The Structure of Robust Controllers for Distributed Parameter Systems

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***Abstract*— Using a very general formulation of the Internal Model Principle for infinite-dimensional systems it is shown that a robust controller tracking/rejecting signals generated by an infinite-dimensional exosystem can be decomposed into a servocompensator and a stabilizing controller. The servocompensator contains an internal model of the exosystem generating the reference and disturbance signals and the stabilizing controller stabilizes the infinite-dimensional closed-loop system. As such the decomposition gives a parametrization of robustly regulating controllers in the time domain. Various ways of stabilizing the closed-loop system are presented.**

***Index Terms*— Infinite-Dimensional Systems, Robust Regulation, Internal Model Principle**

I. Introduction

One of the main results of classical control theory of finite-dimensional linear systems is the Internal Model Principle (IMP) due to Francis and Wonham [1], and Davison [2], [3]. This principle asserts that any error feedback controller which achieves closed loop stability also achieves robust output regulation if and only if the controller contains a suitably duplicated model of the dynamics of the exosystem generating the reference and disturbance signals which the controller is required to track/reject.

The approach of Francis and Wonham is geometric in nature. Davison's approach is analytic and leads to a remarkably simple result showing that a robust controller can be divided into two parts: a servocompensator and a stabilizing controller. The servocompensator contains an internal model of the dynamics of the reference and disturbance signals in the form of a p-copy of the exosystem, where p is the dimension of the output space. The purpose of the stabilizing controller is to stabilize the extended system consisting of the servocompensator and the plant. In this paper this result is extended to infinite-dimensional systems.

We use a characterization of IMP based on the $\mathcal{G}$-conditions (Definition 2 below). Using the $\mathcal{G}$-conditions we show that if the reference and disturbance signals are generated by a infinite-dimensional exosystem, then the controller can be decomposed into a servocompensator and a stabilizing controller generalizing Davison's result to infinite-dimensional plants and exosystems. This also gives a new proof for the finite-dimensional case.

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Finally we show that stabilizing controllers of differing complexity can be found depending on the properties of the system to be regulated.

II. Notation

If X is a Hilbert space, then the inner product of $x, y \in X$ is denoted by $\langle x, y\rangle$. The set of bounded linear operators from the normed space X to the normed space Y is denoted by $\mathcal{L}(X,Y)$ and $\mathcal{L}(X) = \mathcal{L}(X,X)$. The domain, null space, spectrum, point spectrum, and the resolvent set of a linear operator A are denoted by $\mathcal{D}(A)$, $\mathcal{N}(A)$, $\sigma(A)$, $\sigma_p(A)$, and $\rho(A)$, respectively. The resolvent operator of A is $R(\lambda; A) = (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$. The C_0-semigroup $T_A(t)$ generated by A on a Hilbert space X is exponentially stable if there are positive constants M and α such that $\|T_A(t)\| \le Me^{-\alpha t}$ for $t \ge 0$, strongly stable if $T_A(t)x \to 0$ as $t \to \infty$ for every $x \in X$, and weakly stable if $\langle T(t)x, y\rangle \to 0$ as $t \to \infty$ for every $x, y \in X$. In this case we also say that A is exponentially stable, strongly stable, or weakly stable.

III. Preliminaries

A. The Exosystem

The reference and disturbance signals are assumed to be generated by the exosystem

$$\dot{v} = Sv, \qquad v(0) \in W, \tag{1}$$

where $S : \mathcal{D}(S) \subset W \to W$ is a generator of C_0-semigroup $T_S(t)$ on a Hilbert-space W. We assume that the spectrum of S is a pure point spectrum $\sigma_p(S) = \{\, \mathrm{i}\omega_k \mid k \in J \subset \mathbb{Z}\,\}$ with simple and distinct eigenvalues $\omega_k \ne \omega_l$ for $k \ne l$. If the index set J is infinite we also demand that $\inf_{k \ne l} |\omega_k - \omega_l| > 0$.

B. The Plant

The plant P is described by the equations

$$\dot{x} = Ax + Bu + F_s v, \qquad x(0) \in X \tag{2a}$$
$$y = Cx + Du + F_m v, \tag{2b}$$

where the state $x(t) \in X$, the input $u(t) \in U$, the output $y(t) \in Y$ and the signal $v(t) \in W$ is a solution of (1). The spaces X, U and Y are Hilbert spaces. The system operator $A : \mathcal{D}(A) \subset X \to X$ is the generator of a C_0-semigroup $T_A(t)$, all the other operators are bounded: $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$, $D \in \mathcal{L}(U,Y)$, $F_s \in \mathcal{L}(W,X)$, and $F_m \in \mathcal{L}(W,Y)$. We also assume that $\sigma_p(S) \subset \rho(A)$ and that the transfer function of the plant $P(s) = C(sI - A)^{-1}B + D \in \mathcal{L}(U,Y)$ is boundedly invertible for $s \in \sigma_P(S)$.

The reference signal $r : [0,\infty) \to Y$ is given by $r = F_r v$ where $F_r \in \mathcal{L}(W,Y)$. Combining the plant equations (2) and the tracking error $e = y - r = y - F_r v$ we can write the plant equations into the more convenient form:

$$\dot{x} = Ax + Bu + Ev, \qquad x(0) \in X \tag{3a}$$
$$e = Cx + Du + Fv, \tag{3b}$$

where $E = F_s$ and $F = F_m - F_r$.

C. The Controller

The controller is defined by the equations

$$\dot{z} = \mathcal{G}_1 z + \mathcal{G}_2 e, \qquad z(0) \in Z \tag{4a}$$
$$u = Kz, \tag{4b}$$

where $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a C_0-semigroup on the Hilbert space Z, $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$ and $K \in \mathcal{L}(Z,U)$.

D. The Closed-Loop System

Let $X_e = X \times Z$ be the extended state-space, consisting of the plant and controller states, and let $x_e(t) = (x(t), z(t)) \in X_e$ be the extended state. Combining the equations (3) and (4) we get the closed-loop system

$$\dot{x}_e = A_e x_e + B_e v, \qquad x_e(0) \in X_e \tag{5a}$$
$$e = C_e x_e + D_e v, \tag{5b}$$

where $C_e = [C \ DK] \in \mathcal{L}(X_e,Y)$, $D_e = F \in \mathcal{L}(W,Y)$, and $A_e : \mathcal{D}(A_e) = \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1) \subset X_e \to X_e$ and $B_e \in \mathcal{L}(W,X_e)$ are given by

$$A_e = \begin{bmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{bmatrix}, \qquad B_e = \begin{bmatrix} E \\ \mathcal{G}_2 F \end{bmatrix}.$$

E. The Output Regulation Problem

Definition 1: The Output Regulation Problem (ORP) is defined as follows: Design a controller (4) such that

1) The closed-loop system operator A_e generates a stable C_0-semigroup.
2) For all initial states $x_e(0) \in X_e$ and $v(0) \in W$ we have $\lim_{t\to\infty} e(t) = 0$.

In The Robust Output Regulation Problem (RORP) it is required in addition that the controller (4) solves the ORP even if in (3) the operators A, B, C, D, E, and F are perturbed to $A + \Delta_A$, $B + \Delta_B$, $C + \Delta_C$, $D + \Delta_D$, $E + \Delta_E$ and $F + \Delta_F$, respectively, in such a way that the closed-loop system remains stable.

IV. Previous Results

The authors have previously proved the following theorem.

Theorem 1: If A_e generates a strongly/weakly stable C_0-semigroup and there exists an operator $H_{\mathrm{ss}} \in \mathcal{L}(W,X_e)$ which satisfies $H_{\mathrm{ss}}\mathcal{D}(S) \subset \mathcal{D}(A_e)$ and the constrained Sylvester equations

$$H_{\mathrm{ss}} S - A_e H_{\mathrm{ss}} = B_e, \text{ on } \mathcal{D}(S) \tag{6a}$$
$$C_e H_{\mathrm{ss}} + D_e = 0, \tag{6b}$$

then the controller (4) solves the ORP.

Proof: See [4]. ■

If S is a finite-dimensional operator, then exponential stability of A_e is sufficient to guarantee that (6a) has a solution. For an infinite-dimensional operator S necessary and sufficient conditions for the solvability of (6a) are given in [4].

A solution of (6a) does not necessarily satisfy (6b). For this we need the following definition.

Definition 2: The controller $(\mathcal{G}_1, \mathcal{G}_2)$ satisfies the $\mathcal{G}$-conditions if

$$\mathcal{N}(\mathcal{G}_2) = \{0\}, \tag{7a}$$
$$\mathcal{R}(\mathcal{G}_2) \cap \mathcal{R}(sI - \mathcal{G}_1) = \{0\}, \qquad \text{for all } s \in \sigma_p(S). \tag{7b}$$

Now we have the following theorem [4], [5], [6]. The proofs in the last reference are also valid for unbounded control and observation.

Theorem 2: Assume that the controller (4) satisfies the $\mathcal{G}$-conditions. If A_e generates a strongly/weakly stable C_0-semigroup and there exists an operator $H_{\mathrm{ss}} \in \mathcal{L}(W,X_e)$ satisfying $H_{\mathrm{ss}}\mathcal{D}(S) \subset \mathcal{D}(A_e)$ and the Sylvester equation (6a), then the controller solves the RORP.

V. Main Results

A. The Structure of the controller

The following theorem has been proved in [5] under the assumption $\sigma_p(S) \cap \sigma(A_e) = \emptyset$. We give here a proof with the less restrictive assumptions $\sigma_p(S) \cap \sigma_p(A_e) = \emptyset$ and (8) below.

Theorem 3: If $\sigma_p(S) \cap \sigma_p(A_e) = \emptyset$, then the following are equivalent for $s \in \sigma_p(S)$.

1) The operators $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the $\mathcal{G}$-conditions (7) and the inclusion

$$\{0\} \times \mathcal{R}(\mathcal{G}_2) \subset \mathcal{R}(sI - A_e). \tag{8}$$

2) The restriction $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)} : \mathcal{N}(sI - \mathcal{G}_1) \to Y$ is a bijection.

Proof: Throughout the proof let $s \in \sigma_p(S)$. Then $s \notin \sigma_p(A_e)$ and the operator $sI - A_e$ is injective.

1) $\Longrightarrow$ 2).

First we show that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is an injection. Suppose that $z \in \mathcal{N}(sI - \mathcal{G}_1)$ satisfies $P(s)Kz = 0$. Let $x = R(s;A)BKz \in \mathcal{D}(A)$. Then

$$\begin{aligned}(sI - A_e)\begin{bmatrix} x \\ z \end{bmatrix} &= \begin{bmatrix} sI - A & -BK \\ -\mathcal{G}_2 C & sI - \mathcal{G}_1 - \mathcal{G}_2 DK \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \\ &= \begin{bmatrix} BKz - BKz \\ -\mathcal{G}_2 P(s)Kz + (sI - \mathcal{G}_1)z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.\end{aligned} \tag{9}$$

Since $sI - A_e$ is injective we have $z = 0$. Therefore $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is an injection.

Next we show that $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is a surjection. Let $y \in Y$ be arbitrary. Since $(0, -\mathcal{G}_2 y) \in \{0\} \times \mathcal{R}(\mathcal{G}_2)$, it follows from (8) that there is a $(x,z) \in \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1)$ such that

$$\begin{bmatrix} 0 \\ -\mathcal{G}_2 y \end{bmatrix} = (sI - A_e) \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} (sI - A)x - BKz \\ -\mathcal{G}_2(Cx + DKz) + (sI - \mathcal{G}_1)z \end{bmatrix}.$$

We get from the first equation $x = R(s;A)BKz$ and substituting this into the second equation gives

$$-\mathcal{G}_2 y = -\mathcal{G}_2 P(s)Kz + (sI - \mathcal{G}_1)z \iff \mathcal{G}_2(P(s)Kz - y) = (sI - \mathcal{G}_1)z.$$

Since $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the condition (7b) we must have $(sI - \mathcal{G}_1)z = 0$ and $\mathcal{G}_2(P(s)Kz - y) = 0$, and furthermore $P(s)Kz - y = 0$, since $\mathcal{G}_2$ satisfies (7a). Hence $z \in \mathcal{N}(sI - \mathcal{G}_1)$ and $y = P(s)Kz$, so $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is a surjection.

2) $\implies$ 1).

First we prove (7a). Let $y \in \mathcal{N}(\mathcal{G}_2)$. Since $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is a surjection, there is a $z \in \mathcal{N}(sI - \mathcal{G}_1)$ such that $y = P(s)Kz$. Then $\mathcal{G}_2 P(s)Kz = 0$. Choosing $x = R(s;A)BKz \in \mathcal{D}(A)$ we get as in (9) that $(sI - A_e)(x,z) = (0,0)$. Since $sI - A_e$ is injective we have $z = 0$ and therefore $y = 0$.

Next we prove (7b). Let $v \in \mathcal{R}(\mathcal{G}_2) \cap \mathcal{R}(sI - \mathcal{G}_1)$. Then there are $y \in Y$ and $z \in \mathcal{D}(\mathcal{G}_1)$ such that $v = \mathcal{G}_2 y = (sI - \mathcal{G}_1)z$. First we show that there is a $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that $v = \mathcal{G}_2 P(s)Kz_1 = (sI - \mathcal{G}_1)z_1$. Since $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is a surjection, there is a $z_0 \in \mathcal{N}(sI - \mathcal{G}_1)$ such that $P(s)Kz_0 = y - P(s)Kz$. Then $y = P(s)K(z + z_0)$ and since $(sI - \mathcal{G}_1)z = (sI - \mathcal{G}_1)(z + z_0)$, we can choose $z_1 = z + z_0$. Now choosing $x_1 = R(s;A)BKz_1 \in \mathcal{D}(A)$ we get as in (9) that $(sI - A_e)(x_1, z_1) = (0,0)$. Since $sI - A_e$ is injective we have $z_1 = 0$ and therefore $v = 0$.

Finally we prove (8). Let $(0,v) \in \{0\} \times \mathcal{R}(\mathcal{G}_2)$ be arbitrary. Then there is an $y \in Y$ such that $v = \mathcal{G}_2 y$. Since $(P(s)K)|_{\mathcal{N}(sI-\mathcal{G}_1)}$ is a surjection, there is a $z \in \mathcal{N}(sI - \mathcal{G}_1)$ such that $P(s)Kz = -y$. Let $x = R(s;A)BKz \in \mathcal{D}(A)$. Then

$$(sI - A_e) \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} (sI - A)x - BKz \\ -\mathcal{G}_2(Cx + DKz) + (sI - \mathcal{G}_1)z \end{bmatrix} = \begin{bmatrix} BKz - BKz \\ -\mathcal{G}_2(P(s)Kz) \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathcal{G}_2(-y) \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix}.$$

Therefore $(0,v) \in \mathcal{R}(sI - A_e)$ and the proof is complete. ■

It follows from Theorem 3 that under the stated conditions $\mathcal{N}(sI - \mathcal{G}_1)$ is isomorphic to Y, and since Y is nontrivial, we must have $\mathcal{N}(sI - \mathcal{G}_1) \neq \{0\}$ and hence $\sigma_p(S) \subset \sigma_p(\mathcal{G}_1)$. Therefore the controller must contain a copy of the dynamics of the exosystem S and thus satisfies the Internal Model Principle. In particular, if $\dim Y = p$, then $\dim \mathcal{N}(sI - \mathcal{G}_1) = p$ and every eigenvalue of S must have multiplicity at least p as an eigenvalue of $\mathcal{G}_1$. Hence $\mathcal{G}_1$ contains a p-copy of S. Note that this also holds for infinite-dimensional exosystems.

In the following we assume that $\dim Y = p$. The next theorem shows that with a suitable decomposition of the controller state space Z the operator $\mathcal{G}_1$ can be written in a lower triangular form.

Theorem 4: Assume that the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the $\mathcal{G}$-conditions (7) and the equation (8). The space Z can be decomposed into a direct sum $Z = Z_1 \dot{+} Z_2$ so that $\mathcal{G}_1$ can be represented by a lower triangular matrix

$$\begin{bmatrix} G_1 & 0 \\ R_1 & R_2 \end{bmatrix},$$

where $\sigma_p(S) \subset \sigma_p(G_1)$ and each eigenvalue $\mathrm{i}\omega_k \in \sigma_p(G_1)$ has multiplicity p.

Proof: It follows from Theorem 3 that $\mathcal{N}(sI - \mathcal{G}_1)$ is isomorphic to Y for $s \in \sigma_p(S)$. Hence s must be an eigenvalue of $\mathcal{G}_1$ with multiplicity p and the eigenspaces $\mathcal{N}(sI - \mathcal{G}_1)$ have dimension p. Since the eigenspaces are finite-dimensional, $-\mathrm{i}\omega_k$ is an eigenvalue of ${\mathcal{G}_1}^*$ with multiplicity p. Write Z as a direct sum $Z = Z_1 \dot{+} Z_2$ where $Z_1 = \overline{Z}_0$,

$$Z_0 = \sum_{k=-\infty}^{\infty} \mathcal{N}(-\mathrm{i}\omega_k I - {\mathcal{G}_1}^*), \tag{10}$$

and Z_2 is any closed complementary subspace of Z_1.

Next we show that Z_1 is an invariant subspace of ${\mathcal{G}_1}^*$. Let $T_1(\cdot)$ be the semigroup generated by ${\mathcal{G}_1}^*$ and let $t > 0$ be fixed. The subspaces $\mathcal{N}(-\mathrm{i}\omega_k I - {\mathcal{G}_1}^*) \subset \mathcal{D}({\mathcal{G}_1}^*)$ are closed invariant subspaces of ${\mathcal{G}_1}^*$, so it follows from Lemma 2.5.4 of [7] that they are also $T_1(t)$-invariant. Now let $\varepsilon > 0$ and $x \in Z_1$ be arbitrary. Then there is an $y \in Z_0$ such that $\|x - y\| < \varepsilon / \|T_1(t)\|$ and y has the representation

$$y = \sum_{k=-\infty}^{\infty} y_k,$$

where $y_k \in \mathcal{N}(-\mathrm{i}\omega_k I - {\mathcal{G}_1}^*)$ and the series converges unconditionally. Now the boundedness of $T_1(t)$ implies that

$$T_1(t)y = \sum_{k=-\infty}^{\infty} T_1(t)y_k,$$

and the series converges unconditionally. Since $T_1(t)y_k \in \mathcal{N}(-\mathrm{i}\omega_k I - {\mathcal{G}_1}^*)$ we have $T_1(t)y \in Z_0$ and because $\|T_1(t)x - T_1(t)y\| \leq \|T_1(t)\|\|x - y\| < \varepsilon$, we have $T_1(t)x \in \overline{Z}_0 = Z_1$. Therefore Z_1 is $T_1(t)$-invariant, and by Lemma 2.5.3 of [7] it is also ${\mathcal{G}_1}^*$-invariant.

Since Z_1 is ${\mathcal{G}_1}^*$-invariant, the operator ${\mathcal{G}_1}^*$ can be represented as

$${\mathcal{G}_1}^* = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix},$$

where

$$G_{11} = {\mathcal{G}_1}^*|_{Z_1}, G_{12} = P_1 {\mathcal{G}_1}^*|_{Z_2}, G_{22} = P_2 {\mathcal{G}_1}^*|_{Z_2},$$

and P_1 and P_2 are projections onto Z_1 and Z_2, respectively. Clearly

$$\mathcal{G}_1{}^*|_{\mathcal{N}(-\mathrm{i}\omega_k I-\mathcal{G}_1{}^*)} = -\mathrm{i}\omega_k I,$$

where I is a $p\times p$ identity matrix, so $-\mathrm{i}\omega_k$ is an eigenvalue of G_{11} with multiplicity p. Therefore

$$\mathcal{G}_1 = \begin{bmatrix} G_{11}^* & 0 \\ G_{12}^* & G_{22}^* \end{bmatrix},$$

which is of the claimed form. Moreover, $\mathrm{i}\omega_k$ is an eigenvalue of G_{11}^* with multiplicity p.

If the exosystem is finite-dimensional, the series (10) reduces to a finite sum and we can take $Z_1 = Z_0$. ■

Using the decomposition of Z in Theorem 4 we can write $\mathcal{G}_2$ in the form

$$\mathcal{G}_2 = \begin{bmatrix} G_2 \\ R_3 \end{bmatrix},$$

where $G_2 = P_1\mathcal{G}_2$ and $R_3 = P_2\mathcal{G}_2$ and $K = [K_1\ K_2]$ where $K_1 = KP_1$ and $K_2 = KP_2$. Here P_1 and P_2 are the projections defined in the proof of Theorem 4. Then the controller parameters take the form

$$\mathcal{G}_1 = \begin{bmatrix} G_1 & 0 \\ R_1 & R_2 \end{bmatrix}, \quad \mathcal{G}_2 = \begin{bmatrix} G_2 \\ R_3 \end{bmatrix}, \quad K = [K_1\ K_2],$$

which is the same controller stucture as given by Davison in [3]. The parameters (G_1, G_2) define the servocompensator on the state space Z_1

$$\dot{z}_1 = G_1 z_1 + G_2 e, \tag{11a}$$

which contains an internal model of the exosystem S. The parameters $(R_1, R_2, R_3, K_1, K_2)$ define a stabilizing controller on the state space Z_2

$$\dot{z}_2 = R_1 z_1 + R_2 z_2 + R_3 e \tag{11b}$$

$$u = K_1 z_1 + K_2 z_2. \tag{11c}$$

This decomposition of the controller into two parts is very useful for design purposes, since it allows us to design the servocompensator and the stabilizing controller independently of one another.

B. The Stabilizing Controller

Next we show that any choice of the parameters R_k and K_k of the stabilizing controller (11), which stabilize the augmented system consisting of the plant and the servocompensator will also stabilize our original closed-loop system (5). Therefore an alternative view of the stabilizing controller is that its purpose is to stabilize this augmented system.

The augmented system with state space $X_s = X \times Z_1$, output space $Y_s = Y \times Z_1$, state $x_s = (x, z_1) \in X_s$, and output $y_s = (e, z_1) \in Y_s$ is described by the equations

$$\dot{x}_s = A_s x_s + B_s u + E_s v, \tag{12a}$$

$$y_s = C_s x_s + D_s u + F_s v, \tag{12b}$$

where $A_s : \mathcal{D}(A_s) = \mathcal{D}(A) \times Z_1 \subset X_s \to X_s$

$$A_s = \begin{bmatrix} A & 0 \\ G_2 C & G_1 \end{bmatrix}, \quad B_s = \begin{bmatrix} B \\ G_2 D \end{bmatrix}, \quad E_s = \begin{bmatrix} E \\ G_2 F \end{bmatrix}.$$

and

$$C_s = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \quad D_s = \begin{bmatrix} D \\ 0 \end{bmatrix}, \quad F_s = \begin{bmatrix} F \\ 0 \end{bmatrix}.$$

Now suppose that the parameters of the controller (11) are selected so that it stabilizes the system (12). Then the closed-loop system consisting of (11) and (12) has the state space $X_e = X_s \times Z_2 = X \times Z_1 \times Z_2$ and the extended state $x_e = (x_s, z_2) = (x, z_1, z_2) \in X_e$ satisfies

$$\dot{x}_e = \begin{bmatrix} A_s & 0 \\ \tilde{R}C_s & R_2 \end{bmatrix} x_e + \begin{bmatrix} B_s \\ \tilde{R}D_s \end{bmatrix} u + \begin{bmatrix} E_s \\ \tilde{R}F_s \end{bmatrix} v, \tag{13}$$

where $\tilde{R} = [R_3\ R_1]$. Then $\tilde{R}D_s = R_3 D$, $\tilde{R}C_s = [R_3 C\ R_1]$, and $\tilde{R}F_s = R_3 F$. Substituting these into (13) we get

$$\dot{x}_e = \begin{bmatrix} A & 0 & 0 \\ G_2 C & G_1 & 0 \\ R_3 C & R_1 & R_2 \end{bmatrix} x_e + \begin{bmatrix} B \\ G_2 D \\ R_3 D \end{bmatrix} u + \begin{bmatrix} E \\ G_2 F \\ R_3 F \end{bmatrix} v. \tag{14}$$

The control signal u is given by

$$u = K_2 z_2 + K_1 z_1 = K_2 z_2 + [0\ K_1] y_s = [0\ K_1\ K_2] x_e.$$

C. Adding State Feedback to the Controller

Looking at the form of the control signal u, we notice that we can generalize the controller by redefining u as $u = [K_0\ K_1\ K_2] x_e$ and thus allowing state feedback. This gives the closed loop system operator

$$A_e = \begin{bmatrix} A & 0 & 0 \\ G_2 C & G_1 & 0 \\ R_3 C & R_1 & R_2 \end{bmatrix} + \begin{bmatrix} B \\ G_2 D \\ R_3 D \end{bmatrix} \begin{bmatrix} K_0 & K_1 & K_2 \end{bmatrix}$$

$$= \begin{bmatrix} A + BK_0 & BK_1 & BK_2 \\ G_2(C + DK_0) & G_1 + G_2 D K_1 & G_2 D K_2 \\ R_3(C + DK_0) & R_1 + R_3 D K_1 & R_2 + R_3 D K_2 \end{bmatrix}. \tag{15}$$

The original controller (4) did not have a state feedback term, but we can easily add one by redefining (4b) as $u = Kz + K_0 x$. Now it is easy to see that applying the controller (11) to the plant (3) results in the closed-loop system operator (14) (without state feedback) or (15) (with state feedback), which proves our claim that a stabilizing controller stabilizing the augmented system also stabilizes the original system.

Clearly A_e in (15) is otherwise the same as in (14), except A and C are replaced with $A_K = A + BK_0$ and $C_K = C + DK_0$, repsectively. It is well known that for $s \in \rho(A) \cap \rho(A_K)$ the transfer function $P(s$ is invertible iff $P_K(s) = C_K(sI - A_K)^{-1}B + D$ is invertible. Therefore our basic assumption on the invertibility of $P(s)$ on $\sigma_p(S)$ also holds for $P_K(s)$, provided that K_0 is chosen so that $\sigma_p(S) \subset \rho(A_K)$. Also, no essential changes are needed to the results in Section V-A.

Next we investigate various possibilities of stabilizing the closed-loop system with increasing controller complexity. In subsections V-C.1 and V-C.2 below we examine the case where the stabilizing controller is not needed. Then $X_e = X_s$ and we drop the last row and column of A_e and K_2 from u to get the closed loop operator

$$A_{e1} = \begin{bmatrix} A + BK_0 & BK_1 \\ G_2(C + DK_0) & G_1 + G_2DK_1 \end{bmatrix}. \qquad (16)$$

1) Stabilization for a Stable Plant: Suppose that A is exponentially stable and the exosystem is finite-dimensional. Then we have shown in [8] that we do not need state feedback or the stabilizing controller, so we can set $K_0 = 0$ in (16) and the resulting A_{e1} can be stabilized with feedback only from the servocompensator.

2) Stabilization with State Feedback: Next assume that state feedback is possible and the stabilizing controller is not needed, so A_{e1} is given by (16). Set $K_0 = K_{01} + K_{02}$, $C_K = C + DK_{01}$, $A_K = A + BK_{01}$, and let $H \in \mathcal{L}(X, Z_1)$ be the solution of the Sylvester equation

$$\begin{aligned} -G_1H + G_2C_K + HA_K &= 0 \\ \iff G_1H - HA_K &= G_2C_K, \end{aligned}$$

and set $K_{02} = K_1H$. Then A_{e1} can be written as

$$A_{e1} = \begin{bmatrix} A_K + BK_1H & BK_1 \\ G_2(C_K + DK_1H) & G_1 + G_2DK_1 \end{bmatrix}.$$

Applying to A_{e1} the similarity transformation

$$T_H = \begin{bmatrix} I & 0 \\ H & I \end{bmatrix}, \qquad T_H^{-1} = \begin{bmatrix} I & 0 \\ -H & I \end{bmatrix},$$

we get

$$A_{e2} = T_HA_{e1}T_H^{-1} = \begin{bmatrix} A_K & BK_1 \\ 0 & G_1 + B_1K_1 \end{bmatrix},$$

where $B_1 = G_2D + HB$. This is exactly the same operator as $T_H\tilde{A}_{e1}T_H^{-1}$ in [4, p. 15], where it was shown that it can be strongly or weakly stabilized if A is exponentially stabilizable and G_1 can be strongly or weakly stabilized.

3) Stabilization with a Reduced Order Observer: Next assume that state feedback is not available, so we set $K_0 = 0$. We show that taking the stabilizing controller to be an observer for the plant state x, we can stabilize the closed loop system. Let $Z_2 = X$ and consider the observer

$$\begin{aligned} \dot{z}_2 &= Az_2 + Bu + L(\hat{y} - e), \\ \hat{y} &= Cz_2 + Du. \end{aligned}$$

Using $u = K_1z_1 + K_2z_2$ we get

$$\begin{aligned} \dot{z}_2 &= Az_2 + Bu + LCz_2 + LDu - Le \\ &= (A + LC)z_2 + (B + LD)u - Le \\ &= (A + LC + (B + LD)K_2)z_2 + (B + LD)K_1z_1 - Le \end{aligned}$$

Comparing this to (11b) we see that $R_3 = -L$ and

$$R_1 = (B + LD)K_1, \qquad R_2 = A + LC + (B + LD)K_2.$$

Then

$$\begin{aligned} R_1 + R_3DK_1 &= (B + LD)K_1 - LDK_1 = BK_1, \\ R_2 + R_3DK_2 &= A + LC + (B + LD)K_2 - LDK_2 \\ &= A + LC + BK_2, \end{aligned}$$

and the system operator A_e becomes

$$A_e = \begin{bmatrix} A & BK_1 & BK_2 \\ G_2C & G_1 + G_2DK_1 & G_2DK_2 \\ -LC & BK_1 & A + LC + BK_2 \end{bmatrix}.$$

Now applying to A_e the similarity transformation

$$T_1 = \begin{bmatrix} I & 0 & -I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad T_1^{-1} = \begin{bmatrix} I & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

we get

$$\begin{aligned} \tilde{A}_e &= T_1A_eT_1^{-1} \\ &= \begin{bmatrix} A + LC & 0 & 0 \\ G_2C & G_1 + G_2DK_1 & G_2(C + DK_2) \\ -LC & BK_1 & A + BK_2 \end{bmatrix} \\ &= \begin{bmatrix} A + LC & 0 \\ \mathcal{G}_2C & \tilde{A}_{e1} \end{bmatrix}, \end{aligned}$$

where

$$\tilde{A}_{e1} = \begin{bmatrix} G_1 + G_2DK_1 & G_2(C + DK_2) \\ BK_1 & A + BK_2 \end{bmatrix}.$$

But we have $\tilde{A}_{e1} = JA_{e1}J^{-1}$ with K_0 replaced by K_2 and

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Hence $\tilde{A}_{e1}$ can be strongly or weakly stabilized. Therefore, if A is also exponentially detectable, then $\tilde{A}_e$, and hence A_e, can be strongly or weakly stabilized [4].

This is a reduced order observer, since we do not need to observe the servocompensator state, only the plant state.

VI. Conclusion

In this paper we have given a new proof which shows that a robust controller can be decomposed into a servocompensator and a stabilizing controller. The servocompensator contains a p times duplicated internal model of the exosystem generating the reference and disturbance signals, where p is the dimension of the output space. The purpose of the stabilizing controller is to stabilize the system consisting of the plant and the servocompensator.

It is shown that the decomposition of the controller into the servocompensator and the stabilizing compensator allows controllers of varying complexity depending on the properties of the plant.

The case of unbounded control and observation will be dealt with in a future paper using the methods presented in [6].

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