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# Statistical properties of the model parameters in the continuum approach to high-cycle fatigue

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ABSTRACT

This paper is a sequel to papers studying the continuum approach to the high-cycle fatigue model of Ottosen *et al.* First, we study the estimation of fatigue limit and statistical characteristics of the estimates. We have two cases. Either the fatigue limit is a material constant or it is a random variable. Finally, we derive approximate distributions for the parameter estimators of the fatigue model due to Ottosen *et al.*

## 1. Introduction

Mechanical fatigue phenomena occur when a material is subjected to the repeated application of stresses or strains, which produces changes in the material microstructure as well as the initiation, growth and coalescence of microdefects. This degrades the material's properties.

It is customary to distinguish between high-cycle (HCF) and low-cycle fatigue (LCF). In low-cycle fatigue, plastic deformations occur on a macroscopic scale, while when the loading is in a high-cycle fatigue regime, the macroscopic behavior can be considered primarily as elastic. If the loading consists of well defined cycles, the transition between LCF and HCF regimes is typically considered to occur between $10^3 - 10^4$ cycles.

Ottosen et al. [1] proposed a continuum-based HCF model which is based on a moving endurance surface and a set of internal variables characterizing the movement and damage accumulation. The evolution of these internal variables is governed by evolution equations. Such an approach treats multiaxial stress states and arbitrary loading sequences in a unified manner, and heuristic cycle counting techniques are not needed. In Section 2, we give a short review of the method.

To determine the model parameters, we need some Wöhler curves of the material with different mean stresses. Since a Wöhler curve is always a least square fitting of measured values, there is a lot of uncertainty in the initial data and hence also in the model parameters. The statistical properties of Wöhler curves are discussed for example in [2–4]. Hence, a natural research problem is that if we know the statistical properties of Wöhler curves, for example their distributions, how can the corresponding things be computed in the model parameters? In this paper, we provide an answer to this question.

The structure of the paper is the following. In Sections 2 and 3, we give a short introduction to the fatigue model of Ottosen et al. and explain the estimation of parameters. In Section 4, we discuss the statistical properties of Wöhler curves and describe in detail what kind of initial distributions are needed and how they can be measured. In Section, 5 we study the statistical distributions of the model parameters.

## 2. Evolution equation-based continuum model

First, we briefly recall the basic ideas of evolution equation-based fatigue model, represented in [1]. The fundamental idea is to define a so-called endurance surface

$$\beta(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = 0$$

in stress space such that the damage develops when the stress $\boldsymbol{\sigma}$ is outside of the surface. In [1] and in this paper, we use the function of the form

$$\beta(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = \frac{1}{\sigma_{-1}}(\overline{\sigma} + AI_1 - \sigma_{-1}), \tag{1}$$

where $\sigma_{-1}$ and $A$ are positive material parameters, $I_1 = \operatorname{tr}(\boldsymbol{\sigma})$ is the first stress invariant of $\boldsymbol{\sigma}$ and

$$\overline{\sigma} = \sqrt{\tfrac{3}{2}\operatorname{tr}((\boldsymbol{s} - \boldsymbol{\alpha})^2)}, \tag{2}$$

where $\boldsymbol{s} = \boldsymbol{\sigma} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I}$ and $\boldsymbol{I}$ stands for the identity matrix.

The variable $\boldsymbol{\alpha}$ denotes the center of the endurance surface, and it is governed by the evolution equation

$$\dot{\boldsymbol{\alpha}} = \begin{cases} C(\boldsymbol{s} - \boldsymbol{\alpha})\dot{\beta}, & \text{when } \beta, \dot{\beta} \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The fundamental postulate of the continuum model is that the damage increases when the center $\boldsymbol{\alpha}$ moves. The damage development

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is modeled by the damage equation

$$\dot{D} = \begin{cases} g(\beta, D)\dot{\beta}, & \text{if} \quad \beta, \dot{\beta} \geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $g$ is a damage rule function. Usually $D$ is normalized such that the initial condition is $D(0) = 0$ and the failure happens at the time $t_f$ when $D(t_f) = 1$. In the sequel, we will use the damage of rule

$$g(\beta, D) = Ke^{L\beta},$$

as given by Ottosen et al. [1].

## 3. Estimating the material parameters

In this section, we give a short description how to find the material parameters $A$, $\sigma_{-1}$, $C$, $K$ and $L$ from the given material measurements. A detailed description can be found in [1], and we only give the necessary computational details below. Let us start from the parameters $A$ and $\sigma_{-1}$. They are governed by the linear part of the Haigh diagram due to the equation

$$\sigma_a + A\sigma_m - \sigma_{-1} = 0.$$

Hence, $\sigma_{-1}$ is the endurance limit for $\sigma_m = 0$ and $A$ the negative of the slope of the line.

With $C$, $K$ and $L$, we proceed as follows. Assume that we have experimental data $(\sigma_a^{(i)}, \sigma_m^{(i)}, N_{\text{exp}}^{(i)}), i = 1, \ldots, n$, where fatigue failure takes place at the $N_{\text{exp}}^{(i)}$th cycle. Assume further that the uniaxial stress loading varies periodically between $\sigma_2 = \sigma_m + \sigma_a$ and $\sigma_4 = \sigma_m - \sigma_a$. Then Ottosen et al. [1] show that the positions of endurance surface $\alpha_2$ and $\alpha_4$ are given by the equations

$$\begin{cases} \frac{3}{2}\alpha_2 - (A+1)\sigma_2 + \sigma_{-1} - \frac{\sigma_{-1}}{CA}(A+1)\ln\left(\frac{1-\frac{CA}{\sigma_{-1}}\left(\sigma_2 - \frac{3}{2}\alpha_2\right)}{1-\frac{CA}{\sigma_{-1}(A+1)}\left(\sigma_{-1} - \frac{3}{2}A\alpha_4\right)}\right) & = 0 \\ -\frac{3}{2}\alpha_4 - (A-1)\sigma_4 + \sigma_{-1} - \frac{\sigma_{-1}}{CA}(A-1)\ln\left(\frac{1-\frac{CA}{\sigma_{-1}}\left(\sigma_4 - \frac{3}{2}\alpha_4\right)}{1-\frac{CA}{\sigma_{-1}(A-1)}\left(\sigma_{-1} - \frac{3}{2}A\alpha_2\right)}\right) & = 0. \end{cases}$$

A straightforward matrix computation shows that for uniaxial stress loading $\sigma$ we have

$$\beta = \frac{1}{\sigma_{-1}}(|\sigma - \tfrac{3}{2}\alpha| + A\sigma - \sigma_{-1}).$$

Integrating the damage evolution equation, we see that in each cycle the damage increases by

$$\Delta D = \frac{K}{L}\left(\exp(L\beta_2) + \exp(L\beta_4) - 2\right).$$

When $N$ cycles lead to fatigue failure, i.e. $D = 1$, then

$$\frac{1}{N} = \frac{K}{L}\left\{ \exp\left(\frac{L}{\sigma_{-1}}\left[(A+1)\sigma_2 - \frac{3}{2}\alpha_2 - \sigma_{-1}\right]\right) + \exp\left(\frac{L}{\sigma_{-1}}\left[(A-1)\sigma_4 + \frac{3}{2}\alpha_4 - \sigma_{-1}\right]\right) - 2\right\}. \tag{5}$$

Finally, we calibrate the parameters $C, K$ and $L$ by minimizing the sum of squares

$$S(C, K, L) = \sum_{i=1}^{n}\left(\ln N_{\text{exp}}^{(i)} - \ln N^{(i)}\right)^2,$$

where $N^{(i)}$ is a predicted number of cycles given by (5) and $N_{\text{exp}}^{(i)}$ is an experimentally obtained number of cycles.

For physical reasons, the true model parameters are positive. Since the terms $1/N^{(i)}$ are small, some iterates may give a negative value for some terms and consequently a complex value for $\ln N^{(i)}$, which results in a failure of the algorithm.

For these reasons, we have to minimize $S(C, K, L)$ subject to constraints $C \geq 0$, $K \geq 0$, and $L \geq 0$. The state-of-the-art algorithm for this kind of problem is the reflective trust region method. For more information on this algorithm, see Coleman and Li [5].

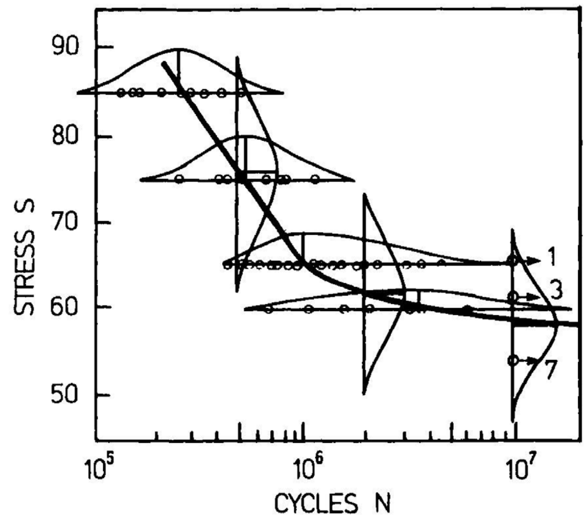


**Fig. 1.** Fatigue life distributions and strength distributions.

## 4. Statistical properties of Wöhler curves

Fatigue test is the main tool for analyzing fatigue lifetime of a material. Here a material specimen is subjected to cyclic loading until the specimen fails. Hence we obtain a set of experimental lifetimes. Since a Wöhler curve gives the stress as a function of the number of lifetime cycles, then the experimental lifetimes are related to an underlying Wöhler curve, see e.g. [6].

To take into account statistical properties, we associate a collection of distributions to a curve. The following Fig. 1 due to Nelson [2] elucidates the situation. In this section, we discuss different kinds of distributions on a Wöhler curve and review their estimation.

### 4.1. Finite lifetime part of a Wöhler curve

The slanted part of a Wöhler curve is called the finite lifetime part. A fundamental problem is to find a distribution of the lifetime cycles for a given amplitude $\sigma_a$ and mean stress $\sigma_m$. For the continuum model, one distribution is not enough, i.e. we need distributions for different amplitudes and mean stresses. Thus, we assume that the following measurements are obtained.

- *Measure a sample of lifetimes*

  $$\{N_j(\sigma_{a_i}, \sigma_{m_i})\}_{j=1}^{k_i}$$

  *for some different amplitude* $\sigma_{a_i}$ *and mean stress* $\sigma_{m_i}$, $i = 1, \ldots, \ell$.

In other words, we measure lifetime samples or datasets for $\ell$ different combinations of amplitude and mean stress. Two such collections are illustrated in Fig. 2.

Thus, the lifetime $N$ is a random variable depending on the mean stress $\sigma_{m_i}$ and the corresponding amplitude $\sigma_{a_i}$. The random variable is denoted by $L_i$. We model these lifetimes with log-normal distribution. Hence, we have log-normally distributed random variables

$$L_i = N \mid \{\sigma_a = \sigma_{a_i}, \sigma_m = \sigma_{m_i}\} \sim \text{LogN}(\mu_i, \nu_i^2),$$

for $i = 1, \ldots, \ell$.

Recall that $L$ is log-normally distributed, denoted by $L \sim \text{LogN}(\mu, \nu^2)$, if and only if $\ln(L) \sim N(\mu, \nu^2)$. In this case, see [7],

$$\text{E}(L) = \exp(\mu + \tfrac{1}{2}\nu^2) \quad \text{and} \quad \text{Var}(L) = \exp(2\mu)(\exp(2\nu^2) - \exp(\nu^2)).$$

The classical maximum likelihood estimators for parameters are

$$\hat{\mu}_i = \frac{1}{k_i}\sum_{j=1}^{k_i}\ln(N_j(\sigma_{a_i}, \sigma_{m_i}))$$

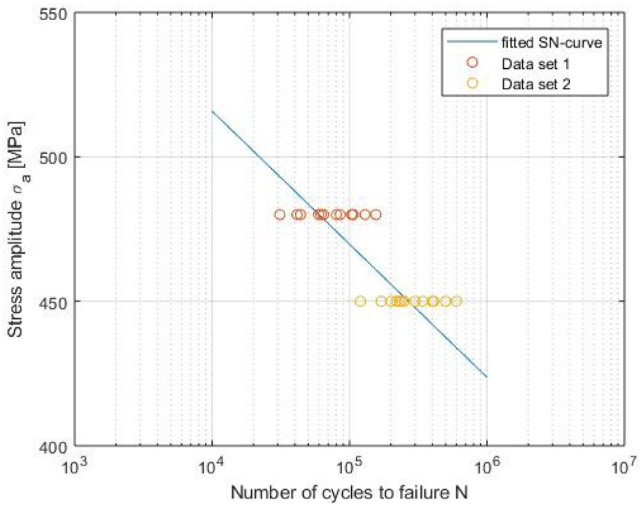


**Fig. 2.** Two collections of lifetimes related to different amplitudes and the same mean stress.

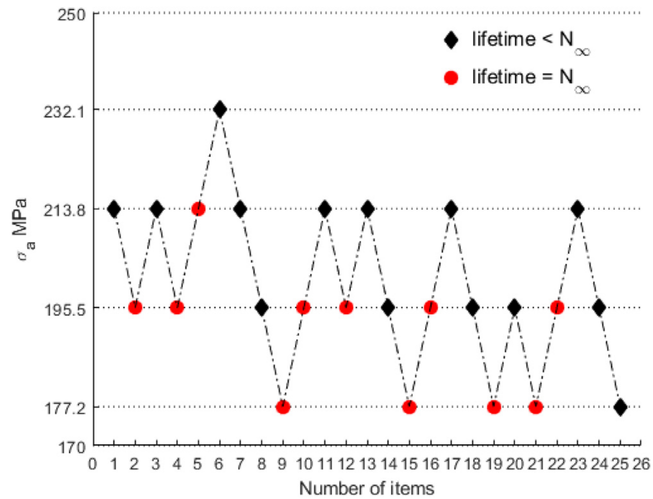


**Fig. 3.** Result of a staircase test.

and

$$\hat{v}_i^2 = \frac{1}{k_i}\sum_{j=1}^{k_i}(\ln(N_j(\sigma_{a_i},\sigma_{m_i})) - \hat{\mu}_i)^2.$$

Hence, $L_i$ is approximatively log-normally distributed, that is $L_i \approx \mathrm{LogN}(\hat{\mu}_i, \hat{v}_i^2)$, for each $i = 1, \dots, \ell$.

### 4.2. Estimating the distribution of fatigue limit

If the stress is uniaxial and periodic with an amplitude $\sigma_a$ and mean stress $\sigma_m$, then, as proven by Ottosen et al. [1], the linear part of the Haigh diagram holds true, i.e.

$$\sigma_a + A\sigma_m - \sigma_{-1} = 0.$$

The amplitude $\sigma_a$ is called a fatigue limit or the endurance limit for mean stress $\sigma_m$. Furthermore, it is well known that this equation is compatible with experimental data.

In this section, we estimate a fatigue limit $\sigma_{el}$ for a fixed mean stress $\sigma_m$ and derive its distribution. Suppose that we have a set of test pieces. For each item, we apply a uniaxial periodic stress with amplitude $\sigma_a$ and mean stress $\sigma_m$ and see whether it has a finite lifetime or not. First, we decide what we mean by a finite lifetime. For this, we fix a quantity $N_\infty$, called an infinite lifetime. If the failure occurs before $N_\infty$ cycles, we say that a specimen has a finite lifetime, otherwise it has an infinite lifetime. Rabb [4] suggests $N_\infty = 10^7$. Next, we choose a ground amplitude $\sigma_{ga}$ and a distance $d$. All admissible amplitudes are of the form $\sigma_a = \sigma_{ga} \pm jd, j = 0, 1, 2, \dots$.

Now the test procedure is the following staircase method as introduced by Dixon and Mood [8]. The first item is tested with amplitude $\sigma_{a_1} = \sigma_{ga}$. Suppose that $i$th item is tested with amplitude $\sigma_{a_i}$. If it has an infinite lifetime, then the next item is tested at a higher amplitude, i.e. $\sigma_{a_{i+1}} = \sigma_{a_i} + d$. Otherwise the amplitude for the next item is lower, i.e. $\sigma_{a_{i+1}} = \sigma_{a_i} - d$. After testing all the specimens, we obtain the following staircase (Fig. 3 given by Rabb [4]). There are 25 test pieces, $\sigma_{ga} = 213.8$ MPa and $d = 18.3$ MPa.

The probability that a test fails is

$$p_i = \mathbb{P}(N \le N_\infty \mid \{\sigma_a = \sigma_{a,i}, \sigma_m = 0\}).$$

Now we propose a model

$$p_i = \frac{1}{\sqrt{2\pi}s_{el}}\int_{-\infty}^{\sigma_{a_i}} \exp\left(-\frac{(t-\sigma_{el})^2}{2s_{el}^2}\right) dt.$$

Changing the integration variable, we get

$$p_i = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\sigma_{a_i}-\sigma_{el}}{s_{el}}} \exp(-\tfrac{1}{2}u^2)\, du = \Phi\left(\frac{\sigma_{a_i}-\sigma_{el}}{s_{el}}\right), \tag{6}$$

where $\Phi$ is the cdf of the standard normal distribution $N(0,1)$. We see that if $\sigma_{a_i}$ is big enough, then $p_i \approx 1$, and if it is small enough, then $p_i \approx 0$.

Let $\sigma_{a_i}, i = 1, \dots, R$, be the admissible amplitudes. Note that $R$ as well as $\sigma_{a_i}, i = 1, \dots, R$, are random entities. Suppose that $n_i$ tests are done at amplitude $\sigma_{a_i}$ and $m_i$ tests failed. If $NF_i$ denotes the number of failed tests, then by the independence of tests and Eq. (6) we see, that $NF_i$ follows a binomial distribution. Hence,

$$\mathbb{P}(NF_i = m_i) = \binom{n_i}{m_i} p_i^{m_i}(1-p_i)^{n_i-m_i}.$$

Let us denote the parameter vector and the data vectors by

$$\theta = \begin{bmatrix}\sigma_{el}\\ s_{el}\end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} n_1\\ \vdots\\ n_R\\ m_1\\ \vdots\\ m_R\end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} R\\ \sigma_{a_1}\\ \vdots\\ \sigma_{a_R}\end{bmatrix}.$$

Since tests are independent, we have the likelihood function

$$L(\theta \mid \mathbf{x}) = \prod_{i=1}^{R}\binom{n_i}{m_i}p_i^{m_i}(1-p_i)^{n_i-m_i} = K\prod_{i=1}^{R}p_i^{m_i}(1-p_i)^{n_i-m_i},$$

where $K$ does not contain model parameters. After discarding the constant term $\ln(K)$, the log-likelihood function looks like

$$l(\theta \mid \mathbf{x}) = \ln(L(\theta \mid \mathbf{x})) = \sum_{i=1}^{R} m_i \ln(p_i) + (n_i - m_i)\ln(1-p_i). \tag{7}$$

The maximum likelihood estimate $\hat{\theta}$ of $\theta$ is obtained as we maximize the log-likelihood function $l(\theta \mid \mathbf{x})$ with respect to $\theta$. The problem has no analytic solution, so we have solve it numerically. Fortunately, any decent book on optimization gives plenty of algorithms for solving the problem. The most popular are the Levenberg–Marquardt and trust region methods.

**Theorem 4.1.** *Let $\hat{\theta}$ be a maximum likelihood estimator of the parameter vector $\theta$. Then under fairly general conditions*

$$\hat{\theta} \approx N(\theta, I_\theta^{-1}),$$

*where* $I_{\theta} = -\mathrm{E}_{\theta}(l''(\theta \mid \mathbf{x}))$ *is the Fischer information matrix. Here* $l''$ *denotes the Hessian of* $l$ *with respect to* $\theta$ *and the expectation is applied to* $\mathbf{x}$ *with model parameters given by* $\theta$*.*

**Proof.** For the exact formulation of the theorem, its proof and discussion about the loose version given here, see Ferguson [9]. □

Computing the partial derivatives of log-likelihood function (7) and taking the expectations, we get

$$I_{\theta} = \begin{bmatrix} \mathrm{E}_{\mathbf{y}}\left(\sum_{i=1}^{R} \frac{\mathrm{E}_{\mathbf{x}|\mathbf{y}}(n_i)}{p_i(1-p_i)}\left(\frac{\partial p_i}{\partial \sigma_{el}}\right)^2\right) & \mathrm{E}_{\mathbf{y}}\left(\sum_{i=1}^{R} \frac{\mathrm{E}_{\mathbf{x}|\mathbf{y}}(n_i)}{p_i(1-p_i)}\frac{\partial p_i}{\partial \sigma_{el}}\frac{\partial p_i}{\partial s_{el}}\right) \\ \mathrm{E}_{\mathbf{y}}\left(\sum_{i=1}^{R} \frac{\mathrm{E}_{\mathbf{x}|\mathbf{y}}(n_i)}{p_i(1-p_i)}\frac{\partial p_i}{\partial \sigma_{el}}\frac{\partial p_i}{\partial s_{el}}\right) & \mathrm{E}_{\mathbf{y}}\left(\sum_{i=1}^{R} \frac{\mathrm{E}_{\mathbf{x}|\mathbf{y}}(n_i)}{p_i(1-p_i)}\left(\frac{\partial p_i}{\partial s_{el}}\right)^2\right) \end{bmatrix},$$

where by Eq. (6)

$$\frac{\partial p_i}{\partial \sigma_{el}} = -\frac{1}{\sigma_{el}}\phi\left(\frac{\sigma_{a_i}-\sigma_{el}}{s_{el}}\right) \quad \text{and} \quad \frac{\partial p_i}{\partial s_{el}} = -\frac{\sigma_{a_i}-\sigma_{el}}{s_{el}^2}\phi\left(\frac{\sigma_{a_i}-\sigma_{el}}{s_{el}}\right)$$

and $\phi$ is the pdf of $N(0,1)$. Here $\mathrm{E}_{\mathbf{x}|\mathbf{y}}$ denotes the conditional expectation with respect to $\mathbf{x}$ as $\mathbf{y}$ is fixed and $\mathrm{E}_{\mathbf{y}}$ denotes the expectation with respect to $\mathbf{y}$.

For detailed computations, see Silvey [10]. Substituting $\hat{\theta}$ for $\theta$, and replacing the expectations by terms computed with observed values, we obtain an approximation $\hat{I}_{\theta}$ of the Fisher information matrix.

Now let $v^2$ be the (1,1) element of the matrix $\hat{I}_{\theta}^{-1}$. Then by Theorem 4.1 we get an approximate distribution of a maximum likelihood estimator of endurance limit

$$\hat{\sigma}_{el} \approx N(\sigma_{el}, v^2)$$

and consequently approximate confidence intervals for $\sigma_{el}$.

**Remark 4.2.** Suppose we have $N$ test items. Let $\mathbf{a}_i$ be a random variable denoting the applied amplitude and $p(\sigma_a) = \Phi(\frac{\sigma_a-\sigma_{el}}{s_{el}})$. By the test procedure we have

$$\mathbb{P}(\mathbf{a}_{i+1} = u \mid \mathbf{a}_i = v) = \begin{cases} p(v), & \text{if } u = v-d \\ 1-p(v), & \text{if } u = v+d \\ 0, & \text{otherwise.} \end{cases}$$

These transition probabilities with initial condition $\mathbf{a}_1 = \sigma_{ga}$ define a Markov chain. Now the applied amplitudes are a realization of the Markov chain. Moreover

$$\begin{aligned} R &= \mathrm{range}\{\mathbf{a}_1, \dots, \mathbf{a}_N\} \\ &= \text{the number of distinct values in the set } \{\mathbf{a}_1, \dots, \mathbf{a}_N\}. \end{aligned}$$

With this formulation, the stochastic process controlling the selection of amplitudes is a random walk in random environments, RWRE for short. RWRE is quite a new area of research in the theory of stochastic processes. For more information, see Peterson [11] and Zeitouni [12].

### *4.3. Random fatigue limit model*

In the staircase method, it is assumed that the fatigue limit is a material constant. As pointed out by Nelson [2], a more realistic approach is to assume that every specimen has its own fatigue limit. In this spirit, Pascual and Meeker [13] introduced a random fatigue limit model (RFLM for short)

$$\ln(N) = \beta_0 + \beta_1 \ln(\sigma_a - \gamma) + \epsilon, \; \gamma < \sigma_a, \tag{8}$$

where $N$ is lifetime, $\sigma_a$ is stress amplitude and $\gamma$ denotes the fatigue limit. Note that there are two random variables: $\gamma$, the random fatigue limit, and $\epsilon$, which describes the variability of the lifetime of specimens with the same fatigue limit. The limitation of RFLM is that it fits on a Wöhler curve only locally. To solve these problems, there are more complicated models (see e.g. [6]). But as a method to study distributions for amplitudes $\gamma < \sigma_a$, the model is useful.

Denote $W = \ln(N)$, $V = \ln(\gamma)$ and $x = \ln(\sigma_a)$. Then RFLM includes the following distributional assumptions.

$$f_V(v \mid \theta) = \frac{1}{\sigma_\gamma}\phi_V\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right),$$

where $\mu_\gamma$ and $\sigma_\gamma$ are location and scale parameters, and

$$f_{W|V}(w \mid v, x, \theta) = \frac{1}{\nu}\phi_{W|V}\left(\frac{w-\beta_0-\beta_1\ln(\exp(x)-\exp(v))}{\nu}\right), \; v < x.$$

Here, the location parameter is $\mu(v,x,\theta) = \beta_0 + \beta_1\ln(\exp(x)-\exp(v))$, the scale parameter is $\nu$ and

$$\theta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \nu \\ \mu_\gamma \\ \sigma_\gamma \end{bmatrix}.$$

In addition, $\phi_V$, and respectively $\phi_{W|V}$, is either the standardized smallest extreme value pdf $\phi_{sev}$ or standardized normal pdf $\phi_n$. Recall that

$$\phi_{sev}(u) = \exp(u - \exp(u)).$$

**Theorem 4.3.** *If the pdf of a random variable* $X$ *is given by*

$$f_X(x) = \frac{1}{\sigma}\phi_{sev}\left(\frac{x-\mu}{\sigma}\right),$$

*then the random variable* $Y = \exp(X)$ *is Weibull distributed with parameters* $\eta = \exp(\mu)$ *and* $\beta = 1/\sigma$*.*

**Proof.** Since $y = g(x) = \exp(x)$ is a bijection from $\mathbb{R}$ onto $\mathbb{R}^+$, then

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\left|(g^{-1})'(y)\right| = \frac{1}{\sigma y}\phi_{sev}\left(\frac{\ln(y)-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma y}\exp\left(\left(\frac{\ln(y)-\mu}{\sigma}\right) - \exp\left(\frac{\ln(y)-\mu}{\sigma}\right)\right), y > 0. \end{aligned}$$

After simplification, we obtain

$$f_Y(y) = \frac{1}{\sigma\exp(\mu)}\left(\frac{y}{\exp(\mu)}\right)^{\frac{1}{\sigma}-1}\exp\left[-\left(\frac{y}{\exp(\mu)}\right)^{\frac{1}{\sigma}}\right], \; y > 0,$$

which completes the proof. □

We see that Weibull distribution can be written as

$$f_Y(y) = \frac{1}{\sigma y}\phi_{sev}\left(\frac{\ln(y)-\mu}{\sigma}\right), \; y > 0.$$

From this form, it directly follows that the pdf of $X = \ln(Y)$ is given by

$$f_X(x) = \frac{1}{\sigma}\phi_{sev}\left(\frac{x-\mu}{\sigma}\right), \; x \in \mathbb{R}.$$

For more information on this topic see Meeker and Escobar [14].

Now we have

$$\begin{aligned} f_W(w \mid x, \theta) &= \int f_{W|V}(w \mid v, x, \theta) f_V(v \mid \theta)\, dv \\ &= \int_{-\infty}^{x} \frac{1}{\nu\sigma_\gamma}\phi_{W|V}\left(\frac{w-\mu(v,x,\theta)}{\nu}\right)\phi_V\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right) dv. \end{aligned}$$

Integrating with respect to $w$ and changing the order of integration gives

$$F_W(w \mid x, \theta) = \int_{-\infty}^{x} \frac{1}{\sigma_\gamma}\Phi_{W|V}\left(\frac{w-\mu(v,x,\theta)}{\nu}\right)\phi_V\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right) dv,$$

where $\Phi_{W|V}$ is the cdf of $W \mid V$. Using this distribution, Pascual and Meeker [13] constructed design curves under different distributional combinations. Design curves are constructed as quantiles of fatigue life at constant log-stresses $x$.

Parameters are estimated as usual by maximizing the likelihood function

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f_W(w_i \mid x_i, \boldsymbol{\theta})^{\delta_i}(1 - F_W(w_i \mid x_i, \boldsymbol{\theta}))^{1-\delta_i},$$

where

$$\delta_i = \begin{cases} 1, & \text{if } N_i < N_\infty, \\ 0, & \text{if } N_i \geq N_\infty, \end{cases}$$

or equivalently the log-likelihood function $l(\boldsymbol{\theta}) = \ln(L(\boldsymbol{\theta}))$.

Recently, Pollak and Palazotto [15] discussed a different kind of procedure for obtaining the design curves for fatigue limit. First, they estimated the shape of the S-N curve and then fitted a Weibull distribution to the residual stresses.

We propose a modification of RFLM called a modified RFML (MRFLM for short). Now confidence curves are estimated directly from conditional distribution $f_{V|W}$. Since $f_{V|W} f_W = f_{V,W} = f_{W|V} f_V$, we have

$$\begin{aligned} f_{V|W}(v \mid w, x, \boldsymbol{\theta}) &= \frac{f_{W|V}(w \mid v, x, \boldsymbol{\theta}) f_V(v \mid \boldsymbol{\theta})}{f_W(w \mid x, \boldsymbol{\theta})} \\ &= \frac{\phi_{W|V}\left(\frac{w-\mu(v,x,\boldsymbol{\theta})}{\nu}\right)\phi_V\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right)}{\int_{-\infty}^{x} \phi_{W|V}\left(\frac{w-\mu(v,x,\boldsymbol{\theta})}{\nu}\right)\phi_V\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right) dv}, \quad v < x. \end{aligned}$$

Now suppose that we select a set of logarithmic stress amplitudes $x_i = \ln(\sigma_{a_i}), i = 1, \ldots, r$, each one with equal probability. Then by total probability theorem

$$f_{V|W}(v \mid w, \boldsymbol{\theta}) = \frac{1}{r}\sum_{i=1}^{r} f_{V|W}(v \mid w, x_i, \boldsymbol{\theta}).$$

It follows that

$$F_{V|W}(z \mid w, \boldsymbol{\theta}) = \mathbb{P}(V \leq z \mid w, \boldsymbol{\theta}) = \frac{1}{r}\sum_{i=1}^{r}\int_{-\infty}^{z} f_{V|W}(v \mid w, x_i, \boldsymbol{\theta})\, dv$$

and

$$\int_{-\infty}^{z} f_{V|W}(v \mid w, x_i, \boldsymbol{\theta})\, dv = \begin{cases} \dfrac{\int_{-\infty}^{z} \phi_{W|V}\left(\frac{w-\mu(x_i,v,\boldsymbol{\theta})}{\nu}\right)\phi_V\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right) dv}{\int_{-\infty}^{x_i} \phi_{W|V}\left(\frac{w-\mu(x_i,v,\boldsymbol{\theta})}{\nu}\right)\phi_V\left(\frac{v-\mu_\gamma}{\sigma_\gamma}\right) dv}, & \text{if } z \leq x_i, \\ 1, & \text{if } z > x_i. \end{cases}$$

If $w$ is fixed, then also $N$ is fixed and we have the cdf

$$\begin{aligned} F_{\gamma|N}(\sigma_a \mid N, \boldsymbol{\theta}) &= \mathbb{P}(\gamma \leq \sigma_a \mid N, \boldsymbol{\theta}) = \mathbb{P}(V \leq \ln(\sigma_a) \mid w, \boldsymbol{\theta}) \\ &= F_{V|W}(\ln(\sigma_a) \mid w, \boldsymbol{\theta}), \end{aligned}$$

from which we are able to compute confidence intervals for $\gamma \mid N$.

**Example 4.4.** As an example, Pollak and Palazetto [15] used a data set of 68 items of dual-phase Ti-6Al-4V titanium alloy. They estimated parameters of two models of their own, say PP1 and PP2, and for comparison parameters of RFLM. For $V$ and $W \mid V$, they selected a sev-normal model. Pascual and Meeker [13] compared different models. They showed among other things that with the Akaike information criterion, the sev–sev model was the worst. For the other three models, the results were coincident. We will use the estimates obtained by Pollak and Palazetto. However, they estimated Weibull parameters instead of sev-parameters. When we make the pertinent changes allowed by Theorem 4.3, we get

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} 4.950 \\ -2.110 \\ 0.16 \\ 6.004 \\ 0.056 \end{bmatrix}.$$

**Table 1**
Comparison of fatigue strengths at $N_\infty = 10^9$.

| Fatigue strength | RFLM | PP1 | PP2 | MRFLM |
|---|---|---|---|---|
| Median | 398 | 413 | 410 | 382 |
| 90% lower bound | 359 | 388 | 380 | 348 |
| 95% lower bound | 344 | 378 | 369 | 343 |
| 99% lower bound | 315 | 356 | 343 | 340 |

As an example, we selected 200 logarithmic amplitudes uniformly on interval $[\ln(340), \ln(430)]$. The confidence bounds of different models are given in Table 1. We see that MRFLM gives the shortest confidence region.

## 5. Distribution of model parameters

In this section, we study the statistical properties of model parameters $A$, $\sigma_{-1}$, $C$, $K$ and $L$. Recall that since the test specimens are different, the observed lifetimes $N_{\text{exp}}^{(i)}$ are values of independent random variables. Consequently, the model parameters are random entities.

### *5.1. Distributions for $A$ and $\sigma_{-1}$*

To compute parameters $A$ and $\sigma_{-1}$ from the equation $\sigma_a + A\sigma_m - \sigma_{-1} = 0$, we need two points $(0, \sigma_{a,1})$ and $(\sigma_{m,2}, \sigma_{a,2})$ from the Haigh diagram. We assume that

$$\sigma_{a,1} \sim N(\mu_1, \nu_1^2) \quad \text{and} \quad \sigma_{a,2} \sim N(\mu_2, \nu_2^2)$$

are independent. Hence,

$$\begin{bmatrix} \sigma_{a,1} \\ \sigma_{a,2} \end{bmatrix} \sim N(\mu, \boldsymbol{\Sigma}),$$

where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \nu_1^2 & 0 \\ 0 & \nu_2^2 \end{bmatrix}.$$

Since $A$ is the negative of the slope of the linear part in the Haigh diagram, we may write the parameter vector $U$ as

$$U = \begin{bmatrix} A \\ \sigma_{-1} \end{bmatrix} = B\begin{bmatrix} \sigma_{a,1} \\ \sigma_{a,2} \end{bmatrix},$$

where

$$B = \begin{bmatrix} \frac{1}{\sigma_{m,2}} & -\frac{1}{\sigma_{m,2}} \\ 1 & 0 \end{bmatrix}.$$

Using properties of multidimensional normal distributions, see Mardia, Kent and Bibby [16], we obtain

$$\begin{bmatrix} A \\ \sigma_{-1} \end{bmatrix} \sim N(B\boldsymbol{\mu}, B\boldsymbol{\Sigma}B^T).$$

Since

$$B\boldsymbol{\mu} = \begin{bmatrix} \frac{\mu_1-\mu_2}{\sigma_{m,2}} \\ \mu_1 \end{bmatrix} \quad \text{and} \quad B\boldsymbol{\Sigma}B^T = \begin{bmatrix} \frac{\nu_1^2+\nu_2^2}{\sigma_{m,2}} & \frac{\nu_1^2}{\sigma_{m,2}} \\ \frac{\nu_1^2}{\sigma_{m,2}} & \nu_1^2 \end{bmatrix}$$

we see that $A$ and $\sigma_{-1}$ are correlated and hence dependent. Writing $A = \begin{bmatrix} 1 & 0 \end{bmatrix} U$ as above, we get

$$A \sim N\left(\frac{\mu_1-\mu_2}{\sigma_{m,2}}, \frac{\nu_1^2+\nu_2^2}{\sigma_{m,2}}\right) \text{ and } \sigma_{-1} \sim N(\mu_1, \nu_1^2), \text{ respectively.}$$

**Example 5.1.** Experimental values of the lifetime for S45C carbon steel was given in [17]. We selected those amplitudes $\sigma_a$ for which $N = 10^7$ and either $\sigma_m = 0$ or $\sigma_m = 209.45$. A Haigh diagram was constructed as a linear fit to these points. The following figure illustrates the situation (see Fig. 4). This gives us $A = 0.2475$ and $\sigma_{-1} = 222.33$. Furthermore, we

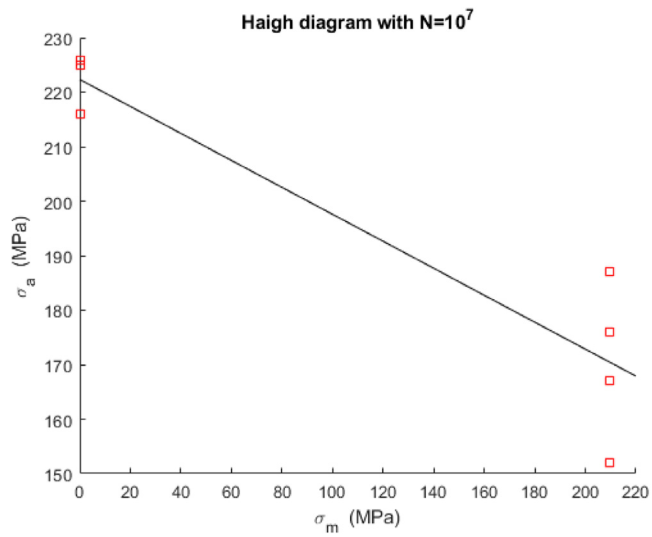


**Fig. 4.** Haigh diagram for S45C steel.

easily compute $v_1^2 = 30.33$ and $v_2^2 = 219$, which yields an approximate covariance matrix for $\hat{U}$

$$\begin{bmatrix} 1.1904 & 0.1448 \\ 0.1448 & 30.3333 \end{bmatrix}.$$

**Remark 5.2.** Parameter estimates for $\mu_i, v_i^2, i = 1,2$ can be obtained as follows. Run the procedure given in Section 4.2 say $p$ times with mean stress $\sigma_m = 0$. Then the mean and sample variance provide estimates $\hat{\mu}_1$ and $\hat{v}_1^2$. Similarly $\hat{\mu}_2$ and $\hat{v}_2^2$ are obtained when the test procedures are executed with mean stress $\sigma_{m,2}$.

### *5.2. Distributions for C, K and L*

This case is theoretically and practically more complicated, because the parameters come from the estimation procedure explained in Section 2.

Assume that $\mathbf{x}_i$ are the amplitude-mean stress pairs corresponding to the lifetime measurements $N_{\text{exp}}^{(i)}$, $i = 1, \ldots, n$. We denote

$$\mathbf{x}_i = \begin{bmatrix} \sigma_{a,i} \\ \sigma_{m,i} \end{bmatrix} \in \mathbb{R}^2 \quad \text{and} \quad \boldsymbol{\theta} = \begin{bmatrix} C \\ K \\ L \end{bmatrix} \in \mathbb{R}^3.$$

**Remark 5.3.** The experimental cycle numbers are obtained as follows. A set of properly prepared test specimens are selected. Each specimen is subjected to a sinusoidal stress until it breaks. Thus we obtain an experimental lifetime. The procedure, as described for instance in [18] and [19], is expensive and demands a lot of expertise. Since specimens are different, then the random variables $N_{\text{exp}}^{(i)}$ are independent.

For theoretical lifetimes $N^{(i)}$ we proceed as follows. Since a Wöhler curve is an ordinary function, see [20,21], we assume that our model, with true parameter values, gives us a value of that function.

Consequently, with fixed parameter values, we may take as an input a stress $\sigma(t) = \sigma_m + \sigma_a \sin(t)$, solve differential Eqs. (3) and (4) until the damage variable $D$ equals to 1. Thus we obtain a lifetime $t_f$ and hence a point on the Wöhler curve. However, with sinusoidal stress Ottosen et al. [1] gave an explicit formula (5) for a cycle number $N$.

Then, the number of cycles given by the HCF model is given by

$$N^{(i)}(\boldsymbol{\theta}) = N(\mathbf{x}_i; \boldsymbol{\theta}), \ \ i = 1, \ldots, n.$$

**Table 2**
Parameters of continuum fatigue model for 20Mn structural steel.

| Material | A | $\sigma_{-1}$ | C | K | L |
|---|---|---|---|---|---|
| 20 Mn steel | 0.3214 | 740 | $4.3078 \cdot 10^{-4}$ | $9.0232 \cdot 10^{-6}$ | 5.1690 |

Let

$$\mathbf{N} = \begin{bmatrix} \ln N_{\text{exp}}^{(1)} \\ \vdots \\ \ln N_{\text{exp}}^{(n)} \end{bmatrix} \quad \text{and} \quad M(\boldsymbol{\theta}) = \begin{bmatrix} \ln N(\mathbf{x}_1; \boldsymbol{\theta}) \\ \vdots \\ \ln N(\mathbf{x}_n; \boldsymbol{\theta}) \end{bmatrix}.$$

For statistical considerations, we propose the model

$$\mathbf{N} = M(\boldsymbol{\theta}^\star) + \epsilon,$$

where $\boldsymbol{\theta}^\star$ is the true value of parameter vector $\boldsymbol{\theta}$ and $\epsilon \sim N(\mathbf{0}, \sigma^2 I)$. An estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}^\star$ is obtained by minimizing the residual sum of squares

$$S(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left( \ln N_{\text{exp}}^{(i)} - \ln N^{(i)} \right)^2 = \|N - M(\boldsymbol{\theta})\|^2.$$

Since $\boldsymbol{\theta}^\star > 0$ we solve a restricted least squares problem

$$\min S(\boldsymbol{\theta}) \text{ subject to } \boldsymbol{\theta} \geq 0.$$

Hence, we have to solve a classical nonlinear regression problem. Nonlinear regression problems are discussed extensively from different points of view by Seber and Wild [22].

**Remark 5.4.** We have assumed that each component of the error term $\epsilon$ has the same variance. This assumption is also used by Paolino et al. [3]. However, Fig. 5 suggests that the error variance may depend on the amplitude. Still, the dataset is too small for a definitive answer.

Nelson [2] studied different possibilities to model the standard deviation. In conclusion, he writes that *there is a need for more sophisticated models that do not have distributions that cross.*

Under the normality assumption, a least squares estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}^\star$ is also a maximum-likelihood estimator, see Seber and Wild [22]. Hence, by Theorem 4.1, $\hat{\boldsymbol{\theta}}$ is approximately normally distributed.

Furthermore, under fairly general regularity conditions, we have

(a) for large $n$ $\hat{\boldsymbol{\theta}} \approx \text{N}(\boldsymbol{\theta}^\star, \sigma^2 B^{-1})$, where $J(\boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}} M(\boldsymbol{\theta}) \in \mathbb{R}^{n \times 3}$ is the Jacobian of $M(\boldsymbol{\theta})$ and $B = J(\boldsymbol{\theta}^\star)^T J(\boldsymbol{\theta}^\star)$,
(b) $\hat{\sigma}^2 = \frac{1}{n-3} S(\hat{\boldsymbol{\theta}})$,
(c) $\hat{\boldsymbol{\theta}}$ and $\hat{\sigma}^2$ are independent.

Finally, we may approximate the covariance matrix of $\hat{\boldsymbol{\theta}}$ by $\hat{\sigma}^2 \hat{B}^{-1}$, where $\hat{B} = J(\hat{\boldsymbol{\theta}})^T J(\hat{\boldsymbol{\theta}})$ and the $j$th column of $J(\hat{\boldsymbol{\theta}})$ is approximated by a forward-difference approximation

$$J(\hat{\boldsymbol{\theta}})_j = \frac{M(\hat{\boldsymbol{\theta}} + \xi_j e_j) - M(\hat{\boldsymbol{\theta}})}{\xi_j}, \ \ j = 1,2,3. \tag{9}$$

**Example 5.5.** We apply the model to 20Mn structural steel. For the estimation of $C$, $K$ and $L$, we used data on the HCF domain given in [19, Figure 16]. Data for $A$ and $\sigma_{-1}$ were obtained from [18, Figures 5 and 6].

Here we have $n = 11$ and the residual sum of squares $SSQ = 1.6486$, which gives a error variance estimate $\hat{\sigma}^2 = 0.2061$. The computed parameter estimates are given in Table 2.

Fig. 5 gives the estimated Wöhler curve.

Applying approximation (9), we get the following approximate covariance matrix of $\boldsymbol{\theta}$

$$\hat{\sigma}^2 \hat{B}(\hat{\boldsymbol{\theta}})^{-1} = \begin{bmatrix} 3.3573 \cdot 10^{-9} & 6.0662 \cdot 10^{-12} & -3.7094 \cdot 10^{-5} \\ 6.0662 \cdot 10^{-12} & 3.0672 \cdot 10^{-14} & -8.0992 \cdot 10^{-8} \\ -3.7094 \cdot 10^{-5} & -8.0992 \cdot 10^{-8} & 0.4200 \end{bmatrix}.$$

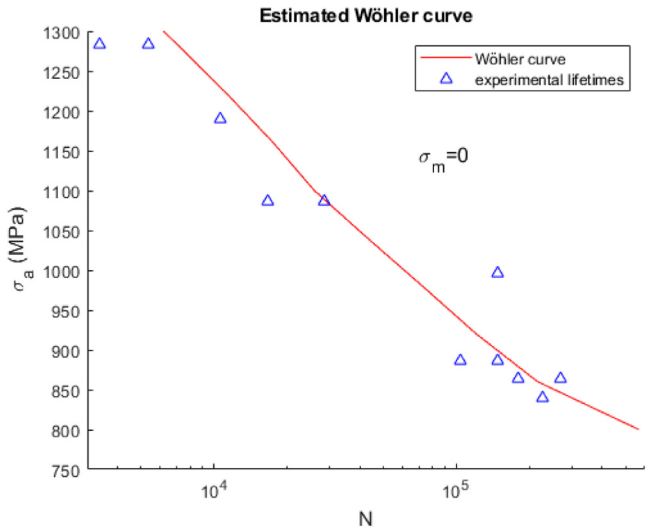


**Fig. 5.** Wöhler curve of 20Mn steel.

The reader should notice that the preceding algorithm produces only one realization from a distribution, which is approximately normal. Moreover, this normal distribution includes unknown parameters $\theta^\star$ and $\sigma^2 B^{-1}$. If we know the distributions of lifetimes, we may simulate the distribution of $\widehat{\theta}$ as follows.

(a) Choose a set of lifetimes $\mathcal{N}_i$, $i = 1, \dots, m$, and parameters $A$ and $\sigma_{-1}$ from their joint distribution.
(b) Compute the corresponding $\widehat{\theta}$.
(c) Repeat (a) and (b) $q$ times. This gives a sample $\{\widehat{\theta}_j\}_{j=1}^q$.
(d) Estimate the parameters of the normal distribution from the sample.

## 6. Conclusions

In this paper, we have discussed the statistical properties of the HCF model of Ottosen et al. [1]. We have studied the statistical properties of Wöhler curves. First, we associated a lifetime distribution for each amplitude. Then, we studied the properties of fatigue limit, both the constant limit and the random limit. For the random limit, we proposed a modified fatigue limit model.

Next, we showed that the parameters of the continuum-based fatigue model are asymptotically normally distributed and gave the parameters of the distributions. The topic was elucidated by examples. Finally, we discussed how the presented theory can be used to simulate those distributions.

Except lifetimes also stress history may contain random variation. In [23] we discuss the stochastic modeling of stress history.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## References

[1] N. Ottosen, R. Stenström, M. Ristinmaa, Continuum approach to high-cycle fatigue modeling, Int. J. Fatigue 30 (6) (2008) 996–1006.
[2] W. Nelson, Accelerated Testing, Wiley, 2004.
[3] D.S. Paolino, A. Tridello, H.S. Geng, G. Chiandussi, M. Rossetto, Dublex S-N fatigue curves: statistical distribution of the transition fatigue life, Frattura ed Integrità Strutturale 30 (2014) 417–423.
[4] R. Rabb, Todennäköisyysteoriaan Pohjautuva Väsymisanalyysi, Books on Demand, 2013, in Finnish.
[5] T.F. Coleman, Y. Li, An interior trust region approach for nonlinear minimization subject to bounds, SIAM J. Optim. 6 (1996) 418–445.
[6] R. Fouchereau, G. Celeux, P. Pamphile, Probabilistic modeling of S-N curves, Int. J. Fatigue 68 (2014) 217–223.
[7] L. Råde, B. Westergren, Mathematics Handbook for Science and Engineering, Studentlitteratur, 1995.
[8] W.J. Dixon, A.M. Mood, A method for obtaining and analyzing sensitivity data, J. Amer. Statist. Assoc. 43 (1948) 109–126.
[9] T.S. Ferguson, A Course in Large Sample Theory, Springer, 1996.
[10] S.D. Silvey, Statistical Inference, Chapman and Hall, 1983.
[11] J. Peterson, Lecture Notes on Random Walks in Random Environments, https://www.math.purdue.edu/peterson/Teaching. (Accessed 5 July 2019).
[12] O. Zeitouni, Random walks in random environments, in: Lecture Notes in Mathematics, 1837, Springer, 2004, pp. 189–312.
[13] F.G. Pascual, W.Q. Meeker, Estimating fatigue curves with the random fatigue-limit model, Technometrics 41 (4) (1999) 277–289.
[14] W.Q. Meeker, L.A. Escobar, Statistical Methods for Reliability Data, Wiley, 1998.
[15] R.D. Pollak, A.N. Palazotto, A comparison of maximum likelihood models for fatigue strength characterization in materials exhibiting a fatigue limit, Probabilistic Engineering Mechanics 24 (2009) 236–241.
[16] K.V. Mardia, J.T. Kent, J.M. Bibby, Multivariate Analysis, Academic Press, 1979.
[17] K. Shiozawa, N.Z. Gakkai, Databook on Fatigue Strength of Metallic Materials, Vol. 1, Elsevier, 1996.
[18] K. Burkart, H. Bomas, H.-W. Zoch, Fatigue of notched case-hardened specimens of steel SAE 5120 in the VHCF regime and application of the weakest-link concept, Int. J. Fatigue 33 (2011) 59–68.
[19] C. Dengo, G. Meneghetti, M. Dabalà, Experimental analysis of bending fatigue strength of plain and notched case-hardened gear steels, Int. J. Fatigue 80 (2015) 145–161.
[20] N.E. Dowling, Mechanical Behavior of Materials, Pearson, 2013.
[21] S. Suresh, Fatigue of Materials, Cambridge University Press, 1998.
[22] G.A.F. Seber, C.J. Wild, Nonlinear Regression, Wiley, 2003.
[23] O. Kaleva, H. Orelma, Modeling stress history as a stochastic process, Int. J. Fatigue 143 (2021) http://dx.doi.org/10.1016/j.ijfatigue.2020.105996.