n} N((\beta \circ \alpha)(d)) \cong N((\beta \circ \alpha)(c))$.

Proof. For 1), we note that $\operatorname{res}_{\overline{S}} N$ is an $R\overline{S}$-module, where $(\operatorname{res}_{\overline{S}} N)(d) = N(d)$ for all $d \in \overline{S}$. We may then apply Proposition 5.11 1) to get the result. For 2), we use Proposition 5.11 2) on the RS-module $\operatorname{res}_S N$. □

5.3 Finitely determined modules

Let M be an $R\mathcal{C}$-module. In [19, p. 24, Def. 4.1], Miller defines an *encoding* of M by a poset $\mathcal{D}$ to be a poset morphism $f : \mathcal{C} \to \mathcal{D}$ and an $R\mathcal{D}$-module N such that $\operatorname{res}_f N \cong M$. Furthermore, as defined in [19, p. 25, Ex. 4.5], an $R\mathbb{Z}^n$-module M is *finitely determined*, if it is has an encoding by the convex projection $\pi : \mathbb{Z}^n \to [a, b]$ for some $a \leq b$ in $\mathbb{Z}^n$. Here the convex projection π takes every point in $\mathbb{Z}^n$ to its closest point in the closed interval

$$[a, b] = \{c \in \mathbb{Z}^n \mid a \leq c \leq b\}.$$

If we write $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, we get a formula for π, where for any $c := (c_1, \ldots, c_n) \in \mathbb{Z}^n$,

$$\pi(c) = (\pi_1(c_1), \ldots, \pi_n(c_n)),$$

with

$$\pi_i(c_i) = \max(a_i, \min(c_i, b_i))$$

for all $i \in \{1, \ldots, n\}$. This definition of a finitely determined module may seem a bit different from the one given in the introduction, but the definitions are equivalent ([19, p. 32, Remark 5.2]). Note that, unlike Miller, we will not require R to be a field, or finitely determined modules to be pointwise finite dimensional.

Remark 5.13. Let M be an $R\mathbb{Z}^n$-module. Then M is finitely determined with the encoding convex projection $\pi\colon \mathbb{Z}^n \to [a,b]$ if and only if $M \cong \operatorname{res}_\pi \operatorname{res}_{[a,b]} M$. Indeed, if $M \cong \operatorname{res}_\pi N$ for some $R[a,b]$-module N, then for all $c \in \mathbb{Z}^n$, we have

$$M(c) \cong (\operatorname{res}_\pi N)(c) = N(\pi(c)) = N(\pi(\pi(c))) \cong M(\pi(c))$$

because for all $c \in \mathbb{Z}^n$, $\pi(\pi(c)) = \pi(c)$.

We would like to show that the notion of finitely determined is consistent with our notion of S-determined, when S is finite. While the requirement that $\operatorname{Supp}(M) \subseteq \uparrow S$ does not necessarily hold for finitely determined modules, we do have the following:

Proposition 5.14. *Let M be an $R\mathbb{Z}^n$-module, and let $a, b \in \mathbb{Z}^n$ such that $a \leq b$. Set $u := (1, 1, \ldots, 1) \in \mathbb{Z}^n$.*

1) *If M is $[a,b]$-determined, then M is finitely determined with the encoding convex projection $\pi\colon \mathbb{Z}^n \to [a-u, b]$.*
2) *If $\operatorname{Supp}(M) \subseteq \uparrow a$ and M is finitely determined with the encoding convex projection $\pi\colon \mathbb{Z}^n \to [a,b]$, then M is $[a+u, b]$-determined.*

Proof. For 1), suppose that M is $[a,b]$-determined. Let $c := (c_1, \ldots, c_n) \in \mathbb{Z}^n$, and let $\pi\colon \mathbb{Z}^n \to [a-u, b]$ be the convex projection. We note that if $c_i < a_i$ for some $i \in \{1, \ldots, n\}$, then also $\pi_i(c_i) < a_i$, so that $c, \pi(c) \notin \operatorname{Supp}(M)$. Otherwise $c \geq a$, in which case $\pi(c) \leq c$ and $\operatorname{mub}([a,b] \cap \downarrow\pi(c)) = \operatorname{mub}([a,b] \cap \downarrow c)$. Thus $M(\pi(c)) \to M(c)$ is an isomorphism by the definition of $[a,b]$-determined modules, and $M \cong \operatorname{res}_\pi \operatorname{res}_{[a-u,b]} M$.

To prove 2), assume that $\operatorname{Supp}(M) \subseteq \uparrow a$ and M is finitely determined with the encoding convex projection $\pi\colon \mathbb{Z}^n \to [a,b]$. Let $c := (c_1, \ldots, c_n) \in \mathbb{Z}^n$. Suppose that $c_i < a_i$ for some $i \in \{1, \ldots, n\}$. From the condition $\operatorname{Supp}(M) \subseteq \uparrow a$, we see that $M(c) = 0$. Since M is finitely determined, we also have $M(c) = M(\pi(c)) = 0$. Thus $M(c) = 0$, if $c_i \leq a_i$ for some $i \in \{1, \ldots, n\}$. If this is not the case, we have $c \geq a + u$. Let $c \leq d$ in C such that $a + u \leq c \leq d$ and $[a+u, b] \cap \downarrow c = [a+u, b] \cap \downarrow d$. This implies that $\pi(c) = \pi(d)$, so $M(c \leq d)$ is an isomorphism. $\square$

Extending to $\overline{\mathbb{Z}}^n$

Let $a \le b$ in $\mathbb{Z}^n$. We will now shift our focus to $R\overline{\mathbb{Z}}^n$-modules. With the notation from Section 5.2, we will view the case $S = [a, b]$. In particular, we have $\alpha = \alpha_{\overline{[a,b]}}$ and $\beta = \beta_{[a,b]}$. Proposition 5.10 gives us formulas for α and β. If $c := (c_1, \dots, c_n) \in \overline{\mathbb{Z}}^n$ and $d := (d_1, \dots, d_n) \in \overline{[a, b]}$, then

$$\alpha(c) = (\alpha_1(c_1), \dots, \alpha_n(c_n)) \quad \text{and} \quad \beta(d) = (\beta_1(d_1), \dots, \beta_n(d_n)).$$

Here $\alpha_i := \alpha_{\overline{S_i}}$ and $\beta_i := \beta_{S_i}$ for all $i \in \{1, \dots, n\}$. Explicitly,

$$\alpha_i(c_i) = \begin{cases} -\infty, & \text{if } c_i < a_i; \\ c_i, & \text{if } a_i \le c_i \le b_i; \\ b_i, & \text{if } c_i > b_i, \end{cases} \quad \text{and} \quad \beta_i(d_i) = \begin{cases} a_i, & \text{if } d_i = -\infty; \\ d_i, & \text{otherwise.} \end{cases}$$

for every $i \in \{1, \dots, n\}$. The next proposition shows us that the composition $\beta \circ \alpha$ is an extension of the convex projection π from $\mathbb{Z}^n$ to $\overline{\mathbb{Z}}^n$.

Proposition 5.15. *Let $\pi \colon \mathbb{Z}^n \to [a, b]$ be the convex projection, and let $c \in \mathbb{Z}^n$. Then*

$$\pi(c) = (\beta \circ \alpha)(c).$$

Proof. Suppose first that $n = 1$. Recall that $\pi(c) = \max(a, \min(c, b))$. Now there are three cases:

- If $c \in [a, b]$, then $(\beta \circ \alpha)(c) = \beta(c) = c = \pi(c)$.
- If $c < a$, then $(\beta \circ \alpha)(c) = \beta(-\infty) = a = \pi(c)$.
- If $c > b$, then $(\beta \circ \alpha)(c) = \beta(b) = b = \pi(c)$.

Suppose next that $n > 1$. Using Proposition 5.10, we may write

$$\alpha(c) = (\alpha_1(c_1), \dots, \alpha_n(c_n)) \quad \text{and} \quad \beta(d) = (\beta_1(d_1), \dots, \beta_n(d_n))$$

for all $d \in \overline{[a, b]}$. Similarly, also recall

$$\pi(c) = (\pi_1(c_1), \dots, \pi_n(c_n)).$$

It now follows from the case $n = 1$ that

$$\begin{aligned}(\beta \circ \alpha)(c) &= \beta(\alpha_1(c_1), \ldots, \alpha_n(c_n)) \\ &= ((\beta_1 \circ \alpha_1)(c_1), \ldots, (\beta_n \circ \alpha_n)(c_n)) \\ &= (\pi_1(c_1), \ldots, \pi_n(c_n)) \\ &= \pi(c).\end{aligned}$$

□

Corollary 5.16. *Let $[a, b] \subseteq \mathbb{Z}^n$, and let N be an $R[a, b]$-module. Then*

$$\overline{\operatorname{res}_\pi N} \cong \operatorname{res}_{\beta\circ\alpha} N.$$

Proof. Let $c \in \overline{\mathbb{Z}}^n$. From the definitions, we get

$$(\overline{\operatorname{res}_\pi N})(c) = \lim_{d \geq c,\ d \in \mathbb{Z}^n} (\operatorname{res}_\pi N)(d) = \lim_{d \geq c,\ d \in \mathbb{Z}^n} N(\pi(d)).$$

Proposition 5.15 and Proposition 5.11 then show us that

$$\lim_{d \geq c,\ d \in \mathbb{Z}^n} N(\pi(d)) = \lim_{d \geq c,\ d \in \mathbb{Z}^n} N((\beta \circ \alpha)(d)) \cong N((\beta \circ \alpha)(c)) = (\operatorname{res}_{\beta\circ\alpha} N)(c).$$

□

Proposition 5.17. *Let M be an $R\mathbb{Z}^n$-module, and let $c \in \overline{[a, b]}$. Set $u := (1, 1, \ldots, 1) \in \mathbb{Z}^n$.*

1) *If M is finitely determined with the encoding convex projection $\pi\colon \mathbb{Z}^n \to [a, b]$, then $\overline{M}(c) \cong M(\beta(c))$.*
2) *If $\overline{M}$ is $\overline{[a + u, b]}$-determined, then $\overline{M}(c) \cong M(\beta(c))$.*

Proof. To show 1), suppose that M is finitely determined with the encoding convex projection $\pi\colon \mathbb{Z}^n \to [a, b]$. Then, by the definition of $\overline{M}$,

$$\overline{M}(c) = \lim_{d \geq c,\ d \in \mathbb{Z}^n} M(d).$$

Because M is finitely determined, we have $M(d) \cong M(\pi(d))$ for all $d \in \mathbb{Z}^n$. This implies that

$$\overline{M}(c) \cong \lim_{d \geq c,\ d \in \mathbb{Z}^n} M(\pi(d)).$$

We may now apply Corollary 5.12 to see that $\overline{M}(c) \cong M(\beta(\alpha(c)))$. Note that $c \in \overline{[a,b]}$ implies $\alpha(c) = c$. Thus $\overline{M}(c) \cong M(\beta(c))$.

Next, to prove 2), let $\overline{M}$ be $\overline{[a+u,b]}$-determined. Since $c \leq \beta(c)$, it is then enough to show that $\overline{[a+u,b]} \cap \downarrow c = \overline{[a+u,b]} \cap \downarrow \beta(c)$. We instantly have $\downarrow c \subseteq \downarrow \beta(c)$. For the other direction, let $d := (d_1, \ldots, d_n) \in \overline{[a+u,b]} \cap \downarrow \beta(c)$. We want to show that $d \leq c$. Recall that we may write $\beta(c) = (\beta_1(c_1), \ldots, \beta_n(c_n))$, where

$$\beta_i(c_i) = \begin{cases} a_i, & \text{if } c_i = -\infty; \\ c_i, & \text{otherwise.} \end{cases}$$

for all $i \in \{1, \ldots, n\}$. Suppose that $i \in \{1, \ldots, n\}$. If $\beta_i(c_i) = c_i$, we have $d_i \leq \beta_i(c_i) = c_i$. Otherwise, if $\beta_i(c_i) = a_i$, we must have $d_i = c_i = -\infty$, because $d_i, c_i \in \overline{[a_i+1, b_i]}$. We conclude that $d \leq c$. □

The next Corollary is a direct consequence of Proposition 5.17.

Corollary 5.18. *Let M be an $R\mathbb{Z}^n$-module. Set $u := (1, 1, \ldots, 1) \in \mathbb{Z}^n$.*

1) If M is finitely determined with the convex projection $\pi \colon \mathbb{Z}^n \to [a,b]$, then

$$\operatorname{res}_{\overline{[a,b]}} \overline{M} \cong \operatorname{res}_\beta \operatorname{res}_{[a,b]} M.$$

2) If $\overline{M}$ is $\overline{[a+u,b]}$-determined, then

$$\operatorname{res}_{\overline{[a,b]}} \overline{M} \cong \operatorname{res}_\beta \operatorname{res}_{[a,b]} M.$$

We will now present the main result of this chapter.

Theorem 5.19. *Let M be an $R\mathbb{Z}^n$-module. Then M is finitely determined if and only if $\overline{M}$ is S-determined for some finite $S \subseteq \overline{\mathbb{Z}}^n$. In particular, we have:*

1) If M is finitely determined with the encoding convex projection $\pi \colon \mathbb{Z}^n \to [a,b]$, then M is $\overline{[a,b]}$-determined;

2) If M is $\overline{[a+u,b]}$-determined, where $u := (1, \ldots, 1) \in \mathbb{Z}^n$, then M is finitely determined with the encoding convex projection $\pi \colon \mathbb{Z}^n \to [a,b]$.

Proof. Suppose first that M is finitely determined. Let $\pi\colon \mathbb{Z}^n \to [a,b]$ be its encoding convex projection. Now, since M is finitely determined, we have $M \cong \operatorname{res}_\pi \operatorname{res}_{[a,b]} M$, as stated in Remark 5.13. By Corollary 5.16 and Corollary 5.18, respectively, we get

$$\overline{M} \cong \overline{\operatorname{res}_\pi \operatorname{res}_{[a,b]} M} \cong \operatorname{res}_{\beta\circ\alpha} \operatorname{res}_{[a,b]} M = \operatorname{res}_\alpha \operatorname{res}_\beta \operatorname{res}_{[a,b]} M \cong \operatorname{res}_\alpha \operatorname{res}_{\overline{[a,b]}} \overline{M}.$$

This means that M is $\overline{[a,b]}$-determined by Proposition 5.6.

Next, suppose that $\overline{M}$ is S-determined for some finite $S \subseteq \overline{\mathbb{Z}}^n$. Since S is finite, we may assume that $\tilde{S} \subseteq \overline{[a+u,b]}$ for some $a,b \in \mathbb{Z}^n$. Recall that if M is T-determined and $T \subseteq T'$, then M is T'-determined. This implies that M is both $\overline{[a+u,b]}$-determined and $\overline{[a,b]}$-determined. Using Proposition 5.6, we see that $\overline{M} \cong \operatorname{res}_\alpha \operatorname{res}_{\overline{[a,b]}} \overline{M}$. Applying Corollary 5.18 and Corollary 5.16, respectively, shows us that

$$\overline{M} \cong \operatorname{res}_\alpha \operatorname{res}_{\overline{[a,b]}} \overline{M} \cong \operatorname{res}_\alpha \operatorname{res}_\beta \operatorname{res}_{[a,b]} M = \operatorname{res}_{\beta\circ\alpha} \operatorname{res}_{[a,b]} M \cong \overline{\operatorname{res}_\pi \operatorname{res}_{[a,b]} M}.$$

Restricting this to $\mathbb{Z}^n$, we get the required result by Remark 5.13. $\square$

Example 5.20. Let M be an $R\mathbb{Z}^2$-module that is defined on objects by

$$M(c) = \begin{cases} R, & \text{if } c \leq (0,0); \\ 0, & \text{otherwise,} \end{cases}$$

for all $c \in \mathbb{Z}^2$, and where a morphism $R \to R$ is always id_R. Then M is finitely determined with the convex projection $\pi\colon \mathbb{Z}^2 \to [(0,0),(1,1)]$. Now, by Theorem 5.19 1), $\overline{M}$ is $\overline{[(0,0),(1,1)]}$-determined. Here $\overline{[(0,0),(1,1)]}$ is the set

$$\{(-\infty,-\infty),(0,-\infty),(-\infty,0),(1,-\infty),(-\infty,1),(0,0),(0,1),(1,0),(1,1)\}.$$

In particular, we have

$$\overline{M}((-\infty,-\infty)) = \overline{M}((-\infty,0)) = \overline{M}((0,-\infty)) = \overline{M}((0,0)) = R,$$

and

$$\overline{M}((-\infty,1)) = \overline{M}((1,-\infty)) = \overline{M}((0,1)) = \overline{M}((1,0)) = \overline{M}((1,1)) = 0.$$

Furthermore, using Theorem 4.15, we see that $\overline{M}$ is $\overline{[0,1]}$-presented. In more concrete terms, we have an exact sequence of $R\overline{\mathbb{Z}}^2$-modules

$$K \to N \to \overline{M} \to 0,$$

where

$$N = R[\mathrm{Mor}_{\overline{\mathbb{Z}}^2}((-\infty, -\infty), -)]$$

and

$$K = R[\mathrm{Mor}_{\overline{\mathbb{Z}}^2}((1, -\infty), -)] \oplus R[\mathrm{Mor}_{\overline{\mathbb{Z}}^2}((-\infty, 1), -)].$$

Admissible join-sublattices

Recall that a subset $L \subseteq \overline{\mathbb{Z}}^n$ is a join-sublattice if $\mathrm{mub}(S) \in L$ for every finite subset $S \subseteq L$. Note that this is equivalent to the condition that $L = \hat{L}$. Given a join-sublattice $L \subseteq \overline{\mathbb{Z}}^n$, following Perling in [23, pp. 16-19, Ch. 3.1], we define the zip-functor

$$\mathrm{zip}_L \colon R\mathbb{Z}^n\text{-}\mathbf{Mod} \to RL\text{-}\mathbf{Mod}$$

and the unzip-functor

$$\mathrm{unzip}_L \colon RL\text{-}\mathbf{Mod} \to R\overline{\mathbb{Z}}^n\text{-}\mathbf{Mod}.$$

Contrary to Perling, we do not assume that R is a field. The zip-functor maps an $R\mathbb{Z}^n$-module M to the RL-module $\mathrm{res}_L \overline{M}$, whereas he unzip-functor maps an RL-module N to an $R\overline{\mathbb{Z}}^n$-module $\mathrm{unzip}_L N$ defined by

$$(\mathrm{unzip}_L N)(c) = \begin{cases} N(\mathrm{mub}(L \cap {\downarrow}c)), & \text{if } L \cap {\downarrow}c \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

for all $c \in \overline{\mathbb{Z}}^n$. Note that $\mathrm{Supp}(\mathrm{unzip}_L N) \subseteq {\uparrow}L$.

Remark 5.21. It turns out that unzip_L is essentially the same thing as res_α, when L is finite and $\alpha := \alpha_L$. There is the slight complication that unzip_L is defined for RL-modules, while res_α is defined for $R\tilde{L}$-modules. We may, however, extend an RL-module N to an $R\tilde{L}$-module $\tilde{N}$ by setting

$$\tilde{N}((-\infty, \dots, -\infty)) = 0,$$

if $(-\infty,\dots,-\infty) \notin L$, and $\tilde{N}(c) = N(c)$, otherwise. Defined in this way, we see that $\operatorname{unzip}_L N \cong \operatorname{res}_\alpha \tilde{N}$.

Given an $R\mathbb{Z}^n$-module M, the join-sublattice L is called *M-admissible* in [23] if the condition $\overline{M} \cong \operatorname{unzip}_L \operatorname{zip}_L M$ is satisfied. This leads to our following proposition.

Proposition 5.22. *Let M be an $R\mathbb{Z}^n$-module, and L a finite join-sublattice. Then L is M-admissible if and only if $\overline{M}$ is L-determined.*

Proof. Let $c \in \overline{\mathbb{Z}}^n$. With the earlier notation, we see that

$$\operatorname{unzip}_L \operatorname{zip}_L M = \operatorname{unzip}_L \operatorname{res}_L \overline{M} \cong \operatorname{res}_\alpha \widetilde{\operatorname{res}_L \overline{M}},$$

where

$$(\operatorname{res}_\alpha \widetilde{\operatorname{res}_L \overline{M}})(c) = \begin{cases} (\operatorname{res}_\alpha \operatorname{res}_{\bar{L}} \overline{M})(c), & \text{if } L \cap {\downarrow} c \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

Assume first that $\overline{M} \cong \operatorname{unzip}_L \operatorname{zip}_L M$. If $L \cap {\downarrow} c = \emptyset$, we have $\overline{M}(c) = 0$ by the definition of unzip_L. But in this case $\alpha(c) \leq c$, so that $L \cap {\downarrow}\alpha(c) = \emptyset$, and using the definition of unzip_L again, we get

$$(\operatorname{res}_\alpha \operatorname{res}_{\bar{L}} \overline{M})(c) = \overline{M}(\alpha(c)) = 0.$$

On the other hand, if there exists an element $d \in L \cap {\downarrow} c$, then $\overline{M}(c) \cong (\operatorname{res}_\alpha \operatorname{res}_{\bar{L}} \overline{M})(c)$ by the above formula. Since these $R\overline{\mathbb{Z}}$-modules are isomorphic on all objects, we have

$$\overline{M} \cong \operatorname{res}_\alpha \operatorname{res}_{\bar{L}} \overline{M},$$

and $\operatorname{Supp}(\overline{M}) \subseteq {\uparrow} L$, so $\overline{M}$ is L-determined by Proposition 5.6.

Conversely, suppose that $\overline{M}$ is L-determined. By Proposition 5.6, we have $\overline{M} \cong \operatorname{res}_\alpha \operatorname{res}_{\bar{L}} \overline{M}$ and $\operatorname{Supp}(\overline{M}) \subseteq {\uparrow} L$. The above formula shows us that

$$(\operatorname{unzip}_L \operatorname{zip}_L M)(c) = (\operatorname{res}_\alpha \operatorname{res}_{\bar{L}} \overline{M})(c)$$

for all $c \in {\uparrow} L$. If $c \notin {\uparrow} L$, then $c \notin \operatorname{Supp}(\overline{M})$, which means that $\overline{M}(c) = 0$. In this case, we also have $(\operatorname{unzip}_L \operatorname{zip}_L M)(c) = 0$ by the definition of unzip_L. Thus we have an isomorphism

$$\overline{M} \cong \operatorname{unzip}_L \operatorname{zip}_L M. \qquad \square$$

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