Remez Exchange Algorithm for Approximating Powers of the Q-Function by Exponential Sums

Islam M. Tanash and Taneli Riihonen
Faculty of Information Technology and Communication Sciences, Tampere University, Finland
e-mail: {islam.tanash, taneli.riihonen}@tuni.fi

***Abstract*—In this paper, we present simple and tight approximations for the integer powers of the Gaussian Q-function, in the form of exponential sums. They are based on optimizing the corresponding coefficients in the minimax sense using the Remez exchange algorithm. In particular, the best exponential approximation is characterized by the alternation of its absolute error function, which results in extrema that alternate in sign and have the same magnitude of error. The extrema are described by a system of nonlinear equations that are solved using Newton–Raphson method in every iteration of the Remez algorithm, which eventually leads to a uniform error function. This approximation can be employed in the evaluation of average symbol error probability (ASEP) under additive white Gaussian noise and various fading models. Especially, we present several application examples on evaluating ASEP in closed forms with Nakagami-m, Fisher–Snedecor $\mathcal{F}$, $\eta-\mu$, and $\kappa-\mu$ channels. The numerical results show that our approximations outperform the existing ones with the same form in terms of the global error. In addition, they achieve high accuracy for the whole range of the argument with and without fading, and it can even be improved further by increasing the number of exponential terms.**

I. Introduction

The Gaussian Q-function and the directly related error function $\mathrm{erf}(\cdot)$ are of fundamental importance to communication theory—and many other statistical sciences—whenever noise and interference or a channel can be modelled as a Gaussian random variable. This importance is reflected by the different applications in statistical performance analysis including the evaluation of error probabilities for various digital modulation schemes and different fading models [1]. The Q-function does not have an exact closed expression and it usually exists as a built-in numerical function in most of the software programs. Nevertheless, many of the Q-function applications encounter complicated integrals of it that cannot be simplified to closed-form expressions in terms of elementary functions.

Therefore, several approximations and bounds are available in [2]–[13]. The authors in [2] and [3] have proposed relatively complicated, but highly accurate, approximations and bounds that are impractical for actual evaluation of systems' performance and more suitable for improving the calculation efficiency. More accurate approximations for the Q-function are provided in [4], [5]. The approximation of the first power in [4] is later simplified in [6] using Taylor series expansion. An accurate polynomial approximation for $Q(x)$ is derived in [7]. A single-term exponential approximation with polynomial argument of the second degree is presented in [8]. The simplest form of exponential approximations and bounds were first proposed by Chiani *et al.* in [9], and other ones are also developed using different approaches in [10]–[13].

The aforementioned approximations and bounds find applications in various communication problems. For example, the approximations in [7] are applied to analytically calculate the average symbol error rate of pulse amplitude modulation in log-normal channels. In [8], the authors derive the probability of detection for an energy detector over a Rayleigh fading channel. Moreover, the exponential approximations in [9] are implemented to compute error probabilities for space–time codes and phase-shift keying.

The aim of this work is to develop new accurate approximations for the Gaussian Q-function and its integer powers by adopting the simple exponential form originally proposed in [9] with acquiring novel, improved coefficients for it. The work in [9] is limited to two-term approximation without methodology for optimally extending it to higher number of terms and integer powers. In particular, we minimize the maximum absolute difference between the exponential sum (3) and $Q^p(x)$ to obtain the best global minimax approximation for any number of terms like we did in [13], but now we avoid complicated nonlinear equations thereof, numerical solving of which is very sensitive to the right choice of initial guesses.

We solve the coefficients by the Remez exchange algorithm and propose a new heuristic method to find the initial guesses needed for it. The resulting approximations render significantly higher accuracy in terms of global error and adequate accuracy for the whole range of the argument when compared to the existing ones of [9]–[11] with the same form and number of terms. The accuracy can even be increased further by increasing the number of exponential terms. Finally, some application examples on evaluating average error probabilities over different generalized fading distributions are provided to validate the high accuracy of the new approximations in comparison to the reference approximations.

II. Problem Formulation

The Gaussian Q-function is defined classically as

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\tfrac{1}{2}t^2\right) \mathrm{d}t. \tag{1}$$

An alternative representation in the polar domain was developed by Craig [14] for communication theory applications as

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$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \exp\left(-\frac{1}{2\sin^2\theta} x^2\right) \, \mathrm{d}\theta, \tag{2}$$

that is valid for $x \geq 0$ only. Indeed, throughout this article, we shall confine our discussions to the domain $x \geq 0$ since the results can be trivially extended to the negative real axis using the relation $Q(x) = 1 - Q(-x)$.

The weighted sum of exponential functions adopted herein for approximating $Q^p(x)$ is written as [9, Eq. (8)]

$$\tilde{Q}_p(x) \triangleq \sum_{n=1}^{N} a_n \exp\left(-b_n x^2\right), \tag{3}$$

that is likewise valid for $x \geq 0$ only. In [9], Chiani *et al.* use the trapezoidal integration rule to find $\{(a_n, b_n)\}_{n=1}^N$ for $N = 2$ by optimizing the center point of (2) to minimize the integral of relative error in an argument range of interest. Moreover, other approximations for any N are also derived using the rectangular rule with non-optimized equispaced points.

Our research problem is to optimize the coefficients of the approximation in the sense of minimax absolute error as

$$\{(a_n^*, b_n^*)\}_{n=1}^N \triangleq \underset{\{(a_n,b_n)\}_{n=1}^N}{\arg\min} \; d_{\max}, \tag{4}$$

in which $d_{\max}$ refers to the global tightness of the approximation $\tilde{Q}_p(x)$ over the range $[0, \infty)$ and is measured as

$$d_{\max} \triangleq \max_{x \geq 0} |d(x)|. \tag{5}$$

The above absolute error function is defined as

$$d(x) \triangleq \tilde{Q}_p(x) - Q^p(x), \tag{6}$$

and it converges to zero when x tends to infinity, i.e., $\lim_{x\to\infty} d(x) = 0$. Thus, in plain words, our goal that is expressed in (4) is to solve the optimized set of coefficients $\{(a_n^*, b_n^*)\}_{n=1}^N$ to minimize $d_{\max}$ given in (5), substitute them in (3), and so obtain increasingly accurate approximations not only for the Q-function but also for its integer powers.

III. Solution by Remez Exchange Algorithm

We solve (4) by applying the famous exchange algorithm established by Evgeny Remez in 1934. The Remez algorithm is an iterative methodology that can be used to derive the best approximation in the minimax sense using different nonlinear approximating functions (that are typically Chebyshev polynomials) and is characterized by the uniform alternation of the corresponding error function [15] as seen in Fig. 1 after the third iteration. In this paper, we use the sum of exponentials defined in (3) as the approximating function to obtain the best unique approximation for the power of the Q-function, since it is a completely monotonic function [16], [17]. The corresponding error function should alternate exactly $2N$ times on $[0, \infty)$ between maximum and minimum values of equal magnitude, resulting in a total of $2N+1$ extrema points. The exponential approximation also results in $2N+1$ unknowns, namely the $2N$ coefficients of (3) and the global error per (5).

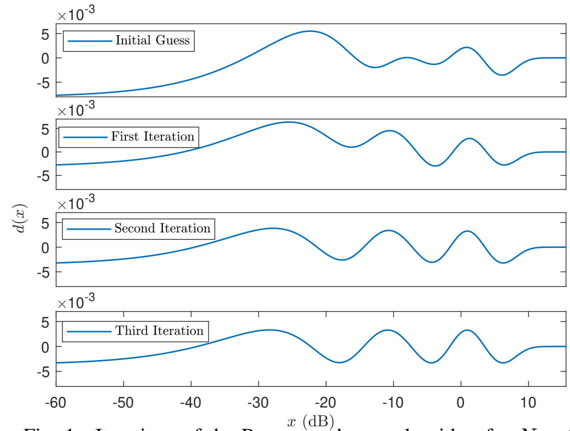


Fig. 1. Iterations of the Remez exchange algorithm for $N = 3$.

A. Algorithm Formulation

The steps for applying the Remez exchange algorithm to approximate the Q-function are summarized in Algorithm 1.

First, we construct a system of $2N + 1$ simultaneous equations that describe the $2N + 1$ extrema of the required uniform error function as

$$\mathbf{f}(\mathbf{r}) \triangleq \begin{bmatrix} f_0(\mathbf{r}) \\ f_1(\mathbf{r}) \\ \vdots \\ f_{2N}(\mathbf{r}) \end{bmatrix} \triangleq \begin{bmatrix} d(x_0) + d_{\max} \\ d(x_1) - d_{\max} \\ \vdots \\ d(x_k) + (-1)^k \, d_{\max} \\ \vdots \\ d(x_{2N}) + (-1)^{2N} \, d_{\max} \end{bmatrix} = \mathbf{0}, \tag{7}$$

where x_k is the abscissa value of the kth extremum of the error function and $\mathbf{r} = [a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, d_{\max}]^T$ is a vector of the unknowns. The first extremum occurs always at $x_0 = 0$, which results in $d(0) = \sum_{n=1}^N a_n - (\frac{1}{2})^p$ since $Q^p(0) = (\frac{1}{2})^p$ and $\tilde{Q}_p(0) = \sum_{n=1}^N a_n$. The adopted exponential approximation results in a nonlinear type of equations, opposing to the linear type which usually occur with the best polynomial approximations and often accompanied with the Remez algorithm whenever it is presented in the literature.

The Newton–Raphson method is a root finding technique that can be regarded as a somewhat ideal solver for this system of nonlinear equations since it is quadratically convergent when approaching the root. It is also an iterative method that requires initial guesses for the unknowns (roots) and we refer to its iterations as the inner iterations to differentiate them from the outer ones of the Remez algorithm. Furthermore, it is based on approximating a continuous and differentiable function by a straight line tangent to it, which results when applied on our system of equations (7) in

$$\mathbf{r}^{(v+1)} = \mathbf{r}^{(v)} - \left[\mathbf{J}^{(v)}\left(\mathbf{r}^{(v)}\right)\right]^{-1} \mathbf{f}\left(\mathbf{r}^{(v)}\right), \tag{8}$$

where v is the inner-iteration counter, and $\mathbf{J}(\cdot)$ is the Jacobian matrix that is calculated as

$$\mathbf{J}(\mathbf{r}) = \begin{bmatrix} \frac{\partial f_0(\mathbf{r})}{\partial r_0} & \frac{\partial f_0(\mathbf{r})}{\partial r_1} & \cdots & \frac{\partial f_0(\mathbf{r})}{\partial r_{2N}} \\ \frac{\partial f_1(\mathbf{r})}{\partial r_0} & \frac{\partial f_1(\mathbf{r})}{\partial r_1} & \cdots & \frac{\partial f_1(\mathbf{r})}{\partial r_{2N}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{2N}(\mathbf{r})}{\partial r_0} & \frac{\partial f_{2N}(\mathbf{r})}{\partial r_1} & \cdots & \frac{\partial f_{2N}(\mathbf{r})}{\partial r_{2N}} \end{bmatrix}, \tag{9}$$

with $[r_0, r_1, \ldots, r_{2N}] = [a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, d_{\max}]$, $\frac{\partial f_k(\mathbf{r})}{\partial a_n} = \exp(-b_n\, x_k^2)$, $\frac{\partial f_k(\mathbf{r})}{\partial b_n} = -a_n\, x_k^2 \exp(-b_n\, x_k^2)$, and $\frac{\partial f_k(\mathbf{r})}{\partial d_{\max}} = (-1)^k$. This procedure is repeated until the differences between the values of $\mathbf{r}$ of two successive iterations are smaller than a predefined threshold value. The Newton–Raphson method is implemented on (7) to find the vector of unknowns in every iteration of the Remez algorithm.

Assuming that we have a reasonably good initial guess for $\{(a_n, b_n)\}_{n=1}^{N}$ that formulates the proposed approximation and enables the construction of the corresponding absolute error function, we can locate extrema points thereof and the value of global error and use them for initializing $\{x_k\}_{k=1}^{2N}$ (but fixing $x_0 = 0$) and $d_{\max}$, respectively. We start the iterative procedure by solving the nonlinear system of equations using the aforementioned Newton–Raphson method, together with the initialized vector of unknowns $\mathbf{r}^{(0)}$. The obtained error function that has the same error value at each of the initial extrema points with alternating signs does not (yet) necessarily give the minimax solution since these points may not be at the extrema of the error function. Therefore, we need to find the new set of $\{x_k\}_{k=1}^{2N}$ by first locating the $2N$ roots of $d(x)$, which we denote by $\{z_i\}_{i=1}^{2N}$ using any root-finding numerical technique such as the bisection method or even the Newton–Raphson method yet again. Then we split the positive x-axis into $2N+1$ sub-intervals as $[0, z_1]$, $[z_1, z_2]$, …, $[z_{2N}, \infty)$.

For each sub-interval, we locate the point at which the error function attains its maximum magnitude by setting $d'(x) = 0$, for which the derivative is defined as

$$d'(x) = -2\sum_{n=1}^{N} a_n\, b_n\, x \exp\left(-b_n x^2\right) + p\,\frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}x^2\right) Q^{p-1}(x). \tag{10}$$

In particular, we numerically find x_k that meets $d'(x_k) = 0$ after substituting the kth sub-interval in (10). If the root does not exist, we take the endpoint that gives the larger absolute value of the two.

Finally, we replace the previous extrema points by the new ones and continue repeating the above steps for a number of iterations until the difference between the previous extrema points and the new ones are below a predefined threshold ϵ.

B. Initial Guesses

Before we can start the Remez method, we must obtain good initial guesses for $\{(a_n, b_n)\}_{n=1}^{N}$. In this subsection, we describe one possible, heuristic method that works for the cases illustrated in this paper. In particular, we focus on finding initial guesses for the first power of the Gaussian Q-function, which we can use as basis for finding initial guesses for higher values of p as will be explained later in this subsection.

Algorithm 1 Remez Exchange Algorithm

Initialize $\{x_k^0\}_{k=1}^{2N}$, ϵ
Set $t \leftarrow 0$, $x_0 \leftarrow 0$
repeat
  Solve (7) for unknowns $\{(a_n, b_n)\}_{n=1}^{N}$, $d_{\max}$ using Newton–Raphson method
  Find $\{z_i\}_{i=1}^{2N}$
  Divide $[0, \infty)$ into $2N+1$ sub-intervals by using $\{z_i\}_{i=1}^{2N}$ as boundaries
  for $k \leftarrow 1$ **to** $2N$ **do**
    Find the root of $d'(x)$ in the kth sub-interval.
    if such root does not exist **then**
      Evaluate $d(x)$ at endpoints and choose the point that gives the maximum
    end if
    Denote the obtained root or point by x_k^{t+1}
  end for
  Set $\{x_k^{t+1}\}_{k=1}^{2N}$ to $\{x_k^{t}\}_{k=1}^{2N}$
  $t \leftarrow t+1$
until $\left|\{x_k^{t}\}_{k=1}^{2N} - \{x_k^{t-1}\}_{k=1}^{2N}\right| < \epsilon$
Best minimax approximation is obtained

For $p = 1$ and lower values of N, we assigned repeatedly different random values for $\{(a_n, b_n)\}_{n=1}^{N}$ and calculated $d(x)$ per (6) for each N. Once we were lucky enough to come across any $d(x)$ that has the correct shape with $2N+1$ extrema (e.g., the initial guess in Fig. 1), $\{x_k\}_{k=1}^{2N}$ and $d_{\max}$ were calculated and used together with the corresponding $\{(a_n, b_n)\}_{n=1}^{N}$ to solve the considered optimization problem (4) using Algorithm 1. This yields in a unique set of the optimized coefficients $\{(a_n^*, b_n^*)\}_{n=1}^{N}$ which gives exactly the required uniform shape (e.g., the third iteration in Fig. 1).

After reaching certain N, we were able to use curve fitting techniques to formulate equations that can give good initial values for $\{b_n\}_{n=1}^{N}$ and $\{z_i\}_{i=1}^{2N}$ for $N = 1, 2, 3, \ldots, 10$. Each b_n-coefficient of the proposed approximations with any N has been assigned an equation of the form $b_n = A_n\, N^{B_n} + C_n$, and A_n, B_n and C_n are given in Table I. Moreover, one equation is formulated to calculate all the initial guesses of z_i, $i = 1, 2, 3, \ldots, 2N$, for any value of N as $z_i = (0.4845\, i^{-1.364} - 29.72)\, N^{(0.003752\, i^{-1.122} + 0.4884)} + (105.9\, i^{0.1924} - 94.83)$. Next, the initial guesses for $\{a_n\}_{n=1}^{N}$ are found by substituting the above calculated initial values in (6) to formulate a system of linear equations describing the absolute error function at its roots as $d(z_i) = \sum_{n=1}^{N} c_n\, a_n - q_i = 0$, where $c_n = \exp(-b_n\, z_i^2)$ and $q_i = Q(z_i)$ are constant. After solving the linear system of equations for the unknowns $\{a_n\}_{n=1}^{N}$, we can easily locate the initial guesses for $d_{\max}$ and $\{x_k\}_{K=0}^{2N}$ from $d(x)$ that is numerically calculated using the initial guesses of $\{(a_n, b_n)\}_{n=1}^{N}$.

On the other hand, for higher values of p, we rely on

TABLE I

THE PARAMETERS OF THE POWER EQUATION THAT IS USED TO FIND AN INITIAL GUESS $b_n = A_n\, N^{B_n} + C_n$ FOR $n = 1, 2, \ldots, N$ WITH $N \leq 10$.

n	A_n	B_n	C_n
1	6.514e−1	−1.075e+0	5.051e−1
2	2.389e+1	−1.658e+0	6.633e−1
3	6.908e+2	−2.481e+0	1.217e+0
4	6.699e+4	−3.983e+0	5.022e+0
5	3.002e+6	−4.959e+0	1.183e+1
6	2.793e+8	−6.244e+0	3.453e+1
7	1.063e+14	−1.129e+1	4.356e+2
8	7.474e+16	−1.315e+1	1.188e+3
9	3.721e+19	−1.478e+1	2.790e+3
10	1.048e+20	−1.384e+1	1.808e+4

the optimized coefficients $\{b_n^*\}_{n=1}^{N}$ and the corresponding $\{x_k\}_{k=0}^{2N}$ of the first power. We have found that they can be used to construct initial guesses for the higher powers through the relations $x_{k,p} = x_k - 2\,p$, $b_{n,p} = (2.25 + 1.65\,(p-2))\,b_n$, and $d_{max,p} = d_{max}$ where we use the subscripts p only herein in this equation to differentiate the coefficients of $p > 1$ from those of the first power. The initial guesses for $\{a_n\}_{n=1}^{N}$ can be easily found using the linear system of equations that solves $d(x_k) = \sum_{n=1}^{N} c_n\, a_n - q_k = (-1)^{k+1} d_{\max}$, where $c_n = \exp(-b_n\, x_k^2)$ and $q_k = Q(x_k)$ are constant. It is worth mentioning that using these relations will directly give all the required initial guesses for $p = 2, 3, 4$. However, for $p \geq 5$, one might need to use the resulted values from applying the above relations as a mean value around which small random variance is introduced; this iterative process is repeated until the correct number of extrema is obtained.

C. Proposed Approximations

The convergence of the algorithm is illustrated in Fig. 1, which shows an example of finding the uniform error function for $N = 3$ that results in seven extrema points. The approximation converges to its minimax behaviour after three iterations starting from a non-uniform error function with the correct number of extrema and ending with all the extrema points having the same value of error.

The new sets of the optimized coefficients of the considered approximation (3) are solved herein for $N = 1, 2, 3, \ldots, 10$ and $p = 1, 2, 3, 4$ in the minimax sense. In particular, we have calculated the required initial guesses using the heuristic method explained in the previous subsection and then applied the iterative Remez algorithm to obtain the uniform exponential approximation. In Fig. 2, we illustrate the achieved global absolute error, $d_{\max}$, in all the considered cases. We can clearly see that as the number of terms increases, the global error decreases resulting in very high accuracy.

IV. APPLICATION EXAMPLES

In general, the ASEP of most of the digital modulation techniques for coherent detection are linear combinations of integrals, whose integrand is the product of powers of the Gaussian Q-function and the fading probability density function (PDF) of the fading channel as follows:

$$I_p(\alpha) \triangleq \int_0^\infty Q^p(\alpha\,\sqrt{\gamma})\, f_\gamma(\gamma)\, \mathrm{d}\gamma, \tag{11}$$

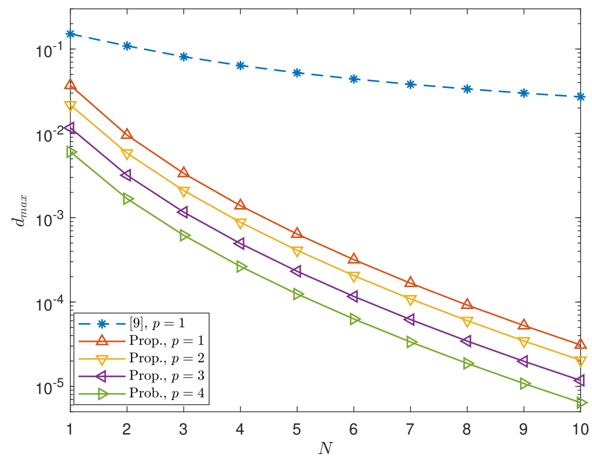


Fig. 2. The global absolute error when $\tilde{Q}_p(x)$ is the minimax approximation of $Q^p(x)$ for $p = 1, 2, 3, 4$, and when $\tilde{Q}_1(x)$ is the non-optimized rectangular rule in [9], both for $N = 1, 2, 3, \ldots, 10$.

where γ is the instantaneous signal-to-noise ratio (SNR), with $f_\gamma(\gamma)$ being its PDF, and α is a constant that depends on the digital modulation and detection techniques. For example, the conditional SEP in coherent detection of quadrature amplitude modulation (QAM) and differentially encoded quadrature phase-shift keying (DE-QPSK) are calculated by [1]

$$P_E = 2\,Q(\sqrt{\gamma}) - Q^2(\sqrt{\gamma}), \tag{12}$$

$$P_E = 4\,Q(\sqrt{\gamma}) - 8\,Q^2(\sqrt{\gamma}) + 8\,Q^3(\sqrt{\gamma}) - 4\,Q^4(\sqrt{\gamma}), \tag{13}$$

respectively, and the corresponding ASEPs in terms of (11) thus become $\bar{P}_E = 2\,I_1(1) - I_2(1)$ for 4-QAM and $\bar{P}_E = 4\,I_1(1) - 8\,I_2(1) + 8\,I_3(1) - 4\,I_4(1)$ for DE-QPSK.

Next we substitute the exponential approximation into (11) to obtain

$$\begin{aligned} I_p(\alpha) &\approx \sum_{n=1}^{N} a_n \int_0^\infty \exp(-b_n\,\alpha^2\,\gamma)\, f_\gamma(\gamma)\mathrm{d}\gamma \\ &= \sum_{n=1}^{N} a_n\, M_\gamma(-b_n \alpha^2), \end{aligned} \tag{14}$$

where $M_\gamma(s) = \int_0^\infty \exp(s\gamma)\, f_\gamma(\gamma)\,\mathrm{d}\gamma$ is the moment generating function (MGF) associated with the random variable γ. In what follows, we derive closed-form expressions for the general ASEP term defined in (11) over different fading channels, namely Nakagami-m, Fisher–Snedecor $\mathcal{F}$, $\eta - \mu$, and $\kappa - \mu$ fading channels.

A. Nakagami-m Fading

For Nakagami-m fading, we substitute the gamma MGF, i.e., $M_\gamma(s) = \left(1 - \frac{s\,\bar{\gamma}}{m}\right)^{-m}$, in (14) which yields directly

$$I_p(\alpha) \approx \sum_{n=1}^{N} a_n \left(1 + \frac{b_n\,\alpha^2 \bar{\gamma}}{m}\right)^{-m}, \tag{15}$$

where $m > 0$ is the fading parameter and $\bar{\gamma}$ is the average SNR. The ASEP of 4-QAM and DE-QPSK over Nakagami-m fading are calculated using (15) and the corresponding absolute error is illustrated in Fig. 3.

B. Fisher–Snedecor $\mathcal{F}$ Fading

Next we find analytical results for (11) with Fisher–Snedecor $\mathcal{F}$ distribution which is used to model the composite effects of both small and large scale fading (shadowing). The former is assumed to follow Nakagami-m distribution, and the latter follows inverse Nakagami-m distribution. We substitute the MGF derived in [18, Eq. 10] in (14), which yields

$$I_p(\alpha) \approx \sum_{n=1}^{N} a_n \, {}_1F_1\left(m; 1-m_s; \frac{b_n \alpha^2 \bar{\gamma} m_s}{m}\right) + \frac{\Gamma(-m_s)}{\beta(m, m_s)} \times \left(\frac{b_n \alpha^2 \bar{\gamma} m_s}{m}\right)^{m_s} {}_1F_1\left(m + m_s; 1 + m_s; \frac{b_n \alpha^2 \bar{\gamma} m_s}{m}\right),$$

where m is the fading severity parameter, $m_s \neq \mathbb{N}$ is the shadowing parameter, $\beta(\cdot,\cdot)$ and ${}_1F_1(\cdot;\cdot;\cdot)$ denote beta and Kummer confluent hypergeometric functions, respectively.

C. Generalized $\eta - \mu$ and $\kappa - \mu$ Fading

Finally, we evaluate the average of arbitrary powers of the Q-function in (11) over $\eta - \mu$ and $\kappa - \mu$ fading channels. The former fits well for non-line-of-sight applications and includes the Nakagami-q (Hoyt) and Nakagami-m fading as special cases while the latter fits better to line-of-sight applications and includes the Rice and Nakagami-m fading as special cases. We calculate their MGFs from their PDFs [19, Eqs. 1, 4] and we substitute them in (14). Thus, under $\eta - \mu$ fading we obtain

$$I_p(\alpha) \approx \frac{2\sqrt{\pi}\, \mu^{\mu+\frac{1}{2}}\, h^{\mu}}{\Gamma(\mu)\, H^{\mu-\frac{1}{2}}\, \bar{\gamma}^{\mu+\frac{1}{2}}} \sum_{u=0}^{\infty} \frac{\Gamma(2\mu + 2u)}{u!\, \Gamma(\mu - \frac{1}{2} + u + 1)} \times \left(\frac{\mu\, H}{\bar{\gamma}}\right)^{\mu - \frac{1}{2} + 2u} \sum_{n=1}^{N} a_n \left(b_n\, \alpha^2 + \frac{2\mu h}{\bar{\gamma}}\right)^{-(2\mu+2u)},$$

where η and μ are the fading parameters, $h = (2 + \eta^{-1} + \eta)/4$ and $H = (\eta^{-1} - \eta)/4$ for Format 1 of the distribution and $h = \frac{1}{(1-\eta^2)}$ and $H = \eta/(1-\eta^2)$ for Format 2. On the other hand, for the $\kappa - \mu$ fading model, we obtain

$$I_p(\alpha) \approx \frac{1}{\exp(\mu\kappa)} \sum_{u=0}^{\infty} \frac{\mu^{\mu+2u}\, \kappa^{u} (1+\kappa)^{\mu+u}}{\bar{\gamma}^{\mu+u}\, \Gamma(u+1)} \sum_{n=1}^{N} a_n \left(b_n\, \alpha^2 + \frac{\mu(1+\kappa)}{\bar{\gamma}}\right)^{-\mu-u}, \tag{16}$$

in which $\kappa > 0$ is the ratio between the total power of the dominant components and the total power of the scattered waves, and $\mu > 0$ is the number of multipath clusters.

V. Numerical Results

Throughout this section, we will be dealing with the absolute error function obtained by subtracting the numerically calculated exact expression of I_p defined in (11), from the approximated one in (14). The same applies for ASEP which is a linear combination of I_p. In Fig. 3, we compare the absolute error calculated from the proposed approximations and the existing ones with the same form, for different values of m. It is observed that our approximation have the least global error and result in a tighter approximation of the ASEP over the whole range of the average SNR for $m = 0.5$ as seen for 4-QAM plot. For higher values of m, some of the existing approximations have higher accuracy with exactly the same number of exponential terms as seen for DE-QPSK plot. However, when increasing number of terms, the accuracy of our approximation increases significantly and outperforms the others for almost the whole range of average SNR. It should be mentioned that increasing the number of exponential terms does not affect the analytical complexity. Moreover, when substituting the reference approximations with two terms in (12) and (13), we get 5-term and 14-term approximations for the ASEP in 4-QAM and DE-QPSK, respectively.

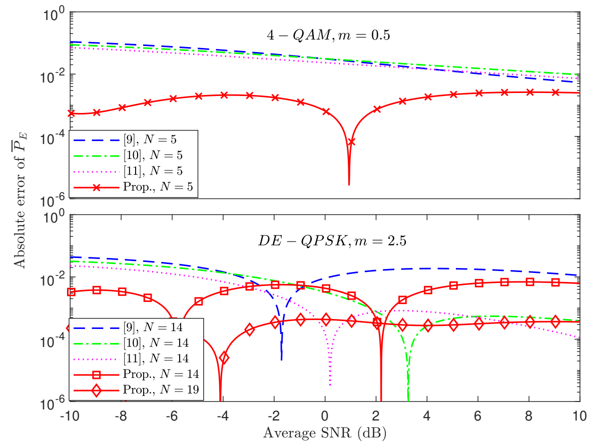
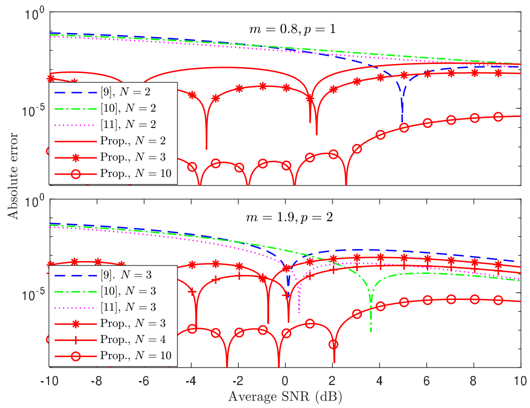


Fig. 3. The absolute error of ASEP for 4-QAM and DE-QPSK over Nakagami-m using the proposed approximation and the reference exponential approximations.

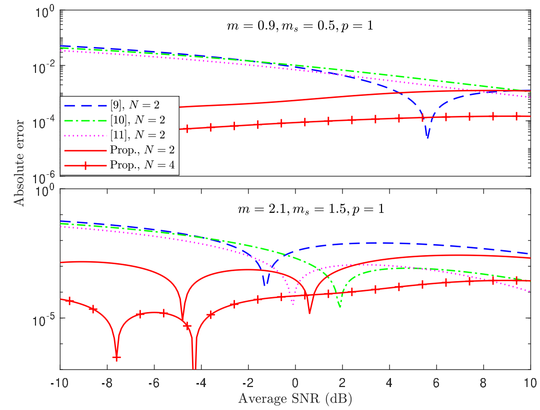
Figure 4 compares the difference between the exact I_p in (11) and its approximations in Nakagami-m, Fisher–Snedecor $\mathcal{F}$, $\eta - \mu$, and $\kappa - \mu$ fading channels presented in (15), before (16), in (16), and in (16), respectively, calculated using the existing and proposed approximations, for different values of the fading parameters and different integer powers. It is seen that the proposed approximations are tight even for lower SNR values, opposing to the existing ones. In particular, our approximation outperforms the others for a wide range of the argument using the same number of exponential terms and its accuracy can be increased even further by increasing the number of terms. The reference approximations are derived for a limited number of terms, namely $N = 2, 3$ or 4 only, whereas our approximations are derived till $N = 10$ to offer higher and adequate accuracy without affecting analytical complexity.

VI. Conclusion

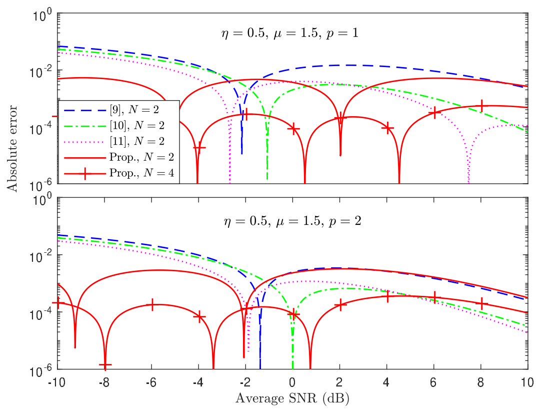
This paper proposed accurate and tractable approximations for the integer powers of the Q-function as a weighted sum of exponential functions. The novel sets of coefficients of the best exponential approximation are optimally solved using the Remez exchange algorithm to obtain uniform alternating absolute error function. We also considered the general problem of evaluating the ASEP over different fading channels, in which we implemented our approximations and showed that they render high accuracy in terms of global error and for the whole



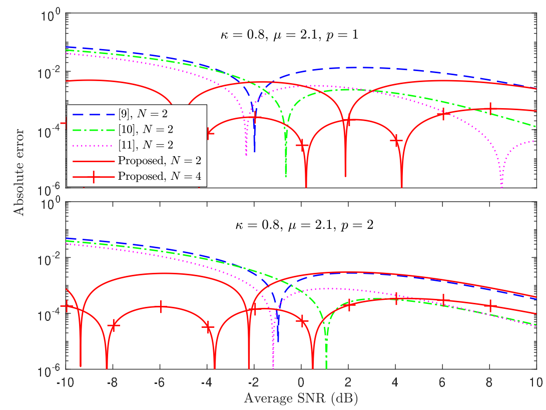
(a) Nakagami-m fading



(b) Fisher–Snedecor $\mathcal{F}$ fading



(c) generalized $\eta - \mu$ fading



(d) generalized $\kappa - \mu$ fading

Fig. 4. The absolute error of $I_p(\alpha)$ over several fading distributions, all with $\alpha = 1$ and for different fading parameters.

range of the argument. Even higher accuracy can be achieved by simply increasing the number of exponential terms.

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