Dynamic polynomial stabilization of a 1D wave equation*

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Abstract

We study observer-based dynamic stabilization of a one-dimensional wave equation with boundary control and distributed observation. The control system we consider is exponentially stabilizable but not exponentially detectable. Consequently, exponential energy decay is not achievable with dynamic output feedback. As our main result we design an observer-based controller which achieves rational decay of energy for a class of initial conditions. The controller design relies on helpful results on polynomial stability of semigroups generated by block operator matrices.

1 Introduction

In this article we design an observer-based stabilizing controller for a one-dimensional controlled wave equation with boundary control and distributed observation. Due to the type of measurement, the controlled PDE is not exponentially detectable, and therefore the degree of stability is crucially limited in the sense that dynamic exponential stabilization is not possible. The model we consider is

$$\text{(1a)}\qquad w_{tt}(\xi,t) = w_{\xi\xi}(\xi,t), \qquad \xi \in (0,1)$$

$$\text{(1b)}\qquad w_t(0,t) = 0, \qquad w_\xi(1,t) = u(t)$$

$$\text{(1c)}\qquad w(\xi,0) = w_0(\xi), \qquad w_t(\xi,0) = w_1(\xi)$$

$$\text{(1d)}\qquad y(t) = \int_0^1 w_t(\xi,t)\overline{c(\xi)}d\xi$$

with a given $c(\cdot) \in L^2(0,1)$. The total energy of the wave system is defined as

$$E(t) = \frac{1}{2}\Big(\int_0^1 |w_t(\xi,t)|^2 d\xi + \int_0^1 |w_\xi(\xi,t)|^2 d\xi\Big).$$

We consider stabilization of (1) in the sense that $E(t) \to 0$ as $t \to \infty$. Instead of exponential stabilization, we design an observer-based dynamic controller that achieves *polynomial stability* [1, 3] and rational convergence of energy for a class of initial conditions of the system and the controller. The controller design and stability results are based on standard Luenberger observer construction and subsequent analysis of the polynomial stability of the closed-loop system.

Dynamic stabilization of boundary controlled one-dimensional wave equations has been studied in several references in the case of exponential closed-loop stability, most notably using backstepping [11, 8], observer design [9], and Active Disturbance Rejection Control [6]. Strong asymptotic stabilization using dynamic controllers has been studied extensively in [5],[12, Ch. 6] for a very general class of infinite-dimensional systems. For wave equations, strong dynamic stabilization has also been considered in [7, 10]. In addition, polynomial stability of semigroups generated by block operator matrices has been studied previously in [13].

The structure of the paper is as follows. In Section 2 we present the observer-based controller for (1) and state the main results. The proofs of the main theorems are presented in Section 3, and in Section 4 we demonstrate our results with numerical simulations. Section 5 contains concluding remarks.

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $D(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $A : X \to X$, then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is $R(\lambda, A) = (\lambda - A)^{-1}$. The inner product on a Hilbert space is denoted by $\langle\cdot,\cdot\rangle$. Throughout the paper we denote $H_l^1(0,1) = \{\, x \in H^1(0,1) \mid x(0) = 0 \,\}$ and $H_r^1(0,1) = \{\, x \in H^1(0,1) \mid x(1) = 0 \,\}$.

2 Dynamic Polynomial Stabilization

The controller we propose is of the form

$$\text{(2a)}\qquad \hat{w}_{tt}(\xi,t) = \hat{w}_{\xi\xi}(\xi,t) - \gamma c(\xi)\int_0^1 [\hat{w}_t(r,t) - w_t(r,t)]\,\overline{c(r)}dr,$$

$$\text{(2b)}\qquad \hat{w}_t(0,t) = 0, \qquad \hat{w}_\xi(1,t) = -\beta\hat{w}_t(1,t)$$

$$\text{(2c)}\qquad \hat{w}(\xi,0) = \hat{w}_0(\xi), \qquad \hat{w}_t(\xi,0) = \hat{w}_1(\xi)$$

$$\text{(2d)}\qquad u(t) = -\beta\hat{w}_t(1,t)$$

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with $\gamma > 0$ and $\beta > 0$. Our main result, stated below in Theorem 2.1, shows that the controller achieves asymptotic decay of the energy $E(t)$, and that under the additional assumption (3) on $c(\cdot)$ in (1d) the energy of the classical solutions of the controlled system decay at a specific rational rate. This rate is determined by the behaviour of the Fourier coefficients of $c(\cdot)$ through (3). It follows from the theory of strongly continuous semigroups that the condition (3) is also necessary for the decay rate in Theorem 2.1. Due to the lack of exponential detectability it is in fact impossible to achieve an energy decay rate which would be applicable for all $(w_0, w_1)^T$ and $(\hat{w}_0, \hat{w}_1)^T$ in $H^1(0,1) \times L^2(0,1)$, and instead the precise rate of $E(t) \to 0$ depends on the initial conditions. It should be noted that the displacement $w(\cdot, t)$ is not required to converge to zero as $t \to \infty$.

THEOREM 2.1. *If $c(\cdot) \in L^2(0,1)$ is such that $\langle c(\cdot), \sin(\pi(k-1/2)\cdot)\rangle_{L^2} \neq 0$ for all $k \in \mathbb{N}$, then with the controller* (2) *the energy of the mild solution of* (1) *satisfies $E(t) \to 0$ as $t \to \infty$ for any initial conditions $w_0, \hat{w}_0 \in H^1(0,1)$ and $w_1, \hat{w}_1 \in L^2(0,1)$.*

If in addition there exist $\alpha, m_\alpha > 0$ such that

$$(3) \quad |\langle c(\cdot), \sin(\pi(k-1/2)\cdot)\rangle_{L^2}|^2 \geq m_\alpha k^{-\alpha}, \quad \forall k \in \mathbb{N},$$

then there exist $M, t_0 > 0$ such that for all initial conditions $w_1, \hat{w}_1 \in H_l^1(0,1)$, $w_0, \hat{w}_0 \in H^2(0,1)$ satisfying $w_0'(1) = \hat{w}_0'(1) = -\beta \hat{w}_1(1)$ we have

$$E(t) \leq \frac{M}{t^{2/\alpha}} \Big(\|w_0''\|_{L^2}^2 + \|w_1'\|_{L^2}^2 + \|\hat{w}_0'' - w_0''\|_{L^2}^2 + \|\hat{w}_1' - w_1'\|_{L^2}^2 \Big)$$

for all $t \geq t_0$.

The wave equation (1) can be written in the form

$$(4a) \quad \dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in X$$

$$(4b) \quad y(t) = Cx(t)$$

on $X = L^2(0,1) \times L^2(0,1)$ with state $x(t) = (w_t(\cdot,t), w_\xi(\cdot,t))^T$ and

$$A = \begin{bmatrix} 0 & \partial_\xi \\ \partial_\xi & 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_0, & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}$$

$$D(A) = \big\{ (x_1, x_2)^T \in H^1(0,1) \times H^1(0,1) \;\big|\; x_1(0) = x_2(1) = 0 \big\}$$

where $B_0 = \delta_1(\cdot) \in (H_l^1(0,1))^*$ and $B_0^* f = f(1)$ for $f \in H_l^1(0,1)$, and $C_0 f = \langle f, c\rangle_{L^2}$. The expressions of B and B^* and the characterization of $D(A)$ can be derived as shown in [16, Sec. 10.1]. The operator $B \in \mathcal{L}(\mathbb{C}, X_{-1})$ is admissible and $C \in \mathcal{L}(X, \mathbb{C})$. Note that $D(A) = H_l^1(0,1) \times H_r^1(0,1)$.

Similarly, with the choice of the state $\hat{x} = (\hat{w}_t, \hat{w}_\xi)^T$ the controller (2) can be written as

$$(5a) \quad \dot{\hat{x}}(t) = (A + BK)\hat{x}(t) + LC(\hat{x}(t) - x(t))$$

$$(5b) \quad u(t) = K\hat{x}(t), \qquad \hat{x}(0) = \hat{x}_0$$

where $L = -\gamma C^* \in \mathcal{L}(\mathbb{C}, X)$ and $K = -\beta \overline{B^*}$, and where $\overline{B^*} = \begin{bmatrix} B_0^*, & 0 \end{bmatrix} \in \mathcal{L}(H_l^1(0,1) \times H^1(0,1), \mathbb{C})$ is the extension of $B^* = \begin{bmatrix} B_0^*, & 0 \end{bmatrix} \in \mathcal{L}(D(A), \mathbb{C})$. The extension is *compatible* in the sense of [15, Def. 5.1.1]. The operators $(A+BK)|_X$ and $A + LC$ generate semigroups $T_K(t)$ and $T_L(t)$, respectively, on X. The former is exponentially stable [4], and the latter is either strongly or polynomially stable depending on the properties of the function $c(\cdot)$ (see the proof of Theorem 2.2 for details). Because of its structure, the stabilizing controller (2) is formally based on the Luenberger observer for (4). The closed-loop dynamics are determined by

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}.$$

If we apply an invertible change of coordinates $(x(t), \hat{x}(t))^T \to (x(t), \hat{x}(t) - x(t))^T$ we arrive at

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ \hat{x}(t) - x(t) \end{bmatrix} = \begin{bmatrix} A + BK & BK \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) - x(t) \end{bmatrix}.$$

Since the states of (4) and (5) are given by $x(t) = (w_t(\cdot,t), w_\xi(\cdot,t))^T$ and $\hat{x}(t) = (\hat{w}_t(\cdot,t), \hat{w}_\xi(\cdot,t))^T$ and $E(t) = \frac{1}{2}\|x(t)\|^2$, Theorem 2.1 follows from the following abstract stability result. The theorem also shows that if $c(\cdot)$ is such that (3) holds, then the observation error $\|\hat{x}(t) - x(t)\|$ converges at a rational rate for a suitable set of initial conditions.

THEOREM 2.2. *If $c(\cdot) \in L^2(0,1)$ is such that $\langle c(\cdot), \sin(\pi(k-1/2)\cdot)\rangle_{L^2} \neq 0$ for all $k \in \mathbb{N}$, then for all $x(0), \hat{x}(0) \in X$*

$$\|x(t)\| \to 0, \qquad \|\hat{x}(t) - x(t)\| \to 0, \qquad \text{as} \quad t \to \infty.$$

If in addition there exist $\alpha, m_\alpha > 0$ such that (3) *holds then the controller achieves polynomial closed-loop stability and there exist $M, t_0 > 0$ such that*

$$\left\| \begin{bmatrix} x(t) \\ \hat{x}(t) - x(t) \end{bmatrix} \right\| \leq \frac{M}{t^{1/\alpha}} \left(\|Ax_0 + BK\hat{x}_0\| + \|A(\hat{x}_0 - x_0)\| \right)$$

for $t \geq t_0$ and for all initial states $x_0, \hat{x}_0 \in H^1(0,1) \times H^1(0,1)$ satisfying $x_0 - \hat{x}_0 \in D(A)$ and $Ax_0 + BK\hat{x}_0 \in X$.

REMARK 1. *If the boundary condition $w_t(0,t) = 0$ is replaced with $w_\xi(0,t) = 0$, the undamped system has a second order eigenvalue at $\lambda = 0$. This eigenvalue would need to be factored out of the state space of the control system, since such an eigenspace cannot be stabilized with rank one control and observation operators.*

3 Proofs of the main results

We begin by proving Theorem 2.2.

Proof of Theorem 2.2. We have $\sigma(A) = \{i\pi(k-1/2)\}_{k\in\mathbb{Z}}$ and the orthonormal eigenvectors of A are given by $\phi_k = (\sin(\pi(k-1/2)\cdot), \cos(\pi(k-1/2)\cdot))^T$ for all $k\in\mathbb{Z}$. The pair (C,A) is approximately controllable if $C\phi_k \neq 0$ for all $k\in\mathbb{Z}$, which is equivalent to the property that $\langle c(\cdot), \sin(\pi(k-1/2)\cdot)\rangle_{L^2} \neq 0$ for all $k\in\mathbb{N}$. Under this assumption the semigroup $T_L(t)$ generated by $A+LC$ with $L=-\gamma C^*$ is strongly stable [2, Cor. 3.2]. The pair (A,B) is exactly controllable, and the semigroup $T_K(t)$ generated by $(A+BK)|_X$ is exponentially stable [4].

Denote $A_K = A+BK$ and $A_L = A+LC$. We will next show that the semigroup $T_c(t)$ generated by

$$A_c = \begin{bmatrix} A_K & BK \\ 0 & A_L \end{bmatrix}$$
$$D(A_c) = \Big\{ (x_1,x_2)^T \in D(\overline{B^*}) \times D(A) \;\Big|\; Ax_1 + BK(x_1+x_2) \in X \Big\}$$

on $X\times X$ is strongly stable. The operator K is admissible for A_L by [16, Thm. 5.4.2] and

$$T_c(t) = \begin{bmatrix} T_K(t) & S(t) \\ 0 & T_L(t) \end{bmatrix},$$

where

$$S(t)x_2 = \int_0^t T_K(t-s)BKT_L(s)x_2 ds, \quad \forall x_2 \in D(A_L).$$

For the strong stability of $T_c(t)$ it is sufficient to show that there exists $M_S \geq 0$ such that for all $x_2\in D(A_L)$ we have $\|S(t)x_2\| \leq M_S\|x_2\|$ and $\|S(t)x_2\| \to 0$ as $t\to\infty$. The admissibilities of B and K imply that there exist $\kappa_B, \kappa_K > 0$ such that for all $x_2 \in D(A_L)$, $f\in L^2(0,1)$ and $t\in[0,1]$

$$\left\| \int_0^t T_K(t-s)Bf(s)ds \right\| \leq \kappa_B \|f\|_{L^2(0,1)},$$
$$\|KT_L(\cdot)x_2\|_{L^2(0,1)} \leq \kappa_K \|x_2\|.$$

Let $x_2 \in D(A_L)$ with $\|x_2\|=1$ and $t>0$ be arbitrary, and denote $t = n + t_0$ for some $n\in\mathbb{N}_0$ and $t_0\in[0,1)$. Let $M_1, M_2, \omega > 0$ be such that $\|T_K(t)\| \leq M_1 e^{-\omega t}$ and $\|T_L(t)\| \leq M_2$ for all $t\geq 0$. If we denote $g(k) = \|T_L(k)x_2\|_X$ for $k\in\mathbb{N}_0$, then $\|KT_L(\cdot)T_L(k)x_2\|_{L^2(0,1)} \leq \kappa_K g(k)$ and

$$\begin{aligned}
&\left\| \int_0^t T_K(t-s)BKT_L(s)x_2 ds \right\| \\
&\leq \left\| \int_0^{t_0} T_K(t_0-r)BKT_L(r)T_L(n)x_2 dr \right\| \\
&+ \sum_{k=0}^{n-1} \left\| T_K(t-k-1) \int_0^1 T_K(1-r)BKT_L(r)T_L(k)x_2 dr \right\| \\
&\leq \kappa_B \|KT_L(\cdot)T_L(n)x_2\|_{L^2(0,1)} \\
&\quad + \kappa_B \sum_{k=0}^{n-1} \|T_K(t-k-1)\| \|KT_L(\cdot)T_L(k)x_2\|_{L^2(0,1)} \\
&\leq \kappa_B\kappa_K \left(g(n) + M_1 e^{\omega} \sum_{k=0}^{n-1} e^{-\omega(n-k)} g(k) \right).
\end{aligned}$$

Since $g(k) \leq M_2\|x_2\| = M_2$ for all k, the upper bound is uniformly bounded with respect n, and since $x_2 \in D(A_L)$ with $\|x_2\|=1$ was arbitrary, we have $\sup_{t\geq 0}\|S(t)\| < \infty$. Since $g(k)\to 0$ as $k\to\infty$, the upper bound also converges to zero as $n\to\infty$, and this together with the uniform boundedness of $\|S(\cdot)\|$ and denseness of $D(A_L)$ implies that that $\|S(t)x_2\| \to 0$ as $t\to\infty$ for all $x_2\in X$.

We will now show that if the additional condition (3) is satisfied, then the semigroup $T_c(t)$ is polynomially stable. For this it is sufficient to show that $i\mathbb{R} \subset \rho(A_c)$ and $\|R(i\omega, A_c)\| = M_c(1+|\omega|^\alpha)$ for some $M_c > 0$ [3]. Under the condition (3) we have from [14, Thm. 6.3] that the semigroup generated by A_L is polynomially stable, and in particular $\|R(i\omega, A_L)\| = M_L(1+|\omega|^\alpha)$ for some $M_L > 0$. For $\omega\in\mathbb{R}$ we have $i\omega \in \rho(A_K)\cap\rho(A_L)$ and the resolvent operator of A_c is given by

$$R(i\omega, A_c) = \begin{bmatrix} R(i\omega, A_K) & R(i\omega, A_K)BKR(i\omega, A_L) \\ 0 & R(i\omega, A_L) \end{bmatrix}.$$

Since $T_K(t)$ is exponentially stable and B is admissible, there exists $M_1 > 0$ such that $\|R(i\omega, A_K)\| \leq M_1$ and $\|R(i\omega, A_K)B\| \leq M_1$ for all $\omega\in\mathbb{R}$. In addition, since K is A_L-admissible and $T_L(t)$ is uniformly bounded, the resolvent identity $KR(i\omega, A_L) = KR(i\omega+1, A_L) + KR(i\omega+1, A_L)R(i\omega, A_L)$ implies that there exists M_2 such that $\|KR(i\omega, A_L)\| \leq M_2(1+\|R(i\omega, A_L)\|)$ for all $\omega\in\mathbb{R}$. Thus for any $x = (x_1, x_2) \in X\times X$ with $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 = 1$ we have (denoting $R_K = R(i\omega, A_K)$ and $R_L = R(i\omega, A_L)$ for brevity)

$$\begin{aligned}
\|R(i\omega, A_c)x\|^2 &= \left\| \begin{bmatrix} R_K x_1 + R_K BKR_L x_2 \\ R_L x_2 \end{bmatrix} \right\|^2 \\
&= \|R_K x_1 + R_K BKR_L x_2\|^2 + \|R_L x_2\|^2 \\
&\leq 2\|R_K\|^2 + 2\|R_K B\|^2 M_2^2 (1+\|R_L\|)^2 + \|R_L\|^2
\end{aligned}$$

which implies $\|R(i\omega, A_c)\| = M_c(1+|\omega|^\alpha)$ for some $M_c > 0$.

Finally, by [3, Thm. 2.4] the resolvent estimate $\|R(i\omega, A_c)\| = M_c(1+|\omega|^\alpha)$ shows that there exists $\tilde{M}, t_0 > 0$ such that for every $t \geq t_0$

$$\left\| \begin{bmatrix} x(t) \\ \hat{x}(t) - x(t) \end{bmatrix} \right\| = \left\| T_c(t) \begin{bmatrix} x_0 \\ \hat{x}_0 - x_0 \end{bmatrix} \right\| \leq \frac{\tilde{M}}{t^{1/\alpha}} \left\| A_c \begin{bmatrix} x_0 \\ \hat{x}_0 - x_0 \end{bmatrix} \right\|.$$

This together with boundedness of LC and the invertibility of A implies the decay rate in the statement of the theorem. □

Proof of Theorem 2.1. The first claim follows directly from the first part of Theorem 2.2, and we will now show that the rational decay rate of $E(t)$ follows from the second part. By Theorem 2.2 we clearly have a rational decay rate for $E(t) = \frac{1}{2}\|x(t)\|^2$ for all initial states such that $(x_0, \hat{x}_0 - x_0) \in D(A_c)$, where

$$D(A_c) = \Big\{ (x_1, x_2)^T \in X_B \times D(A) \;\Big|\; Ax_1 + BK(x_1 + x_2) \in X \Big\}$$

with $X_B := D(A) + \mathcal{R}(A^{-1}B)$. As in [16, Sec. 10.1], by construction of the system (4) and [16, Rem. 10.1.5] we have that the element $x_B = A^{-1}B \in X$ is the unique solution $(f_1, f_2)^T \in H_l^1(0,1) \times H^1(0,1)$ of the boundary value problem

$$f_2'(\xi) = 0, \quad f_1'(\xi) = 0,$$
$$f_1(0) = 0, \quad f_2(1) = -1,$$

i.e. $x_B = (0, -1)^T$. This (or alternatively [16, Rem. 10.1.3]) also implies that $X_B = H_l^1(0,1) \times H^1(0,1)$.

Recall $x_0 = (w_1(\cdot), w_0'(\cdot))^T$ and $\hat{x}_0 = (\hat{w}_1(\cdot), \hat{w}_0'(\cdot))^T$. We have $(x_0, \hat{x}_0 - x_0)^T \in D(A_c)$ if and only if $x_0 \in X_B = H_l^1(0,1) \times H^1(0,1)$, $\hat{x}_0 - x_0 \in D(A) = H_l^1(0,1) \times H_r(0,1)$ and

$$Ax_0 + BK\hat{x}_0 \in L^2(0,1) \times L^2(0,1). \tag{6}$$

The first requirements are satisfied if and only if $w_1, \hat{w}_1 \in H^1(0,1)$ and $w_0, \hat{w}_0 \in H^2(0,1)$ are such that

$$w_1(0) = 0,$$
$$\hat{w}_1(0) = w_1(0) = 0,$$
$$\hat{w}_0'(1) = w_0'(1).$$

Finally, since $x_0 \in X_B$, we can write $x_0 = x_1 + A^{-1}Bu_0$ where $x_1 = (w_1(\xi), w_0'(\xi) - w_0'(1))^T \in D(A)$ and $u_0 = -w_0'(1)$. Now (6) becomes

$$L^2 \times L^2 \ni Ax_0 + BK\hat{x}_0 = Ax_1 + B(u_0 + K\hat{x}_0),$$

which is satisfied if and only if $u_0 = -K\hat{x}_0 = \beta\hat{w}_1(1)$, i.e., if $w_0'(1) = -\beta\hat{w}_1(1)$. These are precisely the conditions in Theorem 2.1.

The decay rate in Theorem 2.1 now follows from Theorem 2.2 since $E(t) = \frac{1}{2}\|x(t)\|^2$, and

$$\|A(\hat{x}_0 - x_0)\|^2 = \|\hat{w}_0'' - w_0''\|_{L^2}^2 + \|\hat{w}_1' - w_1'\|_{L^2}^2$$

$$\|Ax_0 + BK\hat{x}_0\|^2 = \|Ax_1\|^2 = \left\| A \begin{bmatrix} w_1(\cdot) \\ w_0'(\cdot) - w_0'(1) \end{bmatrix} \right\|^2 = \|w_0''\|_{L^2}^2 + \|w_1'\|_{L^2}^2.$$

□

4 Numerical Example

In this section we illustrate the performance of the controller with a numerical simulation. We consider a system with an output profile function $c(\xi) = 10(1-\xi)$ and stabilization parameters $\beta = 0.5$ and $\gamma = 1$. For this function $c(\cdot)$ we can explicitly compute

$$\langle c(\cdot), \sin((k-1/2)\pi\cdot)\rangle_{L^2} = 10 \cdot \frac{(k-1/2)\pi + (-1)^k}{\pi^2(k-1/2)^2}$$

and thus (3) holds for $\alpha = 2$ and for some $m_\alpha > 0$. Theorem 2.1 therefore implies that the controller (2) will achieve strong closed-loop stability and the energy of the solution will converge at a rational rate $E(t) \sim 1/t$ for all initial states $w_1, \hat{w}_1 \in H_l^1(0,1)$, $w_0, \hat{w}_0 \in H^2(0,1)$ satisfying $w_0'(1) = \hat{w}_0'(1) = -0.5\hat{w}_1(1)$. Note that these conditions are especially satisfied for functions $w_1, \hat{w}_1 \in H_l^1(0,1)$, $w_0, \hat{w}_0 \in H^2(0,1)$ satisfying $w_0'(1) = \hat{w}_0'(1) = \hat{w}_1(1) = 0$. For more general initial states the energy $E(t)$ still converges to zero as $t \to \infty$, but the convergence does not have a guaranteed asymptotic rate.

For the simulation, the wave equations (1) and (2) were approximated using a truncated eigenfunction expansion associated to the unstable system (1). The number of modes included in the approximation was $N = 40$. The numerical simulations were completed in Python using the NumPy and SciPy libraries.

Figure 1 depicts the energy of the solution associated to the initial conditions

$$w_0(\xi) = \xi(3\xi/2 - 1) \tag{7a}$$
$$w_1(\xi) = \sin(2\pi\xi) \tag{7b}$$
$$\hat{w}_0(\xi) = \frac{1}{14}(\sin(7(1-\xi)) - \sin(7)) \tag{7c}$$
$$\hat{w}_1(\xi) = \xi(2-\xi), \tag{7d}$$

which satisfy the conditions for $E(t) \sim 1/t$. The behaviour of the controlled wave profile $w(\cdot, t)$ and the displacement error $\hat{w}(\cdot, t) - w(\cdot, t)$ of the observer are depicted in Figure 2.

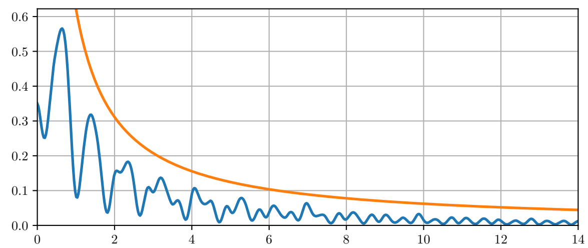

Figure 1: Energy $E(t)$ for the initial conditions (7) and a curve $\tilde{M}/t$.

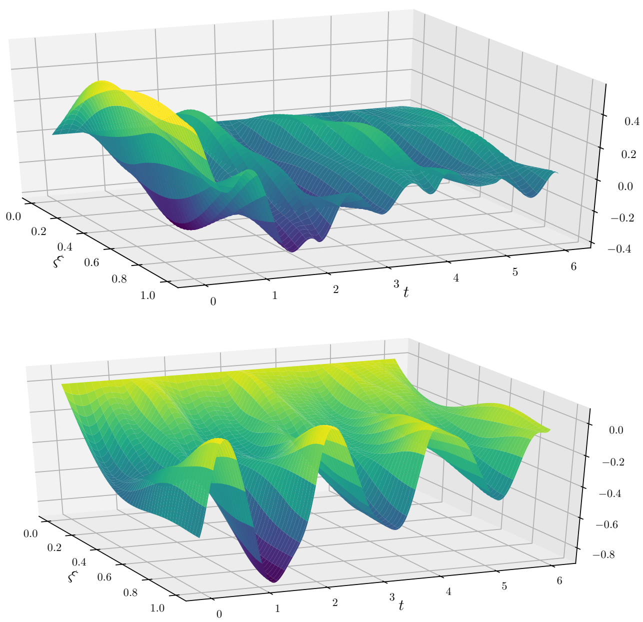

Figure 2: Wave profile $w(\xi, t)$ (top) and observation error $\hat{w}(\xi, t) - w(\xi, t)$ (bottom) for the initial conditions (7).

5 Conclusions

In this paper we have designed a dynamic output feedback controller for a one-dimensional wave equation with boundary control and distributed observation. We have shown that energy of the controlled system decays at a rational rate provided that the Fourier coefficients of the function $c(\cdot)$ determining the output measurement (1d) satisfy an additional condition. The analysis of the spectrum of the closed-loop system shows that the exponent of this decay rate in Theorem 2.1 cannot be improved. In addition, due to the lack of exponential closed-loop stability, it is only possible to present decay rates that hold for specific classes of initial conditions, and there is no decay rate which would be applicable for general inital data $(w_0, w_1)^T$ and $(\hat{w}_0, \hat{w}_1)^T$ in $H^1(0,1) \times L^2(0,1)$.

References

[1] C. J. K. Batty and T. Duyckaerts. Non-uniform stability for bounded semi-groups on Banach spaces. *J. Evol. Equ.*, 8:765–780, 2008.

[2] C. D. Benchimol. A note on weak stabilizability of contraction semigroups. *SIAM J. Control Optim.*, 16(3):373–379, 1978.

[3] A. Borichev and Y. Tomilov. Optimal polynomial decay of functions and operator semigroups. *Math. Ann.*, 347(2):455–478, 2010.

[4] S. Cox and E. Zuazua. The rate at which energy decays in a string damped at one end. *Indiana Univ. Math. J.*, 44(2):545–573, 1995.

[5] R F. Curtain and J. C. Oostveen. Necessary and sufficient conditions for strong stability of distributed parameter systems. *Systems & Control Letters*, 37(1):11–18, 1999.

[6] Hongyinping Feng and Bao-Zhu Guo. Observer design and exponential stabilization for wave equation in energy space by boundary displacement measurement only. *IEEE Trans. Automat. Control*, 62(3):1438–1444, 2017.

[7] Bao-Zhu Guo and Wei Guo. The strong stabilization of a one-dimensional wave equation by non-collocated dynamic boundary feedback control. *Automatica J. IFAC*, 45(3):790–797, 2009.

[8] Bao-Zhu Guo and Feng-Fei Jin. Arbitrary decay rate for two connected strings with joint anti-damping by boundary output feedback. *Automatica J. IFAC*, 46(7):1203–1209, 2010.

[9] Bao-Zhu Guo and Zhi-Chao Shao. Stabilization of an abstract second order system with application to wave equations under non-collocated control and observations. *Systems Control Lett.*, 58(5):334–341, 2009.

[10] Wei Guo and Zhi-Chao Shao. Strong stability of an unstable wave equation by boundary feedback with only displacement observation. *IEEE Trans. Automat. Control*, 57(9):2367–2372, 2012.

[11] M. Krstic, Bao-Zhu Guo, A. Balogh, and A. Smyshlyaev. Output-feedback stabilization of an unstable wave equation. *Automatica J. IFAC*, 44(1):63–74, 2008.

[12] J. Oostveen. *Strongly stabilizable distributed parameter systems*, volume 20 of *Frontiers in Applied Mathematics*. SIAM Philadelphia, PA, 2000.

[13] L. Paunonen. Polynomial stability of semigroups generated by operator matrices. *J. Evol. Equ.*, 14(4):885–911, 2014.

[14] L. Paunonen. Robust controllers for regular linear systems with infinite-dimensional exosystems. *SIAM J. Control Optim.*, 55(3):1567–1597, 2017.

[15] O. Staffans. *Well-Posed Linear Systems*. Cambridge University Press, 2005.

[16] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Birkhäuser Basel, 2009.